# MINIMAL POINTS AND NON-HOLONOMIC CONTROLLABILITY ON COMPACT MANIFOLDS

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*Dedicated to Alexander Plakhov on the occasion of his 65th birthday.*

Abstract. We study the problem of non-holonomic point-to-point controllability for ODEs with drift possessing some recursion property of the flow (it is supposed nonwandering or chain recurrent) and satisfying various versions of the Hörmander condition (also known as Lie bracket generating condition). We show that for the flows on compact manifolds, it suffices to assume the validity of the Hörmander condition on the closure of the set of their minimal points only. Also, we construct a 2-dimensional example of a drift defining a chain recurrent flow and the vector fields defining the non-holonomic constraint, which together satisfy the Hörmander condition, but the flow is not controllable in the direction of the given vector fields.

## 1. Introduction

Let a set of smooth vector fields on a smooth connected manifold be such that the Lie algebra generated by those fields span the tangent bundle of the manifold (this is called Hörmander or Lie bracket generating condition). Then one can reach any given point of the manifold from another given point (one says in this case that global controllability holds) using controls tangent to the planes from the distribution defined by the mentioned vector fields. In other words, if one considers the distribution of planes defined by the given vector fields as a non-holonomic constraint for the control system, then the Hörmander condition suffices for non-holonomic controllability of the latter. This is contents of the celebrated Chow-Rashevskii theorem [5, 13] which is one of the important results in both control theory and differential geometry.

For control systems with a drift, the analogous controllability problem is more complicated. If all the given vector fields defining the non-holonomic constraint span the tangent space $T_xM$ for each $x \in M$, global controllability is provided when all the points of the manifold are chain recurrent for the flow generated by the drift (see [4, corollary C]). The case when these vector fields do not span the tangent space but rather satisfy the Hörmander condition, is much more delicate. In particular, theorem 4.2.7 in [2] (see also [8] and the earlier results [9] and [12]) establish controllability in the case when the drift is divergence-free (and, hence, nonwandering). The overview of principal known results on the subject can be found in [3]. In particular, corollary 31 of [3] (see also corollary 26 of the same paper) says that if the drift is nonwandering (called weakly recurrent in the quoted paper),

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the global controllability with non-holonomic constraint defined by a set of vector fields is guaranteed if these vector fields possibly together with the drift satisfy the Hörmander condition. Moreover, proposition 27 of [3] asserts that when the drift is wandering (i.e. not nonwandering), then global non-holonomic controllability may still hold true under a stronger condition that the vector fields defining the constraint satisfy the Hörmander condition themselves (i.e., without taking into account the drift). In the mentioned proposition, nothing is assumed about the recurrence of the flow, but the controls can be taken big.

Here we suggest a new approach that uses the properties of the flow generated by the drift. We observe that on a compact manifold any pseudotrajectory of a flow passes infinitely many times near the minimal points of this flow (and the time intervals between nearby passages are bounded). Thus, for nonwandering or only chain recurrent drifts it suffices to assume that the Hörmander condition be satisfied just near the set of minimal points of the latter flow rather than on the whole phase space. Both of the results mentioned from [3] are generalized in this sense. It follows that, for example, for Morse-Smale systems, it is enough for the Hörmander condition to be valid on all the stationary points and all the periodic orbits (both finite in number) of the system. We also show that for nonwandering drift the control may be taken arbitrarily small. Both our results (i.e. Theorem 4.2 for nonwandering drifts and Theorem 5.1 for chain recurrent drifts) are rather sharp in view of [14] and [15] which show that already in the driftless case (meaning that all the points of the phase space are stationary hence minimal) the Hörmander condition is not only sufficient but even necessary for controllability when the vector fields defining the non-holonomic constraint are real analytic.

We also show the sharpness of the request for the restricted Hörmander condition in our second result (Theorem 5.1 for only chain recurrent flow). In fact, we provide an example (in Section 6, see Theorem 6.3) of a drift generating a chain recurrent (but wandering) flow such that non-holonomic controllability does not take place if the vector fields defining the constraint do not satisfy the Hörmander condition themselves, even when the Hörmander condition is valid for these vector fields together with the drift. Similar obstructions for local accessibility were considered in [7] and [12]. However, it is know that if the local control is possible, the global one follows [6, corollary 4.5.11].

## 2. Preliminaries

2.1. **Notation.** Let $M$ be a $C^1$-smooth compact connected Riemannian manifold. We always assume it to be equipped with its intrinsic (geodesic) Riemannian distance $d$. The notation $B_\rho(x)$ stands for the ball in $M$ of radius $\rho$ centered at $x \in M$.

2.2. **Some results on topological dynamics.** Let $V$ be a $C^1$-smooth vector field on $M$. Consider the flow $(t,x) \in \mathbb{R} \times M \to \varphi_V(t,x) \in M$ induced by the system

$$\dot{x} = V(x), \tag{2.1}$$

that is, $x(\cdot) = \varphi_V(\cdot, x_0)$ is the solution of (2.1) with initial conditions $x(0) = x_0$.

**Definition 2.1.** *We say that a point $x_0 \in M$ is minimal with respect to the flow $\varphi_V$ if the closure of the orbit*

$$\overline{O_V(x)} = \overline{\{\varphi_V(t,x) : t \in \mathbb{R}\}}$$

*does not contain any proper $\varphi_V$-invariant closed subset.*

Denote by $\mathrm{Min}_V$ the closure of the set of all points minimal with respect to the flow $\varphi_V$. Evidently, any stationary or periodic point is minimal. The converse statement is, generally speaking, false (one may consider the irrational wrapping of the torus where all the points are minimal but not periodic). Besides, any compact $\varphi_V$-invariant set contains a minimal subset (all the points of that subset are minimal), which follows from the Kuratowski-Zorn lemma.

**Definition 2.2.** *We say that a point $x_0 \in M$ is nonwandering for (2.1) (equivalently, for the flow $\varphi_V$), if for any neighborhood $U$ of $x_0$ there exists a couple of points $\{p, q\} \subset U$ and a $T \geq 1$ such that $\varphi_V(T, p) = q$. Equivalently, the point $x_0$ is nonwandering if for every neighborhood $U$ of $x_0$ there exists a $T \geq 1$ such that $\varphi_V(T, U) \bigcap U \neq \emptyset$.*

*Remark* 2.3. In fact, we may assume that the mentioned value $T$ is greater than any positive constant that does not depend on $\delta$.

