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Currents and Curvature Varifolds in Continuum Mechanics

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Dedicated to Nina Nikolaevna Uraltseva.

Abstract. The use in continuum mechanics of weak diffeomorphisms, Cartesian currents and curvature k-varifolds with boundary is discussed. The attention is focused on the analysis of the existence of minimizers for the energies of elastic complex bodies and bodies admitting cracks. The interplay between the analytical features and the physical needs is stressed.

1. Mechanical reasons and some historical remarks

The scenario opens on the problem of finding minimizers (ground states) of the energy of elastic simple Cauchy bodies undergoing large deformations. Ingredients are: (i) a minimalist description of the morphology of a body obtained only through its gross place in the physical space, a geometrical picture defining the so-called Cauchy bodies; (ii) an energy varying along transplacements describing changes of places. Convenience suggests to manage two (isomorphic) copies of the physical ambient space, say $\mathbb{R}^d$ and $\hat{\mathbb{R}}^d$. The first copy hosts a place $\mathcal{B}$, an open, connected set with Lipschitz boundary, that can be in principle occupied by the body and is taken as a reference. One-to-one, differentiable, orientation-preserving maps

$$x \longmapsto u := u\,(x) \in \hat{\mathbb{R}}^d, \quad x \in \mathcal{B},$$

allow one to reach new places $u\,(\mathcal{B})$.

The constitutive assumption of a conservative setting for both material properties and external actions justifies the presumption of the existence of an energy $\mathcal{E}\,(u, \mathcal{B})$. The adjective 'simple', used in specifying the bodies under scrutiny, underlines that the bulk elastic energy depends on u through its spatial first derivative only. The total energy to be analyzed is a 3-form over the first jet of the trivial fiber bundle $\mathcal{B} \times \hat{\mathbb{R}}^d$ with sections defined by the transplacement maps $x \longmapsto u\,(x)$. It reads

$$\mathcal{E}\,(u, \mathcal{B}) := \int_{\mathcal{B}} e\,(x, Du\,(x))\ dx + \int_{\mathcal{B}} \hat{e}\,(u\,(x))\ dx,$$

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where $e(\cdot)$ is the *bulk elastic energy* and $\hat{e}(\cdot)$ the *potential of external actions*. Boundary conditions of Dirichlet type are presumed here for the map $x \longmapsto u(x)$. The prescription of invariance with respect to isometric changes in observers, that is, invariance with respect to the action of the Euclidean group $\hat{\mathbb{R}}^d \ltimes SO(d)$ over $\hat{\mathbb{R}}^d$, implies that $e(\cdot)$ cannot be convex in $Du(x)$, the gradient of deformation. Such a result (see [**39**]) opened the problem of finding minimal physically significant requirements allowing the existence of minimizers of $\mathcal{E}(\cdot, \mathcal{B})$. The notion of polyconvexity of $e(\cdot)$ [**30**] has been the decisive ingredient for proving the existence of minimizers in nonlinear elasticity (see [**4**]). Minimizers have been found to be only locally weak invertible maps. Global invertibility was established later under appropriate Dirichlet boundary conditions [**5**]. Further results in that spirit are in [**37, 38, 33, 32**]. A critical review of the subsequent work can be found in [**6**]. More precisely, $W^{1,p}(\mathcal{B}, \mathbb{R}^3)$ is the Sobolev space hosting naturally the minimizers of $\mathcal{E}(\cdot, \mathcal{B})$. Their existence accrues from classical results, e.g. De Giorgi's semicontinuity result [**20**], once the weak compactness with respect to sequences of minors of the gradient of deformation Du is established. The condition is assured straight away when deformations admit bounded energy in $W^{1,p}$, $p > 3$. In the case $p = 3$, it has been found in [**31**] by using the orientation-preserving property of the deformations. When $p < 3$ another problem arises: in fact, nonnegligible is the presence of transplacement maps with graphs admitting boundaries with projections into the interior of $\mathcal{B}$. Such boundaries describe the formation of 'holes' and/or open 'fractures' of various natures. The need of avoiding the presence of singular sets, a need dictated by the will of describing *purely* elastic deformations, which are deformations with no material threshold allowing a phase transition which might determine plastic flows or rupture of material bonds, opened a research program leading to the notion of *weak diffeomorphisms* defined in [**14**]. Such maps are in essence transplacements satisfying a special condition, the so-called *condition of zero boundary*, which prevents the formation of undesired 'holes' or open 'fractures'. Weak diffeomorphisms are, in fact, orientation-preserving maps which allow the frictionless contact of parts of the external boundary of the body and prevent the self-penetration of the matter. The existence of minimizers of $\mathcal{E}(\cdot, \mathcal{B})$ in the class of weak diffeomorphisms has been established in [**14**].

The definition of weak diffeomorphisms is based, amid other things, on properties of the d-current integrations (*currents*, for short) over the graphs of them. The current associated with the map $x \longmapsto u(x)$ is, in essence, a linear functional, indicated by G_u, over smooth d-forms with compact support in $\mathcal{B} \times \hat{\mathbb{R}}^d$. The boundary ∂G_u of G_u is defined by duality: $\partial G_u(\omega) = G_u(d\omega)$, where ω is a generic $(d-1)$-form with compact support over $\mathcal{B} \times \hat{\mathbb{R}}^d$, provided that $\partial G_u(\omega) = 0$ on $\mathcal{B} \times \hat{\mathbb{R}}^d$. The condition of zero boundary appearing in the definition of weak diffeomorphisms is thus $\partial G_u(\omega) = 0$. The analysis in terms of currents provides us with the weak compactness with respect to the sequences of minors of Du for $p \leq 3$ (see also [**16**]). For $p > 3$, the condition $\partial G_u(\omega) = 0$ is obviously satisfied.

Roughly speaking, the essential idea followed in defining currents is to control the properties of the graphs of the maps under scrutiny by duality, that is, through the properties of functionals involving the graphs themselves and forms over the product space in which the graphs are defined. The technique is of interest per se in the calculus of variations (see the treatise [**17**]). It is useful for the analysis of a number of problems in different settings.

In dealing with nonlinear elasticity of simple bodies, a problem is the possible appearance of a gap phenomenon (see related discussions in [**18**]). Results describing the approximation of weak diffeomorphisms with regular maps are, in fact, not available. A gap phenomenon appears in the case of energies constructed over general manifold-valued maps. The physical setting falling within this scheme is, for example, the one of spin structures admitting Dirichlet energy (also called the Frank energy) when they self-organize and do not undergo gross deformation. The mathematical scenario has deep features. The first result mentioned here dates back to 1964: Every smooth map $\nu : \mathfrak{X} \longrightarrow \mathcal{Y}$ between two compact, differentiable Riemannian manifolds can be deformed in its own homotopy class into a minimizer of the Dirichlet energy

$$\frac{1}{2}\int_{\mathfrak{X}} |D\nu|^2 \ dvol_{\mathfrak{X}},$$

provided that the sectional curvature, $\sigma_{\mathcal{Y}}$, of $\mathcal{Y}$ is nonnegative [**16**]. A distinguished representation of the homotopy class can be constructed this way. The question was whether the condition $\sigma_{\mathcal{Y}} \leq 0$ could be dropped or weakened.

Consider a geodesic ball $B_R(p)$ in $\mathcal{Y}$ and suppose that (i) $B_R(p)$ is disjoint from the cut locus of its centre p, (ii) the radius R satisfies the inequality

$$R < \frac{\pi}{2\sqrt{k}}.$$

The scalar k is determined by the supremum of the sectional curvature of $\mathcal{Y}$ in $B_R(p)$. When such a supremum is negative, k is set equal to 0. A class C_φ of maps $\nu : B^n \longrightarrow B_R(p)$ is defined over the unit ball B^n in $\mathbb{R}^n$. The index φ indicates that the members of C_φ agree with a boundary datum φ over ∂B_R. Results in [**21**] show the existence of a unique harmonic map ν in C_φ which is a minimizer of the Dirichlet energy and is smooth. When $\mathcal{Y}$ is the standard n-dimensional unit sphere S^n in $\mathbb{R}^{n+1}$, minimizers of the Dirichlet energy amid maps with range in a fixed hemisphere, say the upper hemisphere, are smooth. Moreover, if one considers differentiable maps $\nu : B^n \to S^n$, the *equator map*

$$\left(0, \frac{x}{|x|}\right)$$

is a weak harmonic map. It is stable if $n \geq 8$. In fact, every stationary harmonic map from B^n into S^n is regular up to dimension 7 [**19, 37**] according to the fact that the normal vector field to the minimal surface defines a harmonic map. If one considers maps $\nu : B^n \to S^2$, with $n \geq 3$ and no further restriction, the map $\nu(x) = \frac{x}{|x|}$ is also a minimizer of the Dirichlet energy [**9**].

