On the relaxation of polyconvex functionals with linear growth under strict convergence in BV

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October 5, 2025

Abstract

We consider the relaxation of polyconvex functionals with linear growth with respect to the strict convergence in the space of functions of bounded variation. The functionals under relaxation are of the form $F(u,\Omega) := \int_\Omega f(\nabla u)dx$, where $u : \Omega \to \mathbb{R}^m$, and f is polyconvex. In constrast with the case of relaxation with respect to the standard L^1-convergence, in the case that Ω is 2-dimensional, we prove that the sets map $A \mapsto F(u,A)$ for A open, is, for every $u \in BV(\Omega;\mathbb{R}^m)$, $m \geq 1$, the restriction of a Borel measure. This is not true in the case $\Omega \subset \mathbb{R}^n$, with $n \geq 3$. Using the integral representation formula for a special class of functions, we also give a short proof of the existence of Cartesian maps whose relaxed area functional with respect to the L^1-convergence is strictly larger than the area of its graph.

Key words: Polyconvexity, Plateau problem, relaxation, area functional, minimal surfaces, Cartesian maps, integral representation.

AMS (MOS) 2020 Subject Classification: 49J45, 49Q05, 49Q15, 28A05, 28A75, 74B20.

1 Introduction

Polyconvexity arises in non-linear elasticity as in many branches of mechanics of solids, and is a more realistic hypothesis on the energy functional than just convexity [3]. The setting under consideration in this paper is the one where the growth of the involved functional is linear, circumstance in which the standard lower semicontinuity results [31, 33] do not apply.

Given an open bounded set Ω, the prototype example of energy with this growth condition is provided by the area functional that, given a map $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$ smooth enough, computes the n-dimensional Hausdorff measure of the graph $G_u := \{(x,y) \in \Omega \times \mathbb{R}^m : y = u(x)\}$ of u. Thanks to the area formula, the area functional takes the form

$$\mathbb{A}(u,\Omega) := \int_\Omega |\mathcal{M}(\nabla u)|dx, \tag{1.1}$$

where $\mathcal{M}(\nabla u)$ is the vector whose entries are all the determinants of the $k \times k$-submatrices of ∇u, $k = 0,\dots,\min\{n,m\}$ (the 0×0 determinant is conventionally taken as 1). More generally, we consider energies such as

$$F(u,\Omega) = \int_\Omega f(\nabla u)dx, \tag{1.2}$$

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where f is polyconvex, that is, there exists a convex function g such that

$$f(\nabla u) = g(\mathcal{M}(\nabla u)). \tag{1.3}$$

The condition of linear growth considered in [1] is expressed by the relation

$$g(\mathcal{M}(\nabla u)) \geq c_0 |\mathcal{M}(\nabla u)|, \tag{1.4}$$

for some positive constant c_0. Due to the lack of lower semicontinuity of this kind of functionals a relaxation procedure is necessary. This approach has been studied in [1], where the authors considered the L^1-relaxation of F given by

$$\mathcal{F}^{L^1}(u,\Omega) = \inf\{\liminf_{k\to\infty} F(u_k,\Omega) : (u_k) \subset C^1(\Omega;\mathbb{R}^m),\ u_k \to u \text{ in } L^1(\Omega;\mathbb{R}^m)\}, \tag{1.5}$$

and defined for any $u \in L^1(\Omega;\mathbb{R}^m)$. The relaxed functional $\mathcal{F}^{L^1}$ turns out to be L^1-lower semicontinuous and extend the functional F from $C^1(\Omega;\mathbb{R}^m)$ to $L^1(\Omega;\mathbb{R}^m)$. However, the behaviour of $\mathcal{F}^{L^1}$ is extremely wild, due to non-local phenomena that arise already for the relaxed area functional as soon as $n, m > 1$. Apart from the 1-dimensional case ($n = 1$) that is much simpler, assuming $n \geq 2$, there is a big difference between the one codimensional case ($m = 1$) and the higher codimensional one. Indeed, if u is scalar valued, then the functional $\mathcal{F}^{L^1}$ is local and admits an integral representation: In the special case of the relaxed area functional, which we denote by $\mathcal{A}^{L^1}$, it can be proved that the domain of $\mathcal{A}^{L^1}$ is the space $BV(\Omega)$ and that

$$\mathcal{A}^{L^1}(u,\Omega) = \int_\Omega \sqrt{1+|\nabla u|^2}dx + |D^s u|(\Omega), \qquad \forall u \in BV(\Omega), \tag{1.6}$$

where ∇u denotes the approximate gradient of u and $D^s u$ the singular part of the distributional derivative Du of u. A similar expression in terms of the recession function holds in the case of general function g (see [23]).

Instead, the case $m \geq 2$ does not enjoy so good properties: For general $u \in BV(\Omega;\mathbb{R}^m)$ it can be proved only that

$$\begin{aligned} \mathcal{A}^{L^1}(u,\Omega) &\geq \int_\Omega \sqrt{1+|\nabla u|^2}dx + |D^s u|(\Omega), \qquad &\forall u \in BV(\Omega;\mathbb{R}^m),\\ \mathcal{F}^{L^1}(u,\Omega) &\geq \int_\Omega g(\mathcal{M}(\nabla u))dx + c_0|D^s u|(\Omega), \qquad &\forall u \in BV(\Omega;\mathbb{R}^m), \end{aligned} \tag{1.7}$$

and that there exist maps of bounded variations for which $\mathcal{A}^{L^1}$ (and $\mathcal{F}^{L^1}$) is $+\infty$ (see [12,13]). The domain of $\mathcal{A}^{L^1}$, namely the set of maps for which $\mathcal{A}^{L^1}$ is finite, is a subset of $BV(\Omega;\mathbb{R}^m)$, whose precise description is not available. Moreover, it has been proved in [1] that, for some fixed $u \in BV(\Omega;\mathbb{R}^m)$, the set function $A \subset \Omega \to \mathcal{F}^{L^1}(u,A)$ is not in subadditive, and thus $\mathcal{F}^{L^1}$ does not admit any integral representation. This is true also for the area functional, where the non-subadditivity property has been incountered already for two simple examples of functions: The vortex map u_V and the triple junction function u_T. The former is the Sobolev map $u_V(x) = \frac{x}{|x|}$ in the ball $\Omega = B_R(0) \subset \mathbb{R}^2$, the latter $u_T : B_R(0) \subset \mathbb{R}^2 \to \{\alpha,\beta,\gamma\}$ is a piecewise constant map assuming three values that are the three vertices of an equilateral triangle in $\mathbb{R}^2$. For both these functions, suggested by De Giorgi in [24], Acerbi and Dal Maso proved the non-subadditivity property exploiting suitable lower and upper bounds for $\mathcal{A}^{L^1}$. Also, the precise values of $\mathcal{A}^{L^1}(u_V, B_R(0))$ and $\mathcal{A}^{L^1}(u_T, B_R(0))$ were not available at that time, and only recently it has been possible to find them explicitely (see [7–9, 11, 46]). In the last references, it is clear how the nonlocality of $\mathcal{A}^{L^1}(u_V,\cdot)$ and $\mathcal{A}^{L^1}(u_T,\cdot)$ pops up: In the former case, we have

$$\mathcal{A}^{L^1}(u_V, B_R(0)) = \int_{B_R(0)} \sqrt{1+|\nabla u_V|^2}dx + \mathcal{H}^2(C_R), \tag{1.8}$$

where $\mathcal{H}^2(C_R)$ is the 2-dimensional Hausdorff measure of a minimal surface C_R obtained by solving a particular non-parametric Plateau problem with partial free boundary in codimension 1. This object, whose shape is (the half of) a sort of catenoid constrained to contain a segment, is a suitable projection in $\mathbb{R}^3$ of the vertical part of the cartesian current S obtained as limit of the graphs $G_{u_k} \subset \Omega \times \mathbb{R}^2$ of a recovery sequence $(u_k) \subset C^1(B_R;\mathbb{R}^2)$ for $\mathcal{A}^{L^1}(u_V, B_R(0))$ (see [8] for the non-parametric Plateau problem and [7, 9] for the computation of $\mathcal{A}^{L^1}(u_V, B_R(0))$). The radius $R > 0$ represents the height of the catenoid, and hence the area of C_R depends on R, in such a way that $\mathcal{H}^2(C_R) \leq 2\pi R$; for R larger than a certain threshold it happens that $\mathcal{H}^2(C_R) = \pi$. A similar phenomenon is observed for u_T, where the singular contribution in $\mathcal{A}^{L^1}(u_T, B_R(0))$ is provided by the area of three minimal surfaces in $\mathbb{R}^3$ solving a nonparametric Plateau-type problem with partial free boundary. Also in this case, these minimal surfaces have the role of filling the holes in the graph of G_{u_T}, hence arising as vertical parts of the cartesian current obtained as limit of the graphs G_{u_k} of a recovery sequence $(u_k) \subset C^1(B_R;\mathbb{R}^2)$ for $\mathcal{A}^{L^1}(u_T, B_R(0))$ (see [11, 46]).

The relaxed area of u_V and u_T in a ball $B_R(0)$ are the unique non-trivial cases in which $\mathcal{A}^{L^1}(u,\Omega)$ is explicit, and minimal changes in the geometry of the domain or on the choice of the function u makes the computation of $\mathcal{A}^{L^1}(u,\Omega)$ out of reach; in more general cases, only (non-sharp) upper bounds are available, as in [14] for the case of Sobolev maps with values in $\mathbb{S}^1$ (thus generalizing the vortex map) and in [6, 47] for the case of piecewise constant functions taking three values (hence generalizing the triple junction function). In any case, we believe that the vertical parts of cartesian currents obtained as limits of the graphs G_{u_k} of a recovery sequence $(u_k) \subset C^1(B_R;\mathbb{R}^2)$ can be often described, in a similar fashion as for u_T and u_V, as minimal surfaces arising as solutions of non-parametric Plateau problems with partial free boundaries (see [10]) or semicartesian Plateau problems (see [12, 13]).

One of the issue encontered in the analysis of the relaxation in (1.5) is that, when one considers, for $u \in BV(\Omega;\mathbb{R}^m)$, a sequence $(u_k) \subset C^1(\Omega;\mathbb{R}^m)$ realizing the infimum (i.e., a so-called recovery sequence), then the limit of the graphs G_{u_k} in $\Omega \times \mathbb{R}^m$, seen as integral currents, cannot be easily identified. Indeed, it is only known that

$$G_{u_k} \rightharpoonup G_u + V_{\min} =: S_{\min},$$

where $V_{\min}$ is called vertical part, and is such that $\partial V_{\min} = -\partial G_u$. But unless few general properties on $V_{\min}$ (that are common to vertical parts of cartesian currents, see [32]) nothing can be said, a priori, on its geometry. The knowledge of $V_{\min}$ would give rise, at least for the area functional, the trivial lower bound (which follows by lower-semicontinuity of the mass)

$$\mathcal{A}^{L^1}(u,\Omega) \geq |S_{\min}| = |G_u| + |V_{\min}|,$$

where by $|\cdot|$ we indicate the total mass of a current. However, $V_{\min}$ strongly depends on Ω, in general, and this is the main reason of non-locality of $\mathcal{A}^{L^1}$ (and of $\mathcal{F}^{L^1}$).

In contrast, this phenomenon disappears, at least in the case $n = 2$, if one consider the relaxation of F with respect to strict topology in $BV(\Omega;\mathbb{R}^m)$. Namely, let us consider, for $\Omega \subset \mathbb{R}^2$ and for all $u \in BV(\Omega;\mathbb{R}^m)$, the functional

$$\mathcal{F}(u,\Omega) = \inf\{\liminf_{k\to\infty} F(u_k,\Omega) : (u_k) \subset C^1(\Omega;\mathbb{R}^m),\ u_k \to u \text{ strictly in } BV(\Omega;\mathbb{R}^m)\}. \tag{1.9}$$

It is then possible to show that if $u_k \subset C^1(\Omega;\mathbb{R}^m)$ converges to u strictly in $BV(\Omega;\mathbb{R}^m)$ and $\mathbb{A}(u_k,\Omega) < C < +\infty$ for all k, then

$$G_{u_k} \rightharpoonup G_u + V_{\text{strict}} =: S_{\text{strict}} \qquad \text{as currents,} \tag{1.10}$$

where V_{strict} (and hence S_{strict}) is uniquely determined and does not depend on the specific sequence u_k. This result has been proved in [42], where relaxation in (1.9) has been considered for the area functional. The relaxed area functional under strict convergence has been analyzed more in detail in [4, 5, 17, 18]. Due to the more restrictive request that u_k approximate u in the strict topology, it is straightforward that

$$\mathcal{F}(u,\Omega) \geq \mathcal{F}^{L^1}(u,\Omega),$$

and strict inequality often occurs. In fact also the domain of $\mathcal{F}(u,\Omega)$ is strictly smaller than that of $\mathcal{F}^{L^1}(u,\Omega)$ (precisely, there exists $u \in BV(\Omega;\mathbb{R}^m)$ for which $\mathcal{A}^{L^1}(u,\Omega)$ is finite and $\mathcal{A}(u,\Omega)$ is $+\infty$, see [5]).

As a consequence of (1.10), for the relaxed area functional $\mathcal{A}(u,\Omega)$, it holds

$$\mathcal{A}(u,\Omega) \geq |S_{\text{strict}}| = |G_u| + |V_{\text{strict}}| = \int_\Omega |\mathcal{M}(\nabla u)|dx + |V_{\text{strict}}|. \tag{1.11}$$

This provides a natural lower bound for $\mathcal{A}(u,\Omega)$, since V_{strict} is uniquely determined by u. However, it has been observed [42] that also in this case the strict inequality can occur in (1.11), so the lower bound is not optimal (see also [4, 5, 17]). On the other hand, following the analysis of [4, 5, 17], in the case that $\Omega \subset \mathbb{R}^2$, all the phenomena related to non-subadditivity of the set function $A \mapsto \mathcal{A}(u,A)$ seemed to disappear, at least for a suitable class of maps of bounded variation u, so it has been conjectured that actually the set function $A \mapsto \mathcal{A}(u,A)$ is a Borel measure restricted to the class of open sets. This conjecture has been disproved in the case $\Omega \subset \mathbb{R}^3$ in [18], where the authors show that already for the vortex map $u_V(x) = \frac{x}{|x|}$ some similar phenomena as in dimension 2 for $\mathcal{A}^{L^1}$ take place. However it remained an open problem to undestand if in dimension 2 the conjecture is true.

In the present paper we show this conjecture, which actually applies also for the more general polyconvex functionals $\mathcal{F}$:

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^2$ be an open and bounded set, let $m \geq 1$, and let $u \in BV(\Omega;\mathbb{R}^m)$; then the function $A \mapsto \mathcal{F}(u,A)$, defined for all open sets $A \subseteq \Omega$, is the restriction of a Borel measure.*

The above result applies to all polyconvex functionals of the form (1.2) satisfying (1.3) for a general convex function g that is linear or sublinear, in the sense that there exists a positive constant C_g with

$$g(\mathcal{M}(\nabla u)) \leq C_g(|\mathcal{M}(\nabla u)| + 1). \tag{1.12}$$

At the same time, we assume also some coercivity property of g (see (2.7) below), that in the case in which $n = m = 2$, it is expressed as

$$c_g|\det\,(\nabla u)| \leq g(\mathcal{M}(\nabla u)) \tag{1.13}$$

for some positive constant c_g (condition that is weaker than (1.4)). With these two requirements we includes in our analysis the interesting prototype cases of the area functional $g(\mathcal{M}(\nabla u)) =$

$|\mathcal{M}(\nabla u)|$ and of the total variation of the Jacobian functional, i.e., the functional (in the case $n = m = 2$)

$$TVJ(u,\Omega) := \int_\Omega |\det (\nabla u)| dx, \tag{1.14}$$

defined for $u : \Omega \to \mathbb{R}^2$.

In order to show Theorem 1.1 we apply the standard result due to De Giorgi and Letta which characterizes the maps on open sets which are Borel measures (see Theorem 3.1 below). This accounts to check monotonicity, additivity, subadditivity, and inner regularity of the set function $A \mapsto \mathcal{F}(u, A)$, defined for A open. Although additivity on disjoint set is straightforward, notice that already monotocity is non-trivial, due to the fact that, if $B \subset A$, the restriction of a recovery sequence for $\mathcal{F}(u, A)$ to B is not necessarily converging strictly to u on B. So, accurate modifications of recovery sequence are necessary.

A fundamental step to show subadditivity and inner regularity is contained in Proposition 4.6. Under suitable conditions on u and $B \subset\subset A$, it states that if u_k is a recovery sequence for $\mathcal{F}(u, A)$ and $u_k \llcorner \partial B$ strictly converges to $u \llcorner \partial B$, then $u_k \llcorner B$ is a recovery sequence for $\mathcal{F}(u, B)$. To prove Proposition 4.6 we assume that v_k is a recovery sequence for $\mathcal{F}(u, B)$ and we consider a map w_k obtained by glueing v_k and $u_k \llcorner (A \setminus B)$ on a tubular neighborhood of ∂B. We show that this can be done by modifying v_k and $u_k \llcorner (A \setminus B)$ a little bit so that their energy does not increase too much; this is possible thanks to the assumption of strict convergence of u_k to u on ∂B, since Proposition 3.6 allows to reparametrize $u_k \llcorner \partial B$ in such a way that it can be glued to $v_k \llcorner \partial B$ by a tricky interpolation argument. This is a crucial point, which is possible only because the set ∂B is 1-dimensional, and this argument fails in the case $B \subset \mathbb{R}^n$ with $n \geq 3$ (this is related with the fact that the total minimal lifting of u is unique, see [42], that is not true in dimension greater than 2). To apply the previous interpolation between v_k and $u_k \llcorner (A \setminus B)$ we need that $v_k \llcorner \partial B$ also converges to u strictly on ∂B. This is not always true, and requires an ad hoc modification of a recovery sequence v_k for $\mathcal{F}(u, B)$. A key ingredient in order to modify recovery sequences is the fact that strict convergence on an open set $A \subset \mathbb{R}^2$ is inherited on suitable curves $\Gamma \subset A$. This allows to conclude that v_k converges strictly to u on almost every level set of the distance function $d(\cdot, \partial B)$. With ad hoc transformation in tubular neighborhood of ∂B, we can then modify v_k, without changing $F(v_k, B)$ too much, in order that the modified sequence converges strictly to u on ∂B (see Lemma 4.5).

In view of Theorem 1.1 we expect that, at least for the area and total variation of the Jacobian functional, a suitable integral representation is possible. We provide in Section 6 some examples of known results. Using these, it is possible to show that for the standard relaxation of the area functional with respect to the L^1 convergence, the singular contribution is not only due to the presence of holes (or singularities) in the graph of the considered map. Indeed, even if a map $u : \Omega \to \mathbb{R}^2$ is Cartesian (i.e., its graph G_u has not holes, namely $\partial G_u = 0$ as current in $\mathcal{D}_1(\Omega \times \mathbb{R}^2)$), it is possible that the relaxed area $\mathcal{A}^{L^1}(u, \Omega)$ is strictly larger than the 2-dimensional hausdorff measure of G_u (in other words, a singular contribution due to relaxation pops up). This result is given providing a special function u (see Theorem 6.1 in Section 6), but can be easily extended to more a general class; we point out that the aforementioned Theorem was already known and proved with different techniques in [33, Theorem 1, Section 6.5.1].

We finally emphasize that an integral representation of this kind of functionals as in [23] is not possible if we relax with respect to the L^1-topology, due to the lack of sub-additivity of $A \mapsto \mathcal{F}^{L^1}(u, A)$, unless one requires more restrictive growth conditions on g (see for instance [28, 29, 48]).

The structure of the paper is as follows: In the next Section 2 we introduce some standard notation and in its Subsection 2.2 we recall the setting of the problem and give the first results on relaxation. In Section 3 we start with measure theoretic, geometry tools, and preliminary results; further in Section 4 we start by describing how to modify Lipschitz maps in order to cut and paste suitable recovery sequences for $\mathcal{F}(u,\Omega)$. In Section 5 we finally give the proof of Theorem 1.1, exploiting De Giorgi and Letta Theorem, and thus checking that standard conditions of the set map $A \mapsto \mathcal{F}(u,A)$ are satisfied. In Section 6 we exhibit some known result of representation formulas for the area functional (and for the total variation of the Jacobian one); motivated by this, we introduce the double 8-curve map u_φ, which is a 0-homogeneous Cartesian map and we show in Theorem 6.1 that

$$\mathcal{A}^{L^1}(u_\varphi, B_r(0)) > \int_{B_r(0)} \sqrt{1+|\nabla u_\varphi|^2}dx.$$

The paper ends with an Appendix where we collect a couple of standard results used in the manuscript.

2 Notation and Setting

2.1 Notation

In what follows we denote by $\mathcal{L}^n$ the Lebesgue measure and, for $0 \le d \le n$, by $\mathcal{H}^d$ the d-dimensional Hausdorff measure in $\mathbb{R}^n$. Let $A \subseteq \mathbb{R}^n$ be an open set and let $M \ge 1$, we denote by $\mathcal{M}_b(A;\mathbb{R}^M)$ the space of Radon measures with bounded total variation, and if $\mu \in \mathcal{M}_b(A;\mathbb{R}^M)$ we denote by $|\mu|(U)$ its total variation on $U \subseteq A$.

Functions of bounded variation: We will recall the main properties of functions of bounded variation, and we refer to [2] for more detail. Let $A \subseteq \mathbb{R}^n$ be an open set and let $u \in BV(A;\mathbb{R}^m)$ be a map. We denote by Du the distributional derivative of u which splits as

$$Du = \nabla u + D^c u + D^j u,$$

where ∇u is the approximate gradient (i.e. the absolutely continuous part of Du with respect to $\mathcal{L}^n$), $D^c u$ is the Cantor part, and $D^j u$ the jump part of Du. The jump set of u is denoted by $S_u \subset A$ and it is a $(n-1)$-rectifiable set; if ν is a unit vector normal to S_u at $x \in S_u$, then we denote

$$u^+(x) := \operatorname{aplim}_{y\to x,\ (y-x)\cdot\nu>0} u(x), \qquad u^-(x) := \operatorname{aplim}_{y\to x,\ (y-x)\cdot\nu<0} u(x)$$

and so it turns out that

$$D^j u = (u^+ - u^-) \otimes \nu \cdot \mathcal{H}^{n-1} \llcorner S_u.$$

We denote by $|Du|(A)$ the total variation of u in A, that coincides with

$$|Du|(A) = \sup\{\sum_{i=1}^m \int_A u_i \cdot \operatorname{div} \varphi_i dx : \varphi \in C^1_c(A;\mathbb{R}^{m\times n}),\ \|\varphi\|_{L^\infty} \le 1\} \tag{2.1}$$

where φ_i denotes the i-th row of φ.

In the one dimensional case $n=1$ the jump set S_u reduces to an at most countable (possibly empty) subset of A. If $t \in A$ we denote

$$u(t^+) := \lim_{x\to t^+} u(x) \qquad u(t^-) := \lim_{x\to t^-} u(x),$$

so that $D^j u = \sum_{t\in S_u}(u(t)^+ - u(t)^-)\delta_t = \sum_{t\in S_u}(u(t^+) - u(t^-))\delta_t$. In the one dimensional case there exists always a good representative of u that is right-continuous, and its only discontinuity points are those in the jump set.

Definition 2.1. *We say that a sequence $u_k \subset BV(A;\mathbb{R}^m)$ converges to $u \in BV(A;\mathbb{R}^m)$ strictly in $BV(A;\mathbb{R}^m)$ if*

$$u_k \to u \text{ in } L^1(A;\mathbb{R}^m), \qquad |Du_k|(A) \to |Du|(A),$$

when $k \to \infty$.

