NECESSARY AND SUFFICIENT CONDITIONS FOR THE MAZ'YA–SHAPOSHNIKOVA FORMULA IN (FRACTIONAL) SOBOLEV SPACES

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Abstract. We investigate the asymptotic behavior, as $\varepsilon \to 0$, of nonlocal functionals

$$\mathcal{F}_\varepsilon(u) = \iint_{\mathbb{R}^N \times \mathbb{R}^N} \rho_\varepsilon(y-x)\,|u(x)-u(y)|^p \,\mathrm{d}x\,\mathrm{d}y, \quad u \in L^p(\mathbb{R}^N), \quad 1 \leqslant p < \infty,$$

associated with a general family of non-negative measurable kernels $\{\rho_\varepsilon\}_{\varepsilon>0}$. Our primary aim is to single out the weakest moment-type assumptions on the family of kernels $\{\rho_\varepsilon\}_{\varepsilon>0}$ that are necessary and sufficient for the pointwise convergence

$$\lim_{\varepsilon\to 0} \mathcal{F}_\varepsilon(u) = 2\|u\|_{L^p}^p$$

to hold for every u in a prescribed subspace of $L^p(\mathbb{R}^N)$. In the canonical smooth regime of compactly supported functions ($u \in C_c^\infty(\mathbb{R}^N)$), we show that convergence occurs when two optimal conditions are satisfied: (i) a mass escape condition, and (ii) a short-range attenuation effect, expressed by the vanishing as $\varepsilon \to 0$ of the kernels p-moments in any fixed neighborhood of the origin. This general framework recovers the classical Maz'ya–Shaposhnikova theorem for fractional-type kernels and extends the convergence result to a much broader class of interaction profiles, which may be non-symmetric and non-homogeneous. Furthermore, using a density argument that preserves the moment assumptions, we prove that the same necessary and sufficient conditions remain valid in the integer-order Sobolev setting ($u \in W^{1,p}(\mathbb{R}^N)$). Finally, by adapting the method to fractional Sobolev spaces $W^{s,p}(\mathbb{R}^N)$ with $s \in (0,1)$, we recover the Maz'ya–Shaposhnikova formula and extend it under analogous abstract conditions on the family of kernels $\{\rho_\varepsilon\}_{\varepsilon>0}$.

1. Introduction

Given a family of non-negative kernels $\{\rho_\varepsilon\}_{\varepsilon>0}$, where $\rho_\varepsilon : \mathbb{R}^N \to [0,+\infty)$, we consider the associated *nonlocal Dirichlet energies* by

$$\mathcal{F}_\varepsilon(u) := \int_{\mathbb{R}^N \times \mathbb{R}^N} \rho_\varepsilon(y-x)\,|u(x)-u(y)|^p \,\mathrm{d}x\,\mathrm{d}y, \quad u \in L^p(\mathbb{R}^N),$$

for $1 \leqslant p < \infty$ and $N \in \mathbb{N}$. These functionals arise in a wide range of applications, including Lévy-type stochastic processes, peridynamic models in mechanics, image processing, and the study of fractional Sobolev norms.

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In their seminal work [4], Bourgain, Brezis, and Mironescu showed that for the family of *fractional kernels*

$$\rho_\varepsilon(z) = (1-\varepsilon)|z|^{-(N+\varepsilon p)}, \qquad \varepsilon \in (0,1), \tag{1.1}$$

there exists $K_{p,N} > 0$ such that

$$\lim_{\varepsilon\to 1^-} \mathcal{F}_\varepsilon(u) = K_{p,N}\|\nabla u\|^p_{L^p}, \tag{1.2}$$

for every $u \in W^{1,p}(\mathbb{R}^N)$. One year later, Maz'ya and Shaposhnikova in [41], completed the picture by proving that for the family of fractional kernels

$$\rho_\varepsilon(z) = \frac{\varepsilon p}{|\mathbb{S}^{N-1}|}|z|^{-(N+\varepsilon p)}, \qquad \varepsilon \in (0,1), \tag{1.3}$$

one has

$$\lim_{\varepsilon\to 0} \mathcal{F}_\varepsilon(u) = 2\|u\|^p_{L^p}, \tag{1.4}$$

for every $u \in W^{s,p}(\mathbb{R}^N)$, with $s \in (0,1)$. The limit identity (1.4) is now widely known as the *Maz'ya–Shaposhnikova (MS) formula.*

Various extensions of the BBM and MS formulas in Euclidean spaces have been developed, yielding a large and diverse literature. We do not attempt to provide an exhaustive survey; instead, we highlight contributions most closely related to [4, 41].

A Γ-convergence result that interprets Sobolev and BV norms as limits of nonlocal integral functionals was established in [46] (see also [16]). Integral and relaxation characterizations of Sobolev and BV spaces, together with lower semicontinuity results for families of nonlocal functionals converging to Sobolev norms were developed in [37, 38]. The asymptotic behavior of the fractional s-perimeter in the regime $s \searrow 0$ is analyzed in [23], where conditions and examples are provided that clarify when the limit exists and how it relates to volume-type quantities. A sustained line of work has extended BBM/MS-type asymptotics across the full scale of fractional Sobolev spaces $W^{s,p}$, carefully treating both the $s \nearrow 1$ (BBM-type) and the $s \searrow 0$ (MS-type) regimes and addressing endpoint and integrability subtleties; representative works in this direction include [7, 6, 8, 9, 11, 10, 5]. Fourier-analytic and function-space methods (notably in Triebel–Lizorkin and related endpoint spaces) offer alternative — and in some cases sharper — descriptions of fractional norms and delicate endpoint limits; see for instance [11, 10, 5]. Magnetic variants of the BBM/MS formulas have also been developed: several results demonstrate that magnetically weighted Gagliardo-type seminorms converge to the appropriate local magnetic energies in the corresponding limits [44, 43, 45]. Within Micromagnetics, a BBM-type formula incorporating nonlocal antisymmetric exchange interactions (Dzyaloshinskii–Moriya interaction, DMI) has been established in [17] (see also [20] for further results concerning physically relevant kernels). Other generalizations focus on geometric and growth flexibility. BBM/MS-type formulas have been proved in sub-Riemannian and Carnot-type settings and in various metric-measure frameworks by exploiting homogeneous structures, tangent analysis, or volume-growth and Bishop–Gromov–type controls; Orlicz–Sobolev extensions that accommodate non-power growth have also been investigated [13, 15, 27, 32]. Further directions encompass vector- and matrix-valued nonlocal functionals with applications to elasticity and homogenization [39, 35], refined asymptotic expansions for special kernel classes such as radial kernels [26, 25], and a BBM-type representation for functions of bounded deformation (BD) that links nonlocal energies to symmetric gradients [3].

Over the past three decades, significant progress has been made in first-order analysis on metric measure spaces, including the development of first-order Sobolev spaces, functions of bounded variation, and their connections with variational problems and partial differential equations; see, for instance, [1] and the references therein. In [12, Remark 6], Brezis raised a question concerning the relationship between the BBM formula and Sobolev spaces in the setting of metric measure spaces. This issue was subsequently addressed in [21], where new characterizations of Sobolev and BV spaces in PI spaces were provided, inspired by the BBM framework; see also [34, 33]. Related results had previously been obtained in Ahlfors-regular spaces [42], and further investigations were carried out in certain PI spaces locally resembling Euclidean spaces [30]. More recently, [31] showed that the essential assumptions for obtaining (1.4) and (1.2) are Rademacher's theorem and volume growth at infinity. Finally, [32] established a surprising link between the MS formula and the generalized Bishop–Gromov inequality in the framework of metric measure spaces.

Despite the central role of the power-law kernels (1.1) and (1.3), many contemporary applications demand *greater flexibility* in the choice of interaction kernels — for instance, compactly supported or anisotropic kernels, or kernels exhibiting singular behavior different from the classical power-law regime. Motivated by these needs, several works have investigated broader kernel classes and identified conditions under which BBM/MS-type limits continue to hold; see, in particular, the complete characterization of kernels for which (1.2) is valid in [18] (for $p = 2$) and in [28] (for the remaining cases), as well as the radial-kernel analyses in [25, 26].

This naturally raises the following question.

Kernel generality. *For a given subspace $X^p(\mathbb{R}^N) \subseteq L^p(\mathbb{R}^N)$, which families of kernels $\{\rho_\varepsilon\}_{\varepsilon>0}$ ensure that (1.4) holds for every $u \in X^p(\mathbb{R}^N)$?*

In this paper, we offer a systematic and sharp answer to that question. Our main contributions can be summarized as follows.

- **Exact kernel criterion in the smooth setting (Theorem 2.1).** We show that for arbitrary non-negative, measurable kernels ρ_ε, the convergence $\mathcal{F}_\varepsilon(u) \to 2\,\|u\|_{L^p}^p$ for all $u \in C_c^\infty(\mathbb{R}^N)$ is governed by two simple moment-type conditions on the kernels: a mass-escape condition and a short-range attenuation effect, expressed through the vanishing of suitably rescaled p-moments near the origin (see (2.4) or (2.5)). This formulation both unifies and extends the classical fractional-kernel case.

- **Extension to $W^{1,p}(\mathbb{R}^N)$ (Corollary 2.8).** Remarkably, using a density argument that preserves the moment hypotheses, we show that the very same kernel conditions are necessary and sufficient for the convergence $\mathcal{F}_\varepsilon(u) \to 2\,\|u\|_{L^p}^p$ on the whole Sobolev space $W^{1,p}(\mathbb{R}^N)$. In other words, passing from compactly supported smooth functions to the Sobolev class $W^{1,p}(\mathbb{R}^N)$ does not require any stronger assumptions on the kernels.

- **Fractional Sobolev regime (Theorem 3.1).** By suitably adapting our moment assumptions to the weaker $W^{s,p}(\mathbb{R}^N)$ framework with $s \in (0,1)$, we recover the Maz'ya–Shaposhnikova formula for a broad class of kernels that includes the classical fractional family. This yields a genuine extension of the fractional MS result. Furthermore, Theorem 3.7 shows that, under an additional hypothesis on the admissible kernels $\{\rho_\varepsilon\}_{\varepsilon>0}$, the conditions of Theorem 3.1 are not merely sufficient but also necessary for a Maz'ya–Shaposhnikova–type identity.

Beyond these main results, we develop several tools of independent interest and a detailed discussion of the limitations arising in the fractional case, which highlights an open problem about the necessity of our kernel assumptions (see Remark 3.3).

Applications. Although our study is purely analytic, the two kernel properties we single out are tailored to capture several mechanisms relevant in applications. In peridynamics and nonlocal elasticity, one often encounters compactly supported finite-horizon kernels whose effective interaction length may vary with a parameter; such kernels are covered by the mass–escape condition when their support drifts to infinity (see, e.g., [24, 22, 47]. In probability and materials models, long-tailed (Lévy-type) kernels are prototypical examples of mass concentration at infinity and satisfy our hypotheses as in the classical fractional case [2]. Finally, in image processing, anisotropic or non-homogeneous kernels are vital for tasks like image denoising [14, 29, 40].