The dimension of the base and the target manifold play a role in the analysis. Consider, for example, differentiable maps $\nu : B^3 \to S^2$ and their associated Dirichlet energies. Take a Lipschitz map $\varphi : \partial B^3 \to S^2$ and define $W^{1,2}_\varphi\left(B^3, S^2\right)$ to be the space of the $W^{1,2}\left(B^3, S^2\right)$-map with traces agreeing with φ on ∂B^3. It is not empty because at least $\nu(x) := \varphi(x/|x|)$ belongs to it. A gap phenomenon occurs for the Dirichlet energy, namely

$$\inf_{W^{1,2}_\varphi(B^3,S^2)} \int_{B^3} |D\nu(x)|^2 \ dx < \inf_{W^{1,2}_\varphi(B^3,S^2)\cap C^0(B^3,S^2)} \int_{B^3} |D\nu(x)|^2 \ dx,$$

even when $\deg\varphi = 0$ [**20**]. Notice that φ has no smooth extension to B^3 with values in S^2 when $\deg\varphi \neq 0$.

A question is the evaluation of the relaxed energy, that is, the limit value of the Dirichlet energy obtained by using special sequences of maps, selected in some way. Two strategies have been followed. The first one involves sequences $\{\nu_k\}$ in $C^1\left(B^3, S^2\right)$ converging weakly in $W^{1,2}$, namely $\nu_k \rightharpoonup \nu$ in $W^{1,2}\left(B^3, S^2\right)$. In evaluating the relaxed Dirichlet energy as $k \to \infty$, one finds an energy involving the occurrence of point charges with degree equal to ± 1 [**9**]. The second strategy is on the geometric side: sequences $\{\nu_k\}$ in $C^1\left(B^3, S^2\right)$ are once more considered, but their convergence is evaluated in terms of currents. A generic sequence $\{\nu_k\}$ is said to be convergent when the associated sequence of currents converges weakly, that is, when $G_{\nu_k} \rightharpoonup T$. In order to evaluate the relaxed energy one has first to write the energy in terms of currents, precisely in terms of the 3-vector orienting the graph of the generic ν, a 3-vector given by the Radon-Nikodým derivative of the current associated with ν with respect to its total variation. The new representation of the Dirichlet energy is called the *parametric extension*. A *strong density result* in [**15**] shows that the relaxed Dirichlet energy evaluated as before over sequences converging in terms of currents, namely for $G_{\nu_k} \rightharpoonup T$, coincides with its parametric extension to T. Even in this case singularities appear. They are charges connected by line defects where 'fusion' of the spin structures occurs: the spin field becomes multi-valued over the line, at each ν taking as value the entire S^2. A preference between the two methods summarized above is addressed primarily by physical instances. In fact, the selection of a functional space in which analyses are developed is essentially a constitutive prescription when one manages models with some physical meaning.

From the abstract side, refinements and extensions of the theory of the currents, namely the definition and the analysis of entities called the *semi-currents* (see [**3**]) and the *Cartesian currents* in $cart^{2,1}\left(B^n \times \mathcal{Y}\right)$ ([**23**]), are tools for the analysis of Dirichlet energies over differentiable maps $\nu : B^n \to \mathcal{Y}$ and $\nu : \mathfrak{X} \to \mathcal{Y}$, where $\mathfrak{X}$ and $\mathcal{Y}$ are as above. Precisely, semi-currents over the trivial bundle $B^n \times \mathcal{Y}$ with basis B^n are currents evaluated by testing the graphs of the maps under analysis over a compactly supported smooth form having at most two differentials in the fiber $\mathcal{Y}$. They represent limits of sequences of smooth maps $\nu : B^n \to \mathcal{Y}$ with equibounded Dirichlet energies. When such currents have (i) no boundary, (ii) finite Dirichlet energy and (iii) they admit a special representation of the weak limit of sequences of graphs of smooth maps with bounded Dirichlet energies, they are called Cartesian currents in $cart^{2,1}\left(B^n \times \mathcal{Y}\right)$ (see [**23**]). Precisely, consider smooth maps $\nu_k : B^n \to \mathcal{Y}$ and related bounded Dirichlet energies. Denote by $\gamma_1 ... \gamma_R$ a basis of integral cycles in the spherical subgroup $H_2^{sph}\left(\mathcal{Y}\right)$, by $\mathbb{L}_r\left(T\right)$ an $(n-2)$-dimensional current in B^n and by S_T a current which does not vanish only on forms which have nonzero differential of the component of the forms with exactly two differentials over $\mathcal{Y}$. The statement that the limit T of G_{ν_k}, as $k \to \infty$, is in $cart^{2,1}\left(B^n \times \mathcal{Y}\right)$ means that there exists a $W^{1,2}\left(B^n, \mathcal{Y}\right)$-map ν_T such that

$$T = G_{\nu_T} + \sum_{r=1}^{R} \mathbb{L}_r\left(T\right) \times \gamma_r + S_T,$$

when the second integral homology group $H_2\left(\mathcal{Y}\right)$ of $\mathcal{Y}$ is torsion-free. The map ν_T defines uniquely the representation of T through its essential features. In such a representation, the geometrical properties of the target manifold $\mathcal{Y}$ and the dimension of B have an influence. In fact, when $n \geq 3$, an additional requirement is that for any generic 2-dimensional subspace K of $\mathbb{R}^n$, the restriction of T to $K \times \mathcal{Y}$ is a 2-dimensional current in $cart^{2,1}\left((B^n \cap K) \times \mathcal{Y}\right)$.

Weak closedness of $cart^{2,1}\left(B^n \times \mathcal{Y}\right)$ can then be proven (see the details of the proof in [**23**]). Extensions of the strong density result mentioned above can be achieved in terms of Cartesian currents (see [**17, 18**] and the references therein). The final result (see [**18**] and the references therein) is that the relaxed Dirichlet energy evaluated on sequences $\{\nu_k\}$ in $C^1\left(B^n, \mathcal{Y}\right)$ converging weakly (write $\nu_k \rightharpoonup \nu$) in $W^{1,2}\left(B^n, \mathcal{Y}\right)$ equals the infimum of the parametric extension of the energy to Cartesian currents with characteristic map ν_T equal to the limit ν of ν_k.

Additional aspects of the analysis of energies associated with maps between manifolds opened further problems. For example, when $\mathfrak{X}$ is the unit disc and $\mathcal{Y}$ a compact manifold, lack of (sequential) compactness appears as a consequence of invariance with respect to horizontal conformal changes, which are conformal changes of $\mathfrak{X}$ [**35, 40**].

The nontrivial homotopy of the target manifold $\mathcal{Y}$ plays a nontrivial role in the analyses. Consider $\mathcal{Y}$ to be simply connected and such that its second homotopy group $\pi_2(\mathcal{Y})$ is trivial, namely $\pi_2(\mathcal{Y}) = 0$. A problem is to check whether minimizers of the Dirichlet energy on maps $\nu : B^1 \longrightarrow \mathcal{Y}$ are regular. The positive answer was presented in [**24**]: stationary points are regular; the result is a consequence of the absence of the so-called harmonic spheres, which are forbidden by the condition $\pi_2(\mathcal{Y}) = 0$. Stationary harmonic maps have singular sets of dimension at most $n-2$ [**7**]. Nonstationary harmonic maps display moreover scattered discontinuities: in 1995, Rivière found harmonic maps that are everywhere singular [**34**].

The attention then shifted from the Dirichlet energy

$$\int_{\mathfrak{X}} |D\nu|^2 \; dvol_{\mathfrak{X}},$$

then to more complicated quadratic energies such as, for example, Oseen-Frank energy for liquid crystals, the density of which is

$$\begin{aligned} e(\nu, D\nu) &= k_1 (\text{div } \nu)^2 + k_2 (\nu \cdot \text{curl } \nu + q) \\ &\quad + k_3 |\nu \times \text{curl } \nu|^2 + \gamma \left(\text{tr}(D\nu)^2 - (\text{div } \nu)^2\right), \end{aligned}$$

which satisfies the head-to-tail symmetry $\nu \to -\nu$ for the map $\nu : \mathcal{B} \to P^2$, with P^2 the projective plane, and is objective in the sense that it is invariant with respect to the action of the special orthogonal group over P^2. Intrinsic in this last remark is a refinement of the standard notion of observer arising in the structure of the mechanics of complex bodies (see [**27**]). Of course, since the macroscopic deformation is not accounted for in the expression of $e(\nu, D\nu)$, actual and reference shapes are identified through an isomorphism $i : \mathbb{R}^d \to \hat{\mathbb{R}}^d$. For this reason, the differential operator D can be considered as a spatial derivative with respect to the actual places of the material elements. Oseen-Frank energy evaluated along maps $\nu : \mathcal{B} \to S^2$ has been analyzed so far as a natural extension of the prototype Dirichlet energy. As mentioned above, candidates to be minimizers of energies with Dirichlet's or Oseen's-Frank's structures display concentrations of energy over co-dimension 2 sets. A natural manner of analyzing the essential characteristics of these concentrations is to evaluate the graphs of the maps under scrutiny. Precisely, one considers the values that graphs take over forms compactly supported over a trivial bundle with basis $\mathcal{B}$ and typical fiber S^2. Cartesian currents come then once more into play. Minimizers of the energies under analysis are then naturally expressed in terms of them (see [**17**]).