Denote the set of all nonwandering points of (2.1) by $\Omega_V$. It is known that $\mathrm{Min}_V \subset \Omega_V$ (see [10, §3.3]). The converse inclusion is, in general, false.

**Definition 2.4.** *We introduce the notion of a $\delta$-solution and a chain recurrent point of (2.1) as follows.*

- *Given a $\delta > 0$, we say that a continuous piecewise smooth function $x\colon [a, b] \to M$ is a $\delta$-solution of (2.1), if for any $t \in [a, b]$ such that the derivative $\dot{x}$ exists, one has*
$$|\dot{x}(t) - V(x(t))|_{x(t)} \leq \delta,$$
*where $|\cdot|_x$ stands for the Riemannian norm in the tangent space $T_xM$. Equivalently, $x(\cdot)$ is a $\delta$-solution of (2.1), if*
$$\dot{x}(t) = V(x(t)) + u(t, x(t))$$
*for some control function $u$ over $[a, b] \times M$ such that $u(t, \cdot)$ is a vector field over $M$ and*
$$|u(t, x(t))|_{x(t)} \leq \delta.$$
- *We say that a point $x_0 \in M$ is chain recurrent for (2.1) (equivalently, for the flow $\varphi_V$), if for any $\delta > 0$ there exists a $T \geq 1$ and a $\delta$-solution $x\colon [0, T] \to M$ of (2.1) such that*
$$x(0) = x(T) = x_0.$$
*The statement of Remark 2.3 is also applicable here.*

*Remark* 2.5. In Euclidean spaces (proceeding to charts) we can write $u(t)$ instead of $u(t, x(t))$.

Denote the set of all chain recurrent points by $\mathrm{CR}_V$. It is known [1] that
$$\Omega_V \subset \mathrm{CR}_V.$$
The converse inclusion may be wrong: as an example, one can consider the flow defined by the ODE
$$\dot{x} = 1 - \cos x$$
on the unit circle.

**Definition 2.6.** *The ODE (2.1) (or, equivalently, the flow $\varphi_V$ or just the drift $V$) is called*

- nonwandering*, if $\Omega_V = M$,*
- chain recurrent*, if $\mathrm{CR}_V = M$ that is, if for every $p \in M$ and for every $\delta > 0$ there is a $T > 0$ and a $\delta$-solution $x\colon [0,T] \to M$ of (2.1) satisfying $x(0) = x(T) = p$,*
- chain transitive*, if for every couple of points $\{p,q\} \subset M$ and for every $\delta > 0$ there is a $T > 0$ and a $\delta$-solution $x\colon [0,T] \to M$ of (2.1) satisfying $x(0) = p$, $x(T) = q$.*

Clearly, if system (2.1) is nonwandering then it is also chain recurrent. Besides, any chain transitive flow is also chain recurrent.

Given a subset $A \subset M$ and a $\sigma > 0$, we let $(A)_\sigma$ stand for the $\sigma$-neighborhood of $A$. The following statement is valid.

**Lemma 2.7.** *For any $\varepsilon > 0$ there is a $\delta > 0$ and a $T > 0$ such that if $x\colon [0,T] \to M$ is a $\delta$-solution of (2.1), then*

$$\{x(t)\colon t \in [0,T]\} \bigcap (\mathrm{Min}_V)_\varepsilon \neq \emptyset.$$

*Proof.* Suppose the statement of the lemma does not hold. Then there exists an $\varepsilon > 0$ and a sequence of $\delta_k$-pseudotrajectories $x_k\colon [0,T_k] \to M$ such that

(1) $\delta_k > 0$, $\lim_k \delta_k = 0$,
(2) $\lim_k T_k \to +\infty$,
(3) $\{x_k(t)\colon t \in [0,T_k]\} \bigcap (\mathrm{Min}_V)_\varepsilon = \emptyset$.

By the Ascoli-Arcelá theorem, there is a subsequence $x_k$ (not relabeled) converging to a solution $x_*$ of (2.1) uniformly over the compact intervals of time. Then

$$\overline{\{x_*(t)\colon t \geq 0\}} \bigcap (\mathrm{Min}_V)_\varepsilon = \emptyset. \tag{2.2}$$

The $\omega$-limit set for the trajectory of $x_*(t)$ is nonempty in view of the compactness of $M$, hence it contains some minimal points. On the other hand, this set cannot intersect with $(\mathrm{Min}_V)_\varepsilon$ in view of (2.2), and this contradiction proves the lemma. □

The following more or less folkloric lemma will be used in the proofs below.

**Lemma 2.8.** *Let the flow $\varphi_V$ of the system (2.1) defined on a connected Riemannian manifold be chain recurrent. Then it is chain transitive.*

*Proof.* Let $d$ be the Riemannian distance on the Riemannian manifold $M$, $\{p,q\} \subset M$, $p \neq q$ and $\theta \subset M$ be an arc (i.e. an injective curve) with endpoints $p$ and $q$. Assume that $\varepsilon > 0$ is given. Divide $\theta$ into $N$ consecutive points $x_i$, with $x_1 = p$, and $x_N = q$, so that $d(x_i, x_{i+1}) \leq \delta$ for all $i = 1, \ldots, N-1$, and $\delta$ to be chosen later. Once we arrive at the point $x_i$ at the instance $t_i \geq 0$, $i = 1, \ldots, N-1$, we choose the control $u^1_\varepsilon$ with $|u^1_\varepsilon(\cdot)| < \varepsilon/2$ so as to return to the same point $x_i$ after time $T_i > 1$. According to lemma 3.2 from [11] there is a $\tau \in (0, T_i)$, a $\rho > 0$ both depending on $\varepsilon$ and a continuous piecewise continuous control $u^2_\varepsilon$ with $|u^2_\varepsilon(\cdot)| < \varepsilon/2$ different from zero only on $(t_i + T_i - \tau, t_i + T_i)$ such that using the control

$$u_\varepsilon := u^1_\varepsilon + u^2_\varepsilon$$

we arrive at $x_{i+1}$ at the instance $t_{i+1} := t_i + T_i$, once $\delta$ is chosen so as $\delta \in (0, \rho)$. Repeating this procedure, we may construct the way from $p$ to $q$. □