Structure of the paper. Section 2 states our principal results and places them in the context of the existing literature. There, we treat convergence both in the smooth setting and in integer-order Sobolev spaces, proving Theorem 2.1 and Corollary 2.8. Section 3 then addresses the fractional Sobolev case: in Theorem 3.1 we provide a sufficient condition for the validity of the Maz'ya–Shaposhnikova formula, while in Theorem 3.7 we prove that, under an additional assumption on the admissible kernels $\{\rho_\varepsilon\}_{\varepsilon>0}$, these conditions are not only sufficient but also necessary for a Maz'ya–Shaposhnikova-type formula to hold.

2. The MS formula in the smooth and integer-order Sobolev settings

Let $p \in [1,\infty)$ and $N \in \mathbb{N}$. Consider a family of nonnegative measurable kernels $\{\rho_\varepsilon : \mathbb{R}^N \to [0,\infty)\}_{\varepsilon>0}$. For every $\varepsilon > 0$ and every measurable function u, we define the associated nonlocal energy functional

$$\mathcal{F}_\varepsilon(u) := \int_{\mathbb{R}^N\times\mathbb{R}^N} \rho_\varepsilon(y-x)\,|u(x)-u(y)|^p \mathrm{d}x\,\mathrm{d}y, \tag{2.1}$$

with the convention that the above energy takes infinite values whenever u does not belong to its domain:

$$\mathcal{X}^p_\varepsilon(\mathbb{R}^N) := \{u \in L^p(\mathbb{R}^N) \,:\, \mathcal{F}_\varepsilon(u) < +\infty\}. \tag{2.2}$$

Given $\varepsilon_0 > 0$, we further introduce the function space:

$$X^p_{\varepsilon_0}(\mathbb{R}^N) := \bigcap_{\varepsilon \leqslant \varepsilon_0} \mathcal{X}^p_\varepsilon(\mathbb{R}^N). \tag{2.3}$$

Observe that, without further assumptions on the kernels ρ_ε, the spaces $\mathcal{X}^p_\varepsilon(\mathbb{R}^N)$ need not be nested as ε varies, whereas by construction the intersections $X^p_{\varepsilon_0}(\mathbb{R}^N)$ form a decreasing family in ε_0.

Depending on the regularity of the function u, we establish a generalized version of the Maz'ya–Shaposhnikova formula [41].

2.1. The MS formula in the smooth settings. Our first main result concerns functions $u \in C^\infty_c(\mathbb{R}^N)$.

Theorem 2.1. *The following three conditions are equivalent:*

(1) **Uniform moment conditions.** *For every fixed radius $R > 0$, as $\varepsilon \to 0$ the kernels $\{\rho_\varepsilon\}_{\varepsilon>0}$ satisfy*

$$\lim_{\varepsilon\to 0} \int_{|z|>R} \rho_\varepsilon(z)\,\mathrm{d}z = 1 \quad \textit{and} \quad \lim_{\varepsilon\to 0} \int_{|z|<R} |z|^p \rho_\varepsilon(z)\,\mathrm{d}z = 0. \tag{2.4}$$

(2) **Iterated limits conditions.** *The double limits in the order "first $\varepsilon \to 0$, then $R \to \infty$" satisfy*

$$\lim_{R\to\infty}\lim_{\varepsilon\to 0} \int_{|z|>R} \rho_\varepsilon(z)\,\mathrm{d}z = 1 \quad \textit{and} \quad \lim_{R\to\infty}\lim_{\varepsilon\to 0} \int_{|z|<R} |z|^p \rho_\varepsilon(z)\,\mathrm{d}z = 0. \tag{2.5}$$

(3) **Maz'ya–Shaposhnikova formula.** *For every smooth, compactly supported function $u \in C^\infty_c(\mathbb{R}^N)$, $\mathcal{F}_\varepsilon(u)$ is well-defined for $\varepsilon > 0$ sufficiently small, and*

$$\lim_{\varepsilon\to 0} \mathcal{F}_\varepsilon(u) = 2\|u\|^p_{L^p(\mathbb{R}^N)}. \tag{2.6}$$

Moreover, for each fixed $R > 0$, there holds

$$\lim_{\varepsilon\to 0} \int_{|z|<R} \int_{\mathbb{R}^N} \rho_\varepsilon(z)|u(x+z) - u(x)|^p \,\mathrm{d}x\,\mathrm{d}z = 0. \tag{2.7}$$

Theorem 2.1 both characterizes and extends the Maz'ya–Shaposhnikova framework: it gives a precise description of those kernels whose associated *nonlocal* Dirichlet energies converge to the *local* L^p-norm in the regime where only long-range interactions remain relevant. The proof is presented in Section 2.2; before that we collect a few remarks and illustrative examples.

Remark 2.2 (Finite Lévy measures). We preliminarily observe that, for sufficiently small ε, each kernel ρ_ε fulfilling the fixed–radius moment conditions (2.4) is a finite Lévy measure, i.e.

$$\int_{\mathbb{R}^N} \min\{1, |z|^p\} \rho_\varepsilon(z)\,\mathrm{d}z \;<\; +\infty.$$

We refer the reader to [26] for a complete overview of the subject. In particular, the first claim in item (3) is automatically satisfied, for instance, by [26, Proposition 3.12]. In fact, by conditions (2.4), there exist $\varepsilon_0 > 0$ and $c_\rho > 0$, such that

$$\int_{\mathbb{R}^N} \min\{1, |z|^p\}\, \rho_\varepsilon(z)\,\mathrm{d}z < c_\rho \quad \text{for all} \quad 0 < \varepsilon \leqslant \varepsilon_0.$$

Let $u \in W^{1,p}(\mathbb{R}^N)$ and $z \in \mathbb{R}^N$. We denote by $\tau_z u(x) := u(x+z)$ the translation of u by z. Standard Sobolev estimates yield, for every $z \in \mathbb{R}^N$,

$$\|\tau_z u - u\|^p_{L^p} \;\leqslant\; 2^p \|u\|^p_{L^p}, \qquad \|\tau_z u - u\|^p_{L^p} \leqslant |z|^p \|\nabla u\|^p_{L^p}.$$

Hence

$$\|\tau_z u - u\|^p_{L^p} \leqslant 2^p\, \min\{1, |z|^p\}\, \|u\|^p_{W^{1,p}}.$$

Integrating this estimate against $\rho_\varepsilon(z)$ entails

$$\mathcal{F}_\varepsilon(u) \leqslant 2^p \|u\|^p_{W^{1,p}} \int_{\mathbb{R}^N} \min\{1, |z|^p\} \rho_\varepsilon(z)\,\mathrm{d}z < c_\rho \|u\|^p_{W^{1,p}}.$$

Thus $u \in \mathcal{X}^p_\varepsilon(\mathbb{R}^N)$ for every $\varepsilon \leqslant \varepsilon_0$, and the inclusion $W^{1,p}(\mathbb{R}^N) \subset X^p_{\varepsilon_0}(\mathbb{R}^N)$ follows.

Remark 2.3 (On the uniform moment conditions). We refer to the two conditions in (2.4), which ensure the validity of the generalized Maz'ya–Shaposhnikova formula in Theorem 2.1, as the *mass-escape condition* and the *short-range attenuation effect*. The contribution of the nonlocal energy (2.2) that yields the MS formula comes from regions far away from the origin. For this reason, in the case of functions with support on the whole space $\mathbb{R}^N$, it is essential to work with kernels that, in the limit, concentrate their mass at infinity. This motivates the first condition in (2.4), referred to as *mass-escape condition*.

On the other hand, the second condition in (2.4), instead, specifies how singular the kernel can be near the origin. It also requires that the short-range contribution vanish in the limit, ensuring that the total mass is concentrated away from the origin—hence the term short-range attenuation. Note, however, that the kernel is not necessarily singular at the origin a priori. For instance, the kernel

$$\rho_\varepsilon(z) := \begin{cases} \varepsilon & \text{if } z \in B_1(0), \\ \dfrac{\varepsilon p}{|\,\mathbb{S}^{N-1}\,|}\,|z|^{-(N+\varepsilon p)} & \text{if } z \in \mathbb{R}^N \setminus B_1(0). \end{cases}$$

satisfies the conditions (2.4) and therefore guarantees the validity of the MS formula. Further examples on kernels are provided in the Remark below.

Remark 2.4 (A few examples of kernels satisfying (2.4)). We collect below some families of kernels complying with (2.4) (equivalently, with (2)).

(1) Canonical examples are the *fractional kernels* $\rho_\varepsilon(z) := \frac{\varepsilon p}{|\mathbb{S}^{N-1}|}\,|z|^{-(N+\varepsilon p)}$, which indeed satisfy for each fixed $R>0$

$$\int_{|z|>R} \rho_\varepsilon(z)\,\mathrm{d}z = R^{-\varepsilon p} \xrightarrow{\varepsilon\to 0} 1$$

and

$$\int_{|z|<R} |z|^p\,\rho_\varepsilon(z)\,\mathrm{d}z = \varepsilon p \frac{R^{p-\varepsilon p}}{p-\varepsilon p} \xrightarrow{\varepsilon\to 0} 0.$$

(2) Another interesting family of examples can be constructed as follows. For any $\phi \in C_c^\infty(\mathbb{R}^N)$, such that $\phi \geqslant 0$, and $\mathrm{supp}(\phi) \subset B(0,1)$, with $\int_{\mathbb{R}^N}\phi = 1$, we define $\rho_\varepsilon(z) := \phi(z - a_\varepsilon e_1)$, where $a_\varepsilon \to \infty$ as $\varepsilon \to 0$ and e_1 is the first element of the canonical basis of $\mathbb{R}^N$. It is straightforward to see that this family of kernels satisfies (2.4).