The topology induced by the strict convergence is metrizable and we denote by d_s the distance associated with it: Specifically, for $u, v \in BV(A;\mathbb{R}^m)$ we set

$$d_s(u,v) := \|u-v\|_{L^1} + \big||Du|(A) - |Dv|(A)\big|. \tag{2.2}$$

With this notation $u_k \to u$ strictly in $BV(A;\mathbb{R}^m)$ if and only if $d_s(u_k,u) \to 0$.

We recall the following approximation result:

Theorem 2.2. *Let $A \subset \mathbb{R}^n$ be a bounded open set, and let $u \in BV(\Omega;\mathbb{R}^m)$. Then there exists a sequence $(v_k) \subseteq C^\infty(A;\mathbb{R}^m)$ such that $v_k \to u$ strictly in $BV(A;\mathbb{R}^m)$.*

Inspecting the proof of the Theorem above (see, e.g., [2]), the following remark is in order:

Remark 2.3. *The previous Theorem is obtained by a local argument of mollification and then using a unity partition. In particular, if u is Lipschitz continuous in A, then*

$$v_k \to u \qquad \text{weakly* in } W^{1,\infty}(A;\mathbb{R}^m) \text{ and strongly in } W^{1,p}(A;\mathbb{R}^m),$$

for all $p < \infty$, and the functions v_k are Lipschitz continuous with Lipschitz constant less than or equal to the one of u.

Multi-indeces and sub-determinants: Let $n, m \geq 2$ be fixed integers; multi-indeces $\alpha \subseteq \{1,\dots,n\}$ and $\beta \subseteq \{1,\dots,m\}$ are two ordered sets, possibly empty. We denote by $|\cdot|$ the cardinality; by $\overline{\alpha}$ we denote the complementary of α, i.e. $\overline{\alpha} := \{1,\dots,n\} \setminus \alpha$, and similarly $\overline{\beta} := \{1,\dots,m\} \setminus \beta$. Given a $m \times n$ matrix $A = (a_{ij})$, $i \in \{1,\dots,m\}$, $j \in \{1,\dots,n\}$, and given α, β multi-indeces as above such that $|\alpha| + |\beta| = n$, we denote by

$$M^{\beta}_{\overline{\alpha}}(A),$$

the determinant of the submatrix of A whose columns are indexed in $\overline{\alpha}$ and lines in β, multiplied by $\theta(\alpha)$, the sign of the permutation $(\alpha, \overline{\alpha}) \in S(n)$ (with the convention that $M^{\varnothing}_{\varnothing}(A) = 1$). In the specific case of our interest, if $n = 2$ and $A = \nabla u$, with $u : \mathbb{R}^2 \to \mathbb{R}^m$ a sufficiently smooth map, it holds

$$M^{\varnothing}_{\varnothing}(A) = 1 \qquad M^{i}_{j}(\nabla u) = (-1)^j \frac{\partial u_i}{\partial x_j} \qquad M^{i_1 i_2}_{12}(\nabla u) = \frac{\partial u_{i_1}}{\partial x_1}\frac{\partial u_{i_2}}{\partial x_2} - \frac{\partial u_{i_2}}{\partial x_1}\frac{\partial u_{i_1}}{\partial x_2}.$$

We denote by $\{e_1,\dots,e_n\}$ the canonical basis of 1-vectors of $\mathbb{R}^n$, and by $\{\varepsilon_1,\dots,\varepsilon_m\}$ that of the target space $\mathbb{R}^m$. The dual basis of 1-covectors are denoted by $\{dx_1,\dots,dx_n\}$ and $\{dy_1,\dots,dy_m\}$, respectively. If $\alpha \subseteq \{1,\dots,n\}$ and $\beta \subseteq \{1,\dots,m\}$ are ordered sets as above, we denote by e_α and ε_β the k-vector and h-vector defined as

$$e_\alpha := e_{\alpha_1} \wedge \cdots \wedge e_{\alpha_k} \qquad \text{if } \alpha = \{\alpha_1,\dots,\alpha_k\}, \tag{2.3}$$

$$\varepsilon_\beta := \varepsilon_{\beta_1} \wedge \cdots \wedge \varepsilon_{\beta_h} \qquad \text{if } \beta = \{\beta_1,\dots,\beta_h\}, \tag{2.4}$$

where $k = |\alpha|$, $h = |\beta|$, so in the case $n = 2$ it holds

$$e_\varnothing = 1, \qquad e_\alpha = e_j \quad \text{if } \alpha = \{j\}, \qquad e_{12} = e_1 \wedge e_2. \tag{2.5}$$

Next we introduce the n-vector associated to a C^1 map $u : \mathbb{R}^n \to \mathbb{R}^m$

$$\mathcal{M}(\nabla u) := \sum_{|\alpha|+|\beta|=n} M_{\bar\alpha}^{\beta}(\nabla u) e_\alpha \wedge \varepsilon_\beta,$$

where the sum takes place over all multi-indeces $\alpha \subseteq \{1, \dots, n\}$ and $\beta \subseteq \{1, \dots, m\}$ with $|\alpha| + |\beta| = n$.

Currents, Graphs, and Cartesian maps: We introduce some notation in the theory of currents and Cartesian currents. We refer to [38] and [32] for more details. For an open set $A \subset \mathbb{R}^n$ we denote by $\mathcal{D}^k(A)$ the space of (compactly supported in A) smooth k-forms and by $\mathcal{D}_k(A)$ the space of k-dimensional currents, where $0 \le k \le n$. Given $T \in \mathcal{D}_k(A)$ we denote by $|T|_U$ the mass of T in U, whenever $U \subset A$ is open. Given $T \in \mathcal{D}_k(A)$ with $k \ge 1$, its boundary $\partial T \in \mathcal{D}_{k-1}(A)$ is defined by

$$\partial T(\omega) := T(d\omega) \qquad \forall \omega \in \mathcal{D}^{k-1}(A),$$

where $d\omega$ denotes the external differential of ω. In the case $k = 0$ by convention it is $\partial T = 0$. Whenever $F : A \to B$ is a Lipschitz map between open sets, and $T \in \mathcal{D}_k(A)$, the symbol $F_\sharp T \in \mathcal{D}_k(B)$ denotes the push-forward of T by F.

We say that a current $T \in \mathcal{D}_k(A)$ is rectifiable if there exist a $\mathcal{H}^k$-rectifiable set[1] S, a simple unit k-vector $\tau(x)$ for $\mathcal{H}^k$-a.e. $x \in S$, and a measurable function $\theta : S \to \mathbb{R}$ with

$$T(\omega) = \int_S \theta(x) \langle \omega(x), \tau(x) \rangle \, d\mathcal{H}^k(x), \qquad \omega \in \mathcal{D}^k(A).$$

Here $\langle \cdot, \cdot \rangle$ denotes the duality pairing between k-covectors and k-vectors. A rectifiable current $T \in \mathcal{D}_k(A)$ is said integral if θ takes integer values, τ is tangent to S, and $|T|_A < +\infty$, $|\partial T|_A < +\infty$. In the special case in which $k = n$ and $S = E$ is a finite subset of $\mathbb{R}^n$, we denote by $[\![E]\!]$ the standard integration over E defined as the rectifiable n-current with $\theta = 1$ and $\tau = e_1 \wedge \cdots \wedge e_n$, the standard orientation of $\mathbb{R}^n$. Precisely

$$[\![E]\!](\omega) = \int_E \langle \omega(x), e_1 \wedge \cdots \wedge e_n \rangle \, dx, \qquad \omega \in \mathcal{D}^n(\mathbb{R}^n).$$

If E is a finite perimeter set with finite Lebesgue measure, then $[\![E]\!]$ turns out to be an integral current.

Given a map $u \in C^1(A; \mathbb{R}^m)$ we introduce its graph $G_u \subseteq A \times \mathbb{R}^m$ as

$$G_u = \{(x, y) \in A \times \mathbb{R}^m : y = u(x)\}$$

and we use the map $\mathrm{Id} \times u : A \to A \times \mathbb{R}^m$, $(\mathrm{Id} \times u)(x) := (x, u(x))$, to parametrize it. G_u is identified in a natural way with an integral current given by integration over it. More precisely, denoting this current by $[\![G_u]\!]$, its standard orientation is given by $\mathcal{M}(\nabla u)/|\mathcal{M}(\nabla u)|$, the multiplicity θ is always 1, and so for all n-form $\omega \in \mathcal{D}^n(A \times \mathbb{R}^m)$ it holds

$$[\![G_u]\!](\omega) := (\mathrm{Id} \times u)_\sharp [\![A]\!] = \int_A \langle \omega(x, u(x)), \mathcal{M}(\nabla u(x)) \rangle \, dx.$$

[1] S is said $\mathcal{H}^k$-rectifiable if there are (at most) countably many Lipschitz maps $\phi_h : \mathbb{R}^k \to \mathbb{R}^n$ such that

$$S \subseteq N \cup \bigcup_{h=0}^{+\infty} \phi_h(\mathbb{R}^k), \qquad \mathcal{H}^k(N) = 0.$$

It is seen that $[\![G_u]\!]$ hass mass that coincides with the $\mathcal{H}^n$-measure of G_u, and is given by

$$|[\![G_u]\!]| = \mathbb{A}(u, A) = \int_A |\mathcal{M}(\nabla u)|\, dx.$$

It turns out, thanks to the regularity of u, that $[\![G_u]\!]$ is boundaryless.

We now want to extend the definitions above for maps $u \in BV(A, \mathbb{R}^m)$. To this aim we denote by $R_u \subseteq A$ the set of regular points of u, namely the points x that are Lebesgue points for u and ∇u, moreover $u(x)$ coincides with its Lebesgue value and u is approximately differentiable at x. We denote

$$G_u^R := \{(x, y) \in R_u \times \mathbb{R}^2 : y = u(x)\}.$$

Also G_u^R is $\mathcal{H}^n$-rectifiable and we define

$$\mathcal{G}_u := [\![G_u^R]\!] = (\mathrm{Id} \times u)_\sharp [\![R_u]\!].$$

It holds that

$$|\mathcal{G}_u| = \int_A |\mathcal{M}(\nabla u)|\, dx,$$

where ∇u is the approximate gradient of u. In general $\mathcal{G}_u$ has non-trivial boundary. In the special case that $\partial \mathcal{G}_u = 0$ in $\mathcal{D}_{n-1}(A \times \mathbb{R}^m)$ we say that u is a Cartesian map.

2.2 Relaxation and the setting of the problem

In what follows $\Omega \subset \mathbb{R}^2$ will be our reference domain, an open bounded set. Let $N := 1 + 2m + m(m+1)/2$ and let $g : \mathbb{R}^N \to [0, +\infty)$ be convex; we will assume that there is a constant $C_g > 0$ such that for all $A \in \mathbb{R}^N$,

$$|g(A)| \le C_g(|A| + 1). \tag{2.6}$$

Furthermore, we assume not degeneracy of the functional through the folllowing condition

$$|g(A)| \ge c_g \sum_{\substack{i,j=1 \\ i \ne j}}^{m} |M_{12}^{ij}(A)|, \tag{2.7}$$

for a general positive constant c_g. In the case that $m = 2$ the above condition is equivalent to (1.13). As a consequence of the growth condition (2.6) and of the convexity of g, the subdifferential ∂g satisfies

$$\|\partial g\|_{L^\infty} \le C_g. \tag{2.8}$$

For all $u \in C^1(\Omega; \mathbb{R}^m) \cap BV(\Omega; \mathbb{R}^m)$ we define

$$F(u, \Omega) := \int_\Omega g(\mathcal{M}(\nabla u))dx. \tag{2.9}$$

The first observation is that F is lower semicontinuous with respect to the strict convergence on the space of regular maps:

Theorem 2.4. *The functional F in* (2.9) *is lower semicontinuous on $C^1(\Omega; \mathbb{R}^m) \cap BV(\Omega; \mathbb{R}^m)$ with respect to the strict convergence in $BV(\Omega; \mathbb{R}^m)$.*

Proof. It is enough to apply [31, Theorem 2.6]. Alternatively we can give a short proof as follows: We have to show that if $u_k \in C^1(\Omega;\mathbb{R}^m) \cap BV(\Omega;\mathbb{R}^m)$ strictly converges to $u \in C^1(\Omega;\mathbb{R}^m) \cap BV(\Omega;\mathbb{R}^m)$, then $F(u,\Omega) \leq \liminf_{k\to\infty} F(u_k,\Omega)$. We can then suppose without loss of generality that $F(u_k,\Omega)$ tends to a finite limit. Thanks to the strict convergence we deduce that

$$\nabla u_k \rightharpoonup \nabla u \text{ weakly star in } \mathcal{M}_b(\Omega;\mathbb{R}^{2m}).$$

Furthermore, condition (2.7) ensures that $\|M_{12}^{i_1 i_2}(\nabla u)\|_{L^1}$ is uniformly bounded, and so we can suppose that

$$M_{12}^{i_1 i_2}(\nabla u_k) \rightharpoonup m^{i_1 i_2} \text{ weakly star in } \mathcal{M}_b(\Omega), \tag{2.10}$$

for some measure $m^{i_1 i_2}$, for all i_1, i_2 with $1 \leq i_1 < i_2 \leq m$. On the other hand, using [42] (see also [25, Theorem 2.5 and Remark 2.8]) we conclude that $m^{i_1 i_2} = M_{12}^{i_1 i_2}(\nabla u_k)$. In particular, we have found

$$\mathcal{M}(\nabla u_k) \rightharpoonup \mathcal{M}(\nabla u) \text{ weakly star in } \mathcal{M}_b(\Omega;\mathbb{R}^N).$$

So the conclusion follows from the standard result [31, Lemma 2.1]. □

Next, we are concerned with the relaxation $\mathcal{F}$ of the functional (2.9) in the space $BV(\Omega;\mathbb{R}^m)$ with respect to the strict convergence; this is given by (1.9), where the approximating functions u_k are obviously taken in $C^1(\Omega;\mathbb{R}^m) \cap BV(\Omega;\mathbb{R}^m)$, since we approximate u in the strict topology. The following result is straightforward:

Theorem 2.5. *The functional $\mathcal{F}(\cdot,\Omega)$ is lower semicontinuous on the space $BV(\Omega;\mathbb{R}^m)$ with respect to the strict convergence.*

Notice that the functional $\mathcal{F}$ extends F in (2.9), as a consequence of Theorem 2.4.

Definition 2.6. *Let $u \in BV(\Omega;\mathbb{R}^m)$. We call $(v_k) \subset C^1(\Omega;\mathbb{R}^m) \cap BV(\Omega;\mathbb{R}^m)$ a recovery sequence for $\mathcal{F}(u,\Omega)$ whenever $v_k \to u$ strictly and*

$$\mathcal{F}(u,\Omega) = \lim_{k\to\infty} F(v_k,\Omega).$$

For any $u \in BV(\Omega;\mathbb{R}^m)$ a recovery sequence always exists, by definition of relaxation.

Let $u \in \mathrm{Lip}(\Omega;\mathbb{R}^m)$: By Remark 2.3, there exists a sequence $(v_k) \subset C^1(\Omega;\mathbb{R}^m) \cap BV(\Omega;\mathbb{R}^m)$ such that $v_k \to u$ strictly in $BV(\Omega;\mathbb{R}^m)$ and

$$\begin{aligned} &\nabla v_k \to \nabla u && \text{strongly in } L^1(\Omega;\mathbb{R}^{m\times 2}), \\ &M_{12}^{ij}(\nabla v_k) \to M_{12}^{ij}(\nabla u) && \text{strongly in } L^1(\Omega), \end{aligned}$$

for all $i, j \in \{1,\dots,m\}$. Up to a subsequence these convergences take place also poitwise a.e., and by (1.12) we can apply Lebesgue dominated convergence theorem to conclude

$$F(v_k,\Omega) \to F(u,\Omega) := \int_\Omega g(\nabla u)dx, \tag{2.11}$$

where in the right-hand side, with a little abuse of notation, we have extended the definition of F in (2.9) to Lipschitz functions. This in particular implies $\mathcal{F}(u,\Omega) \leq F(u,\Omega)$ for all Lipschitz functions u. On the other hand, we also have the following:

Theorem 2.7. *Let $u \in \mathrm{Lip}(\Omega;\mathbb{R}^m)$; then*

$$\mathcal{F}(u,\Omega) = \int_\Omega g(\mathcal{M}(\nabla u))dx.$$

Proof. Thanks to the previous discussion we have only to show that $F(u,\Omega) \le \mathcal{F}(u,\Omega)$. If $v_k \in C^1(\Omega;\mathbb{R}^m)$ is a recovery sequence for $F(u,\Omega)$, then

$$\begin{aligned} \nabla v_k &\to \nabla u && \text{weakly star in } \mathcal{M}_b(\Omega;\mathbb{R}^{m\times 2}),\\ M^{ij}_{12}(\nabla v_k) &\to M^{ij}_{12}(\nabla u) && \text{weakly star in } \mathcal{M}_b(\Omega), \end{aligned}$$

for all $i,j \in \{1,\dots,m\}$. The first convergence is trivial, whereas the second one follows from [25, Theorem 2.5 and Remark 2.8] (see also [42]), since in this case condition (2.7) and the strict convergence ensure that $|\mathcal{M}(\nabla v_k)|$ are uniformly bounded in $L^1(\Omega)$. Hence from standard lower semicontinuity results (see, e.g., [31, Lemma 2.1]) we conclude $F(u,\Omega) \le \liminf_{k\to\infty} F(v_k,\Omega) = \mathcal{F}(u,\Omega)$. □

Theorem 2.8. *Let $K \subset \Omega$ be a compact set and assume that $u \in BV(\Omega;\mathbb{R}^m)$ is continuous on Ω, is of class C^1 on $\Omega \setminus K$ and is Lipschitz continuous on K. Then*

$$\mathcal{F}(u,\Omega) \le F(u,\Omega) := \int_\Omega g(\mathcal{M}(\nabla u))dx. \tag{2.12}$$

Proof. Let us extend u to 0 outside Ω. We choose an open neighborhood U of K so that $U \subset\subset \Omega$, and a cut-off map $\varphi \in C^\infty_c(U)$ such that $0 \le \varphi \le 1$, $\varphi = 1$ on K. For all $k \ge 1$ such that $\frac{1}{k} < \operatorname{dist}(K,\partial\Omega)$ we define

$$u_k = \varphi(u \star \rho_k) + (1-\varphi)u, \tag{2.13}$$

where $\rho_k(x) = k^2\rho(kx)$ and ρ is a smooth convolution kernel. The function $u \star \rho_k$ strictly converges to u, and so it is easily checked that $u_k \in C^1(\Omega;\mathbb{R}^m)$ strictly converge to u in $BV(\Omega;\mathbb{R}^m)$. Now, $u \llcorner U$ belongs to $W^{1,\infty}(U;\mathbb{R}^m)$, and by definition of u_k we can easily show that $u_k \to u$ strongly in $W^{1,p}(U;\mathbb{R}^m)$ for all $p < +\infty$. This implies that $F(u_k,U) \to F(u,U)$. In particular,

$$F(u,\Omega) = F(u,\Omega\setminus\overline{U}) + F(u,U) = \lim_{k\to\infty}\big(F(u,\Omega\setminus\overline{U}) + F(u_k,U)\big) = \lim_{k\to\infty} F(u_k,\Omega).$$

This implies $\mathcal{F}(u,\Omega) \le F(u,\Omega)$. □

3 Tools and preliminary results

3.1 Properties of measures

In order to prove our main result Theorem 1.1 we will employ the classical theorem named after De Giorgi and Letta, which we collect here in a form specialized for our setting (see [2, Theorem 1.53] for the general formulation and its proof). We denote by $\mathcal{U}(\Omega)$ the family of open subsets of Ω.

Theorem 3.1 (De Giorgi-Letta)**.** *Let $\Omega \subset \mathbb{R}^2$ be an open set and assume that $\mu : \mathcal{U}(\Omega) \to [0,+\infty]$ is a function so that $\mu(\varnothing) = 0$. If*

(i) μ is non-decreasing, i.e., $\mu(B) \le \mu(A)$ for all $A,B \in \mathcal{U}(\Omega)$, $B \subseteq A$;

(ii) μ is additive, i.e., $\mu(A\cup B) = \mu(A) + \mu(B)$ for all $A,B \in \mathcal{U}(\Omega)$, $A \cap B = \varnothing$;

(iii) μ is sub-additive, i.e., $\mu(A) \le \mu(B_1) + \mu(B_2)$ for all $A,B_1,B_2 \in \mathcal{U}(\Omega)$, $A \subseteq B_1 \cup B_2$;

(iv) μ is inner regular, i.e., for all $A \in \mathcal{U}(\Omega)$ it holds

$$\mu(A) = \sup\{\mu(B) : B \in \mathcal{U}(\Omega),\ B \subset\subset A\};$$

Then μ is the restriction to $\mathcal{U}(\Omega)$ of a Borel measure $\overline{\mu} : \mathcal{B}(\Omega) \to [0, +\infty]$.

We will often use the following result due to Reshetnyak (see [2, Theorem 2.39]):

Theorem 3.2. *Let $M \geq 1$ and let μ, μ_k be Radon measures in $A \subseteq \mathbb{R}^n$ taking values in $\mathbb{R}^M$. Suppose that $\mu_k \rightharpoonup \mu$ weakly star as measures and that $|\mu_k|(A) \to |\mu|(A)$. Then*

$$\int_A f\left(x, \frac{\mu_k}{|\mu_k|}(x)\right) d|\mu_k|(x) \to \int_A f\left(x, \frac{\mu}{|\mu|}(x)\right) d|\mu|(x)$$

as $k \to \infty$ for all continuous and bounded functions $f : A \times S^{M-1} \to \mathbb{R}$.

We will also need the following property valid for strictly converging Borel measures μ_k, μ.

Lemma 3.3. *Suppose that $\mu_k \rightharpoonup \mu$ weakly star as measures and $|\mu_k|(A) \to |\mu|(A)$, and let $B \subset A$ be open. Then if $|\mu|(A \cap \partial B) = |\mu_k|(A \cap \partial B) = 0$ for all k, it holds*

$$|\mu_k|(B) \to |\mu|(B).$$

Proof. By lower semicontinuity of the total variation on open sets and thanks to the hypothesis $\mu(A \cap \partial B) = 0$ we have

$$\begin{aligned} |\mu|(A) = |\mu|(B) + |\mu|(A \setminus \overline{B}) &\leq \liminf_{k\to\infty} |\mu_k|(B) + \liminf_{k\to\infty} |\mu_k|(A \setminus \overline{B}) \\ &\leq \liminf_{k\to\infty} |\mu_k|(A) = \lim_{k\to\infty} |\mu_k|(A) = |\mu|(A), \end{aligned}$$

so all the inequalities are equalities and in particular $|\mu|(B) = \liminf_{k\to\infty} |\mu_k|(B)$. Since the same holds for every subsequence of μ_k, we easily infer that the liminf is indeed a limit. □

We also collect the following result which can be found in [33, Proposition 1, Section 1.3.4].

Proposition 3.4. *Let A be open and bounded and let h be a positive integer. Let $V_k, V \in L^1(A; \mathbb{R}^h)$ be such that $V_k \rightharpoonup V$ weakly star in $L^1(A; \mathbb{R}^h)$ and moreover*

$$\int_A \sqrt{1 + |V_k|^2} dx \to \int_A \sqrt{1 + |V|^2} dx$$

as $k \to +\infty$. Then $V_k \to V$ strongly in $L^1(A; \mathbb{R}^h)$.

