ON THE GEOMETRIC PROPERTIES OF MULTI-OPERATOR TWO-PHASE ELLIPTIC MEASURE

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Abstract. We provide a structural characterization of a given boundary using two-phase elliptic measure in a multi-operator setting, extending to this novel setting results of Kenig, Preiss & Toro, Toro & Zhao and Azzam & Mourgoglou, including a partial answer to Bishop's question regarding the validity of Oksendal's conjecture under the assumption of mutual absolute continuity of the elliptic measures. Our techniques rely on a reduction to a multi-operator two-phase free-boundary problem combined with an extension of the powerful tools introduced by Preiss in his Density Theorem.

Contents

1. Intro

1.1. The multi-operator setting and new results. In this work, we study the structure of the points of mutual absolute continuity for a pair of elliptic measures $\omega^{\pm}$ corresponding to two *different* divergence-form operators on complementary NTA domains Ω^+ and $\Omega^- = \mathbb{R}^n \setminus \overline{\Omega^+}$. Identifying the relationship between the structure of a boundary and the behavior of its elliptic measure has a long history, starting with the classical case of harmonic measure, i.e. when the operator in question is the Laplacian. We provide an overview of the history of the problem in Section 1.2 below. Our setting herein falls into the regime of *two-phase* elliptic measure problems. In contrast to the one-phase setting, one does not relate the behavior of the elliptic measure and surface measure of the boundary, but instead considers the relative behavior between the two elliptic measures for the complementary domains.

More precisely, our setting is as follows. Consider complementary domains $\Omega^+ \subset \mathbb{R}^n$ and $\Omega^- = \mathbb{R}^n \setminus \overline{\Omega^+}$, with $n \geq 3$. Suppose that Ω^+ is a two-sided local NTA domain (see Section

2). Recall that[1] if $\Omega^\pm$ are two-sided NTA with $\partial\Omega^+ = \partial\Omega^- =: \partial\Omega$ then the Dirichlet problem for an elliptic divergence-form operator with any continuous boundary data admits a solution in $W^{1,2}_{\text{loc}}(\Omega^+) \cap C(\overline{\Omega^+})$, and the elliptic measure associated to this Dirichlet problem is well-defined. For each choice of sign, let $L_{A^\pm}u = \operatorname{div}(A^\pm \nabla u)$ be the second order linear elliptic differential operator with symmetric and uniformly elliptic coefficients $A^\pm$, where $A^\pm$ are 2-quasicontinuous matrix-valued functions on $\mathbb{R}^n$. Recall that a function $f : \mathbb{R}^n \to \mathbb{R}$ is called 2-quasicontinuous if for all $\varepsilon > 0$ there exists an open set V so that $\operatorname{cap}_2(V) \le \varepsilon$ and $f|_{\mathbb{R}^n \setminus V}$ is continuous, see [EG15, Sections 4.7, 4.8].

Remark 1.1. We work with 2-quasicontinuous functions $A^\pm : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ precisely so that for $\omega^\pm$-a.e. p we know $\lim_{r\to 0} \frac{1}{\omega^\pm(B(p,r))} \int |A^\pm - A^\pm(p)| d\omega^\pm = 0$, as soon as $\Omega^\pm$ are Wiener regular. See [HKM06, Theorem 11.14] for more details.

Remark 1.2. Since we expect tools from complex analysis will provide stronger answers with simpler techniques to the questions answered herein, we do not consider the case $n = 2$ even though the results remain valid in that case under the assumption that the coefficients are continuous outside a set of zero logarithmic capacity (see for instance the difference between Theorem 1.4 and Theorem 1.5).

Whenever $A^\pm$ and $\Omega^\pm$ are understood, we allow $\omega^\pm$ to denote the elliptic measures with suppressed poles $x^\pm \in \Omega^\pm$ (see Section 2 for more details).