It is interesting to point out that the compactness of the support of ϕ can be removed in certain cases, for instance if we choose $\phi(x) = (2\pi)^{-N/2}e^{-|x|^2/2}$ and $a_\varepsilon = 1/\varepsilon$. In fact, more generally, the two conditions in (2.4) are satisfied as soon as $\|\phi\|_{L^1(\mathbb{R}^N)} = 1$. To see this, we argue as follows.
We start proving

$$\lim_{\varepsilon\to 0}\int_{|z|<R} |z|^p\,\rho_\varepsilon(z)\,\mathrm{d}z = 0.$$

Clearly,

$$0 \leqslant \int_{|z|<R} |z|^p\,\rho_\varepsilon(z)\,\mathrm{d}z \leqslant R^p \int_{|z|<R} \rho_\varepsilon(z)\,\mathrm{d}z.$$

Thus, it suffices to show that

$$\lim_{\varepsilon\to 0}\int_{|z|<R} \rho_\varepsilon(z)\,\mathrm{d}z = 0.$$

To prove this, set $a_\varepsilon := 1/\varepsilon$ and make the change of variables $y = z - a_\varepsilon e_1$. We obtain

$$\int_{|z|<R} \rho_\varepsilon(z)\,\mathrm{d}z = \int_{|y+a_\varepsilon e_1|<R} \phi(y)\,\mathrm{d}y.$$

For each fixed $y \in \mathbb{R}^N$ we have $|y + a_\varepsilon e_1| \to \infty$ as $\varepsilon \to 0$, so the indicator $\mathbf{1}_{\{|y+a_\varepsilon e_1|<R\}}$ converges pointwise to 0. Because $0 \leqslant \mathbf{1}_{\{|y+a_\varepsilon e_1|<R\}}\phi(y) \leqslant \phi(y)$ and $\phi \in L^1(\mathbb{R}^N)$ with $\int_{\mathbb{R}^N}\phi = 1$, we apply the Dominated Convergence Theorem to get

$$\lim_{\varepsilon\to 0}\int_{|y+a_\varepsilon e_1|<R} \phi(y)\,\mathrm{d}y = 0.$$

Combining this with the previous inequalities, we infer the conclusion.
Let us now prove

$$\lim_{\varepsilon\to 0}\int_{|z|>R} \rho_\varepsilon(z)\,\mathrm{d}z = 1.$$

As before, with the change of variables $y = z - a_\varepsilon e_1$, we get

$$\int_{|z|>R} \rho_\varepsilon(z)\,\mathrm{d}z = \int_{|y+a_\varepsilon e_1|>R} \phi(y)\,\mathrm{d}y.$$

For each fixed y the indicator $\mathbf{1}_{\{|y+a_\varepsilon e_1|>R\}}$ converges pointwise to 1 as $\varepsilon \to 0$. Since $0 \leqslant \mathbf{1}_{\{|y+a_\varepsilon e_1|>R\}}\phi(y) \leqslant \phi(y)$ and ϕ is integrable, the Dominated Convergence Theorem yields

$$\lim_{\varepsilon\to 0} \int_{|y+a_\varepsilon e_1|>R} \phi(y)\,\mathrm{d}y = \int_{\mathbb{R}^N} \phi(y)\,\mathrm{d}y = 1,$$

which proves the claim.

(3) An alternative example with a slightly stronger singularity in the origin is given by:

$$\rho_\varepsilon(z) := \begin{cases} \frac{\varepsilon^2 p^2}{|\mathbb{S}^{N-1}|} \log(1/|z|)|z|^{-(N+\varepsilon p)} & \text{in } B_1(0), \\ \frac{\varepsilon p}{|\mathbb{S}^{N-1}|}\,|z|^{-(N+\varepsilon p)} & \text{in } \mathbb{R}^N \setminus B_1(0). \end{cases}$$

For this example, it is easier to show that conditions (2.5) are satisfied. Indeed, in establishing (2.5) we can always assume that $R > 1$. We argue as follows. If $R > 1$ then

$$\int_{|z|>R} \rho_\varepsilon(z)\,\mathrm{d}z = \varepsilon p \int_R^\infty r^{-\varepsilon p - 1}\,\mathrm{d}r = \frac{1}{R^{\varepsilon p}} \to 1 \quad \text{as } \varepsilon \to 0.$$

This shows that the first condition in (2.5) is satisfied.

We now prove that the second condition in (2.5) is fulfilled. Again, we can assume $R > 1$. We obtain

$$\int_{|z|<R} |z|^p \rho_\varepsilon(z)\,\mathrm{d}z = A_\varepsilon + B_\varepsilon := \int_{1\leqslant|z|<R} |z|^p\rho_\varepsilon(z)\,\mathrm{d}z + \int_{|z|<1} |z|^p\rho_\varepsilon(z)\mathrm{d}z.$$

Passing to polar coordinates, we find

$$A_\varepsilon = \varepsilon p \int_1^R r^{p-\varepsilon p-1}\,\mathrm{d}r = \frac{\varepsilon p}{p - \varepsilon p}(R^{p-\varepsilon p} - 1) \to 0 \text{ as } \varepsilon \to 0$$

$$B_\varepsilon = (\varepsilon p)^2 \int_0^1 \log(1/r)\, r^{p-1-\varepsilon p}\,\mathrm{d}r = \frac{\varepsilon^2}{(1-\varepsilon)^2} \to 0 \text{ as } \varepsilon \to 0,$$

where for the computation of B_ε we simply used that for $\alpha > 0$ a primitive of $r^\alpha \log(r)$ is given by the function $\frac{r^{\alpha+1}}{\alpha+1}\big(\log r - \frac{1}{\alpha+1}\big)$.

2.2. Proof of Theorem 2.1. In the following, for every $R > 0$, we will often use the decomposition

$$\begin{aligned} \mathcal{F}_\varepsilon(u) = &\int_{|z|>R} \rho_\varepsilon(z) \int_{x\in\mathbb{R}^N} |u(x+z) - u(x)|^p \mathrm{d}x\,\mathrm{d}z \\ &+ \int_{|z|<R} \rho_\varepsilon(z) \int_{x\in\mathbb{R}^N} |u(x+z) - u(x)|^p \mathrm{d}x\,\mathrm{d}z \\ =:&\, I_{\varepsilon,R}[u] + II_{\varepsilon,R}[u]. \end{aligned} \tag{2.8}$$

The proof of Theorem 2.1 rests on the following auxiliary result.

Lemma 2.1. *Let $R > 0$. The following statements are equivalent:*

(1)
$$\lim_{\varepsilon\to 0}\int_{|z|<R} |z|^p \rho_\varepsilon(z)\,\mathrm{d}z = 0. \tag{2.9}$$

(2) *For every $u \in C_c^\infty(\mathbb{R}^N)$*
$$\lim_{\varepsilon\to 0} II_{\varepsilon,R}[u] = \lim_{\varepsilon\to 0}\int_{|z|<R}\int_{\mathbb{R}^N} \rho_\varepsilon(z)|u(x+z)-u(x)|^p \mathrm{d}x\,\mathrm{d}z = 0. \tag{2.10}$$

Proof. **(1 ⇒ 2).** For any $u \in C_c^\infty(\mathbb{R}^N)$, classical Sobolev inequalities give
$$\int_{\mathbb{R}^N} |u(x+z)-u(x)|^p \mathrm{d}x \leqslant |z|^p \|\nabla u\|^p_{L^p(\mathbb{R}^N)}.$$
Therefore
$$0 \leqslant II_{\varepsilon,R}[u] \leqslant \|\nabla u\|^p_{L^p(\mathbb{R}^N)} \int_{|z|<R} |z|^p \rho_\varepsilon(z)\,\mathrm{d}z,$$
and the right-hand side vanishes as $\varepsilon \to 0$ by assumption.

(2 ⇒ 1). We begin by decomposing the integral over $\{0 <| z |< R\}$ into $\{0 <| z |< \delta\}$ and $\{\delta <| z |< R\}$, with $\delta \in (0, R)$ to be chosen later. We handle these two regions separately.
We first estimate II_ε in the **small range** $0 < |z| < \delta$. Let $u_1 \in C_c^\infty\left(\mathbb{R}^N\right)$. A first-order Taylor expansion of ∇u_1 at $x \in \mathbb{R}^N$ yields, for each $|z| < \delta$,
$$\left|\int_0^1 \nabla u_1(x+tz)\cdot \frac{z}{|z|}\,\mathrm{d}t\right|^p \geqslant \frac{1}{2^{p-1}}\left|\nabla u_1(x)\cdot\frac{z}{|z|}\right|^p - \delta^p \left\|\nabla^2 u_1\right\|^p_{L^\infty}.$$
It follows that
$$\begin{aligned} II_{\varepsilon,\delta}[u_1] &:= \int_{|z|<\delta} |z|^p\rho_\varepsilon(z)\int_{\mathbb{R}^N}\left|\int_0^1 \nabla u_1(x+tz)\cdot\frac{z}{|z|}\,\mathrm{d}t\right|^p \mathrm{d}x\,\mathrm{d}z \\ &\geqslant \int_{|z|<\delta}|z|^p\rho_\varepsilon(z)\int_{B_{1+\delta}}\left(\frac{1}{2^{p-1}}\left|\nabla u_1(x)\cdot\frac{z}{|z|}\right|^p - \delta^p\|\nabla^2 u_1\|^p_{L^\infty}\right)\mathrm{d}x\,\mathrm{d}z. \end{aligned}$$
Next, we observe that if we choose u_1 to be a radial bump, i.e., $u_1(x) = g(|x|)$ for a suitable function $g \in C_c^\infty(\mathbb{R}_+)$ such that $\operatorname{supp} g \subseteq [0, 1)$, then, $\nabla u_1(x) = g'(|x|)\frac{x}{|x|}$ for every $x \neq 0$ and with a change of variables, we find that
$$II_{\varepsilon,\delta}[u_1] \geqslant \alpha_p(u_1)\int_{|z|<\delta}|z|^p\rho_\varepsilon(z)\,\mathrm{d}z \tag{2.11}$$
with
$$\alpha_p(u_1) := \left(\frac{1}{2^{p-1}}\int_{B_{1+\delta}}\left|g'(|x|)\frac{x_1}{|x|}\right|^p \mathrm{d}x\right) - |B_{1+\delta}|\delta^p\|\nabla^2 u_1\|^p_{L^\infty}. \tag{2.12}$$

In particular, if we choose the radial profile $g(s) = \frac{1}{2}\,s^2$ for $s \leqslant \frac{1}{2}$, then in $B_{1/2}$ we have $g'(|x|)\frac{x_1}{|x|} = x_1$, and therefore one obtains the lower bound

$$\alpha_p(u_1) \geqslant \frac{1}{2^{p-1}} \int_{B_{1/2}} |x_1|^p \mathrm{d}x - |B_{1+\delta}|\delta^p \|\nabla^2 u_1\|^p_{L^\infty},$$

which is strictly positive provided δ is chosen sufficiently small. Combining this positivity with assumption 2 and (2.11), we conclude that

$$\lim_{\varepsilon\to 0} \int_{|z|<\delta} |z|^p \rho_\varepsilon(z)\,\mathrm{d}z = 0. \tag{2.13}$$