The mixture of the result mentioned above has allowed the construction of a tool for the investigation of the existence of ground states in elastic complex bodies. They are bodies with complicated material texture admitting changes (mutations) which have a drastic influence on the overall mechanical behavior. Such an influence is exerted through interactions that can be hardly represented only by means of standard stresses. Quasicrystals, ferroelectrics, polymeric bodies or magnetizable materials are paradigmatic examples. A multifield description of their morphology is called upon. It is intrinsically multi-scale and multidimensional. In fact, besides the standard transplacement maps, describing the macroscopic deformation, a manifold-valued map $\nu : \mathcal{B} \to \mathcal{M}$ comes into play to account for the main features of the material morphology *inside* the generic material element (the essential elements of the mechanics of complex bodies can be found in [**11, 26, 29**]). The existence result of minimizers of the pertinent elastic energy in large deformations has been presented in [**28**]. The essential features of the procedure are summarized below because they are a synthesis of both the results in finite elasticity of simple bodies in terms of Cartesian currents and the ones on the energies associated with maps between manifolds. What one obtains is not properly the trivial superposition of the two aspects because there is energetic coupling between macroscopic deformation and changes in the material morphology at low scales in space. It is recognized that minimizers are weak diffeomorphisms in terms of transplacements and Sobolev maps in terms of morphological descriptors. Macroscopic deformations are orientation-preserving maps that avoid the interpenetration of matter and the formation of holes. Weak diffeomorphisms minimizing the energy have graphs without boundary, in fact.

However, fractures occur in nature. When a variational description of cracked bodies is adopted by following [**19**], minimality of the energy at every time among all virtual crack-transplacement fields at that time is required. When time evolution occurs, energy conservation also has to be imposed throughout. The difficulty of managing crack geometries in finding minimizers has suggested also the convenient simplification of identifying cracks with the jump sets of displacement fields. BV functions have then been involved [**1, 14**] as candidates to be minimizers of the elastic energy of simple bodies. A difficulty arises: theorems allowing the selection of fields with discontinuity sets describing reasonable (physically significant) crack patterns seem to be not available, at least in the current literature; a detailed review on this topic can be found in [**8**]. In any case, this kind of approach seems not to be able to account for partially opened cracks. In fact, in this case, the transplacement field is continuous across the closed part of the crack although the material bonds are broken. A way to account for these aspects is to resort to k-dimensional varifolds, which are measures on a fiber bundle based on $\mathcal{B}$ and constructed taking as a prototype fiber a Grassmanian of k-planes selected over $\mathcal{B}$. Cracks are then intended as supports of varifolds which describe the lateral surfaces of the cracks and the tips. If the treatment is restricted to the case of simple Cauchy bodies for the sake of simplicity, an extended notion of weak diffeomorphisms has to be used in order to represent the main properties of transplacement maps allowing cracks. The possibility of boundaries in the graphs of such maps must be considered. Each boundary accounts for a jump in the transplacement field. To assure that such jumps occur only within the cracked part of $\mathcal{B}$, the mass of the relevant boundary current is assumed to be bounded by the measure of the support of the varifolds accounting for the potential crack patterns. Energy along the margins of the cracks is considered, in accordance with Griffith's original suggestion. Minimizers of the overall energy; i.e., the sum of bulk, surface and tip contributions, are then pairs of transplacement maps and varifolds. The control of minimizing sequences is assured by presuming that the surface energy depends on the curvature of the crack, rather than the sole area as in Griffith's model. For $\mathcal{B} \subset \mathbb{R}^3$, the resulting energy is then

$$\begin{aligned}\mathcal{E}(u, V, \mathcal{B}) \quad : \quad &= \int_{\mathcal{B}} e(x, Du)\ dx + \sum_{k=1}^{2} \alpha_k \int_{\mathcal{G}_k(\mathcal{B})} \left|\mathbf{A}_{(k)}\right|^{p_k}\ dV_k \\ &+ \sum_{k=1}^{2} \beta_k \mathbf{M}(V_k) + \gamma \mathbf{M}(\partial V_1),\end{aligned}$$

where $e(x, Du)$ is the bulk elastic ener[illegible] the potential of external bulk actions, namely $\tilde{e}(u)$, can be added, and the term third addendum is the sum of the surface and tip energies which are just proportional to the measure of the lateral surfaces of the crack and the tip. These ingredients correspond to the standard Griffith's approach. The extension with respect to Griffith's approach is constituted by the presence of the second and the third addenda. Precisely, the second addendum accounts for the dependence of the surface and line energies on the curvature, $\mathbf{A}_{(k)}$ is in fact the curvature of the k-dimensional varifold V_k, while the third addendum measures the energy at possible corners along the tip. The functional dependence on $\mathbf{A}_{(k)}$ is intended as a measure of the macroscopic influence of low scale effects in the debonding process. The symbol $\mathbf{M}(\cdot)$ represents the 'mass' of the varifold introduced in parentheses. Existence results of pairs of extended weak diffeomorphisms and varifolds minimizing the energy introduced above have been collected in [**22**]. The minimization process is over a class of bodies. It is just this circumstance that renders compatible the variational description sketched above with the intrinsic dissipative nature of the fracture mechanics. An advantage of the formulation is the possibility of deriving the weak form of balance equations even when one knows only that the fracture set is just a k-dimensional rectifiable set.

Extensions to the case of complex bodies seems to be available under additional conditions on the surface energy, which accounts for the morphological descriptor fields.

The fast bird's eye view of the functional methods to analyze various types of energies describing crucial offsprings of condensed matter physics presented so far has not touched on evolution phenomena. The analysis of the conservation laws arising in the general model-building framework of the mechanics of complex bodies is still an open issue. Open also is the description of the evolution of cracks in terms of varifolds.

2. A first tool: Cartesian currents

The reasons for considering graphs of maps and the associated currents, in analyzing the energies of simple and complex deformable bodies, have been mentioned above. Some details may clarify those remarks.

Let $u : \mathcal{B} \to \hat{\mathbb{R}}^d$ be a member of $W^{1,1}(\mathcal{B}, \hat{\mathbb{R}}^d)$. Denote by $M(F)$ the so-called d-vector collecting all the minors of F (namely, the minors of $Du(x)$). $M(F)$ is then at each $x \in \mathcal{B}$ an element of $\Lambda_d(\mathbb{R}^d \times \hat{\mathbb{R}}^d)$, the space of d-vectors with the associated dual space $\Lambda^d(\mathbb{R}^d \times \hat{\mathbb{R}}^d)$. Forms over $\mathcal{B}$ are then maps $\omega : \mathcal{B} \to \Lambda^d(\mathbb{R}^d \times \hat{\mathbb{R}}^d)$. Their space is indicated by $\mathcal{D}^d(\mathcal{B} \times \hat{\mathbb{R}}^d)$. The map assigning to every point x the d-vector $M(F(x))$ is of course of the type $\mathcal{B} \to \Lambda_d(\mathbb{R}^d \times \hat{\mathbb{R}}^d)$.

It is possible to construct the *d-current integration* G_u over the graph of u, by taking only the rectifiable part of the graph of u (see [**17**]). Precisely, G_u is the linear functional on smooth d-forms ω with compact support in $\mathcal{B} \times \hat{\mathbb{R}}^d$ defined by

$$G_u := \int_{\mathcal{B}} \langle \omega(x, u(x)), M(Du(x)) \rangle \ dx.$$

The *boundary current* associated with G_u is indicated by ∂G_u and defined by $\partial G_u(\omega) := G_u(d\omega)$, $\omega \in \mathcal{D}^{d-1}(\mathcal{B} \times \hat{\mathbb{R}}^d)$ with $\mathcal{D}^{d-1}(\mathcal{B} \times \hat{\mathbb{R}}^d)$ the space of $(d-1)-$forms with compact support in $\mathcal{B} \times \hat{\mathbb{R}}^d$. Details on the nature and the properties of Cartesian currents can be found in [**17**].

