FREE DISCONTINUITY PROBLEMS IN CALCULUS OF VARIATIONS

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In this paper I deal with some results and still open questions which I have pointed out during the Meeting in Paris in J.L. Lions' honour, taking into account also the results of some papers published a short time later.

Recently many authors have emphasized the interest of some *free discontinuity problems* both in Information Theory (see [23]) and in Mathematical Physics (see [7], [10], [16], [25]).

We would regard as *free discontinuity problems* in a given open set $\Omega \subseteq \mathbb{R}^n$ those problems for which the solutions (as in the next Theorem 1) are pairs (K, u), where K is a closed set and $u \in C^1(\Omega \backslash K)$. Whenever K is an essential boundary (or union of essential boundaries) we retain, following the literature (e.g. [2], [5]), the term *free boundary problems*. We instead regard as *minimal boundary problems* (following [14], [1], [24]) those variational problems where for the admissible pairs (K, u) is required that u be constant in each connected component of $\Omega \backslash K$.

A result on the existence of a solution of a free discontinuity problem is the following theorem (proved in [13]).

Theorem 1. *Let $n \in \mathbb{N}$, $n \geq 2$, and let $\Omega \subseteq \mathbb{R}^n$ be an open set, $1 \leq q < +\infty$, $0 < \lambda < +\infty$, $0 < \mu < +\infty$, $g \in L^q(\Omega) \cap L^\infty(\Omega)$, then there exists at least one pair (K, u) minimizing the functional G defined for every closed set $K \subset \mathbb{R}^n$ and for every $u \in C^1(\Omega \backslash K)$ by*

$$G(K, u) = \int_{\Omega \backslash K} |\nabla u|^2 dx + \mu \int_{\Omega \backslash K} |u - g|^q dx + \lambda \mathcal{H}_{n-1}(K \cap \Omega), \tag{1}$$

where $\mathcal{H}_{n-1}$ is the $(n-1)$-dimensional Hausdorff measure.

The functional (1) is a simple example of the functionals which occur in free discontinuity problems ; this problem is different from a free boundary problem because the set K considered in Theorem 1 is not necessarily an essential boundary or union of essential boundaries.

It is reasonable to foresee that the qualitative results established in studying the functional (1) may be later extended to the study of many other free discontinuity problems, of which we will give some examples further on.

With regard to the functional (1), we point out a first regularity property which immediately follows from the conclusion of [13].

Remark 1. Let (K, u) be a minimizing pair of the functional (1). Then

(i) $K \cap \Omega$ is $(\mathcal{H}_{n-1}, n-1)$ rectifiable, i.e. (as in [17]) there exists a sequence of C^1 surfaces $\{S_h\}$ such that

$$\mathcal{H}_{n-1}(K \backslash (\bigcup_h S_h)) = 0 \ ;$$

(ii) there exists a minimizing pair (K', u') such that $K' \subseteq K$, $\mathcal{H}_{n-1}(K \setminus K') = 0$, $\overline{K' \cap \Omega} = K'$, $u = u'$ in $\Omega \setminus K$ and, for every $x \in K' \cap \Omega$,

$$\liminf_{\rho \to 0} \rho^{1-n} \mathcal{H}_{n-1}(K' \cap B_\rho(x)) > 0.$$

We note that, given (K, u), the pair (K', u') is uniquely determined by the conditions (ii).

For further regularity results concerning the pairs (K, u) minimizing the functional (1), it seems interesting to prove (or disprove) the following conjectures.

Conjecture 1. If (K, u) is a minimizing pair of the functional (1), then there exists p such that $2 < p < 4$ and

$$\int_{\Omega' \setminus K} |\nabla u|^p \, dx < +\infty$$

for every open set $\Omega' \subset\subset \Omega$ (i.e. $\bar{\Omega}'$ compact and $\bar{\Omega}' \subset \Omega$).

Conjecture 2. If (K, u) is a minimizing pair of the functional (1) and if we set

$$E(x) = \cap \{C ; C \subseteq \mathbb{R} \text{ closed set}, \lim_{y \to x} \operatorname{dist}(u(y), C) = 0\},$$

then, for every $x \in \Omega$, $E(x)$ has a finite cardinality $\alpha(x) \leq n + 1$.