We now turn to estimating II_ε in the **intermediate range** $\delta < |z| < R$. We choose $u_2 \in C_c^\infty(\mathbb{R}^N)$ such that $\operatorname{supp} u_2 \subseteq B_{\delta/2}(0)$. We find that

$$\begin{aligned}
II_{\varepsilon,R}[u_2] &\geqslant \int_{\delta<|z|<R} \int_{|x|>\delta/2} \rho_\varepsilon(z)|u_2(x+z) - u_2(x)|^p \mathrm{d}x\,\mathrm{d}z \\
&= \int_{\delta<|z|<R} \int_{|x|>\delta/2} \rho_\varepsilon(z)|u_2(x+z)|^p \mathrm{d}x\,\mathrm{d}z \\
&\geqslant \frac{1}{R^p} \int_{\delta<|z|<R} |z|^p \rho_\varepsilon(z) \int_{|x-z|>\delta/2} |u_2(x)|^p \mathrm{d}x\,\mathrm{d}z \\
&= \frac{1}{R^p} \int_{\delta<|z|<R} |z|^p \rho_\varepsilon(z) \int_{(\mathbb{R}^N\setminus B_{\delta/2}(z))\cap B_{\delta/2}(0)} |u_2(x)|^p \mathrm{d}x\,\mathrm{d}z.
\end{aligned}$$

Observe that for $|z| > \delta$, we have $(\mathbb{R}^N \setminus B_{\delta/2}(z)) \cap B_{\delta/2}(0) \equiv B_{\delta/2}(0)$. Therefore

$$\begin{aligned}
II_{\varepsilon,R}[u_2] &\geqslant \frac{1}{R^p} \int_{\delta<|z|<R} |z|^p \rho_\varepsilon(z) \int_{B_{\delta/2}(0)} |u_2(x)|^p \mathrm{d}x\,\mathrm{d}z \\
&= \frac{\|u_2\|^p_{L^p}}{R^p} \int_{\delta<|z|<R} |z|^p \rho_\varepsilon(z)\,\mathrm{d}z.
\end{aligned}$$

Under assumption 2, taking the limit on both sides as $\varepsilon \to 0$ forces

$$\lim_{\varepsilon\to 0} \int_{\delta<|z|<R} |z|^p \rho_\varepsilon(z)\,\mathrm{d}z = 0. \tag{2.14}$$

The implication ($2 \Rightarrow 1$) now follows by combining (2.13) and (2.14). □

Proof of Theorem 2.1. **(1 ⇒ 2).** Trivial by taking the limits in the prescribed order.

(2 ⇒ 3). First, by monotonicity in $R > 0$, the second condition in equation (2.5) is equivalent to the requirement

$$\lim_{\varepsilon\to 0} \int_{|z|<R} |z|^p \rho_\varepsilon(z)\,\mathrm{d}z = 0 \quad \text{for every } R > 0. \tag{2.15}$$

Let $u \in C_c^\infty(\mathbb{R}^N)$ and decompose the nonlocal energy as in (2.8), i.e., $\mathcal{F}_\varepsilon(u) = I_{\varepsilon,R}[u] + II_{\varepsilon,R}[u]$. By Lemma 2.1, for each fixed $R > 0$, we have that

$$\lim_{\varepsilon\to 0} II_{\varepsilon,R}[u] = \lim_{\varepsilon\to 0} \int_{|z|<R} \int_{\mathbb{R}^N} \rho_\varepsilon(z)|u(x+z) - u(x)|^p \,\mathrm{d}x\,\mathrm{d}z = 0. \tag{2.16}$$

Next, choose R so large that $\operatorname{supp} u \subset B_{R/2}$. Then,

$$\begin{aligned} I_{\varepsilon,R}[u] &= \int_{|z|>R} \rho_\varepsilon(z) \left(\int_{|x|<R/2} |u(x+z) - u(x)|^p \mathrm{d}x + \int_{|x|>R/2} |u(x+z)|^p \mathrm{d}x \right) \mathrm{d}z \\ &= \int_{|z|>R} \rho_\varepsilon(z) \left(\int_{|x|<R/2} |u(x)|^p \mathrm{d}x + \int_{|x|>R/2} |u(x+z)|^p \mathrm{d}x \right) \mathrm{d}z \\ &= 2\, \|u\|_{L^p}^p \int_{|z|>R} \rho_\varepsilon(z)\,\mathrm{d}z, \end{aligned} \tag{2.17}$$

where we used that the shifted support $x+z$ remains outside $B_{R/2}$ when $|z| > R$ and $|x| < R/2$, i.e., $|x+z| \geqslant |z| - |x| \geqslant R/2$.

Combining (2.16) and (2.17), for every $R > 0$ such that $\operatorname{supp} u \subset\subset B_{R/2}$, we get that

$$\lim_{\varepsilon\to 0} \mathcal{F}_\varepsilon(u) = 2\|u\|_{L^p}^p \lim_{\varepsilon\to 0} \int_{|z|>R} \rho_\varepsilon(z)\,\mathrm{d}z.$$

Finally, letting $R \to \infty$ and using the first condition in item 2, yields that

$$\lim_{\varepsilon\to 0} \mathcal{F}_\varepsilon(u) = 2\|u\|_{L^p}^p \lim_{R\to\infty} \lim_{\varepsilon\to 0} \int_{|z|>R} \rho_\varepsilon(z)\,\mathrm{d}z = 2\|u\|_{L^p}^p$$

for every $u \in C_c^\infty(\mathbb{R}^N)$, as claimed. This concludes the proof of the implication (2 $\Rightarrow$ 3).

(3 $\Rightarrow$ 1). Let $u \in C_c^\infty(\mathbb{R}^N)$. By hypothesis, we have

$$\lim_{\varepsilon\to 0} \mathcal{F}_\varepsilon(u) = 2\|u\|_{L^p}^p \tag{2.18}$$

and for every fixed $R > 0$,

$$\lim_{\varepsilon\to 0} II_{\varepsilon,R}[u] = 0. \tag{2.19}$$

According to Lemma 2.1, the previous relation entails the second relation in item 1, i.e., that

$$\lim_{\varepsilon\to 0} \int_{|z|<R} |z|^p \rho_\varepsilon(z)\,\mathrm{d}z = 0 \quad \text{for every } R > 0. \tag{2.20}$$

It remains to show the complementary tail condition

$$\lim_{\varepsilon\to 0} \int_{|z|>R} \rho_\varepsilon(z)\,\mathrm{d}z = 1 \quad \text{for every } R > 0.$$

To this end, fix $R > 0$ and choose $u \in C_c^\infty(\mathbb{R}^N)$ with $\operatorname{supp} u \subset B_{R/2}$. A direct computation—identical to that which produced equation (2.17)—gives

$$\mathcal{F}_\varepsilon(u) = 2\|u\|_{L^p}^p \int_{|z|>R} \rho_\varepsilon(z)\,\mathrm{d}z + II_{\varepsilon,R}[u].$$

Passing to the limit as $\varepsilon \to 0$ and invoking (2.18)-(2.19) shows

$$2\|u\|_{L^p}^p = 2\|u\|_{L^p}^p \lim_{\varepsilon\to 0} \int_{|z|>R} \rho_\varepsilon(z)\,\mathrm{d}z,$$

from which the tail condition follows at once. This completes the proof of Theorem 2.1. □

We conclude this section with three corollaries showing that, under further integrability assumptions on the kernels, condition (2.7) can be dropped.

Corollary 2.5. *Under the same assumptions as in Theorem 2.1, assume in addition that $\int_{\mathbb{R}^N} \rho_\varepsilon(x)\,\mathrm{d}x = 1$ for every $\varepsilon > 0$. Then, the condition (2.7) in Theorem 2.1 is not needed, since it follows automatically from the convergence assumption. Precisely, under the uniform normalization condition*

$$\int_{\mathbb{R}^N} \rho_\varepsilon(x)\,\mathrm{d}x = 1, \quad \forall \varepsilon > 0,$$

the Maz'ya–Shaposhnikova (MS) limit $\lim_{\varepsilon\to 0} \mathcal{F}_\varepsilon(u) = 2\|u\|_{L^p}^p$ for all $u \in C_c^\infty(\mathbb{R}^N)$ implies the fixed-radius moment conditions

$$\lim_{\varepsilon\to 0} \int_{|z|>R} \rho_\varepsilon(z)\,\mathrm{d}z = 1 \quad \textit{and} \quad \lim_{\varepsilon\to 0} \int_{|z|\leqslant R} |z|^p \rho_\varepsilon(z)\,\mathrm{d}z = 0$$

for all $R > 0$.

Proof. It is convenient to introduce some notation. For a function $u \in C_c^\infty(\mathbb{R}^N)$, set

$$A_u(z) := \int_{\mathbb{R}^N} |u(x+z) - u(x)|^p\,\mathrm{d}x, \quad g_u(z) := 2\|u\|_{L^p}^p - A_u(z).$$

Observe that A_u is continuous in z, $0 \leqslant A_u(z) \leqslant 2^p\|u\|_{L^p}^p$, and if $\operatorname{supp} u \subset B(0, R_0)$ for some $R_0 > 0$, then $A_u(z) = 2\|u\|_{L^p}^p$ for $|z| > 2R_0$; hence g_u is continuous, non-negative, and compactly supported.

Step 1: Towards the mass escape condition (2.5). We show that for every fixed $R > 0$,

$$\lim_{\varepsilon\to 0} \int_{|z|\leqslant R} \rho_\varepsilon(z)\,\mathrm{d}z = 0, \qquad \text{or equivalently} \qquad \lim_{\varepsilon\to 0} \int_{|z|>R} \rho_\varepsilon(z)\,\mathrm{d}z = 1.$$

From the MS hypothesis and the normalization $\int_{\mathbb{R}^N} \rho_\varepsilon = 1$ we have, for every fixed $u \in C_c^\infty(\mathbb{R}^N)$,

$$\lim_{\varepsilon\to 0} \int_{\mathbb{R}^N} \rho_\varepsilon(z) A_u(z)\,\mathrm{d}z = 2\|u\|_{L^p}^p \iff \lim_{\varepsilon\to 0} \int_{\mathbb{R}^N} \rho_\varepsilon(z) g_u(z)\,\mathrm{d}z = 0. \tag{2.21}$$

Let $R > 0$ be fixed. Choose numbers $0 < \rho < r$ such that $r > R + \rho$, and set $s := r - \rho$. Note that $s > R$ by construction. Let $u \in C_c^\infty(\mathbb{R}^N)$ be a radial function satisfying

$$\operatorname{supp} u \subset \overline{B(0,r)}, \qquad u \geqslant 0, \qquad u > 0 \text{ on } B(0,r). \tag{2.22}$$

As a first step, we want to prove that g_u is strictly positive on the compact set $\overline{B(0,s)}$, so that g_u attains a strictly positive minimum in there:

$$m_u(s) := \min_{|z|\leqslant s} g_u(z) > 0. \tag{2.23}$$

For that, we analyze the term

$$A_u(z) = \int_{\mathbb{R}^N} |u(x+z) - u(x)|^p \,\mathrm{d}x.$$

By the elementary lemma (if $a, b > 0$ and $p \geqslant 1$ then $|a-b|^p < a^p + b^p$), in order to conclude that $g_u(z)$ is strictly positive on $\overline{B(0,s)}$, it is sufficient to prove that $u(\cdot)$ and $u(\cdot+z)$ are *both strictly positive* on a set of positive Lebesgue measure, namely, as we now show, on $B(0,\rho)$. Indeed, after that, we have

$$\int_{\mathbb{R}^N} |u(x+z) - u(x)|^p \,\mathrm{d}x < \int_{\mathbb{R}^N} u^p(x+z) + u^p(x) \,\mathrm{d}x = 2\|u\|_{L^p}^p \quad \text{for all } |z| \leqslant s,$$

that is $g_u(z) > 0$ for every $|z| \leqslant s$, from which (2.23) follows. Thus, we have to prove that for every $|z| \leqslant s$ both $u(\cdot)$ and $u(\cdot+z)$ are *strictly positive* on $B(0,\rho)$. This is indeed the case. For every $|z| \leqslant s$ and every $|x| < \rho$ we have

$$|x| < r, \quad |x+z| < r.$$

Hence $u(x) > 0$ and $u(x+z) > 0$. Thus, for every $|z| \leqslant s$, the functions $u(\cdot)$ and $u(\cdot+z)$ are *strictly positive* on the open set $B(0,\rho)$ of positive Lebesgue measure.