Currents, as recalled above, are essential tools to define weak diffeomorphisms, the role of which in finding minimizers for the energy of deformable bodies, even simple nonlinear elastic Cauchy bodies, has been introduced in the previous section. Their formal definition reads as follows:

Definition 1. A map $u \in W^{1,1}(\mathcal{B}, \hat{\mathbb{R}}^d)$ is said to be a **weak diffeomorphism** (in short $u \in dif^{1,1}(\mathcal{B}, \hat{\mathbb{R}}^d)$), when

- ***(i)*:** $\det Du(x) > 0$ *for almost every* $x \in \mathcal{B}$,
- ***(ii)*:** $|M(Du)| \in L^1(\mathcal{B})$,
- ***(iii)*:** $\partial G_u = 0$ *on* $\mathcal{D}^{d-1}(\mathcal{B} \times \hat{\mathbb{R}}^d)$,
- ***(iv)*:** *for any* $f \in C_c^\infty(\mathcal{B} \times \hat{\mathbb{R}}^d)$,

$$\int_{\mathcal{B}} f(x,u(x)) \det Du(x) \ dx \leq \int_{\hat{\mathbb{R}}^d} \sup_{x \in \mathcal{B}} f(x, w) \ dw.$$

The conditions listed in the definition above have explicit physical meaning which allows weak diffeomorphisms to be candidates for the description of standard deformations. Condition (i) is the standard requirement assuring that the map $x \longmapsto u(x)$ is orientation-preserving. The subsequent requirement prescribes that all the minors of Du are bounded in $L^1(\mathcal{B})$. For example, it implies that the average volume change be not infinite. In this way, if one thinks of u as a transplacement mapping, one is prescribing that global extreme deformations are prevented. The third condition imposes that the graph of the map $x \longmapsto u(x)$ has no boundary current inside $\mathcal{B} \times \hat{\mathbb{R}}^d$. In other words, such a condition imposes that fractures do not occur. The condition may appear more clear when one thinks of the standard deformation of a body from its reference macroscopic place $\mathcal{B}$ to the actual place $\mathcal{B}_a := u(\mathcal{B})$ as a d-dimensional surface S in $\mathbb{R}^d \times \hat{\mathbb{R}}^d$ having one-to-one projections into both factors of the cross product space. Elements of the tangent bundle of S then describe infinitesimal deformations. At each point of S, namely the pair (x, y), where now $y = u(x)$, the tangent $d-$vector to S at (x, y) can be written in terms of $Du(x)$, adj$Du(x)$ and $\det Du(x)$, or, alternatively, in terms of $D\hat{u}(y)$, adj$D\hat{u}(y)$ and $\det D\hat{u}(y)$, where $\hat{u}$ is the inverse deformation with $x = \hat{u}(y)$. It is then evident that the tangent d-vector to S is associated with $M(Du)$. The current G_u is then a global *dual* way to account for the multilinear algebra over the tangent bundle of S. If the current has a boundary, this means that there are regions of S

where all $(d-1)$-forms are not exact (recall that $\partial G_u := G_u(d\omega)$), which means that there are homological discontinuities in $\mathcal{B}$.

Condition (iv) in Definition 1 has been proposed in [**14**] and is a generalization of another condition suggested in [**13**]. It permits us to move $\mathcal{B}$ along u into a region $\mathcal{B}_a$ in such a way that self-contact between parts of the boundary $\partial\mathcal{B}$ is allowed while self-penetration is excluded.

The structure properties of $dif^{1,1}$ are summarized in the theorem reported without proof below.

THEOREM 1 ([**14, 17**]).

(1) *(Closure) Let $\{u_k\}$ be a sequence with $u_k \in dif^{1,1}(\mathcal{B}, \hat{\mathbb{R}}^d)$ for any k. If*

$$u_k \rightharpoonup u \qquad \text{and} \qquad M(Du_k) \rightharpoonup v$$

weakly in L^1, then $v = M(Du)$ a.e. and $u \in dif^{1,1}(\mathcal{B}, \hat{\mathbb{R}}^d)$.

(2) *(Compactness) Let $\{u_k\}$ be a sequence with $u_k \in W^{1,r}(\mathcal{B}, \hat{\mathbb{R}}^d)$, $r > 1$. Consider u_k as weak diffeomorphisms. Assume that there exists a constant $C > 0$ and a convex function $\vartheta : [0, +\infty) \to \mathbb{R}^+$ such that $\vartheta(t) \to +\infty$ as $t \to 0^+$, and*

$$\|M(Du_k)\|_{L^r(\mathcal{B})} \le C, \qquad \int_{\mathcal{B}} \vartheta(\det Du_k(x))\, dx \le C.$$

By taking subsequences $\{u_j\}$ with $u_j \rightharpoonup u$ in $W^{1,r}(\mathcal{B}, \hat{\mathbb{R}}^d)$, one gets $u_j \to u$ in $L^r(\mathcal{B})$, $M(Du_j) \rightharpoonup M(Du)$ in L^r and $\int_{\mathcal{B}} \vartheta(\det Du(x))\, dx \le C$. In particular, u is a weak diffeomorphism.

Although weak diffeomorphisms can appropriately describe standard deformations, for technical reasons the attention will be focused in the sequel on a subspace of $dif^{1,1}(\mathcal{B}, \hat{\mathbb{R}}^d)$, namely

$$dif^{r,1}(\mathcal{B}, \hat{\mathbb{R}}^d) := \left\{ u \in dif^{1,1}(\mathcal{B}, \hat{\mathbb{R}}^d) \,|\, |M(Du)| \in L^r(\mathcal{B}) \right\},$$

for some $r > 1$.

As remarked in the introduction, the existence of gap phenomena in relaxing energies defined over jet bundles on the class of weak diffeomorphisms is still an open problem.

3. Currents in the mechanics of complex bodies

As mentioned in the initial section, the combined use of the properties of weak diffeomorphisms and Sobolev maps allows one to find minimizers of the elastic energies in complex bodies, simple Cauchy bodies being obtained when the macroscopic influence of the material complexity is neglected.

Bodies are called *complex* when changes in their material texture at various subscales (from nano-to-mesoscopic level) prominently influence the gross behavior through peculiar interactions generated by the mutations of the substructure. Their list includes liquid crystals, bodies with dense distributions of microcracks, quasi-periodic alloys, materials with polarization (ferroelectrics or magnetoelastic bodies), various types of composites and bodies with strain-gradient effects. Microstructures can be exploited, even invented anew, to reach predetermined goals. An essential problem in their description is that of bridging scales even from the atomic to the macroscopic level, translating through the continuum limit the prominent aspects of the microstructural features. Inner dimensions are exploited. They are the dimension of the manifold of substructural shapes which is used to represent the peculiar features of the material microstructure.

The work developed so far on the foundations of the mechanics of complex bodies has underlined a wide family of scientific questions of theoretical and applied character (see [**10, 11, 12, 26, 29, 15, 36, 27**] and the references therein). In classical field theories the starting point is to consider a body as an abstract set, the elements of which are called the *material elements*. A model of a body starts then from the attribution of geometrical structure to such a set. Essentially, at least a rough idea of the material element must be at our disposal. The clear definition of the nature of the generic material element is not a trivial task. However, in the standard format of continuum mechanics, the problem is overcome by describing the geometry of every material element only through the place in space it occupies. The description is the minimalist one. One considers the material element as a monad in Leibniz' words, that is, a windowless box. No interest is shown for the geometry of the structure of the material element and its changes. Information on it are known just at the level of constitutive structures.

The analysis of complex bodies alters the standard paradigms of continuum mechanics. Events at low scales influence, in fact, the gross behavior. To take into account these effects the first step is to furnish a more detailed representation of the material elements. They have to be considered as *systems* rather than monads. The geometry of the inner microstructure must be represented. In fact, one selects only some prominent geometric features and some morphological descriptor of the inner geometry. Their choice is a structural part of the modeling process. The description is *multifield*, so it is *intrinsically multiscale* and *multidimensional*. The standard *transplacement field*

$$x \longmapsto u := u(x) \in \hat{\mathbb{R}}^d, \quad x \in \mathcal{B},$$

pictures macroscopic *deformations*, while the *morphological descriptor field*

$$x \longmapsto \nu := \nu(x) \in \mathcal{M}, \quad x \in \mathcal{B},$$

describes the inner geometry of the microstructure. $\mathcal{M}$ is the so-called *manifold of substructural shapes*.

Standard requirements are assumed to hold: $x \longmapsto u$ is essentially an orientation-preserving piecewise C^1-diffeomorphism, and the region $u(\mathcal{B})$ has the same geometrical structural properties of $\mathcal{B}$. The field $x \longmapsto \nu$ is assumed to be piecewise differentiable. In order to construct the essential features of the mechanics of complex bodies, it is not necessary to select some specific manifold $\mathcal{M}$. The only necessary assumption is that $\mathcal{M}$ is just a finite-dimensional differentiable manifold (preferably without boundary).

