FIRST ORDER EQUATION ON RANDOM MEASURES AS SUPERPOSITION OF WEAK SOLUTIONS TO THE MCKEAN-VLASOV EQUATION

ALESSANDRO PINZI

Abstract. The goal of this paper is to define an evolution equation for a curve of random probability measures $(M_t)_{t\in[0,T]} \subset \mathcal{P}(\mathcal{P}(\mathbb{R}^d))$ associated to a non-local drift $b : [0,T] \times \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}^d$ and a non-local diffusion term $a : [0,T] \times \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \to \mathrm{Sym}_+(\mathbb{R}^{d\times d})$. Then, we show that any solution to such an equation on random measures can be lifted twice: to a superposition of solutions to a non-linear Kolmogorov-Fokker-Planck equation and to a superposition of weak solutions to the McKean-Vlasov equations. Finally, we exploit this nested superposition result to show how existence and uniqueness can be transferred from the equation on random measures to the associated non-linear Kolmogorov-Fokker-Planck equation and to the McKean-Vlasov equation, assuming uniqueness of the linearized version of KFP. As a tool, we will introduce integral metrics over the spaces of probability measures $\mathcal{P}(\mathbb{R}^d)$ and $\mathcal{P}(\mathcal{P}(\mathbb{R}^d))$ in duality with smooth functions.

Keywords: random measure, superposition principle, McKean-Vlasov, Kolmogorov-Fokker-Planck, integral metrics.
2020 MSC: 60G57, 35R15, 35Q84, 60H15.

1. Introduction

The superposition principle plays an important role in many evolution problems: its first version was proved by L. Ambrosio (see [AGS08, Theorem 8.2.1]), and relates the Eulerian (continuity equation) and the Lagrangian description (system of ODEs) of the flow led by a vector field $v : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$. It was introduced to study the well-posedness of ODEs (in a selection sense) under non-smooth assumptions on the vector field [Amb04; AC08], extending the celebrated work by R. Di Perna and P.L. Lions [DL89]. It is a useful tool in many other contexts (e.g. optimal transport), and for this reason, it has been extended to more abstract spaces (see e.g. [ST17]).

Let us recall the following two versions of the superposition principle:

(A) the *stochastic superposition principle* proved in [Fig08; Tre16; BRS21], that in the spirit of the Ambrosio's superposition principle, relates weak solutions of a stochastic differential to the associated linear Kolmogorov-Fokker-Planck equation (see also §2.3);

(B) the *nested superposition principle* proved in [PS25a], where the authors defined an abstract continuity equation for a curve of random measures $(M_t)_{t\in[0,T]} \subset \mathcal{P}(\mathcal{P}(\mathbb{R}^d))$ that can be written either as superposition of solutions to a non-local continuity equation $\partial_t\mu_t + \mathrm{div}(b_t(\cdot,\mu_t)\mu_t) = 0$ or as superposition of solutions to an interacting particle systems $dX_t = b_t(X_t, \mathrm{Law}(X_t))dt$.

Bocconi University, Department of Decision Sciences, via Roentgen 1, 20136 Milano (Italy). Email: alessandro.pinzi@phd.unibocconi.it. *Orcid*: https://orcid.org/0009-0007-9146-5434.

In this paper, we want to prove a new nested superposition principle for random measures (which in particular is an extension of (B)) that relies on the stochastic superposition principle (A). To be more specific, consider a non-local vector field $b : [0,T] \times \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}^d$ and a non-local diffusion term $\sigma : [0,T] \times \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}^{d\times m}$, and define $a := \sigma\sigma^\top$. The term non-local highlights their dependence on the measure variable. For any $\mu \in \mathcal{P}(\mathbb{R}^d)$, define the operator

$$L^{\mu}_{b_t,a_t}\phi(x) := \sum_{i=1}^{d} b^i_t(x,\mu)\cdot \partial_i\phi(x) + \frac{1}{2}\sum_{i,j=1}^{d} a^{i,j}_t(x,\mu)\partial_{i,j}\phi(x),$$

for all $\phi \in C^2_b(\mathbb{R}^d)$. The stochastic superposition principle shows there is a correspondence (possibly non 1-1) between curves of probability measures $(\mu_t)_{t\in[0,T]} \subset \mathcal{P}(\mathbb{R}^d)$ that solve (in a distributional sense) the non-linear Kolmogorov-Fokker-Planck equation

$$\partial_t\mu_t = (L^{\mu_t}_{b_t,a_t})^*\mu_t, \tag{1.1}$$

and weak solutions to the McKean-Vlasov equation (or equivalently solutions of the associated non-linear martingale problem [SV06])

$$dX_t = b_t(X_t, \mathrm{Law}(X_t))dt + \sigma_t(X_t, \mathrm{Law}(X_t))dW_t. \tag{1.2}$$

We will introduce an evolution equation over random measures $\mathcal{P}(\mathcal{P}(\mathbb{R}^d))$ that can be seen either as a superposition of solutions to (1.1) or as a superposition of weak solutions to (1.2). To this aim, we define an operator $\mathcal{K}_{b_t,a_t}$ acting on cylinder functions $\mathrm{Cyl}^{1,2}_b(\mathcal{P}(\mathbb{R}^d))$ (see Definition 3.4, they will play the role of smooth test functions), so that the equation $\partial_t M_t = \mathcal{K}^*_{b_t,a_t}M_t$ is well defined (see Definition 4.1), where $(M_t)_{t\in[0,T]} \subset \mathcal{P}(\mathcal{P}(\mathbb{R}^d))$. Then, the main result of this paper can be summarized as follows.

Theorem 1.1. *Let $b : [0,T]\times\mathbb{R}^d\times\mathcal{P}(\mathbb{R}^d) \to \mathbb{R}^d$ and $a : [0,T]\times\mathbb{R}^d\times\mathcal{P}(\mathbb{R}^d) \to \mathrm{Sym}_+(\mathbb{R}^{d\times d})$ be Borel measurable maps. Let $\boldsymbol{M} = (M_t)_{t\in[0,T]} \in C([0,T], \mathcal{P}(\mathcal{P}(\mathbb{R}^d)))$ be such that*

$$\partial_t M_t = \mathcal{K}^*_{b_t,a_t}M_t \quad \text{and} \quad \int_0^T\int_{\mathcal{P}(\mathbb{R}^d)}\int_{\mathbb{R}^d} \frac{|b_t(x,\mu)|}{1+|x|} + \frac{|a_t(x,\mu)|}{1+|x|^2}d\mu(x)dM_t(\mu)dt < +\infty. \tag{1.3}$$

Then, there exist $\Lambda \in \mathcal{P}(C([0,T],\mathcal{P}(\mathbb{R}^d)))$ and $\mathfrak{L} \in \mathcal{P}(\mathcal{P}(C([0,T],\mathbb{R}^d)))$ liftings of $\boldsymbol{M}$, in the sense that $(\mathfrak{e}_t)_\sharp\Lambda = M_t$ and $(E_t)_\sharp\mathfrak{L} = M_t$ (see (4.16)) for all $t\in[0,T]$ satisfying:

(1) Λ-a.e. $\boldsymbol{\mu} = (\mu_t)_{t\in[0,T]} \in C([0,T],\mathcal{P}(\mathbb{R}^d))$ solves (1.1) and it satisfies

$$\int\int_0^T\int_{\mathbb{R}^d}\frac{|b_t(x,\mu_t)|}{1+|x|} + \frac{|a_t(x,\mu_t)|}{1+|x|^2}d\mu_t(x)dtd\Lambda(\boldsymbol{\mu}) < +\infty; \tag{1.4}$$

(2) $\mathfrak{L}$-a.e. $\lambda \in \mathcal{P}(C_T(\mathbb{R}^d))$ is a martingale solution of (1.2) (see Definition 2.5) and it satisfies

$$\int\int\int_0^T \frac{|b_t(\gamma_t,(\mathrm{e}_t)_\sharp\lambda)|}{1+|\gamma_t|} + \frac{|a_t(\gamma_t,(\mathrm{e}_t)_\sharp\lambda)|}{1+|\gamma_t|^2}dtd\lambda(\gamma)d\mathfrak{L}(\lambda) < +\infty; \tag{1.5}$$

(3) $E_\sharp\mathfrak{L} = \Lambda$ (see (4.16)) and there exists a map $G_{b,a} : C([0,T],\mathcal{P}(\mathbb{R}^d)) \to \mathcal{P}(C([0,T],\mathbb{R}^d))$ such that $(G_{b,a})_\sharp\Lambda = \mathfrak{L}$ and $E(G_{b,a}(\boldsymbol{\mu})) = \boldsymbol{\mu}$ for Λ-a.e. $\boldsymbol{\mu}$.

Conversely:

(i) given $\mathfrak{L} \in \mathcal{P}(\mathcal{P}(C([0,T],\mathbb{R}^d)))$ satisfying the condition in (2), then $E_\sharp\mathfrak{L} \in \mathcal{P}(C([0,T],\mathcal{P}(\mathbb{R}^d)))$ satisfies the condition in (1);

(ii) given $\Lambda \in \mathcal{P}(C([0,T],\mathcal{P}(\mathbb{R}^d)))$ *satisfying the condition in (1), then the curve of random measures defined by* $M_t := (\mathfrak{e}_t)_\sharp \Lambda$*, satisfies* (1.3).

This theorem is followed by a uniqueness scheme: indeed, under suitable assumptions, we can prove that uniqueness for the equation on random measures is equivalent to uniqueness of (1.1) and (1.2) (see Proposition 5.6).

The advantages of this approach are many:

- the equation $\partial_t M_t = \mathcal{K}^*_{b_t,a_t} M_t$ shares interesting properties. Firstly, it is linear in $M_.$, making it an infinite-dimensional linearized version of the Kolmogorov-Fokker-Planck equation. Secondly, it is a first order equation, in the sense that the operator $\mathcal{K}_{b_t,a_t}$ satisfies the Leibniz rule $\mathcal{K}_{b_t,a_t}(FG) = F\mathcal{K}_{b_t,a_t}G + G\mathcal{K}_{b_t,a_t}F$;
- in principle, it can be used to study well-posedness for (1.1) and (1.2), in a selection sense, under low-regularity for the coefficients, in analogy with the original works by Ambrosio (for the deterministic case) and Figalli-Trevisan (for the stochastic case).

Of course, there is no free lunch: the first thing to do to pursue the previous way for well-posedness is to find a good class of curves of random measures in which to look for existence and uniqueness, under suitable assumptions on the coefficients, that must be understood as well. To find such a class seems to be a challenging problem, due to the infinite-dimensionality of the space $\mathcal{P}(\mathbb{R}^d)$ and the lack of a nice reference measure on it, which prevents the application of standard finite-dimensional techniques. This aspect is worth further investigating, for example endowing the space $\mathcal{P}(\mathbb{R}^d)$ with a Gaussian-regular measure (see [PS25b]).

The well-posedness for McKean-Vlasov equation attracted many researchers in recent years [BR24; CD22]. In [BR24], the authors grouped the results of a series of papers in which they studied the McKean-Vlasov equation, mainly exploiting the theory developed by A. Figalli and D. Trevisan. In particular, in [BR23], they showed how the uniqueness of a linearized version of the McKean-Vlasov equation is useful to gather uniqueness for the original equation. In §5, Assumption 5.3 is a fundamental assumption on uniqueness of the linearized McKean-Vlasov, and it is exploited as in [BR23] to prove the uniqueness scheme.

Other approaches for the well-posedness of the McKean-Vlasov equation (recovering also strong uniqueness) are through uniform continuity assumptions on the coefficients [De 20; PR25; CF22]. In particular, some Lipschitz/Hölder continuity assumptions are done w.r.t. the state and the measure variable. To do so, several distances are considered in the measure variable. In §3, we introduce a new natural distance, in duality with C^ℓ functions, and the case with $\ell = 2$ will be an important tool for the proof of Theorem 1.1. It would be interesting to understand if other metric-like discrepancies of order 2 between probability measures (e.g. [HT19; BMQ25; BB24]) may give a metric characterization of the equation $\partial_t M_t = \mathcal{K}^*_{b_t,a_t} M_t$, as in [PS25a, Section 3 & 4].

A similar result to our main theorem was already obtained in [LSZ22], where they considered a more general equation, putting a stochastic noise at the level of equation (1.1), raising an operator $\mathcal{K}$, at the level of random measures, of diffusive nature. The novelties of this paper are both in the approach, that relies on a measurable selection argument (see §4) and in the integrability assumptions; indeed, we only require integrability of

$$\frac{b_t(x,\mu)}{1+|x|} \quad \text{and} \quad \frac{a_t(x,\mu)}{1+|x|^2}, \tag{1.6}$$

while they could perform the liftings only under L^p-assumptions of a and b, with $p > 1$ (see also [Reh23]). Moreover, we also show the uniqueness equivalences under the uniqueness assumption for the linearized Kolmogorov-Fokker-Planck equation, that allows one to recover uniqueness under uniformly (w.r.t. the variable μ) Lipschitz assumption in the variable x (see Lemma 5.5).

Outline of the paper. In **Section 2**, we fix the main spaces we are going to use in the paper, in particular fixing some natural topologies on them. Then, we recall the stochastic superposition principle, introducing the linear Kolmogorov-Fokker-Planck equation and the martingale problem. It will be the notion of solution that we use for stochastic differential equations, that is equivalent to weak solutions.

In **Section 3**, we introduce the distance D_ℓ over probability measures, in duality with C_0^ℓ functions with controlled norm. We will see that they are complete metrics that induce the narrow topology. With the same idea, we define distances over $\mathcal{P}(\mathcal{P}(\mathbb{R}^d))$ in duality with smooth cylinder functions (see Definitions 3.4 and 3.5) and we show that they all induce the narrow over narrow topology over $\mathcal{P}(\mathcal{P}(\mathbb{R}^d))$ (see §2.1.4).

In **Section 4**, we introduce the operator $\mathcal{K}_{b_t,a_t}$ acting on cylinder functions and associated to the non-local drift term $b : [0,T] \times \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}^d$ and the non-local diffusion term $a : [0,T] \times \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \to \mathrm{Sym}_+(\mathbb{R}^{d\times d})$. Then we prove the nested stochastic superposition principle, Theorem 1.1, in several steps: first, we show Claims (i) and (ii); then, we prove the existence of Λ satisfying Claim (1), embedding $\mathcal{P}(\mathbb{R}^d)$ (endowed with the distance D_2 introduced in Section 3) in $\mathbb{R}^\infty$, in which we transfer the equation to use the superposition principle in $\mathbb{R}^\infty$ proved in [AT14]; then, we show the existence of $\mathfrak{L}$ that satisfies Claims (2) and (3), using a measurable selection argument. The non-trivial part is to prove the measurability of the set $\mathrm{KFP}(b,a)$ and $\mathrm{MP}(b,a)$ (see Definition 4.3).

Finally, in **Section 5**, we show how uniqueness can be transferred from the equation on random measures to the non-linear Kolmogorov-Fokker-Planck and to weak solutions of the McKean-Vlasov equation.

2. Preliminaries

2.1. **Canonical topology over spaces of measures and the space of continuous curves.**

Let (X,τ) be a Polish space, i.e. for which there exists a distance d that induces the topology τ and makes it a complete and separable space. The collection $\mathcal{B}(X)$ denotes the Borel σ-algebra of X, i.e. the σ-algebra generated by the topology τ.

2.1.1. *Space of continuous curves.* The space of continuous curves from $[a,b]$ to X, denoted by $C([a,b],Y)$ is naturally endowed with the *compact-open topology*. Its topology can be metrized as well, resulting as Polish (see [Sri08, Theorem 2.4.3] for separability): indeed, considering a distance d over X that induces its topology τ, the compact open topology is metrized by the sup-distance over curves, i.e.

$$D_d(\boldsymbol{x}_1,\boldsymbol{x}_2) := \sup_{t\in[0,T]} d(\boldsymbol{x}_1(t),\boldsymbol{x}_2(t)) \quad \forall \boldsymbol{x}_1,\boldsymbol{x}_2 \in C([a,b];X). \tag{2.1}$$

It will be useful, in this context, to consider a bounded distance $\hat{d}$ that induces the compact-open topology. In particular, given any distance d inducing τ, defining the truncated one, i.e. $\hat{d} = d\wedge 1$, we have that the compact-open topology over $C([a,b],X)$ is induced by $D_{\hat{d}}$ as well.

Fixing a distance d that induces the topology of X, the set of absolutely continuous curves denoted by $AC([a,b],X)$ is defined as the collection of continuous curves $\boldsymbol{x} = (x_t)_{t\in[a,b]}$ for which there exists a function $g : (a,b) \to [0,+\infty]$ that is in $L^1(a,b)$ and satisfying

$$d(x_s,x_t) \le \int_s^t g(r)dr \qquad \text{for all } a \le s < t \le b.$$

For absolutely continuous curves, there always exists the metric derivative, i.e.

$$|\dot{\boldsymbol{x}}|_d(t) := \lim_{h\to 0} d(x_{t+h}, x_t)$$

exists for a.e. $t \in (a,b)$. Moreover $|\dot{\boldsymbol{x}}|(\cdot) \in L^1(a,b)$ and it is (pointwise) the smallest function g that can be considered in the definition.

When $a = 0$, the space $C([0,b],X)$ will be denoted as $C_b(X)$.

2.1.2. *Spaces of measures.* We will denote with $\mathcal{P}(X)$ the space of Borel probability measures over X. More generally, we denote by $\mathcal{M}_+(X)$, $\mathcal{M}(X)$ and $\mathcal{M}(X;\mathbb{R}^n)$, respectively, the space of finite positive measures, signed measures and $\mathbb{R}^n$-valued measures, in both the last two cases with finite total variation, where for a given $\nu \in \mathcal{M}(X;\mathbb{R}^n)$, its total variation is

$$|\nu|(A) := \sup\left\{ \sum_{n=1}^{+\infty} |\nu(A_n)| \ : \ \bigcup A_n = A,\ A_i \cap A_j = \emptyset \text{ as } i \neq j \right\}. \tag{2.2}$$

Notice that $\mathcal{P}(X) \subset \mathcal{M}_+(X) \subset \mathcal{M}(X)$. In particular, we will endow $\mathcal{M}(Y;\mathbb{R}^n)$ with the narrow topology, so that $\mathcal{P}(X)$ and $\mathcal{M}_+(X)$ are endowed with the subspace topology, and actually are closed subsets.

The narrow topology over $\mathcal{M}(X;\mathbb{R}^n)$ is the smallest topology for which the functional

$$\mathcal{M}(X;\mathbb{R}^n) \ni \nu \mapsto \int_X \phi(x)\cdot d\nu(x)$$

is continuous for all $\phi \in C_b(X;\mathbb{R}^n)$, i.e. bounded and continuous functions over X taking values in $\mathbb{R}^n$. It is important to recall that:

(a) the spaces $\mathcal{P}(X)$ and $\mathcal{M}_+(X)$, endowed with the narrow topology, are Polish;
(b) the space $\mathcal{M}(X;\mathbb{R}^n)$ is not Polish, but is still a Lusin space (see e.g. [PS25a, Remark 2.4]).

Recall the push-forward operation defined over positive measures: if $f : (Z_1,\mathcal{F}_1) \to (Z_2,\mathcal{F}_2)$ is a function between two generic measurable spaces and μ is a measure defined over $(Z_1,\mathcal{F}_1)$, then $f_\sharp\mu$ is a measure over $(Z_2,\mathcal{F}_2)$ defined as

$$f_\sharp\mu(E) := \mu(f^{-1}(E)) \quad \forall E \in \mathcal{F}_2.$$

Notice that, together with what we introduced §2.1.1, we fixed canonical topologies over $\mathcal{P}(C([a,b],X))$ and $C([a,b],\mathcal{P}(X))$.

The following measurability result will be useful in the following, and for its proof we refer for example to [PS25a, Appendix D].

Lemma 2.1. *Let $g : X \to [0,+\infty]$ and $f : X \to \mathbb{R}^n$ be Borel measurable maps. Then:*

(i) the map

$$G : \mathcal{M}_+(Y) \to [0,+\infty], \quad G(\mu) := \int_Y g\ d\mu$$

is Borel. In particular, the set $\{\mu \in \mathcal{M}_+(Y)\ :\ \int_X g d\mu < +\infty\}$ is Borel measurable;

(ii) for any $p \geq 1$, the set

$$\{(\mu,\nu) \in \mathcal{M}_+(Y) \times \mathcal{M}(Y;\mathbb{R}^n)\ :\ f \in L^p(\mu),\ \nu = f\mu\}$$

is Borel, considering the product topology over $\mathcal{M}_+(Y) \times \mathcal{M}(Y;\mathbb{R}^n)$.

2.1.3. 1-*Wasserstein distance.* Let d be a bounded distance that induces the prescribed topology over X. The associated 1-Wasserstein distance, denoted W_1, over the space of probability measures is defined as

$$W_{1,d}(\mu_1,\mu_2) := \inf\left\{\int_{X\times X} d(x_1,x_2)d\pi(x_1,x_2) \ : \ \pi\in\Pi(\mu_1,\mu_2)\right\} \quad \text{for all } \mu_1,\mu_2\in\mathcal{P}(X),$$

where $\Pi(\mu_1,\mu_2)$ is the collection of all the transport plans π between μ_1 and μ_2, i.e. all the probability measures $\pi\in\mathcal{P}(X\times X)$ satisfying $\pi(A\times X)=\mu_1(A)$ and $\pi(X\times B)=\mu_2(B)$ for all $A,B\in\mathcal{B}(X)$. If the distance considered is not bounded, we can always replace it by $\hat{d}=d\wedge 1$, that is still a distance that induces the same topology.