Applying (2.21) to the test function u in (2.22), and using (2.23), we get that

$$0 = \lim_{\varepsilon\to 0} \int_{\mathbb{R}^N} \rho_\varepsilon(z) g_u(z) \,\mathrm{d}z \geqslant \limsup_{\varepsilon\to 0} \int_{|z|\leqslant s} \rho_\varepsilon(z) g_u(z) \,\mathrm{d}z \geqslant m_u(s) \limsup_{\varepsilon\to 0} \int_{|z|\leqslant s} \rho_\varepsilon(z) \,\mathrm{d}z.$$

Since $m_u(s) > 0$ we deduce

$$\lim_{\varepsilon\to 0} \int_{|z|\leqslant s} \rho_\varepsilon(z) \,\mathrm{d}z = 0.$$

Because $s > R$ and $\rho_\varepsilon \geqslant 0$, we also get

$$0 \leqslant \limsup_{\varepsilon\to 0} \int_{|z|\leqslant R} \rho_\varepsilon(z) \,\mathrm{d}z \leqslant \lim_{\varepsilon\to 0} \int_{|z|\leqslant s} \rho_\varepsilon(z) \,\mathrm{d}z = 0,$$

which proves the desired mass escape

$$\lim_{\varepsilon\to 0} \int_{|z|\leqslant R} \rho_\varepsilon(z) \,\mathrm{d}z = 0, \quad \text{and hence} \quad \lim_{\varepsilon\to 0} \int_{|z|> R} \rho_\varepsilon(z) \,\mathrm{d}z = 1.$$

Step 2: Short-range attenuation effect. Let $R > 0$ be fixed. Choose a function $u \in C_c^\infty(\mathbb{R}^N)$ with $\operatorname{supp} u \subset B(0, R/2)$. For any $|z| > R$, the supports of $u(\cdot+z)$ and $u(\cdot)$ are disjoint, so $A(z) = 2\|u\|_{L^p}^p$ for any $|z| > R$. It follows that

$$\mathcal{F}_\varepsilon(u) = 2\|u\|_{L^p}^p \int_{|z|>R} \rho_\varepsilon(z) \,\mathrm{d}z + \int_{|z|\leqslant R} \rho_\varepsilon(z) A(z) \,\mathrm{d}z.$$

We take the limit as $\varepsilon \to 0$. By the MS formula (2.6) and the mass escape property from Step 1, we get

$$\lim_{\varepsilon\to 0}\int_{|z|\leqslant R}\rho_\varepsilon(z)A(z)\,\mathrm{d}z = 0.$$

By Lemma (2.1) we conclude. □

We omit the proofs of the next two corollaries, for they are a direct consequence of the proof of Theorem 2.1.

Corollary 2.6. *Assume that the kernels $\{\rho_\varepsilon\}$ further satisfy*

$$\lim_{R\to\infty}\lim_{\varepsilon\to 0}\int_{|z|>R}\rho_\varepsilon(z)\,\mathrm{d}z = 1.$$

Then,

$$\lim_{R\to\infty}\lim_{\varepsilon\to 0}\int_{|z|<R}|z|^p\rho_\varepsilon(z)\,\mathrm{d}z = 0$$

if and only if for every smooth, compactly supported function $u \in C_c^\infty(\mathbb{R}^N)$, $\mathcal{F}_\varepsilon(u)$ is well-defined for ε sufficiently small, and

$$\lim_{\varepsilon\to 0}\mathcal{F}_\varepsilon(u) = 2\|u\|^p_{L^p(\mathbb{R}^N)}.$$

Corollary 2.7. *Assume that the kernels $\{\rho_\varepsilon\}$ further satisfy*

$$\lim_{R\to\infty}\lim_{\varepsilon\to 0}\int_{|z|<R}|z|^p\rho_\varepsilon(z)\,\mathrm{d}z = 0.$$

Then,

$$\lim_{R\to\infty}\lim_{\varepsilon\to 0}\int_{|z|>R}\rho_\varepsilon(z)\,\mathrm{d}z = 1,$$

if and only if for every smooth, compactly supported function $u \in C_c^\infty(\mathbb{R}^N)$, $\mathcal{F}_\varepsilon(u)$ is well-defined for ε sufficiently small, and

$$\lim_{\varepsilon\to 0}\mathcal{F}_\varepsilon(u) = 2\|u\|^p_{L^p(\mathbb{R}^N)}.$$

2.3. Corollary: The MS formula in Sobolev spaces of integer order. In Theorem 2.1, we have generalized the Maz'ya–Shaposhnikova identity (2.6) to a significantly wider family of kernels by working within the space $C_c^\infty(\mathbb{R}^N)$. Although this smooth, compactly supported setting strengthens the implication ($3 \Rightarrow 1$), it appears at first glance, to weaken the converse implication ($1 \Rightarrow 3$). In the present section, we demonstrate that no generality is lost under this restriction: one may equivalently replace $C_c^\infty(\mathbb{R}^N)$ with the Sobolev space $W^{1,p}(\mathbb{R}^N)$, again, under the broader hypotheses on the kernels stated in equation (2.4). This extension is a direct consequence of the well-posedness of the corresponding energy functionals established in Remark 2.2, combined with Theorem 2.1. In Section 3, we will further show that, for the special case of fractional kernels, our main result recovers the classical Maz'ya–Shaposhnikova formula for the fractional Sobolev space $W^{s,p}(\mathbb{R}^N)$.

Corollary 2.8 (Sobolev setting)**.** *The following three conditions are equivalent:*

(1) **Uniform moment conditions**. *For every fixed radius $R > 0$, as $\varepsilon \to 0$ the kernels $\{\rho_\varepsilon\}_{\varepsilon>0}$ satisfy*

$$\lim_{\varepsilon\to 0}\int_{|z|>R}\rho_\varepsilon(z)\,\mathrm{d}z = 1 \quad \textit{and} \quad \lim_{\varepsilon\to 0}\int_{|z|<R}|z|^p\rho_\varepsilon(z)\,\mathrm{d}z = 0. \tag{2.24}$$

(2) **Iterated limits conditions.** *The double limits in the order "first $\varepsilon \to 0$, then $R \to \infty$" satisfy*

$$\lim_{R\to\infty}\lim_{\varepsilon\to 0}\int_{|z|>R}\rho_\varepsilon(z)\,\mathrm{d}z = 1 \quad \textit{and} \quad \lim_{R\to\infty}\lim_{\varepsilon\to 0}\int_{|z|<R}|z|^p\rho_\varepsilon(z)\,\mathrm{d}z = 0. \tag{2.25}$$

(3) **Maz'ya–Shaposhnikova formula in $W^{1,p}(\mathbb{R}^N)$.** *For every $u \in W^{1,p}(\mathbb{R}^N)$, the energy $\mathcal{F}_\varepsilon(u)$ is well-defined for $\varepsilon > 0$ sufficiently small, and*

$$\lim_{\varepsilon\to 0}\mathcal{F}_\varepsilon(u) = 2\|u\|^p_{L^p(\mathbb{R}^N)}. \tag{2.26}$$

Moreover, for each $R > 0$, there holds

$$\lim_{\varepsilon\to 0}\int_{|z|<R}\int_{\mathbb{R}^N}\rho_\varepsilon(z)|u(x+z)-u(x)|^p\,\mathrm{d}x\,\mathrm{d}z = 0. \tag{2.27}$$

Proof. **(1 $\Rightarrow$ 2).** Trivial by taking the limits in the prescribed order.

(2 $\Rightarrow$ 3) The well-posedness of the associated energy functionals established in Remark 2.2 immediately yields the first conclusion in item 3. To prove the remaining statements, we first note that, by the monotonicity in $R > 0$, the second limit relation in equation (2.25) is equivalent to

$$\lim_{\varepsilon\to 0}\int_{|z|<R}|z|^p\rho_\varepsilon(z)\,\mathrm{d}z = 0 \quad \text{for every } R > 0. \tag{2.28}$$

Let $u \in W^{1,p}(\mathbb{R}^N)$ and decompose the nonlocal energy as in (2.8), i.e., $\mathcal{F}_\varepsilon(u) = I_{\varepsilon,R}[u] + II_{\varepsilon,R}[u]$. To estimate $II_{\varepsilon,R}[u]$, we observe that for fixed $R > 0$ there holds

$$\begin{aligned}\lim_{\varepsilon\to 0} II_{\varepsilon,R}[u] &= \lim_{\varepsilon\to 0}\int_{|z|<R}\int_{\mathbb{R}^N}\rho_\varepsilon(z)|u(x+z)-u(x)|^p\,\mathrm{d}x\,\mathrm{d}z \\ &\leqslant \|\nabla u\|^p_{L^p(\mathbb{R}^N)}\lim_{\varepsilon\to 0}\int_{|z|<R}|z|^p\rho_\varepsilon(z)\,\mathrm{d}z = 0. \end{aligned} \tag{2.29}$$

Thus,

$$\limsup_{\varepsilon\to 0}\mathcal{F}_\varepsilon(u) = \lim_{R\to\infty}\limsup_{\varepsilon\to 0} I_{\varepsilon,R}[u], \tag{2.30}$$
$$\liminf_{\varepsilon\to 0}\mathcal{F}_\varepsilon(u) = \lim_{R\to\infty}\liminf_{\varepsilon\to 0} I_{\varepsilon,R}[u]. \tag{2.31}$$

To estimate $I_{\varepsilon,R}[u]$ we argue by density. Let $(u_n) \in C_c^\infty(\mathbb{R}^N)$ be a sequence such that $u_n \to u$ in $L^p(\mathbb{R}^N)$ as $n \to \infty$.