3.2 Lipschitz and BV curves

Given a Lipschitz map $\varphi : [a, b] \to \mathbb{R}^m$, we denote by $L_\varphi := \int_a^b |\dot{\gamma}| d\tau$ its total variation and we introduce the quantity

$$s_\varphi(t) = \frac{1}{L_\varphi + (b - a)} \int_a^t (|\dot{\varphi}| + 1) d\tau, \qquad \forall t \in [a, b]. \tag{3.1}$$

This is a strictly increasing and continuous function, so we let $t_\varphi : [0, 1] \to [a, b]$ be its inverse $t_\varphi = s_\varphi^{-1}$, which satisfies

$$\dot{t}_\varphi(s) = \frac{L_\varphi + (b - a)}{|\dot{\varphi}(t_\varphi(s))| + 1} \qquad \text{for a.e. } s \in [0, 1]. \tag{3.2}$$

In particular $\dot{t}_\varphi(s) \le L_\varphi + (b-a)$ for a.e. $s \in [0,1]$. A similar definition applies to a function $\gamma \in BV([a,b];\mathbb{R}^m)$, for which we denote $L_\gamma := |\dot\gamma|([a,b])$ and

$$s_\gamma(t) = \frac{1}{L_\gamma + (b-a)}\big(|\dot\gamma|([a,t)) + (t-a)\big), \qquad \forall t \in [a,b], \tag{3.3}$$

which defines a strictly increasing map with jumps set S_γ, the jump set of γ; moreover

$$s_\gamma(t_1) - s_\gamma(t_2) \ge \frac{t_1 - t_2}{L_\gamma + (b-a)}, \qquad 0 \le t_2 \le t_1 \le 1,$$

and so it follows that if $t_\gamma := s_\gamma^{-1} : [0,1] \to [a,b]$ is the inverse of s_γ that is constant on $[s_\gamma(t^-), s_\gamma(t^+)]$ for all $t \in S_\gamma$, we have

$$t_\gamma(s_1) - t_\gamma(s_2) = |t_\gamma(s_1) - t_\gamma(s_2)| \le (s_1 - s_2)(L_\gamma + (b-a)), \qquad 0 \le s_2 \le s_1 \le 1.$$

Hence t_γ is Lipschitz continuous with Lipschitz constant $L_\gamma + (b-a)$.

Definition 3.5. *Given $\gamma \in BV([a,b];\mathbb{R}^m)$ we define $\overline{\gamma} : [0,1] \to \mathbb{R}^m$ as*

$$\overline{\gamma}(s) = \begin{cases} \frac{\gamma(t^+)(s - s_\gamma(t^-)) + \gamma(t^-)(s_\gamma(t^+) - s)}{s_\gamma(t^+) - s_\gamma(t^-)} & \text{if } s \in [s_\gamma(t^-), s_\gamma(t^+)], \\ \gamma(t_\gamma(s)) & \text{otherwise.} \end{cases} \tag{3.4}$$

When $\gamma = \varphi$ is Lipschitz continuous, it simply holds $\overline{\varphi}(s) = \varphi(t_\varphi(s))$, a Lipschitz continuous map satisfying

$$\left|\frac{d}{ds}\overline{\varphi}(s)\right| = \left|\dot\varphi(t_\varphi(s))\dot{t}_\varphi(s)\right| \le L_\varphi + (b-a), \qquad \text{for a.e. } s \in [0,1]. \tag{3.5}$$

The same is true for $\overline{\gamma}$ when $\gamma \in BV([a,b];\mathbb{R}^m)$; we will obtain this as a consequence of the following result.

Proposition 3.6. *Let $\gamma \in BV([a,b];\mathbb{R}^m)$ and let $(\varphi_k) \subset Lip([a,b];\mathbb{R}^m)$ be a sequence of maps converging strictly to γ as $k \to \infty$. The functions $\overline{\varphi}_k := \varphi_k \circ t_{\varphi_k} : [0,1] \to \mathbb{R}^m$ are Lipschitz continuous with uniformly bounded Lipschitz constants and*

$$\begin{aligned} &\overline{\varphi}_k \to \overline{\gamma} \qquad \text{strictly in } BV([0,1];\mathbb{R}^m) \text{ and weakly star in } W^{1,\infty}([0,1];\mathbb{R}^m), \\ &s_{\varphi_k} \to s_\gamma \qquad \text{strictly in } BV([a,b]), \\ &t_{\varphi_k} \to t_\gamma \qquad \text{weakly star in } W^{1,\infty}([0,1]). \end{aligned} \tag{3.6}$$

Moreover there exists a function $a_\gamma : \mathbb{R}^+ \to \mathbb{R}^+$ depending only on γ and such that $a_\gamma(t) \to 0$ when $t \to 0^+$, and

$$\|s_\varphi - s_\gamma\|_{L^1} + \|\overline{\varphi} - \overline{\gamma}\|_{L^\infty} \le a_\gamma(d_s(\varphi,\gamma)),$$

for all $\varphi \in Lip([a,b];\mathbb{R}^m)$.

Proposition 3.6 can be obtained by inspecting and slightly modifying the arguments leading to [17, Lemma 2.10] and [5, Lemma 2.7]. For the reader convenience and for the sake of completeness we give the proof.

Proof. Let us denote $L_\gamma := |\dot\gamma|([a,b])$, and $L_k := |\dot\varphi_k|([a,b])$ the total variations of γ and φ_k respectively. To shortcut the notation we denote $s_{\varphi_k} : [a,b] \to [0,1]$ in (3.1) by $s_k = s_{\varphi_k}$

and its inverse $t_{\varphi_k} : [0,1] \to [a,b]$ in (3.2) as $t_k = t_{\varphi_k}$. Moreover we recall the definition of $s_\gamma \in BV([a,b])$ given in (3.3).

Step 1: Convergence of s_{φ_k} and t_{φ_k}. Thanks to the strict convergence of φ_k to γ, it is easy to see that $s_k \to s_\gamma$ pointwise a.e. and strictly in $BV([a,b])$. In particular, if γ is continuous at $t \in [a,b]$, then $s_k(t) \to s_\gamma(t)$. Moreover, s_γ is strictly increasing, and its jump set coincides with the jump set S_γ of γ.

As for t_k, due to the fact that its Lipschitz constant is less than or equal to $L_k + (b-a)$, and since $L_k \to L_\gamma$, we conclude that there is a Lipschitz function $\tau : [0,1] \to [a,b]$ such that, up to a subsequence,

$$t_k \rightharpoonup \tau \qquad \text{weakly star in } W^{1,\infty}([0,1]),$$

and hence also pointwise on $[0,1]$. We claim that $\tau = t_\gamma = s_\gamma^{-1}$, and so, by uniqueness of the limit, we will also infer that the whole sequence t_k converges to t_γ.

Notice that τ is non-decreasing and continuous, and maps $[0,1]$ onto $[a,b]$; let then $\sigma \in [0,1]$ be so that $\tau(\sigma) \notin S_\gamma$. For any $\varepsilon > 0$ we can find $0 < \delta \leq \varepsilon$ so that $I_\delta = (\tau(\sigma) - \delta, \tau(\sigma) + \delta)$ enjoies $|\dot\gamma|(I_\delta) < \varepsilon$, and in addition $\tau(\sigma) - \delta \notin S_\gamma$ and $\tau(\sigma) + \delta \notin S_\gamma$. The last condition implies that $|\dot\varphi_k|(I_\delta) \to |\dot\gamma|(I_\delta)$, and so

$$\begin{aligned}\lim_{k\to\infty} |s_k(t_k(\sigma)) - s_k(\tau(\sigma))| &= \lim_{k\to\infty} \frac{1}{L_k + (b-a)} \left| \int_{\tau(\sigma)}^{t_k(\sigma)} |\dot\varphi_k| + 1 dr \right| \\ &\leq \frac{1}{L_\gamma + (b-a)} \lim_{k\to\infty} \int_{I_\delta} |\dot\varphi_k| + 1 dr \leq \frac{3\varepsilon}{L_k + (b-a)}.\end{aligned}$$

By arbitrariness of ε we conclude that

$$s_k(\tau(\sigma)) \to s_k(t_k(\sigma)) = \sigma \qquad \text{as } k \to \infty. \tag{3.7}$$

On the other hand $s_k(\tau(\sigma)) \to s_\gamma(\tau(\sigma))$, so we conclude $s_\gamma(\tau(\sigma)) = \sigma$ for all σ with $\tau(\sigma) \notin S_\gamma$. This implies that $\tau(\sigma) = t_\gamma(\sigma)$ for any σ such that $\tau(\sigma) \notin S_\gamma$, but now, since τ is continuous non-decresing and so is t_γ (which in addition is constant on the connected components of $t_\gamma^{-1}(S_\gamma)$), necessarily $\tau(\sigma) = t_\gamma(\sigma)$ for all $\sigma \in [0,1]$.

Step 2: Convergence of $\overline\varphi_k$. Recalling that

$$\left|\frac{d}{ds}\overline\varphi_k(s)\right| \leq L_k + b - a \qquad \text{for a.e. } s \in [0,1],$$

and since $L_k \to L_\gamma$ as $k \to +\infty$, $\overline\varphi_k$ are uniformly bounded in $W^{1,\infty}([0,1];\mathbb{R}^m)$, and so, up to a subsequence, they converge weakly star to some limit $\zeta \in W^{1,\infty}([0,1];\mathbb{R}^m)$ with

$$\left|\frac{d}{ds}\zeta(s)\right| \leq L_\gamma + b - a \qquad \text{for a.e. } s \in [0,1]. \tag{3.8}$$

We have to prove that this limit is $\overline\gamma$, indipendently from the subsequence; as a consequence it will follow that the full sequence $\overline\varphi_k$ converges to $\overline\gamma$.

To this purpose we fix

$$\sigma \in [0,1] \setminus \left(\cup_{t\in S_\gamma} [s_\gamma(t^-), s_\gamma(t^+)]\right);$$

this is equivalent to require that $t_\gamma(\sigma) \notin S_\gamma$. Thus we write

$$\begin{aligned}|\overline\varphi_k(\sigma) - \overline\gamma(\sigma)| &= |\varphi_k(t_k(\sigma)) - \gamma(t_\gamma(\sigma))| \leq |\varphi_k(t_k(\sigma)) - \varphi_k(t_\gamma(\sigma))| + |\varphi_k(t_\gamma(\sigma)) - \gamma(t_\gamma(\sigma))| \\ &\leq \left| \int_{t_k(\sigma)}^{t_\gamma(\sigma)} |\dot\varphi_k| + 1\, dr \right| + |\varphi_k(t_\gamma(\sigma)) - \gamma(t_\gamma(\sigma))| \\ &= (L_k + (b-a))(s_k(t_k(\sigma)) - s_k(t_\gamma(\sigma))) + |\varphi_k(t_\gamma(\sigma)) - \gamma(t_\gamma(\sigma))|\end{aligned}$$

and thanks to (3.7) and the fact that $\varphi_k \to \gamma$ pointwise a.e. on $[a,b] \setminus S_\gamma$, we conclude that

$$\varphi_k(\sigma) \to \overline{\gamma}(\sigma) \qquad \text{for a.e. } \sigma \in [0,1] \setminus \left(\cup_{t \in S_\gamma} [s_\gamma(t^-), s_\gamma(t^+)]\right).$$

Therefore we conclude $\zeta = \overline{\gamma}$ a.e. on $[0,1] \setminus \left(\cup_{t\in S_\gamma}[s_\gamma(t^-), s_\gamma(t^+)]\right)$. We want to show that $\zeta(s)$ coincides with the first line in (3.4) when $s \in [s_\gamma(t^-), s_\gamma(t^+)]$, for some $t \in S_\gamma$.

If $t \in S_\gamma$, there are sequences $t_j^- \to t^-$ and $t_j^+ \to t^+$ as $j \to \infty$, such that $t_j^\pm$ are continuity points of γ (and of s_γ). In particular $\gamma(t_j^\pm) = \overline{\gamma}(s_\gamma(t_j^\pm)) \to \overline{\gamma}(s_\gamma(t)^\pm)$ as $j \to \infty$, so

$$\overline{\gamma}(s_\gamma(t)^\pm) = \gamma(t^\pm).$$

Moreover, since $s_\gamma(t)^+ = s_\gamma(t)^- + \frac{1}{L_\gamma+b-a}|\dot{\gamma}|(\{t\})$ we deduce that

$$s_\gamma(t)^+ - s_\gamma(t)^- = \frac{1}{L_\gamma + b - a}|\gamma(t^+) - \gamma(t^-)| = \frac{1}{L_\gamma + b - a}|\overline{\gamma}(s_\gamma(t)^+) - \overline{\gamma}(s_\gamma(t)^-)|.$$

We conclude that the curve $\overline{\gamma} \llcorner [s(t^-), s(t^+)]$ is a curve connecting $\overline{\gamma}(s_\gamma(t)^-)$ to $\overline{\gamma}(s_\gamma(t)^+)$ on an interval of length $\frac{1}{L_\gamma+b-a}|\overline{\gamma}(s_\gamma(t)^+) - \overline{\gamma}(s_\gamma(t)^-)|$; by (3.8) this curve must necessarily be the constant speed parametrization of the segment with endpoints $\overline{\gamma}(s_\gamma(t)^-)$ and $\overline{\gamma}(s_\gamma(t)^+)$, namely $\zeta(s)$ coincides with the interpolation in (3.4). We conclude then also the first thesis in (3.6).

Step 3: To prove the last statement, we set

$$a_\gamma(t) := \sup\{\|s_\varphi - s_\gamma\|_{L^1} + \|\overline{\varphi} - \overline{\gamma}\|_{L^\infty} : \varphi \in \mathrm{Lip}([a,b];\mathbb{R}^m),\ d_s(\varphi,\gamma) \le t\}.$$

Assume by contradiction that there exists a sequence of positive numbers $t_k \searrow 0$ such that $\lim_{k\to\infty} a_\gamma(t_k) > 0$. Then, by definition of a_γ we can find functions $\psi_k \in \mathrm{Lip}([a,b];\mathbb{R}^m)$ such that $d_s(\psi_k,\gamma) \le t_k$ and

$$\lim_{k\to\infty} (\|s_{\psi_k} - s_\gamma\|_{L^1} + \|\overline{\varphi}_k - \overline{\gamma}\|_{L^\infty}) > 0.$$

This is a clear contradiction with (3.6), hence the thesis follows. □

Corollary 3.7. *Let $\gamma \in BV([a,b];\mathbb{R}^m)$, then $\overline{\gamma}$ is Lipschitz continuous with Lipschitz constant $L_\gamma + (b-a)$.*

Proof. It is sufficient to approximate γ in the strict topology of $BV([a,b];\mathbb{R}^m)$ by Lipschitz maps, and the thesis follows from Proposition 3.6. □

Interpolation between Lipschitz curves: Let $h > 0$ be fixed and let $[a,b]$, $a < b$, be an interval. For Lipschitz maps $\varphi, \psi : [a,b] \to \mathbb{R}^m$ we introduce the following interpolations: $\Phi_{\varphi,\psi} : [a,b] \times [0,h] \to \mathbb{R}^m$ given by

$$\Phi_{\varphi,\psi}(t,r) := \varphi\Big(t_\varphi\big(s_\varphi(t)\frac{r}{h} + s_\psi(t)\frac{h-r}{h}\big)\Big), \tag{3.9}$$

that satisfies $\Phi_{\varphi,\psi}(t,h) = \varphi(t)$ and $\Phi_{\varphi,\psi}(t,0) = \varphi(t_\varphi \circ s_\psi(t))$, and the mapping $\Psi_{\varphi,\psi} : [a,b] \times [0,h] \to \mathbb{R}^m$ defined by

$$\Psi_{\varphi,\psi}(t,r) := \varphi\Big(t_\varphi\big(s_\psi(t)\big)\Big)\frac{h-r}{h} + \psi\Big(t_\psi\big(s_\psi(t)\big)\Big)\frac{r}{h} = \overline{\varphi}(s_\psi(t))\frac{h-r}{h} + \overline{\psi}(s_\psi(t))\frac{r}{h}, \tag{3.10}$$

where we recall $\overline{\varphi}(s) = \varphi \circ t_\varphi(s)$ and $\overline{\psi}(s) = \psi \circ t_\psi(s)$. This satisfies $\Psi_{\varphi,\psi}(t,0) = \overline{\varphi}(s_\psi(t)) = \Phi_{\varphi,\psi}(t,0)$ and $\Psi_{\varphi,\psi}(t,h) = \overline{\psi}(s_\psi(t)) = \psi(t)$. We compute the derivatives of $\Phi_{\varphi,\psi}$ and $\Psi_{\varphi,\psi}$ and for a.e. $(t,r) \in [a,b] \times [0,h]$ we find

$$\begin{aligned}
\frac{\partial}{\partial t}\Phi_{\varphi,\psi}(t,r) &= \dot{\varphi}\Big(t_\varphi\big(s_\varphi(t)\frac{r}{h} + s_\psi(t)\frac{h-r}{h}\big)\Big)\dot{t}_\varphi\big(s_\varphi(t)\frac{r}{h} + s_\psi(t)\frac{h-r}{h}\big)\big(\dot{s}_\varphi(t)\frac{r}{h} + \dot{s}_\psi(t)\frac{h-r}{h}\big),\\
\frac{\partial}{\partial r}\Phi_{\varphi,\psi}(t,r) &= \dot{\varphi}\Big(t_\varphi\big(s_\varphi(t)\frac{r}{h} + s_\psi(t)\frac{h-r}{h}\big)\Big)\dot{t}_\varphi\big(s_\varphi(t)\frac{r}{h} + s_\psi(t)\frac{h-r}{h}\big)\frac{s_\varphi(t) - s_\psi(t)}{h},\\
\frac{\partial}{\partial t}\Psi_{\varphi,\psi}(t,r) &= \Big(\dot{\varphi}\Big(t_\varphi\big(s_\psi(t)\big)\Big)\dot{t}_\varphi\big(s_\psi(t)\big)\frac{h-r}{h} + \dot{\psi}\Big(t_\psi\big(s_\psi(t)\big)\Big)\dot{t}_\psi\big(s_\psi(t)\big)\frac{r}{h}\Big)\,\dot{s}_\psi(t)\\
&= \Big(\frac{h-r}{h}\dot{\overline{\varphi}}(s_\psi(t)) + \frac{r}{h}\dot{\overline{\psi}}(s_\psi(t))\Big)\,\dot{s}_\psi(t),\\
\frac{\partial}{\partial r}\Psi_{\varphi,\psi}(t,r) &= \frac{1}{h}\,\Big(\psi\Big(t_\psi\big(s_\psi(t)\big)\Big) - \varphi\Big(t_\varphi\big(s_\psi(t)\big)\Big)\Big) = \frac{\overline{\psi}(s_\psi(t)) - \overline{\varphi}(s_\psi(t))}{h},
\end{aligned}$$

that, by (3.1) and (3.2), lead to the following estimates

$$\begin{aligned}
\left|\frac{\partial}{\partial t}\Phi_{\varphi,\psi}(t,r)\right| &\leq (L_\varphi + (b-a))\left|\dot{s}_\varphi(t)\frac{r}{h} + \dot{s}_\psi(t)\frac{h-r}{h}\right|\\
&\leq (L_\varphi + (b-a))\left(\frac{|\dot{\varphi}(t)|+1}{L_\varphi + (b-a)} + \frac{|\dot{\psi}(t)|+1}{L_\psi + (b-a)}\right),\\
\left|\frac{\partial}{\partial r}\Phi_{\varphi,\psi}(t,r)\right| &\leq \frac{L_\varphi + (b-a)}{h}|s_\psi(t) - s_\varphi(t)|;
\end{aligned}$$

furthermore we also have

$$\frac{\partial}{\partial t}\Phi_{\varphi,\psi}(t,r) \wedge \frac{\partial}{\partial r}\Phi_{\varphi,\psi}(t,r) = \det\ (\nabla\Phi_{\varphi,\psi}(t,r)) = 0, \tag{3.11}$$

for almost every $(t,r) \in [a,b] \times [0,h]$, due to the fact that the image of $\Phi_{\varphi,\psi}$ is one dimensional. Finally we can estimate on $D := [a,b] \times [0,h]$ the integral

$$\begin{aligned}
\int_D |\nabla\Phi_{\varphi,\psi}(t,r)|dtdr &\leq (L_\varphi + (b-a))\int_D \frac{|\dot{\varphi}(t)|+1}{L_\varphi + (b-a)} + \frac{|\dot{\psi}(t)|+1}{L_\psi + (b-a)} + \frac{|s_\psi(t) - s_\varphi(t)|}{h}dtdr\\
&= 2h(L_\varphi + (b-a)) + (L_\varphi + (b-a))\int_a^b |s_\psi(t) - s_\varphi(t)|dt. \qquad (3.12)
\end{aligned}$$

As for $\Psi_{\varphi,\psi}$, by the estimates

$$\begin{aligned}
\left|\frac{\partial}{\partial t}\Psi_{\varphi,\psi}(t,r)\right| &= \left|\frac{h-r}{h}\dot{\overline{\varphi}}(s_\psi(t)) + \frac{r}{h}\dot{\overline{\psi}}(s_\psi(t))\right|\dot{s}_\psi(t) \leq (L_\varphi + L_\psi + (b-a))\dot{s}_\psi(t),\\
\left|\frac{\partial}{\partial r}\Psi_{\varphi,\psi}(t,r)\right| &\leq \frac{|\overline{\psi}(s_\psi(t)) - \overline{\varphi}(s_\psi(t))|}{h},
\end{aligned}$$

we can write

$$\begin{aligned}
\int_D |\nabla\Psi_{\varphi,\psi}(t,r)|dtdr &\leq \int_D (L_\varphi + L_\psi + (b-a))\dot{s}_\psi(t) + \frac{|\overline{\psi}(s_\psi(t)) - \overline{\varphi}(s_\psi(t))|}{h}dtdr\\
&= (L_\varphi + L_\psi + (b-a))h + \int_a^b |\overline{\psi}(s_\psi(t)) - \overline{\varphi}(s_\psi(t))|\dot{s}_\psi(t)dt\\
&= (L_\varphi + L_\psi + (b-a))h + \int_0^1 |\overline{\psi}(s) - \overline{\varphi}(s)|ds, \qquad (3.13)
\end{aligned}$$

where we have used that $\int_a^b \dot{s}_\psi(t)dt = 1$. Finally

$$\begin{aligned}\int_D |\frac{\partial}{\partial t}\Psi_{\varphi,\psi}(t,r) \wedge \frac{\partial}{\partial r}\Psi_{\varphi,\psi}(t,r)|dtdr &\le (L_\varphi + L_\psi + (b-a)) \int_D \frac{|\overline{\psi}(s_\psi(t)) - \overline{\varphi}(s_\psi(t))|}{h} \dot{s}_\psi(t)dtdr \\ &= (L_\varphi + L_\psi + (b-a)) \int_a^b |\overline{\psi}(s_\psi(t)) - \overline{\varphi}(s_\psi(t))| \dot{s}_\psi(t)dt \\ &= (L_\varphi + L_\psi + (b-a)) \int_0^1 |\overline{\psi}(s) - \overline{\varphi}(s)|ds. \end{aligned} \tag{3.14}$$

3.3 Tubular neighborhoods of regular curves

Given a set $A \subset \mathbb{R}^2$ we denote by $\text{dist}(x, A)$ the distance from x to A, and by $\text{dist}^\pm(x, A)$ the signed distance from x to A, defined as

$$\text{dist}^\pm(x, A) := \begin{cases} \text{dist}(x, A) & \text{if } x \in A^c, \\ -\text{dist}(x, A^c) & \text{if } x \in A, \end{cases}$$

where $A^c := \mathbb{R}^2 \setminus A$. We consider the following regularity assumption (R) of a set A:

(R) We assume that A is a connected bounded open set with boundary of class C^3.

If $A \subset \mathbb{R}^2$ satisfies (R), then ∂A consists of finitely many loops Γ_i, $i = 0, 1, \dots, N$, of class C^3, labeled so that, if E_i denotes the bounded connected component of $\mathbb{R}^2 \setminus \Gamma_i$, then

$$A = E_0 \setminus (\cup_{i=1}^N E_i). \tag{3.15}$$

Notice that the presence of a unique big component E_0 is due to the hypothesis that A is connected[2].