Recall that a non-zero Radon measure ν is a *tangent measure* to μ at $a \in \mathbb{R}^n$, denoted $\nu \in \operatorname{Tan}(\mu, a)$, if there exists $c_i > 0$ and $r_i \downarrow 0$ so that the pushforwards $T_{a,r_i}[\mu]$ of μ under the maps $T_{a,r_i}(x) := \frac{x-a}{r_i}$ satisfy $\lim_{i\to\infty} c_i T_{a,r_i}[\mu] = \nu$ in the weak-$*$ sense [Pre87]. We denote the space of $(n-1)$-dimensional flat measures in $\mathbb{R}^n$ by

$$\mathcal{F} = \{c\mathcal{H}^{n-1} \llcorner \pi \mid \pi \in G(n-1, n)\}, \tag{1.1}$$

where $G(m, n)$ denotes the Grassmannian of m-dimensional linear subspaces of $\mathbb{R}^n$.

Our main result is the following.

Theorem 1.3. *Suppose that $\Omega^\pm$ are complementary NTA domains with common boundary $\partial\Omega$, that $A^\pm : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ are 2-quasicontinuous and uniformly elliptic matrix-valued functions, and that $L^\pm_A = -\operatorname{div}(A^\pm \nabla \cdot)$ on $\Omega^\pm$. If*

$$F_1 = \left\{ y \in \partial\Omega : 0 < \frac{d\omega^-}{d\omega^+}(y) < +\infty \right\},$$

then there exists a full measure subset of F_1, denoted F^*, so that $\partial\Omega = F^* \sqcup S \sqcup N$ and

(i) $\operatorname{Tan}(\omega^\pm, p) \subset \mathcal{F}$ *for all* $p \in F^*$, $\dim_{\mathcal{H}}(F^*) \le n - 1$, *and on* F^*, $\omega^+ \ll \omega^- \ll \omega^+$;
(ii) $\omega^+ \perp \omega^-$ *on* S;
(iii) $\omega^\pm(N) = 0$.

1.2. **Motivating background and history of harmonic measure.** The motivation for our study of the relationship between regularity properties of the two-phase multi-operator elliptic measures and structure of the boundary stem from powerful analogous results for two-phase harmonic measure, that is, the case where $A^\pm = \text{id}$. Let us briefly describe the history of this problem, starting with the story in the plane. The following is a combination of results due to Makarov, McMillan, Pomerenke, and Choi; see [GM05] for the precise references.

Theorem 1.4. *If $\partial\Omega \subset \mathbb{R}^2$ is a Jordan curve and ω the corresponding harmonic measure, then one can write $\partial\Omega$ as a disjoint union $\partial\Omega = G \sqcup S \sqcup N$, with*

(1) $\omega(N) = 0$;
(2) $\mathcal{H}^1(S) = 0$;
(3) $\limsup_{r\to 0} r^{-1}\omega(B(p, r)) = +\infty$ *and* $\liminf_{r\to 0} r^{-1}\omega(B(p, r)) = 0$ *for ω-a.e.* $p \in S$;
(4) *there is a geometric description of the points in S as twist points*[2];
(5) $\omega \ll \mathcal{H}^1 \ll \omega$ *on* G;

[1] See, for instance, [KPT09].
[2] See [GM05] for a definition of twist points.

(6) *Every point in G is a cone point for Ω, that is, the vertex of a cone contained in Ω;*
(7) *ω- a.e. and $\mathcal{H}^1$- a.e. cone point for Ω is in G*
(8) *For ω-a.e. $p \in G$, $\lim_{r\to 0} r^{-1}\omega(B(p,r))$ exists and takes values in $(0,\infty)$.*

Note that property (8) implies that $\omega \llcorner G$ is rectifiable by Preiss' Density Theorem [Pre87].

We recall the Hausdorff dimension of ω, denote $\dim_{\mathcal{H}}(\omega)$ is defined by

$$\dim_{\mathcal{H}}(\omega) = \inf\{s : \exists E \subset \partial\Omega \text{ with } \mathcal{H}^s(E) = 0 \text{ and } \omega(E\cap K) = \omega(\partial\Omega\cap K)\,\forall \text{ compact } K \subset \mathbb{R}^n\}.$$

Through a slight abuse of notation, we will also write $\dim_{\mathcal{H}}(E)$ to denote the Hausdorff dimension of a set E.

In the plane, Makarov established that for simply connected domains, $\dim_{\mathcal{H}}(\Omega) = 1$ [Mak85], answering Oskendal's Conjecture in the plane, concerning which type of domains $\dim_{\mathcal{H}}(\omega) = 1$ for. For general domains in $\mathbb{R}^2$, it is only known that $\dim_{\mathcal{H}}(\omega) \le 1$, [Car85], [JW88].