We first notice that for every $\tau > 0$ there holds (see [19, Lemma 2])

$$I_{\varepsilon,R}[u] \leqslant (1+\tau)^{p-1} I_{\varepsilon,R}[u_n] + \left(\frac{1+\tau}{\tau}\right)^{p-1} I_{\varepsilon,R}[u-u_n], \tag{2.32}$$

$$I_{\varepsilon,R}[u] \geqslant \frac{1}{(1+\tau)^{p-1}} I_{\varepsilon,R}[u_n] - \frac{1}{\tau^{p-1}} I_{\varepsilon,R}[u-u_n]. \tag{2.33}$$

We observe that

$$\begin{aligned} I_{\varepsilon,R}[u-u_n] &= \int_{|z|>R} \rho_\varepsilon(z) \int_{x\in\mathbb{R}^N} |u(x+z) - u_n(x+z) + u_n(x) - u(x)|^p \mathrm{d}x \, \mathrm{d}z \\ &\leqslant 2^{p-1} \int_{|z|>R} \rho_\varepsilon(z) \int_{x\in\mathbb{R}^N} |u(x+z) - u_n(x+z)|^p \mathrm{d}x \, \mathrm{d}z \\ &\qquad + 2^{p-1} \int_{|z|>R} \rho_\varepsilon(z) \int_{x\in\mathbb{R}^N} |u(x) - u_n(x)|^p \mathrm{d}x \, \mathrm{d}z \\ &= 2^p \|u-u_n\|_{L^p}^p \int_{|z|>R} \rho_\varepsilon(z) \, \mathrm{d}z. \end{aligned} \tag{2.34}$$

Proceeding as in the proof of Theorem 2.1 (cf. (2.17)), we find that for every $n \in \mathbb{N}$,

$$I_{\varepsilon,R}(u_n) = 2\|u_n\|_{L^p}^p \int_{|z|>R_n} \rho_\varepsilon(z) \, \mathrm{d}z, \tag{2.35}$$

for every $R > 2R_n$, where the latter radius is chosen so that $\operatorname{supp} u_n \subseteq B_{R_n}$.

By (2.32)–(2.33), together with (2.34) and (2.35), we obtain that the following estimates hold

$$I_{\varepsilon,R}[u] \leqslant \left(2(1+\tau)^{p-1} \|u_n\|_{L^p}^p + 2^p \left(\frac{1+\tau}{\tau}\right)^{p-1} \|u-u_n\|_{L^p}^p \right) \int_{|z|>R} \rho_\varepsilon(z) \, \mathrm{d}z, \tag{2.36}$$

$$I_{\varepsilon,R}[u] \geqslant \left(\frac{2}{(1+\tau)^{p-1}} \|u_n\|_{L^p}^p - \frac{2^p}{\tau^{p-1}} \|u-u_n\|_{L^p}^p \right) \int_{|z|>R} \rho_\varepsilon(z) \, \mathrm{d}z, \tag{2.37}$$

for every $R > 2R_n$. Assume now that n is big enough, so that

$$\left(\frac{2}{(1+\tau)^{p-1}} \|u_n\|_{L^p}^p - \frac{2^p}{\tau^{p-1}} \|u-u_n\|_{L^p}^p \right) \geqslant 0. \tag{2.38}$$

From the previous two relations, we obtain

$$\limsup_{\varepsilon\to 0} I_{\varepsilon,R}[u] \leqslant \left(2(1+\tau)^{p-1}\|u_n\|_{L^p}^p + 2^p\left(\frac{1+\tau}{\tau}\right)^{p-1}\|u-u_n\|_{L^p}^p\right) \cdot \limsup_{\varepsilon\to 0}\int_{|z|>R}\rho_\varepsilon(z)\,\mathrm{d}z, \tag{2.39}$$

$$\liminf_{\varepsilon\to 0} I_{\varepsilon,R}[u] \geqslant \left(\frac{2}{(1+\tau)^{p-1}}\|u_n\|_{L^p}^p - \frac{2^p}{\tau^{p-1}}\|u-u_n\|_{L^p}^p\right) \cdot \liminf_{\varepsilon\to 0}\int_{|z|>R}\rho_\varepsilon(z)\,\mathrm{d}z, \tag{2.40}$$

for every $R > 2R_n$. Taking the limit for $R\to\infty$, we infer

$$\limsup_{R\to\infty}\limsup_{\varepsilon\to 0} I_{\varepsilon,R}[u] \leqslant \left(2(1+\tau)^{p-1}\|u_n\|_{L^p}^p + 2^p\left(\frac{1+\tau}{\tau}\right)^{p-1}\|u-u_n\|_{L^p}^p\right) \cdot \limsup_{R\to\infty}\limsup_{\varepsilon\to 0}\int_{|z|>R}\rho_\varepsilon(z)\,\mathrm{d}z, \tag{2.41}$$

$$\liminf_{R\to\infty}\liminf_{\varepsilon\to 0} I_{\varepsilon,R}[u] \geqslant \left(\frac{2}{(1+\tau)^{p-1}}\|u_n\|_{L^p}^p - \frac{2^p}{\tau^{p-1}}\|u-u_n\|_{L^p}^p\right) \cdot \liminf_{R\to\infty}\liminf_{\varepsilon\to 0}\int_{|z|>R}\rho_\varepsilon(z)\,\mathrm{d}z. \tag{2.42}$$

Thus, passing first to the limit for $n\to\infty$ and then for $\tau\to 0$ we conclude

$$\limsup_{R\to\infty}\limsup_{\varepsilon\to 0} I_{\varepsilon,R}[u] \leqslant 2\|u\|_{L^p}^p \limsup_{R\to\infty}\limsup_{\varepsilon\to 0}\int_{|z|>R}\rho_\varepsilon(z)\,\mathrm{d}z, \tag{2.43}$$

$$\liminf_{R\to\infty}\liminf_{\varepsilon\to 0} I_{\varepsilon,R}[u] \geqslant 2\|u\|_{L^p}^p \liminf_{R\to\infty}\liminf_{\varepsilon\to 0}\int_{|z|>R}\rho_\varepsilon(z)\,\mathrm{d}z, \tag{2.44}$$

which, in turn, by (2.25) yields

$$\lim_{R\to+\infty}\lim_{\varepsilon\to 0} I_{\varepsilon,R}[u] = 2\|u\|_{L^p}^p. \tag{2.45}$$

Finally, combining (2.29) and (2.45), we infer that

$$\lim_{\varepsilon\to 0}\mathcal{F}_\varepsilon(u) = 2\|u\|_{L^p}^p \lim_{R\to\infty}\lim_{\varepsilon\to 0}\int_{|z|>R}\rho_\varepsilon(z)\,\mathrm{d}z = 2\|u\|_{L^p}^p. \tag{2.46}$$

This proves the implication ($2 \Rightarrow 3$).

($3 \Rightarrow 1$). Since relation (2.26) holds in particular for any $u \in C_c^\infty(\mathbb{R}^N)$, the implication ($3 \Rightarrow 1$) proved in Theorem 2.1 applies. This completes the proof. □

3. The case of fractional Sobolev spaces

In this section, we analyze nonlocal energy functionals acting on fractional Sobolev spaces $W^{s,p}(\mathbb{R}^N)$ with exponent $s \in (0,1)$. Unlike the integral-order case, the weaker regularity of functions in $W^{s,p}(\mathbb{R}^N)$ requires a corresponding adaptation of the hypotheses on the kernel family $\{\rho_\varepsilon\}_{\varepsilon>0}$. Under these modified assumptions, one obtains a direct analog of the classical Maz'ya–Shaposhnikova convergence formula, now valid in the fractional regime.

Theorem 3.1 (Fractional Maz'ya–Shaposhnikova formula)**.** *Let $p \in [1,\infty)$ and $s \in (0,1)$. Suppose the kernels $\rho_\varepsilon : \mathbb{R}^N \to [0,\infty)$ satisfy*

$$\lim_{R\to\infty}\lim_{\varepsilon\to 0}\int_{|z|>R}\rho_\varepsilon(z)\,\mathrm{d}z = 1 \quad \textit{and} \quad \lim_{R\to\infty}\lim_{\varepsilon\to 0}\int_{|z|<R}|z|^{sp}\rho_\varepsilon(z)\,\mathrm{d}z = 0. \tag{3.1}$$

Then, for every $u \in W^{s,p}(\mathbb{R}^N)$,

$$\lim_{\varepsilon\to 0}\mathcal{F}_\varepsilon(u) = 2\|u\|^p_{L^p(\mathbb{R}^N)}. \tag{3.2}$$

Moreover, for each $R > 0$,

$$\lim_{\varepsilon\to 0} II_\varepsilon[u] = \lim_{\varepsilon\to 0}\int_{|z|<R}\int_{\mathbb{R}^N}\rho_\varepsilon(z)|u(x+z)-u(x)|^p\,\mathrm{d}x\,\mathrm{d}z = 0. \tag{3.3}$$

Before proceeding with the proof we make some remarks.

Remark 3.2. Note that, as we observed in Theorem 2.1, by monotonicity in $R > 0$, the second condition in (3.1) is equivalent to

$$\lim_{\varepsilon\to 0}\int_{|z|<R}|z|^{sp}\rho_\varepsilon(z)\,\mathrm{d}z = 0 \quad \text{for every } R > 0.$$

However, no similar simplification applies to the first condition: although

$$\lim_{\varepsilon\to 0}\int_{|z|>R}\rho_\varepsilon(z)\,\mathrm{d}z = 1 \quad \text{for every } R > 0 \tag{3.4}$$

implies the first limit in (3.1), the converse need not hold a priori.

Remark 3.3 (An open question)*.* The classical family of *fractional* kernels

$$\rho_\varepsilon(z) = \frac{\varepsilon p}{|\mathbb{S}^{N-1}|}|z|^{-(N+\varepsilon p)} \quad \text{with } \varepsilon \in (0,1) \tag{3.5}$$

satisfies condition (3.1). Indeed, for each $s \in (0,1)$, one checks directly that if $\varepsilon \leqslant s$, then as $\varepsilon \to 0$ both the limit relations in (3.1) are satisfied. Thus, Theorem 3.1 indeed recovers the classical Maz'ya–Shaposhnikova convergence formula while allowing any kernel family satisfying the more abstract conditions (3.1). In this sense, our Theorem 3.1 is a genuine generalization of the fractional Maz'ya–Shaposhnikova formula.

Moreover, when u has higher regularity—either $u \in C^\infty$ (Theorem 2.1) or $u \in W^{1,p}$ (Corollary 2.8)—one can accommodate kernels that are even *more singular* at the origin. This reflects the fact that greater function smoothness permits weaker tail assumptions on $\{\rho_\varepsilon\}_{\varepsilon>0}$.