Sets with C^3-boundary and tubular neighborhoods: Let $A \subset \mathbb{R}^2$ be a set satisfying (R). For $\delta \in (0, 1)$ small enough there exists a tubular neighborhood T_δ of ∂A, given by

$$T_\delta := \{x \in \mathbb{R}^2 : \text{dist}(x, \partial A) < \delta\}.$$

We parametrize T_δ with $(t, r) \in \partial A \times (-\delta, \delta)$ so that

$$\partial A_r := \{x \in \mathbb{R}^2 : \text{dist}^\pm(x, A) = r\}$$

consists of $N + 1$ curves Γ_r^i of class C^2, namely

$$\Gamma_r^0 := \{x \in \mathbb{R}^2 : \text{dist}^\pm(x, E_0) = r\} \qquad \Gamma_r^i := \{x \in \mathbb{R}^2 : \text{dist}^\pm(x, E_i) = -r\}.$$

We denote $T_\delta = \cup_{i=1}^N T_\delta^i$ where T_δ^i is a δ-neighborhood of Γ_i, namely

$$T_\delta^i = \{x \in \mathbb{R}^2 : \text{dist}(x, \Gamma_i) < \delta\}.$$

For simplicity[3], let us assume that the number N of holes in A is zero, i.e., A is simply connected; there is $\gamma \in C^3([a, b]; \mathbb{R}^2)$ a Jordan curve parametrized by arc-length enclosing the open bounded connected and simply-connected set A, $\Gamma = \gamma([a, b])$. We will denote

$$T_\delta^+ = \{x \in \mathbb{R}^2 : \text{dist}^\pm(x, A) \in (0, \delta)\}, \qquad T_\delta^- = \{x \in \mathbb{R}^2 : \text{dist}^\pm(x, A) \in (-\delta, 0)\},$$

[2]If A instead has $K > 1$ connected components, then every component enjoys a decomposition as (3.15).
[3]The following argument applies to all connected components of Γ in the general case.

the external and inner tubular neighborhoods of ∂A. By the tubular neighborhood theorem, there exists a bi-Lipschitz bijection $\mathcal{T}_\delta : [a,b) \times (-\delta,\delta) \to T_\delta$, such that

$$|\det(\nabla \mathcal{T}_\delta(t,r))| = 1 + R_\delta(t,r),$$

where $\|R_\delta\|_{L^\infty} = o(1) \to 0$ as $\delta \to 0$. Indeed one sets, for all $(t,r) \in [a,b) \times (-\delta,\delta)$,

$$\mathcal{T}_\delta(t,r) := \gamma(t) + r\dot\gamma(t)^\perp, \tag{3.16}$$

where $v^\perp = (-v_2, v_1)$, and it holds

$$\begin{aligned} &\frac{\partial}{\partial t}\mathcal{T}_\delta(t,r) = \dot\gamma(t) + r\ddot\gamma(t)^\perp, && \frac{\partial}{\partial r}\mathcal{T}_\delta(t,r) = \dot\gamma(t)^\perp, \\ &\det(\nabla \mathcal{T}_\delta) = 1 + r\dot\gamma(t)\cdot\ddot\gamma(t)^\perp =: 1 + R_\delta(t,r), && |R_\delta(t,r)| \le C_\gamma |r| \le C_\gamma \delta, \end{aligned}$$

where, here and below, we denote by $C_\gamma > 0$ a constant depending on γ but independent of δ (and which might change from line to line). Notice also that since γ is of class C^3, $\nabla \mathcal{T}_\delta$ is of class C^1, and (since $\delta \in (0,1)$)

$$|\nabla \mathcal{T}_\delta(t,r)| \le |\dot\gamma(t)| + r|\ddot\gamma(t)| \le C_\gamma + C_\gamma \delta \le C_\gamma,$$

Let $h \in (0,\delta)$. For $x \in T_h$ we have $\nabla \mathcal{T}_h^{-1}(x) = \big(\nabla \mathcal{T}_h(\mathcal{T}_h^{-1}(x))\big)^{-1}$, so

$$\det(\nabla \mathcal{T}_h^{-1}(x)) = \frac{1}{\det\big(\nabla \mathcal{T}_h(\mathcal{T}_h^{-1}(x))\big)} = \frac{1}{1 + R_h(\mathcal{T}_h^{-1}(x))} = 1 - \frac{R_h(\mathcal{T}_h^{-1}(x))}{1 + R_h(\mathcal{T}_h^{-1}(x))},$$

and, if h is small enough, we conclude

$$\det(\nabla \mathcal{T}_h^{-1}(x)) = 1 + R'_h(x), \qquad \|R'_h\|_{L^\infty} \le C_\gamma h. \tag{3.17}$$

Eventually, using that for a invertible matrix A one has $A^{-1} = \operatorname{cof}(A)^T(\det A)^{-1}$, we conclude

$$\begin{aligned} &\nabla \mathcal{T}_h^{-1}(x) = \operatorname{cof}\big(\nabla \mathcal{T}_h(\mathcal{T}_h^{-1}(x))\big)^T (1 + R'_h(x)), \\ &|\nabla \mathcal{T}_h^{-1}(x)| \le C_\gamma + C_\gamma h \le C_\gamma, \end{aligned} \tag{3.18}$$

for all $x \in T_h$ so $\mathcal{T}_h$ is bi-Lipschitz with a constant depending only on γ.

Restriction of BV-functions on curves: As above, let A satisfy (R), assume that A is simply connected, and let $\gamma \in C^3([a,b];\mathbb{R}^2)$ be an arc-length parametrization of a Jordan curve $\Gamma = \partial A$. Let T_δ be a tubular neighborhood of Γ, $\delta \in (0,1)$ small enough. Let $\widehat{\zeta} : [a,b] \times (-\delta,\delta) \to \mathbb{R}^2$ be the map

$$\widehat{\zeta}(t,r) := \frac{\frac{\partial \mathcal{T}_\delta}{\partial t}(t,r)}{|\frac{\partial \mathcal{T}_\delta}{\partial t}(t,r)|} = \frac{\dot\gamma(t) + r\ddot\gamma(t)^\perp}{|\dot\gamma(t) + r\ddot\gamma(t)^\perp|}, \tag{3.19}$$

that is the oriented unit vector tangent to Γ_r at the point $\gamma(t,r)$. Using that γ parametrizes by arc-length, a tedious but straightforward computation shows that the map

$$\zeta(x) := \widehat{\zeta}(\mathcal{T}_\delta^{-1}(x)), \qquad x \in T_\delta, \tag{3.20}$$

satisfies $\zeta \in C^1(T_\delta;\mathbb{S}^1)$ and is divergence free[4].

[4] We can also see this as follows: ζ is a unit vector such that $\zeta^\perp$ is orthogonal to the level sets of the signed distance function $d^\pm$ from Γ. In particular, since the distance function has gradient of length 1 almost everywhere, $\zeta^\perp$ coincides with $\nabla d^\pm$ almost everywhere. If follows that $\operatorname{div} \zeta = \operatorname{Curl} \zeta^\perp = \operatorname{Curl} \nabla d^\pm = 0$.

Definition 3.8. *Let $r \in (-\delta, \delta)$ and $\varphi : \Gamma_r \to \mathbb{R}^m$; we say that $\varphi \in C^1(\Gamma_r; \mathbb{R}^m)$ if $\varphi(\mathcal{T}_\delta(\cdot, r)) : [a, b) \to \mathbb{R}^m$ is of class C^1.*

Remark 3.9. *Given $\varphi \in C^1(\Gamma_r; \mathbb{R}^m)$ we can extend it on T_δ by defining $\overline{\varphi}(t, r') := \varphi(\gamma(t) + r\dot{\gamma}(t)^\perp)$ for all $r' \in (-\delta, \delta)$ and $t \in [a, b)$. The function $\overline{\varphi} \circ \mathcal{T}_\delta^{-1}(x)$ defined for all $x \in T_\delta$ is then an extension of φ and is of class C^1. Indeed, clearly $\overline{\varphi} \in C^1([a, b) \times (-\delta, \delta))$, and so $\overline{\varphi} \circ \mathcal{T}_\delta^{-1} \in C^1(T_\delta)$ because $\mathcal{T}_\delta^{-1}$ is of class C^1. In particular, we conclude that every function $\varphi \in C^1(\Gamma_r; \mathbb{R}^m)$ is the restriction to Γ_r of a function of class $C^1(T_\delta; \mathbb{R}^m)$. Since it is also easy to see that every function of class $C^1(T_\delta; \mathbb{R}^m)$ has a C^1 restriction on Γ_r as in Definition 3.8, we conclude that $\varphi \in C^1(\Gamma_r; \mathbb{R}^m)$ if and only if it is the restriction of a function $\widehat{\varphi} \in C^1(T_\delta; \mathbb{R}^m)$ on Γ_r.*

Definition 3.10. *Let $u : \Gamma_r \to \mathbb{R}^m$, we say that $u \in BV(\Gamma_r; \mathbb{R}^m)$ if*

$$\sup\{\int_{\Gamma_r} u \cdot \Big(\sum_{j=1}^2 D_j(\varphi\zeta_j)\Big) d\mathcal{H}^1 : \varphi \in C^1(T_\delta; \mathbb{R}^m), |\varphi| \leq 1\} < +\infty.$$

We denote the supremum above by $|D_\zeta u|(\Gamma_r)$.

Exploiting that ζ is divergence-free, we can write

$$|D_\zeta u|(\Gamma_r) = \sup\{\int_{\Gamma_r} u \cdot D_\zeta \varphi d\mathcal{H}^1 : \varphi \in C^1(T_\delta; \mathbb{R}^m), |\varphi| \leq 1\},$$

where $D_\zeta \varphi := \sum_{j=1}^2 D_j \varphi \zeta_j$. Recalling that $\mathcal{T}_\delta(\cdot, r)$ is a parametrization of Γ_r, if $u \in BV(\Gamma_r; \mathbb{R}^m)$ we see that

$$\begin{aligned}\int_a^b |\frac{d}{dt} u(\mathcal{T}_\delta(t, r))| dt &= \sup\{\int_a^b \frac{d}{dt} u(\mathcal{T}_\delta(t, r)) \cdot \psi(\mathcal{T}_\delta(t, r)) dt : \psi \in C^1(\Gamma_r; \mathbb{R}^m), |\psi| \leq 1\} \\ &= \sup\{\int_a^b u(\mathcal{T}_\delta(t, r)) \cdot \frac{d}{dt}\psi(\mathcal{T}_\delta(t, r)) dt : \psi \in C^1(\Gamma_r; \mathbb{R}^m), |\psi| \leq 1\}\end{aligned}$$

and, up to extending ψ to T_δ as in Remark 3.9, we have

$$\frac{d}{dt}\psi(\mathcal{T}_\delta(t, r)) = \nabla\psi(\mathcal{T}_\delta(t, r)) \frac{\partial \mathcal{T}_\delta}{\partial t}(t, r) = \nabla\psi(\mathcal{T}_\delta(t, r))\widehat{\zeta}(t, r)|\frac{\partial \mathcal{T}_\delta}{\partial t}(t, r)|,$$

so we conclude

$$\int_a^b |\frac{d}{dt} u(\mathcal{T}_\delta(t, r))| dt = \sup\{\int_{\Gamma_r} u \cdot D_\zeta \psi d\mathcal{H}^1 : \psi \in C^1(\Gamma_r; \mathbb{R}^m), |\psi| \leq 1\} = |D_\zeta u|(\Gamma_r). \tag{3.21}$$

Remark 3.11. *Equality (3.21) in particular implies that if $u_k, u \in BV(\Gamma_r; \mathbb{R}^m)$ are such that*

$$u_k \to u \qquad \text{strictly in } BV(\Gamma_r; \mathbb{R}^m),$$

then also

$$u_k(\mathcal{T}_\delta(\cdot, r)) \to u(\mathcal{T}_\delta(\cdot, r)) \qquad \text{strictly in } BV([a, b]; \mathbb{R}^m),$$

and viceversa. More precisely, for all $r \in (-\delta, \delta)$ and any $v \in BV(\Gamma_r; \mathbb{R}^m)$ it holds

$$|D_\zeta v|(\Gamma_r) = |D_t(v \circ \mathcal{T}_\delta(\cdot, r))|(a, b),$$

and there are two positive constants c_δ, C_δ depending only on Γ and δ such that

$$c_\delta \|u \circ \mathcal{T}_\delta(\cdot, r)\|_{L^1([a,b])} \leq \|u\|_{L^1(\Gamma_r)} \leq C_\delta \|u \circ \mathcal{T}_\delta(\cdot, r)\|_{L^1([a,b])}.$$

This follows from the bi-Lipschitz property of $\mathcal{T}_\delta$ and on the fact that $|\frac{d}{dt} T_\delta(\cdot, r)|$ is close to 1, for $r \in (-\delta, \delta)$.

Given $v : T_\delta \to \mathbb{R}^m$ a Lipschitz map, by coarea formula we can write

$$\int_{T_\delta} |\nabla v \zeta| dx = \int_{-\delta}^{\delta} \int_{\Gamma_r} |\nabla v \zeta| d\mathcal{H}^1 dr = \int_{-\delta}^{\delta} \int_{\Gamma_r} |D_\zeta v| d\mathcal{H}^1 dr,$$

and since ζ is a unit oriented tangent vector to Γ_r, $\nabla v \zeta = \sum_{j=1}^{2} D_j v \zeta_j$ represents the tangential derivative $D_\zeta v$ of v to Γ_r. Now, $\mathcal{T}_\delta(\cdot, r)$ is a parametrization from $[a, b]$ of Γ_r, so we write

$$\begin{aligned}\int_a^b |\frac{d}{dt} v(\mathcal{T}_\delta(t,r))| dt &= \int_a^b |\nabla v(\mathcal{T}_\delta(t,r)) \frac{d\mathcal{T}_\delta}{dt}(t,r)| dt \\ &= \int_a^b |\nabla v(\mathcal{T}_\delta(t,r)) \zeta(t,r)| |\frac{d\mathcal{T}_\delta}{dt}(t,r)| dt = \int_{\Gamma_r} |D_\zeta v| d\mathcal{H}^1, \end{aligned} \tag{3.22}$$

and we conclude

$$\int_{T_\delta} |D_\zeta v| dx = \int_{-\delta}^{\delta} \int_a^b |\frac{d}{dt} v(\mathcal{T}_\delta(t,r))| dt dr. \tag{3.23}$$

In the following lemma we discuss how strict convergence is inehrited on curves.

Lemma 3.12. *Let $u_k : T_\delta \to \mathbb{R}^m$ be Lipschitz maps and let $u \in BV(T_\delta; \mathbb{R}^m)$ be such that*

$$u_k \to u \qquad \text{strictly in } BV(T_\delta; \mathbb{R}^m).$$

Then, for a.e. $r \in (-\delta, \delta)$ the function $u \llcorner \Gamma_r$ belongs to $BV(\Gamma_r; \mathbb{R}^m)$ and (up to a non-relabelled subsequence) $u_k \llcorner \Gamma_r$ converge strictly in $BV(\Gamma_r; \mathbb{R}^m)$ to $u \llcorner \Gamma_r$.

Proof. By Reshetniak Theorem 3.2 we have, as $k \to \infty$,

$$\int_{T_\delta} |D_\zeta u_k| dx = \int_{T_\delta} |\nabla u_k \zeta| dx \to \int_{T_\delta} \Big| \frac{Du}{|Du|} \zeta \Big| d|Du|. \tag{3.24}$$

The quantity in the right-hand side is equal to

$$\begin{aligned}\int_{T_\delta} \Big| \frac{Du}{|Du|} \zeta \Big| d|Du| &= \sup\{ \int_{T_\delta} \sum_{j=1}^{2} \varphi \cdot \frac{D_j u}{|Du|} \zeta_j d|Du| : \varphi \in C^1(T_\delta; \mathbb{R}^m), |\varphi| \le 1\} \\ &= \sup\{ \int_{T_\delta} \sum_{j=1}^{2} \zeta_j \varphi \cdot dD_j u : \varphi \in C^1(T_\delta; \mathbb{R}^m), |\varphi| \le 1\} \\ &= \sup\{ \int_{T_\delta} u \cdot (\nabla \varphi \zeta) dx : \varphi \in C^1(T_\delta; \mathbb{R}^m), |\varphi| \le 1\}\end{aligned}$$

where in the last equality we have used the divergence-free property of ζ. Therefore, by (3.24), we conclude

$$\lim_{k \to \infty} \int_{T_\delta} |\nabla u_k \zeta| dx = \sup\{ \int_{T_\delta} u \cdot D_\zeta \varphi dx : \varphi \in C^1(T_\delta; \mathbb{R}^m), |\varphi| \le 1\}. \tag{3.25}$$

On the other hand

$$\int_{T_\delta} |\nabla u_k \zeta| dx = \int_{-\delta}^{\delta} \int_a^b |\frac{d}{dt} u_k(\mathcal{T}_\delta(t,r))| dt dr,$$

whence

$$\lim_{k\to\infty}\int_{-\delta}^{\delta}\int_a^b |\frac{d}{dt}u_k(\mathcal{T}_\delta(t,r))|dtdr = \sup\{\int_{T_\delta} u\cdot D_\zeta\varphi dx : \varphi\in C^1(T_\delta;\mathbb{R}^m), |\varphi|\le 1\}. \tag{3.26}$$

Now, by Fatou Lemma

$$\lim_{k\to\infty}\int_{-\delta}^{\delta}\int_a^b |\frac{d}{dt}u_k(\mathcal{T}_\delta(t,r))|dtdr \ge \int_{-\delta}^{\delta}\liminf_{k\to\infty}\int_a^b |\frac{d}{dt}u_k(\mathcal{T}_\delta(t,r))|dtdr \tag{3.27}$$

and we know from the strict convergence of u_k to u that for a.e. $r\in(-\delta,\delta)$ the trace $u_k\llcorner\Gamma_r$ converges to $u\llcorner\Gamma_r$ in $L^1(\Gamma_r;\mathbb{R}^m)$. This implies that, for a.e. $r\in(-\delta,\delta)$

$$\begin{aligned}\liminf_{k\to\infty}\int_a^b |\frac{d}{dt}u_k(\mathcal{T}_\delta(t,r))|dt &\ge \int_a^b |\frac{d}{dt}u(\mathcal{T}_\delta(t,r))|dt\\ &= \sup\{\int_{\Gamma_r} u\cdot D_\zeta\varphi d\mathcal{H}^1 : \varphi\in C^1(T_\delta;\mathbb{R}^m), |\varphi|\le 1\}\end{aligned} \tag{3.28}$$

where we have used (3.21); so that

$$\begin{aligned}\int_{-\delta}^{\delta}\liminf_{k\to\infty}\int_a^b |\frac{d}{dt}u_k(\mathcal{T}_\delta(t,r))|dtdr &\ge \int_{-\delta}^{\delta}\sup\{\int_{\Gamma_r} u\cdot\nabla\varphi\zeta d\mathcal{H}^1 : \varphi\in C^1(T_\delta;\mathbb{R}^m), |\varphi|\le 1\}dr\\ &\ge \sup\{\int_{-\delta}^{\delta}\int_{\Gamma_r} u\cdot\nabla\varphi\zeta d\mathcal{H}^1 dr : \varphi\in C^1(T_\delta;\mathbb{R}^m), |\varphi|\le 1\}.\end{aligned} \tag{3.29}$$

We have found then, from (3.26), that the inequalities in (3.27) and (3.29) are all equalities. In particular, equality in (3.28) holds for a.e. $r\in(-\delta,\delta)$, and denoting

$$f(r) := \int_a^b |\frac{d}{dt}u(\mathcal{T}_\delta(t,r))|dt \qquad f_k(r) := \int_a^b |\frac{d}{dt}u_k(\mathcal{T}_\delta(t,r))|dt$$

equality (3.27) implies that

$$\lim_{k\to\infty}\int_{-\delta}^{\delta} f_k(r)dr = \int_{-\delta}^{\delta} f(r)dr, \qquad \liminf_{k\to\infty} f_k(r) = f(r).$$

Thus Lemma 7.1 in the Appendix entails that $f_k\to f$ in $L^1((-\delta,\delta))$, and there is a subsequence such that for a.e. $r\in(-\delta,\delta)$

$$f_k(r)\to f(r),$$

that is the thesis. □

Transformations in tubular neighborhoods: Let $\Gamma := \gamma([a,b])$ be a Jordan curve parametrized by arc-length by $\gamma\in C^3([a,b];\mathbb{R}^2)$, and enclosing the simply-connected set A satisfying (R); let $\delta\in(0,1)$ be small enough and let T_δ be a tubular neighborhood of Γ. We want to define a bijection between T_δ and itself, which will be needed to modify suitable recovery sequences u_k for the involved functional. To this aim, we first introduce for $c\in(0,\delta)$ fixed, and $n\in\mathbb{N}$, $n>\frac{2}{\delta}$, the map

$$\Upsilon_{\delta,n,c} : [a,b]\times[-\delta,\delta]\to[a,b]\times[-\delta,\delta], \qquad \Upsilon_{\delta,n,c}(t,r) = (t,\tau_{\delta,n,c}(r)),$$

where $\tau_{\delta,n,c}$ is the piecewise affine interpolant such that $\tau_{\delta,n,c}(-\delta) = -\delta$, $\tau_{\delta,n,c}(-\frac{c}{n}) = 0$, and $\tau_{\delta,n,c}(\delta) = \delta$, namely

$$\tau_{\delta,n,c}(r) = \begin{cases} \frac{n\delta r + c\delta}{n\delta - c} & \text{for } r \in [-\delta, -\frac{c}{n}), \\ \frac{n\delta r + c\delta}{n\delta + c} & \text{for } r \in [-\frac{c}{n}, \delta]. \end{cases}$$

For all $(t,s) \in [a,b] \times [-\delta, \delta]$ we write

$$\Upsilon_{\delta,n,c}(t,s) = (t,s) + (0, \tau_{\delta,n,c}(s) - s), \qquad \text{with } |(0, \tau_{\delta,n,c}(s) - s)| \leq \frac{C}{n}, \tag{3.30}$$

for a constant $C > 0$ independent of δ and $n > \frac{2}{\delta}$. Computing $\nabla \Upsilon_{\delta,n,c}$, we write

$$\nabla \Upsilon_{\delta,n,c} = \mathrm{Id} + M_{\delta,n,c}, \qquad M_{\delta,n,c} := \begin{pmatrix} 0 & 0 \\ 0 & \dot{\tau}_{\delta,n,c} - 1 \end{pmatrix}, \tag{3.31}$$

in such a way that $|M_{\delta,n,c}| \leq \frac{C}{n}$ almost everywhere (here C is a positive constant independent of $n > \frac{2}{\delta}$ and δ). Analogously, it is immediately checked that

$$\nabla \Upsilon^{-1}_{\delta,n,c} = \mathrm{Id} + M'_{\delta,n,c}, \qquad \text{with } |M'_{\delta,n,c}| \leq \frac{C}{n}, \tag{3.32}$$

and for all $(t,s) \in [a,b] \times [-\delta, \delta]$ we have $\Upsilon^{-1}_{\delta,n,c}(t,s) = (t, \tau^{-1}_{\delta,n,c}(s))$, so we may write

$$\Upsilon^{-1}_{\delta,n,c}(t,s) = (t,s) + (0, \tau^{-1}_{\delta,n,c}(s) - s), \qquad \text{with } |(0, \tau^{-1}_{\delta,n,c}(s) - s)| \leq \frac{C}{n}. \tag{3.33}$$

We now define, for $\delta \in (0,1)$ as above and $n \in \mathbb{N}$, $n > \frac{2}{\delta}$, the following transformation

$$\Sigma_{\delta,n,c} : \overline{T}_\delta \to \overline{T}_\delta, \qquad \Sigma_{\delta,n,c} := \mathcal{T}_\delta \circ \Upsilon_{\delta,n,c} \circ \mathcal{T}_\delta^{-1}. \tag{3.34}$$