In $\mathbb{R}^n$ for $n \ge 3$, Bourgain [Bou87] showed there exists a constant $\beta_n > 0$ so that $\dim_{\mathcal{H}}(\omega) \le n - \beta_n$. Wolff showed that Oksendal's conjecture fails [Wol95] for $n \ge 3$ by constructing "Wolff snowflakes" for which one can have $\dim_{\mathcal{H}}(\omega) < n-1$ and $\dim_{\mathcal{H}}(\omega) > n-1$. Wolff's examples are two-sided NTA domains. Lewis, Verchota & Vogel [LVV05] re-examined Wolff's snowflakes and discovered one can simultaneously have $\dim_{\mathcal{H}}(\omega^{\pm}) < n-1$ or $\dim_{\mathcal{H}}(\omega^{\pm}) > n-1$. However, due to the Alt-Caffarelli-Friedman monotonicity formula, if $\omega^+ \ll \omega^- \ll \omega^+$ then $\dim_{\mathcal{H}}(\omega^{\pm}) \ge n-1$. This final observation led Bishop [Bis92] to ask whether $\omega^{\pm}(E) > 0$ and $\omega^+ \ll \omega^- \ll \omega^+$ ensure that $\dim_{\mathcal{H}}(E) = n-1$, where $\dim_{\mathcal{H}}(E)$ denotes the Hausdorff dimension of E. Bishop's question was fully answered positively in [AMT17a].

In the absence of complex analytic tools, studying the geometry of $\partial\Omega$ for rough domains $\Omega \subset \mathbb{R}^n$ for $n \ge 3$ in terms of the behavior of harmonic measure relies heavily on "two-phase" techniques for the harmonic measure. The following slightly weaker version of a boundary decomposition theorem holds for harmonic measure in $\mathbb{R}^n$, $n \ge 3$.

Theorem 1.5 ([KPT09]). *Let $n \ge 3$. If $\Omega^{\pm} \subset \mathbb{R}^n$ are complementary NTA domains and $\Omega^{\pm}$ are the interior and exterior harmonic measures, then $\partial\Omega = \Gamma \sqcup S \sqcup N$ where*

(1) *$\omega^+ \perp \omega^-$ on S and $\omega^{\pm}(N) = 0$.*
(2) *On Γ, $\omega^+ \ll \omega^- \ll \omega^+$ and $\dim_{\mathcal{H}} \Gamma \le n-1$, and if $\omega^{\pm}(\Gamma) > 0$ then $\dim_{\mathcal{H}} \Gamma = n-1$.*
(3) *If $\mathcal{H}^n \llcorner \partial\Omega$ is a Radon measure then Γ is $(n-1)$-rectifiable and $\omega^{\pm} \ll \mathcal{H}^{n-1} \ll \omega^{\pm}$ on Γ*
(4) *For every point $p \in \Gamma$, $\mathrm{Tan}(\omega^{\pm}, p) \subset \mathcal{F}$.*

Recent results [DM21, Per25] demonstrate that in the one-phase elliptic measure setting, sufficient regularity of the elliptic coefficients are necessary at the boundary for the relationship between geometry of a boundary and absolute continuity of the elliptic measure to hold. The conclusion of Theorem 1.5 was extended to the case of two-phase elliptic [TZ21, AM13] measure for a *single* elliptic operator whose coefficients have sufficient regularity across the boundary $\partial\Omega$. One can thus view our main result Theorem 1.3 as an answer to the question: does the above relationship between the regularity of interior- and exterior- elliptic measure and the geometry of the boundary remain true in the *multi-operator* setting, where the coefficients of the corresponding elliptic operators may have a discontinuity across the boundary?