Conjecture 3. Under the assumptions of conjecture 2, set $S = \{x \in \Omega \; ; \; \alpha(x) = 2\}$, then

$$\mathcal{H}_{n-1}((K \cap \Omega) \setminus S) = 0.$$

Remark 2. If conjecture 3 is true, by remark 1, the set K' given by (ii) satisfies also the condition $K' = \bar{S}$.

Conjecture 4. If, under the assumptions of conjecture 2, $x \in K \cap \Omega$, $\alpha(x) = 1$ and $E(x) = \{\eta\}$, then for almost all $y \in \mathbb{R}^n$ the limit

$$\lim_{\rho \to 0} \rho^{-1/2}(u(x + \rho y) - \eta)$$

exists and is finite (see [23], [9]).

Conjecture 5. If $2 \leq n \leq 7$ the set S considered in the conjecture 3 is a C^1 surface, whereas, if $n \geq 8$, the set S is a C^1 surface up to a singular set of Hausdorff dimension not exceeding $n - 8$ (see [18]).

Conjecture 6. In the hypotheses of conjectures 2, 3, if the minimizing pair (K, u) satisfies the condition $K = \bar{S}$, then $\mathcal{H}_{n-2}((K \setminus S) \cap \Omega') < +\infty$ for every open set $\Omega' \subset\subset \Omega$.

Beside these conjectures, which must be proved (or disproved) under the assumptions of Theorem 1, we may consider a strengthening of Theorem 1 itself whenever some regularity conditions on the open set Ω are required.

For example, if in addition to all the hypotheses of Theorem 1 we assume Ω to be bounded and the existence of a closed set M such that $\mathcal{H}_{n-1}(M) = 0$ and $\partial\Omega \setminus M$ is a C^1 surface, then the previous conjectures 2-3-6 must be modified as follows.

Conjecture 2*. For every $x \in \bar{\Omega}\backslash M$, $E(x)$ has a finite cardinality $\alpha(x)$.

Conjecture 3*. Set $S = \{x \in \Omega \; ; \; \alpha(x) = 2\}$, if $K = \bar{S}$ then $\mathcal{H}_{n-1}(K\backslash S) = 0$.

Conjecture 6*. If the minimizing pair (K, u) satisfies the condition $K = \bar{S}$ and moreover $\mathcal{H}_{n-2}(M) < +\infty$, then $\mathcal{H}_{n-2}(K\backslash S) < +\infty$.

We remark that the proof of all these conjectures should be very interesting even in the case $n = 2$; among other things this would provide a positive answer to a problem posed by D. Mumford and J. Shah (see [23], §1).

In addition to Theorem 1, which can be seen as an existence theorem for a Neumann type problem, we can also look for some existence theorems for Dirichlet type problems. An example of such results is given by the following theorem (proved in [8]).

Theorem 2. *Let $n \in \mathbb{N}$, $n \geq 2$, and let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set with $\mathcal{H}_{n-1}(\partial\Omega) < +\infty$; assume that a closed set M exists such that $\mathcal{H}_{n-1}(M) = 0$ and $\partial\Omega\backslash M$ is a C^1 surface ; let $w \in C^1(\partial\Omega\backslash M) \cap L^\infty(\partial\Omega\backslash M)$ and $0 < \lambda < +\infty$. Then there exists at least one pair (K, u) minimizing the functional G defined for every closed set $K \subset \bar{\Omega}$ and for every $u \in C^1(\Omega\backslash K) \cap C^0(\bar{\Omega}\backslash(M \cup K))$ with $u = w$ on $\partial\Omega\backslash(M \cup K)$ by*

$$G(K, u) = \int_{\Omega\backslash K} |\nabla u|^2 dx + \lambda \mathcal{H}_{n-1}(K). \tag{2}$$

Naturally also for the minimizing pairs given by Theorem 2 we can study the regularity questions (in the interior and on the boundary) similar to the ones before quoted for the minimizing pairs given by Theorem 1.

For the behaviour on the boundary, different situations should occur in the two cases, and it is hard to compare *a priori* their difficulties.

As concerns the regularity results in the interior of Ω, we can formulate again Conjectures 1-6, and we can hope that the lack of the term $\int_{\Omega\backslash K} |u - g|^q dx$ in the functional (2) makes this investigation slightly easier than the one for the functional (1).