2.3. **The Hörmander condition.** Let $M$ be a $C^\infty$ smooth manifold and $X_j$, $j = 1, \ldots, m$ be vector fields over $M$. We consider the set $\mathcal{X}$ of all $X_j$ and all their continuous Lie brackets (if the vector fields are smooth enough so that the respective brackets will be defined), i.e.

$$\mathcal{X} := \{X_i\}_{i=1}^m \bigcup \{[X_i, X_j]\}_{i,j=1}^m \bigcup \{[[X_i, X_j], X_k]\}_{i,j,k=1}^m \cup \ldots$$

**Definition 2.9.** *The set of vector fields $\{X_j\}_{j=1^m}$ is said to satisfy the Hörmander condition, if*

$$\text{span}\,\{Y(x) \colon Y \in \mathcal{X}\} = T_x M.$$

*for every $x \in M$, where $T_x M$ stands for the tangent space to $M$ at $x$.*

The celebrated Chow-Rashevskii theorem asserts that if $M$ is connected (not necessarily compact) and the set of vector fields $\{X_j\}_{j=1^m}$ satisfies the Hörmander condition, then for every couple of points $\{p, q\} \subset M$ there is a piecewise smooth curve $x \colon [0, T] \to M$ such that

$$\begin{aligned} &\dot{x}(t) = u_1(t) X_1(x(t)) + \ldots + u_m(t) X_m(x(t)), \\ &x(0) = p, \quad x(T) = q \end{aligned}$$

for some $T > 0$ and some continuous piecewise smooth functions $u_j \colon [0, T] \to \mathbb{R}$, $j = 1, \ldots, m$.

Moreover, the curve $x(t)$ may be selected so that for any $t$ for which the derivative $\dot{x}(t)$ exists, one of the functions $u_j$ equals 1 or $-1$ while all others vanish.

## 3. Control affine systems

In this paper, we aim to study the controllability of the control affine systems of the form

$$\dot{x}(t) = V(x(t)) + u_1(t) X_1(x(t)) + \ldots + u_m(t) X_m(x(t)), \tag{3.1}$$

where $V$, $X_1, \ldots, X_m$ are the given smooth vector fields on $M$. Namely, given a couple of points $\{p, q\} \subset M$, we are interested in finding the piecewise continuous functions $u_j \colon [0, T] \to \mathbb{R}$, $j = 1, \ldots, m$ for some $T > 0$ such that there is a solution $x \colon [0, T] \to M$ of (3.1) satisfying

$$x(0) = p, \quad x(T) = q.$$

If such functions can be found, then we will call (3.1) controllable in $M$. If, moreover, for every $\delta > 0$ one can find such functions $u_j$ that also satisfy

$$|u_j(t)| \le \delta, \qquad j = 1, \ldots, m$$

for all $t \in [0, T]$, then (3.1) will be called controllable with arbitrarily small controls.

In the paper [3], the following two statements are proven (corollaries 31 and 32, respectively).

**Lemma 3.1.** *Let the set of vector fields $\{X_1, \ldots, X_m\}$ satisfy the Hörmander condition in $M$. Then the system (3.1) is controllable on $M$.*

**Lemma 3.2.** *Let the system (2.1) be nonwandering and the set of vector fields*

$$\{V, X_1, \ldots, X_m\}$$

*satisfy the Hörmander condition in $M$. Then the system (3.1) is controllable on $M$ with arbitrarily small controls.*

Given two positive numbers $\varepsilon$ and $\tau$, we consider the set $\mathcal{A}_{\varepsilon,\tau,p} \subset M$ of all the points $q$ such that the boundary value problem for some system (3.1) with all $|u_j(t)| \le \varepsilon$ and boundary conditions

$$x(0) = p, \qquad x(\theta) = q$$

is solvable for a $\theta \in (0, \tau]$. We also consider the sets

$$\mathcal{A}_{\varepsilon,p} := \bigcup_{\tau>0} \mathcal{A}_{\varepsilon,\tau,p}, \quad \mathcal{A}_p := \bigcup_{\varepsilon>0} \mathcal{A}_{\varepsilon,p},$$

that is the set of all the points attainable from $p$ using controls bounded in the uniform norm by $\varepsilon$, and using controls with arbitrary norm, respectively.

We quote a classical Control Theory statement also given in [3].

**Lemma 3.3.** *(Krener's theorem). Let the set of vector fields*

$$\{V, X_1, \ldots, X_m\}$$

*satisfy the Hörmander condition on $M$. Then for any $\tau > 0$ and any $\varepsilon > 0$, the point $p$ belongs to the closure of the interior of the set $\mathcal{A}_{\varepsilon,\tau,p}$. In particular, the latter interior is non-empty.*

In the next two sections, we generalize the results of Lemmata 3.1 and 3.2. In particular, we show that it suffices to assume that the Hörmander condition is satisfied on a neighborhood of the set of minimal points.

## 4. The nonwandering drift

We show now that for controllability of systems with nonwandering drift, it is enough that the Hörmander condition be satisfied only over the closure of the set of all minimal points.

To this aim we formulate first the following technical lemma which will serve both here and in the next section where we will be considering systems with only chain recurrent drift but under a restricted Hörmander conditon.