Here, the discussion of standard and generalized measures of deformations, the representation of macroscopic and microscopic actions through the external power, the generalized notions of observers and their changes (notions which correspond to a nontrivial sliding in the standard paradigm), the invariance requirements of the external power (or the companion relative power) leading to balance equations obtained independently of constitutive prescription are not recalled (relevant comments can be found in [**11, 26, 27**]). The attention is focused only on elastic complex bodies. Their behavior is governed by an energy of the type

$$\mathcal{E}\left(u,\nu,\mathcal{B}\right):=\int_{\mathcal{B}}e\left(x,u\left(x\right),Du\left(x\right),\nu\left(x\right),D\nu\left(x\right)\right)\ dx,$$

where

$$e\left(x,u,Du,\nu,D\nu\right)=\hat{e}\left(x,Du,\nu,D\nu\right)-\hat{w}\left(u,\nu\right),$$

with $\hat{e}\left(x,Du,\nu,D\nu\right)$ the elastic energy and $\hat{w}\left(u,\nu\right)$ the potential of external bulk actions; it admits the additive decomposition $\hat{w}\left(u,\nu\right)=\hat{w}_1\left(u\right)+\hat{w}_2\left(\nu\right)$. Equilibrium states are described by minimizers of such an energy. Conditions assuring their existence are sketched below.

Consider the energy density e as a map

$$e:\mathcal{B}\times\hat{\mathbb{R}}^d\times\mathcal{M}\times M^+_{d\times d}\times M_{N\times d}\rightarrow\bar{\mathbb{R}}^+$$

with values $e\left(x,u,F,\nu,N\right)$, where $F:=Du\left(x\right)$, $N:=D\nu\left(x\right)$, and assume that the properties (H1) and (H2) listed below hold.

(H1): e is such that there exists a Borel function

$$Pe:\mathcal{B}\times\hat{\mathbb{R}}^d\times\mathcal{M}\times\Lambda_d(\mathbb{R}^d\times\hat{\mathbb{R}}^d)\times M_{N\times d}\rightarrow\bar{\mathbb{R}}^+,$$

with values $Pe\left(x,u,\nu,\xi,N\right)$, which is

(a) l. s. c. in $\left(u,\nu,\xi,N\right)$ for a.e. $x\in\mathcal{B}$,
(b) convex in $\left(\xi,N\right)$ for any $\left(x,u,\nu\right)$,
(c) such that $Pe\left(x,u,\nu,M\left(F\right),N\right)=e\left(x,u,\nu,F,N\right)$ for any list of entries

$$\left(x,u,\nu,F,N\right)\quad\text{with}\quad\det F>0.$$

In terms of Pe, the energy functional becomes

$$\mathcal{E}\left(u,\nu,\mathcal{B}\right)=\int_{\mathcal{B}}Pe\left(x,u\left(x\right),\nu\left(x\right),M\left(F\right),N\right)\ dx. \tag{3.1}$$

(H2): The energy density e satisfies the growth condition

$$e\left(x,u,\nu,F,N\right)\geq C_1\left(\left|M\left(F\right)\right|^r+\left|N\right|^s\right)+\vartheta\left(\det F\right) \tag{3.2}$$

for any $\left(x,u,\nu,F,N\right)$ with $\det F>0$, $r,s>1$, $C_1>0$ constants and $\vartheta:\left(0,+\infty\right)\rightarrow\mathbb{R}^+$ a convex function such that $\vartheta\left(t\right)\rightarrow+\infty$ as $t\rightarrow0^+$.

In essence, the assumption that Pe is convex in $\left(M\left(F\right),N\right)$ for any $\left(x,y,\nu\right)$ is an assumption of stability of the material. It accounts for a possible interplay between the gradient of the gross deformation and the inhomogeneity of the distribution of the microstructure within the body. In fact, the inhomogeneity, that is, the way in which the microstructure varies from place to place, is measured by the gradient of the morphological descriptor. The growth condition (3.2) has a constitutive nature. It prescribes that the energy admits a polynomial lower bound which has the typical structure of a decomposed energy of Ginzburg-Landau type. It describes only interactions between neighboring material elements and does not account for the energy associated with the interactions inside every material element. For this reason, with (3.2) one is presuming in a sense that substructural events within the generic material element may only increase the overall energy.

Dirichlet boundary conditions for u and ν are imposed here over portions of the boundary $\partial\mathcal{B}$ indicated by $\partial\mathcal{B}_u$ and $\partial\mathcal{B}_\nu$, respectively. In fact, it is assumed that the field $x\longmapsto u\left(x\right)$ takes assigned values $x\longmapsto u_0\left(x\right)$ over $\partial\mathcal{B}_u$ while $x\longmapsto\nu\left(x\right)$ is prescribed to be $x\longmapsto\nu_0\left(x\right)$ over $\partial\mathcal{B}_\nu$.

Under these conditions, existence of minimizers for the energy can be investigated in the space

$$\mathcal{W}_{r,s} := \left\{ (u,\nu) \,|u \in dif^{r,1}(\mathcal{B},\hat{\mathbb{R}}^d),\ \nu \in W^{1,s}(\mathcal{B},\mathcal{M}) \right\}.$$

From the closure theorem for weak diffeomorphisms reported above and standard semicontinuity results, an existence theorem can be derived.

THEOREM 2 ([28]). *Under the hypotheses (H1) and (H2), if there is a pair* $(u_0,\nu_0) \in \mathcal{W}_{r,s}$ *such that* $\mathcal{E}(u_0,\nu_0) < +\infty$, *the functional* $\mathcal{E}$ *achieves the minimum value in the classes*

$$\mathcal{W}^d_{r,s} := \{(u,\nu) \in \mathcal{W}_{r,s} | u = u_0 \text{ on } \partial\mathcal{B}_u, \nu = \nu_0 \text{ on } \partial\mathcal{B}_\nu\}$$

and

$$\mathcal{W}^c_{r,s} := \left\{(u,\nu) \in \mathcal{W}_{r,s} \mid \partial G_u = \partial G_{u_0} \text{ on } \mathcal{D}^2(\mathbb{R}^3 \times \hat{\mathbb{R}}^3), \nu = \nu_0 \text{ on } \partial\mathcal{B}_\nu \right\}.$$

Like the constraints on the structure of the energy, even the choice of the functional space $\mathcal{W}_{r,s}$ has a constitutive nature. Other possible choices of the functional space can be made.

For example, one can imagine another lower bound for the energy density. Any choice of lower bounds has a constitutive nature. The second one adopted here is indicated by (H3).

(H3): The energy density satisfies the growth condition

$$e(x,u,\nu,F,N) \geq C_2(|F|^{d-1} + |\text{adj }F|^{d/d-1} + |N|^s) + \vartheta(\det F) \tag{3.3}$$

for any (x,u,ν,F,N) with $\det F > 0$, $C_2 > 0$ a constant and $\vartheta : (0,+\infty) \to \mathbb{R}^+$ as above.

Consider the energy as defined over the functional class

$$\begin{aligned} \mathcal{W}_{d,\frac{d}{d-1},s} \quad : \quad = &\left\{ (u,\nu) \,|u \in W^{1,d-1}(\mathcal{B},\hat{\mathbb{R}}^d), \text{adj}(Du) \in L^{d/d-1}, \right. \\ &\left. \text{(iv.) in Def. 1 holds}, \nu \in W^{1,s}(\mathcal{B},\mathcal{M}) \right\}. \end{aligned}$$

By taking into account the $L \log L$ estimate in [31] (see also [35]), one can find a relevant existence result.

THEOREM 3 ([28]). *Under the assumptions (H1) and (H3) reported above, the functional* $\mathcal{E}$ *achieves its minimum value in the class*

$$\mathcal{W}^d_{d,\frac{d}{d-1},s} := \left\{ (u,\nu) \in \mathcal{W}_{d,\frac{d}{d-1},s} \mid u = u_0 \text{ on } \partial\mathcal{B}_u, \nu = \nu_0 \text{ on } \partial\mathcal{B}_\nu \right\},$$

provided that there exists a pair $(u_0,\nu_0) \in \mathcal{W}_{d,\frac{d}{d-1},s}$ *such that* $\mathcal{E}(u_0,\nu_0,\mathcal{B}) < +\infty$.

The presence of the function ϑ in the lower bounds selected in (H2) and (H3) is justified by the need of avoiding physically undesired behaviors such as the extreme deformations obtained by letting $\det F$ go to zero on a set of positive measure.

Different special structural choices of the energy $\mathcal{E}(u,\nu,\mathcal{B})$ can be made. A couple of examples are reported here.