This map sends the set $\mathcal{T}_\delta([a,b], -\frac{c}{n})$ to the curve Γ. Moreover there is a constant C_γ, depending only on γ, such that

$$|\Sigma_{\delta,n,c}(x) - x| \leq \frac{C_\gamma}{n}, \qquad \forall x \in \overline{T}_\delta. \tag{3.35}$$

This follows from (3.30) and the Lipschitz continuity of $\mathcal{T}_\delta$. It is convenient also to introduce

$$\Sigma^-_{\delta,n,c} : \overline{T}^-_\delta \setminus T_{\frac{c}{n}} \to \overline{T}^-_\delta, \qquad \Sigma^-_{\delta,n,c} := (\mathcal{T}_\delta \circ \Upsilon_{\delta,n,c} \circ \mathcal{T}_\delta^{-1}) \llcorner (\overline{T}^-_\delta \setminus T_{\frac{c}{n}}), \tag{3.36}$$

the restriction of $\Sigma_{\delta,n,c}$ to $\overline{T}^-_\delta \setminus T_{\frac{c}{n}}$. For a.e. $x \in T_\delta$, we have

$$\nabla \Sigma_{\delta,n,c}(x) = \nabla \mathcal{T}_\delta(\Upsilon_{\delta,n,c} \circ \mathcal{T}_\delta^{-1}(x)) \nabla \Upsilon_{\delta,n,c}(\mathcal{T}_\delta^{-1}(x)) \nabla \mathcal{T}_\delta^{-1}(x), \tag{3.37}$$

and writing $\nabla \mathcal{T}_\delta(\Upsilon_{\delta,n,c} \circ \mathcal{T}_\delta^{-1}(x)) = \nabla \mathcal{T}_\delta\Big(\mathcal{T}_\delta^{-1}(x) + (\Upsilon_{\delta,n,c} \circ \mathcal{T}_\delta^{-1}(x) - \mathcal{T}_\delta^{-1}(x))\Big)$, we get

$$\nabla \mathcal{T}_\delta(\Upsilon_{\delta,n,c} \circ \mathcal{T}_\delta^{-1}(x)) = \nabla \mathcal{T}_\delta(\mathcal{T}_\delta^{-1}(x)) + \rho_{\delta,n,c}(x), \tag{3.38}$$

where, by using the Lipschitz continuity of $\nabla \mathcal{T}_\delta$ (it is of class C^1) and by (3.30), the matrix

$$\rho_{\delta,n,c}(x) := \nabla \mathcal{T}_\delta\Big(\mathcal{T}_\delta^{-1}(x) + (\Upsilon_{\delta,n,c} \circ \mathcal{T}_\delta^{-1}(x) - \mathcal{T}_\delta^{-1}(x))\Big) - \nabla \mathcal{T}_\delta(\mathcal{T}_\delta^{-1}(x))$$

enjoies, for a.e. $x \in T_\delta$,

$$|\rho_{\delta,n,c}(x)| \le \frac{C_\gamma}{n} \tag{3.39}$$

(here and below, unless explicitely stated, C_γ is a positive constant independent of $n > \frac{2}{\delta}$ and δ, but depending on γ). Plugging (3.31) and (3.38) into (3.37) we obtain

$$\begin{aligned}\nabla\Sigma_{\delta,n,c}(x) &= (\nabla\mathcal{T}_\delta(\mathcal{T}_\delta^{-1}(x)) + \rho_{\delta,n,c}(x))(\mathrm{Id} + M_{\delta,n,c}(\mathcal{T}_\delta^{-1}(x)))\nabla\mathcal{T}_\delta^{-1}(x)\\ &= \mathrm{Id} + \nabla\mathcal{T}_\delta(\mathcal{T}_\delta^{-1}(x))M_{\delta,n,c}(\mathcal{T}_\delta^{-1}(x))\nabla\mathcal{T}_\delta^{-1}(x) + \rho_{\delta,n,c}(x)(\mathrm{Id} + M_{\delta,n,c}(\mathcal{T}_\delta^{-1}(x)))\nabla\mathcal{T}_\delta^{-1}(x)\\ &=: \mathrm{Id} + \sigma_{\delta,n,c}(x),\end{aligned} \tag{3.40}$$

where we have used that $\nabla\mathcal{T}_\delta(\mathcal{T}_\delta^{-1}(x)) = (\nabla\mathcal{T}_\delta^{-1}(x))^{-1}$ and, thanks to (3.31), (3.39), and the Lipschitz continuity of $\nabla\mathcal{T}_\delta$, we have

$$|\sigma_{\delta,n,c}(x)| \le \frac{C_\gamma}{n}. \tag{3.41}$$

Finally, by (3.40), we have also, for n large enough

$$\det\,(\nabla\Sigma_{\delta,n,c}(x)) = 1 + d_{\delta,n,c}(x), \qquad \text{with } \|d_{\delta,n,c}\|_{L^\infty} \le \frac{C_\gamma}{n}, \tag{3.42}$$

and a similar expression holds for $\det\,(\nabla\Sigma_{\delta,n,c}(x)^{-1})$, namely

$$\det\,(\nabla\Sigma_{\delta,n,c}(x)^{-1}) = 1 + \widehat{d}_{\delta,n,c}(x), \qquad \text{with } \|\widehat{d}_{\delta,n,c}\|_{L^\infty} \le \frac{C_\gamma}{n}. \tag{3.43}$$

In what follows we will sometimes employ also the map $\widehat{\Sigma}_{\delta,n,c}$, defined as $\Sigma_{\delta,n,c}$ but with $\mathcal{T}_\delta$ replaced by $\widehat{\mathcal{T}_\delta}$ given by

$$\widehat{\mathcal{T}_\delta}(t,r) = \mathcal{T}_\delta(t,-r),$$

for all $(t,r) \in [a,b] \times (-\delta,\delta)$. Namely

$$\widehat{\Sigma}_{\delta,n,c} : T_\delta \to T_\delta, \qquad \Sigma_{\delta,n,c} := \widehat{\mathcal{T}_\delta} \circ \Upsilon_{\delta,n,c} \circ \widehat{\mathcal{T}_\delta}^{-1}. \tag{3.44}$$

We will consider $\Sigma^+_{\delta,n,c} : \overline{T}^+_\delta \setminus T_{\frac{c}{n}} \to \overline{T}^+_\delta$ defined as

$$\Sigma^+_{\delta,n,c} := (\widehat{\mathcal{T}_\delta} \circ \Upsilon_{\delta,n,c} \circ \widehat{\mathcal{T}_\delta}^{-1}) \llcorner (\overline{T}^+_\delta \setminus T_{\frac{c}{n}}). \tag{3.45}$$

For $\widehat{\Sigma}_{\delta,n,c}$, $\Sigma^-_{\delta,n,c}$, and $\Sigma^+_{\delta,n,c}$ similar estimates as in (3.39), (3.41), and (3.42) hold true. Eventually, using that $\Upsilon^{-1}_{\delta,n,c}$ satisfies (3.32) and (3.33), the same holds also for $\widehat{\Sigma}^{-1}_{\delta,n,c}$, $\widehat{\mathcal{T}_\delta}^{-1}$, $(\Sigma^-_{\delta,n,c})^{-1}$, and $(\Sigma^+_{\delta,n,c})^{-1}$. Specifically, we will write

$$\begin{aligned}&\nabla\Sigma^\pm_{\delta,n,c}(x) = \mathrm{Id} + \sigma^\pm_{\delta,n,c}(x), && \|\sigma^\pm_{\delta,n,c}\|_{L^\infty} \le \frac{C_\gamma}{n},\\ &\det\,(\nabla\Sigma^\pm_{\delta,n,c}(x)) = 1 + d^\pm_{\delta,n,c}(x), && \|d^\pm_{\delta,n,c}\|_{L^\infty} \le \frac{C_\gamma}{n},\\ &\nabla(\Sigma^\pm_{\delta,n,c})^{-1}(x) = \mathrm{Id} + \widehat{\sigma}^\pm_{\delta,n,c}(x), && \|\widehat{\sigma}^\pm_{\delta,n,c}\|_{L^\infty} \le \frac{C_\gamma}{n},\\ &\det\,(\nabla(\Sigma^\pm_{\delta,n,c})^{-1}(x)) = 1 + \widehat{d}^\pm_{\delta,n,c}(x), && \|\widehat{d}^\pm_{\delta,n,c}\|_{L^\infty} \le \frac{C_\gamma}{n},\end{aligned} \tag{3.46}$$

where $\sigma^\pm_{\delta,n,c} : \overline{T}^\pm_\delta \setminus T_{\frac{c}{n}} \to \mathbb{R}^{2\times 2}$, $d^\pm_{\delta,n,c} : \overline{T}^\pm_\delta \setminus T_{\frac{c}{n}} \to \mathbb{R}$, $\widehat{\sigma}^\pm_{\delta,n,c} : \overline{T}^\pm_\delta \to \mathbb{R}^{2\times 2}$, and $\widehat{d}^\pm_{\delta,n,c} : \overline{T}^\pm_\delta \to \mathbb{R}$ are suitable functions.

3.4 Composition of maps with planar transformations

In this section we use the planar transformations introduced in the previous one to modify suitable functions defined on planar domains.

Interpolations between maps on Jordan curves: Let $\Gamma := \gamma([a,b])$, $\gamma \in C^3([a,b];\mathbb{R}^2)$, be a Jordan curve parametrized by arc-length. Recalling the functions in (3.9) and (3.10), for two given Lipschitz maps $\varphi, \psi : [a,b] \to \mathbb{R}^2$ with $\varphi(a) = \varphi(b)$ and $\psi(a) = \psi(b)$, we define the interpolation $H_{\varphi,\psi,h} : \overline{T}_h \to \mathbb{R}^2$ as

$$H_{\varphi,\psi,h} := \begin{cases} \Phi_{\varphi,\psi} \circ \mathcal{T}_h^{-1} & \text{in } \overline{T}_h^+ \\ \Psi_{\varphi,\psi} \circ \widehat{\mathcal{T}}_h^{-1} & \text{in } \overline{T}_h^-, \end{cases} \tag{3.47}$$

where $0 < h \le \delta$ and $\delta \in (0,1)$ is small enough. The interpolation $H_{\varphi,\psi,h}$ turns out to be Lipschitz continuous.

For $r, s \in (-\delta, \delta)$ fixed, recalling that the map $\mathcal{T}_\delta(\cdot, r) : [a,b] \to \Gamma_r$ is a parametrization of the curve Γ_r, it follows that if u, v are Lipschitz maps defined on T_δ and $\varphi = u \circ \mathcal{T}_\delta^{-1}(\cdot, r)$ and $\psi = v \circ \mathcal{T}_\delta^{-1}(\cdot, s)$ then $H_{\varphi,\psi,h}$ interpolates in $\overline{T}_h$ between $u \llcorner \Gamma_r$ and $v \llcorner \Gamma_s$.

Let us estimate the gradient and Jacobian determinant of $H_{\varphi,\psi,h}$ in T_h^+: recalling that $\mathcal{T}_h$ is bi-Lipschitz with constant depending only on γ, since $\nabla H_{\varphi,\psi,h}(x) = \nabla \Phi_{\varphi,\psi}(\mathcal{T}_h^{-1}(x)) \nabla \mathcal{T}_h^{-1}(x)$, for a.e. $x \in \overline{T}_h^+$, one has

$$\begin{aligned} &|\nabla H_{\varphi,\psi,h}(x)| \le |\nabla \Phi_{\varphi,\psi}(\mathcal{T}_h^{-1}(x))||\nabla \mathcal{T}_h^{-1}(x)| \le C_\gamma |\nabla \Phi_{\varphi,\psi}(\mathcal{T}_h^{-1}(x))|, \\ &\det\big(\nabla H_{\varphi,\psi,h}(x)\big) = \det\big(\nabla \Phi_{\varphi,\psi}(\mathcal{T}_h^{-1}(x))\big) \det\,(\nabla \mathcal{T}_h^{-1}(x)). \end{aligned}$$

Once again, here and below we denote by $C_\gamma > 0$ a constant depending on γ, but independent of δ, φ, and ψ. Hence, setting $D = [a,b] \times [0,h]$, one has

$$\begin{aligned} \int_{T_h^+} |\nabla H_{\varphi,\psi,h}(x)| dx &\le C_\gamma \int_{T_h^+} |\nabla \Phi_{\varphi,\psi}(\mathcal{T}_h^{-1}(x))| dx = C_\gamma \int_D |\nabla \Phi_{\varphi,\psi}(t,r)| |\det\,(\nabla \mathcal{T}_h(t,r))| dt dr \\ &\le C_\gamma \int_D |\nabla \Phi_{\varphi,\psi}(t,r)| dt dr, \end{aligned} \tag{3.48}$$

and analogously on $\overline{T}_h^-$

$$\int_{T_h^-} |\nabla H_{\varphi,\psi,h}(x)| dx \le C_\gamma \int_{T_h^-} |\nabla \Psi_{\varphi,\psi}(\widehat{\mathcal{T}}_h^{-1}(x))| dx \le C_\gamma \int_D |\nabla \Psi_{\varphi,\psi}(t,r)| dt dr. \tag{3.49}$$

Therefore, exployting (3.12) and (3.13) we conclude

$$\begin{aligned} \int_{T_h} |\nabla H_{\varphi,\psi,h}(x)| dx &\le 2C_\gamma h(L_\varphi + b - a) + C_\gamma (L_\varphi + b - a) \int_a^b |s_\psi(t) - s_\varphi(t)| dt \\ &\quad + C_\gamma (L_\varphi + L_\psi + b - a) h + C_\gamma \int_0^{L_\psi} |\overline{\psi}(s) - \overline{\varphi}(s)| ds \\ &\le C_{\gamma, L_\varphi, L_\psi} (h + \|s_\psi - s_\varphi\|_{L^1} + \|\overline{\psi} - \overline{\varphi}\|_{L^\infty}), \end{aligned} \tag{3.50}$$

where the constant $C_{\gamma, L_\varphi, L_\psi}$ is independent of δ, depends on γ, L_φ, and L_ψ, but is uniformly bounded by a constant C_γ (depending only on γ) as soon as

$$L_\varphi + L_\psi \le C,$$

for an absolute constant $C > 0$ (notice that $b-a$ coincides with the length of Γ and hence we include the dependence on $b-a$ in C_γ). Regarding the Jacobian determinant, using (3.11) and (3.14), we find out that

$$\int_{T_h^+} |\det(\nabla H_{\varphi,\psi,h})(x)|dx = 0, \tag{3.51}$$

$$\begin{aligned}\int_{T_h^-} |\det(\nabla H_{\varphi,\psi,h})(x)|dx &\leq \int_D |\det\big(\nabla\Psi_{\varphi,\psi}(t,r)\big)||\det(\nabla\mathcal{T}_h^{-1}(\mathcal{T}_h(t,r)))||\det(\nabla\mathcal{T}_h(t,r))|dtdr\\ &\leq (L_\varphi + L_\psi + b - a)C_\gamma \int_0^1 |\overline{\psi}(s) - \overline{\varphi}(s)|ds \leq C_{\gamma,L_\varphi,L_\psi}\|\overline{\psi} - \overline{\varphi}\|_{L^\infty}.\end{aligned}$$

Estimates for the gradient and Jacobian of composition of maps: Let $A \subset \mathbb{R}^2$ be an open set and let $B \subset\subset A$ satisfy (R) and be simply-connected. Let $\gamma \in C^3([a,b];\mathbb{R}^2)$ an arc-length parametrization of $\Gamma := \partial B$. If B is not simply-connected, we will apply the following discussion to each loop forming ∂B. Let $\delta \in (0,1)$ be small enough and let T_δ be a tubular neighborhood of Γ. For a map $v \in \mathrm{Lip}(T_\delta;\mathbb{R}^2)$ we consider

$$u := v \circ \Sigma_{\delta,n,c}$$

whose gradient and Jacobian determinant satisfy

$$\begin{aligned}&\nabla u(x) = \nabla v(\Sigma_{\delta,n,c}(x))\nabla\Sigma_{\delta,n,c}(x) = \nabla v(\Sigma_{\delta,n,c}(x)) + \nabla v(\Sigma_{\delta,n,c}(x))\sigma_{\delta,n,c}(x)\\ &\det(\nabla u(x)) = \det(\nabla v(\Sigma_{\delta,n,c}(x))) + \det(\nabla v(\Sigma_{\delta,n,c}(x)))d_{\delta,n,c}(x),\end{aligned} \tag{3.52}$$

for a.e. $x \in T_\delta$, where we have used (3.40) and (3.42). In particular we deduce

$$\begin{aligned}\int_{T_\delta} |\nabla u(x) - \nabla v(x)|dx &\leq \int_{T_\delta} |\nabla v(\Sigma_{\delta,n,c}(x)) - \nabla v(x)|dx + \frac{C_\gamma}{n}\int_{T_\delta} |\nabla v(\Sigma_{\delta,n,c}(x))|dx\\ &\leq \beta_v(\frac{1}{n}) + \frac{C_\gamma}{n}(1 + \frac{C_\gamma}{n})\int_{T_\delta} |\nabla v|dx\end{aligned} \tag{3.53}$$

where in the last inequality we have used (3.41), (3.42), and where $\beta_v(\frac{1}{n}) := \int_{T_\delta} |\nabla v(\Sigma_{\delta,n,c}(x)) - \nabla v(x)|dx$. Arguing similarly, we can also estimate

$$\begin{aligned}\int_{T_\delta} |\det(\nabla u) - \det(\nabla v)|dx &\leq \int_{T_\delta} |\det(\nabla v(\Sigma_{\delta,n,c}(x))) - \det(\nabla v(x))|dx\\ &\quad + \frac{C_\gamma}{n}(1 + \frac{C_\gamma}{n})\int_{T_\delta} |\det(\nabla v)|dx\\ &= \eta_v(\frac{1}{n}) + \frac{C_\gamma}{n}(1 + \frac{C_\gamma}{n})\int_{T_\delta} |\det(\nabla v)|dx,\end{aligned} \tag{3.54}$$

where $\eta_v(\frac{1}{n}) := \int_{T_\delta} |\det(\nabla v(\Sigma_{\delta,n,c}(x))) - \det(\nabla v(x))|dx$. Notice that both the quantities $\beta_v(\frac{1}{n})$ and $\eta_v(\frac{1}{n})$ tend to 0 as $n \to \infty$, thanks to the fact that $\Sigma_{\delta,n,c}(x) \to x$ uniformly.

Analogously, if we define $u^- : \overline{T}_\delta^- \setminus T_{\frac{c}{n}} \to \mathbb{R}^2$ and $u^+ : \overline{T}_\delta^+ \setminus T_{\frac{c}{n}} \to \mathbb{R}^2$ as

$$u^\pm := v \circ \Sigma_{\delta,n,c}^\pm$$

respectively, then we will have

$$\begin{aligned}\int_{T_\delta^-\setminus T_{\frac{c}{n}}} |\nabla u^- - \nabla v| dx &\leq \int_{T_\delta^-\setminus T_{\frac{c}{n}}} |\nabla v(\Sigma_{\delta,n,c}(x)) - \nabla v(x)| dx + \frac{C_\gamma}{n}\int_{T_\delta^-\setminus T_{\frac{c}{n}}} |\nabla v(\Sigma_{\delta,n,c}(x))| dx \\ &\leq \beta_v^-(\frac{1}{n}) + \frac{C_\gamma}{n}(1+\frac{C_\gamma}{n})\int_{T_\delta^-} |\nabla v| dx \\ \int_{T_\delta^+\setminus T_{\frac{c}{n}}} |\nabla u^+ - \nabla v| dx &\leq \beta_v^+(\frac{1}{n}) + \frac{C_\gamma}{n}(1+\frac{C_\gamma}{n})\int_{T_\delta^+} |\nabla v| dx, \end{aligned} \tag{3.55}$$

and

$$\begin{aligned}\int_{T_\delta^-\setminus T_{\frac{c}{n}}} |\det(\nabla u^-) - \det(\nabla v)| dx &\leq \eta_v^-(\frac{1}{n}) + \frac{C_\gamma}{n}(1+\frac{C_\gamma}{n})\int_{T_\delta^-} |\det(\nabla v)| dx, \\ \int_{T_\delta^+\setminus T_{\frac{c}{n}}} |\det(\nabla u^+) - \det(\nabla v)| dx &\leq \eta_v^+(\frac{1}{n}) + \frac{C_\gamma}{n}(1+\frac{C_\gamma}{n})\int_{T_\delta^+} |\det(\nabla v)| dx. \end{aligned} \tag{3.56}$$

Also in this case the quantities $\beta_v^\pm(\frac{1}{n})$ and $\eta_v^\pm(\frac{1}{n})$ tend to zero as $n \to \infty$.

4 Main properties of recovery sequences for $\mathcal{F}$

Let $\Gamma := \gamma([a,b])$, $\gamma \in C^3([a,b];\mathbb{R}^2)$, be a Jordan curve parametrized by arc-length and let T_δ be a tubular neighborhood of it, for $\delta \in (0,1)$ small enough.

Definition 4.1 (The function ψ_u). *If $u \in BV(T_\delta;\mathbb{R}^m)$ we define the function $\psi_u : (-\delta,\delta) \to \mathbb{R}$ as*

$$\psi_u(r) = |u \llcorner \Gamma_r|_{BV(\Gamma_r)} = |D_\zeta u|(\Gamma_r), \tag{4.1}$$

for all $r \in (-\delta,\delta)$ and where D_ζ is the tangential distributional derivative of u to Γ_r (see Definition (3.10)).

The function ψ_u turns out to be measurable and, since $u \in BV(T_\delta;\mathbb{R}^m)$, by coarea formula it belongs to $L^1((-\delta,\delta))$. Indeed, if $u_k \in C^1(T_\delta;\mathbb{R}^m)$ converges strictly to u in $BV(T_\delta;\mathbb{R}^m)$ (see Remark 2.3) then by Lemma 3.12 and Fatou Lemma we can write

$$\int_{-\delta}^{\delta} |D_\zeta u|(\Gamma_r) dr \leq \liminf_{k\to\infty} \int_{-\delta}^{\delta} |D_\zeta u_k|(\Gamma_r) dr = \liminf_{k\to\infty} \int_{T_\delta} |\nabla u_k \zeta| dx = |D_\zeta u|(T_\delta) < +\infty,$$

where we have used (3.25).

The following result is a crucial lemma which has the role of estimating the errors of energy when one wants to glue two Lipschitz maps along a Jordan curve.

Lemma 4.2. *Let $A \subset \mathbb{R}^2$ be a bounded open set and let $B \subset A$ be a open subset whose boundary is $\partial B =: \Gamma \subset A$ is a closed Jordan curve of class C^3; let $u \in BV(A;\mathbb{R}^m)$ be such that*

$$|Du|(\partial B) = 0, \qquad u \llcorner \partial B \in BV(\partial B;\mathbb{R}^m),$$

let $v^+, v^- \in \mathrm{Lip}_{\mathrm{loc}}(A;\mathbb{R}^m)$ be two maps and let $\delta > 0$ be small so that T_δ is a tubular neighborhood of Γ. Then there exists a function $\omega_\Gamma : \mathbb{R}^+ \to \mathbb{R}^+$ depending on Γ and on $u \llcorner \Gamma$ (but independent *of $v^\pm$) with $\lim_{t\to 0^+}\omega_\Gamma(t)=0$ and the such that following holds: for all $\varepsilon>0$ there exists a function $w\in \mathrm{Lip_{loc}}(A;\mathbb{R}^m)$ with*

$$\begin{aligned}
&w=v^- \text{ in } B\setminus T_\delta \text{ and } w=v^+ \text{ in } A\setminus B\setminus T_\delta,\\
&\|w-v^-\|_{L^1(T_\delta\cap B)}\le 3\|v^--u\|_{L^1(T_\delta\cap B)}+r,\\
&\|w-v^+\|_{L^1(T_\delta\cap (A\setminus B))}\le 3\|v^+-u\|_{L^1(T_\delta\cap (A\setminus B))}+r,\\
&\int_B |\nabla w-\nabla v^-|dx\le r, \qquad \int_{A\setminus B}|\nabla w-\nabla v^+|dx\le r,\\
&F(w,B)\le F(v^-,B)+r,\quad F(w,A\setminus\overline{B})\le F(v^+,A\setminus\overline{B})+r,\\
&r\le \varepsilon+\omega_\Gamma\big(d_s(v^+\llcorner\Gamma,u\llcorner\Gamma)+d_s(v^-\llcorner\Gamma,u\llcorner\Gamma)\big).
\end{aligned}\tag{4.2}$$

Moreover, if $v^+,v^-\in \mathrm{Lip}(A;\mathbb{R}^m)$ then $w\in\mathrm{Lip}(A;\mathbb{R}^m)$.