**Lemma 4.1.** *Let the system (2.1) be chain recurrent. If there is a $\sigma > 0$ (resp. a $\sigma > 0$ and an $\varepsilon > 0$) such that for every $x_0 \in \mathrm{Min}_V$ and every ball $B_\sigma(x_0) \subset M$, for every couple of points $\{y_0, y_1\} \subset B_\sigma(x_0)$ one has $y_1 \in \mathcal{A}_{y_0}$ (resp. $y_1 \in \mathcal{A}_{\varepsilon,y_0}$), then $\mathcal{A}_z = M$ (resp. $\mathcal{A}_{\varepsilon,z} = M$) for all $z \in M$, i.e. the system is controllable (resp. controllable with controls $|u_j| \le \varepsilon$).*

*Proof.* We will prove both claims at once, so fix an $\varepsilon > 0$, if appropriate. Take a $\delta > 0$ and a $T > 0$ according to Lemma 2.7 with $\sigma/2$ in place of $\varepsilon$, i.e., so that for every $\delta$-solution $\psi(\cdot)$ of (2.1) we have

$$\{\psi(t) \colon t \in [0, T]\} \bigcap (\mathrm{Min}_V)_{\sigma/2} \neq \emptyset. \tag{4.1}$$

Of course, without loss of generality, we may increase $T$ if necessary to have $T > 2$. We may now further decrease $\delta > 0$ taking it so small that additionally for every $\delta$-solution $\psi(\cdot)$ of (2.1) and every $t \in [-4T, 4T]$ we have

$$d(\psi(t), \varphi_V(t, \psi(0))) \le \sigma/2. \tag{4.2}$$

Consider now an arbitrary couple of points $\{p, q\} \subset M$. The chain recurrence property implies chain transitivity by Lemma 2.8, and thus there exists a $\delta$-solution

$\psi\colon [0,\tau] \to M$ of (2.1) such that $\psi(0) = p$, $\psi(\tau) = q$. Since $\tau$ may be taken as large as we like, we assume $\tau > T$ and set

$$N := \left\lfloor \frac{\tau}{T} \right\rfloor,$$

so that $N \geq 1$ and $NT \leq \tau \leq (N+1)T$ and hence

$$T \leq \frac{\tau}{N} \leq 2T.$$

Thus, in view of (4.1) we can find points $y_j \in \mathrm{Min}_V$ and instants of time $t_j \in [(j-1)\tau/N, j\tau/N]$ such that

$$d(\psi(t_j), y_j) \leq \sigma/2 \tag{4.3}$$

for all $j = 1, \ldots, N$.

Now we construct the control using the following algorithm (see Fig. 1).

- We start at the point $p$ and let the control functions $u_k(t) := 0$ for $t \in [0, t_1]$, $k - 1, \ldots, m$, that is, we just follow the flow induced by the drift $V$ until the instance $t_1$ thus arriving at a point $x_1 := \varphi_V(t_1, p)$. Note that
$$0 \leq t_1 \leq \tau/N \leq 2T,$$
and therefore $d(x_1, \psi(t_1)) \leq \sigma/2$ by (4.2) so that by (4.3) we get
$$d(x_1, y_1) \leq \sigma.$$
Thus both $x_1 \in B_\sigma(y_1)$ and $\psi(t_1) \in B_\sigma(y_1)$, and therefore by the assumptions we may use the control functions $u_1, \ldots, u_m$ (with $|u_j| \leq \varepsilon$ if appropriate) to get from $x_1$ to $\psi(t_1)$.
- Similarly, for every $j = 1, \ldots, N-1$ once we arrive at a point $\psi(t_j)$ at some instance $t'_j$ we let the control functions
$$u_1(t) = \ldots = u_m(t) := 0 \quad \text{for } t \in [t'_j, t'_j + (t_{j+1} - t_j)],$$
that is, we just follow the flow induced by the drift $V$ for $t_{j+1} - t_j$ instances of time, thus arriving at a point
$$x_{j+1} := \varphi_V(t_{j+1} - t_j, \psi(t_j)).$$
One clearly has
$$t_{j+1} - t_j \leq 2\frac{\tau}{N} \leq 4T,$$
so that $d(x_{j+1}, \psi(t_{h+1})) \leq \sigma/2$ by (4.2), hence by (4.3) one has
$$d(x_{j+1}, \psi(t_{j+1})) \leq \sigma.$$
Thus both $x_{j+1} \in B_\sigma(y_{j+1})$ and $\psi(t_{j+1}) \in B_\sigma(y_{j+1})$, and therefore we may use the control functions $u_1, \ldots, u_m$ (with $|u_j| \leq \varepsilon$ if appropriate) to get from $x_{j+1}$ to $\psi(t_{j+1})$.
- Finally, once we arrive at a point $\psi(t_N)$ at some instance $t'_N$, setting
$$x_* := \varphi_V(-(\tau - t_N), q),$$
we get $d(x_*, \psi(t_N)) \leq \sigma/2$ by (4.2) since
$$0 \leq \tau - t_N \leq \frac{\tau}{N} \leq 2T.$$
Thus by (4.3) one has
$$d(x_*, y_N) \leq \sigma,$$

which implies that both $x_* \in B_\sigma(y_N)$ and $\psi(t_N) \in B_\sigma(y_N)$, and therefore we may use the control functions $u_1, \ldots, u_m$ (with $|u_j| \le \varepsilon$ if appropriate) to get from $\psi(t_N)$ to $x_*$. It remains then to set $u_1, \ldots, u_m$ to zero, i.e. follow the flow of $V$ for $\tau - t_N$ instances of time to arrive finally at $q$.

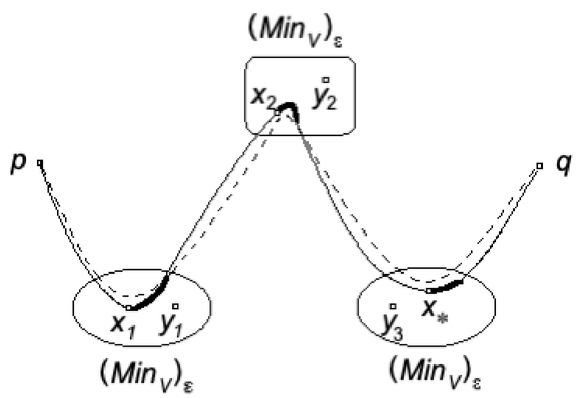


FIGURE 1. Control in neighborhoods of the minimal sets. The dashed line stands for $\psi(t)$, the regular lines for exact trajectories, and the bold lines for trajectories of systems with a control

Note that the control functions we constructed are all zero unless we had to connect points in small neighborhoods of some minimal point of (2.1), and hence they are bounded by $\varepsilon$ everywhere. □

**Theorem 4.2.** *Let the system* (2.1) *be nonwandering and suppose that the set of vector fields* $\{V, X_1, \ldots, X_m\}$ *satisfies the Hörmander condition on* $\mathrm{Min}_V$*. Then the system* (3.1) *is controllable with arbitrarily small controls.*

*Proof.* Fix an arbitrary $\varepsilon > 0$. Under the conditions of the theorem being proven we have that for any point $x_0 \in \mathrm{Min}_V$ there is a neighborhood $U(x_0)$ such that the set $\{V, X_1, \ldots, X_m\}$ satisfies the Hörmander condition in $U(x_0)$. Thus by Lemma 4.3 (given below) there us a $\sigma > 0$ such that for every couple of points $\{y_0, y_1\} \subset B_\sigma(x_0)$ one has $y_1 \in \mathcal{A}_{\varepsilon, y_0}$. Since $\mathrm{Min}_V$ is compact (as a closed subset of compact $M$), we may assume without loss of generality $\sigma$ to be independent of $x_0$. Thus conditions of Lemma 4.1 are satisfied, and invoking the latter we get that the system is controllable with controls $|u_j| \le \varepsilon$. Since $\varepsilon > 0$ is arbitrary, then the system is controllable with arbitrarily small controls as claimed. □

The following statement on local controllability was used in the above proof.