Despite this generality, the fractional result lacks the full equivalence found in the integral-order case. It remains an open question whether the condition

$$\lim_{\varepsilon\to 0} II_{\varepsilon,R}[u] = 0 \quad \forall R>0,\ \forall u\in W^{s,p}(\mathbb{R}^N),$$

already implies the second requirement in (3.1) without placing further restrictions on the class of kernels. A positive answer would provide a complete converse to Theorem 3.1, mirroring the equivalences of Theorem 2.1. In Theorem 3.7 below, we take a first step in that direction by proving the converse under additional hypotheses on the admissible kernels; our partial converse for admissible kernels pinpoints where those difficulties arise and indicates natural directions for further work.

In the following lemma, we verify that the nonlocal energies are well-defined on $W^{s,p}(\mathbb{R}^N)$ under (3.1).

Lemma 3.1 (Well-Definedness of the Nonlocal Energies in $W^{s,p}$)**.** *Let $p\in[1,\infty)$ and $s\in(0,1)$. Assume that the kernels $\{\rho_\varepsilon\}$ satisfy the two limit relations in (3.1). Then, there exists $\varepsilon_0>0$ such that, for all $0<\varepsilon<\varepsilon_0$ the continuous embedding* (cf. (2.3))

$$W^{s,p}(\mathbb{R}^N) \hookrightarrow X^p_{\varepsilon_0}(\mathbb{R}^N) = \bigcap_{\varepsilon\leqslant\varepsilon_0} \mathcal{X}^p_\varepsilon(\mathbb{R}^N) \tag{3.6}$$

holds. More precisely, there is a constant $c_\rho>0$, depending only on $\{\rho_\varepsilon\}$, such that for all $0<\varepsilon\leqslant\varepsilon_0$

$$\mathcal{F}_\varepsilon(u) \leqslant c_\rho \|u\|^p_{W^{s,p}} \quad \forall\, u\in W^{s,p}(\mathbb{R}^N).$$

Proof. Let us denote by $[u]^p_{W^{s,p}}$ the fractional seminorm, defined by

$$[u]^p_{W^{s,p}} := \iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\,\mathrm{d}x\,\mathrm{d}y \tag{3.7}$$

and by $\|u\|_{W^{s,p}} := \|u\|_{L^p} + [u]_{W^{s,p}}$ the associated fractional norm.

We divide the proof into two steps.

Step 1. First, we show that for every $z\in\mathbb{R}^N$, there holds

$$\|\tau_z u - u\|_{L^p} \leqslant c|z|^s \|u\|_{W^{s,p}} \tag{3.8}$$

for some positive constant $c>0$ that depends only on s,p. For that, recall (see e.g., [36, Lemma 6.14]) that there exists a constant $c>0$, that depends only on s,p, such that for all $0<|z|<1/2$ there holds

$$\|\tau_z u - u\|_{L^p} \leqslant c|z|^s [u]_{W^{s,p}}. \tag{3.9}$$

Now, if $|z|\geqslant 1/2$ then $(2|z|)^s\geqslant 1$ and, therefore,

$$\|\tau_z u - u\|_{L^p} \leqslant 2\|u\|_{L^p} \leqslant 2^{s+1}|z|^s\|u\|_{L^p} \leqslant 2^{s+1}|z|^s\|u\|_{W^{s,p}}. \tag{3.10}$$

Combining (3.9) and (3.10) we get (3.8).

Step 2. We now show (3.6). By assumptions (3.1), there exists some $\varepsilon_0>0$ such that for every $\varepsilon\leqslant\varepsilon_0$

$$\int_{\mathbb{R}^N} \min\{1,|z|^{sp}\}\rho_\varepsilon(z)\,\mathrm{d}z < +\infty.$$

Let $u \in W^{s,p}(\mathbb{R}^N)$. Then, for every $z \in \mathbb{R}^N$, using (3.8), the following inequalities hold

$$\|\tau_z u - u\|_{L^p} \leqslant 2\|u\|_{L^p(\mathbb{R}^N)} \leqslant (2+c)\|u\|_{W^{s,p}},$$

$$\|\tau_z u - u\|_{L^p} \leqslant (2+c)|z|^s\|u\|_{W^{s,p}}.$$

Therefore, combining the two previous estimates, we get

$$\|\tau_z u - u\|_{L^p} \leqslant (2+c)\min\{1, |z|^s\}\|u\|_{W^{s,p}}.$$

Elevating both sides to the power p and integrating in z against $\rho_\varepsilon(z)\mathrm{d}z$, we obtain that

$$\mathcal{F}_\varepsilon(u) \leqslant \|u\|^p_{W^{s,p}} \int_{\mathbb{R}^N} \min\{1, |z|^{sp}\}\rho_\varepsilon(z)\,\mathrm{d}z < +\infty$$

for every $\varepsilon \leqslant \varepsilon_0$. This proves that $u \in \mathcal{X}^p_\varepsilon(\mathbb{R}^N)$ for every $\varepsilon \leqslant \varepsilon_0$. □

3.1. Proof of Theorem 3.1. In this subsection, we prove Theorem 3.1.

By monotonicity in $R > 0$, the second condition in (3.1) is equivalent to

$$\lim_{\varepsilon\to 0}\int_{|z|<R} |z|^{sp}\rho_\varepsilon(z)\,\mathrm{d}z = 0 \quad \text{for every } R > 0. \tag{3.11}$$

Fix $u \in W^{s,p}(\mathbb{R}^N)$ and any $R > 0$. Decompose the nonlocal energy as in (2.8), i.e., $\mathcal{F}_\varepsilon(u) = I_{\varepsilon,R}[u] + II_{\varepsilon,R}[u]$. Recalling the translation-difference estimate (3.8), we have that

$$\begin{aligned} \lim_{\varepsilon\to 0} II_{\varepsilon,R}[u] &= \lim_{\varepsilon\to 0}\int_{|z|<R}\int_{\mathbb{R}^N} \rho_\varepsilon(z)|u(x+z)-u(x)|^p\,\mathrm{d}x\,\mathrm{d}z \\ &\leqslant c\|u\|^p_{W^{s,p}} \lim_{\varepsilon\to 0}\int_{|z|<R} |z|^{sp}\rho_\varepsilon(z)\,\mathrm{d}z = 0. \end{aligned} \tag{3.12}$$

Thus,

$$\limsup_{\varepsilon\to 0}\mathcal{F}_\varepsilon(u) = \lim_{R\to\infty}\limsup_{\varepsilon\to 0} I_{\varepsilon,R}[u],$$
$$\liminf_{\varepsilon\to 0}\mathcal{F}_\varepsilon(u) = \lim_{R\to\infty}\liminf_{\varepsilon\to 0} I_{\varepsilon,R}[u].$$

Next, to estimate $I_{\varepsilon,R}[u]$ we argue by density as in the proof of Corollary 2.8. Let $(u_n) \in C^\infty_c(\mathbb{R}^N)$ be a sequence such that $u_n \to u$ in $W^{s,p}(\mathbb{R}^N)$ as $n \to \infty$, (see, e.g., [36, Theorem 6.66]). We have that for every $\tau > 0$ there holds (see again [19, Lemma 2])

$$I_{\varepsilon,R}[u] \leqslant (1+\tau)^{p-1} I_{\varepsilon,R}[u_n] + \left(\frac{1+\tau}{\tau}\right)^{p-1} I_{\varepsilon,R}[u-u_n], \tag{3.13}$$

$$I_{\varepsilon,R}[u] \geqslant \frac{1}{(1+\tau)^{p-1}} I_{\varepsilon,R}[u_n] - \frac{1}{\tau^{p-1}} I_{\varepsilon,R}[u-u_n]. \tag{3.14}$$

The conclusion then follows by arguing exactly as in the proof of Corollary 2.8.

3.2. Towards necessary conditions in the fractional Sobolev-spaces setting. In this final subsection, we show that, under an extra hypothesis on the admissible kernels $\{\rho_\varepsilon\}$, the conditions introduced in Theorem 3.1 are not only sufficient but also necessary for the validity of a Maz'ya–Shaposhnikova type formula.

We begin by defining the class of kernels that we will consider.

Definition 3.4 (Admissible families of kernels)**.** A family of non-negative measurable kernels $\{\rho_\varepsilon\}_{\varepsilon>0}$ belongs to the class $\mathcal{A}_{s,p}$, for $p > 1$ and $0 < s < 1$, if the following normalization condition holds:

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^N}(1\wedge |z|^{sp})\rho_\varepsilon(z)\,\mathrm{d}z = 1,$$

where $1\wedge |z|^{sp} = \min(1, |z|^{sp})$.

A family of kernels $\{\rho_\varepsilon\}_{\varepsilon>0}$ satisfies the *uniform moment conditions* if for every fixed radius $R > 0$, the following two limits hold as $\varepsilon \to 0$:

(1) *Mass Escape:*

$$\lim_{\varepsilon\to 0}\int_{|z|>R}\rho_\varepsilon(z)\,\mathrm{d}z = 1. \tag{3.15}$$

(2) *Short-Range Attenuation:*

$$\lim_{\varepsilon\to 0}\int_{|z|<R}|z|^{sp}\rho_\varepsilon(z)\,\mathrm{d}z = 0. \tag{3.16}$$

Remark 3.5. Fractional-type kernels of the form

$$\rho_\varepsilon(z) := \frac{\varepsilon p}{|\mathbb{S}^{N-1}|}\,|z|^{-(N+\varepsilon p)}$$

belong to $\mathcal{A}_{s,p}$ for every $0 < s < 1$ and every $p > 1$. Indeed, we compute

$$\frac{\varepsilon p}{|\mathbb{S}^{N-1}|}\int_{\mathbb{R}^N}\frac{1\wedge |z|^{sp}}{|z|^{N+\varepsilon p}}\,\mathrm{d}z = \frac{s}{s-\varepsilon}\xrightarrow{\varepsilon\to 0} 1.$$

The main result of this section is Theorem 3.7, stated below. In it, we show that when the analysis is restricted to kernels in $\mathcal{A}_{s,p}$, the implication in Theorem 3.1 can be sharpened to an equivalence. This gives a precise characterization of those kernels in $\mathcal{A}_{s,p}$ for which the MS formula holds: they are exactly the kernels that satisfy the uniform moment conditions. Before moving further in this direction, let us clarify the relationships among the concepts introduced above. Our first observation is that the uniform moment conditions are strictly stronger than simple membership in the admissible class $\mathcal{A}_{s,p}$. The following result makes this statement precise:

Lemma 3.2. *If a family of kernels $\{\rho_\varepsilon\}_{\varepsilon>0}$ satisfies the two uniform moment conditions, then it necessarily belongs to the class of admissible fractional kernels $\mathcal{A}_{s,p}$.*