Proof. Assume that $\gamma:[a,b]\to\mathbb{R}^2$ is an arc-length parametrization of the loop Γ. Let us consider the corresponding map $\mathcal{T}_\delta$ in (3.16) and let T_δ^- and T_δ^+ denote the inner and outer parts of T_δ with respect to B, i.e., $T_\delta^-=\overline{B}\cap T_\delta$, $T_\delta^+=T_\delta\setminus B$. Then we set, for any $n\ge 1$,

$$w_n:=\begin{cases} v^-\circ\Sigma^-_{\delta,n,c} & \text{in } T_\delta^-\setminus T_{\frac{c}{n}},\\ v^+\circ\Sigma^+_{\delta,n,c} & \text{in } T_\delta^+\setminus T_{\frac{c}{n}},\\ v^- & \text{in } B\setminus T_\delta,\\ v^+ & \text{in } A\setminus B\setminus T_\delta,\end{cases}\tag{4.3}$$

where we recall the maps $\Sigma^-_{\delta,n,c}$ and $\Sigma^+_{\delta,n,c}$ in (3.36) and (3.45), with $c\in(0,\delta)$ fixed. We have to define w_n in $T_{\frac{c}{n}}$: we set

$$\widetilde{\varphi}:=(v^-\circ\Sigma^-_{\delta,n,c})\llcorner\Gamma_{-\frac{c}{n}},\qquad \widetilde{\psi}:=(v^+\circ\Sigma^+_{\delta,n,c})\llcorner\Gamma_{\frac{c}{n}},$$

and recalling (3.9), (3.10), and (3.47), we define

$$w_n:=H_{\varphi,\psi,\frac{c}{n}}\qquad\qquad \text{in } T_{\frac{c}{n}},\tag{4.4}$$

where $\varphi,\psi:[a,b]\to\mathbb{R}^m$ are given by

$$\varphi=\widetilde{\varphi}\circ\mathcal{T}_\delta(\cdot,-\frac{c}{n})\qquad\qquad \psi=\widetilde{\psi}\circ\mathcal{T}_\delta(\cdot,\frac{c}{n}).\tag{4.5}$$

By definition of $\widetilde{\varphi}$ and $\widetilde{\psi}$, using (3.36) and (3.45), φ and ψ can be equivalently written as

$$\begin{aligned}
\varphi(t)&=v^-\circ\mathcal{T}_\delta\circ\Upsilon_{\delta,n,c}(t,-\frac{c}{n})=v^-(\mathcal{T}_\delta(t,0))=v^-(\gamma(t))\\
\psi(t)&=v^+\circ\mathcal{T}_\delta\circ\Upsilon_{\delta,n,c}(t,\frac{c}{n})=v^+(\mathcal{T}_\delta(t,0))=v^+(\gamma(t)).
\end{aligned}$$

In this way we have that w_n is Lipschitz continuous in $T_{\frac{c}{n}}$ and

$$w_n=\widetilde{\varphi}\text{ on }\Gamma_{-\frac{c}{n}},\qquad\qquad w_n=\widetilde{\psi}\text{ on }\Gamma_{\frac{c}{n}}.$$

Moreover w_n turns out to be globally Lipschitz in A if so are v^+ and v^-. Let us estimate the gradient and Jacobian determinant integral of w_n in T_δ: by (3.55) we have

$$\begin{aligned}
\int_{T_\delta^-\setminus T_{\frac{c}{n}}}|\nabla w_n-\nabla v^-|dx&\le\beta^-_{v^-}(\frac{1}{n})+\frac{C_\gamma}{n}(1+\frac{C_\gamma}{n})\int_{T_\delta^-}|\nabla v^-|dx,\\
\int_{T_\delta^+\setminus T_{\frac{c}{n}}}|\nabla w_n-\nabla v^+|dx&\le\beta^+_{v^+}(\frac{1}{n})+\frac{C_\gamma}{n}(1+\frac{C_\gamma}{n})\int_{T_\delta^+}|\nabla v^+|dx,
\end{aligned}\tag{4.6}$$

and in particular there is a constant $C_\gamma > 0$ (depending on γ, but independent of n) such that

$$\int_{T_\delta \setminus T_{\frac{c}{n}}} |\nabla w_n| dx \leq \beta^-_{v^-}(\frac{1}{n}) + \beta^+_{v^+}(\frac{1}{n}) + \frac{C_\gamma}{n}\Big(\int_{T^-_\delta} |\nabla v^-| dx + \int_{T^+_\delta} |\nabla v^+| dx\Big), \tag{4.7}$$

Furthermore, on account of (3.56), it follows, for all $i, j \in \{1, \ldots, m\}$, $i \neq j$,

$$\begin{aligned}&\int_{T^-_\delta \setminus T_{\frac{c}{n}}} |M^{ij}_{12}(\nabla w_n) - M^{ij}_{12}(\nabla v^-)| dx + \int_{T^-_\delta \setminus T_{\frac{c}{n}}} |M^{ij}_{12}(\nabla w_n) - M^{ij}_{12}(\nabla v^-)| dx \\ &\leq \eta^-_{v^-}(\frac{1}{n}) + \eta^+_{v^+}(\frac{1}{n}) + \frac{C_\gamma}{n}\Big(\int_{T^-_\delta} |M^{ij}_{12}(\nabla v^-)| dx + \int_{T^+_\delta} |M^{ij}_{12}(\nabla v^+)| dx\Big). \end{aligned} \tag{4.8}$$

As for the integral on $T_{\frac{c}{n}}$, using (3.50) and (3.51), we have for all $i, j \in \{1, \ldots, m\}$, $i \neq j$,

$$\begin{aligned}\int_{T_{\frac{c}{n}}} |\nabla w_n| dx &\leq C_{\gamma, L_\varphi, L_\psi}(\frac{1}{n} + \|s_\psi - s_\varphi\|_{L^1} + \|\overline{\psi} - \overline{\varphi}\|_{L^\infty}) \\ &\leq C_{\gamma, L_\varphi, L_\psi}(\frac{1}{n} + \|s_\psi - s_\sigma\|_{L^1} + \|s_\varphi - s_\sigma\|_{L^1} + \|\overline{\psi} - \overline{\sigma}\|_{L^\infty} + \|\overline{\varphi} - \overline{\sigma}\|_{L^\infty}) \\ \int_{T_{\frac{c}{n}}} |M^{ij}_{12}(\nabla w_n)| dx &\leq C_{\gamma, L_\varphi, L_\psi}\|\overline{\psi} - \overline{\varphi}\|_{L^\infty} \leq C_{\gamma, L_\varphi, L_\psi}\big(\|\overline{\psi} - \overline{\sigma}\|_{L^\infty} + \|\overline{\varphi} - \overline{\sigma}\|_{L^\infty}\big). \end{aligned} \tag{4.9}$$

Here we have set $\sigma := u \circ \gamma$ and denoted by $\overline{\sigma}$ the generalized curve in (3.4). By Proposition 3.6 we find a function a_γ such that, up to enlarging the constant $C_{\gamma, L_\varphi, L_\psi}$ if necessary,

$$\int_{T_{\frac{c}{n}}} |M^{ij}_{12}(\nabla w_n)| dx + \int_{T_{\frac{c}{n}}} |\nabla w_n| dx \leq C_{\gamma, L_\varphi, L_\psi}\big(\frac{1}{n} + a_\gamma(d_s(\varphi, \sigma) + d_s(\psi, \sigma))\big). \tag{4.10}$$

We observe that inequalities (4.6), (4.8), and (4.10) entail

$$\int_{T^-_\delta} |\mathcal{M}(\nabla v^-) - \mathcal{M}(\nabla w_n)| dx + \int_{T^+_\delta} |\mathcal{M}(\nabla v^+) - \mathcal{M}(\nabla w_n)| dx \leq o(1) + C_\gamma a_\gamma(d_s(\varphi, \sigma) + d_s(\psi, \sigma))$$

for some quantity $o(1)$ tending to 0 as $n \to \infty$. These estimates together with (2.8) entail

$$\begin{aligned} F(w_n, B) = \int_B g(\mathcal{M}(\nabla w_n)) &\leq \int_B g(\mathcal{M}(\nabla v^-)) - \partial g(\mathcal{M}(\nabla v^-))(\mathcal{M}(\nabla v^-) - \mathcal{M}(\nabla w)) dx \\ &\leq F(v^-, B) + C_g \int_{T^-_\delta} |\mathcal{M}(\nabla v^-) - \mathcal{M}(\nabla w)| dx =: F(v^-, B) + r', \end{aligned} \tag{4.11}$$

with $r' \leq C_g(o(1) + C_\gamma a_\gamma(d_s(\varphi, \sigma) + d_s(\psi, \sigma))) =: C_g o(1) + \omega_\Gamma(d_s(\varphi, \sigma) + d_s(\psi, \sigma))$. A similar reasoning for the set $A \setminus B$ leads to

$$F(w, A \setminus B) = F(v^+, A \setminus B) + r'',$$

with $r'' \leq C_g o(1) + \omega_\Gamma(d_s(\varphi, \sigma) + d_s(\psi, \sigma))$. So if we take n large enough, we have obtained the last but one line in (4.2). Also the forth inequality in (4.2) easily follows from (4.6) and (4.9). It remains to estimate the L^1-norms. Owing to the explicit expressions of $\Phi_{\varphi,\psi}$ and $H_{\varphi,\psi,\frac{c}{n}}$ in

(3.9) and (3.47), denoting $h = \frac{c}{n}$, we write

$$
\begin{aligned}
\int_{T^-_{\frac{c}{n}}} |w_n| dx &\le (1+\frac{C_\gamma}{n}) \int_a^b \int_0^h |\varphi\Big(t_\varphi\big(s_\varphi(t)\frac{r}{h} + s_\psi(t)\frac{h-r}{h}\big)\Big)| dr dt \\
&= (1+\frac{C_\gamma}{n}) \int_a^b \int_0^h |\overline{\varphi}\big(s_\varphi(t)\frac{r}{h} + s_\psi(t)\frac{h-r}{h}\big)| dr dt \\
&\le \frac{c}{n}(b-a)(1+\frac{C_\gamma}{n})\big(\|\overline{\varphi}-\overline{\sigma}\|_{L^\infty} + \|\overline{\sigma}\|_{L^\infty}\big) \\
&\le \frac{C_\gamma}{n}\big(a_\gamma(d_s(\varphi,\sigma)) + \|\sigma\|_{L^\infty}\big),
\end{aligned}
$$

where the first inequality follows from (3.17) and the last one from Proposition 3.6. Analogously

$$
\begin{aligned}
\int_{T^+_{\frac{c}{n}}} |w_n| dx &\le (1+\frac{C_\gamma}{n}) \int_a^b \int_0^h |\overline{\varphi}(s_\psi(t))\frac{h-r}{h} + \overline{\psi}(s_\psi(t))\frac{r}{h}| dr dt \\
&\le \frac{C_\gamma}{n}\big(\|\overline{\varphi}-\overline{\sigma}\|_{L^\infty} + \|\overline{\psi}-\overline{\sigma}\|_{L^\infty} + 2\|\overline{\sigma}\|_{L^\infty}\big) \\
&\le \frac{C_\gamma}{n}\big(a_\gamma(d_s(\varphi,\sigma)) + a_\gamma(d_s(\psi,\sigma)) + \|\sigma\|_{L^\infty}\big).
\end{aligned}
$$

At the same time we have

$$
\begin{aligned}
&\int_B |w_n - v^-| dx = \int_{T_\delta \setminus T_{\frac{c}{n}}} |v^- \circ \Sigma^-_{\delta,n,c} - v^-| dx \\
&\le \int_{T_\delta \setminus T_{\frac{c}{n}}} |v^- \circ \Sigma^-_{\delta,n,c} - u \circ \Sigma^-_{\delta,n,c}| dx + \int_{T_\delta \setminus T_{\frac{c}{n}}} |u \circ \Sigma^-_{\delta,n,c} - u| dx + \int_{T_\delta \setminus T_{\frac{c}{n}}} |u - v^-| dx \\
&\le (1+\frac{C_\gamma}{n})\|v^- - u\|_{L^1(T^-_\delta)} + \int_{T_\delta \setminus T_{\frac{c}{n}}} |u \circ \Sigma^-_{\delta,n,c} - u| dx + \|v^- - u\|_{L^1(T^-_\delta)} \\
&\le 3\|v^- - u\|_{L^1(T^-_\delta)} + \int_{T_\delta} |u \circ \Sigma_{\delta,n,c} - u| dx,
\end{aligned}
$$

(for $n > 1/C_\gamma$) and a similar inequality holds for $\int_{A\setminus B} |w_n - v^+| dx$. Hence, the second and third inequalities in (4.2) follow from the last three expressions, noticing that we can choose n big enough so that

$$
\frac{C_\gamma}{n}\|\sigma\|_{L^\infty} \le \varepsilon, \qquad \int_{T_\delta} |u \circ \Sigma_{\delta,n,c} - u| dx \le \varepsilon,
$$

where the last condition can be obtained because $u \circ \Sigma_{\delta,n,c} \to u$ in $L^1(T_\delta;\mathbb{R}^m)$ as $n \to \infty$. $\square$

Being the construction leading to the result above local, it can be easily extended to more general open set B as follows:

Corollary 4.3. *Let A be a bounded open set and let $B \subset A$ be an open subset with boundary $\partial B \subset A$ a finite union of closed curves of class C^3; let $u \in BV(A;\mathbb{R}^m)$ be such that*

$$
|Du|(\partial B) = 0, \qquad u \llcorner \partial B \in BV(\partial B;\mathbb{R}^m),
$$

let $v^+, v^- \in \mathrm{Lip}_{\mathrm{loc}}(A;\mathbb{R}^m)$ be two maps and let $\delta > 0$ be small so that T_δ is a tubular neighborhood of $\Gamma := \partial B$. Then for all $\varepsilon > 0$ there exists $w \in \mathrm{Lip}_{\mathrm{loc}}(A;\mathbb{R}^m)$ such that the first five lines of (4.2) hold, together with

$$r \leq \varepsilon + \sum_{i=0}^{N} \omega_{\Gamma^i}\big(d_s(v^+ \llcorner \Gamma^i, u \llcorner \Gamma^i) + d_s(v^- \llcorner \Gamma^i, u \llcorner \Gamma^i)\big), \tag{4.12}$$

where $\omega_{\Gamma^i} : \mathbb{R}^+ \to \mathbb{R}^+$ are functions depending on Γ^i and on $u \llcorner \Gamma^i$ respectively, such that $\lim_{t\to 0^+} \omega_{\Gamma^i}(t) = 0$. Also, if $v^+, v^- \in \mathrm{Lip}(A;\mathbb{R}^m)$ then $w \in \mathrm{Lip}(A;\mathbb{R}^m)$.

A straightforward consequence of the previous result is the following:

Corollary 4.4. *Let A be a bounded open set and let $B \subset A$ be an open subset with boundary ∂B a finite union of closed curves of class C^3; let $u \in BV(A;\mathbb{R}^m)$ be such that*

$$|Du|(\partial B) = 0, \qquad u \llcorner \partial B \in BV(\partial B;\mathbb{R}^m),$$

and let $(u_k), (v_k) \subset \mathrm{Lip}_{\mathrm{loc}}(A;\mathbb{R}^m)$ be two sequences of maps such that

$$\begin{aligned} &v_k \to u \;\; \text{and} \;\; u_k \to u \qquad \text{strictly in } BV(A;\mathbb{R}^m), \\ &u_k \llcorner \Gamma^i \to u \llcorner \Gamma^i \;\; \text{and} \;\; v_k \llcorner \Gamma^i \to u \llcorner \Gamma^i \qquad \text{strictly in } BV(\Gamma^i;\mathbb{R}^m), \end{aligned}$$

where $\Gamma = \cup_{i=0}^{N} \Gamma^i$ is the decomposition of Γ in simple Jordan curves Γ^i. Then there exists a sequence $(w_k) \subset \mathrm{Lip}_{\mathrm{loc}}(A;\mathbb{R}^m)$ such that

$$\begin{aligned} &w_k \to u \text{ strictly in } BV(A;\mathbb{R}^m), \\ &\liminf_{k\to\infty} F(w_k, B) \leq \liminf_{k\to\infty} F(v_k, B), \\ &\liminf_{k\to\infty} F(w_k, A \setminus \overline{B}) \leq \liminf_{k\to\infty} F(u_k, A \setminus \overline{B}). \end{aligned}$$

Furthermore, if u_k and v_k are Lipschitz continuous on A, so is w_k.

We now use the previous result to modify suitable recovery sequences.

Lemma 4.5. *Let A be a bounded open set and let $B \subset\subset A$ be an open subset whose boundary is $\partial B =: \Gamma$ a finite union of closed curves of class C^3. Let $u \in BV(A;\mathbb{R}^m)$ be given and assume that 0 is a regular value for the function ψ in (4.1). Then there exists a recovery sequence $(v_k) \subset \mathrm{Lip}(B;\mathbb{R}^m)$ for $\mathcal{F}(u, B)$ such that $v_k \llcorner \Gamma \to u \llcorner \Gamma$ strictly in $BV(\Gamma;\mathbb{R}^m)$.*

With a little abuse of notation, we have call v_k a recovery sequence even if, in general, $v_k \notin C^1(B;\mathbb{R}^m)$. With this we mean that

$$\lim_{k\to\infty} F(v_k, B) = \mathcal{F}(u, B).$$

Proof. Let $(u_k) \subset C^1(B;\mathbb{R}^m)$ be a recovery sequence for $\mathcal{F}(u, B)$, let T_δ be a tubular neighborhood of Γ, with $\delta \in (0, 1)$ small enough. We will modify u_k in T_δ^- in order to produce v_k. To do so, we again assume that Γ consists of a unique loop (the same argument applied to each component of Γ covers the general case). Let Σ^-_{δ,n,c_n} be the map in (3.36), where we consider the numbers $c_n \in (0, \delta)$ in such a way that

$$\lim_{n\to\infty} \psi(-\frac{c_n}{n}) = \psi(0), \tag{4.13}$$

and, at the same time, for all $n > 0$ fixed

$$u_k \llcorner \Gamma_{-\frac{c_n}{n}} \to u \llcorner \Gamma_{-\frac{c_n}{n}} \qquad \text{strictly in } BV(\Gamma_{-\frac{c_n}{n}}; \mathbb{R}^m).$$

This choice is possible thanks to the hypothesis that 0 is regular for ψ, and since the convergence above holds on Γ_t, for a.e. $t \in (-\delta, 0)$. Then we define

$$v_{k,n}(x) := u_k\left((\Sigma^-_{\delta,n,c_n})^{-1}(x)\right), \qquad x \in T^-_\delta.$$

Notice that $(\Sigma^-_{\delta,n,c_n})^{-1} : \overline{T}^-_\delta \to \overline{T}^-_\delta \setminus T_{\frac{c_n}{n}}$ is such that $(\Sigma^-_{\delta,n,c_n})^{-1}(\Gamma) = \Gamma_{-\frac{c_n}{n}}$, and so, writing $x = \mathcal{T}_\delta(t, 0)$ for $x \in \Gamma$, $t \in [a, b]$, we have

$$v_{k,n}(\mathcal{T}_\delta(t, 0)) = u_k(\mathcal{T}_\delta \circ \Upsilon^{-1}_{\delta,n,c_n}(t, 0)) = u_k(\mathcal{T}_\delta(t, \tau^{-1}_{\delta,n,c_n}(0))) = u_k(\mathcal{T}_\delta(t, -\frac{c_n}{n})),$$

for all $t \in [a, b]$. In particular Remark 3.11 implies that, for all $n > 0$ fixed

$$v_{k,n} \circ \mathcal{T}_\delta(\cdot, 0) \to u \circ \mathcal{T}_\delta(\cdot, -\frac{c_n}{n}) \qquad \text{strictly in } BV([a, b]; \mathbb{R}^m).$$

We can then find, for all $k > 0$, a natural number $n_k > 0$ such that $n_k \nearrow +\infty$ (as $k \to \infty$) and satisfying

$$\int_\Gamma |\nabla v_{k,n_k} \zeta| d\mathcal{H}^1 = \int_a^b |\frac{d}{dt} v_{k,n_k}(\mathcal{T}_\delta(t, 0))| dt \le |D_\zeta u(\mathcal{T}_\delta(\cdot, -\frac{c_{n_k}}{n_k}))|([a, b]) + \frac{1}{k},$$

where ζ appears in (3.20) which, we recall, is the unit oriented tangent vector to Γ. Recalling (3.21), we also have

$$|D_\zeta u(\mathcal{T}_\delta(\cdot, -\frac{c_{n_k}}{n_k}))|([a, b]) = |D_\zeta u|(\Gamma_{-\frac{c_{n_k}}{n_k}})$$

so we readily infer, thanks to (4.13) and the lower semicontinuity of the variation, that

$$\lim_{k \to \infty} \int_\Gamma |\nabla v_{k,n_k} \zeta| d\mathcal{H}^1 = |D_\zeta u|(\Gamma),$$

and therefore the function $v_k := v_{k,n_k}$ satisfies

$$v_k \llcorner \Gamma \to u \llcorner \Gamma \qquad \text{strictly in } BV(\Gamma; \mathbb{R}^m).$$

To conclude the proof we need to show that v_k is still a recovery sequence for $\mathcal{F}(u, B)$. Notice that, since u_k are Lipschitz in $B \setminus T_{\frac{c_n}{n}}$ and Σ^-_{δ,n,c_n} is bi-Lipschitz, also $v_{k,n}$ are Lipschitz continuous on B. In T^-_δ, arguing as in (3.52), it holds, for $i, j \in \{1, \dots, m\}$, $i \neq j$,

$$\begin{aligned} &\nabla v_k(x) = \nabla u_k(\Sigma^{-1}_{\delta,n_k,c_{n_k}}(x)) \nabla \Sigma^{-1}_{\delta,n_k,c_{n_k}}(x) = \nabla u_k(\Sigma_{\delta,n_k,c_{n_k}}(x)) + \nabla u_k(\Sigma_{\delta,n_k,c_{n_k}}(x)) \widehat{\sigma}^-_{\delta,n_k,c_{n_k}}(x), \\ &M^{ij}_{12}(\nabla v_k(x)) = M^{ij}_{12}(\nabla u_k(\Sigma_{\delta,n_k,c_{n_k}}(x))) + M^{ij}_{12}(\nabla u_k(\Sigma_{\delta,n_k,c_{n_k}}(x))) \widehat{d}^-_{\delta,n_k,c_{n_k}}(x), \end{aligned} \tag{4.14}$$

thanks to (3.46). We then introduce the vector

$$\widetilde{\mathcal{M}}(\nabla u_k(\Sigma^{-1}_{\delta,n_k,c_{n_k}}(x))) := \big(1, \nabla u_k(\Sigma^{-1}_{\delta,n_k,c_{n_k}}(x)), M_{12}(\nabla u_k(\Sigma^{-1}_{\delta,n_k,c_{n_k}}(x)))\big). \tag{4.15}$$

where to shotcut the notation, we have denoted $M_{12}(\nabla w) \in \mathbb{R}^{m(m+1)/2}$ the vector with entries $M_{12}^{ij}(\nabla w)$, $i, j \in \{1, \dots, m\}$, $i \neq j$. Using (3.46) we infer

$$|\mathcal{M}(\nabla v_k) - \widetilde{\mathcal{M}}(\nabla u_k(\Sigma^{-1}_{\delta,n_k,c_{n_k}}(x)))| \leq \frac{C_\gamma}{n_k}\Big|\big(0, \nabla u_k(\Sigma^{-1}_{\delta,n_k,c_{n_k}}(x)), M_{12}(\nabla u_k(\Sigma^{-1}_{\delta,n_k,c_{n_k}}(x)))\big)\Big|.$$