**Lemma 4.3.** *Fix an arbitrary* $\varepsilon > 0$*. Let the flow* $\varphi_V$ *be nonwandering, and the point* $x_0$ *be such that the vector fields*

$$\{V, X_1, \ldots, X_m\}$$

*satisfy the Hörmander condition in a neighborhood* $U(x_0)$ *of* $x_0$*. Then there is a* $\sigma > 0$ *such that for every couple of points* $\{y_0, y_1\} \subset B_\sigma(x_0)$ *in the ball* $B_\sigma(x_0) \subset M$ *one has* $y_1 \in \mathcal{A}_{\varepsilon, y_0}$ *(in particular,* $x_0$ *belongs to the interior of the set* $\mathcal{A}_{\varepsilon, x_0}$*).*

*Proof.* We divide the proof into several steps.

**Step 1.** Consider the set of vector fields

$$\mathcal{Y} := \{V, -V\} \cup \{\pm V \pm \varepsilon X_k\}_{k=1}^m.$$

The set $\mathcal{Y}$ still satisfies the Hörmander condition. Thus by the classical Chow-Rashevskii theorem (theorem 17 of [3]) every $x_0 \in \mathbb{R}^d$ admits a neighborhood $U^0(x_0)$ with any two points of $U^0(x_0)$ being linked by a piecewise smooth curve, such that tangent lines to that curve (where they exist) are parallel to one of the vector fields from $\mathcal{Y}$. Up to restricting $U(x_0)$ we may suppose without loss of generality $U(x_0) = U^0(x_0)$.

**Step 2.** Now we claim that for any point $x_1 \in U(x_0) \bigcap \mathcal{A}_{\varepsilon,x_0}$ and any $\theta > 0$ the points $\varphi_V(-\theta, x_1)$ and $\varphi_{V\pm\varepsilon X_i}(-\theta, x_1)$ belong to the closure of the set $\mathcal{A}_{\varepsilon,x_1}$ and, hence, to the closure of $\mathcal{A}_{\varepsilon,x_0}$.

**Step 2.1.** We prove the claim for $\varphi_V(-\theta, x_1)$. Indeed, by Krener's theorem (Lemma 3.3), the interior of the set $\mathcal{A}_{\varepsilon,x_1}$ is non-empty and intersects with the open ball $B_\delta(x_1)$ for any $\delta > 0$. Since the flow $\varphi_V$ is nonwandering, for any $\delta > 0$ there exists a $T_1 > \theta$ and two points $x_2 \in \mathcal{A}_{\varepsilon,x_1} \cap B_\delta(x_1)$ and $x_3 = \varphi_V(T_1, x_2)$ such that $x_3 \in \mathcal{A}_{\varepsilon,x_1} \cap B_\delta(x_1)$.

We observe now that for any $\theta > 0$ the point $x_{3,\theta} := \varphi_V(T_1 - \theta, x_3)$ belongs to the set $\mathcal{A}_{\varepsilon,x_1}$. On the other hand, $x_{3,\theta} = \varphi_V(-\theta, x_3)$ and since $x_3 \in B_\delta(x_1)$, we get $x_{3,\theta} \in \varphi_V(-\theta, B_\delta(x_1))$. Choosing $\delta$ sufficiently small we thus get the point $x_{3,\theta}$ as close to $\varphi_V(-\theta, x_1)$ as we like, which proves the claim.

**Step 2.2.** We now approximate the points $\varphi_{V\pm\varepsilon X_j}(-\theta, x_1)$ (we will do this for $\varphi_{V-\varepsilon X_j}(-\theta, x_1)$, the argument for $\varphi_{V+\varepsilon X_j}(-\theta, x_1)$ being completely symmetric). To this aim, we suggest a construction resembling Euler's numerical method. First, we do it for small values of $\theta$ and then iterate. First of all, we observe that there exists a $C > 0$ depending only on $V$, $X_j$ and $\varepsilon \in \mathbb{R}$, such that the distance between points

$$x_{4,\theta} := \varphi_{V+\varepsilon X_j}(\theta, \varphi_V(-2\theta, x_1)) \quad \text{and } x_{5,\theta} := \varphi_{V-\varepsilon X_j}(-\theta, x_1)$$

does not exceed $C\theta^2/2$. To prove this, it suffices to consider the Taylor decompositions for $x_{4,\theta}$ and $x_{5,\theta}$ as functions of $\theta$, observing that the zero and first order in $\theta$ terms of these decompositions coincide.

Then, similarly to Step 2.1, we take the points $x_2 \in \mathcal{A}_{\varepsilon,x_1} \cap B_\delta(x_1)$ and $x_3 = \varphi_V(T_1, x_2)$ such that $x_3 \in \mathcal{A}_{\varepsilon,x_1} \cap B_\delta(x_1)$ and $T_1 > 2\theta$. We may take $\delta$ so small that the distance between $x_{4,\theta}$ and the point

$$x_{6,\theta} := \varphi_{V+\varepsilon X_j}(\theta, \varphi_V(-2\theta, x_3))$$

be not greater than $C\theta^2/2$. Observe that $x_{6,\theta} \in \mathcal{A}_{\varepsilon,x_1}$ and the distance between $x_{5,\theta}$ and $x_{6,\theta}$ is not greater than $C\theta^2$.