Proof. To establish membership in $\mathcal{A}_{s,p}$, we must evaluate the limit of the admissibility integral. We decompose this integral over the regions $|z| < 1$ and $|z| \geqslant 1$:

$$\int_{\mathbb{R}^N}(1\wedge |z|^{sp})\rho_\varepsilon(z)\,\mathrm{d}z = \int_{|z|<1}|z|^{sp}\rho_\varepsilon(z)\,\mathrm{d}z + \int_{|z|\geqslant 1}\rho_\varepsilon(z)\,\mathrm{d}z.$$

We take the limit as $\varepsilon \to 0$ and apply the two moment conditions with the specific radius $R = 1$. The first term vanishes by the short-range attenuation condition:

$$\lim_{\varepsilon\to 0}\int_{|z|<1} |z|^{sp}\rho_\varepsilon(z)\,\mathrm{d}z = 0.$$

The second term converges to 1 by the mass escape condition:

$$\lim_{\varepsilon\to 0}\int_{|z|\geqslant 1} \rho_\varepsilon(z)\,\mathrm{d}z = 1.$$

Summing the limits, we get that the family of kernels belongs to $\mathcal{A}_{s,p}$. □

Remark 3.6 (The reverse implication does not hold). A family of kernels may be admissible ($\in \mathcal{A}_{s,p}$) without fulfilling the moment conditions. To illustrate this, it is enough to consider a kernel family whose mass is concentrated at the origin. Precisely, let $\phi \geqslant 0$ be a $C_c^\infty(\mathbb{R}^N)$ function with $\operatorname{supp}\phi \subset B(0,1)$ and normalized such that

$$\int_{B(0,1)} |w|^{sp}\phi(w)\,\mathrm{d}w = 1.$$

For every $\varepsilon > 0$, define $\rho_\varepsilon(z) := \varepsilon^{-N-sp}\phi(z/\varepsilon)$. The support of this kernel is $\operatorname{supp}\rho_\varepsilon \subseteq B(0,\varepsilon)$. By a change of variables $z = \varepsilon w$, we can verify the admissibility condition:

$$\begin{aligned}\lim_{\varepsilon\to 0}\int_{\mathbb{R}^N}(1\wedge |z|^{sp})\rho_\varepsilon(z)\,\mathrm{d}z &= \lim_{\varepsilon\to 0}\int_{|z|<\varepsilon}|z|^{sp}\varepsilon^{-N-sp}\phi\left(\frac{z}{\varepsilon}\right)\,\mathrm{d}z\\ &= \lim_{\varepsilon\to 0}\int_{|w|<1}|\varepsilon w|^{sp}\varepsilon^{-N-sp}\phi(w)(\varepsilon^N\,\mathrm{d}w)\\ &= \int_{|w|<1}|w|^{sp}\phi(w)\,\mathrm{d}w = 1.\end{aligned}$$

So, $\{\rho_\varepsilon\}$ belongs to $\mathcal{A}_{s,p}$. However, the uniform moment conditions fail. For any fixed $R > 0$, we can choose $\varepsilon < R$. Then the support of ρ_ε is entirely contained within $B(0,R)$. This means:

$$\int_{|z|>R}\rho_\varepsilon(z)\,\mathrm{d}z = 0 \quad \text{for all } \varepsilon < R.$$

The limit is 0, which violates the mass-escape condition ($\lim \to 1$).

On the other hand, if $\{\rho_\varepsilon\}_{\varepsilon>0}$ is in the class $\mathcal{A}_{s,p}$, then the uniform moment conditions become equivalent.

Lemma 3.3. *Let $\{\rho_\varepsilon\}$ be a family of non-negative measurable kernels in $\mathbb{R}^N$. If the family belongs to the class $\mathcal{A}_{s,p}$ then the uniform moment conditions* (3.15) *and* (3.16) *are equivalent.*

Proof. By assumption, the family of kernels is admissible, i.e.,

$$1 = \lim_{\varepsilon\to 0}\int_{\mathbb{R}^N}(1\wedge |z|^{sp})\rho_\varepsilon(z)\,\mathrm{d}z. \tag{3.17}$$

((3.15) $\Rightarrow$ 3.16). The Mass Escape condition (3.15) implies that for any $0 < R_1 < R_2$ we have

$$\lim_{\varepsilon\to 0}\int_{R_1<|z|<R_2} \rho_\varepsilon(z)\,\mathrm{d}z = \lim_{\varepsilon\to 0}\int_{|z|>R_1} \rho_\varepsilon(z) - \lim_{\varepsilon\to 0}\int_{|z|>R_2} \rho_\varepsilon(z)\,\mathrm{d}z = 0. \tag{3.18}$$

In particular,

$$\lim_{\varepsilon\to 0}\int_{1<|z|<R} |z|^{sp}\rho_\varepsilon(z)\,\mathrm{d}z \leqslant R^{sp}\lim_{\varepsilon\to 0}\int_{1<|z|<R} \rho_\varepsilon(z)\,\mathrm{d}z = 0. \tag{3.19}$$

Now, observe that it is sufficient to prove that (3.16) holds for every $R > 1$. But for $R > 1$ we have

$$\begin{aligned}\int_{|z|<R} |z|^{sp}\rho_\varepsilon(z)\,\mathrm{d}z &= \int_{|z|<1} (1\wedge|z|^{sp})\rho_\varepsilon(z)\,\mathrm{d}z + \int_{1<|z|<R} |z|^{sp}\rho_\varepsilon(z)\,\mathrm{d}z\\ &= \int_{\mathbb{R}^N} (1\wedge|z|^{sp})\rho_\varepsilon(z)\,\mathrm{d}z - \int_{|z|>1}\rho_\varepsilon(z)\,\mathrm{d}z + \int_{1<|z|<R} |z|^{sp}\rho_\varepsilon(z)\,\mathrm{d}z,\end{aligned}$$

from which, taking into account (3.17), (3.19), and condition (3.15) applied with $R = 1$, passing to the limit for $\varepsilon \to 0$ we get condition (3.16).

((3.16) $\Rightarrow$ (3.15)). The Short-Range Attenuation condition (3.16) implies that for any $0 < R_1 < R_2$ we have

$$\lim_{\varepsilon\to 0}\int_{R_1<|z|<R_2} |z|^{sp}\rho_\varepsilon(z)\,\mathrm{d}z = \lim_{\varepsilon\to 0}\int_{|z|<R_2} |z|^{sp}\rho_\varepsilon(z) - \lim_{\varepsilon\to 0}\int_{|z|<R_1} |z|^{sp}\rho_\varepsilon(z)\,\mathrm{d}z = 0. \tag{3.20}$$

In particular, for any $R < 1$ we have

$$\lim_{\varepsilon\to 0}\int_{R<|z|<1} \rho_\varepsilon(z)\,\mathrm{d}z \leqslant R^{-sp}\lim_{\varepsilon\to 0}\int_{R<|z|<1} |z|^{sp}\rho_\varepsilon(z)\,\mathrm{d}z = 0. \tag{3.21}$$

Now, observe that it is sufficient to prove that (3.15) holds for every $R < 1$. But for $R < 1$ we have

$$\begin{aligned}\int_{|z|>R} \rho_\varepsilon(z)\,\mathrm{d}z &= \int_{R<|z|<1} \rho_\varepsilon(z)\,\mathrm{d}z + \int_{|z|>1} (1\wedge|z|^{sp})\rho_\varepsilon(z)\,\mathrm{d}z\\ &= \int_{R<|z|<1} \rho_\varepsilon(z)\,\mathrm{d}z + \int_{\mathbb{R}^N} (1\wedge|z|^{sp})\rho_\varepsilon(z)\,\mathrm{d}z - \int_{|z|<1} |z|^{sp}\rho_\varepsilon(z)\,\mathrm{d}z,\end{aligned}$$

from which, taking into account (3.17), (3.21), and condition (3.16) applied with $R = 1$, passing to the limit for $\varepsilon \to 0$ we get condition (3.15) . □

We can now state and prove the main result of this section.

Theorem 3.7 (Conditional fractional Sobolev setting)**.** *Let $p \in [1,\infty)$ and $s \in (0,1)$. Assume that $\{\rho_\varepsilon\} \subset \mathcal{A}_{s,p}$. The following three statements are equivalent:*

(1) ***(Uniform moment conditions)*** *For every fixed radius $R > 0$, as $\varepsilon \to 0$ the kernels satisfy*

$$\lim_{\varepsilon\to 0} \int_{|z|>R} \rho_\varepsilon(z)\,\mathrm{d}z = 1 \quad \text{and} \quad \lim_{\varepsilon\to 0} \int_{|z|<R} |z|^{sp} \rho_\varepsilon(z)\,\mathrm{d}z = 0. \tag{3.22}$$

(2) ***(Iterated-limits conditions)*** *The iterated limits taken in the order "first $\varepsilon \to 0$, then $R \to \infty$" satisfy*

$$\lim_{R\to\infty} \lim_{\varepsilon\to 0} \int_{|z|>R} \rho_\varepsilon(z)\,\mathrm{d}z = 1 \quad \text{and} \quad \lim_{R\to\infty} \lim_{\varepsilon\to 0} \int_{|z|<R} |z|^{sp} \rho_\varepsilon(z)\,\mathrm{d}z = 0. \tag{3.23}$$

(3) ***(Maz'ya–Shaposhnikova formula in $W^{s,p}(\mathbb{R}^N)$)*** *For every $u \in W^{s,p}(\mathbb{R}^N)$ the nonlocal energy $\mathcal{F}_\varepsilon(u)$ is well-defined for all sufficiently small ε, and*

$$\lim_{\varepsilon\to 0} \mathcal{F}_\varepsilon(u) = 2\|u\|^p_{L^p(\mathbb{R}^N)}. \tag{3.24}$$

Moreover, for each fixed $R > 0$,

$$\lim_{\varepsilon\to 0} \int_{|z|<R} \int_{\mathbb{R}^N} \rho_\varepsilon(z)\,|u(x+z) - u(x)|^p\,\mathrm{d}x\,\mathrm{d}z = 0. \tag{3.25}$$

Proof The implication (1 ⇒ 2) follows in the same way as the analogous implication in Theorem 2.1, while the implication (2 ⇒ 3) is an immediate consequence of Theorem 3.1. Thus, it suffices to prove the converse implication (3 ⇒ 1).

Assume that (3.24) and (3.25) hold. Since $C_c^\infty(\mathbb{R}^N) \subset W^{s,p}(\mathbb{R}^N)$, by Theorem 2.1, the first relation in (3.22) already holds; that is, for every $R > 0$ one has

$$\lim_{\varepsilon\to 0} \int_{|z|>R} \rho_\varepsilon(z)\,\mathrm{d}z = 1. \tag{3.26}$$

But then, Lemma 3.3 completes the proof of the implication (3 ⇒ 1), and hence of the theorem. □

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