In particular, in the multi-operator setting, we find a partial answer to Bishop's question concerning the validity of Oksendal's conjecture under the assumption of mutual absolute continuity of the pair of elliptic measures. Namely, we show that the set of points of mutual absolute continuity has dimension at most $n-1$ and we make progress toward a geometric characterization by showing that on a full measure subset of these points the elliptic measures $\omega^{\pm}_{A^{\pm}}$ have flat tangents. The reverse inequality to fully answer Bishop's question, as in the second conclusion of Theorem 1.5 (2), is still wide open since the original proof uses the ACF monotonicity formula. Although an analogue of such a monotonicity formula exists in the multi-operator setting (see [ST23]), it does not share the property of characterizing flat regions of the boundary as in the single-operator case, thus preventing it from being useful in the study of the regularity properties of the boundary; cf. Remark 4.21.

Two notable differences between the Theorem 1.4 and 1.5 are that, first, mutual absolute continuity of the harmonic and surface measure on the "good set" is established in the plane and not in Theorem 1.5 without any additional assumption and, second, only in the plane is there a geometric characterization of the "good set" and the "singular set" in terms of the surface measure. Both the mutual absolute continuity of harmonic and surface measure and a geometric characterization of the "good set" are handled in the remarkable work of [AMT17a], which can be summarized as follows.

Theorem 1.6. [AMT17a] *Let $n \geq 3$. If $\Omega^{\pm} \subset \mathbb{R}^n$ are complementary NTA domains, then $\partial\Omega = G \sqcup S \sqcup N$, where*

(1) *$\omega^+ \perp \omega^-$ on S and $\omega^{\pm}(N) = 0$*
(2) *G is $(n-1)$-rectifiable and $\omega^{\pm} \ll \mathcal{H}^{n-1} \ll \omega^{\pm}$ on G.*

Note that a characterization of S in terms of surface measure is not possible in general without additional assumptions such as uniform upper $(n-1)$ density control (see [Bad12]), as demonstrated via a counterexample by the same authors in [AMT17b].

The reliance on the Riesz transform in [AMT17a] (in particular the results of [GST18]) in proving Theorem 1.6 again leaves a full geometric characterization of the set of mutually absolutely continuous points currently out of reach in the setting herein, though we conjecture that such a strengthened characterization should also hold true.

1.3. **Overview of proof of Theorem 1.3 and structure of article.** The following class of measures identifies the free-boundary problem that our tangent measures satisfy at a.e. point in F_1.

Definition 1.7. For any two elliptic matrices $A^{\pm}$, we write $\nu \in \mathcal{D}(A^+, A^-)$ to mean that ν is a Radon measure with $0 \in \operatorname{spt}\nu$, and the following properties:

(1) $\mathbb{R}^n \setminus \operatorname{spt} \nu = \Omega^+ \cup \Omega^-$ for unbounded complementary NTA domains $\Omega^{\pm}$ with fixed NTA constants.
(2) There exist non-negative functions $u^{\pm} > 0$ on $\Omega^{\pm}$ and supported on $\overline{\Omega^{\pm}}$ respectively, so that
$$\int \varphi \, d\nu = \int_{\mathbb{R}^n} \nabla u^{\pm} \cdot (A^{\pm} \nabla \varphi) \qquad \forall \varphi \in C_c^{\infty}(\mathbb{R}^n).$$
The functions $u^{\pm}$ are referred to as Green's functions with pole at infinity for the triple $(\Omega^{\pm}, \nu, L_{A^{\pm}})$.

Indeed, in Theorem 3.3 we prove a more detailed version of the following theorem:

Theorem 1.8. *Suppose that $\Omega^{\pm}$, L_A and $\omega^{\pm}$ are as above. For $\omega^{\pm}$-a.e. point p in the set F_1 defined in Theorem 1.3,*
$$\operatorname{Tan}(\omega^{\pm}, p) \subset \mathcal{D}(A^+(p), A^-(p)).$$

The proof of Theorem 1.3, much like its analogues in the case of harmonic measure [KPT09] or a single elliptic operator [AM19], relies on a procedure first introduced by Preiss in his resolution of the density theorem. We will now briefly summarize the procedure of Preiss and how to use tangent measures, but with the specific focus on proving the claim that $\operatorname{Tan}(\omega^{\pm}, p) \subset \mathcal{F}$ in Theorem 1.3. Therefore, this summary also provides a broad outline of the rest of the paper. We wish to apply a key lemma regarding the connectedness of the space of tangent measures, Lemma 2.20, in the setting where $\mathcal{M} = \mathcal{D}(A_1, A_2)$. We encourage the reader to look at that lemma before reading the following three-step overview of what remains:

(1) For $\omega^{\pm}$-a.e. $p \in F_1$, there are elliptic matrices A_1, A_2 so that $\operatorname{Tan}(\omega^{\pm}, p) \subset \mathcal{D}(A_1, A_2)$; this is the contents of Theorem 3.3.
(2) The pair $(\mathcal{F}, \mathcal{D}(A_1, A_2))$ satisfy (P_i). This is the hardest step, and in confirming it we use Lemma 2.18 to equivalently show that the pair $(\mathcal{F}, \mathcal{D}(\mathrm{id}, \Lambda(p)^{-1} A_2 \Lambda(p))$ satisfy (P_i), where $\Lambda(p) = \sqrt{A_1}$. This is shown in Section 4.
(3) After the first two steps we can apply Lemma 2.20, but risk the case that the dichotomy tells us we have no flat tangents. We resolve this concern, by applying [AM13, Proposition 4.4] (which we restate with additional details in Lemma 4.6) and Theorem 2.17 to conclude that since the Λ-*tangents* for $\omega^{\pm}$-a.e. point $p \in F_1$ satisfy $\operatorname{Tan}_{\Lambda}(\omega^{\pm}, p) \subset \mathcal{D}(\mathrm{id}, \Lambda(p)^{-1} A_2 \Lambda(p))$ then for $\omega^{\pm}$-a.e. p, $\operatorname{Tan}_{\Lambda}(\mu, p) \cap \mathcal{F} \neq \emptyset$, thus

ruling out case (ii) in the dichotomy and proving the claim that for $\omega^{\pm}$-a.e. $p \in F_1$, $\mathrm{Tan}(\omega^{\pm}, p) \subset \mathcal{F}$.

While the experienced reader may notice that we passed between tangent measures and Λ-tangents in our outline of the paper above, this is done formally via Lemma 2.18 and is useful to perform an appropriate linear change of variables to simplify the structure of our free boundary problem to match the form as that previously studied in [Fel97, AM13, CDSS18].

Step (2) involves demonstrating that if a blown up boundary is sufficiently close to flat, then it is in fact flat. In the case $A^{\pm} = \mathrm{id}$ the free boundary problem reduces significantly to the case where the function $u = u^+ - u^-$ for the blown up Greens functions $u^{\pm}$ is a harmonic polynomial, as previous used in [KPT09, AM19] (see also [BET17]). Therein, such a rigidity result can thus be derived via a Liouville-type theorem for harmonic polynomials. Here, however, the coefficients $A^{\pm}(p)$ differ on each side of $\partial\Omega$ and thus one cannot generally hope that the blown-up boundary is the nodal set of an elliptic polynomial. We instead replace this with the delicate study of the aforementioned multi-operator free boundary problem, for which we can obtain an analogous Liouville-type, adapting ground-breaking techniques of Caffarelli [Caf87, Caf89, Caf88], which had been already observed to extend to such a setting in the works [Fel97, AM13]. The overall idea is a two-step procedure: first show that flatness of the boundary implies the boundary is Lipschitz, and then prove that Lipschitz regularity of the boundary in turn implies the boundary is $C^{1,\alpha}$. Although this is a local argument, which has since been superseded by the more flexible approach of De Silva [DS11], we crucially rely on proving rigidity in a global setting, which seems out of reach with De Silva's techniques. We refer the reader to Section 4.3 for a more detailed discussion of the approach. Note that in the special case where the jump in the coefficients is characterized by a scalar function, namely $A^+(p) = cA^-(p)$, one may consider a suitably weighted difference of u^+ and u^- in order to once again reduce to the elliptic polynomial case; see [KLS17, KLS19]. In that setting, the same procedure as in [AM19] carries through without a hitch, leaving it uninteresting in our setting. In particular, [KLS17, KLS19] make use of the Alt-Caffarelli-Friedman monotonicity formula and Almgren's frequency function, neither of which we can use in this setting.

Acknowledgments. The authors wish to thank Tatiana Toro for introducing this problem to them, hosting them at SLMath (formerly MSRI) and for a number of valuable discussions and insights. They also thank Max Engelstein, Dennis Kriventsov, and Steve Hofmann for fruitful discussions, as well as all of the staff at SLMath for their hospitality.