The 1-Wasserstein distance associated with the truncated distance $\hat{d}$, i.e. $W_{1,\hat{d}}$, induces the narrow topology over $\mathcal{P}(X)$.

Another important feature of the 1-Wasserstein distance is its dual formulation: for all $\mu_1,\mu_2\in\mathcal{P}(\mathbb{R}^d)$, it holds

$$W_{1,\hat{d}}(\mu_1,\mu_2) = \sup\left\{\int_X \phi d\mu_1 - \int_X \phi d\mu_2 \ : \ \phi\in C_b(X),\ \phi \text{ 1-Lipschitz w.r.t. } \hat{d}\right\}. \tag{2.3}$$

We will make use of this formulation to define different distances inducing the narrow topology over $\mathcal{P}(\mathbb{R}^d)$ (see §3.1): in this particular case it holds

$$W_{1,|\cdot|\wedge 1}(\mu_1,\mu_2) = \sup\left\{\int_{\mathbb{R}^d} \phi d\mu_1 - \int_{\mathbb{R}^d} \phi d\mu_2 \ : \ \phi\in C^1_0(\mathbb{R}^d),\ \|\nabla\phi\|_\infty\le 1\right\},$$

where $C^1_0(\mathbb{R}^d)$ denotes the class of C^1-functions such that $\phi(x)\to 0$ and $\nabla\phi(x)\to 0$ when $|x|\to+\infty$.

2.1.4. *Space of random measures.* An important space in this paper is the one of random measures over $\mathbb{R}^d$, i.e. the space of probability measures over probability measures $\mathcal{P}(\mathcal{P}(\mathbb{R}^d))$. Since $\mathcal{P}(\mathbb{R}^d)$ with the narrow topology is Polish, we may endow $\mathcal{P}(\mathcal{P}(\mathbb{R}^d))$ with the topology described in the previous subsection, that we will call narrow over narrow topology. Moreover, given any bounded distance D over $\mathcal{P}(\mathbb{R}^d)$ that induces the narrow topology, we know that $W_{1,D}$ induces the narrow over narrow topology.

2.2. **A note on filtrations and measurability.** In this subsection, we introduce what is usually called the natural filtration over the space of continuous curves $C_T(X):=C([0,T];X)$, where X is a Polish space. In particular, the following lemma shows that it is equivalently generated by the collection of all the evaluations at time $s\in[0,t]$ or just by the restriction of the curve in the interval $[0,t]$, considering the topologies introduced in the previous subsection.

Lemma 2.2. *Let (X,τ) be a Polish space and consider*

$$\mathrm{e}_s : C_T(X)\to X,\quad \mathrm{e}_s(\gamma):=\gamma(s),\quad |_{[0,s]} : C_T(X)\to C_s(X),\quad |_{[0,s]}(\gamma)=\gamma|_{[0,s]}.$$

Endow $C_t(X)$ with its natural topology and the associated Borel σ-algebra for any $t\in[0,T]$. Then, for all $t\in[0,T]$, the smallest σ-algebra on $C_T(X)$ that makes measurable e_s for all $s\in[0,t]$ coincides with the smallest σ-algebra that makes measurable $|_{[0,t]}$. Such σ-algebra will be indicated with $\mathcal{F}_t$, and the collection $(\mathcal{F}_t)_{t\in[0,T]}$ is commonly called the natural filtration of $C_T(X)$.

Proof. Let $\mathcal{F}_t:=\sigma(\mathrm{e}_s : s\in[0,T])$ and $\tilde{\mathcal{F}}_t=\sigma(|_{[0,t]})$. The inclusion $\mathcal{F}_t\subseteq\tilde{\mathcal{F}}_t$ is immediate, since $\mathrm{e}_s=\mathrm{e}^{(t)}_s\circ|_{[0,t]}$ for all $s\in[0,t]$, where $\mathrm{e}^{(t)}_s : C_t(X)\to X$ is the evaluation at time $s\le t$ for curves defined up to time t.

Regarding the other inclusion, endow X with a distance d that makes it complete and separable, and $C_t(X)$ with the sup distance, and notice that for all balls $B_r(\tilde\gamma)$, with $\tilde\gamma \in C_t(X)$, it holds

$$\begin{aligned}
|_{[0,t]}^{-1}(B_r(\tilde\gamma)) &= \{\gamma \in C_T(X) \ : \ \max_{s\in[0,t]} d(\gamma(s),\tilde\gamma(s)) < r\} \\
&= \bigcup_{k\geq 1} \{\gamma \in C_T(X) \ : \ \max_{s\in[0,t]} d(\gamma(s),\tilde\gamma(s)) \leq r - 1/k\} \\
&= \bigcup_{k\geq 1} \{\gamma \in C_T(X) \ : \ \sup_{q\in[0,t]\cap\mathbb{Q}} d(\gamma(q),\tilde\gamma(q)) \leq r - 1/k\} \\
&= \bigcup_{k\geq 1} \bigcap_{q\in[0,t]\cap\mathbb{Q}} \{\gamma \in C_T(X) \ : \ d(\gamma(q),\tilde\gamma(q)) \leq r - 1/k\} \\
&= \bigcup_{k\geq 1} \bigcap_{q\in[0,t]\cap\mathbb{Q}} \{\gamma \in C_T(X) \ : \ \mathrm{e}_q(\gamma) \in \overline{B}_{r-1/k}(\tilde\gamma(q))\} \in \mathcal{F}_t.
\end{aligned}$$

Since $C_t(X)$ is separable (see [Sri08, Theorem 2.4.3]), it has a countable basis for the topology made of balls, and then we can conclude that $|_{[0,t]}^{-1}(A) \in \mathcal{F}_t$ for all $A \subset C_t(X)$ open. Then $|_{[0,t]}$ is $\mathcal{F}_t$-measurable, i.e. $\tilde{\mathcal{F}}_t \subseteq \mathcal{F}_t$. □

Lemma 2.3. *Let $t \in [0,T]$ and $\mathcal{G}_t$ a countable basis for the topology of $C_t(\mathbb{R}^d)$. Define the set of $\mathcal{F}_t$-measurable simple functions*

$$\mathcal{V}_t := \mathrm{Span}_{\mathbb{Q}} \left\{ \mathbb{1}_{E_t^{-1}(A)} \ : \ A \in \mathcal{A}_t \right\}, \quad \textit{where} \quad \mathcal{A}_t := \left\{ \bigcup_{k=1}^n B_k \ : \ n \in \mathbb{N}, \ B_k \in \mathcal{G}_t \right\}. \tag{2.4}$$

Then, for any probability measure $\alpha \in \mathcal{P}(C_T(\mathbb{R}^d))$, the set $\mathcal{S}_t$ is dense in $L^p(\alpha,\mathcal{F}_t)$ for any $p \in [1,+\infty)$.

Proof. It suffices to show that we can approximate $\mathbb{1}_{\tilde B}$ for any $B \in \mathcal{F}_t$. In particular, there exists $B \in \mathcal{B}(C_t(\mathbb{R}^d))$ such that $\tilde B = |_{[0,t]}^{-1}(B)$ and $\mathbb{1}_{\tilde B}(\gamma) = \mathbb{1}_B(|_{[0,t]}(\gamma))$.
Consider now the space $L^p\big(C_t(\mathbb{R}^d), (E_t)_\sharp\alpha, \mathcal{B}(C_t(\mathbb{R}^d))\big)$. By outer regularity, it is not hard to prove that $\mathcal{A}_t' := \mathrm{Span}_{\mathbb{Q}}\{\mathbb{1}_A \ : \ A \in \mathcal{A}_t\}$ is dense in $L^p\big(C_t(\mathbb{R}^d), \mathcal{B}(C_t(\mathbb{R}^d)), (|_{[0,t]})_\sharp\alpha\big)$. Then, there exists $f_n = \sum_{j=1}^N q_{j,n}\mathbb{1}_{A_{j,n}}$, with $N \in \mathbb{N}$, $q_{j,n} \in \mathbb{Q}$ and $A_{j,n} \in \mathcal{A}_t$ such that

$$\begin{aligned}
0 \leftarrow \int_{C_t(\mathbb{R}^d)} |f_n - \mathbb{1}_B|^p d(|_{[0,t]})_\sharp\alpha &= \int_{C_T(\mathbb{R}^d)} \left| \sum_{j=1}^N q_{j,n}\mathbb{1}_{A_{j,n}}(|_{[0,t]}(\gamma)) - \mathbb{1}_B(|_{[0,t]}(\gamma)) \right|^p d\alpha(\gamma) \\
&= \int_{C_T(\mathbb{R}^d)} \left| \sum_{j=1}^N q_{j,n}\mathbb{1}_{|_{[0,t]}^{-1}(A_{j,n})}(\gamma) - \mathbb{1}_{\tilde B}(\gamma) \right|^p d\alpha(\gamma) = \int_{C_T(\mathbb{R}^d)} |g_n(\gamma) - \mathbb{1}_{\tilde B}(\gamma)|^p d\alpha(\gamma),
\end{aligned}$$

with $g_n := \sum_j q_{j,n}\mathbb{1}_{|_{[0,t]}^{-1}(A_{j,n})} \in \mathcal{V}_t$. □

2.3. **Martingale problem, KFP equation and superposition principle in $\mathbb{R}^d$.** In this subsection, working in the Euclidean space, we recall the definition of the martingale problem associated with a second order operator L, introduced in [SV06], the associated Kolmogorov-Fokker-Planck equation on probability measures, and the superposition principle that links them, for which we rely on [Fig08], [Tre16].

Here, we will deal with Borel functions

$$a : [0,T] \times \mathbb{R}^d \to \mathrm{Sym}_+(\mathbb{R}^{d\times d}), \quad b : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d, \tag{2.5}$$

where $\mathrm{Sym}_+(\mathbb{R}^{d\times d})$ are the positive definite square matrices of size d. To them, we can associate for any $t \in [0,T]$ the differential operator

$$L_{b_t,a_t}\phi(x) := b_t(x)\cdot\nabla\phi(x) + \frac{1}{2}a_t(x) : \nabla^2\phi(x) \quad \forall \phi \in C^2_b(\mathbb{R}^d), \tag{2.6}$$

where $A : B := \sum_{i,j} A_{ij}B_{ij}$ is the scalar product between matrices. Then, we can define the KFP equations and the martingale problem associated to the operator L.

Definition 2.4. *Let $\boldsymbol{\mu} = (\mu_t)_{t\in[0,T]} \in C_T(\mathcal{P}(\mathbb{R}^d))$. We say that the Kolmogorov-Fokker-Planck equation $\partial_t\mu_t = L^*_{b_t,a_t}\mu_t$ is satisfied if*

$$\int_0^T \int_{B_R} |b_t(x)| + |a_t(x)| d\mu_t(x)dt < +\infty \quad \forall R > 0, \tag{2.7}$$

and for all $\xi \in C^1_c((0,T))$ and $\phi \in C^2_c(\mathbb{R}^d)$ it holds

$$\int_0^T \xi'(t)\int_{\mathbb{R}^d} \phi(x)d\mu_t(x)dt = -\int_0^T \xi(t)\int_{\mathbb{R}^d} L_{b_t,a_t}\phi(x)d\mu_t(x)dt. \tag{2.8}$$

Notice that, as shown in [Tre16, Remark 2.3], it is not restrictive to assume that the curve $\boldsymbol{\mu}$ is narrowly continuous, and it actually satisfies

$$\int_s^t\int \partial_t f(r,x)d\mu_r(x)dr = \int f(t,x)d\mu_t(x) - \int f(s,x)d\mu_s(x) - \int_s^t\int L_{b_r,a_r}f(r,x)d\mu_r(x)dr, \tag{2.9}$$

for all $0 \le s < t \le T$ and $f \in C^{1,2}_c([0,T]\times\mathbb{R}^d)$.

Definition 2.5. *Let $\lambda \in \mathcal{P}(C_T(\mathbb{R}^d))$. We say that it is a solution to the martingale problem associated to L, if*

$$\int_{C_T(\mathbb{R}^d)}\int_0^T \big(|b_t(\gamma_t)| + |a_t(\gamma_t)|\big)\mathbb{1}_{B_R}(\gamma_t)\, dtd\lambda(\gamma) < +\infty \quad \forall R > 0, \tag{2.10}$$

and for all $\xi \in C^1_c(0,T)$ and $\phi \in C^2_c(\mathbb{R}^d)$ it holds

$$[0,T] \ni t \mapsto X^{\xi\phi}_t(\gamma) := \xi(t)\phi(\gamma_t) - \int_0^t \xi'(r)\phi(\gamma_r) + \xi(r)L_r\phi(\gamma_r)dr \tag{2.11}$$

is a martingale in the filtered space $\big(C_T(\mathbb{R}^d), (\mathcal{F}_t)_{t\in[0,T]}, \lambda\big)$, where $\mathcal{F}_t = \sigma(e_r : r \in [0,t])$.

Again, using density of the span of separated variables functions, it is not difficult to see that if λ satisfies (2.10) and (2.11), then it holds that

$$[0,T] \ni t \mapsto X^f_t(\gamma) := f(t,\gamma_t) - f(0,\gamma_0) - \int_0^t \partial_t f(r,\gamma_r) + L_{b_r,a_r}f(r,\gamma_r)dr \tag{2.12}$$

is a martingale for each $f \in C^{1,2}_c([0,T]\times\mathbb{R}^d)$.

In this work, we will always deal with solutions of some martingale problem, but it is important to recall their connection with solutions to stochastic differential equations. Let $\sigma : [0,T]\times\mathbb{R}^d \to \mathbb{R}^{d\times m}$, $m \ge 1$, be such that $a = \sigma\sigma^\top$. Then λ is a martingale solution associated with L_{b_t,a_t} if and only if there exists a probability space $(\Omega,\mathcal{F},\mathbb{P})$ and two continuous processes $(X_t)_{t\in[0,T]}$, $(B_t)_{t\in[0,T]}$ defined on it, and respectively taking values in $\mathbb{R}^d$ and $\mathbb{R}^m$, such that B is a Brownian motion, the law of X is λ and it satisfies

$$dX_t = b_t(X_t)dt + \sigma_t(X_t)dB_t \quad \text{and} \quad \forall R > 0 \quad \mathbb{E}\left[\int_0^T \big(|b_t(X_t)| + |a_t(X_t)|\big)\mathbb{1}_{B_R}(X_t)dt\right] < +\infty. \tag{2.13}$$

In particular, uniqueness of solutions of the martingale problem is equivalent to the uniqueness in law of solutions to (2.13).

In the following lemma, we evaluate the quadratic variation of these martingales, whose proof was already given in [Tre16, Corollary A.4], but we show it here for completeness.

Lemma 2.6. *For any $f \in C_c^{1,2}([0,T]\times\mathbb{R}^d)$ and any martingale solution λ associated to L_t, the quadratic variation of the martingale X_t^f is given by*

$$[X^f]_t(\gamma) = \int_0^t \nabla f(s,\gamma_s)^\top a_s(\gamma_s)\nabla f(s,\gamma_s)ds. \tag{2.14}$$

Proof. Let $\alpha_t(\gamma) := \int_0^t \nabla f(s,\gamma_s)^\top a_s(\gamma_s)\nabla f(s,\gamma_s)ds$. It suffices to show that $(X_t^f)^2 - \alpha_t$ is a local martingale. The key observation is the following equality:

$$L_{b_t,a_t}(f^2)(t,x) - 2f(t,x)L_{b_t,a_t}f(t,x) = \nabla f(t,\gamma_t)^\top a_t(\gamma_t)\nabla f(t,\gamma_t). \tag{2.15}$$

Let

$$X_t := X_t^f, \quad \varphi_t(\gamma) := f(t,\gamma(t)), \quad \ell_t(\gamma) := \big(\partial_t f + L_{b_t,a_t} f\big)(t,\gamma(t)), \quad N_t := X_t^{f^2}.$$

Then, it holds

$$\begin{aligned}
X_t^2 =& \varphi_t^2 + \int_0^t\int_0^t \ell_s\ell_r dr ds - 2\varphi_t\int_0^t \ell_s ds\\
=& N_t + \int_0^t \big(\partial_t f^2 + L_{b_s,a_s} f^2\big)(s,\gamma(s))ds + 2\int_0^t \ell_s\left(\int_s^t \ell_r dr - \varphi_t\right)ds\\
&\pm 2\int_0^t f(s,\gamma(s))\big(\partial_t f + L_{b_s,a_s} f\big)(s,\gamma(s))ds\\
\overset{(2.15)}{=}& N_t - 2\int_0^t \ell_s\left(\varphi_t - \varphi_s - \int_s^t \ell_r dr\right)ds + \int_0^t \alpha_s ds.
\end{aligned}$$

We conclude observing that $N_t = X_t^{f^2}$ is a martingale, and applying [Tre14, Lemma 2.10] we have that $t \mapsto \int_0^t \ell_s\left(\varphi_t - \varphi_s - \int_s^t \ell_r dr\right)ds$ is a local martingale, since it is not hard to prove that $\int_0^T\int |\ell_t(\gamma)|d\lambda(\gamma)dt < +\infty$. □

In [Fig08; Tre16; BRS21], they proved the so-called *superposition principle*, linking solutions to the martingale problem with solutions of the Kolmogorov-Fokker-Planck equation. Here we state its more general version, specifically referring to [BRS21, Theorem 1.1].

Theorem 2.7. *Let $b : [0,T]\times\mathbb{R}^d \to \mathbb{R}^d$ and $a : [0,T]\times\mathbb{R}^d \to \mathrm{Sym}_+(\mathbb{R}^{d\times d})$ be Borel functions, and $L_t := L_{b_t,a_t}$ their associated operator as in (2.6). Let $(\mu_t)_{t\in[0,T]} \in C_T(\mathcal{P}(\mathbb{R}^d))$ be a solution to $\partial_t\mu_t = L_t^*\mu_t$ satisfying, together with (2.7), the additional integrability assumption*

$$\int_0^T\int_{\mathbb{R}^d} \frac{|a_t(x)| + |\langle b_t(x), x\rangle|}{1+|x|^2} d\mu_t(x)dt < +\infty. \tag{2.16}$$

Then there exists $\lambda \in \mathcal{P}(C_T(\mathbb{R}^d))$ solution of the martingale problem associated with L, such that $\mu_t = (\mathrm{e}_t)_\sharp\lambda$.
Conversely, if $\lambda \in \mathcal{P}(C_T(\mathbb{R}^d))$ is a solution of the martingale problem, then $\mu_t^\lambda := (\mathrm{e}_t)_\sharp\lambda$ is a solution for the Kolmogorov-Fokker-Planck equation.

Notice that (2.16) is satisfied if the following stronger assumption holds:

$$\int_0^T \int_{\mathbb{R}^d} \frac{|b_t(x)|}{1+|x|} + \frac{|a_t(x)|}{1+|x|^2} d\mu_t(x)dt < +\infty. \tag{2.17}$$

3. Integral metrics

In this section, we introduce new integral metrics (see [Zol84]) over the space of probability measures and of random measures of $\mathbb{R}^d$. They seem to be quite natural to work in our setting, but we think they could have independent interest. Anyway, we didn't find any previous literature about them.

3.1. **Integral metrics over $\mathcal{P}(\mathbb{R}^d)$ in duality with smooth functions.** We will use the notation $C_0^\ell(\mathbb{R}^d)$ to denote functions $\phi \in C^\ell(\mathbb{R}^d)$ such that ϕ and all its derivatives of order at most ℓ tend to 0 as $|x| \to +\infty$. Moreover,

$$\|\phi\|_{C^\ell} := \max_{j=0,\dots,\ell} \|\nabla^{\otimes j}\phi\|_\infty = \max_{j=0,\dots,\ell} \sup_{x\in\mathbb{R}^d} |\nabla^{\otimes j}\phi(x)|,$$

where $|\nabla^{\otimes j}\phi(x)|$ is the 2-norm of all the entries of the tensor.

Definition 3.1 (Integral metric over positive measures). *Let $\ell \geq 1$ be an integer number. For all $\mu, \nu \in \mathcal{M}_+(\mathbb{R}^d)$ we define*

$$D_{C_0^\ell}(\mu,\nu) := \sup\left\{ \left| \int_{\mathbb{R}^d} \phi d\mu - \int_{\mathbb{R}^d} \phi d\nu \right| \ : \ \phi \in C_0^\ell(\mathbb{R}^d), \ \|\phi\|_{C^\ell} \leq 1 \right\}. \tag{3.1}$$

Shortly, we will more often use the notation D_ℓ for $D_{C_0^\ell}$.

Proposition 3.2. *The following properties hold for any $\ell \geq 1$:*

(1) D_ℓ is a distance over $\mathcal{M}_+(\mathbb{R}^d)$;
(2) D_ℓ is a complete metric that metrizes narrow convergence. In particular, $(\mathcal{P}(\mathbb{R}^d), D_\ell)$ is a complete and separable metric space;
(3) there exists a countable subset $\mathcal{S}_\ell \subset C_c^\ell(\mathbb{R}^d)$ satisfying $\|\phi\|_{C^\ell} \leq 1$ for any $\phi \in \mathcal{S}_\ell$, such that

$$D_{C_0^\ell}(\mu,\nu) = \sup\left\{ \left| \int_{\mathbb{R}^d} \phi d\mu - \int_{\mathbb{R}^d} \phi d\nu \right| \ : \ \phi \in \mathcal{S}_\ell \right\}. \tag{3.2}$$

Proof. (1) The symmetry and the triangular inequality are trivial. If $D_\ell(\mu,\nu) = 0$, then $\int \phi d\mu = \int \phi d\nu$ for all $\phi \in C_0^\ell(\mathbb{R}^d)$. Using a regularization argument, we obtain that $\int \phi d\mu = \int \phi d\nu$ for any bounded Borel functions, which implies $\mu = \nu$.