(1) Neglect macroscopic deformation and consider ν to be a scalar coinciding, for example, with the volume fraction of a given phase in a two-phase material. The density

$$\zeta\left(\nu^2 - 1\right)^2 + \varsigma\,|N|^2,$$

with ζ and ς two constants, can be selected. With ν in $\tilde{\mathbb{R}}^m$, the previous density becomes $\left(|\nu|^2 - 1\right)^2 + \mu |N|^2$. Both densities are of Ginzburg-Landau type.

(2) Assume the existence of an internal constraint of the type

$$\nu = \nu(F).$$

In this case the microstructure is called *latent* (see [**10**]). The energy density becomes that of a second-grade Cauchy body, that is,

$$e\left(x, u, Du, D^2u\right).$$

The special choice

$$e\left(x, u, Du, D^2u\right) = \left(|Du|^2 - 1\right)^2 + \varsigma^2 \left|D^2u\right|^2$$

is of Aviles-Giga type. It can be obtained even when (a) the macroscopic deformation is neglected and (b) the morphological descriptor ν coincides with the spatial derivative of some field, namely $\nu = D\phi$, with $x \longmapsto \phi(x)$ a differentiable map.

The existence theorems reported above apply to wide classes of complex bodies. An extended list of prominent physical examples can be found in [**28**].

The first variation of the energy $\mathcal{E}(u_0, \nu_0, \mathcal{B})$ can be obtained in different ways.

In the presence of regular minimizers admitting tangential derivatives, since the energy density presented above is in essence a 3-form over the first jet bundle of a bundle $\mathcal{Y}$ over $\mathcal{B}$, namely $\pi : \mathcal{Y} \to \mathcal{B}$, with π the canonical projection and the typical fiber $\pi^{-1}(x) = \hat{\mathbb{R}} \times \mathcal{M}$, one can use the standard *vertical lift* of the first jet bundle (canonical injection) to determine common Euler-Lagrange equations. Elements of them are representations of standard and microstructural interactions arising within the body. In particular, the derivative of the energy with respect to F represents the standard Piola-Kirchhoff stress, the derivative with respect to N the so-called microstress measuring contact interactions due to inhomogeneous microstructural changes, and the derivative of $\hat{e}$ with respect to ν indicates a self-action within every material element. The derivatives of $\hat{w}$ measure the external bulk actions over the body in its whole (gravitational action) and the microstructure (e.g., electric fields determining the polarization in a body).

The tangential derivative of maps in $\mathcal{W}_{r,s}$ or $\mathcal{W}_{d,\frac{d}{d-1},s}$ does not always exist. Moreover, regularity theorems seem not to be available. However, *horizontal variations* can be computed. They are determined by diffeomorphisms of $\mathcal{B}$ into itself such that their restriction over the boundary $\partial\mathcal{B}$ coincides with the identity. Horizontal variations lead to balance equations called the balance of configurational forces, the ones displaying the balance of actions on defects in solids, at least in the conservative setting when dissipative actions are not present. When smooth minimizers are at our disposal, the balance of configurational forces coincides also with the pullback in the reference place $\mathcal{B}$ of the balances of standard and microstructural actions. For nonsmooth minimizers, the two classes of balance equations, namely the ones obtained by vertical variations and the one following from horizontal variations, have a different nature. Examples stressing this difference can be found in [**21**], vol. I, pp. 152-153.

4. Another tool: varifolds

Consider the dimension d of the ambient space greater than or equal to 2. For a positive integer k, $1 \leq k \leq d$, the Grassmann manifold of k-planes through the origin in $\mathbb{R}^d$ is indicated by $\mathcal{G}_{k,d}$ and is also identified with the set of the projectors $\Pi : \mathbb{R}^d \to \mathbb{R}^d$ onto k-planes. They have the well-known characterizing properties $\Pi^2 = \Pi$, $\Pi^2 = \Pi$, $\operatorname{Rank} \Pi = k$. Every projector is an element of a compact subset of $\mathbb{R}^d \otimes \mathbb{R}^d$.

It is then possible to construct a bundle $\mathcal{G}_k(\mathcal{B})$ with a natural projection $\pi : \mathcal{G}_k(\mathcal{B}) \to \mathcal{B}$ and typical fiber coinciding with the Grassmann manifold $\mathcal{G}_{k,d}$.

Definition 2. A k-**varifold** over $\mathcal{B}$ is a nonnegative Radon measure V over the bundle $\mathcal{G}_k(\mathcal{B})$.

In short-hand notation one writes $V \in \mathsf{M}(\mathcal{G}_k(\mathcal{B}))$. Let $\pi_\#$ be the projection of measures over $\mathcal{B}$ associated with the natural projection $\pi : \mathcal{G}_k(\mathcal{B}) \to \mathcal{B}$. The projection $\pi_\#$ allows one to define the *weighed measure* of the varifold V, which is the Radon measure over $\mathcal{B}$ defined by $\mu_V := \pi_\# V$. Such a measure defines the *mass* $\mathbf{M}(V)$ of the varifold through the relation

$$\mathbf{M}(V) := V(\mathcal{G}_k(\mathcal{B})) = \mu_V(\mathcal{B}).$$

It is rather immediate to construct varifolds over a subset $\mathfrak{b}$ of $\mathcal{B}$. For the purpose of the analysis developed here, some assumptions on the structure of $\mathfrak{b}$ are necessary. In fact, $\mathfrak{b}$ is individuated by a measure which is absolutely continuous with integer density θ with respect to the k-dimensional Hausdorff measure $\mathcal{H}^k$ in $\mathbb{R}^d$. It is then assumed that (i) $\mathfrak{b}$ is an $\mathcal{H}^k$-measurable, k-rectifiable subset of $\mathcal{B}$ and (ii) the density θ belongs to $L^1(\mathfrak{b}, \mathcal{H}^k)$ and takes integer values. All these assumptions avoid the selection of too many exotic subsets $\mathfrak{b}$. For example, they assure that for almost every $x \in \mathfrak{b}$, there exists the approximated (in the sense of geometric measure theory, see [**17**]) tangent space $T_x\mathfrak{b}$ to $\mathfrak{b}$ at x.

Under these conditions a varifold associated with the triple $(\mathfrak{b}, \theta, \mathcal{H}^k)$ can be defined through its action over the space of compactly supported C^0 functions over the fiber bundle $\mathcal{G}_k(\mathcal{B})$. Such a measure is indicated by $V_{\mathfrak{b},\theta}(\varphi)$ and is defined by

$$V_{\mathfrak{b},\theta}(\varphi) := \int_{\mathcal{G}_k(\mathcal{B})} \varphi(x, \Pi)\ dV_{\mathfrak{b},\theta}(x, \Pi) := \int_{\mathfrak{b}} \theta(x)\, \varphi(x, \Pi)\ d\mathcal{H}^k,$$

for any $\varphi \in C^0(\mathcal{G}_k(\mathcal{B}))$. The second-rank tensor $\Pi(x)$ is the projection onto $T_x\mathfrak{b}$ at all x's where $T_x\mathfrak{b}$ is defined. This circumstance justifies the assumption of having at disposal a set $\mathfrak{b}$ admitting (an at least approximated) tangent space at almost every x. $V_{\mathfrak{b},\theta}$ is called the *integer rectifiable k-varifold associated with* $(\mathfrak{b}, \theta, \mathcal{H}^k)$.

When one selects vector-valued measures in the space of Radon measures over $\mathcal{G}_k(\mathcal{B})$, it is done with the aim of defining a special class of varifolds: the *varifolds with curvature* [**23, 25**]. The generalized curvature associated with $\mathfrak{b}$ is a third-rank tensor A which is at every $x \in \mathfrak{b}$ the value of a tensor field defined over $\mathcal{G}_k(\mathcal{B})$ and taking values over $\mathbb{R}^{d*} \otimes \mathbb{R}^d \otimes \mathbb{R}^{d*}$.

Definition 3. A varifold V is called a **curvature k-varifold with boundary** if

(1) *V is an integer, rectifiable k-varifold $V_{\mathfrak{b},\theta}$ associated with the triple* $(\mathfrak{b}, \theta, \mathcal{H}^k)$,

(2) *there exists a function* $A \in L^1\left(\mathcal{G}_k(\mathcal{B}), \mathbb{R}^{d*} \otimes \mathbb{R}^d \otimes \mathbb{R}^{d*}\right)$, *in components* $A_j^{\ell i}$, *and a vector Radon measure* $\partial V \in \mathsf{M}\left(\mathcal{G}_k(\mathcal{B}), \mathbb{R}^d\right)$, *the so-called* ***varifold boundary measure****, such that, for every* $\varphi \in C_c^\infty(\mathcal{G}_k(\mathcal{B}))$, *one gets*

$$\int_{\mathcal{G}_k(\mathcal{B})} \left(\Pi D_x \varphi + A^t D_\Pi \varphi + AI\varphi\right) \, dV(x,\Pi) = -\int_{\mathcal{G}_k(\mathcal{B})} \varphi \, d\partial V(x,\Pi).$$

In the previous formula, I is the second-rank unit tensor. Indices are saturated in such a way that $(\Pi D_x \varphi + A^t D_\Pi \varphi + AI\varphi)$ is a vector.