We may study also mixed problems having as particular cases the problems considered in the Theorems 1 and 2.

Together with the previous regularity problems, also with regard only to the existence of solutions, many interesting problems are still open whenever one weakens the condition $g \in L^\infty(\Omega)$ in Theorem 1. For example one could begin by asking if Theorem 1 is still true under the assumption $g \in L^q(\Omega) \cap L^\infty_{\text{loc}}(\Omega)$ or, more generally, $g \in L^q(\Omega) \cap L^r_{\text{loc}}(\Omega)$ with $nq \leq r \leq +\infty$.

Beside the problems just posed we may consider many other free discontinuity problems.

For instance the problems posed in Theorems 1 and 2 for scalar functions u may be studied for vector functions u under suitable constraints. In particular we may formulate the following conjecture related to the theory of liquid crystals (see [7], [10], [16], [25]).

Conjecture 7. There exists the minimum of the functional (1) when $u \in [C^1(\Omega\backslash K)]^n$ and $|u| \equiv 1$.

Moreover we may consider also functionals where higher order derivatives appear ; for example we may formulate the following conjecture.

Conjecture 8. Given $g \in L^2(\Omega) \cap L^\infty(\Omega)$, there exist minimizing pairs for the functionals G_1 and G_2 defined for every closed set $K \subseteq \mathbb{R}^N$ and for every $u \in C^2(\Omega \backslash K)$ respectively by

$$G_1(K,u) = \int_{\Omega\backslash K} |\Delta u|^2 dx + \int_{\Omega\backslash K} |u-g|^2 dx + \mathcal{H}_{n-1}(K \cap \Omega),$$
$$G_2(K,u) = \int_{\Omega\backslash K} |\Delta u|^2 dx + \int_{\Omega\backslash K} |u-g|^2 dx + \mathcal{H}_{n-2}(K \cap \Omega).$$

We remark that the free discontinuity problems should be regarded as a possible schematization of problems in mathematical physics where both volume forces and surface tensions are present (see [10), [16], [25]).

To obtain plausible schematizations we must enlarge considerably the class of the functionals considered before.

As a first step, we may consider functionals more general than (1), i.e. functionals of the kind

$$\int_{\Omega\backslash K} f(x,u,\nabla u)dx + \int_{K\cap\Omega} \phi(x) d\mathcal{H}_{n-1}, \tag{3}$$

where $K \subset \mathbb{R}^n$ is a closed set and $u \in [C^1(\Omega\backslash K)]^m$, $m \in \mathbb{N}$. But this enlargement still seems to be inadequate for studying many problems in mathematical physics which appear to require further generalizations.

To this aim, we must recall the notion of approximate limits and of traces (see [12] and also [17]).

For $n,m \in \mathbb{N}, \Omega \subseteq \mathbb{R}^n$ open set, let $u : \Omega \to \mathbb{R}^m$ be a Borel function ; for $x \in \Omega$, $z \in \tilde{\mathbb{R}}^m = \mathbb{R}^m \cup \{\infty\}$ we set $z = \operatorname{aplim}_{y\to x} u(y)$ (*approximate limit* of u at x) if

$$g(z) = \lim_{\rho\to 0} |B_\rho|^{-1} \int_{B_\rho} g(u(x+y))dy$$

for every $g \in C^0(\tilde{\mathbb{R}}^m)$, where $B_\rho = \{y \in \mathbb{R}^n; |y| < \rho\}$.

For $x \in \Omega, z \in \tilde{\mathbb{R}}^m, v \in \partial B_1$, we set $z = tr^+(x,u,v)$ (*exterior trace* of the function u at x in the direction v), if for every $g \in C^0(\tilde{\mathbb{R}}^m)$

$$g(z) = \lim_{\rho\to 0} |B_\rho|^{-1} \int_{B_\rho \cap \{y; <v,y>>0\}} g(u(x+y))dy$$

where $< v,y >$ denotes the scalar product in $\mathbb{R}^n$. We define the *interior trace* as $tr^-(x,u,v) = tr^+(x,u,-v)$.