(2) Let $\mu_n \in \mathcal{M}_+(\mathbb{R}^d)$ be a Cauchy sequence w.r.t. D_ℓ. We first prove that μ_n is tight, i.e. for all $\varepsilon > 0$ there exists $R > 0$ such that $\mu_n(B_R^c) \leq \varepsilon$ for any $n \in \mathbb{N}$. Indeed, by contradiction, assume that there exists $\varepsilon > 0$ such that for all $j \in \mathbb{N}$ there exists $n(j) \in \mathbb{N}$ for which $\mu_{n(j)}(B_j^c) > \varepsilon$. In particular, there exists $R(j) \gg j$ such that $\mu_{n(j)}(B_{R(j)} \setminus B_j) \geq \frac{\varepsilon}{2}$. Now, consider a function $\phi_j \in C_c^\ell(\mathbb{R}^d)$ satisfying

$$\phi_j \equiv 1 \text{ on } B_{R(j)} \setminus B_j, \quad \phi_j \equiv 0 \text{ on } B_{j/2} \cup B_{2R(j)}^c, \quad \|\phi_j\|_{C^\ell} \leq 1.$$

Such functions exist at least for $j \geq m$, where m is big enough and depends only on the required regularity ℓ. Then, consider $\bar{n} \in \mathbb{N}$ such that $D_\ell(\mu_n, \mu_{\bar{n}}) \leq \varepsilon/4$ for all $n \geq \bar{n}$. So,

$$\frac{\varepsilon}{4} \geq \liminf_{j\to+\infty} D_\ell(\mu_{n(j)}, \mu_{\bar{n}}) \geq \liminf_{j\to+\infty} \int \phi_j d\mu_{n(j)} - \int \phi_j d\mu_{\bar{n}} \geq \frac{\varepsilon}{2} - \limsup_{j\to+\infty} \int \phi_j d\mu_{\bar{n}} = \frac{\varepsilon}{2},$$

obtaining a contradiction. Having tightness, we know that there exists $\mu \in \mathcal{M}_+(\mathbb{R}^d)$ such that $\mu_n \to \mu$ narrowly. It is easy to see that $\mu_n \to \mu$ narrowly implies $D_\ell(\mu_n, \mu) \to 0$. Indeed, if $\mu_n \to \mu$ narrowly, we know that the truncated Wasserstein distance $W_{1,|\cdot|\wedge 1}(\mu_n, \mu) \to 0$, so considering $\pi_n \in \Gamma(\mu_n, \mu)$ realizing it, it holds

$$\begin{aligned} D_1(\mu_n,\mu) &= \sup_{\|\phi\|_{C^1_0}\le 1} \left| \int \phi d\mu_n - \int \phi d\mu \right| \le \sup_{\|\phi\|_{C^1_0}\le 1} \int |\phi(x)-\phi(y)| d\pi_n(x,y) \\ &\le \int |x-y| \wedge 2 d\pi_n(x,y) \le 2W_{1,|\cdot|\wedge 1} \to 0. \end{aligned}$$

Then, notice that for any $\ell > 1$ it holds

$$D_\ell(\mu_n,\mu) \le D_1(\mu_n,\mu).$$

In particular, we proved that $\mu_n \to \mu$ narrowly if and only if $D_\ell(\mu_n, \mu) \to 0$.

(3) It suffices to notice that the Banach space $\big(C^\ell_0(\mathbb{R}^d), \|\cdot\|_{C^\ell}\big)$ is separable. □

We can also consider an integral metric adding stronger conditions on the test functions. We define it just for the case $\ell = 2$, but it can be easily generalized. Let $C^2_{0,w}(\mathbb{R}^d)$ be the space of *weighted C^2-functions*, defined as the closure of $C^2_c(\mathbb{R}^d)$ with respect to the norm

$$\|\phi\|_{C^2_{0,w}} := \|\phi\|_\infty + \|(1+|\cdot|)\nabla\phi\|_\infty + \|(1+|\cdot|^2)\nabla^{\otimes 2}\phi\|_\infty. \tag{3.3}$$

The space $C^2_{0,w}(\mathbb{R}^d)$ is Banach and separable. We can then consider the *weighted integral metric* defined by

$$\begin{aligned} D_{2,w}(\mu,\nu) =& D_{C^2_{0,w}}(\mu,\nu) := \sup\left\{ \left| \int_{\mathbb{R}^d} \phi d\mu - \int_{\mathbb{R}^d} \phi d\nu \right| \ :\ \phi \in C^2_{0,w}(\mathbb{R}^d),\ \|\phi\|_{C^2_{0,w}} \le 1 \right\} \\ =& \sup\left\{ \left| \int_{\mathbb{R}^d} \phi d\mu - \int_{\mathbb{R}^d} \phi d\nu \right| \ :\ \phi \in C^2_c(\mathbb{R}^d),\ \|\phi\|_{C^2_{0,w}} \le 1 \right\}. \end{aligned} \tag{3.4}$$

The analogous of Proposition 3.2 can be proved for this distance.

Proposition 3.3. *The following hold:*

1. *(1) $D_{2,w}$ is a distance over $\mathcal{M}_+(\mathbb{R}^d)$;*
2. *(2) $D_{2,w}$ is a complete metric that metrizes narrow convergence. In particular, $(\mathcal{P}(\mathbb{R}^d), D_{2,w})$ is a complete and separable metric space;*
3. *(3) there exists a countable subset $\mathcal{S}_{2,w} \subset C^2_c(\mathbb{R}^d)$ satisfying $\|\phi\|_{C^2_{0,w}} \le 1$ for any $\phi \in \mathcal{S}_{2,w}$, such that*

$$D_{2,w}(\mu,\nu) = \sup\left\{ \left| \int_{\mathbb{R}^d} \phi d\mu - \int_{\mathbb{R}^d} \phi d\nu \right| \ :\ \phi \in \mathcal{S}_{2,w} \right\}. \tag{3.5}$$

Proof. The proof of Claims (1) and (3) is the same as the one of Proposition 3.2. The proof of Claim (2) follows its same line, but we must be careful in the choice of the test function considered to obtain a contradiction. So, as before consider $\mu_n \in \mathcal{M}_+(\mathbb{R}^d)$ a Cauchy sequence with respect to $D_{2,w}$. We are done if we prove that it is tight. By contradiction, assume that there exists $\varepsilon > 0$ such that for all $j \in \mathbb{N}$ there exists $n(j) \in \mathbb{N}$ for which $\mu_{n(j)}(B^c_j) > \varepsilon$. In particular, there exists $R(j) \gg j$ such that $\mu_{n(j)}\big(B_{R(j)} \setminus B_j\big) \ge \frac{\varepsilon}{2}$ for all $j \in \mathbb{N}$.
Now, consider $\rho \in C^2([0,2])$ such that $\rho(r) = 0$ for all $r \in [0,1]$, $\rho(r) = 1$ in a neighborhood of

$r = 2$, ρ non-decreasing. We use it to build the following test functions $\phi_j \in C_c^2(\mathbb{R}^d)$:

$$\phi_j(x) = \rho_j(|x|) := \begin{cases} \rho\left(\frac{2|x|}{j}\right) & \text{if } |x| \le j, \\ 1 & \text{if } |x| \in (j, R(j)) \\ \rho\left(3 - \frac{|x|}{R(j)}\right) & \text{if } |x| \in [R(j), 3R(j)) \\ 0 & \text{if } |x| \ge 3R(j). \end{cases} \tag{3.6}$$

Computing $\rho_j'(r)$ and $\rho_j''(r)$, one can show that

$$|\rho_j'(r)| \le \frac{3\|\rho'\|_\infty}{r} \mathbb{1}_{(\frac{j}{2},+\infty)}(r), \quad |\rho_j''(r)| \le \frac{9\|\rho''\|_\infty}{r^2} \mathbb{1}_{(\frac{j}{2},+\infty)}(r).$$

This takes us to the estimates on ϕ_j:

$$|\nabla \phi_j(x)| \le \frac{3\|\rho'\|_\infty}{|x|} \mathbb{1}_{(\frac{j}{2},+\infty)}(|x|), \quad |\nabla^{\otimes_2} \phi_j(x)| \le \frac{3\sqrt{d-1}\|\rho'\|_\infty + 9\|\rho''\|_\infty}{|x|^2} \mathbb{1}_{(\frac{j}{2},+\infty)}(|x|).$$

Fixing then $C := 18\|\rho''\|_\infty + 6\sqrt{d-1}\|\rho'\|_\infty$, independent of j, and considering the test functions $f_j(x) := \frac{\phi_j(x)}{C}$, it holds that f_j is a competitor in the supremum for $D_{2,w}$, satisfying $f_j(x) = \frac{1}{C}$ for all $|x| \in (j, R(j))$, $0 \le f_j \le \frac{1}{C}$ and $f_j(x) = 0$ for all $|x| \le \frac{j}{2}$. Fixing then $\bar{n} \in \mathbb{N}$ such that $D_{2,w}(\mu_n, \mu_{\bar{n}}) \le \frac{\varepsilon}{4C}$ for all $n \ge \bar{n}$, we reach a contradiction:

$$\frac{\varepsilon}{4C} \ge \liminf_{j\to+\infty} D_{2,w}(\mu_{n(j)}, \mu_{\bar{n}}) \ge \liminf_{j\to+\infty} \int f_j d\mu_{n(j)} - \int f_j d\mu_{\bar{n}} \ge \frac{\varepsilon}{2C} - \limsup_{j\to+\infty} \int f_j d\mu_{\bar{n}} = \frac{\varepsilon}{2C}.$$

□

Thanks to these distances, we introduce a natural correspondence between the probability measures and the space $\mathbb{R}^\infty$, for which we recall that there are two natural metrics to consider:

- the metric

$$D_\infty(x, y) := \sup_{n\in\mathbb{N}} |x_n - y_n| \wedge 1, \tag{3.7}$$

  inducing the uniform convergence;
- the topology τ_w induced by the element-wise convergence, i.e. $x \to y$ if $x_n \to y_n$ for all $n \in \mathbb{N}$, induced by the distance

$$d_\infty(x, y) := \sum_{n\in\mathbb{N}} \frac{|x_n - y_n| \wedge 1}{2^n}. \tag{3.8}$$

  The topological space $(\mathbb{R}^\infty, \tau_w)$ is Polish.

Such a space is strictly related to the space of probability measures when endowed with the smooth metrics D_ℓ: fix $\ell \ge 1$ and say that $\mathcal{S}_\ell = \{\varphi_{1,\ell}, \varphi_{2,\ell}, \dots\}$ as the countable subset given by Proposition (3.2), (3). Then

$$\begin{aligned} \iota_\ell : \mathcal{P}(\mathbb{R}^d) &\to \mathbb{R}^\infty \\ \mu &\mapsto (L_{\varphi_{1,\ell}}(\mu), L_{\varphi_{2,\ell}}(\mu), \dots) \end{aligned} \tag{3.9}$$

is an isometry between $(\mathcal{P}(\mathbb{R}^d), D_\ell)$ and $(\iota_\ell(\mathcal{P}(\mathbb{R}^d)), D_\infty)$. We are mainly interested in the case $\ell = 2$ and this construction will be fundamental for the proof of our main theorem, and for this reason, we introduce also the map

$$\begin{aligned} \iota_{2,w} &: \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}^\infty \\ &\mu \mapsto (L_{\varphi_1^{(w)}}(\mu), L_{\varphi_2^{(w)}}(\mu), \dots), \end{aligned} \tag{3.10}$$

where $\mathcal{S}_{2,w} = \{\varphi_1^{(w)}, \varphi_2^{(w)}, \dots\}$ is the countable set of test functions introduced in Proposition 3.3. Anyway, in the next subsection, we define similar distances on the space of random measures, for which we need this correspondence for generic $\ell \geq 1$.

3.2. **Integral metrics over $\mathcal{P}(\mathcal{P}(\mathbb{R}^d))$ in duality with smooth cylinder functions.** We define smooth integral metrics over $\mathcal{P}(\mathcal{P}(\mathbb{R}^d))$, and the main result of this subsection is to prove that they induce its natural topology, using the identification with $\mathbb{R}^\infty$ presented above. These distances will not play a role in the following, but are in the same spirit of the ones presented above and we believe they have an independent interest.

Definition 3.4 (Cylinder functions)**.** *Let $h, \ell \geq 0$. A functional $F : \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}$ is said to belong to $\mathrm{Cyl}_c^{h,\ell}(\mathcal{P}(\mathbb{R}^d))$ if there exists $k \in \mathbb{N}$, $\Psi \in C_c^h(\mathbb{R}^k)$ and $\Phi = (\phi_1, \dots, \phi_k) \in C_c^\ell(\mathbb{R}^d, \mathbb{R}^k)$ such that*

$$F(\mu) = \Psi\left(L_\Phi(\mu)\right), \quad L_\Phi(\mu) = \big(L_{\phi_1}(\mu), \dots, L_{\phi_k}(\mu)\big), \quad L_{\phi_i}(\mu) := \int_{\mathbb{R}^d} \phi_i(x) d\mu(x). \tag{3.11}$$

If $\Phi \in C_b^\ell(\mathbb{R}^d, \mathbb{R}^k)$ and $\Psi \in C_b^h(\mathbb{R}^k)$, then we say that $F \in \mathrm{Cyl}_b^{h,\ell}(\mathcal{P}(\mathbb{R}^d))$.

Notice that, if $\Psi \circ L_\Phi \in \mathrm{Cyl}_c^{k,\ell}(\mathcal{P}(\mathbb{R}^d))$, we can also consider $\Psi \in C_b^k(\mathbb{R}^k)$. Moreover, $\mathrm{Cyl}_c^{h,\ell}(\mathcal{P}(\mathbb{R}^d)) \subset \mathrm{Cyl}_b^{h,\ell}(\mathcal{P}(\mathbb{R}^d))$.

Given a function $\Psi \in C_b^h(\mathbb{R}^k)$, for all $h \geq 0$ define the seminorms

$$[\![\Psi]\!]_{C^h} := \sup_{y \in \mathbb{R}^k} \left(\sum_{i_1=1}^k \cdots \sum_{i_h=1}^k |\partial_{i_1} \dots \partial_{i_h} \Psi(y)| \right),$$

where we use the convention that $[\![\Psi]\!]_{C^0} = \|\Psi\|_\infty$.

Definition 3.5. *Let $h, \ell \geq 1$. For any $M, N \in \mathcal{P}(\mathcal{P}(\mathbb{R}^d))$, define*

$$\begin{aligned} \mathfrak{D}_{h,\ell,\mathcal{P}}(M, N) := \sup \Bigg\{ \int_{\mathcal{P}(\mathbb{R}^d)} F d(M - N) \,:\, & k \geq 1,\, F = \Psi \circ L_\Phi \in \mathrm{Cyl}_c^{h,\ell}(\mathcal{P}(\mathbb{R}^d)), \\ & \Psi \in C_0^h(\mathbb{R}^k),\, \Phi = (\phi_1, \dots, \phi_k) \in C_0^l(\mathbb{R}^d; \mathbb{R}^k), \\ & [\![\Psi]\!]_{C^j} \leq 1 \text{ for all } j \leq h,\, \|\phi_i\|_{C^\ell} \leq 1 \Bigg\}. \end{aligned} \tag{3.12}$$

For all $h, \ell \geq 1$, $\mathfrak{D}_{h,\ell,\mathcal{P}}$ is a distance: positivity, symmetry, and triangular inequality are straightforward, while $\mathfrak{D}_{h,\ell,\mathcal{P}}(M, N) = 0 \implies M = N$ is a byproduct of the proof of Proposition 3.8.

Before proceeding, let us notice that, thanks to the map ι_ℓ defined in (3.9), this distance is strictly related to the distance over $\mathcal{P}(\mathbb{R}^\infty)$ defined as

$$\mathfrak{D}_{h,\mathbb{R}^\infty}(\widetilde{M}, \widetilde{N}) := \sup \left\{ \int_{\mathbb{R}^\infty} \Psi \circ \pi_k d(\widetilde{M} - \widetilde{N}) \,:\, k \geq 1,\, \Psi \in C_0^h(\mathbb{R}^k),\, [\![\Psi]\!]_{C^j} \leq 1 \,\text{for}\, j \leq h \right\}, \tag{3.13}$$

for all $\widetilde{M}, \widetilde{N} \in \mathcal{P}(\mathbb{R}^\infty)$.

Lemma 3.6. *For $h, \ell \geq 1$, in (3.12) we can consider the supremum only for $\Phi = (\varphi_{1,\ell}, \dots, \varphi_{k,\ell})$, where $\{\varphi_{1,\ell}, \varphi_{2,\ell}, \dots\} = \mathcal{S}_\ell$ from Proposition 3.2, i.e.*

$$\mathfrak{D}_{h,\ell,\mathcal{P}}(M,N) := \sup\Big\{ \int_{\mathcal{P}(\mathbb{R}^d)} F d(M-N) \,:\, k \geq 1,\, F = \Psi(L_{\varphi_{1,\ell}}, \dots, L_{\varphi_{k,\ell}}), \quad \Psi \in C_0^h(\mathbb{R}^k),\, [\![\Psi]\!]_{C^j} \leq 1 \ \forall j \leq h \Big\}. \tag{3.14}$$

In particular, it holds

$$\mathfrak{D}_{h,\ell,\mathcal{P}}(M,N) = \mathfrak{D}_{h,\mathbb{R}^\infty}((\iota_\ell)_\sharp M, (\iota_\ell)_\sharp N), \tag{3.15}$$

where ι_ℓ is defined as in (3.9).

Proof. (3.14) immediately follows from the density of $\mathcal{S}_\ell$ in $C_0^\ell(\mathbb{R}^d)$ and then (3.15) is an easy consequence of (3.14) and the definitions of $\mathfrak{D}_{h,\ell,\mathcal{P}}$, $\mathfrak{D}_{h,\mathbb{R}^\infty}$ and ι_ℓ. □

A first easy but fundamental lemma is the following.

Lemma 3.7. *Let $\ell \geq 1$. Then, the distance $\mathfrak{D}_{1,\ell,\mathcal{P}}$ is metrically equivalent to the Wasserstein distance built on D_ℓ, in fact*

$$W_{1,D_\ell}(M,N) \leq \mathfrak{D}_{1,\ell,\mathcal{P}}(M,N) \leq 2W_{1,D_\ell}(M,N) \quad \forall M,N \in \mathcal{P}(\mathcal{P}(\mathbb{R}^d)). \tag{3.16}$$

In particular, $\mathfrak{D}_{1,\ell,\mathcal{P}}$ induces the narrow over narrow topology of $\mathcal{P}(\mathcal{P}(\mathbb{R}^d))$.

Proof. From Lemma 3.6, we know that $\mathfrak{D}_{1,\ell,\mathcal{P}}(M,N) = \mathfrak{D}_{1,\mathbb{R}^\infty}((\iota_\ell)_\sharp M, (\iota_\ell)_\sharp N)$. By [PS25a, Lemma C.4], we have

$$W_{1,D_\infty}(\widetilde{M}, \widetilde{N}) = \sup\Big\{ \int_{\mathbb{R}^\infty} \Psi \circ \pi_k d(\widetilde{M} - \widetilde{N}) : k \in \mathbb{N},\, \Psi \in C_c^1(\mathbb{R}^k),\, \|\Psi\|_\infty \leq 1/2,\, [\![\Psi]\!]_{C^1} \leq 1 \Big\}. \tag{3.17}$$

We conclude by noticing that

$$\mathfrak{D}_{1,\ell,\mathcal{P}}(M,N) = \mathfrak{D}_{1,\mathbb{R}^\infty}((\iota_\ell)_\sharp M, (\iota_\ell)_\sharp N), \qquad W_{1,D_\ell}(M,N) = W_{1,D_\infty}((\iota_\ell)_\sharp M, (\iota_\ell)_\sharp N),$$

because ι_ℓ is an isometry between $(\mathcal{P}(\mathbb{R}^d), D_\ell)$ and $(\iota_\ell(\mathcal{P}(\mathbb{R}^d)), D_\infty)$. □

Finally, we can state the main result of this section, that quantifies the discrepancies between the distances $\mathfrak{D}_{h,\ell,\mathcal{P}}$ when h varies, from which it follows that they all induce the narrow on narrow topology over $\mathcal{P}(\mathcal{P}(\mathbb{R}^d))$.

Proposition 3.8. *Let $h \geq 2$ and $\ell \geq 1$. Then, there exists a constant $C > 0$ such that for all $\varepsilon \in (0, C)$ it holds*

$$\mathfrak{D}_{h,\ell,\mathcal{P}}(M,N) \leq \mathfrak{D}_{h-1,\ell,\mathcal{P}}(M,N) \leq \frac{C}{\varepsilon}\mathfrak{D}_{h,\ell,\mathcal{P}}(M,N) + 2\varepsilon \quad \text{for all } M,N \in \mathcal{P}(\mathcal{P}(\mathbb{R}^d)). \tag{3.18}$$

In particular, $(\mathcal{P}(\mathcal{P}(\mathbb{R}^d)), \mathfrak{D}_{h,\ell,\mathcal{P}})$ is a complete metric space, and its induced topology is the narrow over narrow one. Moreover, the constant C is independent from h and ℓ as well.