Now take any value $\theta > 0$ and approximate the point $x_{5,\theta} = \varphi_{V-\varepsilon X_j}(-\theta, x_1)$ as follows. For an $N \in \mathbb{N}$ set $\xi_0 := x_1$. Then we recursively define the points $\xi_i$ as follows: every $\xi_i$ belongs to $\mathcal{A}_{\varepsilon,\xi_{i-1}}$ (and, consequently, to $\mathcal{A}_{\varepsilon,x_1}$) and is such that the distance between the points $\xi_i$ and $\varphi_{V-\varepsilon X_j}(-\theta/N, \xi_{i-1})$ is not greater than $C\theta^2/N^2$. In other words, one may take $\xi_1 := x_{6,\theta/N}$, and every $\xi_i$ is constructed from $\xi_{i-1}$ in the same way as $x_{6,\theta/N}$ is constructed from $x_1$. Let $d(\cdot,\cdot)$ be the distance on the manifold $M$. Given the value $\theta > 0$, we take the constant $L > 0$ so that for any pair of points $z_0, z_1 \in M$, any $j = 1, \ldots, m$ and any $\tau \in [0, \theta]$ we have

$$d(\varphi_{V-\varepsilon V_j}(-\tau, z_0), \varphi_{V-\varepsilon V_j}(-\tau, z_1)) \le L d(z_0, z_1),$$

where for the sake of brevity we denote $\varphi := \varphi_{V-\varepsilon V_j}$. The latter formula implies

$$d\left(\varphi\left(-\frac{(i-1)\theta}{N}, \xi_{N-i+1}\right), \varphi\left(-\frac{i\theta}{N}, \xi_{N-i}\right)\right) = \\ d\left(\varphi\left(-\frac{(i-1)\theta}{N}, \xi_{N-i+1}\right), \varphi\left(-\frac{(i-1)\theta}{N}, \varphi\left(-\frac{\theta}{N}, \xi_{N-i}\right)\right)\right) \leq \\ Ld\left(\xi_{N-i+1}, \varphi\left(-\frac{\theta}{N}, \xi_{N-i}\right)\right) \leq \frac{LC\theta^2}{N^2}.$$

Consequently,

$$d(\xi_N, \varphi_{V-\varepsilon X_j}(-\theta, x_1)) = d(\xi_N, \varphi_{V-\varepsilon X_j}(-\theta, \xi_0)) \leq \\ \sum_{i=1}^{N} d\left(\varphi_{V-\varepsilon V_j}\left(-\frac{(i-1)\theta}{N}, \xi_{N-i+1}\right), \varphi_{V-\varepsilon V_j}\left(-\frac{i\theta}{N}, \xi_{N-i}\right)\right) \leq \frac{LC\theta^2}{N}$$

which can be taken arbitrarily small provided $N$ is large.

**Step 3.** By Steps 1 and 2, there exists a $\sigma > 0$ such that the set $\mathcal{A}_{\varepsilon,x_0}$ is dense in $B_\sigma(x_0)$. We prove that in fact every point $y_0 \in B_\sigma(x_0)$ belongs to the set $\mathcal{A}_{\varepsilon,x_1}$. Consider to this aim the reversed-time problem

$$\dot{x} = -V(x) + \sum_{j=1}^{m} u_j(t) X_m(x) \tag{4.4}$$

with initial conditions $x(0) = y_0$ and the same assumptions on the control functions $u_j(t)$. The Hörmander condition is evidently satisfied for the vector fields $\{-V, X_1, \ldots, X_m\}$. Given $y_0$ and $\varepsilon$, let $\widetilde{\mathcal{A}}_{\varepsilon,y_0}$ be the analogue of the set $\mathcal{A}_{\varepsilon,y_0}$ for system (4.4).

By the Krener's theorem (Lemma 3.3), applied to (4.4), the closure of the interior of the set $\widetilde{\mathcal{A}}_{\varepsilon,y_0}$ contains the point $y_0$, hence in particular the latter interior is non-empty. But due to the density of $\mathcal{A}_{\varepsilon,x_0}$ this interior of $\widetilde{\mathcal{A}}_{\varepsilon,y_0}$ contains a point $z_0 \in B_\sigma(x_0) \cap \mathcal{A}_{\varepsilon,x_0}$. Evidently, $y_0 \in \mathcal{A}_{\varepsilon,z_0} \subset \mathcal{A}_{\varepsilon,x_0}$, which shows the claim of this step.

**Step 4.** Applying now the claim of Step 3 to the reversed-time problem (4.4), we get that $x_0$ belongs to the interior of $\widetilde{\mathcal{A}}_{\varepsilon,x_0}$. Thus there is a $\sigma > 0$ such that for every couple of points $\{y_0, y_1\} \subset B_\sigma(x_0)$ one has $y_0 \in \widetilde{\mathcal{A}}_{\varepsilon,x_0}$, hence $x_0 \in \mathcal{A}_{\varepsilon,y_0}$, and, by Step 3, $y_1 \in \mathcal{A}_{\varepsilon,x_0}$, and therefore $y_1 \in \mathcal{A}_{\varepsilon,y_0}$ concluding the proof. □

## 5. The chain recurrent drift

Now, we only assume that the flow $\varphi_V$ is chain recurrent and show that, for the controllability of the respective system, it suffices to have the validity of the restricted Hörmander condition (only in vector fields $\{X_1, \ldots, X_m\}$) on the closure of the set of minimal points. Here, however, we do not prove that the control functions $u_j$ can be chosen small. Moreover, in our construction, they are large compared to the drift.

**Theorem 5.1.** *Let the system (2.1) be chain recurrent and the set of vector fields $\{X_1, \ldots, X_m\}$ satisfy the Hörmander condition on $\mathrm{Min}_V$. Then (3.1) is controllable in the whole manifold $M$.*

*Proof.* As in the proof of Theorem 4.2 it suffices to verify the validity of conditions of Lemma 4.1 and then to invoke the latter. To do this we use Lemma 5.2 below instead of Lemma 4.3. □

The following local statement (in fact, very similar to proposition 27 of [3]) has been used in the above proof.

