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FUKAYA-YAMAGUCHI CONJECTURE IN DIMENSION FOUR

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ABSTRACT. Fukaya and Yamaguchi [7] conjectured that if M^n is a manifold with nonnegative sectional curvature, then the fundamental group is uniformly virtually abelian. In this short note we observe that the conjecture holds in dimensions up to four.

1. STATEMENT OF THE MAIN RESULT

Let M^n be a smooth manifold with nonnegative sectional curvature and $n \leq 4$. Then there exists an abelian subgroup $A \leq \pi_1(M)$ of the fundamental group with universally bounded index $[\pi_1(M) : A] \leq C(n)$.

It is interesting to note that the above fails if one only assumes $\text{Ric} \geq 0$, see [12] and more recently [3] for examples in the closed case. In the case of $\text{Ric} \geq 0$ the fundamental group may not even be finitely generated, see for instance [1, 2]. However, it is unknown if the infinitely generated component must be abelian; for instance, it is unknown if there exists a normal abelian subgroup $A \leq \pi_1(M)$ such that the quotient $\pi_1(M)/A$ is finitely generated?

2. PROOF OF THE MAIN RESULT

By the Cheeger-Gromoll soul theorem [5], M^n deformation retracts onto a compact totally geodesic submanifold. In particular, it is homotopically equivalent to a compact manifold with nonnegative sectional curvature of dimension $\leq n$. Hence we can assume without loss of generality that M^n is compact.

We break the proof down into two basic cases, which is whether or not the universal cover \tilde{M}^n is compact or noncompact. Let us first deal with the case when \tilde{M}^n is compact, where in fact we will prove a slightly more general result about effective actions. This generalization will prove useful in the noncompact context:

Let (\tilde{M}^n, g) be a simply connected manifold with $n \leq 4$, $\text{sec} \geq -1$ and $\text{diam} \leq D$. Then any finite group Γ acting smoothly and effectively on \tilde{M} admits an abelian subgroup $A \leq \Gamma$ which is generated by at most $C(n, D)$ elements and whose index is uniformly bounded $[\Gamma : A] \leq C(n, D)$. Note that if $\text{sec} \geq 0$ and \tilde{M}^n is compact, then we may rescale in order to assume $\text{diam}(\tilde{M}^n) \leq 1$. In particular, we have that A is generated by at most $C(n)$ generators with $[\Gamma : A] \leq C(n)$.

Proof of Lemma 2. Let us first observe that in the case $n = 2$ we have that $M^2 = S^2$ and in the case $n = 3$ we have that $M^3 = S^3$ as they are simply connected closed manifolds¹. In the case $n = 4$, let us recall that the Euler characteristic of a simply connected four manifold M^4 satisfies

$$\chi(M^4) = 2 + b_2 \geq 2 > 0. \quad (1)$$

In particular, a simply connected four manifold has positive Euler characteristic. Let us now appeal to the results of Mundet i Riera [10]. In the cases where M is either an integral homotopy sphere or has nonzero Euler characteristic, we have that $\text{Diff}(M)$ is a Jordan space. More precisely and effectively, by [10, Theorem 1.2] we have that if Γ is any smooth effective action on M then there exists an abelian group $A \leq \Gamma$ and C depending only on the dimensions of M and $H^*(M, \mathbb{Z})$ such that:

- i) A is generated by at most C elements;
- ii) $[\Gamma : A] \leq C$.

If we combine with Gromov's betti number estimates [8, Theorem 0.2B], which bounds for us the dimension of $H^*(M, \mathbb{Z})$, this finishes the proof of Lemma 2.

$\text{Isom}(N)$. The mappings $\rho_k : \text{Isom}(\tilde{M}) \rightarrow \text{Isom}(\mathbb{R}^k)$ and $\rho_N : \text{Isom}(\tilde{M}) \rightarrow \text{Isom}(N)$ are clearly homomorphisms.