Proof. For simplicity, we will denote $\varphi_i := \varphi_{i,\ell}$ for any $i \in \mathbb{N}$ and $L_{k,\boldsymbol{\varphi}}(\mu) := (L_{\varphi_1}(\mu), \dots, L_{\varphi_k}(\mu))$. The first inequality in (3.18) is trivial, so we concentrate on the second one. To this aim, we introduce the following auxiliary distances for all $r > 0$: for all $M,N \in \mathcal{P}(\mathcal{P}(\mathbb{R}^d))$

$$\mathfrak{D}^{(r)}_{h,\ell,\mathcal{P}}(M,N) := \sup\Big\{ \int_{\mathcal{P}} \Psi \circ L_{k,\boldsymbol{\varphi}} d(M-N) \,:\, k \geq 1,\, \Psi \in C_c^h(\mathbb{R}^k), \quad [\![\Psi]\!]_{C^j} \leq 1 \ \ \forall j \leq h-1,\ [\![\Psi]\!]_{C^h} \leq r \Big\}. \tag{3.19}$$

The main idea is that $\mathfrak{D}^{(r)}_{h,\ell,\mathcal{P}} \to \mathfrak{D}_{h-1,\ell,\mathcal{P}}$ as $r \to +\infty$ and the limit can be uniformly quantified. Indeed, consider any competitor $\Psi \in C_c^{h-1}(\mathbb{R}^k)$ for the supremum that defines $\mathfrak{D}_{h-1.\ell,\mathcal{P}}$ (see (3.14)) and let $\rho^{(k)} \in C_c^\infty(\mathbb{R}^k)$ satisfy: $\operatorname{supp}\rho^{(k)} \subset B(0,1)$, $\rho^{(k)} \geq 0$, $\int \rho^{(k)} d\mathcal{L}^k = 1$, $\rho^{(k)}(y) = \omega_k f(|y|)$ for some $f \in C_c^1([0,1))$ and $\omega_k \in (0,+\infty)$. Notice that the constant ω_k is the normalizing constant, so that $\int \rho^{(k)} d\mathcal{L}^k = 1$, which implies that $\omega_k = (\int_0^1 t^{k-1} f(t)dt)^{-1}$. In particular, let us carefully choose the function f so that we have a bound from above of ω_k: let $f \in C_c^1([0,1))$ such that it is non-increasing, $f(t) = 1$ for all $t \in [0, \frac{1}{2}]$, $f \geq 0$. Consider now, $t_k \in (\frac{1}{2}, 1)$ such that $f(t_k) = \frac{1}{k}$, then

$$\frac{1}{\omega_k} = \int_0^1 t^{k-1} f(t)dt \geq \int_0^{t_k} t^{k-1} f(t)dt \geq \frac{1}{k}\int_0^{t_k} t^{k-1}dt = \frac{t_k^k}{k^2} \geq \frac{1}{k^2 2^k},$$

so that $\omega_k \leq k^2 2^k$.

Consider now $\Psi_\varepsilon = \Psi * \rho^{(k)}_\varepsilon(x)$, where $\rho^{(k)}_\varepsilon(x) = \frac{1}{\varepsilon^k}\rho^{(k)}(x/\varepsilon)$, for which the following properties hold:

- $\|\Psi - \Psi_\varepsilon\|_\infty \leq \varepsilon$ for all $\varepsilon > 0$;
- $\|\Psi_\varepsilon\|_\infty \leq \|\Psi\|_{C^0} \leq 1$ and more generally, since all the derivatives of order at most $h-1$ can be transferred on the function Ψ, we have $[\![\Psi_\varepsilon]\!]_{C^j} \leq [\![\Psi]\!]_{C^j} \leq 1$ for all $j \leq h-1$, since for all $y \in \mathbb{R}^k$ and $j \leq h-1$

$$\sum_{i_1,\dots,i_j=1}^k |(\partial_{i_1}\dots\partial_{i_j}\Psi) * \rho_\varepsilon(y)| \leq \int_{\mathbb{R}^k} \sum_{i_1,\dots,i_j=1}^k |\partial_{i_1}\dots\partial_{i_j}\Psi(y')|\rho_\varepsilon(y-y')dy' \leq [\![\Psi]\!]_{C^j}$$

- regarding $\rho^{(k)}$, it holds

$$\partial_i\rho^{(k)}_\varepsilon(x) = \frac{1}{\varepsilon^{k+1}}\partial_i\rho^{(k)}(x/\varepsilon), \quad \|\partial_i\rho^{(k)}\|_\infty \leq \omega_k\|f'\|_\infty,$$

  so that for all $y \in \mathbb{R}^k$, it holds

$$\begin{aligned}
\sum_{i_1,\dots,i_h=1}^k |\partial_{i_1}\dots\partial_{i_h}\Psi_\varepsilon(y)| &= \sum_{i_1=1}^k \sum_{i_2,\dots,i_h=1}^k |(\partial_{i_2}\dots\partial_{i_h}\Psi) * (\partial_{i_1}\rho_\varepsilon)(y)| \\
&\leq \sum_{i_1=1}^k \sum_{i_2,\dots,i_h=1}^k \int_{\mathbb{R}^k} |\partial_{i_2}\dots\partial_{i_h}\Psi(y-y')|\frac{1}{\varepsilon^{k+1}}\left|\partial_{i_1}\rho\left(\frac{y'}{\varepsilon}\right)\right| dy' \\
&= \int_{\mathbb{R}^k}\left(\sum_{i_2,\dots,i_h=1}^k |\partial_{i_2}\dots\partial_{i_h}\Psi(y-y')|\right)\left(\sum_{i_1=1}^k \frac{1}{\varepsilon^{k+1}}\left|\partial_{i_1}\rho\left(\frac{y'}{\varepsilon}\right)\right|\right) dy' \\
&\leq [\![\Psi]\!]_{C^{h-1}}\frac{1}{\varepsilon}\sum_{i_1=1}^k\int_{B(0,1)} |\partial_{i_1}\rho(y')|dy' \leq \frac{kV_k\omega_k\|f'\|_\infty}{\varepsilon} \leq \frac{k^3 2^k V_k\|f'\|_\infty}{\varepsilon},
\end{aligned}$$

  where V_k is the k-dimensional volume of the unit ball of $\mathbb{R}^k$, and the last inequality follows from the bound $\omega_k \leq k^2 2^k$. Recalling that $V_k = \frac{\pi^{k/2}}{\Gamma(k/2+1)}$, the quantity $k^3 2^k V_k \to 0$ as $k \to +\infty$, which implies that there exists a constant $C > 0$ such that $k^3 2^k V_k\|f'\| \leq C$ for all $k \in \mathbb{N}$ (such a sequence attains its maximum value for $k = 30$). Putting everything together, for all $\varepsilon > 0$, it holds

$$[\![\Psi_\varepsilon]\!]_{C^h} \leq \frac{C}{\varepsilon}.$$

Notice that for all $M, N \in \mathcal{P}(\mathcal{P}(\mathbb{R}^d))$ it holds

$$\left| \int_{\mathcal{P}} \Psi \circ L_{k,\boldsymbol{\varphi}} dM - \int_{\mathcal{P}} \Psi_\varepsilon \circ L_{k,\boldsymbol{\varphi}} dM \right| \le \varepsilon$$

and

$$\mathfrak{D}^{(r)}_{h,\ell,\mathcal{P}}(M,N) \le r \mathfrak{D}_{h,\ell,\mathcal{P}}(M,N) \quad \forall r > 1,$$

since if $\tilde{\Psi}$ is a competitor for $\mathfrak{D}^{(r)}_{h,\ell,\mathcal{P}}(M,N)$, then $\tilde{\Psi}/r$ is a competitor $\mathfrak{D}_{h,\ell,\mathcal{P}}(M,N)$. Then, fixing $\varepsilon \in (0, C)$, (3.18) follows from the following chain of inequalities:

$$\begin{aligned}
\mathfrak{D}_{h-1,\ell,\mathcal{P}}(M,N) = \sup & \left\{ \int_{\mathcal{P}} \Psi \circ L_{k,\boldsymbol{\varphi}} d(M-N) \,:\, k \ge 1,\, \Psi \in C^{h-1}_c(\mathbb{R}^k),\ [\![\Psi]\!]_{C^j} \le 1 \ \ \forall j \le h-1 \right\} \\
\le & 2\varepsilon + \sup \left\{ \int_{\mathcal{P}} \Psi_\varepsilon \circ L_{k,\boldsymbol{\varphi}} d(M-N) \,:\, k \ge 1,\, \Psi \in C^{h-1}_c(\mathbb{R}^k),\ [\![\Psi]\!]_{C^j} \le 1 \ \ \forall j \le h-1 \right\} \\
\le & 2\varepsilon + \sup \left\{ \int_{\mathcal{P}} \tilde{\Psi} \circ L_{k,\boldsymbol{\varphi}} d(M-N) \,:\, k \ge 1,\, \tilde{\Psi} \in C^{h}_c(\mathbb{R}^k),\ [\![\tilde{\Psi}]\!]_{C^j} \le 1 \ \ \forall j \le h-1,\ [\![\tilde{\Psi}]\!]_{C^h} \le \frac{C}{\varepsilon} \right\} \\
= & 2\varepsilon + \mathfrak{D}^{(C\varepsilon^{-1})}_{h,\ell,\mathcal{P}}(M,N) \le 2\varepsilon + \frac{C}{\varepsilon} \mathfrak{D}_{h,\ell,\mathcal{P}}(M,N).
\end{aligned}$$

We conclude proving the last statement by induction on h. If $h = 1$, it follows from Lemma 3.7. Let $h \ge 2$ and assume that $(\mathcal{P}(\mathcal{P}(\mathbb{R}^d)), \mathfrak{D}_{h-1,\ell,\mathcal{P}})$ is a complete metric space that induces the narrow over narrow topology. We conclude proving that $(M_n)_{n\in\mathbb{N}} \subset \mathcal{P}(\mathcal{P}(\mathbb{R}^d))$ is a Cauchy sequence for $\mathfrak{D}_{h,\ell,\mathcal{P}}$ if and only if it is so for $\mathfrak{D}_{h-1,\ell,\mathcal{P}}$. The sufficiency follows from the first inequality in (3.18). On the other hand, assume $(M_n)_{n\in\mathbb{N}} \subset \mathcal{P}(\mathcal{P}(\mathbb{R}^d))$ is a Cauchy sequence w.r.t. $\mathfrak{D}_{h,\ell,\mathcal{P}}$. Let $\varepsilon > 0$ and consider $\bar{n} \in \mathbb{N}$ such that $\mathfrak{D}_{h,\ell,\mathcal{P}}(M_n, M_m) \le \frac{\varepsilon^2}{8C}$ for all $n, m \ge \bar{n}$. Then, using (3.18) with $\varepsilon/4$, it follows that for all $m, n \ge \bar{n}$

$$\mathfrak{D}_{h-1,\ell,\mathcal{P}}(M_n, M_m) \le \frac{4C}{\varepsilon} \mathfrak{D}_{h,\ell,\mathcal{P}}(M_n, M_m) + \frac{\varepsilon}{2} \le \varepsilon.$$

At this point, it is easy to conclude that $\mathfrak{D}_{h,\ell,\mathcal{P}}$ is a complete distance and it induces the same topology of $\mathfrak{D}_{h-1,\ell,\mathcal{P}}$. □

4. Equation on random measures and nested superposition principle

In this section, we introduce an evolution equation for random measures associated with the Borel functions

$$a : [0,T] \times \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \to \mathrm{Sym}_+(\mathbb{R}^{d\times d}), \quad b : [0,T] \times \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}^d. \tag{4.1}$$

According to the Kolmogorov-Fokker-Planck equation and the martingale problem presented in Section 2.3, we introduce the operator

$$L_{b,a}\phi(t,x,\mu) := \frac{1}{2} a_t(x,\mu) : \nabla^{\otimes_2}\phi(x) + b_t(x,\mu) \cdot \nabla\phi(x), \tag{4.2}$$

for any $\phi \in C^2_b(\mathbb{R}^d)$. It is also useful to define the following objects: given $\boldsymbol{\mu} = (\mu_t)_{t\in[0,T]} \in C_T(\mathcal{P}(\mathbb{R}^d))$ and $\lambda \in \mathcal{P}(C_T(\mathbb{R}^d))$, for any $\phi \in C^2_b(\mathbb{R}^d)$ let

$$\begin{aligned}
L^{\boldsymbol{\mu}}_{b_t,a_t}\phi(x) &:= \frac{1}{2} a_t(x,\mu_t) : \nabla^{\otimes_2}\phi(x) + b_t(x,\mu_t) \cdot \nabla\phi(x), \\
L^{\lambda}_{b_t,a_t}\phi(x) &:= \frac{1}{2} a_t(x,(\mathrm{e}_t)_\sharp\lambda) : \nabla^{\otimes_2}\phi(x) + b_t(x,(\mathrm{e}_t)_\sharp\lambda) \cdot \nabla\phi(x).
\end{aligned} \tag{4.3}$$

Notice that, for any $\phi \in C_b^2$, it holds

$$|L_{b,a}\phi(t,x,\mu)| \leq \frac{1}{2}|\nabla^{\otimes_2}\phi||a(t,x,\mu)| + |\nabla\phi||b(t,x,\mu)|, \tag{4.4}$$

and similarly for the operator written in the forms $L^{\boldsymbol{\mu}}_{b_t,a_t}\phi$ and $L^{\lambda}_{b_t,a_t}\phi$.

The non-local nature of the operator L leads to an operator that acts on cylinder functions: for all $t \in [0,T]$ we define the operator $\mathcal{K}_{b_t,a_t}$ acting on cylinder functions $\mathrm{Cyl}_b^{1,2}(\mathcal{P}(\mathbb{R}^d))$ as

$$\begin{aligned}\mathcal{K}_{b_t,a_t}F(x,\mu) &:= \sum_{i=1}^k \partial_i\Psi(L_\Phi(\mu))L_{b,a}\phi_i(t,x,\mu)\\ &= \sum_{i=1}^k \partial_i\Psi(L_\Phi(\mu))\Big(b_t(x,\mu)\cdot\nabla\phi_i(x) + \frac{1}{2}a_t(x,\mu) : \nabla^{\otimes_2}\phi_i(x)\Big),\end{aligned} \tag{4.5}$$

for all $F = \Psi \circ L_{\boldsymbol{\Phi}} \in \mathrm{Cyl}_b^{1,2}(\mathcal{P}(\mathbb{R}^d))$. It is well defined, i.e. it does not depend on the representation chosen for the cylinder function F, since it is uniquely determined by the expressions

$$\sum_{i=1}^k \partial_i\Psi(L_\Phi(\mu))\nabla\phi_i(x) = \nabla_x\left(\frac{d^+}{d\varepsilon}|_{\varepsilon=0}\ F((1-\varepsilon)\mu + \varepsilon\delta_x)\right),$$

$$\sum_{i=1}^k \partial_i\Psi(L_\Phi(\mu))\nabla^{\otimes_2}\phi_i(x) = \nabla_x^{\otimes_2}\left(\frac{d^+}{d\varepsilon}|_{\varepsilon=0}\ F((1-\varepsilon)\mu + \varepsilon\delta_x)\right).$$

Definition 4.1. *Let a and b as in (4.1) and $\boldsymbol{M} = (M_t)_{t\in[0,T]} \in C_T(\mathcal{P}(\mathcal{P}(\mathbb{R}^d)))$. We say that it solves the equation $\partial_t M_t = \mathcal{K}^*_{b_t,a_t}M_t$ if*

$$\int_0^T\int_{\mathcal{P}}\int_{B_R} |a(t,x,\mu)| + |b(t,x,\mu)|d\mu(x)dM_t(\mu)dt < +\infty \quad \forall R > 0, \tag{4.6}$$

and for any $F = \Psi\circ L_\Phi \in \mathrm{Cyl}_c^{1,2}(\mathcal{P}(\mathbb{R}^d))$ and $\xi \in C_c^1(0,T)$ it holds

$$\int_0^T \xi'(t)\int_{\mathcal{P}} F(\mu)dM_t(\mu)dt = -\int_0^T \xi(t)\int_{\mathcal{P}}\int_{\mathbb{R}^d}\mathcal{K}_{b_t,a_t}F(x,\mu)d\mu(x)dM_t(\mu)dt. \tag{4.7}$$

The assumption that the curve $t \mapsto M_t \in \mathcal{P}(\mathcal{P}(\mathbb{R}^d))$ is continuous is not restrictive for our purposes, as we will always assume the stronger integrability condition

$$\int_0^T\int_{\mathcal{P}}\int_{\mathbb{R}^d}\frac{|a(t,x,\mu)|}{1+|x|^2} + \frac{|b(t,x,\mu)|}{1+|x|}d\mu(x)dM_t(\mu)dt < +\infty. \tag{4.8}$$

Lemma 4.2. *Let a and b be as above and $\boldsymbol{M} = (M_t)_{t\in[0,T]} \subset \mathcal{P}(\mathcal{P}(\mathbb{R}^d))$ such that $[0,T] \ni t \mapsto M_t \in \mathcal{P}(\mathcal{P}(\mathbb{R}^d))$ is Borel measurable and satisfies (4.8) and (4.7). Then, there exists a curve $(\tilde{M}_t)_{t\in[0,T]} \in AC_T(\mathcal{P}(\mathcal{P}(\mathbb{R}^d)), \mathcal{W}_{1,D_{2,w}})$ such that $\tilde{M}_t = M_t$ for a.e. $t \in [0,T]$. In particular, $(\tilde{M}_t)_{t\in[0,T]} \in C_T(\mathcal{P}(\mathcal{P}(\mathbb{R}^d)))$ and is the unique continuous representative for $\boldsymbol{M}$.*

Proof. For any $F = \Psi\circ L_\Phi \in \mathrm{Cyl}_c^{1,2}$, the function $t \mapsto \int F(\mu)dM_t(\mu)$ is in $W^{1,1}(0,T)$, with distributional derivative

$$t \mapsto \int_{\mathcal{P}}\sum_{i=1}^k \partial_i\Psi(L_\Phi(\mu))\int_{\mathbb{R}^d} L_{b,a}\phi_i(t,x,\mu)d\mu(x)dM_t(\mu) \in L^1(0,T).$$

Then, there exists $I_F \subset (0,T)$ such that for any $s,t \in I_F$

$$\int F(\mu)dM_t(\mu) - \int F(\mu)dM_s(\mu) = \int_s^t \int_{\mathcal{P}} \sum_{i=1}^k \partial_i \Psi(L_\Phi(\mu)) \int_{\mathbb{R}^d} L_{b,a}\phi_i(r,x,\mu)d\mu(x)dM_r(\mu)dr.$$

Consider now $I := \bigcap_{F\in\mathcal{C}} I_F$ with $\mathcal{C}$ satisfying the following properties:

- $F = \Psi \circ L_\Phi \in \mathcal{C}$ implies that $\Psi \in \mathcal{F}_k$ for some $k \in \mathbb{N}$ countable such that $\Psi : \mathbb{R}^k \to \mathbb{R}$ is continuously differentiable satisfying $\|\Psi\|_\infty, [\![\Psi]\!]_{C^1} \le 1$ and
$$W_{1,D_\infty}(\widetilde{M}, \widetilde{N}) = \sup_{k\in\mathbb{N}} \sup_{\Psi\in\mathcal{F}_k} \int \Psi(\pi^k(\underline{x}))d\big(\widetilde{M} - \widetilde{N}\big)(\underline{x}).$$
See [PS25a, Remark C.7] for the existence of the $\mathcal{F}_k$'s;
- $F = \Psi \circ L_\Phi$ implies that $\Phi = (\varphi_1, \dots, \varphi_k)$ for some $k \in \mathbb{N}$ and $\mathcal{S}_{2,w} = \{\varphi_1, \varphi_2, \varphi_3, \dots\}$ introduced in Proposition 3.3, (3). Using the notation introduced in §3.2, in this case we write $L_\Phi = L_{k,\boldsymbol{\varphi}}$.