Therefore, exployting (2.8), (3.42), and the convexity of g, we can estimate

$$\begin{aligned}
F(v_k, T_\delta^-) &\leq \int_{T_\delta^-} |g(\mathcal{M}(\nabla v_k)) - g\big(\widetilde{\mathcal{M}}(\nabla u_k(\Sigma^{-1}_{\delta,n_k,c_{n_k}}(x)))\big)|dx + \int_{T_\delta^-} g\big(\widetilde{\mathcal{M}}(\nabla u_k(\Sigma^{-1}_{\delta,n_k,c_{n_k}}(x)))\big)dx \\
&\leq \frac{C_g C_\gamma}{n_k} \int_{T_\delta^-} \Big|\big(0, \nabla u_k(\Sigma^{-1}_{\delta,n_k,c_{n_k}}(x)), M_{12}(\nabla u_k(\Sigma^{-1}_{\delta,n_k,c_{n_k}}(x)))\big)\Big|dx \\
&\quad + \int_{T_\delta^-} g\big(\widetilde{\mathcal{M}}(\nabla u_k(\Sigma^{-1}_{\delta,n_k,c_{n_k}}(x)))\big)dx \\
&= \frac{C_g C_\gamma}{n_k} \int_{T_\delta^- \setminus \overline{T}_{\frac{c_{n_k}}{n_k}}} \Big|\big(0, \nabla u_k(y), M_{12}(\nabla u_k(y))\big)\Big| |\det\,(\nabla \Sigma_{\delta,n_k,c_{n_k}}(y))|dy \\
&\quad + \int_{T_\delta^- \setminus \overline{T}_{\frac{c_{n_k}}{n_k}}} g\big(\widetilde{\mathcal{M}}(\nabla u_k(y))\big)|\det\,(\nabla \Sigma_{\delta,n_k,c_{n_k}}(y))|dy \\
&\leq \frac{C_g C_\gamma}{n_k}(1 + \frac{C_\gamma}{n_k}) \int_{T_\delta^- \setminus \overline{T}_{\frac{c_{n_k}}{n_k}}} \Big|\big(0, \nabla u_k(y), M_{12}(\nabla u_k(y))\big)\Big|dy \\
&\quad + (1 + \frac{C_\gamma}{n_k}) \int_{T_\delta^- \setminus \overline{T}_{\frac{c_{n_k}}{n_k}}} g\big(\widetilde{\mathcal{M}}(\nabla u_k(y))\big)dy \\
&\leq \frac{C_g C_\gamma}{n_k}(1 + \frac{C_\gamma}{n_k})\,(|\nabla u_k|(A) + |M_{12}(\nabla u_k)|(A)) + (1 + \frac{C_\gamma}{n_k})F(u_k, T_\delta^-)
\end{aligned}$$

and so, thanks to (2.7), we conclude, for k large enough,

$$F(v_k, T_\delta^-) \leq F(u_k, T_\delta^-) + \frac{C_{\gamma,g}}{n_k}\left(|\nabla u_k|(A) + F(u_k, A)\right),$$

for a constant $C_{\gamma,g} > 0$ depending on γ, g, but independent on u_k and k. As a consequence, using that u_k is a recovery sequence and that it is converging to u strictly in $BV(\Omega;\mathbb{R}^m)$, we are led to

$$\limsup_{k\to\infty} F(v_k, T_\delta^-) \leq \lim_{k\to\infty} F(u_k, T_\delta^-),$$

which means that v_k is a recovery sequence as well, thanks to the fact that v_k still converges to u strictly in $BV(\Omega;\mathbb{R}^m)$ (how it is easily checked from (4.14)). □

Proposition 4.6. *Let A be a bounded open set and let $B \subset\subset A$ be a open subset whose boundary is $\partial B =: \Gamma \subset A$ a finite union of closed curves of class C^3. Let $T_\delta \subset A$ be a tubular neighborhood of Γ, let $\psi : (-\delta, \delta) \to \mathbb{R}$ be the function defined in* (4.1)*, and assume that* 0 *is a regular value for ψ. Let $(u_k) \subset C^1(A;\mathbb{R}^m)$ be a recovery sequence for $\mathcal{F}(u; A)$ such that $u_k \llcorner \Gamma \to u \llcorner \Gamma$ strictly in $BV(\Gamma;\mathbb{R}^m)$; then $u_k \llcorner B$ is a recovery sequence for $\mathcal{F}(u; B)$.*

Proof. We prove the assertion arguing by contradiction, so assume that u_k is not a recovery sequence for $\mathcal{F}(u, B)$; we can then extract a subsequence such that there exists the limit

$$\lim_{k\to\infty} F(u_k, B) > \mathcal{F}(u, B).$$

Let $(v_k) \subset C^1(B;\mathbb{R}^m)$ be a recovery sequence for $\mathcal{F}(u,B)$ so that

$$\mathcal{F}(u,B) = \lim_{k\to\infty} F(v_k,B) < \lim_{k\to\infty} F(u_k,B).$$

According to Lemma 4.5, we can suppose that $v_k \llcorner \Gamma \to u \llcorner \Gamma$ strictly in $BV(\Gamma;\mathbb{R}^m)$. Therefore, the same being true for $u_k \llcorner \Gamma$, we are in the hypotheses of Corollary 4.4, and we can find a sequence $w_k \in \mathrm{Lip}_{\mathrm{loc}}(A;\mathbb{R}^m)$ (these are Lipschitz continuous in neighborhood $U \subset\subset A$ of B, and C^1 on $A \setminus \overline{U}$) such that

$$\begin{aligned}\lim_{k\to\infty} F(w_k,A) &= \lim_{k\to\infty} F(w_k,A\setminus\overline{B}) + \lim_{k\to\infty} F(w_k,B) = \lim_{k\to\infty} F(u_k,A\setminus\overline{B}) + \lim_{k\to\infty} F(v_k,B)\\ &< \lim_{k\to\infty} F(u_k,A\setminus\overline{B}) + \lim_{k\to\infty} F(u_k,B) = \lim_{k\to\infty} F(u_k,A) = \mathcal{F}(u,A),\end{aligned}$$

that is absurd. The thesis follows. □

5 Proof of Theorem 1.1: Monotonicity, inner regularity and sub-addivitity

This Section is devoted to the proof of Theorem 1.1. To this purpose we need to use Theorem 3.1, and so we will check that hypohteses (i)-(iv) of that theorem are satisfied. We start with the following technical result:

Proposition 5.1. *Let $A \subset \Omega$ be open and let $(u_k) \subset \mathrm{Lip}_{\mathrm{loc}}(\Omega;\mathbb{R}^m)$ be a sequence such that $u_k \to u$ strictly in $BV(\Omega;\mathbb{R}^m)$; then there exists a sequence $(w_j) \subset \mathrm{Lip}_{\mathrm{loc}}(A;\mathbb{R}^m)$ such that the following holds:*

(i) $w_j \to u$ *strictly in* $BV(A;\mathbb{R}^m)$;

(ii) $\liminf_{j\to\infty} F(w_j,A) \le \liminf_{k\to\infty} F(u_k,A)$.

Moreover, if u_k are either Lipschitz continuous or of class C^1 on A, then the sequence $(w_j) \subset \mathrm{Lip}_{\mathrm{loc}}(A;\mathbb{R}^m)$ also satisfies

(iii) $F(w_j,A) \ge \mathcal{F}(w_j,A)$, *for all* $j \ge 1$.

Proof. Step 1: (Setup) As $A \subset \mathbb{R}^2$ is bounded, we consider the set $\Sigma_n \subseteq A$ defined by

$$\Sigma_n := \{x \in A : \mathrm{dist}(x,A^c) = \eta_n\},$$

where the numbers η_n, are chosen so that for all $n \ge 1$ it holds $0 < \eta_{n+1} < \eta_n$, and Σ_n is a finite union of Lipschitz loops $\Sigma_n = \cup_{i=1}^{N_n} \Sigma_n^i$ (see Lemma 7.2 in Appendix). We assume that Σ_n^i is a unique Jordan curve for all $i = 1,\dots,N_n$. Let

$$d_n := \min\{\mathrm{dist}(\Sigma_n^i,\Sigma_n^j),\ 0 \le i < j \le N_n\}, \tag{5.1}$$

and for all $i = 1,\dots,N_n$ we choose a simple loop $\widehat{\Gamma}_n^i$ of class C^4 such that

$$\widehat{\Gamma}_n^i \subset \{x \in A :\ \mathrm{dist}(x,A^c) \in (\eta_{n+1},\eta_n),\ \mathrm{dist}(x,\Sigma_n^i) < \frac{d_n}{4}\},$$

and in such a way that the region enclosed by $\widehat{\Gamma}_n^i$ and Σ_n^i is an annulus type open set contained in $\{x \in A :\ \mathrm{dist}(x,A^c) \in (\eta_{n+1},\eta_n)\}$. For all i, we denote by $\widehat{H}_n^i$ this annulus so that

$$\widehat{H}_n^i \subset \{x \in A :\ \mathrm{dist}(x,A^c) \in (\eta_{n+1},\eta_n)\}, \qquad \partial\widehat{H}_n^i = \widehat{\Gamma}_n^i \cup \Sigma_n^i.$$

Furthermore we consider tubular neighborhoods $T_{\widehat{\delta}_n^i}$ of $\widehat{\Gamma}_n^i$ with $\widehat{\delta}_n^i > 0$ so small in order that

$$T_{\widehat{\delta}_n^i} \subset \{x \in A : \operatorname{dist}(x, A^c) \in (\eta_{n+1}, \eta_n),\ \operatorname{dist}(x, \Sigma_n^i) < \frac{d_n}{2}\}.$$

Notice that, thanks to our choice of the parameters, it turns out that the open sets $T_{\widehat{\delta}_n^i}$, $n \in \mathbb{N}$, $i = 1, \dots, N_n$, are mutually disjoint.

Let now $(u_k) \subset \mathrm{Lip}_{\mathrm{loc}}(\Omega; \mathbb{R}^m)$ be a sequence as in the statement. For all $n \geq 1$ and all $i = 1, \dots, N_n$ we choose a positive number $r_n^i < \widehat{\delta}_n^i/2$ such that, setting, as usual,

$$(\widehat{\Gamma}_n^i)_r := \{x \in T_{\widehat{\delta}_i^n} : \operatorname{dist}(x, \widehat{\Gamma}_n^i) = r\}$$

the following conditions hold:

(a) $|Du|((\widehat{\Gamma}_n^i)_{r_n^i}) = 0$ and $u \llcorner (\widehat{\Gamma}_n^i)_{r_n^i}$ belongs to $BV((\Gamma_n^i)_{r_n^i}; \mathbb{R}^m)$;

(b) Setting $\widehat{\psi}_n^i(r) := |u \llcorner (\widehat{\Gamma}_n^i)_r|_{BV} = |D_\zeta u|((\widehat{\Gamma}_n^i)_r)$ then r_n^i is a regular value for $\widehat{\psi}_n^i$;

(c) $u_k \llcorner (\widehat{\Gamma}_n^i)_{r_n^i} \to u \llcorner (\widehat{\Gamma}_n^i)_{r_n^i}$ strictly in $BV((\widehat{\Gamma}_n^i)_{r_n^i}; \mathbb{R}^m)$.

We observe that the loops $(\widehat{\Gamma}_n^i)_{r_n^i}$ are of class C^3 and we denote

$$\Gamma_n^i := (\widehat{\Gamma}_n^i)_{r_n^i};$$

let H_n^i be the annulus type region enclosed by Σ_n^i and Γ_n^i, so that $H_n^i \subset \widehat{H}_n^i$. In this way conditions (a), (b), and (c), are satisfied for Γ_n^i replacing $(\widehat{\Gamma}_n^i)_{r_n^i}$ and 0 is a regular value for $\psi_n^i(r) := |u \llcorner (\Gamma_n^i)_r|_{BV}$; finally, since $r_n^i < \widehat{\delta}_n^i/2$ the tubular neighborhoods $T_{\delta_n^i}$ of Γ_n^i, with $\delta_n^i := \widehat{\delta}_n^i/2$, for $n > 0$, $i = 1, \dots, N_n$ are all mutually disjoint.

For any integer $n > 0$ fixed, we consider the open set B_n defined as

$$B_n := \overline{A}_n \cup \bigcup_{i=1}^{N_n} H_n^i.$$

In this way and by definition of H_n^i, we see that for all $n > 1$ it holds

$$B_n \subset\subset A, \qquad A_n \subset B_n \subset A_{n+1}.$$

Step 2: We now fix a natural number $j > 0$ and for all $n \geq 1$ we consider the functions $\omega_{\Gamma^i} := \omega_{\Gamma_n^i}$ appearing in the right-hand side of (4.12); then we choose a number $a_n > 0$ so that

$$\sum_{i=1}^{N_n} \omega_{\Gamma^i}(t) < \frac{1}{j 2^{n+1}} \qquad \text{for all } t < a_n. \tag{5.2}$$

For $n = 1$ we consider the set B_1 and owing to conditions (a), (b), and (c), we choose a natural number $k_{1,j} > 0$ so that

(1) $d_s(u_{k_{1,j}} \llcorner \Gamma_1^i, u \llcorner \Gamma_1^i) < \frac{a_1}{2}$, for all $i = 1, \dots, N_1$;

(2) $\|u_{k_{1,j}} - u\|_{L^1(B_1)} + ||Du_{k_{1,j}}|(B_1) - |Du|(B_1)| < \frac{1}{4j}$;

(3) $F(u_{k_{1,j}}, B_1) \leq \liminf_{k \to \infty} F(u_k, B_1) + \frac{1}{4j}$.

Next, for every $n > 1$ we choose $k_{n,j} > k_{n-1,j}$ so that the following holds

(1*) $d_s(u_{k_{n,j}} \llcorner \Gamma^i_n, u \llcorner \Gamma^i_n) < \frac{a_n}{2}$, $\forall i = 1, \dots, N_n$ and $d_s(u_{k_{n,j}} \llcorner \Gamma^i_{n-1}, u \llcorner \Gamma^i_{n-1}) < \frac{a_{n-1}}{2}$, $\forall i = 1, \dots, N_{n-1}$;

(2*) $\|u_{k_{n,j}} - u\|_{L^1(B_n \setminus B_{n-1})} + ||Du_{k_{n,j}}|(B_n \setminus B_{n-1}) - |Du|(B_n \setminus B_{n-1})| < \frac{1}{j2^{n+1}}$;

(3*) $F(u_{k_{n,j}}, B_n \setminus \overline{B}_{n-1}) \leq \liminf_{k\to\infty} F(u_k, B_n \setminus \overline{B}_{n-1}) + \frac{1}{j2^{n+1}}$.

Conditions (2) and (2*) can be obtained because $u_k \to u$ strictly in $BV(\Omega; \mathbb{R}^m)$, and thanks to the hypothesis that $|Du|$ does not concentrate on ∂B_n, for any $n \geq 1$ (so the strict convergence is inerhited on $B_n \setminus \overline{B}_{n-1}$).

Step 3: We now proceed to glue the maps $u_{k_{n,j}}$ along the tubes $T_{\delta^i_n}$ exploiting Lemma 4.2. More precisely, for all $n \geq 1$ we apply Corollary 4.3 with A, B replaced by $B_{n+1} \setminus \overline{B}_{n-1}$ and $B_n \setminus \overline{B}_{n-1}$, respectively, $\delta = \delta_n := \min\{\delta^i_n,\ i = 1, \dots, N_n\}$, and $\varepsilon = \frac{1}{j2^{n+1}}$, $v^- = u_{k_{n,j}}$, $v^+ = u_{k_{n+1,j}}$. This provides us with a map $w_{n,j} \in \mathrm{Lip}(B_{n+1} \setminus \overline{B}_{n-1}; \mathbb{R}^m)$, (here we have set $B_0 = \varnothing$ to deal with the case $n = 1$) such that

$$
\begin{aligned}
& w_{n,j} = u_{k_{n,j}} \text{ in } B_n \setminus \overline{B}_{n-1} \setminus \overline{T}_{\delta_n} \text{ and } w_{n,j} = u_{k_{n+1,j}} \text{ in } B_{n+1} \setminus \overline{B}_n \setminus \overline{T}_{\delta_n}, \\
& \|w_{n,j} - u_{k_{n,j}}\|_{L^1(T_{\delta_n} \cap B_n)} \leq 3\|u_{k_{n,j}} - u\|_{L^1(T_{\delta_n} \cap B_n)} + r_{n,j}, \\
& \|w_{n,j} - u_{k_{n+1,j}}\|_{L^1(T_{\delta_n} \cap (B_{n+1} \setminus \overline{B}_n))} \leq 3\|u_{k_{n+1,j}} - u\|_{L^1(T_{\delta_n} \cap (B_{n+1} \setminus \overline{B}_n))} + r_{n,j}, \\
& \int_{B_n \setminus \overline{B}_{n-1}} |\nabla w_{n,j} - \nabla u_{k_{n,j}}| dx \leq r_{n,j}, \\
& \int_{B_{n+1} \setminus \overline{B}_n} |\nabla w_{n,j} - \nabla u_{k_{n+1,j}}| dx \leq r_{n,j}, \\
& F(w_{n,j}; B_n \setminus \overline{B}_{n-1}) \leq F(u_{k_{n,j}}; B_n \setminus \overline{B}_{n-1}) + r_{n,j}, \\
& F(w_{n,j}; B_{n+1} \setminus \overline{B}_n) \leq F(u_{k_{n+1,j}}; B_{n+1} \setminus \overline{B}_n) + r_{n,j}, \\
& r_{n,j} \leq \frac{1}{j2^{n+1}} + \sum_{i=0}^{N_n} \omega_{\Gamma^i} \big(d_s(u_{k_{n,j}} \llcorner \Gamma^i_n, u \llcorner \Gamma^i_n) + d_s(u_{k_{n+1,j}} \llcorner \Gamma^i_n, u \llcorner \Gamma^i_n) \big) \leq \frac{1}{j2^n},
\end{aligned}
\tag{5.3}
$$

where the last inequality is obtained in view of (1) and (1*), thanks to (5.2). Due to the first line, we can now define $w_j \in \mathrm{Lip}_{\mathrm{loc}}(A; \mathbb{R}^m)$ as

$$
w_j := w_{n,j} \qquad \text{on} \qquad U_n := (B_n \setminus \overline{B}_{n-1} \setminus \overline{T}_{\delta_{n-1}}) \cup T_{\delta_n}.
$$

We can now estimate

$$
\begin{aligned}
\|w_j - u\|_{L^1(A)} &\leq \sum_{n=1}^{\infty} \|w_{n,j} - u_{k_{n,j}}\|_{L^1(T_{\delta_n} \cap B_n)} + \|w_{n,j} - u_{k_{n+1,j}}\|_{L^1(T_{\delta_n} \cap (B_{n+1} \setminus \overline{B}_n))} \\
&\leq \sum_{n=1}^{\infty} 2r_{n,j} + 3\|u - u_{k_{n,j}}\|_{L^1(T_{\delta_n} \cap B_n)} + 3\|u - u_{k_{n+1,j}}\|_{L^1(T_{\delta_n} \cap (B_{n+1} \setminus \overline{B}_n))}
\end{aligned}
$$

where we have used (5.3), and thanks to (2) and (2*) we conclude

$$
\|w_j - u\|_{L^1(A)} \leq \frac{5}{j}. \tag{5.4}
$$

A similar argument applied to the forth and fifth lines in (5.3) and again based on (2) and (2*) leads to

$$|Dw_j|(A) \le \sum_{n=1}^{\infty} |Du_{k_{n,j}}|(T_{\delta_n} \cap B_n)| + |Du_{k_{n+1,j}}|(T_{\delta_n} \cap B_{n+1})| + 2r_{n,j} \le |Du|(A) + \frac{3}{j}. \quad (5.5)$$

Finally, arguing analogously, thanks to the first, sixth, and seventh line in (5.3) and to (3) and (3*) we conclude

$$\begin{aligned} F(w_j, A) &= \sum_{n=1}^{\infty} F(w_j, U_n \cap B_n) + F(w_j, T_{\delta_n} \setminus \overline{B}_n) \\ &\le \sum_{n=1}^{\infty} F(u_{k_{n,j}}, B_n \setminus \overline{B}_{n-1}) + F(u_{k_{n+1,j}}, B_{n+1} \setminus \overline{B}_n) + 2r_{n,j} \\ &\le \frac{2}{j} + \liminf_{k\to\infty} F(u_k, A). \end{aligned} \quad (5.6)$$

To conclude the proof, it is sufficient to observe that the sequence w_j converges, as $j \to \infty$, to u in $L^1(A;\mathbb{R}^m)$ thanks to (5.4); by (5.5), the previous convergence is strict in $BV(A;\mathbb{R}^m)$. Moreover, (5.6) implies (ii). To achieve the last assertion, it is enough to observe that w_j can be obtained as strict limit in $BV(\Omega;\mathbb{R}^m)$ of the functions $\widetilde{w}_{j,m}$ given by

$$\widetilde{w}_{j,m} := \begin{cases} w_j & \text{on } U_m, \\ u_{k_{m+1,j}} & \text{on } A \setminus \overline{U}_m. \end{cases} \quad (5.7)$$

For these maps it is immediately checked that $F(\widetilde{w}_{j,m}, A) \to F(w_j, A)$. So by Theorem 2.7 (if u_k are Lipschitz) and Theorem 2.8 (if u_k are of class C^1) we deduce that

$$F(w_j, A) \ge \liminf_{k\to\infty} \mathcal{F}(w_{j,m}, A) \ge \mathcal{F}(w_j, A),$$

where we have used Theorem 2.5 in the last inequality. □

Remark 5.2. *We emphasize that inequality $F(w_j, A) \le \mathcal{F}(w_j, A)$ is not guaranteed in general, at this stage. This is due to the fact that w_j is not globally Lipschitz in A but only locally, and neither of class C^1. We will prove that for a locally Lipschitz function u there holds $F(u, A) = \mathcal{F}(u, A)$ only in after the proof of Theorem 1.1.*

Corollary 5.3. *Assume the hypotheses of Proposition 5.1 and let $B_0 = \varnothing$, and B_n $(n \ge 1)$ be the sets in Step 1 of its proof. Then the sequence w_j also satisfies, for all $n \ge 1$*

$$F(w_j, B_n \setminus B_{n-1}) \le \liminf_{k\to\infty} F(u_k, B_n \setminus B_{n-1}) + \frac{1}{j2^{n+1}}. \quad (5.8)$$

If moreover u_k is a recovery sequence for $\mathcal{F}(u, A)$, then there holds:

$$\begin{aligned} &\lim_{j\to\infty} F(w_j, A) = \mathcal{F}(u, A), \\ &\lim_{j\to\infty} F(w_j, B_n \setminus \overline{B}_{n-1}) = \mathcal{F}(u, B_n \setminus \overline{B}_{n-1}) \qquad \forall n \ge 1, \\ &\lim_{j\to\infty} F(w_j, B_n) = \mathcal{F}(u, B_n) \qquad \forall n \ge 1. \end{aligned} \quad (5.9)$$

Proof. Inequality (5.8) follows from the definition of w_j, expressions (5.3) and conditions (3) and (3*) in the proof of Proposition 5.1. If u_k is a recovery sequence for $\mathcal{F}(u,A)$, then conditions (a), (b), and (c), in Step 1 of that proof ensure, thanks to Proposition 4.6, $u_k \llcorner (B_n \setminus \overline{B}_{n-1})$ is a recovery sequence for $\mathcal{F}(u, B_n \setminus \overline{B}_{n-1})$ and at the same time $u_k \llcorner B_n$ is a recovery sequence for $\mathcal{F}(u,B_n)$; then the first line in (5.9) follows from (ii) and (iii) of Proposition 5.1, and similarly the second and third lines follow from (5.8). □

We are now in a position to check conditions (i)-(iv) of Theorem 3.1; we start with the monotonicity condition (i):

Theorem 5.4. *(Monotonicity) Let $B \subseteq A$ be bounded open sets and let $u \in BV(A;\mathbb{R}^m)$; then*

$$\mathcal{F}(u,B) \leq \mathcal{F}(u,A).$$

Proof. Let $(u_k) \subset C^1(A;\mathbb{R}^m)$ be a recovery sequence for $\mathcal{F}(u,A)$. According to Proposition 5.1 (applied to the case $A = \Omega$ and B in place of A) there exists a sequence $(w_j) \subset \mathrm{Lip}_{\mathrm{loc}}(B;\mathbb{R}^m)$ such that

$$\liminf_{j\to\infty} F(w_j,B) \leq \liminf_{k\to\infty} F(u_k,B) \leq \liminf_{k\to\infty} F(u_k,A) = \mathcal{F}(u,A).$$

Since $\mathcal{F}(u,B) \leq \liminf_{j\to\infty} F(w_j,B)$ thanks to (iii) in Proposition 5.1, we have concluded. □

As additivity (ii) is trivial, we proceed to verify (iv) of Theorem 3.1, and then go to (iii).