So let us prove that N is compact. To prove the claim let $\hat{M} \subseteq \tilde{M}$ be a fundamental domain. As M is compact we have also that \hat{M} is compactly supported. Now if N is not compact then we can find a sequence $x_j = (0, y_j) \in \mathbb{R}^k \times N$ with $d(x_0, x_j) = d(y_0, y_j) = 2r_j \rightarrow \infty$. Let $x_j(t) = (0, y_j(t)) : [-r_j, r_j] \rightarrow \tilde{M}$ be a geodesic between x_0 and x_j . By the definition of the fundamental domain, we can act on $x_j(t)$ by a deck transformation $\gamma_j = \gamma_j^k \times \gamma_j^N \in \pi_1(M)$ so that the resulting minimizing geodesic $\sigma_j(t) = \gamma_j \cdot x_j(t) = (\gamma_j^k(0), \gamma_j^N \circ x_j(t)) = (\sigma_j^k, \sigma_j^N(t)) : [-r_j, r_j] \rightarrow \tilde{M}$ satisfies $\sigma_j(0) \in \hat{M}$, where $\sigma_j^k \in \mathbb{R}^k$ and $\sigma_j^N(t)$ is a geodesic in N . As $\sigma_j(0)$ is compactly supported we can pass to a subsequential limit $\sigma_j \rightarrow \sigma = (\sigma^k, \sigma^N(t)) : (-\infty, \infty) \rightarrow \tilde{M}$ to get a line $\sigma^N(t)$ in N . We can now apply the Toponogov splitting [11] to see that we have the isometric splitting $N = \mathbb{R} \times N_1$, which is a contradiction to our assumption that N did not split any Euclidean factors. Thus N is compact and this proves Claim 1.

For the next step in the proof let us begin by considering the two groups $\Gamma_k = \rho_k(\Gamma) = \rho_k(\pi_1(M)) \leq \text{Isom}(\mathbb{R}^k)$ and $\Gamma_N^0 = \ker \rho_k \leq \Gamma$. Note that Γ_N^0 is normal in Γ , and by observing that Γ_N^0 fixes the \mathbb{R}^k factor we can naturally embed it $\Gamma_N^0 \leq \text{Isom}(N)$. Note that as Γ is a discrete cocompact action on \tilde{M} and N is compact, we have that Γ_k is a discrete cocompact action on \mathbb{R}^k . It follows from the Bieberbach theorem that there exists an abelian group, in fact a lattice, $A_k \leq \Gamma_k$ with at most $K(k)$ generators for which $[\Gamma_k : A_k] \leq C(k)$.

We set $\Gamma' = \rho_k^{-1}(A_k) \leq \Gamma$, which we observe is a $C(k)$ -finite index subgroup with normal subgroup $\Gamma_N^0 \leq \Gamma'$ and at most $K = K(n)$ generators. We have the short exact sequence

$$0 \rightarrow \Gamma_N^0 \rightarrow \Gamma' \rightarrow A_k \rightarrow 0. \quad (3)$$

As A_k is abelian, we have that the commutator subgroup must satisfy $[\Gamma', \Gamma'] \leq \Gamma_N^0$. Let us now define $\Gamma'_N := \rho_N(\Gamma') \leq \text{Isom}(N)$ to be the image group with $\overline{\Gamma'_N} \leq \text{Isom}(N)$ its closure. Note that $\overline{\Gamma'_N}$ is a compact Lie group.

Claim 2: The connected component of the identity of $\overline{\Gamma'_N} \leq \text{Isom}(N)$ is a torus $T^\ell \leq \overline{\Gamma'_N}$, and there exists a finite normal subgroup $\Sigma \leq \overline{\Gamma'_N}$ such that T^ℓ and Σ generate $\overline{\Gamma'_N}$.