A.S. is grateful for the generous support of Dr. Max Rössler, the Walter Haefner Foundation and the ETH Zürich Foundation. This research was partially conducted during the period A.S. served as a Clay Research Fellow.

M.G. is supported by the European Research Council (ERC) under the European Union's Horizon Europe research and innovation programme (grant agreement No 101087499).

Part of this work was funded by the grant DMS-1853993.

2. Preliminaries

2.1. **Notation and conventions.** Unless otherwise stated, balls $B(x, r)$ of radius r and center x are assumed to be open, and $n \geq 3$ is a natural number. When the center of a given ball $B(0, r)$ is the origin, we instead often write B_r. A *domain* is an open connected set, and p denotes a point in the boundary of a given domain. We use $\Omega, \Omega^+, \Omega^-$ to denote domains. We assume $\Omega^{\pm}$ are complementary domains, that is $\Omega^- = \mathbb{R}^n \setminus \overline{\Omega^+}$. We will often denote "boundary balls" $B(p, r) \cap \partial\Omega$ by $\Delta(p, r)$.

We denote by $\delta_{\Omega}(x)$ the distance from x to $\partial\Omega$. When the domain is understood, we simply write $\delta(x)$.

We refer to m-dimensional linear subspaces of $\mathbb{R}^n$ as m-planes, and $G(m, n)$ denotes the space of m-planes in $\mathbb{R}^n$. We will be taking $m = n - 1$ throughout, and our $(n-1)$-planes will be often denoted by π. We denote the pushforward of a Radon measure μ under a Borel map T by $T[\mu]$.

2.2. **NTA domains.** We will work under the following assumptions on our domains Ω, Ω^+ and Ω^- throughout, as introduced in [Jon82]. The first is a quantitative openness known as the *corkscrew condition.*

Definition 2.1 (Corkscrew Condition)**.** Let $C_0, R_0 > 0$. We say that an open set $\Omega \subset \mathbb{R}^n$ satisfies the (C_0, R_0) interior corkscrew condition if for every $p \in \partial\Omega$ and $r \in (0, R_0)$ there exists a point $x_{p,r}$, called the interior corkscrew point for $\Delta(p, r) = \Omega \cap B(p, r)$, so that $B(x_{p,r}, r/C_0) \subset B(p, r) \cap \Omega$.

We say that an open set $\Omega \subset \mathbb{R}^n$ satisfies the (C_0, R_0) two-sided corkscrew condition if both Ω and $\mathbb{R}^n \setminus \overline{\Omega}$ satisfy the (C_0, R_0) interior corkscrew condition.

The next assumption is a quantitative notion of connectedness known as the *Harnack chain condition.*

Definition 2.2 (Harnack Chain Condition)**.** We say that a domain Ω satisfies the (C_1, R_1)-Harnack chain condition if for every $0 < \rho \leq R_1$, $\Lambda \geq 1$, and every pair of points $x, x' \in \Omega$ with $\min\{\delta_\Omega(x), \delta_\Omega(x')\} \geq \rho$ and $|x - x'| < \Lambda\rho$ there is a chain of balls $B^1, \dots, B^N \subset \Omega$ with $N \leq C_1(\log_2 \Lambda + 1)$ so that:

(1) $x \in B^1$, $x' \in B_N$
(2) $B^k \cap B^{k+1} \neq \emptyset$ for all $k = 1, \dots, N - 1$
(3) $C_1^{-1} \operatorname{diam}(B^k) \leq \operatorname{dist}_{\mathcal{H}}[B^k, \partial\Omega] \leq C_1 \operatorname{diam}(B^k)$,

where $\operatorname{dist}_{\mathcal{H}}$ denotes the Hausdorff distance. The chain of balls is called a *Harnack chain.*

Definition 2.3 (NTA domains)**.** We say that an open set $\Omega \subset \mathbb{R}^n$ is a non-tangentially accessible domain (NTA domain) with constants (C_2, R_2) if it satisfies the (C_2, R_2)-Harnack chain condition and the (C_2, R_2) two-sided corkscrew condition.