**Lemma 5.2.** *Let $x_0 \in M$ be such that the set of vector fields*

$$\{X_1, \ldots, X_m\}$$

*satisfy the Hörmander condition in a neighborhood $U(x_0)$ of $x_0$. Then there is a $\sigma > 0$ such that for every couple of points $\{y_0, y_1\} \subset B_\sigma(x_0)$ in the ball $B_\sigma(x_0) \subset M$ there is an $\varepsilon > 0$ (possibly not small, and depending on $y_1$, $y_0$) such that one has $y_1 \in \mathcal{A}_{\varepsilon,y_0}$ (in particular, $x_0$ belongs to the interior of the set $\mathcal{A}_{x_0}$).*

*Proof.* For a $\gamma \geq 0$ consider the problem

$$\dot{x} = \gamma V(x) + \sum_{j=1}^{m} u_j(t) X_m(x), \tag{5.1}$$

and the respective attainable set $\mathcal{A}^\gamma_{\varepsilon,p} \subset M$ of all the points $q$ such that the boundary value problem for some system (5.1) with all $|u_j(t)| \leq \varepsilon$ and boundary conditions

$$x(0) = p, \qquad x(\theta) = q$$

is solvable for some $\theta > 0$. Since by reparameterizing time we get the equivalence of (5.1) and

$$\dot{x} = V(x) + \frac{1}{\gamma}\sum_{j=1}^{m} u_j(t) X_m(x), \tag{5.2}$$

we have

$$\mathcal{A}^\gamma_{\varepsilon,p} = \mathcal{A}_{\varepsilon/\gamma,p}.$$

In the same way we consider time-reversed problem

$$\dot{x} = -\gamma V(x) + \sum_{j=1}^{m} u_j(t) X_m(x), \tag{5.3}$$

and the respective attainable set $\widetilde{\mathcal{A}}^\gamma_{\varepsilon,p} \subset M$ of all the points $q$ such that the boundary value problem for some system (5.1) with all $|u_j(t)| \leq \varepsilon$ and boundary conditions $x(0) = p$, $x(\theta) = q$ for some $\theta$, and note that $q \in \widetilde{\mathcal{A}}^\gamma_{\varepsilon,p}$, if and only if $p \in \mathcal{A}^\gamma_{\varepsilon,q}$. We omit the reference to $\gamma$ when $\gamma = 1$, since in this case we deal with our original system and its time-reversal, and write just $\mathcal{A}_{\varepsilon,p}$ and $\widetilde{\mathcal{A}}_{\varepsilon,p}$ instead of $\mathcal{A}^1_{\varepsilon,p}$ and $\widetilde{\mathcal{A}}^1_{\varepsilon,p}$ respectively.

First of all, we observe that due to the Chow-Rashevskii theorem (theorem 17 of [3]), applied to the set of vector fields $\{X_1, \ldots, X_m\}$, we have that $x_0$ is an interior point of the set $\mathcal{A}^0_{1,x_0}$. In view of continuous dependence of the solutions of the ODE on parameters (in this case on $\gamma$), we have that $\cup_{\gamma>0}\mathcal{A}^\gamma_{1,x_0}$ is dense in some ball $B_\sigma(x_0) \subset U(x_0) \subset M$. Consider an arbitrary $y_0 \in B_\sigma(x_0)$ and $\delta > 0$. By the Krener's theorem (Lemma 3.3), applied to (5.3), the closure of the interior of the set $\widetilde{\mathcal{A}}^\delta_{\varepsilon,y_0}$ contains $y_0$, hence in particular the latter interior is non-empty. But due to the density of $\cup_{\gamma>0}\mathcal{A}^\gamma_{1,x_0}$ this interior of $\widetilde{\mathcal{A}}^\delta_{1,y_0}$ contains a point $z_0 \in B_\sigma(x_0) \cap \mathcal{A}^\gamma_{1,x_0}$ for some $\gamma > 0$. But $y_0 \in \mathcal{A}_{\varepsilon,z_0}$ hence $y_0 \in \mathcal{A}_{\varepsilon,z_0}$ and $z_0 \in \mathcal{A}_{\varepsilon,x_0}$ with $\varepsilon := \max\{\gamma, \delta\}$, so that $y_0 \in \mathcal{A}_{\varepsilon,x_0}$. In other words, $B_\sigma(x_0) \subset \mathcal{A}_{\varepsilon,x_0}$.

Applying the latter result to the time-reversed system (i.e. to (5.3) with $\gamma := 1$), we get $B_\sigma(x_0) \subset \widetilde{\mathcal{A}}_{\varepsilon,x_0}$. Thus taking $\{y_0, y_1\} \subset B_\sigma(x_0)$, we get $y_0 \in \widetilde{\mathcal{A}}_{\varepsilon,x_0}$, so that $x_0 \in \mathcal{A}_{\varepsilon,y_0}$, and $y_1 \in \mathcal{A}_{\varepsilon,x_0}$, and hence $y_1 \in \mathcal{A}_{\varepsilon,y_0}$ concluding the proof. □

## 6. A non-controllable system with a chain recurrent drift

It is quite natural to ask whether the system with a chain-recurrent drift $V$ and the set of vector fields $\{V, X_1, \ldots, X_m\}$ (including the drift) satisfying the Hörmander condition is controllable. The counterexample given in the next section shows that the answer could be negative even if the Hörmander condition is fulfilled everywhere.

Let $M$ be the two-dimensional flat torus defined as usual as the square $[0, 2\pi] \times [0, 2\pi]$ with opposite sides identified. Consider the system of ordinary differential equations on $M$

$$\begin{cases} \dot{x} = 0, \\ \dot{y} = (1 - \cos x) \sin y \end{cases} \tag{6.1}$$

Let $z := (x, y)^T$; $V(z) := (0, (1-\cos x) \sin y)^T$ be the vector field of the system (6.1), and $\varphi_V$ be the respective flow (see Fig. 2), where the superscript $T$ stands for the matrix transposition.