As a matter of notation, the subclass of curvature varifolds with curvature field $(x,\Pi) \longmapsto A(x,\Pi)$ in $L^p\left(\mathcal{G}_k(\mathcal{B}), \mathbb{R}^{d*} \otimes \mathbb{R}^d \otimes \mathbb{R}^{d*}\right)$, with $p \geq 1$, is indicated in what follows by $CV_k^p(\mathcal{B})$. Moreover, in what follows, $\nabla^{\mathfrak{b}}$ indicates the gradient along $\mathfrak{b}$. The symbol $\Pi_\#$ represents the projector acting over vector measures.

Essential properties of curvature varifolds are discussed in [**23, 25**]. Some of them are summarized in the remarks below where V is a k-varifold with boundary ∂V and curvature $A \in L^1(\mathcal{G}_k(\mathcal{B}))$.

(1) The curvature tensor satisfies the following relations:

$$A_j^{\ell i} = A_\ell^{ji}, \quad A_j^{ji} = 0, \quad A_j^{\ell i} = \Pi_h^\ell A_j^{hi} + \Pi_j^h A_h^{\ell i}, \quad A_j^{\ell h} \Pi_h^i = A_j^{\ell i}, \quad V - a.e.$$

(2) The vector $H^i(x) := A_j^{\ell j}(x, \Pi(x))$ has the meaning of generalized mean curvature for b, and is $\mu_V - a.e.$ x perpendicular to $T_x b$.

(3) The projection map $x \longmapsto \Pi(x)$ is $\mu_V - a.e.$ approximately differentiable and

$$\left(\nabla^{\mathfrak{b}} \Pi_j^\ell(x)\right)^i = A_j^{\ell i}(x, \Pi(x)) \quad \mu_V - a.e.\ x.$$

(4) The support of $|\partial V|$ is contained in the support of V, also $|\partial V| \perp V$, and ∂V is tangential to b in the sense that $\Pi_{\# j}^i (\partial V)^j = (\partial V)^i$ as measures.

(5) V is a varifold with locally bounded first variation and generalized mean curvature in the sense of Allard with generalized mean curvature vector $H(x)$ and generalized boundary $\pi_\# \partial V$. So, Allard's regularity and compactness theorems apply. In particular, it has been shown in [**23**] that if $V = V_{\mathfrak{b},\theta} \in CV_k^p(\mathcal{B})$, with $p > k$, then it is locally the graph of a multivalued function of class $C^{1,\alpha}$, $\alpha = 1 - \frac{p}{k}$, far from ∂V.

Theorem 4 (Compactness [**25**]). *For* $1 < p < \infty$, *consider a sequence* $\{V_r\} \subset CV_k^p(\mathcal{B})$ *of curvature varifolds with boundary and the corresponding sequences of curvatures* $\{A_r\}$ *and boundary measures* $\{\partial V_r\}$. *For every open set* $\Omega \Subset \mathcal{B}$ *and for every* r, *assume the existence of a constant* $c(\Omega)$, *depending on* Ω, *such that*

$$\mu_{V_r}(\Omega) + |\partial V_r|(\mathcal{G}_k(\mathcal{B})) + \int_{\mathcal{G}_k(\mathcal{B})} \left|A^{(r)}\right|^p \, dV_r \leq c(\Omega).$$

Under these conditions, there exists a subsequence $\left\{V^{(r_s)}\right\}$ *of* $\left\{V^{(r)}\right\}$ *and a* k*-varifold* $V \in CV_k^p(\mathcal{B})$, *with curvature* A *and boundary* ∂V, *such that*

$$V_{r_s} \rightharpoonup V, \quad A_{r_s} dV_{r_s} \rightharpoonup A dV, \quad \partial V_{r_s} \rightharpoonup \partial V,$$

in the sense of measures, as $r_s \to \infty$. *Moreover, for any convex and lower semicontinuous function* $f : \mathbb{R}^{d*} \otimes \mathbb{R}^d \otimes \mathbb{R}^{d*} \to [0, +\infty]$, *one gets*

$$\int_{\mathcal{G}_k(\mathcal{B})} f(A) \, dV \leq \liminf_{r_s \to \infty} \int_{\mathcal{G}_k(\mathcal{B})} f(A_{r_s}) \, dV_{r_s}.$$

5. An extended class of weak diffeomorphisms

A new class of weak diffeomorphims with boundary controlled by a varifold can be defined. They are useful to describe cracks or dislocations in continuum mechanics. An essential example of their physical meaning will be presented in the ensuing section.

The basic idea is to have first at disposal a class of curvature varifolds with boundary $\{V_k\}_{k=1}^{d-1}$ such that (i) each element of the class is in $CV_k^{p_k}(\mathcal{B})$ with $p_k > 1$ and (ii) for $k = 2, ..., d-1$,

$$\pi_\# |\partial V_k| \leq \mu_{V_{k-1}}.$$

This last relation is not exotic: it is the weak version of the standard relation occurring between a manifold and its boundary. Such a relation is here expressed in terms of varifolds stratified over different dimensions and supported by rectifiable sets. Once this family of varifolds has been selected, one aims to choose a special class of maps which admit jump sets contained in the supports of the selected varifolds and describe standard deformations outside these sets. Such maps are called the *extended weak diffeomorphisms with boundary controlled by stratified varifolds*. Essentially they are maps satisfying all items in the definition of weak diffeomorphism but the requirement to be without boundary. Boundaries in the graphs of these maps are admitted but they have to satisfy conditions expressed in terms of the selected family of stratified varifolds. These conditions are of two types and are classified as **type 1** and **type 2** below. Their formal definitions are reported in what follows.

DEFINITION 4. Assigned a class $\{V_k\}_{k=1}^{d-1}$ of curvature varifolds with boundary, an **extended weak diffeomorphism with controlled boundary of type** 1 is an a.e. approximately differentiable map which satisfies the conditions (i), (ii), (iv) in Definition 1, and

$$\pi_\# |\partial G_u| \leq \sum_{j=1}^{d-1} \mu_{V_k}.$$

The condition above means that the Green formulas hold true outside the support of the stratified varifolds involved in the previous definition. Moreover, it indicates also that the boundary current ∂G_u has finite mass, and that u belongs to the class $SBV_0(\mathcal{B}, \hat{\mathbb{R}}^d)$ (see [**2**]).

DEFINITION 5. Assigned a class $\{V_k\}_{k=1}^{d-1}$ of curvature varifolds with boundary, an **extended weak diffeomorphism with controlled boundary of type 2** is an a.e. approximately differentiable map which satisfies the conditions (i), (ii), (iv) in Definition 1, and

$$\pi_\# |\partial G_u| \leq \sum_{j=1}^{d-1} \mu_{V^{(j)}} + \pi_\# |\partial V_1|.$$

Comments on the different physical situations described by the two classes are reported in the ensuing section. Here the attention is mainly focused on the class of extended diffeomorphisms with controlled boundary of type 1. To affirm that a map $u : \mathcal{B} \to \hat{\mathbb{R}}^d$ belongs to this class, one writes just $u \in dif^{1,1}(\mathcal{B}, V, \hat{\mathbb{R}}^d)$. The structural properties of $dif^{1,1}(\mathcal{B}, V, \hat{\mathbb{R}}^d)$ are collected in the ensuing theorem.

THEOREM 5. *Consider a sequence of varifolds* $\{V_k\}$ *on* $\mathcal{B}$, *chosen in* $CV_1^p(\mathcal{B})$, $p > 1$, *and with equibounded variation, i.e.* $\sup_k \mu_{V_k}(\mathcal{B}) < \infty$. *Take a sequence* $\{u_k\}$ *such that* $u_k \in dif^{1,1}(\mathcal{B}, V_k, \hat{\mathbb{R}}^d)$. *Assume that there exist* $u \in L^1(\mathcal{B}, \hat{\mathbb{R}}^d)$, $\upsilon \in L^1(\mathcal{B}, \Lambda_d(\mathbb{R}^d \times \hat{\mathbb{R}}^d))$, *and* $V \in CV_1^p(\mathcal{B})$, $p > 1$, *such that* $u_k \rightharpoonup u$, $M(Du_k) \rightharpoonup \upsilon$, *and* $V_k \rightharpoonup V$ *as measures. The identity* $\upsilon = M(Du)$ *holds. Moreover, if* $\det Du > 0$ *a.e., one also finds that* $u \in dif^{1,1}(\mathcal{B}, V, \hat{\mathbb{R}}^d)$.