We remark that if $\operatorname{aplim}_{y\to x} u(y)$ exists, then

$$tr^+(x,u,v) = tr^-(x,u,-v) = \operatorname{aplim}_{y\to x} u(y) \qquad \text{for every } v \in \partial B_1.$$

On the other hand, if for $x \in \Omega$ and $v \in \partial B_1$ there exist $tr^+(x,u,v)$ and $tr^-(x,u,-v)$ and they are equal, then u has approximate limit at x. Moreover, if $\operatorname{aplim}_{y\to x} u(y)$ does not exist and if for $v,v' \in \partial B_1$ there exist $tr^\pm(x,u,v)$ and $tr^\pm(x,u,v')$, then necessarily $v = \pm v'$.

Now let $f : \Omega \times \mathbb{R}^m \times \mathbb{R}^{nm} \to [0,+\infty]$ and $\phi : \Omega \times \partial B_1 \times \tilde{\mathbb{R}}^m \times \tilde{\mathbb{R}}^m \to [0,+\infty]$ be Borel functions with

$$\begin{cases} \phi(x,v,a,b) = \phi(x,-v,b,a) & \text{for every } x \in \Omega, v \in \partial B_1, a \in \tilde{\mathbb{R}}^m, b \in \tilde{\mathbb{R}}^m, \\ \phi(x,v,a,a) = \phi(x,v',a,a) & \text{for every } x \in \Omega, v \in \partial B_1, v' \in \partial B_1, a \in \tilde{\mathbb{R}}^m. \end{cases} \tag{4}$$

We can consider the functional

$$\int_{\Omega\setminus K} f(x,u,\nabla u)dx + \int_{K\cap\Omega} \phi(x,\upsilon,u^+,u^-)d\mathcal{H}_{n-1} \tag{5}$$

defined for every pair (K,u) satisfying the following conditions :

(i) $K \subset \mathbb{R}^n$ is closed,

(ii) $u \in [C^1(\Omega\setminus K)]^m$,

(iii) for $\mathcal{H}_{n-1}$ almost every $x \in K$ there exist υ, u^+ and u^- with $u^+ = tr^+(x,u,\upsilon)$ and $u^- = tr^-(x,u,\upsilon)$.

The conditions (4) ensure that the functional (5) is independent of the choice of the normal υ.

For some results related to the study of functionals like (5) we refer to [3], [4].

We note that beside the functionnals of the type (5), we may consider the functionals of the kind

$$\int_{\Omega\setminus K} f(x,u)dx + \int_{K\cap\Omega} \phi(x,\upsilon,u^+,u^-)d\mathcal{H}_{n-1} \tag{6}$$

defined for every closed set $K \subset \mathbb{R}^n$ and for every $u \in [C^1(\Omega\setminus K)]^m$ with $\nabla u \equiv 0$ in $\Omega\setminus K$.

The study of these functionals brings into the consideration *minimal boundary type problems* which, even if the methods and the results are different, show a sort of "parallelism" with the free discontinuity problems of the kind (5) (see [15]). For example a result "parallel" to Theorem 1 is the following theorem (proved in [11]).

Theorem 3. *Let $n \in \mathbb{N}$, $n \geq 2$, and let $\Omega \subseteq \mathbb{R}^n$ be an open set, $1 \leq q < +\infty$, $0 < \lambda < +\infty$, $g \in L^q(\Omega) \cap L^\infty(\Omega)$, then there exists at least one pair (K,u) minimizing the functional*

$$\int_{\Omega\setminus K} |u-g|^q dx + \lambda\mathcal{H}_{n-1}(K\cap\Omega) \tag{7}$$

defined for every closed set $K \subset \mathbb{R}^n$ and for every $u \in C^1(\Omega\setminus K)$ with $\nabla u \equiv 0$ in $\Omega\setminus K$.

We can consider many conjectures "parallel" to the conjectures related to Theorem 1 ; for example a conjecture "parallel" to the conjecture 2 is the following.

Conjecture 9. If (K,u) is a minimizing pair of the functional (7), then for every $x \in \Omega$ there exists an open set $\Omega' \subset\subset \Omega$ such that $x \in \Omega'$ and in $\Omega'\setminus K$ the function u takes only a finite number of values.

Remark 3. For $n = 2$, under regularity conditions for the open set Ω and the assumption of continuity in $\bar{\Omega}$ for g, the functional (7) has been widely studied in [23] and [22].