These properties imply that $\mathcal{C}$ is countable, so $I \subset (0,T)$ has full Lebesgue measure. In particular, defining $\iota := \iota_{2,w}$ as in (3.10), for any $s,t \in I$ it holds

$$\begin{aligned}
W_{1,D_{2,w}}(M_s, M_t) = W_{1,D_\infty}(\iota_\sharp M_s, \iota_\sharp M_t) &= \sup_{k\in\mathbb{N}} \sup_{\Psi\in\mathcal{F}_k} \int \Psi(\pi^k(\underline{x}))d\big(\iota_\sharp M_s - \iota_\sharp M_t\big)(\underline{x}) \\
&= \sup_{k\in\mathbb{N}} \sup_{\Psi\in\mathcal{F}_k} \int \Psi(L_{\varphi_1}(\mu), \dots, L_{\varphi_k}(\mu))d\big(M_s - M_t\big)(\mu) \\
&= \sup_{k\in\mathbb{N}} \sup_{\Psi\in\mathcal{F}_k} \int_s^t \int_{\mathcal{P}} \sum_{i=1}^k \partial_i\Psi(L_{k,\boldsymbol{\varphi}}(\mu)) \int_{\mathbb{R}^d} L_{b,a}\varphi_i(r,x,\mu)d\mu(x)dM_r(\mu)dr \\
&\le \int_s^t \int_{\mathcal{P}} \int_{\mathbb{R}^d} \frac{|a(r,x,\mu)|}{1+|x|^2} + \frac{|b(r,x,\mu)|}{1+|x|} d\mu(x)dM_r(\mu)dr,
\end{aligned}$$

thanks to (4.4) and the facts that for all $\Psi \in \mathcal{F}_k$, it holds $[\![\Psi]\!]_{C^1} \le 1$ and for all $\varphi_i \in \mathcal{S}_{2,w}$ we have $\|(1+|x|)\nabla\varphi_i\|_\infty \le 1$ and $\|(1+|x|^2)\nabla^{\otimes 2}\varphi_i\|_\infty \le 1$. In particular, by completeness of $(\mathcal{P}(\mathcal{P}(\mathbb{R}^d)), W_{1,D_{2,w}})$, there exists $(\tilde{M}_t) \in AC_T(\mathcal{P}(\mathcal{P}(\mathbb{R}^d)), W_{1,D_{2,w}})$, which satisfies the requirements by construction. □

Let us highlight some features of the operator $\mathcal{K}$: for simplicity we get rid of the time variable, considering two Borel measurable maps $a : \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \to \mathrm{Sym}_+(\mathbb{R}^{d\times d})$ and $b : \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}^d$ and the operator

$$\mathcal{K}_{b,a}F(x,\mu) := \sum_{i=1}^k \partial_i\Psi(L_\Phi(\mu))\Big(b(x,\mu)\cdot\nabla\phi_i(x) + \frac{1}{2}a(x,\mu) : \nabla^{\otimes 2}\phi_i(x)\Big),$$

for all $F = \Psi \circ L_\Phi \in \mathrm{Cyl}_b^{1,2}(\mathcal{P}(\mathbb{R}^d))$. It is not hard to verify that the operator $\mathcal{K}_{b,a}$ satisfies the Leibniz rule, that is

$$\mathcal{K}_{b,a}(FG) = F\mathcal{K}_{b,a}G + G\mathcal{K}_{b,a}F, \tag{4.9}$$

which is a prerogative of first-order operators. Indeed, we may see the equation $\partial_t M_t = \mathcal{K}^*_{b_t,a_t} M_t$ as a first order equation on the space of probability measures, but to do so it is important to understand that the natural geometry to consider on $\mathcal{P}(\mathbb{R}^d)$ cannot be the one given by the Wasserstein on Wasserstein distance, but it should be a second order metric that naturally puts probability measures in duality with two times differentiable functions, as it can be seen in the proof of Lemma 4.2 or later in the proof of the superposition result. To conclude, in $\mathcal{P}(\mathcal{P}(\mathbb{R}^d))$

we see two different levels for the possible geometries to put on it: the inner and the outer one, as it can be seen in Definition 3.5.

We proceed by showing how this equation on random measures is linked to the KFP equation and the martingale problem introduced in Section 2.3. In particular, we link the following objects through a superposition principle:

(1) a curve of random measures $\boldsymbol{M} = (M_t)_{t\in[0,T]} \in C_T(\mathcal{P}(\mathcal{P}(\mathbb{R}^d)))$ solution of $\partial_t M_t = \mathcal{K}^*_{b_t,a_t} M_t$, according to Definition 4.1;

(2) a probability measure $\Lambda \in \mathcal{P}(C_T(\mathcal{P}(\mathbb{R}^d)))$ such that

$$\int\int_0^T \int \frac{|b(t,x,\mu_t)|}{1+|x|} + \frac{|a(t,x,\mu_t)|}{1+|x|^2} d\mu_t(x) d\Lambda dt(\boldsymbol{\mu}) < +\infty \tag{4.10}$$

and Λ-a.e. $\boldsymbol{\mu} = (\mu_t)_{t\in[0,T]}$ is a solution of $\partial_t \mu_t = (L^{\boldsymbol{\mu}}_{b_t,a_t})^* \mu_t$, according to Definition 2.4;

(3) a random measure over curves $\mathfrak{L} \in \mathcal{P}(\mathcal{P}(C_T(\mathbb{R}^d)))$ satisfying

$$\int\int\int_0^T \frac{|b(t,\gamma_t,(\mathrm{e}_t)_\sharp\lambda)|}{1+|\gamma_t|} + \frac{|a(t,\gamma_t,(\mathrm{e}_t)_\sharp\lambda)|}{1+|\gamma_t|^2} dt d\lambda(\gamma) d\mathfrak{L}(\lambda) < +\infty \tag{4.11}$$

and $\mathfrak{L}$-a.e. $\lambda \in \mathcal{P}(C_T(\mathbb{R}^d))$ is a solution to the martingale problem associated to the operator $L^\lambda_{a_t,b_t}$, according to Definition 2.5.

To this aim, it will be crucial to define the following subsets.

Definition 4.3. *Let* $b : [0,T]\times\mathbb{R}^d\times\mathcal{P}(\mathbb{R}^d) \to \mathbb{R}^d$ *and* $a : [0,T]\times\mathbb{R}^d\times\mathcal{P}(\mathbb{R}^d) \to \mathrm{Sym}_+(\mathbb{R}^{d\times d})$ *be Borel measurable functions. The set* $\mathrm{KFP}(b,a) \subset C_T(\mathcal{P}(\mathbb{R}^d))$ *is the subset of the solutions to the non-local Kolmogorov-Fokker-Planck equation, i.e.*

$$\mathrm{KFP}(b,a) := \left\{ \boldsymbol{\mu} = (\mu_t) \ : \ \int_0^T\int_{\mathbb{R}^d} \frac{|b_t(x,\mu_t)|}{1+|x|} + \frac{|a_t(x,\mu_t)|}{1+|x|^2} d\mu_t(x) dt < +\infty, \ \partial_t\mu_t = (L^{\boldsymbol{\mu}}_{b_t,a_t})^*\mu_t \right\}. \tag{4.12}$$

The set $\mathrm{MP}(b,a) \subset \mathcal{P}(C_T(\mathbb{R}^d))$ *is the subset of solutions to the non-local martingale problem, i.e.*

$$\begin{aligned}\mathrm{MP}(b,a) := \Big\{ \lambda \in \mathcal{P}(C_T(\mathbb{R}^d)) \ : \ &\int\int_0^T \frac{|b_t(\gamma_t,(\mathrm{e}_t)_\sharp\lambda)|}{1+|\gamma_t|} + \frac{|a_t(\gamma_t,(\mathrm{e}_t)_\sharp\lambda)|}{1+|\gamma_t|^2} dt d\lambda(\gamma) < +\infty, \\ &\lambda \text{ is a sol. of the martingale problem associated to } L^\lambda_{b_t,a_t} \Big\}.\end{aligned} \tag{4.13}$$

It is important to stress here the relation between a probability measure $\lambda \in \mathrm{MP}(b,a)$ and solutions of some stochastic differential equation. As pointed out in §2.3, if the map a arises from some $\sigma : [0,T]\times\mathbb{R}^d\times\mathcal{P}(\mathbb{R}^d) \to \mathbb{R}^{d\times m}$, $m \geq 1$, as $a = \sigma\sigma^\top$, then $\lambda \in \mathrm{MP}(b,a)$ if and only if there exists a probability space $(\Omega,\mathcal{F},\mathbb{P})$, an $\mathbb{R}^m$-valued Brownian motion $(B_t)_{t\in[0,T]}$ on it and an $\mathbb{R}^d$-valued continuous process $(X_t)_{t\in[0,T]}$ whose law coincides with λ, and in particular $(\mathrm{e}_t)_\sharp\lambda = \mathrm{Law}(X_t)$ for all $t \in [0,T]$, so that it satisfies

$$dX_t = b_t(X_t, \mathrm{Law}(X_t))dt + \sigma_t(X_t,\mathrm{Law}(X_t))dB_t \tag{4.14}$$

and

$$\mathbb{E}\left[\int_0^T \frac{|b_t(X_t,\mathrm{Law}(X_t))|}{1+|X_t|} + \frac{|a_t(X_t,\mathrm{Law}(X_t))|}{1+|X_t|^2}\right] < +\infty \tag{4.15}$$

Now, we show that there is a natural hierarchy between the objects presented above. Let us first introduce the following operators:

$$
\begin{aligned}
&E : \mathcal{P}(C_T(\mathbb{R}^d)) \to C_T(\mathcal{P}(\mathbb{R}^d)), \quad E(\lambda) := ((\mathrm{e}_t)_\sharp \lambda)_{t\in[0,T]},\\
&E_t : \mathcal{P}(C_T(\mathbb{R}^d)) \to \mathcal{P}(\mathbb{R}^d), \quad E_t(\lambda) = (\mathrm{e}_t)_\sharp \lambda, \qquad (4.16)\\
&\mathfrak{e}_t : C_T(\mathcal{P}(\mathbb{R}^d)) \to \mathbb{R}^d, \quad \mathfrak{e}_t(\boldsymbol{\mu}) = \mu_t.
\end{aligned}
$$

Proposition 4.4. *Let $b : [0,T]\times\mathbb{R}^d\times\mathcal{P}(\mathbb{R}^d) \to \mathbb{R}^d$ and $a : [0,T]\times\mathbb{R}^d\times\mathcal{P}(\mathbb{R}^d) \to \mathrm{Sym}_+(\mathbb{R}^{d\times d})$ be Borel measurable maps.*
The functional E maps $\mathrm{MP}(b,a)$ into $\mathrm{KFP}(b,a)$. In particular, if $\mathfrak{L} \in \mathcal{P}(\mathcal{P}(C_T(\mathbb{R}^d)))$ is concentrated over $\mathrm{MP}(b,a)$, then $\Lambda := E_\sharp \mathfrak{L} \in \mathcal{P}(C_T(\mathcal{P}(\mathbb{R}^d)))$ is concentrated over $\mathrm{KFP}(b,a)$.

Proof. Let $\lambda \in \mathrm{MP}(b,a)$ and $\mu_t := (\mathrm{e}_t)_\sharp\lambda$. The integrability condition is clearly satisfied. On the other hand, notice that, directly by definition $L^\lambda_{b_t,a_t}\phi(x) = L^{\boldsymbol{\mu}}_{b_t,a_t}\phi(x)$ for any $x\in\mathbb{R}^d$ and $\phi \in C^2_c(\mathbb{R}^d)$. Then, for any $\xi \in C^1_c(0,T)$ and $\phi\in C^2_c(\mathbb{R}^d)$, for any $t\in[0,T]$ define, as in (2.11),

$$
X^{\xi,\phi,\lambda}_t(\gamma) := \xi(t)\phi(\gamma_t) - \int_0^t \xi'(r)\phi(\gamma_r) + \xi(r)L^\lambda_{b_r,a_r}\phi(\gamma_r)dr.
$$

Since $\lambda\in\mathrm{MP}(b,a)$, $t\mapsto X^{\xi,\phi,\lambda}_t$ is a martingale in the filtered space $(C_T(\mathbb{R}^d),(\mathcal{F}_t)_{t\in[0,T]},\lambda)$ (see Definition 2.5), then

$$
\int_0^T\int_{\mathbb{R}^d}\xi'(t)\phi(x) + \xi(t)L^{\boldsymbol{\mu}}_{b_t,a_t}\phi(x)d\mu_t(x)dt = \int -X^{\xi,\phi,\lambda}_T(\gamma)d\lambda = \int -X^{\xi,\phi,\lambda}_0(\gamma)d\lambda = 0.
$$

□

Proposition 4.5. *Let $b : [0,T]\times\mathbb{R}^d\times\mathcal{P}(\mathbb{R}^d) \to \mathbb{R}^d$ and $a : [0,T]\times\mathbb{R}^d\times\mathcal{P}(\mathbb{R}^d) \to \mathrm{Sym}_+(\mathbb{R}^{d\times d})$ be Borel measurable functions.*
Let $\Lambda \in \mathcal{P}(C_T(\mathcal{P}(\mathbb{R}^d)))$ be satisfying (4.10) and concentrated over $\mathrm{KFP}(b,a)$. Then the curve of random measures $\boldsymbol{M} \in C_T(\mathcal{P}(\mathcal{P}(\mathbb{R}^d)))$, defined by $M_t := (\mathfrak{e}_t)_\sharp\Lambda$, satisfies (4.8) and solves the equation $\partial_t M_t = \mathcal{K}^*_{b_t,a_t}M_t$, according to Definition 4.1.

Proof. First of all, thanks to (4.10) and Fubini's theorem, it holds

$$
\begin{aligned}
&\int_0^T\int_{\mathcal{P}}\int_{\mathbb{R}^d}\frac{|a(t,x,\mu)|}{1+|x|^2} + \frac{|b(t,x,\mu)|}{1+|x|}d\mu(x)dM_t(\mu)dt =\\
&\qquad = \int\int_0^T\int_{\mathbb{R}^d}\frac{|a(t,x,\mu_t)|}{1+|x|^2} + \frac{|b(t,x,\mu_t)|}{1+|x|}d\mu_t(x)dtd\Lambda(\boldsymbol{\mu}) < +\infty.
\end{aligned}
$$

Now, let $\xi\in C^1_c(0,T)$ and $F = \Psi\circ L_\Phi \in \mathrm{Cyl}^{1,2}_c(\mathcal{P}(\mathbb{R}^d))$, then

$$
\begin{aligned}
\int_0^T\xi'(t)\int F(\mu)dM_t(\mu)dt &= \int_0^T\xi'(t)\int F(\mu_t)d\Lambda(\boldsymbol{\mu})dt\\
&= \int\int_0^T\xi'(t)\Psi(L_\Phi(\mu_t))dtd\Lambda(\boldsymbol{\mu})\\
&= -\int\int_0^T\xi(t)\sum_{i=1}^k\partial_i\Psi(L_\Phi(\mu_t))\frac{d}{dt}\left(\int_{\mathbb{R}^d}\phi_i(x)d\mu_t(x)\right)dtd\Lambda(\boldsymbol{\mu})\\
&= -\int\int_0^T\xi(t)\sum_{i=1}^k\partial_i\Psi(L_\Phi(\mu_t))\left(\int_{\mathbb{R}^d}L^{\boldsymbol{\mu}}_{b_t,a_t}\phi_i(x)d\mu_t(x)\right)dtd\Lambda(\boldsymbol{\mu})
\end{aligned}
$$

$$= -\int\int_0^T \xi(t)\sum_{i=1}^k \partial_i\Psi(L_\Phi(\mu_t))\left(\int_{\mathbb{R}^d} L_{b,a}\phi_i(t,x,\mu_t)d\mu_t(x)\right)dtd\Lambda(\boldsymbol{\mu})$$
$$= -\int_0^T\int \xi(t)\sum_{i=1}^k \partial_i\Psi(L_\Phi(\mu_t))\left(\int_{\mathbb{R}^d} L_{b,a}\phi_i(t,x,\mu_t)d\mu_t(x)\right)d\Lambda(\boldsymbol{\mu})dt$$
$$= -\int_0^T\int \xi(t)\sum_{i=1}^k \partial_i\Psi(L_\Phi(\mu))\int_{\mathbb{R}^d} L_{b,a}\phi_i(t,x,\mu)d\mu(x)dM_t(\mu)dt$$
$$= -\int_0^T \xi(t)\int \mathcal{K}_{b_t,a_t}F(x,\mu)d\mu(x)dM_t(\mu)dt.$$

□

Corollary 4.6. *Let $b:[0,T]\times\mathbb{R}^d\times\mathcal{P}(\mathbb{R}^d)\to\mathbb{R}^d$ and $a:[0,T]\times\mathbb{R}^d\times\mathcal{P}(\mathbb{R}^d)\to\mathrm{Sym}_+(\mathbb{R}^{d\times d})$ be Borel measurable functions.*
Let $\mathfrak{L}\in\mathcal{P}(\mathcal{P}(C_T(\mathbb{R}^d)))$ be satisfying (4.11) *and concentrated over* $\mathrm{MP}(b,a)$*. Then the curve of random measures $\boldsymbol{M}\in C_T(\mathcal{P}(\mathcal{P}(\mathbb{R}^d)))$, defined by $M_t:=(E_t)_\sharp\Lambda$, satisfies* (4.8) *and solves the equation $\partial_t M_t=\mathcal{K}^*_{b_t,a_t}M_t$, according to Definition 4.1.*

Proof. Noticing that $E_t=E\circ\mathfrak{e}_t$, it follows from Propositions 4.4 and 4.5. □

To summarize: we proved that there is a natural hierarchy between the objects $\boldsymbol{M}$, Λ and $\mathfrak{L}$ listed above, that is $\mathfrak{L}\implies\Lambda\implies\boldsymbol{M}$. In the next subsections, we will show that we can also go in the opposite direction.

4.1. **Superposition principle: from $\boldsymbol{M}$ to Λ.** Here, we show that the result of Proposition 4.5 can be inverted, i.e. given a curve of random measures $\boldsymbol{M}\in C_T(\mathcal{P}(\mathcal{P}(\mathbb{R}^d)))$ solving the equation $\partial_t M_t=\mathcal{K}^*_{b_t,a_t}M_t$, we show it is possible to obtain a measure $\Lambda\in\mathcal{P}(C_T(\mathcal{P}(\mathbb{R}^d)))$ concentrated over $\mathrm{KFP}(b,a)$ such that $(\mathfrak{e}_t)_\sharp\Lambda=M_t$ for all $t\in[0,T]$.

Theorem 4.7. *Let $\boldsymbol{M}=(M_t)_{t\in[0,T]}\in C_T(\mathcal{P}(\mathcal{P}(\mathbb{R}^d)))$. Let $b:[0,T]\times\mathbb{R}^d\times\mathcal{P}(\mathbb{R}^d)\to\mathbb{R}^d$ and $a:[0,T]\times\mathbb{R}^d\times\mathcal{P}(\mathbb{R}^d)\to\mathrm{Sym}_+(\mathbb{R}^{d\times d})$ be Borel measurable functions and assume that* (4.8) *and* (4.7) *are satisfied. Then, there exists (possibly non-unique) $\Lambda\in\mathcal{P}(C_T(\mathcal{P}(\mathbb{R}^d)))$ satisfying:*

(1) $(\mathfrak{e}_t)_\sharp\Lambda=M_t$ *for all* $t\in[0,T]$*;*
(2)
$$\int\int_0^T\int\frac{|b(t,x,\mu_t)|}{1+|x|}+\frac{|a(t,x,\mu_t)|}{1+|x|^2}d\mu_t(x)dtd\Lambda(\boldsymbol{\mu})<+\infty;$$
(3) Λ *is concentrated over* $\mathrm{KFP}(b,a)$*, in particular Λ-a.e. $\boldsymbol{\mu}=(\mu_t)_{t\in[0,T]}$ solves the Kolmogorov-Fokker-Planck equation $\partial_t\mu_t=(L^{\boldsymbol{\mu}}_{b_t,a_t})^*\mu_t$.*

Proof. **Step 1**: let $\mathcal{S}_{2,w}=\{\varphi_1,\varphi_2,\dots\}$ as in Proposition 3.3, (3) and define $\iota=\iota_{2,w}:\mathcal{P}(\mathbb{R}^d)\to\mathbb{R}^\infty$ as in (3.10), keeping in mind that it is an isometry between $(\mathcal{P}(\mathbb{R}^d),D_{C^2_{0,w}})$ and $(\iota(\mathcal{P}(\mathbb{R}^d)),D_\infty)$. Let $\widetilde{M_t}:=\iota_\sharp M_t$ for any $t\in[0,T]$. Then, $(\widetilde{M_t})_{t\in[0,T]}\in AC_T(\mathcal{P}(\mathbb{R}^\infty),W_{1,D_\infty})$, indeed considering any $\Pi_{t,s}\in\Gamma(M_t,M_s)$ optimal for the distance $\mathcal{W}_{1,D_{2,w}}$, then

$$W_{1,D_\infty}(\widetilde{M_t},\widetilde{M_s})\le\int D_\infty(\underline{x},\underline{y})d(\iota,\iota)_\sharp\Pi_{t,s}(\underline{x},\underline{y})=\int D_{2,w}(\mu,\nu)d\Pi_{t,s}(\mu,\nu)=\mathcal{W}_{1,D_{2,w}}(M_t,M_s),$$

so we conclude thanks to Lemma 4.2.