Theorem 5.5. *(Inner regularity) Let $A \subset \mathbb{R}^2$ be a bounded open set; then*

$$\mathcal{F}(u;A) = \sup\{\mathcal{F}(u;B) : B \textit{ is an open set and } B \subset\subset A\}. \tag{5.10}$$

Proof. Step 1: We consider the same setting in Step 1 of the proof of Proposition 5.1. In particular, we fix a recovery sequence u_k for $\mathcal{F}(u,A)$, and assume that, for all $n \geq 1$, and $i = 1,\dots,N_n$,

(a) $|Du|(\Gamma_n^i) = 0$ and $u \llcorner \Gamma_n^i$ belongs to $BV(\Gamma_n^i;\mathbb{R}^m)$;

(b) Setting $\psi_n^i(r) := |u \llcorner (\Gamma_n^i)_r|_{BV} = |D_\varsigma u|((\Gamma_n^i)_r)$ then 0 is a regular value for ψ_n^i;

(c) $u_k \llcorner \Gamma_n^i \to u \llcorner \Gamma_n^i$ strictly in $BV(\Gamma_n^i;\mathbb{R}^m)$.

By standard arguments one sees that

$$\sup\{\mathcal{F}(u;B) : B \text{ is an open set and } B \subset\subset A\} = \sup\{\mathcal{F}(u;B_n) : n \geq 1\}. \tag{5.11}$$

Indeed, let $B \subset\subset A$; by compactness of $\overline{B}$ one has $\mathrm{dist}(B,A^c) > 0$ and so there exists n such that $B \subset A_n \subset B_n$. So, by monotonicity the inequality sign $\geq$ holds in (5.11), and the converse being obvious, the claim follows.

We fix $\varepsilon > 0$ arbitrary, and prove that there exists n_ε such that

$$\mathcal{F}(u,B_{n_\varepsilon}) \geq \mathcal{F}(u,A) - \varepsilon. \tag{5.12}$$

This will imply the thesis by arbitrariness of $\varepsilon > 0$.

Step 2: Condition (c) ensures that, thanks to Proposition 4.6, $u_k \llcorner B_n$ and $u_k \llcorner (B_{n+1} \setminus \overline{B}_n)$ are still recovery sequences for $\mathcal{F}(u,B_n)$ and $\mathcal{F}(u,B_{n+1} \setminus \overline{B}_n)$ respectively, for all $n \geq 1$. This implies that

$$\mathcal{F}(u,B_n) = \lim_{k\to\infty} F(u_k,B_n) = \sum_{i=1}^n \lim_{k\to\infty} F(u_k,B_i \setminus \overline{B}_{i-1}) = \sum_{i=1}^n \mathcal{F}(u,B_i \setminus \overline{B}_{i-1}),$$

where once more we have set $B_0 = \varnothing$. Since, by monotonicity, for all $n \geq 1$ we have $\mathcal{F}(u, B_n) \leq \mathcal{F}(u, A)$, we conclude

$$\sum_{i=1}^{\infty} \mathcal{F}(u, B_i \setminus \overline{B}_{i-1}) \leq \mathcal{F}(u, A). \tag{5.13}$$

Fix $\varepsilon > 0$; by (5.13) the series in the left-hand side is convergent, and so we can fix $n_\varepsilon > 0$ so that

$$\sum_{i=n_\varepsilon+1}^{\infty} \mathcal{F}(u, B_i \setminus \overline{B}_{i-1}) \leq \varepsilon, \tag{5.14}$$

We consider the sequence w_j provided by Corollary 5.3, that, for all $i \geq 1$, is a recovery sequence for $\mathcal{F}(u, B_i \setminus \overline{B}_{i-1})$ and for $\mathcal{F}(u, B_{n_\varepsilon})$. From (5.8) we deduce that

$$\begin{aligned}
\mathcal{F}(u, A) &= \lim_{j\to\infty} F(w_j, B_{n_\varepsilon}) + \lim_{j\to\infty} \sum_{i=n_\varepsilon+1}^{\infty} F(w_j, B_i \setminus \overline{B}_{i-1}) \\
&\leq \mathcal{F}(u, B_{n_\varepsilon}) + \lim_{j\to\infty} \Big(\sum_{i=n_\varepsilon+1}^{\infty} \liminf_{k\to\infty} F(u_k, B_i \setminus \overline{B}_{i-1}) + \frac{1}{j 2^{n+1}} \Big) \\
&\leq \mathcal{F}(u, B_{n_\varepsilon}) + \lim_{j\to\infty} \Big(\sum_{i=n_\varepsilon+1}^{\infty} \mathcal{F}(u, B_i \setminus \overline{B}_{i-1}) + \frac{1}{j} \Big) \\
&= \mathcal{F}(u, B_{n_\varepsilon}) + \varepsilon.
\end{aligned}$$

By arbitrariness of $\varepsilon > 0$ we conclude. $\square$

Theorem 5.6. *(Sub-additivity) Let $u \in BV(\Omega; \mathbb{R}^m)$ be given. Then for all open sets $A_1, A_2, A \subset \Omega$ with $A \subseteq A_1 \cup A_2$ it holds*

$$\mathcal{F}(u, A) \leq \mathcal{F}(u, A_1) + \mathcal{F}(u, A_2).$$

Proof. Let $u_k \subset C^1(\Omega; \mathbb{R}^m)$ be a recovery sequence for $\mathcal{F}(u, A_1 \cup A_2)$. Starting from the set A, we build, as in Step 1 of the proof of Proposition 5.1, the sets $B_n \subset\subset A$, $n \geq 1$. By definition

$$B_n \subset A_{n+1} = \{x \in A : \operatorname{dist}(x, A^c) > \eta_{n+1}\} \tag{5.15}$$

and taking into account that $\partial B_n = \cup_{i=1}^{N_n} \Gamma_n^i$ enjoies properties (a), (b), and (c) in the proof of Theorem 5.5, we immediately obtain that $u_k \llcorner B_n$ is a recovery sequence for $\mathcal{F}(u, B_n)$. Then we fix $\varepsilon > 0$; owing to the inner regularity, Theorem 5.5, and thanks to (5.11), we choose $n_\varepsilon > 0$ so that

$$\mathcal{F}(u, B_{n_\varepsilon}) \geq \mathcal{F}(u, A) - \varepsilon. \tag{5.16}$$

Next we proceed once again along the lines of Step 1 of Proposition 5.1 for the sets A_1 and A_2, obtaining sets B_n^1 and B_n^2, $n \geq 1$, for which

$$\begin{aligned}
B_n^1 \subset A_{n+1}^1 = \{x \in A_1 : \operatorname{dist}(x, A_1^c) > \eta_{n+1}^1\} \subset B_{n+1}^1, \\
B_n^2 \subset A_{n+1}^2 = \{x \in A_2 : \operatorname{dist}(x, A_2^c) > \eta_{n+1}^2\} \subset B_{n+1}^2,
\end{aligned}$$

for suitable infinitesimal decreasing sequences of numbers η^1_n and η^2_n (which may differ from η_n). We therefore choose $\overline{n}$ big enough so that $\eta^1_{\overline{n}+1}, \eta^2_{\overline{n}+1} < \eta_{n_\varepsilon+1}$, and so we check that

$$B_{n_\varepsilon} \subset A_{n_\varepsilon+1} \subset A^1_{\overline{n}+1} \cup A^2_{\overline{n}+1} \subset B^1_{\overline{n}+1} \cup B^2_{\overline{n}+1}. \tag{5.17}$$

Here the second inclusion is true since $A \subseteq A_1 \cup A_2$, and so

$$\begin{aligned} &\{x \in A : \operatorname{dist}(x, A^c) > \eta_{n_\varepsilon+1}\} \subseteq \{x \in A : \operatorname{dist}(x, (A_1 \cup A_2)^c) > \eta_{n_\varepsilon+1}\} \\ &\quad \subseteq \{x \in A_1 : \operatorname{dist}(x, (A_1 \cup A_2)^c) > \eta_{n_\varepsilon+1}\} \cup \{x \in A_2 : \operatorname{dist}(x, (A_1 \cup A_2)^c) > \eta_{n_\varepsilon+1}\}; \end{aligned}$$

now since $\operatorname{dist}(x, (A_1 \cup A_2)^c) = \min\{\operatorname{dist}(x, A_1^c), \operatorname{dist}(x, A_2^c)\}$, we also have

$$\begin{aligned} &\{x \in A_1 : \operatorname{dist}(x, (A_1 \cup A_2)^c) > \eta_{n_\varepsilon+1}\} \cup \{x \in A_2 : \operatorname{dist}(x, (A_1 \cup A_2)^c) > \eta_{n_\varepsilon+1}\} \\ &\quad \subseteq \{x \in A_1 : \operatorname{dist}(x, A_1^c) > \eta_{n_\varepsilon+1}\} \cup \{x \in A_2 : \operatorname{dist}(x, A_2^c) > \eta_{n_\varepsilon+1}\} \\ &\quad \subseteq \{x \in A_1 : \operatorname{dist}(x, A_1^c) > \eta^1_{\overline{n}+1}\} \cup \{x \in A_2 : \operatorname{dist}(x, A_2^c) > \eta^2_{\overline{n}+1}\} = A^1_{\overline{n}+1} \cup A^2_{\overline{n}+1}. \end{aligned}$$

From (5.17) we can finally write, for all k,

$$F(u_k, B_{n_\varepsilon}) \leq F(u_k, B^1_{\overline{n}+1}) + F(u_k, B^2_{\overline{n}+1}),$$

and so passing to the limit as $k \to \infty$ we end up with

$$\mathcal{F}(u, B_{n_\varepsilon}) \leq \mathcal{F}(u, B^1_{\overline{n}+1}) + \mathcal{F}(u, B^2_{\overline{n}+1}) \leq \mathcal{F}(u, A_1) + \mathcal{F}(u, A_2), \tag{5.18}$$

the second iequality following from monotonicity of $\mathcal{F}(u, \cdot)$. This implies the thesis thanks to (5.16) and the arbitrariness of ε. ☐

6 Examples of representation formulas

In this section we revise some examples showing how the area functional relaxed with respect to strict topology is representable in an integral form.

First we see that for a locally Lipschitz map $u \in BV(\Omega; \mathbb{R}^m)$ there holds the standard expression

$$\mathcal{F}(u, \Omega) = F(u, \Omega).$$

This is because, since $\mathcal{F}(u, \cdot)$ is the restriction of a Borel measure, we have

$$\mathcal{F}(u, \Omega) = \lim_{n\to\infty} \mathcal{F}(u, \Omega_n) = \lim_{n\to\infty} \int_{\Omega_n} g(\mathcal{M}(\nabla u))dx = \int_\Omega g(\mathcal{M}(\nabla u))dx,$$

for any increasing sequence of sets $\Omega_n \subset\subset \Omega$ with $\cup_n \Omega_n = \Omega$. Notice that the second equality above follows since u is Lipschitz on Ω_n and we can apply Theorem 2.7.

We now pass to a more involved example: Consider a rectangle $R := (a, b) \times (c, d) \subset \mathbb{R}^2$, let $h \in (c, d)$ and let $S := (a, b) \times h$. Let $R^+ := (a, b) \times (h, d)$, $R^- := (a, b) \times (c, h)$, and $u \in BV(R; \mathbb{R}^2)$ be a map such that $u^\pm := u \llcorner R^\pm$ are Lipschitz continuous. In this case the relaxed area $\mathcal{A}(u, R)$ has been proved to be [5]

$$\mathcal{A}(u, R) := \mathbb{A}(u, R^+) + \mathbb{A}(u, R^-) + \int_{(a,b)\times(0,1)} |\partial_t X^{\text{aff}} \wedge \partial_s X^{\text{aff}}| dtds, \tag{6.1}$$

where X^{aff} is the affine interpolation between the traces of $u^\pm$ on S, namely

$$X^{\text{aff}}(t, s) := (t, su^+(t, h) + (1 - s)u^-(t, h)), \qquad \forall (t, s) \in (a, b) \times (0, 1). \tag{6.2}$$

This result can be extended to piecewise Lipschitz maps with jump forming a network (namely a graph consisting of finitely many C^2-curves meeting at finitely many junctions points, see [5]). A similar representation formula holds for this kind of maps, where however there appears also the singular contribution of solutions of suitable plateau problems accounting for the junctions points (see [5, Theorem 1.1]).

Another important case is the one of Sobolev maps with values in $\mathbb{S}^1$, $u \in W^{1,1}(\Omega; \mathbb{S}^1)$. In this case, if $\det(\nabla u) = \pi \sum_{i=1}^{\infty}(\delta_{x_i} - \delta_{y_i})$ (see [14] and references therin), then the measure $\mu(A) := \mathcal{A}(u, A)$ takes the form

$$\mu = \sqrt{1 + |\nabla u|^2} \cdot \mathcal{L}^2 + \pi \sum_{i=1}^{\infty} (\delta_{x_i} + \delta_{y_i}).$$

For general maps of bounded variation u an explicit expression of μ is not known at the present stage. This will be object of future research.

6.1 A Cartesian map with singular relaxed area

We consider a Lipschitz curve $\varphi : \mathbb{S}^1 \to \mathbb{R}^2$ and, for $\Omega = B_r$, $r > 0$, the 0-homogeneous map $u_\varphi : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$u_\varphi(x) = \varphi(\frac{x}{|x|}), \qquad x \in \Omega \setminus \{0\}. \tag{6.3}$$

It is easy to see that the graph of u_φ, treated as a 2-integral current $\mathcal{G}_{u_\varphi} \in \mathcal{D}_2(\Omega \times \mathbb{R}^2)$, satisfies

$$\partial \mathcal{G}_{u_\varphi} = \delta_0 \times \varphi_\sharp [\![\mathbb{S}^1]\!] \qquad \text{in } \mathcal{D}_1(\Omega \times \mathbb{R}^2),$$

where $\varphi_\sharp [\![\mathbb{S}^1]\!]$ is the integration over the image of φ, i.e., the push-forward by φ of the standard integration over the unit circle $\mathbb{S}^1$. According to the results in [4] (see also [17]) it holds

$$\mathcal{A}(u_\varphi, \Omega) = \int_\Omega \sqrt{1 + |\nabla u_\varphi|^2} dx + \mathcal{P}(\varphi), \tag{6.4}$$

where $\mathcal{P}(\varphi)$ corresponds to the area of a disk-type solution of the planar Plateau problem with boundary $\varphi(\mathbb{S}^1)$. Specifically

$$\mathcal{P}(\varphi) := \inf\{\int_{B_1} |\partial_{x_1}\Phi \wedge \partial_{x_2}\Phi| dx : \Phi = \varphi \text{ on } \partial B_1,\ \Phi \in \mathrm{Lip}(B_1; \mathbb{R}^2)\}. \tag{6.5}$$

This Plateau problem can be singular, in the sense that the contour $\varphi(\mathbb{S}^1)$ of the minimal disk can have self-intersection and overlappings (see [19–21, 36] for this kind of Plateau problem and generalization). It is interesting to observe that this singular contribution is related with the presence of the Jacobian determinant in the integrand of our functional. Indeed, a similar contribution appears when we consider the total variation of the Jacobian (see [4,17]), relaxation with respect to the strict convergence in BV of (1.14):

$$\mathcal{TVJ}(u_\varphi, \Omega) = \mathcal{P}(\varphi), \tag{6.6}$$

(compare with the results in [45] and [26]).

We now make a specific choice for φ: Let Γ_1 and Γ_2 be two circumferences tangent to each other at the origin 0. If α_i denotes a constant speed parametrization of Γ_i starting from 0, we consider the concatenation

$$\varphi := \alpha_1 \star \alpha_2 \star \alpha_1^{-1} \star \alpha_2^{-1}, \tag{6.7}$$

that is a Lipschitz closed curve running the 8-shaped figure consisting of $\Gamma_1 \cup \Gamma_2$ two times, the first with opposite orientation of the second time. Due to this, it turns out that the current $\varphi_\sharp[\![\mathbb{S}^1]\!]$ is null, so that u_φ is a Cartesian map, namely

$$\partial \mathcal{G}_{u_\varphi} = 0 \qquad\qquad \text{in } \mathcal{D}_1(\Omega \times \mathbb{R}^2).$$

At the same time (6.4) still holds, and $\mathcal{P}(\varphi)$ is nonzero; indeed it turns out that $\mathcal{P}(\varphi)$ coincides with two times the area of the smaller circle between Γ_1 and Γ_2 (see [19, 45]).

We now give a short proof of the following interesting observation (see [33, Theorem 1, Section 6.5.1]):

Theorem 6.1. *Let $r > 0$ and $u_\varphi : B_r(0) \to \mathbb{R}^2$ the Cartesian map in* (6.3) *with φ be the double eight curve in* (6.7)*. Then, it holds*

$$\mathcal{A}^{L^1}(u_\varphi, B_r) > \int_{B_r} \sqrt{1 + |\nabla u_\varphi|^2} dx. \tag{6.8}$$

In other words we have found a Cartesian map whose area functional, even if relaxed with respect to the L^1-topology, is strictly greater than the area of its graph.

Proof. Assume by contradition that for some $\overline{r} > 0$ it holds

$$\mathcal{A}^{L^1}(u_\varphi, B_{\overline{r}}) = \int_{B_{\overline{r}}} \sqrt{1 + |\nabla u_\varphi|^2} dx.$$

Let $(u_k) \subset C^1(B_{\overline{r}}; \mathbb{R}^2)$ be a recovery sequence for $\mathcal{A}^{L^1}(u_\varphi, B_{\overline{r}})$ and denote $V_k := \nabla u_k$. We have

$$\limsup_{k\to\infty} \int_{B_{\overline{r}}} \sqrt{1 + |V_k|^2} dx \le \lim_{k\to\infty} \int_{B_{\overline{r}}} \sqrt{1 + |V_k|^2 + |\det\,(\nabla u_k)|^2} = \int_{B_{\overline{r}}} \sqrt{1 + |\nabla u_\varphi|^2} dx$$

and, on the other hand, by lower semicontinuty

$$\liminf_{k\to\infty} \int_{B_{\overline{r}}} \sqrt{1 + |V_k|^2} dx \ge \int_{B_{\overline{r}}} \sqrt{1 + |\nabla u_\varphi|^2} dx.$$

So $\lim_{k\to\infty} \int_{B_{\overline{r}}} \sqrt{1 + |V_k|^2} dx = \int_{B_{\overline{r}}} \sqrt{1 + |\nabla u_\varphi|^2} dx$; hence by Proposition 3.4 we conclude $V_k = \nabla u_k \to \nabla u_\varphi$ strongly in $L^1(B_{\overline{r}})$. But strong convergence of gradients implies strict convergence in $BV(B_{\overline{r}}; \mathbb{R}^2)$, so by (6.4) we arrive at

$$\liminf_{k\to\infty} \mathbb{A}(u_k, B_{\overline{r}}) \ge \mathcal{A}(u_\varphi, B_{\overline{r}}) = \int_{B_{\overline{r}}} \sqrt{1 + |\nabla u_\varphi|^2} dx + \mathcal{P}(\varphi) > \int_{B_{\overline{r}}} \sqrt{1 + |\nabla u_\varphi|^2} dx,$$

a contradiction. □

7 Appendix

We collect here some useful results for the above discussion.

Lemma 7.1. *Let $A \subset \mathbb{R}$ be a bounded open set and let $f_k, f \in L^1(A)$ be non-negative functions such that*

$$\lim_{k\to\infty} \int_A f_k dx = \int_A f dx, \qquad\qquad f(x) = \liminf_{k\to\infty} f_k(x) \qquad a.e.\ x \in A.$$

Then $f_k \to f$ in $L^1(A)$.

Proof. We prove that $\psi_k := f_k - f$ tends to 0 in $L^1(A)$. To this aim, we denote by $\psi_k^+ = \psi_k \vee 0$ and $\psi_k^- = (-\psi_k) \vee 0$ the positive and negative parts of ψ_k, respectively, so that it is enough to show that they both tends to 0 in $L^1(A)$. As for the negative part, we readily see that $\psi_k^- = (f - f_k) \vee 0 \leq f$, and moreover from $f(x) = \liminf_{k\to\infty} f_k(x)$ we deduce that $\limsup_{k\to\infty} f(x) - f_k(x) = 0$, hence $\lim_{k\to\infty} \psi_k^- = 0$ a.e. on A. Therefore, by Dominated Convergence Theorem $\psi_k^- \to 0$ in $L^1(A)$.

This also allows to treat the positive part, since we know that $0 = \lim_{k\to\infty} \int_A \psi_k dx = \lim_{k\to\infty} \int_A \psi_k^+ dx$, which implies $\psi_k^+ \to 0$ in $L^1(A)$. The thesis is achieved. □

The following result can be found in [30]:

Lemma 7.2. *Let $U \subset \mathbb{R}^2$ be a relatively compact set; then for a.e. $t > 0$ the set*

$$\Gamma_t := \{x \in \mathbb{R}^2 : \operatorname{dist}(x, U) = t\},$$

consists of finitely many Lipschitz curve.

Proof. This follows from the fact that for a.e. t the set $U_t := \{x \in \mathbb{R}^2 : \operatorname{dist}(x, U) < t\}$ is an open set with Lipschitz boundary. □

Acknowledgements: The author wishes to thank Domenico Mucci for fruitful exchanges. He is also grateful with Giovanni Bellettini and Simone Carano, with whom the analysis of the topic began. The author is member of the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM), and joins the project CUP E5324001950001. He also acknowledges the financial support of PRIN 2022PJ9EFL "Geometric Measure Theory: Structure of Singular Measures, Regularity Theory and Applications in the Calculus of Variations". The latter has been funded by the European Union under NextGenerationEU, project number CUP B53D23009400006. Views and opinions expressed are however those of the author(s) only and do not necessarily reflect those of the European Union or The European Research Executive Agency. Neither the European Union nor the granting authority can be held responsible for them.

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