To prove the claim let us first observe that $[\Gamma'_N, \Gamma'_N] \leq \Gamma_N^0 \leq \text{Isom}(N)$, and hence as the commutators are continuous we have that $[\overline{\Gamma'_N}, \overline{\Gamma'_N}] \leq \Gamma_N^0 \leq \text{Isom}(N)$. In particular, $\overline{\Gamma'_N}/\Gamma_N^0$ is an abelian compact Lie group. As it is abelian we have that its connected component is a torus and we can write $\overline{\Gamma'_N}/\Gamma_N^0 = \hat{T}^\ell \times \hat{\Gamma}_A$, where $\hat{\Gamma}_A$ is itself a finite abelian group; the finiteness of $\hat{\Gamma}_A$ is due to compactness of $\text{Isom}(N)$. The point to emphasize is that there exists in $\overline{\Gamma'_N}/\Gamma_N^0$ a finite abelian group $\{e\} \times \hat{\Gamma}_A$ with an element in each connected component of $\overline{\Gamma'_N}/\Gamma_N^0$. By lifting $\{e\} \times \hat{\Gamma}_A$ to $\overline{\Gamma'_N}$ under the quotient map by Γ_N^0 , we have the (normal, because $\hat{\Gamma}_A$ is normal) finite group $\Sigma \leq \overline{\Gamma'_N}$, which satisfies the short exact sequence

$$0 \rightarrow \Gamma_N^0 \rightarrow \Sigma \rightarrow \hat{\Gamma}_A \rightarrow 0. \quad (4)$$

As $\hat{\Gamma}_A$ contains an element in each connected component of $\bar{\Gamma}'_N/\Gamma_N^0$, we have that Σ contains an element in each connected component of $\bar{\Gamma}'_N$. As the identity component of $\bar{\Gamma}'_N/\Gamma_N^0$ is a torus \hat{T}^ℓ , we necessarily have that the identity component of $\bar{\Gamma}'_N$ is a torus T^ℓ , as claimed.

We have two final properties to check before we can finish the proof of the Theorem. The first is to see that Σ is virtually abelian, the second is to see that T^ℓ and Σ commute:

Claim 3: There exists an abelian subgroup $A_\Sigma \leq \Sigma$ with $K(n)$ generators and finite index $[\Sigma : A_\Sigma] \leq C(n)$.

Observe that $\Sigma \leq \bar{\Gamma}'_N \leq \text{Isom}(N)$ is in particular a finite group with an effective action on N , and hence we may apply Lemma 2 to conclude the existence of $A_\Sigma \leq \Sigma$, as claimed.

Claim 4: The subgroups T^ℓ and $\Sigma \leq \bar{\Gamma}'_N \leq \text{Isom}(N)$ commute.

As Σ is normal in $\bar{\Gamma}'_N$, we have for each $t \in T^\ell$ and $\sigma \in \Sigma$ that $t\sigma t^{-1} \in \Sigma$. By continuity considerations, as Σ is finite and T^ℓ is connected, we then immediately get that $t\sigma t^{-1} = \sigma$. In particular, T^ℓ and Σ commute, as claimed.

Let us now finish the proof of the Theorem. Let $\bar{A} \leq \bar{\Gamma}'_N$ be the closed group generated by T^ℓ and $A_\Sigma \leq \Sigma$. Note by Claim 3 that $[\bar{\Gamma}'_N : \bar{A}] \leq C(n)$, and as A_Σ and T^ℓ commute we have that \bar{A} is abelian. Let us define the homomorphism

$$\rho : \Gamma' \xrightarrow{\rho_N} \Gamma'_N \rightarrow \bar{\Gamma}'_N, \quad (5)$$

and then define

$$A := \rho^{-1}(\bar{A}) \leq \Gamma'. \quad (6)$$

Note that A is abelian as we can identify $A \leq \text{Isom}(\mathbb{R}^k) \times \text{Isom}(N)$ and its projections to each factor are abelian. Further, it follows that $[\Gamma' : A] = [\bar{\Gamma}'_N : \bar{A}] \leq C(n)$. When we combine this with the previous estimate $[\Gamma : \Gamma'] \leq C(n)$ we get $[\Gamma : A] \leq C(n)$, which finishes the proof of the Theorem. \square

In [9, Section 6] the authors discuss an alternative strategy, due to Wilking, to perform the reduction to compact universal covers in the context of the Fukaya-Yamaguchi conjecture. Their approach relies on [13, Corollary 6.3].

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