Step 2: on $\mathbb{R}^\infty$ we define, component-wisely, the vector field

$$v_t^{(k)}(\underline{x}) := \begin{cases} \int_{\mathbb{R}^d} b_t(x,\mu)\cdot\nabla\varphi_k(x) + \frac{1}{2}a_t(x,\mu) : \nabla^{\otimes_2}\varphi_k(x)d\mu(x) & \text{if } \underline{x} = \iota(\mu) \\ 0 & \text{otherwise.} \end{cases} \tag{4.17}$$

Then, for any $\xi \in C_c^1(0,T)$ and $\tilde{F} \in \mathrm{Cyl}_b^1(\mathbb{R}^\infty)$, i.e. $\tilde{F}(\underline{x}) = \Psi(x_1,\dots,x_k)$ for some $k \in \mathbb{N}$ and $\Psi \in C_b^1(\mathbb{R}^k)$, it holds

$$\begin{aligned}
&\int_0^T \xi'(t)\int \tilde{F}(\underline{x})d\widetilde{M}_t(\underline{x})dt = \int_0^T\int_{\mathcal{P}} \Psi(L_{\phi_1(\mu)},\dots,L_{\phi_k}(\mu))dM_t(\mu)dt \\
&= -\int_0^T \xi(t)\int\sum_{i=1}^k \partial_i\Psi(L_\Phi(\mu))\int_{\mathbb{R}^d} b_t(x,\mu)\cdot\nabla\phi_i(x) + \frac{1}{2}a_t(x,\mu) : \nabla^{\otimes_2}\phi_i(x)d\mu(x)dM_t(\mu)dt \\
&= -\int_0^T \xi(t)\int\sum_{i=1}^k \partial_i\Psi(L_\Phi(\mu))v_t^{(i)}(\iota(\mu))dM_t(\mu)dt \\
&= -\int_0^T \xi(t)\int \nabla\tilde{F}(\underline{x})\cdot v_t(\underline{x})d\widetilde{M}_t(\underline{x})dt,
\end{aligned}$$

which means that $\widetilde{M}_t$ solves the continuity equation $\partial_t\widetilde{M}_t + \mathrm{div}_{\mathbb{R}^\infty}(v_t\widetilde{M}_t) = 0$ (see [AT14, Section 7]. Then there exists a measure $\widetilde{\Lambda} \in \mathcal{P}(C_T(\mathbb{R}^\infty,\tau_w))$, where τ_w is the element-wise convergence topology over $\mathbb{R}^\infty$ (see (3.8)), satisfying

- $(\mathrm{e}_t)_\sharp\widetilde{\Lambda} = \widetilde{M}_t$ for any $t \in [0,T]$;
- $\widetilde{\Lambda}$-a.e. $\tilde{\gamma} \in AC_T(\mathbb{R}^\infty,\tau_w)$, i.e. each component is in $AC_T(\mathbb{R})$, and it solves

$$\partial_t\tilde{\gamma}^{(i)} = v_t^{(i)}(\tilde{\gamma}_t) \quad \forall i \in \mathbb{N}, \text{ for a.e. } t \in [0,T].$$

Step 3: we prove that $\widetilde{\Lambda}$-a.e. $\tilde{\gamma}$ is such that $\tilde{\gamma}(t) \in \iota(\mathcal{P}(\mathbb{R}^d))$ for all $t \in [0,T]$. To this aim, it is sufficient to prove two things:

- $\widetilde{\Lambda}$-a.e. $\tilde{\gamma}$ is such that $\tilde{\gamma}(t) \in \iota(\mathcal{P}(\mathbb{R}^d))$ for all $t \in [0,T]\cap\mathbb{Q}$;
- $\widetilde{\Lambda}$-a.e. $\tilde{\gamma}$ is in $AC_T(\mathbb{R}^\infty, D_\infty)$.

Then, we conclude simply by completeness of $(\mathbb{R}^\infty, D_\infty)$ and the closedness of $\iota(\mathcal{P}(\mathbb{R}^d))$ in it. The first part is simply obtained noticing that for all $t \in [0,T]\cap\mathbb{Q}$ it holds

$$\widetilde{\Lambda}\big(\tilde{\gamma} \ : \ \tilde{\gamma}(t) \in \iota(\mathcal{P}(\mathbb{R}^d))\big) = \widetilde{M}_t(\iota(\mathcal{P}(\mathbb{R}^d))) = 1.$$

Regarding the second statement, for $\widetilde{\Lambda}$-a.e. $\tilde{\gamma}$ and for all $s,t \in [0,T]$ it holds

$$D_\infty(\tilde{\gamma}(t),\tilde{\gamma}(s)) = \sup_{n\in\mathbb{N}}|\tilde{\gamma}_n(t)-\tilde{\gamma}_n(s)|\wedge 1 \le \sup_{n\in\mathbb{N}}\int_s^t |v_r^{(n)}(\tilde{\gamma}(r))|dr \le \int_s^t \sup_{n\in\mathbb{N}}|v_r^{(n)}(\tilde{\gamma}(r))|dr$$

and it holds

$$\begin{aligned}
&\int\int_0^T \sup_{n\in\mathbb{N}}|v_r^{(n)}(\tilde{\gamma}(r))|drd\widetilde{\Lambda}(\tilde{\gamma}) = \int_0^T\int \sup_{n\in\mathbb{N}}|v_r^{(n)}(\underline{x})|d\widetilde{M}_r(\underline{x})dr \\
&\le \int_0^T\int \sup_{n\in\mathbb{N}}\int_{\mathbb{R}^d}\big|b_r(x,\mu)\cdot\nabla\varphi_n(x)\big| + \frac{1}{2}\big|a_r(x,\mu) : \nabla^{\otimes_2}\varphi_n(x)\big|d\mu(x) \\
&\le \int_0^T\int\int \frac{|b_r(x,\mu)|}{1+|x|} + \frac{|a_r(x,\mu)|}{1+|x|^2}d\mu(x)dM_r(\mu)dr < +\infty.
\end{aligned}$$

In particular, for $\widetilde{\Lambda}$-a.e. $\tilde{\gamma}$, the term $\int_0^T \sup_n |v_r^{(n)}(\tilde{\gamma}(r))|dr < +\infty$, which concludes the proof of the claim.

Step 4: the function

$$\begin{aligned}\Theta : C_T(\iota(\mathcal{P}(\mathbb{R}^d)), D_\infty) \to & C_T(\mathcal{P}(\mathbb{R}^d)),\\ \tilde{\gamma} \mapsto & [t \mapsto \iota^{-1}(\tilde{\gamma}(t))]\end{aligned}$$

is well-defined, together with $\Lambda := \Theta_\sharp \widetilde{\Lambda} \in \mathcal{P}(C_T(\mathcal{P}(\mathbb{R}^d)))$, thanks to the previous step. Let's verify that Λ satisfies the requirements. The fact that $(\mathfrak{e}_t)_\sharp \Lambda = M_t$ is straightforward from the definition of $\widetilde{M_t}$ and $(\mathrm{e}_t)_\sharp \widetilde{\Lambda} = \widetilde{M_t}$. Then, it follows

$$\int \int_0^T \int \frac{|b_t(x,\mu_t)|}{1+|x|} + \frac{|a_t(x,\mu_t)|}{1+|x|^2} d\mu_t(x) dt d\Lambda(\boldsymbol{\mu}) = \int_0^T \int \int \frac{|b_t(x,\mu)|}{1+|x|} + \frac{|a_t(x,\mu)|}{1+|x|^2} d\mu(x) dM_t(\mu) dt < +\infty.$$

Regarding the last part, let $\xi \in C_c^1(0,T)$ and $\varphi_k \in \mathcal{S}_{2,w}$. Then, consider a function $\Psi \in C_c^1(\mathbb{R})$ such that $\Psi(x) = x$ for all $x \in [-1,1]$, so that, defining $\tilde{F}(\underline{x}) = \Psi(x_k) \in \mathrm{Cyl}_c^1(\mathbb{R}^\infty)$, it holds

$$\begin{aligned}
&\int \left| \int_0^T \xi'(t) \int \varphi_k(x) d\mu_t(x) dt + \int_0^T \xi(t) \int L^{\boldsymbol{\mu}}_{b_t,a_t} \varphi_k(x) d\mu_t(x) dt \right| d\Lambda(\boldsymbol{\mu}) \\
&= \int \Bigg| \int_0^T \xi'(t) \Psi\left(\int \varphi_k(x) d\mu_t(x) \right) dt \\
&\quad + \int_0^T \xi(t) \Psi'(L_{\varphi_k}(\mu_t)) \int b_t(x,\mu) \cdot \nabla \varphi_k(x) + \frac{1}{2} a_t(x,\mu) : \nabla^{\otimes 2} \varphi_k(x) d\mu_t(x) dt \Bigg| d\Lambda(\boldsymbol{\mu}) \\
&= \int \left| \int_0^T \xi'(t) \tilde{F}(\tilde{\gamma}(t)) dt + \int_0^T \xi(t) \nabla \tilde{F}(\tilde{\gamma}(t)) \cdot v_t(\tilde{\gamma}(t)) dt \right| d\widetilde{\Lambda}(\tilde{\gamma}) = 0.
\end{aligned}$$

Selecting $\xi \in \mathcal{A} \subset C_c^1(0,T)$, with $\mathcal{A}$ dense subset in the unit ball of $C_0^1(0,T)$ w.r.t. the norm $\|\cdot\|_{C^1}$, this implies that for Λ-a.e. $\boldsymbol{\mu} = (\mu_t)_{t\in[0,T]}$, it holds

$$\xi'(t) \int \varphi(x) d\mu_t(x) dt = -\int_0^T \xi(t) \int L^{\boldsymbol{\mu}}_{b_t,a_t} \varphi(x) d\mu_t(x) dt \quad \forall \xi \in \mathcal{A},\ \forall \varphi \in \mathcal{S}_{2,w}.$$

By density of $\mathcal{A}$ in $C_0^1(0,T)$ and $\mathcal{S}_{2,w}$ in $C^2_{0,w}(\mathbb{R}^d)$ (that in particular contains $C_c^2(\mathbb{R}^d)$), for Λ-a.e. $\boldsymbol{\mu}$ it holds $\partial_t \mu_t = (L^{\boldsymbol{\mu}}_{b_t,a_t})^* \mu_t$. □

4.2. **Nested superposition principle: from Λ to $\mathfrak{L}$.** The goal here is to invert the result of Proposition 4.4, defining a measure $\mathfrak{L} \in \mathcal{P}(\mathcal{P}(C_T(\mathbb{R}^d)))$ concentrated over $\mathrm{MP}(b,a)$ given a measure $\Lambda \in \mathcal{P}(C_T(\mathcal{P}(\mathbb{R}^d)))$ concentrated over $\mathrm{KFP}(b,a)$, so that $E_\sharp \mathfrak{L} = \Lambda$. The strategy is to use a measurable selection argument to define a map $G : \mathrm{KFP}(b,a) \to \mathrm{MP}(b,a)$ that is a right-inverse for E and that we can use to define $\mathfrak{L} := G_\sharp \Lambda$. To do so, it is crucial to prove the Borel measurability of the subsets $\mathrm{KFP}(b,a) \subset C_T(\mathcal{P}(\mathbb{R}^d))$ and $\mathrm{MP}(b,a) \subset \mathcal{P}(C_T(\mathbb{R}^d))$.

Proposition 4.8. *Let $b : [0,T] \times \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}^d$ and $a : [0,T] \times \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \to \mathrm{Sym}_+(\mathbb{R}^{d\times d})$ be Borel measurable functions. Then, the subset $\mathrm{KFP}(b,a) \subset C_T(\mathcal{P}(\mathbb{R}^d))$ is Borel.*

Proof. Let $Y := [0,T] \times \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d)$ and define

$$\begin{aligned}\hat{B} := \Bigg\{ (\hat{\mu}, \hat{\nu}, \hat{\eta}) \in \mathcal{M}_+(Y) \times \mathcal{M}(Y;\mathbb{R}^d) \times \mathcal{M}(Y;\mathbb{R}^{d\times d}) \text{ s.t. } \frac{b}{1+|x|} \in L^1(\hat{\mu};\mathbb{R}^d), \\ \frac{a}{1+|x|^2} \in L^1(\hat{\mu};\mathbb{R}^{d\times d}),\ \hat{\nu} = \frac{b}{1+|x|}\hat{\mu},\ \hat{\eta} = \frac{a}{1+|x|^2}\hat{\mu} \Bigg\}\end{aligned} \tag{4.18}$$

and

$$\widehat{\mathrm{KFP}}(b,a) := \Big\{ (\hat\mu,\hat\nu,\hat\eta) \in \hat B \text{ s.t. } \forall \xi \in C_c^1(0,T),\ \forall \phi \in C_c^2(\mathbb{R}^d)\ \int_Y \xi'(t)\phi(x)d\hat\mu(t,x,\mu) = \tag{4.19}$$
$$= -\int_Y \xi(t)(1+|x|)\nabla\phi(x)\cdot d\hat\nu(t,x,\mu) - \frac{1}{2}\int_Y \xi(t)(1+|x|^2)\nabla^{\otimes_2}\phi(x) : d\hat\eta(t,x,\mu)\Big\}.$$

We endow $\mathcal{M}_+(Y)$ and $\mathcal{M}(Y;\mathbb{R}^n)$ with the narrow topology, so that $\mathcal{M}_+(Y)\times\mathcal{M}(Y;\mathbb{R}^d)\times\mathcal{M}(Y;\mathbb{R}^{d\times d})$ is endowed with the product topology. Then, we claim that $\widehat{\mathrm{KFP}}(b,a)$ is a Borel subset of the product: indeed the integral equality is a closed condition, since all the integrals involved in (4.19) are continuous functions, and $\hat B$ is Borel thanks to Lemma 2.1. Now, define the maps

$$\begin{aligned} &\pi^1 : \mathcal{M}_+(Y)\times\mathcal{M}(Y;\mathbb{R}^d)\times\mathcal{M}(Y;\mathbb{R}^{d\times d}) \to \mathcal{M}_+(Y), \quad \pi^1(\hat\mu,\hat\nu,\hat\eta) = \hat\mu \\ &\kappa : C_T(\mathcal{P}(\mathbb{R}^d)) \to \mathcal{M}_+(Y), \quad \kappa(\boldsymbol{\mu}) = dt\otimes(\mu_t\otimes\delta_{\mu_t}) = \int_0^T \delta_t\otimes\mu_t\otimes\delta_{\mu_t}dt \end{aligned} \tag{4.20}$$

The projection map $\pi^1|_{\hat B}$ is continuous and injective, thus it maps Borel sets in Borel sets (see e.g. [Bog07] for a detailed description or [PS25a, Appendix A] for a quick overview). TIt can be easily seen that the function κ is continuous and injective. Then, we conclude proving that $\mathrm{KFP}(b,a) = \kappa^{-1}\big(\pi^1(\mathrm{K\hat{F}P}(b,a))\big)$:

- let $(\mu_t)_{t\in[0,T]} \in \mathrm{KFP}(b,a)$, then $(\hat\mu,\hat\nu,\hat\eta) := \big(\kappa(\boldsymbol{\mu}), \frac{b}{1+|x|}\kappa(\boldsymbol{\mu}), \frac{a}{1+|x|^2}\kappa(\boldsymbol{\mu})\big) \in \widehat{\mathrm{KFP}}(b,a)$, indeed clearly the conditions on the densities are verified and

$$\int_Y \frac{|b(t,x,\mu)|}{1+|x|} + \frac{|a(t,x,\mu)|}{1+|x|^2} d\big(\kappa(\boldsymbol{\mu})\big)(t,x,\mu) = \int_0^T \int_{\mathbb{R}^d} \frac{|b(t,x,\mu_t)|}{1+|x|} + \frac{|a(t,x,\mu_t)|}{1+|x|^2} d\mu_t(x)dt < +\infty,$$

$$\begin{aligned} 0 &= \int_0^T \xi'(t)\int \phi(x)d\mu_t(x)dt + \int_0^T \xi(t)\int \nabla\phi(x)\cdot b(t,x,\mu_t)d\mu_t(x)dt \\ &\quad + \frac{1}{2}\int_0^T \xi(t)\int \nabla^{\otimes_2}\phi(x) : a(t,x,\mu_t)d\mu_t(x)dt \\ &= \int_0^T\int_{\mathcal{P}}\int_{\mathbb{R}^d} \xi'(t)\phi(x)d\mu(x)d\delta_{\mu_t}(\mu)dt + \int_0^T\int_{\mathcal{P}}\int_{\mathbb{R}^d} \xi(t)\nabla\phi(x)\cdot b(t,x,\mu)d\mu(x)d\delta_{\mu_t}(\mu)dt \\ &\quad + \frac{1}{2}\int_0^T\int_{\mathcal{P}}\int_{\mathbb{R}^d} \xi(t)\nabla^{\otimes_2}\phi(x) : a(t,x,\mu)d\mu_t(x)d\delta_{\mu_t}(\mu)dt \\ &= \int_Y \xi'(t)\phi(x)d\big(\kappa(\boldsymbol{\mu})\big)(t,x,\mu) + \int_Y \xi(t)\nabla\phi(x)\cdot b(t,x,\mu)d\big(\kappa(\boldsymbol{\mu})\big)(t,x,\mu) \\ &\quad + \frac{1}{2}\int_Y \xi(t)\nabla^{\otimes_2}\phi(x) : a(t,x,\mu)d\big(\kappa(\boldsymbol{\mu})\big)(t,x,\mu) \\ &= \int_Y \xi'(t)\phi(x)d\hat\mu(t,x,\mu) + \int_Y \xi(t)(1+|x|)\nabla\phi(x)\cdot d\hat\nu(t,x,\mu) \\ &\quad + \frac{1}{2}\int_Y \xi(t)(1+|x|^2)\nabla^{\otimes_2}\phi(x) : d\hat\eta(t,x,\mu), \end{aligned}$$

  for all $\xi \in C_c^1(0,T)$ and $\phi \in C_c^2(\mathbb{R}^d)$;

- vice versa, let $\boldsymbol{\mu} \in \kappa^{-1}\big(\pi^1(\widehat{\mathrm{KFP}}(b,a))\big)$, which means that $\big(\kappa(\boldsymbol{\mu}), \frac{b}{1+|x|}\kappa(\boldsymbol{\mu}), \frac{a}{1+|x|^2}\kappa(\boldsymbol{\mu})\big) \in \widehat{\mathrm{KFP}}(b,a)$, and we conclude that $\boldsymbol{\mu} \in \mathrm{KFP}(b,a)$ thanks to the same computations made in the previous case. □

Proposition 4.9. *Let $b : [0,T] \times \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}^d$ and $a : [0,T] \times \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \to \mathrm{Sym}_+(\mathbb{R}^{d\times d})$ be Borel measurable functions. Then, the subset $\mathrm{MP}(b,a) \subset \mathcal{P}(C_T(\mathbb{R}^d))$ is Borel.*

Proof. **Step 1**: let us start by fixing the notation. Let $\lambda \in \mathrm{MP}(b,a)$, by (4.13) and Definition 2.5, for all $\xi \in C_c^1(0,T)$ and $\phi \in C_c^2(\mathbb{R}^d)$ it holds

$$t \mapsto X_t^{\xi\phi,\lambda}(\gamma) = X_t^{\xi\phi}(\gamma,\lambda) := \xi(t)\phi(\gamma_t) - \int_0^t \xi'(r)\phi(\gamma_r) + \xi(r)L^\lambda_{b_r,a_r}\phi(\gamma_r)dr$$

is a martingale in the filtered space $\big(C_T(\mathbb{R}^d), (\mathcal{F}_t)_{t\in[0,T]}, \mathcal{F}_T, \lambda\big)$. The integrability conditions ensure that $X_t^{\xi\phi} \in L^1(\lambda)$, so that the martingale condition can be rewritten as follows:

$$\begin{gathered}\forall \xi \in C_c^1(0,T),\ \forall \phi \in C_c^2(\mathbb{R}^d),\ \forall\, 0 \le s < t \le T,\ \forall H : C_T(\mathbb{R}^d) \to [0,1]\ \mathcal{F}_s\text{-meas.}\\ \int_{C_T(\mathbb{R}^d)} H(\gamma)\big(X_t^{\xi\phi,\lambda}(\gamma) - X_s^{\xi\phi,\lambda}(\gamma)\big)d\lambda(\gamma) = 0.\end{gathered} \tag{4.21}$$

To prove measurability of $\mathrm{MP}(b,a)$, it is first necessary to show that such a condition can be asked to hold for a countable number of ξ, ϕ, s, t, H. Before proceeding, it will be useful the following (uniform in time) estimate, that follows from (4.4):

$$\begin{aligned}|X_t^{\xi\phi,\lambda}(\gamma)| &\le (T+1)\|\xi\|_{C^1}\|\phi\|_\infty\\ &+ \|\xi\|_\infty \int_0^T |b(r,\gamma_r,(\mathrm{e}_r)_\sharp\lambda)||\nabla\phi| + |a(r,\gamma_r,(\mathrm{e}_r)_\sharp\lambda)||\nabla^{\otimes_2}\phi|dr\\ &\le (T+3)\|\xi\|_{C^1}\|\phi\|_{C^2_{0,w}}\left(1 + \left\|\frac{b^\lambda(\cdot,\gamma)}{1+|\gamma(\cdot)|}\right\|_{L^1(0,T)} + \left\|\frac{a^\lambda(\cdot,\gamma)}{1+|\gamma(\cdot)|^2}\right\|_{L^1(0,T)}\right),\end{aligned} \tag{4.22}$$

where $b^\lambda(t,\gamma) := b(t,\gamma_t,(\mathrm{e}_t)_\sharp\lambda)$ and similarly for a^λ. In particular, for every $\xi \in C_c^1(0,T)$ and $\phi \in C_c^2(\mathbb{R}^d)$, the right hand side is finite for λ-a.e. γ because $\lambda \in \mathrm{MP}(b,a)$.