PROOF. The assumptions imply that $\mathbf{M}(G_{u_r}) + \mathbf{M}(\partial G_{u_r}) \leq C$ independently of r. In particular, the sequence $\{u_k\}$ is equibounded in $BV(\mathcal{B}, \hat{\mathbb{R}}^d)$ so that, by passing eventually to subsequences, $\{u_k\}$ converges strongly in L^1 and a.e. to u, and G_{u_k} converges to a current S. Moreover, S is an integer multiplicity rectifiable current by the Federer-Fleming compactness theorem. For a more direct proof, see [17]. It then follows that $\upsilon = M(Du)$ a.e. and $S = G_u$ (see [17]). Properties (ii), (iii), and (iv) in Definition 1 hold true. If u satisfies (i), then $u \in dif^{1,1}(\mathcal{B}, V, \hat{\mathbb{R}}^d)$.

In particular, the subclass

$$dif^{p,1}(\mathcal{B}, V, \hat{\mathbb{R}}^d) := \left\{ u \in dif^{1,1}(\mathcal{B}, V, \hat{\mathbb{R}}^d) \mid |M(Du)| \in L^p(\mathcal{B}) \right\}$$

will be useful in the next section.

6. Describing cracks in term of varifolds

Varifolds are an essential tool for describing low-dimensional defects in solids such as discontinuity surfaces, dislocations, cracks.

Assume also that a crack pattern can occur in a Cauchy body which is elastic-brittle. The basic idea is to describe the crack pattern through a family of varifolds of various co-dimensions. Consider the reference configuration $\mathcal{B}$ to be selected in $\mathbb{R}^3$ for the sake of simplicity. Imagine a smooth single crack in the actual configuration of the body which has as pre-image in $\mathcal{B}$ a piece of a certain surface $\mathcal{C}$ which can be assumed smooth just to visualize the situation. A two-dimensional varifold can be used to describe the surface. The Grassmanian is constructed by using the tangent planes to the surface, and the surface itself is the support $\mathfrak{b}$ of the varifold. The boundary of $\mathcal{C}$ is then the support of the boundary of the varifold. A sketch of the situation is described in Figure 1. It is possible to give a special status to a part of the boundary of $\mathcal{C}$, namely the part inside the body (the dashed line in Figure 1), that, is the *tip* of the crack, with the aim of assigning a line energy to it. In this case a one-dimensional varifold can be assigned. It should have support including the tip. Part of the support of such a varifold could also describe line defects such as dislocations occurring ahead of the crack tip. In evaluating the equilibrium of such a body one has two unknowns: (i) the family of varifolds describing the crack pattern, and (ii) the transplacement field. The latter is selected in $dif^{p,1}(\mathcal{B}, V, \hat{\mathbb{R}}^d)$ to assure that the boundary of its graph can be projected over the support of the varifold only. In this way one wants to select a transplacement describing a standard deformation outside the crack pattern.

Of course, a generic crack path is more complicated than the one described above (see also Figure 1). Moreover, it is not necessary that the ambient space be three-dimensional. The treatment proposed here can be set in an ambient space with higher dimension, say d.

Energy is assigned to the lateral margins of the crack and to the tip. As an extension of the classical Griffith scheme, the crack energy depends on the curvature

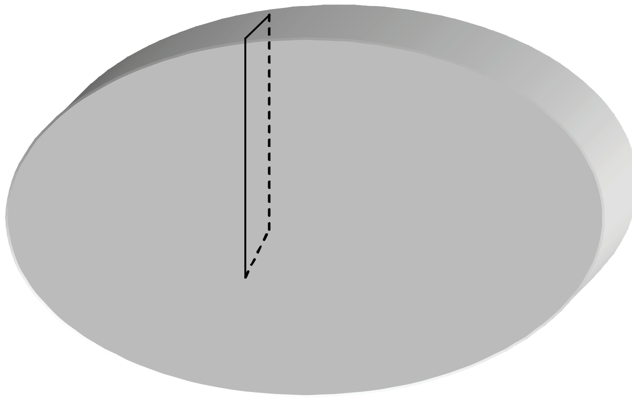

FIGURE 1. Elastic-brittle simple body with a crack which has a planar pre-image in the reference configuration depicted above.

of the crack. Special concrete examples justifying such an extension can be found in [**22**]. The energy presented in the introduction is written for the sake of simplicity in a three-dimensional ambient space to favour the physical visualization of the meaning of the various terms. Its extension to higher-dimensional spaces is however immediate. In this case, the geometry of the crack pattern is then described by a family $\{V_k\}_{k=1}^{d-1}$ of varifolds stratified over supports at various dimensions. Such a family of varifolds is characterized by the ensuing formal definition.

DEFINITION 6. A family $\{V_k\}_{k=1}^{d-1}$ of curvature varifolds in $CV_k^{p_k}(\mathcal{B})$, with $p_k > 1$, is called **stratified** when

$$\pi_{\#}\left|\partial V_k\right| \leq \mu_{V_{k-1}},\ \forall k = 2, ..., d-1.$$

Stratified cracks describe naturally the geometry of crack patterns in a body placed in $\hat{\mathbb{R}}^d$. The associated energy, written in accordance with the remarks above, reads

$$\begin{aligned}\mathcal{E}\left(u, \{V_k\}, \mathcal{B}\right) \quad : \quad &= \int_{\mathcal{B}} e\left(x, Du\right)\ dx + \sum_{k=1}^{d-1} \alpha_k \int_{\mathcal{G}_k(\mathcal{B})} \left|\mathbf{A}_{(k)}\right|^{p_k}\ dV_k \\ &+ \sum_{k=1}^{d-1} \beta_k \mathbf{M}\left(V_k\right) + \gamma \mathbf{M}\left(\partial V_1\right).\end{aligned}$$

Physical convenience suggests the introduction of a family of *comparison varifolds* $\left\{\tilde{V}_k\right\}_{k=1}^{d-1}$ such that for any k one gets $\tilde{V}_k \in CV_k^{p_k}(\mathcal{B})$ and $\mu_{\tilde{V}_k} \leq \mu_{V_k}$. The assignment of $\left\{\tilde{V}_k\right\}$ does not mean that one is considering a preexisting crack pattern because the comparison varifold family can be null. However, when an initial crack exists, the condition assures that the competitors in the minimizing procedure can only extend from the initial crack.

The functional setting in which one tries to find minimizers of $\mathcal{E}(u,V,\mathcal{B})$ can then be specified. The space

$$\mathcal{A}_{q,p,K,\{\tilde{V}_k\}}(\mathcal{B}) \quad : \quad = \Big\{(u,\{V_k\}) \mid V_k \in CV_k^{p_k}(\mathcal{B}), u \in dif^{q,1}(\mathcal{B},V_k,\hat{\mathbb{R}}^d), \\ \{V_k\} \text{ is stratified}, \|u\|_{L^\infty(\mathcal{B})} \le K, \mu_{\tilde{V}_k} \le \mu_{V_k}, \forall k=1,...,d-1\Big\}$$

is then the natural ambient in which the existence of minimizers of the energy $\mathcal{E}(u,V,\mathcal{B})$ can be investigated. In particular, the subspace

$$\mathcal{A}^{u_0}_{q,p,K,\{\tilde{V}_k\}}(\mathcal{B}) := \Big\{(u,\{V_k\}) \in \mathcal{A}_{q,p,K,\{\tilde{V}_k\}}(\mathcal{B}) \mid u(x) = u_0(x), x \in \partial\mathcal{B}_u\Big\},$$

with $\partial\mathcal{B}_u$ the part of the boundary of the body where the transplacement field is prescribed, plays a role.

Theorem 6. *Assume $K > 0$, $q, p_k > 1$, and $\tilde{V}_k \in CV_k^{p_k}(\mathcal{B})$ for any k. If there exists $(u_0, \{V_k^0\}) \in \mathcal{A}^{u_0}_{q,p,K,\{\tilde{V}_k\}}(\mathcal{B})$ such that $\mathcal{E}(u_0,\{V_k^0\},\mathcal{B}) < +\infty$, then $\mathcal{E}(u,\{V_k\},\mathcal{B})$ attains there the minimum value.*

Further details, proofs and the evaluation of the first variation of $\mathcal{E}(u,V,\mathcal{B})$ can be found in [**22**].

A final remark deserves mention: in fact, in managing an energy such as $\mathcal{E}(u,V,\mathcal{B})$, one is in essence considering *a cracked body such as a complex body*, the difference with the format described in the previous sections resting in the nature of the morphological descriptor which is now a measure. $CV_k^p(\mathcal{B})$ plays here the role of the manifold of substructural shapes.

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