We observe that beside the still open qualitative questions about the regularity of the solutions for the free discontinuity problems, there is also open the problem of finding effective numerical methods suitable for the determination of the solutions given in Theorems 1 and 2, at least in the case $n = 2$. To this end a useful attempt could be made to look for a minimizer in classes of pairs (K,u) which already satisfy the conjectures 1-6 or 2*-3*-6* and possibly some of the minimum conditions (necessary for regular enough minimizers) pointed out in the papers [23], [9], [6].

We must warn that the previous numerical questions seem to be very difficult also because of the lack of uniqueness for the problems considered in the present paper. It seems indeed interesting to look for some reasonable hypotheses which, added to those of Theorems 1 or 2, allow to obtain a uniqueness result too.

Beside this brief description of results and conjectures about the free discontinuity problems, we now give a very rapid sketch of the way pursued in [13] for proving Theorem 1. Probably this method can be extended to similar problems.

At first we have joined to the functional (1) a new functional $\bar{G}$ defined on a suitable class of special bounded variation functions (the class $SBV_{\text{loc}}(\Omega)$) and we have proved the existence of at least one minimizer in $SBV_{\text{loc}}(\Omega)$ for $\bar{G}$; then we have proved that a minimizer of $\bar{G}$ has some degree of regularity and finally that it gives a pair minimizing the functional G.

In order to define the class $SBV_{\text{loc}}(\Omega)$, we begin by recalling some definitions about the bounded variation functions (see for instance [19], [20]).

For every $u \in L^1_{\text{loc}}(\Omega)$ we define (see [21])

$$\int_\Omega |Du| = \sup\Big\{\int_\Omega u \text{ div } \phi \, dx \ ; \ \phi \in [C^1_0(\Omega)]^n, \ |\phi| \le 1\Big\}.$$

We say that u belongs to $BV(\Omega)$ iff $u \in L^1(\Omega)$ and $\int_\Omega |Du| < +\infty$; this is equivalent to requiring that $u \in L^1(\Omega)$ and that its distributional derivative Du be a bounded vector measure (see e.g.[21]).

If $u \in BV(\Omega)$, then there exists a.e. in Ω its approximate gradient ∇u, which coincides with a Radon-Nikodym derivative of Du with respect to Lebesgue measure, and

$$\int_\Omega |\nabla u| dx = \inf\Big\{\int_{\Omega\setminus K} |Du| \ : \ K \subset \Omega \text{ compact, meas}(K) = 0\Big\}.$$

We say that a function u belongs to $SBV(\Omega)$ iff $u \in BV(\Omega)$ and

$$\int_\Omega |\nabla u| dx = \inf\Big\{\int_{\Omega\setminus K} |Du| \ : \ K \subset \Omega \text{ compact, } \mathcal{H}_{n-1}(K) < +\infty\Big\}.$$

By $SBV_{\text{loc}}(\Omega)$ we denote the space of all functions which belong to $SBV(\Omega')$ for every open set $\Omega' \subset\subset \Omega$.

By a semicontinuity theorem of L. Ambrosio (see [4]) we prove the existence of a minimizer w for the functional $\bar{G}$, defined for every $u \in SBV_{\text{loc}}(\Omega)$ by

$$\bar{G}(u) = \int_\Omega |\nabla u|^2 dx + \mu \int_\Omega |u-g|^q dx + \lambda H_{n-1}(S_u \cap \Omega)$$

where $S_u = \big\{x \in \Omega \ ; \ \operatorname*{aplim}_{t\to x} u(y) \text{ does not exist } \big\}$.

Then we state some regularity properties for a minimizer w of $\bar{G}$. Namely, if for every $x \in \Omega\setminus S_w$ we set $\tilde{w}(x) = \operatorname*{aplim}_{y\to x} w(y)$, then

$$\tilde{w} \in C^1(\Omega\setminus\bar{S}_w) \text{ and } \mathcal{H}_{n-1}((\bar{S}_w \cap \Omega\setminus S_w) = 0.$$

At this point we infer that $(\bar{S}, \tilde{w})$ gives the minimum of G and

$$G(\bar{S}_w, \tilde{w}) = \bar{G}(w).$$

Thus Theorem 1 is achieved.

Acknowledgement. I whish to thank M. Carriero and A. Leaci for their useful collaboration in the preparation of this paper.

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