Step 2: it is sufficient to check (4.21) for a countable amount of ξ and ϕ. Indeed, consider $\mathcal{D}_T \subset C_0^1(0,T)$ and $\mathcal{D}_{2,w} \subset C^2_{0,w}(\mathbb{R}^d)$ dense and countable subsets. Notice that, by linearity of the integral and of the operator $L^\lambda_{b_t,a_t}$, for any $\xi_1,\xi_2 \in C_c^1(0,T)$ and $\phi_1,\phi_2 \in C_c^2(\mathbb{R}^d)$ it holds

$$X_t^{\xi_1\phi_1+\xi_2\phi_2,\lambda}(\gamma) = X_t^{\xi_1\phi_1,\lambda}(\gamma) + X_t^{\xi_2\phi_2,\lambda}(\gamma) \quad \forall \lambda \in \mathcal{P}(C_T(\mathbb{R}^d)),\ \gamma \in C_T(\mathbb{R}^d).$$

Assume that (4.21) holds for all $\xi \in \mathcal{D}_T$ and $\phi \in \mathcal{D}_{2,w}$. Then, for any $\xi \in C_c^1(0,T)$ and $\phi \in C_c^2(\mathbb{R}^d)$, consider $\xi_n \in \mathcal{D}_T$ and $\phi_n \in \mathcal{D}_{2,w}$ such that $\xi_n \to \xi$ and $\phi_n \to \phi$ uniformly in their respective domains, so that for all $0 \le s < t \le T$ and $H : C_T(\mathbb{R}^d) \to [0,1]$ $\mathcal{F}_s$-measurable, it holds

$$\begin{aligned}\left|\int H(\gamma)\big(X_t^{\xi\phi,\lambda}(\gamma) - X_s^{\xi\phi,\lambda}(\gamma)\big)d\lambda(\gamma)\right| &= \left|\int H(\gamma)\big(X_t^{\xi\phi-\xi_n\phi_n,\lambda}(\gamma) - X_s^{\xi\phi-\xi_n\phi_n,\lambda}(\gamma)\big)d\lambda(\gamma)\right|\\ &\le \int |X_t^{\xi(\phi-\phi_n)+(\xi-\xi_n)\phi_n}(\gamma)| + |X_s^{\xi(\phi-\phi_n)+(\xi-\xi_n)\phi_n}(\gamma)|d\lambda(\gamma)\\ &\le (T+3)\left(\Big(\sup_{n\in\mathbb{N}}\|\phi_n\|_{C^2_{0,w}}\Big)\|\xi_n-\xi\|_{C^1} + \|\xi\|_{C^1}\|\phi_n-\phi\|_{C^2_{0,w}}\right).\end{aligned}$$

$$\cdot\left(1+\int\int_0^T \frac{|b(t,\gamma_t,(\mathrm{e}_t)_\sharp\lambda)|}{1+|\gamma_t|}+\frac{|a(t,\gamma_t,(\mathrm{e}_t)_\sharp\lambda)|}{1+|\gamma_t|^2}dtd\lambda(\gamma)\right)\to 0.$$

Step 3: it is sufficient to check (4.21) for $s<t$ with $s,t\in\mathbb{Q}\cap[0,T]$. Indeed, consider any $0\le s<t\le T$ and assume that (4.21) holds for two sequences of times $s_n<t_n$ satisfying $s_n\searrow s$, $t_n\to t$. Then, consider any $\xi\in\mathcal{D}_T$, $\phi\in\mathcal{D}_{2,w}$ and $H:C_T(\mathbb{R}^d)\to[0,1]$ $\mathcal{F}_s$-measurable, so that H is also $\mathcal{F}_{s_n}$-measurable for any $n\in\mathbb{N}$. By continuity of ξ and ϕ, and by $\int_0^T|L^\lambda_{b_r,a_r}\phi(\gamma_r)|dr<+\infty$ for λ-a.e. γ, it holds that $X^{\xi\phi,\lambda}_{s_n}\to X^{\xi\phi,\lambda}_s$ and $X^{\xi\phi,\lambda}_{t_n}\to X^{\xi\phi,\lambda}_t$ for λ-a.e. γ, dominated by the right-hand side of (4.22), which is in $L^1(\lambda)$. Finally, we can apply dominated convergence theorem to obtain

$$0=\int_{C_T(\mathbb{R}^d)}H(\gamma)\big(X^{\xi\phi,\lambda}_{t_n}(\gamma)-X^{\xi\phi,\lambda}_{s_n}(\gamma)\big)d\lambda(\gamma)\to\int_{C_T(\mathbb{R}^d)}H(\gamma)\big(X^{\xi\phi,\lambda}_{t}(\gamma)-X^{\xi\phi,\lambda}_{s}(\gamma)\big)d\lambda(\gamma).$$

Step 4: it is sufficient to check (4.21) with a countable family of step functions $\mathcal{H}_s$, that depends only on $s\in[0,T]\cap\mathbb{Q}$. First of all, notice that for all ξ, ϕ and λ, thanks to the Burkholder-Davis-Gundy inequality (see [Kal97, Theorem 18.7]) and Lemma 2.6, we have the following (uniform in time) estimates for the $L^2(\lambda)$-norm of $X^{\xi\phi,\lambda}_t$,

$$\begin{aligned}\int|X^{\xi\phi,\lambda}_t|^2d\lambda&\le\int\sup_{t\in[0,T]}|X^{\xi\phi,\lambda}_t|^2d\lambda\le C\int[X^{\xi\phi,\lambda}]_Td\lambda\\ &=C\int\int_0^T\xi^2(t)\nabla\phi^\top(\gamma_t)\cdot a_t(\gamma_t,(\mathrm{e}_t)_\sharp\lambda)\cdot\nabla\phi(\gamma_t)dtd\lambda(\gamma)\\ &\le C\|\xi\|^2_\infty\|(1+|x|)\nabla\phi\|^2_\infty\int\int_0^T\frac{|a_t(\gamma_t,(\mathrm{e}_t)_\sharp\lambda)|}{1+|\gamma_t|^2}dtd\lambda(\gamma)<+\infty.\end{aligned}\tag{4.23}$$

Now, for any $s\in[0,T]$ consider the subset $\mathcal{V}_s$ given by Lemma 2.3, that is countable and dense in $L^2(\lambda,\mathcal{F}_s)$ for all $\lambda\in\mathcal{P}(C_T(\mathbb{R}^d))$. Given $\xi\in\mathcal{D}_T$, $\phi\in\mathcal{D}_{2,w}$, $0\le s<t\le T$ rational times, it suffices to check (4.21) for any $H\in\mathcal{V}_s$. Indeed, let $H:C_T(\mathbb{R}^d)\to[0,1]$ $\mathcal{F}_s$-measurable; in particular, $H\in L^2(\lambda,\mathcal{F}_s)$, so it exists $H_n\in\mathcal{V}_s$ such that $\|H_n-H\|_{L^2(\lambda)}\to 0$, so that

$$\begin{aligned}\left|\int_{C_T(\mathbb{R}^d)}(H_n-H)\big(X^{\xi\phi.\lambda}_t-X^{\xi\phi,\lambda}_s\big)d\lambda\right|&\le\|H_n-H\|_{L^2(\lambda)}\|X^{\xi\phi,\lambda}_t-X^{\xi\phi,\lambda}_s\|_{L^2(\lambda)}\\ &\le\|H_n-H\|_{L^2(\lambda)}2\sqrt{C\|\xi\|^2_\infty\|(1+|x|)\nabla\phi\|^2_\infty\int\int_0^T\frac{|a_t(\gamma_t,(\mathrm{e}_t)_\sharp\lambda)|}{1+|\gamma_t|^2}dtd\lambda(\gamma)}\to 0,\end{aligned}$$

and if $\int H_n(X^{\xi\phi,\lambda}_t-X^{\xi\phi,\lambda}_s)d\lambda=0$ for all $n\in\mathbb{N}$, we also have (4.21) for a generic H that is bounded and $\mathcal{F}_s$-measurable.

Step 5: to recap, we can write $\mathrm{MP}(b,a)$ as

$$\mathrm{MP}(b,a)=\bigcap_{\substack{\xi\in\mathcal{D}_T,\\ \phi\in\mathcal{D}_{\mathbb{R}^d}}}\ \bigcap_{\substack{s,t\in[0,T]\cap\mathbb{Q},\\ s<t}}\ \bigcap_{H\in\mathcal{V}_s}\mathrm{MP}(b,a;\xi,\phi,s,t,H),\tag{4.24}$$

where

$$\mathrm{MP}(b,a;\xi,\phi,s,t,H) := \Big\{ \lambda \in \mathcal{P}(C_T(\mathbb{R}^d)) \ : \tag{4.25}$$
$$\int\int_0^T \frac{|b_t(\gamma_t,(\mathrm{e}_t)_\sharp\lambda)|}{1+|\gamma_t|} + \frac{|a_t(\gamma_t,(\mathrm{e}_t)_\sharp\lambda)|}{1+|\gamma_t|^2} dt d\lambda(\gamma) < +\infty,$$
$$\int_{C_T(\mathbb{R}^d)} H(\gamma)\big(X_t^{\xi\phi,\lambda}(\gamma) - X_s^{\xi\phi,\lambda}(\gamma)\big) d\lambda(\gamma) = 0 \Big\}.$$

So, we are left with the proof of Borel measurability of the sets $\mathrm{MP}(b,a;\xi,\phi,s,t,H)$. Define the Polish space $Z := C_T(\mathbb{R}^d) \times \mathcal{P}(C_T(\mathbb{R}^d))$ endowed with the product topology, and the natural injection from $\mathcal{P}(C_T(\mathbb{R}^d))$ to $\mathcal{P}(Z)$ as

$$\mathfrak{K} : \mathcal{P}(C_T(\mathbb{R}^d)) \to \mathcal{P}(Z), \quad \mathfrak{K}(\lambda) := \lambda \otimes \delta_\lambda. \tag{4.26}$$

Then, for any ξ, ϕ, s, t and H as in (4.24) define

$$\widehat{\mathrm{MP}} := \Big\{ \hat\lambda \in \mathcal{M}_+(Z) \ : \ \int_Z \int_0^T \frac{|b(r,\gamma_r,(\mathrm{e}_r)_\sharp\lambda)|}{1+|\gamma_r|} + \frac{|a(r,\gamma_r,(\mathrm{e}_r)_\sharp\lambda)|}{1+|\gamma_r|^2} dr \ d\hat\lambda(\gamma,\lambda) < +\infty,$$
$$\int_Z H(\gamma)\big(X_t^{\xi\phi,\lambda}(\gamma) - X_s^{\xi\phi,\lambda}(\gamma)\big) d\hat\lambda(\gamma,\lambda) = 0 \Big\}.$$

Since, for all $r \in [0,T]$, the map $Z \ni (\gamma,\lambda) \mapsto (\gamma_r, (\mathrm{e}_r)_\sharp\lambda)$ is continuous (and in particular measurable) for all $r \in [0,T]$, then also the map

$$(\gamma,\lambda) \mapsto \int_0^T \frac{|b(r,\gamma_r,(\mathrm{e}_r)_\sharp\lambda)|}{1+|\gamma_r|} + \frac{|a(r,\gamma_r,(\mathrm{e}_r)_\sharp\lambda)|}{1+|\gamma_r|^2} dr$$

is measurable (see, e.g., [Bog07, §3.4]). Moreover, looking at the definition of $X_t^{\xi\phi,\lambda}(\gamma)$, it is not hard to realize that also

$$(\gamma,\lambda) \mapsto X_t^{\xi\phi,\lambda}(\gamma)$$

is measurable for any $t \in [0,T]$. Then, thanks to Lemma 2.1, the set $\widehat{\mathrm{MP}}$ is Borel. Then, we conclude noticing that $\mathrm{MP}(b,a;\xi,\phi,s,t,H) = \mathfrak{K}^{-1}(\widehat{\mathrm{MP}})$. □

We are ready to prove the main theorem of this section, that we call *nested superposition principle for SDE.*

Theorem 4.10 (Nested superposition principle for SDE)**.** *Let $b : [0,T] \times \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}^d$ and $a : [0,T] \times \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \to \mathrm{Sym}_+(\mathbb{R}^{d\times d})$ be Borel measurable. Then there exists a Souslin-Borel measurable map $G_{b,a} : \mathrm{KFP}(b,a) \to \mathcal{P}(C_T(\mathbb{R}^d))$ satisfying $\mathrm{Im}(G_{b,a}) \subset \mathrm{MP}(b,a)$ and $E \circ G_{b,a}(\boldsymbol{\mu}) = \boldsymbol{\mu}$ for all $\boldsymbol{\mu} \in \mathrm{KFP}(b,a)$, i.e. $G_{b,a}$ is a right-inverse for $E|_{\mathrm{MP}(b,a)}$.*
In particular, if $\Lambda \in \mathcal{P}(C_T(\mathcal{P}(\mathbb{R}^d)))$ is concentrated over $\mathrm{KFP}(b,a)$, then $\mathfrak{L} := (G_{b,a})_\sharp\Lambda \in \mathcal{P}(\mathcal{P}(C_T(\mathbb{R}^d)))$ is well-defined, it is concentrated over $\mathrm{MP}(b,a)$ and it satisfies $E_\sharp\mathfrak{L} = \Lambda$.

Proof. The restriction map

$$E|_{\mathrm{MP}(b,a)} : \mathrm{MP}(b,a) \to \mathrm{KFP}(b,a)$$

is well-defined, thanks to Proposition 4.4. Moreover, because of the finite dimensional superposition for SDE, i.e. Theorem 2.7, it is surjective. Then, thanks to Propositions 4.8 and 4.9, we can apply [Bog07, Theorem 6.9.1] to obtain a Souslin-Borel measurable map

$G_{b,a} : \mathrm{KFP}(b,a) \to \mathcal{P}(C_T(\mathbb{R}^d))$ satisfying the requirements. Then, thanks to the universal measurability of Souslin sets (see [Bog07, Chapter 6]), the measure $\mathfrak{L} := (G_{b,a})_\sharp \Lambda$ is a well-defined Borel measure, and by the properties of $G_{b,a}$ it satisfies the requirements. □

Finally, putting all the results of this section together, we have a proof for the nested stochastic superposition principle, Theorem 1.1.

Proof of Theorem 1.1. The existence of Λ and property (1) come from Theorem 4.7. Then, the existence of $\mathfrak{L}$, together with the properties (2) and (3), is a consequence of Theorem 4.10. On the other hand, (i) and (ii) follow, respectively, from Proposition 4.4 and Proposition 4.5. □

5. Uniqueness scheme

In this section, we show how uniqueness can be transferred between the main objects of Theorem 1.1, $\boldsymbol{M} \in C_T(\mathcal{P}(\mathcal{P}(\mathbb{R}^d)))$, $\Lambda \in \mathcal{P}(C_T(\mathcal{P}(\mathbb{R}^d)))$ and $\mathfrak{L} \in \mathcal{P}(\mathcal{P}(C_T(\mathbb{R}^d)))$.

A first result in this direction is a consequence of the nested superposition principle for SDEs.

Lemma 5.1. *Let $b : [0,T] \times \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}^d$ and $a : [0,T] \times \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \to \mathrm{Sym}_+(\mathbb{R}^{d\times d})$ Borel measurable maps. The following are equivalent:*

1. *for all $\overline{M} \in \mathcal{P}(\mathcal{P}(\mathbb{R}^d))$ there exists at most one curve $\boldsymbol{M} = (M_t)_{t\in[0,T]} \in C_T(\mathcal{P}(\mathcal{P}(\mathbb{R}^d)))$ satisfying* (1.3) *and $M_0 = \overline{M}$;*
2. *for all $\overline{M} \in \mathcal{P}(\mathcal{P}(\mathbb{R}^d))$ there exists at most one $\Lambda \in \mathcal{P}(C_T(\mathcal{P}(\mathbb{R}^d)))$ concentrated over* $\mathrm{KFP}(b,a)$ *and satisfying* (1.4) *and $(\mathfrak{e}_0)_\sharp \Lambda = \overline{M}$;*
3. *given $\mathfrak{L}^1, \mathfrak{L}^2 \in \mathcal{P}(\mathcal{P}(C_T(\mathbb{R}^d)))$ concentrated over* $\mathrm{MP}(b,a)$ *and satisfying* (1.5) *and $(E_0)_\sharp \mathfrak{L}^1 = (E_0)_\sharp \mathfrak{L}^2$, then $E_\sharp \mathfrak{L}^1 = E_\sharp \mathfrak{L}^2$ as elements of $\mathcal{P}(C_T(\mathcal{P}(\mathbb{R}^d)))$.*

In particular, if one of the above conditions holds with existence for some $\overline{M}$, then existence for the other conditions is satisfied as well and for $\overline{M}$-a.e. $\overline{\mu} \in \mathcal{P}(\mathbb{R}^d)$ the following hold:

(a) *there exists a unique curve of measures $\boldsymbol{\mu} = (\mu_t)_{t\in[0,T]} \in \mathrm{KFP}(b,a)$ with $\mu_0 = \overline{\mu}$;*
(b) *there exists $\lambda \in \mathrm{MP}(b,a)$ with $(\mathrm{e}_0)_\sharp \lambda = \overline{\mu}$ and given $\lambda^1, \lambda^2 \in \mathrm{MP}(b,a)$ satisfying $(\mathrm{e}_0)_\sharp \lambda^1 = (\mathrm{e}_0)_\sharp \lambda^2 = \overline{\mu}$, then $(\mathrm{e}_t)_\sharp \lambda^1 = (\mathrm{e}_t)_\sharp \lambda^2$ for all $t \in [0,T]$.*

Proof. (2) $\Longrightarrow$ (3): thanks to Proposition 4.4, $\Lambda^1 := E_\sharp \mathfrak{L}^1$ and $\Lambda^2 := E_\sharp \mathfrak{L}^2$ are concentrated over $\mathrm{KFP}(b,a)$, and because of (1.5), they satisfy (1.4) as well. Then, $\Lambda^1 = \Lambda^2$ if by uniqueness assumption.

(3) $\Longrightarrow$ (2): assume that $\Lambda^1, \Lambda^2 \in \mathcal{P}(C_T(\mathcal{P}(\mathbb{R}^d)))$ satisfy the conditions in (2). Thanks to Theorem 4.10, there exist $\mathfrak{L}^1, \mathfrak{L}^2 \in \mathcal{P}(\mathcal{P}(C_T(\mathbb{R}^d)))$ satisfying the conditions in (3), and we conclude by noticing that $\Lambda^1 = E_\sharp \mathfrak{L}^1 = E_\sharp \mathfrak{L}^2 = \Lambda^2$.

(2) $\Longrightarrow$ (1): let $\boldsymbol{M}^1, \boldsymbol{M}^2 \in C_T(\mathcal{P}(\mathcal{P}(\mathbb{R}^d)))$ satisfying the conditions in (1). We can lift both to $\Lambda^1, \Lambda^2 \in \mathcal{P}(C_T(\mathcal{P}(\mathbb{R}^d)))$ thanks to Theorem 4.7. It is easy to see that both the liftings satisfy the conditions in (2), so that they coincide, from which it follows $M^1_t = (\mathfrak{e}_t)_\sharp \Lambda^1 = (\mathfrak{e}_t)_\sharp \Lambda^2 = M^2_t$ for all $t \in [0,T]$.

(1) $\Longrightarrow$ (2): let $\overline{M} \in \mathcal{P}(\mathcal{P}(\mathbb{R}^d))$ and assume Λ^1, Λ^2 satisfies the conditions in (2). Disintegrating both of them with respect to $\mathfrak{e}_0$, we obtain

$$\Lambda^i = \int_{\mathcal{P}(\mathbb{R}^d)} \Lambda^i_{\overline{\mu}} d\overline{M}(\overline{\mu}), \qquad \text{for } i = 1, 2,$$

where, for $\overline{M}$-a.e. $\overline{\mu}$, say $\overline{\mu} \in \mathcal{N}^c$, $\Lambda_{\overline{\mu}} \in \mathcal{P}(C_T(\mathcal{P}(\mathbb{R}^d)))$ is concentrated over $\mathrm{KFP}(b,a) \cap \{\boldsymbol{\mu} : \mu_0 = \overline{\mu}\}$, that in particular is non-empty. We are done if we show that, for all $\overline{\mu} \in \mathcal{N}^c$, such a set is a singleton. Assume $\boldsymbol{\mu}^1, \boldsymbol{\mu}^2 \in \mathrm{KFP}(b,a) \cap \{\boldsymbol{\mu} : \mu_0 = \overline{\mu}\}$, and consider the curves of random measures $M^i_t := \delta_{\mu^i_t}$, for $i = 1, 2$, that both solves the equation $\partial_t M^i_t = \mathcal{K}^*_{b_t,a_t} M^i_t$. Using the assumption with initial random measure $\delta_{\overline{\mu}}$, we conclude that $\boldsymbol{\mu}^1 = \boldsymbol{\mu}^2$.

We conclude noticing that (a) and (b) are byproducts of this argument. □

As already observed in [Tre16, pp. 11], the presence of the diffusion term does not allow us to conclude uniqueness of martingale solutions only assuming the solution of the associated Kolmogorov-Fokker-Planck equation is uniquely determined by the starting measure. Here, also the non-local nature of the problem is an obstacle for proving the uniqueness of $\mathfrak{L} \in \mathcal{P}(\mathcal{P}(C_T(\mathbb{R}^d)))$ assuming uniqueness of $\Lambda \in \mathcal{P}(C_T(\mathcal{P}(\mathbb{R}^d)))$. In particular, we need to adapt [Tre16, Proposition 2.6] to the following.

Lemma 5.2. *Let $b : [0, T] \times \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}^d$ and $a : [0, T] \times \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \to \mathrm{Sym}_+(\mathbb{R}^{d\times d})$ be Borel measurable maps. Let $0 < s < T$ and $\lambda \in \mathrm{MP}(b, a)$. Then:*

- *the measure $(|_{[s,T]})_\sharp\lambda \in \mathcal{P}(C([s, T], \mathbb{R}^d))$ is a martingale solution associated to $L^\lambda_{b_t,a_t}$ in the interval $[s, T]$;*
- *let $\rho : C([0, T], \mathbb{R}^d) \to [0, +\infty)$ be a bounded probability density with respect to λ and assume it is $\mathcal{F}_s$-measurable. Then $(\rho\lambda)|_{[s,T]} := (|_{[s,T]})_\sharp(\rho\lambda) \in \mathcal{P}(C([s, T], \mathbb{R}^d))$ is a martingale solution associated to the operator $L^\lambda_{b_t,a_t}$ in the interval $[s, T]$.*

Proof. It is an immediate consequence of [Tre16, Proposition 2.6]. □

In view of the previous lemma, we make the following assumption on the coefficients b and a.