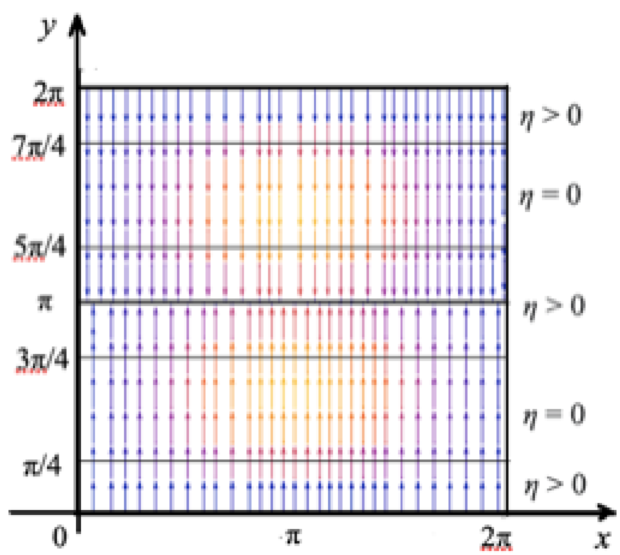


Figure 2. The drift vector field of (6.1) and the function $\eta$

All points $(0, y)$, $y \in [0, 2\pi]$, $(x, 0)$, $(x, \pi)$, $x \in [0, 2\pi]$ are stationary, and the system (6.1) has no other nonwandering points. Nevertheless, the following statement is true.

**Lemma 6.1.** *The flow induced by the system (6.1) is chain transitive (and, hence, chain recurrent).*

*Proof.* We show how a point $(x_1, y_1)$ may be attained from a point $(x_0, y_0)$ with controls with arbitrarily small uniform norm.

*Step 1.* The set of stationary points of (6.1) is the union of circles

$$S := S_1 \cup S_2 \cup S_3, \quad \text{where}$$
$$S_1 := \{(0, y)\colon y \in [0, 2\pi]\}, S_2 := \{(x, 0)\colon x \in [0, 2\pi]\}, S_3 := \{(x, \pi)\colon x \in [0, 2\pi]\}.$$

Every point of $S_1$ can be reached from any other point of $S_1$ in finite time by applying an arbitrarily small control along the vertical direction. Similarly, every point of $S_2$ can be reached from any other point of $S_2$ in finite time by applying an arbitrarily small control along the horizontal direction, and the same is true for $S_3$. Therefore, since $S$ is connected, every point of $S$ can be reached from any other point of $S$ in finite time by applying an arbitrarily small control.

*Step 2.* If a point $(x_0, y_0)$ is not stationary, then its $\omega$=limit set is a singleton $\{(x_0, \pi)\}$. We can reach any neighborhood of the latter point from $(x_0, y_0)$ without any control (that is, just following the flow of $V$) and then apply a small control along the vertical direction to reach the point $(x_0, \pi) \in S$ in finite time.

*Step 3.* A non-stationary point $(x_1, y_1)$ with $y_1 \in (\pi, 2\pi)$ can be reached in finite time from the point $(2\pi, y_1) \in S$ by a small control along the vertical direction and then following the flow of $V$. Analogously, if $y_1 \in (0, \pi)$, then a nonstationary point $(x_1, y_1)$ can be reached in finite time from the point $(0, y_1) \in S$ by a small control along the vertical direction and then following the flow of $V$.

*Step 4.* Combining Steps 1, 2, and 3 we get that every point of $M$ can be reached in finite time from any other point of $M$ by arbitrarily small controls, which proves the lemma. □

Now, we consider a function $\eta \in C^\infty(S^1)$ such that

$$\eta(x) > 0 \quad \text{for } x \in [0, \pi/4) \bigcup (3\pi/4, 5\pi/4) \bigcup (7\pi/4, 2\pi]$$

(see Fig. 2). Introduce the vector fields on the torus by the following formulas:

$$X_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad X_2 = \begin{pmatrix} 0 \\ \eta(y) \end{pmatrix}.$$

**Lemma 6.2.** *The triple $\{V, X_1, X_2\}$ satisfies the Hörmander condition on all the torus $M$.*

*Proof.* For any $(x, y) \in M$, such that $x \neq 0$, $y \neq 0$, $y \neq \pi$, the vectors $V(x, y)$ and $X_1(x, y)$ form the basis of the tangent space $T_{(x,y)}M$. The vectors $X_1(x, y)$ and $X_2(x, y)$ span the tangent space at the points $(x, 0)$ and $(x, \pi)$ for any $x \in [0, 2\pi)$.

Let us calculate now the Lie bracket

$$[V, X_1](x, y) = DX_1(x, y)V(x, y) = \frac{\partial V}{\partial x}(x, y) = \begin{pmatrix} 0 \\ \sin x \sin y \end{pmatrix}.$$

This vector field vanishes at $x = 0$. However, the consequent Lie bracket

$$[[V, X_1], X_1](x, y) = \frac{\partial}{\partial x}[V, X_1](x, y) = \begin{pmatrix} 0 \\ \cos x \sin y \end{pmatrix}$$

is nonzero for $x = 0$ and $y \notin \{0, \pi\}$. This proves the lemma. □

Nevertheless, the following statement is true.

**Theorem 6.3.** *The system $\dot{z} = V(z) + u_1(t)X_1(z) + u_2(t)X_2(z)$, where $z := (x, y)^T$, is not controllable.*

*Proof.* Suppose that there exist piecewise continuous functions $u_1, u_2\colon [0,T] \to \mathbb{R}$ such that the problem

$$\begin{cases} \dot z(t) = V(z(t)) + u_1(t)X_1(z(t)) + u_2(t)X_2(z(t)), \\ z(0) = (0,\pi), \qquad z(T) = (0,0) \end{cases}$$

is solvable for some $T > 0$. Consider the solution $z(\cdot) = (x(\cdot), y(\cdot))^T$. Since $y(0) = \pi$, $y(T) = 0$, at least one of the following statements is true:

(1) there is $t_0 \in (0,T)$ such that $y(t_0) \in (\pi/4, 3\pi/4)$, $\dot y(t_0) < 0$ or
(2) there is a $t_0 \in (0,T)$ such that $y(t_0) \in (5\pi/4, 7\pi/4)$, $\dot y(t_0) > 0$.

Without loss of generality, we assume that the first one is true (if the second inclusion takes place, the proof is similar). Then $\eta(y(t_0)) = 0$ and, consequently, $X_2(z(t_0)) = 0$. Since the second component of the vector field $X_1$ is zero, we have

$$\dot y(t_0) = (1 - \cos x(t_0)) \sin y(t_0) \geq 0$$

which contradicts our assumption. This proves the lemma. □

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