Assumption 5.3. *For all $\boldsymbol{\mu} \in \mathrm{KFP}(b, a)$, for all $s \in [0, T]$ and for all $\tilde{\mu} \ll \mu_s$ such that $\tilde{\mu} \in \mathcal{P}(\mathbb{R}^d)$ and $\frac{d\tilde{\mu}}{d\mu_s} \in L^\infty(\mu_s)$, there exists at most one $\tilde{\boldsymbol{\mu}} \in C([s, T], \mathcal{P}(\mathbb{R}^d))$ such that $\tilde{\mu}_s = \tilde{\mu}$, $\tilde{\mu}_t \ll \mu_t$ for all $t \in [s, T]$, $\frac{d\tilde{\mu}_t}{d\mu_t} \in L^\infty(\mu_t)$ and $\partial_t\tilde{\mu}_t = (L^{\boldsymbol{\mu}}_{b_t,a_t})^*\tilde{\mu}_t$.*

We are asking for uniqueness for the linearized Kolmogorov-Fokker-Planck problems associated to $L^{\boldsymbol{\mu}}_{b_t,a_t}$, fixing $\boldsymbol{\mu} \in C_T(\mathcal{P}(\mathbb{R}^d))$. It is quite a natural strategy to study the uniqueness of the linearized version of the KFP equation to then obtain uniqueness of the non-linear one (see e.g. [BR23]).

The previous assumption can be restated in terms of uniqueness for suitable equations on random measures, as the following shows.

Lemma 5.4. *The following are equivalent:*

(i) Assumption 5.3;

(ii) for all $\boldsymbol{\mu} \in \mathrm{KFP}(b, a)$, for all $s \in [0, T]$ and for all $\tilde{M} \in \mathcal{P}(\mathcal{P}(\mathbb{R}^d))$ concentrated over $\{\mu \in \mathcal{P}(\mathbb{R}^d) : \mu \ll \mu_s, \frac{d\mu}{d\mu_s} \in L^\infty(\mu_s)\}$, there exists at most one $(\tilde{M}_t)_{t\in[s,T]} \in C([s, T], \mathcal{P}(\mathcal{P}(\mathbb{R}^d)))$ satisfying $\tilde{M}_s = \tilde{M}$ and $\partial_t\tilde{M}_t = \mathcal{K}^*_{b^{\boldsymbol{\mu}}_t,a^{\boldsymbol{\mu}}_t}\tilde{M}_t$ in $[s, T]$, where $b^{\boldsymbol{\mu}}_t(x) := b(t, x, \mu_t)$ and similarly for $a^{\boldsymbol{\mu}}_t$ (in particular, they do not depend on the variable μ, since it is fixed by the given curve).

Proof. (ii) $\Longrightarrow$ (i): it follows by considering the starting random measures $\tilde{M} := \delta_{\tilde{\mu}}$.

(i) $\Longrightarrow$ (ii): it is a consequence of Theorem 4.7 with the coefficients $b^{\boldsymbol{\mu}}$ and $a^{\boldsymbol{\mu}}$. □

The following shows a case in which Assumption 5.3 is satisfied.

Lemma 5.5. *Let $b : [0, T] \times \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}^d$ and $a : [0, T] \times \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \to \mathrm{Sym}_+(\mathbb{R}^{d\times d})$ be Borel measurable maps. Assume that b is bounded and that there exists a bounded map*

$\sigma : [0,T] \times \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}^{d\times d}$ such that $a = \sigma\sigma^\top$. Moreover, assume that there exists $L > 0$ such that

$$|\sigma(t,x,\mu) - \sigma(t,y,\mu)| + |b(t,x,\mu) - b(t,y,\mu)| \le L|x-y|, \tag{5.1}$$

for all $(t,\mu) \in [0,T] \times \mathcal{P}(\mathbb{R}^d)$ and $x, y \in \mathbb{R}^d$. Then b and a satisfy Assumption 5.3.

Proof. It is an immediate consequence of [SV06, Theorem 6.4]. □

More generally, the assumptions of the previous lemma can be relaxed to whatever case implies that for all $\boldsymbol{\mu} \in C_T(\mathcal{P}(\mathbb{R}^d))$, the coefficients defined by the maps $(t,x) \mapsto b(t,x,\mu_t)$ and $(t,x) \mapsto a(t,x,\mu_t)$ imply the well-posedness of the Kolmogorov-Fokker-Planck equation associated with them. Finally, we can refine Lemma 5.1. We adopt the obvious notation $\mathrm{KFP}_{[s,T]}(b,a)$ for the curves of probability measures $(\mu_t)_{t\in[s,T]}$ that solves $\partial_t\mu_t = (L^{\boldsymbol{\mu}}_{b_t,a_t})^*\mu_t$ in $[s,T]$ and satisfy the integrability condition in (4.12) integrating between s and T. The similar notation $\mathrm{MP}_{[s,T]}(b,a)$ is used for the set of martingale solutions.

Proposition 5.6. *Let $b : [0,T]\times\mathbb{R}^d\times\mathcal{P}(\mathbb{R}^d) \to \mathbb{R}^d$ and $a : [0,T]\times\mathbb{R}^d\times\mathcal{P}(\mathbb{R}^d) \to \mathrm{Sym}_+(\mathbb{R}^{d\times d})$ be Borel measurable maps. Assume they satisfy Assumption 5.3. Then the following are equivalent:*

(1) for all $s \in [0,T]$ and $\overline{M} \in \mathcal{P}(\mathcal{P}(\mathbb{R}^d))$ there exists a unique curve $\boldsymbol{M} = (M_t)_{t\in[s,T]} \in C([s,T],\mathcal{P}(\mathcal{P}(\mathbb{R}^d)))$ satisfying $\partial_t M_t = \mathcal{K}^*_{b_t,a_t} M_t$ in $[s,T]$, the integrability condition in (1.3) *(integrating between s and T) and $M_s = \overline{M}$;*

(2) for all $s \in [0,T]$ and for all $\overline{M} \in \mathcal{P}(\mathcal{P}(\mathbb{R}^d))$ there exists a unique $\Lambda \in \mathcal{P}(C([s,T],\mathcal{P}(\mathbb{R}^d)))$ concentrated over $\mathrm{KFP}_{[s,T]}(b,a)$, and satisfying (1.4) *(integrating between s and T) and $(\mathfrak{e}_s)_\sharp\Lambda = \overline{M}$;*

(2') for all $s \in [0,T]$ and $\overline{\mu} \in \mathcal{P}(\mathbb{R}^d)$ there exists a unique curve of measures $\boldsymbol{\mu} = (\mu_t)_{t\in[s,T]} \in \mathrm{KFP}_{[s,T]}(b,a)$, with $\mu_s = \overline{\mu}$;

(3) for all $s \in [0,T]$ and for all $\overline{M} \in \mathcal{P}(\mathcal{P}(\mathbb{R}^d))$ there exists a unique $\mathfrak{L} \in \mathcal{P}(\mathcal{P}(C([s,T],\mathbb{R}^d)))$ concentrated over $\mathrm{MP}_{[s,T]}(b,a)$, and satisfying (1.5) *(integrating between s and T) and $(E_s)_\sharp\mathfrak{L}^1 = \overline{M}$;*

(3') for all $s \in [0,T]$ and $\overline{\mu} \in \mathcal{P}(\mathbb{R}^d)$ there exists a unique $\lambda \in \mathrm{MP}_{[s,T]}(b,a)$, with $(\mathrm{e}_s)_\sharp\lambda = \overline{\mu}$.

Proof. By Lemma 5.1, the following are trivial: (1) $\iff$ (2), (3) $\implies$ (2) and (2) $\implies$ (2').

(3') $\implies$ (3): the existence follows noticing that for all $s \in [0,T]$ the map $\mathcal{G}_s : \overline{\mu} \mapsto \lambda^{\overline{\mu}} \in \mathcal{P}(C([s,T],\mathbb{R}^d))$ is well-defined (and Borel measurable as a consequence of [Bog07, Theorem 6.7.3]), where $\lambda^{\overline{\mu}}$ is the martingale solution given by (3'). Then, $\mathfrak{L} := (\mathcal{G}_s)_\sharp\overline{M} \in \mathcal{P}(\mathcal{P}(C([s,T],\mathbb{R}^d)))$ satisfies the requirements. Regarding the uniqueness, let $\mathfrak{L}$ be as in (3) and consider its disintegration with respect to the map E_s, so that

$$\mathfrak{L} = \int_{\mathcal{P}(\mathbb{R}^d)} \mathfrak{L}_{\overline{\mu}} d\overline{M}(\overline{\mu}),$$

and by (3') it follows that $\mathfrak{L}_{\overline{\mu}} = \delta_{\lambda^{\overline{\mu}}}$.

(2') $\implies$ (2): it follows from a similar reasoning.

(2') $\implies$ (3'): fix $s \in [0,T)$, $\overline{\mu} \in \mathcal{P}(\mathbb{R}^d)$ and $\lambda^1, \lambda^2 \in \mathrm{MP}_{[s,T]}(b,a)$, with $(\mathrm{e}_s)_\sharp\lambda^1 = (\mathrm{e}_s)_\sharp\lambda^2 = \overline{\mu}$. Thanks to Lemma 5.1, we know that $\mu_t := (\mathrm{e}_t)_\sharp\lambda^1 = (\mathrm{e}_t)_\sharp\lambda^2$ for all $t \in [0,T]$, so that they are martingale solutions with respect to the same operator $L^{\boldsymbol{\mu}}_{b_t,a_t}$, where $\boldsymbol{\mu} = (\mu_t)_{t\in[s,T]} \in \mathrm{KFP}_{[s,T]}(b,a)$. Now, we prove by induction on $n \in \mathbb{N}$ that $\forall s \le t_1 < \cdots < t_n \le T$, $\forall A_1,\dots,A_n \in \mathcal{B}(\mathbb{R}^d)$

$$\lambda^1(\mathrm{e}_{t_1} \in A_1,\dots,\mathrm{e}_{t_n} \in A_n) = \lambda^2(\mathrm{e}_{t_1} \in A_1,\dots,\mathrm{e}_{t_n} \in A_n), \tag{5.2}$$

that will make us conclude that $\lambda^1 = \lambda^2$. When $n = 1$, (5.2) follows by the fact that the marginals of λ^1 and λ^2 coincide. Now, consider $s \leq t_1 < \cdots < t_n < t_{n+1} \leq T$ and $\forall A_1, \ldots, A_n, A_{n+1} \in \mathcal{B}(\mathbb{R}^d)$. By induction,

$$\alpha := \lambda^1(\mathrm{e}_{t_1} \in A_1, \ldots, \mathrm{e}_{t_n} \in A_n) = \lambda^2(\mathrm{e}_{t_1} \in A_1, \ldots, \mathrm{e}_{t_n} \in A_n).$$

If $\alpha = 0$, then we are done. Otherwise, define the function $\rho : C([s,T];\mathbb{R}^d) \to [0, +\infty)$ as

$$\rho(\gamma) := \frac{1}{\alpha} \prod_{i=1}^{n} \mathbb{1}_{A_i}(\mathrm{e}_{t_i}(\gamma)),$$

that corresponds to the density of λ^1 and λ^2 conditioned with respect to $\{\mathrm{e}_{t_1} \in A_1, \ldots, \mathrm{e}_{t_n} \in A_n\}$. Notice that ρ is $\mathcal{F}_{t_n}$-measurable and consider the measures $\eta^1 := (|_{[t_n,T]})_\sharp(\rho\lambda^1)$ and $\eta^2 := (|_{[t_n,T]})_\sharp(\rho\lambda^2)$. We can conclude that $(\mathrm{e}_{t_n})_\sharp\eta^1 = (\mathrm{e}_{t_n})_\sharp\eta^2$, exploiting (5.2) with a general Borel set $B \in \mathcal{B}(\mathbb{R}^d)$ in place of A_n. Moreover, $(\mathrm{e}_t)_\sharp\eta^1$ and $(\mathrm{e}_t)_\sharp\eta^2$ are absolutely continuous w.r.t. μ_t for all $t \in [t_n, T]$ and their density is controlled by $1/\alpha$. Then, thanks to Lemma 5.2 and Assumption 5.3, we can assess that $(\mathrm{e}_t)_\sharp(\rho\lambda^1) = (\mathrm{e}_t)_\sharp(\rho\lambda^2)$ for all $t \in [t_n, T]$. In particular, it holds

$$\begin{aligned} \frac{\lambda^1(\mathrm{e}_{t_1} \in A_1, \ldots, \mathrm{e}_{t_n} \in A_n, \mathrm{e}_{t_{n+1}} \in A_{n+1})}{\alpha} &= \left[(\mathrm{e}_{t_{n+1}})_\sharp(\rho\lambda^1)\right](A_{n+1}) \\ = \left[(\mathrm{e}_{t_{n+1}})_\sharp(\rho\lambda^2)\right](A_{n+1}) &= \frac{\lambda^2(\mathrm{e}_{t_1} \in A_1, \ldots, \mathrm{e}_{t_n} \in A_n, \mathrm{e}_{t_{n+1}} \in A_{n+1})}{\alpha}. \end{aligned}$$

□

Remark 5.7. *The previous proposition can be localized, in the sense that we may substitute* $\overline{M} \in \mathcal{P}(\mathcal{P}(\mathbb{R}^d))$ *and* $\overline{\mu} \in \mathcal{P}(\mathbb{R}^d)$, *respectively, with* $\overline{M}$ *concentrated over a given set* $\mathcal{A} \subset \mathcal{P}(\mathbb{R}^d)$ *and* $\overline{\mu} \in \mathcal{A}$.

Acknowledgments. The author warmly thanks Giuseppe Savaré and Dario Trevisan for their fruitful suggestions and the encouragement in pursuing the results presented in this paper.

References

[AC08] Luigi Ambrosio and Gianluca Crippa. "Existence, uniqueness, stability and differentiability properties of the flow associated to weakly differentiable vector fields". In: *In: Transport Equations and Multi-D Hyperbolic Conservation Laws, Lecture Notes of the Unione Matematica Italiana* 5 (2008). DOI: 10.1007/978-3-540-76781-7_1.

[AGS08] Luigi Ambrosio, Nicola Gigli, and Giuseppe Savaré. *Gradient flows in metric spaces and in the space of probability measures*. Second. Lectures in Mathematics ETH Zürich. Birkhäuser Verlag, Basel, 2008, pp. x+334. ISBN: 978-3-7643-8721-1.

[Amb04] Luigi Ambrosio. "Transport equation and Cauchy problem for *BV* vector fields". In: *Invent. Math.* 158.2 (2004), pp. 227–260. ISSN: 0020-9910. DOI: 10.1007/s00222-004-0367-2.

[AT14] Luigi Ambrosio and Dario Trevisan. "Well-posedness of Lagrangian flows and continuity equations in metric measure spaces". In: *Anal. PDE* 7.5 (2014), pp. 1179–1234. ISSN: 2157-5045. DOI: 10.2140/apde.2014.7.1179.

[BB24] Karol Bołbotowski and Guy Bouchitté. "Kantorovich-Rubinstein duality theory for the Hessian". In: *arXiv, arXiv:2412.00516* (2024).

[BMQ25] Giovanni Brigati, Jan Maas, and Filippo Quattrocchi. "Kinetic Optimal Transport (OTIKIN)–Part 1: Second-Order Discrepancies Between Probability Measures". In: *arXiv, arXiv:2502.15665* (2025).

[Bog07] V. I. Bogachev. *Measure theory. Vol. I, II.* Springer-Verlag, Berlin, 2007, Vol. I: xviii+500 pp., Vol. II: xiv+575. ISBN: 978-3-540-34513-8; 3-540-34513-2. DOI: 10.1007/978-3-540-34514-5.

[BR23] Viorel Barbu and Michael Röckner. "Uniqueness for nonlinear Fokker–Planck equations and for McKean–Vlasov SDEs: The degenerate case". In: *Journal of Functional Analysis* 285.4 (2023), p. 109980. ISSN: 0022-1236. DOI: https://doi.org/10.1016/j.jfa.2023.109980.

[BR24] Viorel Barbu and Michael Röckner. *Nonlinear Fokker-Planck flows and their probabilistic counterparts.* Springer, 2024. ISBN: 978-3-031-61733-1. DOI: 10.1007/978-3-031-61734-8.

[BRS21] Vladimir I Bogachev, Michael Röckner, and Stanislav V Shaposhnikov. "On the Ambrosio–Figalli–Trevisan superposition principle for probability solutions to Fokker–Planck–Kolmogorov equations". In: *Journal of Dynamics and Differential Equations* 33 (2021), pp. 715–739. DOI: 10.1007/s10884-020-09828-5.

[CD22] Louis-Pierre Chaintron and Antoine Diez. "Propagation of chaos: A review of models, methods and applications. ˆ. Applications". In: *Kinetic and Related Models* 15.6 (2022), vol I 895–1015, vol II 1017–1173. ISSN: 1937-5093. DOI: 10.3934/krm.2022017;10.3934/krm.2022018.

[CF22] Paul-Eric Chaudru de Raynal and Noufel Frikha. "Well-posedness for some non-linear SDEs and related PDE on the Wasserstein space". In: *Journal de Mathématiques Pures et Appliquées* 159 (2022), pp. 1–167. ISSN: 0021-7824. DOI: https://doi.org/10.1016/j.matpur.2021.12.001.

[De 20] PE Chaudru De Raynal. "Strong well posedness of McKean–Vlasov stochastic differential equations with Hölder drift". In: *Stochastic Processes and their Applications* 130.1 (2020), pp. 79–107.

[DL89] R. J. DiPerna and P.-L. Lions. "Ordinary differential equations, transport theory and Sobolev spaces". In: *Invent. Math.* 98.3 (1989), pp. 511–547. ISSN: 0020-9910. DOI: 10.1007/BF01393835.

[Fig08] Alessio Figalli. "Existence and uniqueness of martingale solutions for SDEs with rough or degenerate coefficients". In: *Journal of Functional Analysis* 254.1 (2008), pp. 109–153. ISSN: 0022-1236. DOI: 10.1016/j.jfa.2007.09.020.

[HT19] Martin Huesmann and Dario Trevisan. "A Benamou–Brenier formulation of martingale optimal transport". In: *Bernoulli* (2019). DOI: 10.3150/18-BEJ1069.

[Kal97] Olav Kallenberg. *Foundations of modern probability.* Vol. 1. Springer, 1997. ISBN: 978-3-030-61873-5. DOI: 10.1007/978-3-030-61871-1.

[LSZ22] Daniel Lacker, Mykhaylo Shkolnikov, and Jiacheng Zhang. "Superposition and mimicking theorems for conditional McKean–Vlasov equations". In: *Journal of the European Mathematical Society* 25.8 (2022), pp. 3229–3288. DOI: 10.4171/JEMS/1266.

[PR25] Andrea Pascucci and Alessio Rondelli. "McKean–Vlasov stochastic equations with Hölder coefficients". In: *Stochastic Processes and their Applications* 182 (2025), p. 104564. ISSN: 0304-4149. DOI: https://doi.org/10.1016/j.spa.2025.104564.

[PS25a] Alessandro Pinzi and Giuseppe Savarè. "Nested superposition principle for random measures and the geometry of the Wasserstein on Wasserstein space". In: *arXiv* (2025).

[PS25b] Alessandro Pinzi and Giuseppe Savarè. "Totally convex functions and solution to the L^2-Monge problem for random measures". In: *arXiv* (2025).

[Reh23] Marco Rehmeier. "Linearization and a superposition principle for deterministic and stochastic nonlinear Fokker-Planck-Kolmogorov equations". In: *Annali Scuola Normale Superiore-Classe di Scienze* (2023), pp. 1705–1739.

[Sri08] Sashi Mohan Srivastava. *A course on Borel sets*. Vol. 180. Springer Science & Business Media, 2008. ISBN: 978-0-387-98412-4. DOI: 10.1007/b98956.

[ST17] Eugene Stepanov and Dario Trevisan. "Three superposition principles: currents, continuity equations and curves of measures". In: *J. Funct. Anal.* 272.3 (2017), pp. 1044–1103. ISSN: 0022-1236. DOI: 10.1016/j.jfa.2016.10.025.

[SV06] Daniel W Stroock and SR Srinivasa Varadhan. *Multidimensional diffusion processes*. Vol. 233. Springer Science & Business Media, 2006.

[Tre14] Dario Trevisan. *Well-posedness of Diffusion Processes in Metric Measure Spaces*. PhD thesis cvgmt preprint, http://cvgmt.sns.it/paper/4363/. 2014.

[Tre16] Dario Trevisan. "Well-posedness of multidimensional diffusion processes with weakly differentiable coefficients". In: *Electronic Journal of Probability* 21 (2016), pp. 1–41. DOI: 10.1214/16-EJP4453.

[Zol84] V. M. Zolotarev. "Probability Metrics". In: *Theory of Probability & Its Applications* 28.2 (1984), pp. 278–302. DOI: 10.1137/1128025.