When we discuss an NTA domain, we will typically neglect to specifically mention the associated constants (C_2, R_2), which are referred to as the *NTA constants*, since they will always be fixed. If Ω is unbounded, we require that $\mathbb{R}^n \setminus \partial\Omega$ consists of two non-empty connected components. If Ω is unbounded, then $R_0 = \infty$ is allowed.

Sometimes authors make the distinction between one-sided and two-sided NTA domains. In such cases, our definition coincides with two-sided NTA domains.

Let $\operatorname{dist}_{\mathcal{H}}(A, B)$ denote the Hausdorff distance between two non-empty compact sets $A, B \subset \mathbb{R}^n$, that is, for non-empty compact sets $A, B \subset \mathbb{R}^n$,

$$\operatorname{dist}_{\mathcal{H}}[A, B] := \sup\{d(a, B) : a \in A\} + \sup\{d(b, A) : b \in B\}.$$

For a closed set $\Gamma \subset \mathbb{R}^n$, for any $p \in \Gamma$ and any $r > 0$, we define

$$\Theta_\Gamma(p, r) = \frac{1}{r} \inf \left\{ \operatorname{dist}_{\mathcal{H}} \left[(\Gamma - p) \cap \overline{B_r}, \pi \cap \overline{B_r} \right] : \pi \in G(n-1, n) \right\} \tag{2.1}$$

and

$$\beta_\Gamma(p, r) = \frac{1}{r} \inf_{\pi \in G(n-1,n)} \sup_{q \in \Gamma \cap B(p,r)} \operatorname{dist}(q, p + \pi) \tag{2.2}$$

We recall the following lemma from [KPT09], which will be useful to obtain the dimension estimate on F^* in Theorem 1.3.

Lemma 2.4 ([KPT09, Lemma 2.4])**.** *Suppose that* $\Gamma \subset \mathbb{R}^n$ *is a closed set with the property that*

$$\lim_{r \to 0} \beta_\Gamma(p, r) = 0 \qquad \forall\, p \in \Gamma\,.$$

Then $\dim_{\mathcal{H}}(\Gamma) \leq n - 1\,.$

2.3. **Elliptic measure background.** Suppose that $A \in L^\infty(\Omega; \mathbb{R}^{n \times n})$ is a Λ_0-uniformly elliptic symmetric matrix for some $\Lambda_0 \geq 1$, that is $A^t = A$ and

$$\Lambda_0^{-1} |\xi|^2 \leq \langle \xi, A\xi \rangle \leq \Lambda_0 |\xi|^2 \qquad \text{for every } \xi \in \mathbb{R}^n \setminus \{0\}.$$

If Ω is an NTA domain (see 2.3) then in particular Ω is Wiener regular (see [Wie23]) and so the work [LSW63] guarantees that it is also regular for $L_A = -\operatorname{div}(A\nabla\cdot)$ and so the elliptic measure for (Ω, L_A) with pole at x is well-defined. That is, for $x \in \Omega$ there is a unique probability measure ω^x supported on $\partial\Omega$ so that if $L_A u = 0$ on Ω and $u = f \in C(\partial\Omega) \cap W^{1,2}(\Omega)$ then $u(x) = \int f d\omega^x$. If Ω is an unbounded domain, we may also consider ω to be the elliptic measure with pole at infinity, which is the unique measure so that for all $\varphi \in C_c^\infty(\mathbb{R}^n)$,

$$\int \varphi \, d\omega = \int_\Omega \varphi \, L_A u$$

where u is the corresponding Green's function with pole at infinity (see Theorem 2.6 below).

Note that when discussing regularity properties of $\omega^\pm$ up to an $\omega^\pm$-negligible set, we may without loss of generality fix such poles $x^\pm$ arbitrarily, so long as $\Omega^\pm$ are Wiener regular, since any pair of elliptic measures in a fixed domain associated to a given uniformly elliptic operator are mutually absolutely continuous, due to the Harnack inequality. Indeed, suppose that Ω is Wiener regular, let $x, y \in \Omega$ and suppose that K is a compact set such that $\omega^x(K) = 0$. Then for all $\varepsilon > 0$ there exists an open neighborhood U_ε of K with $\omega^x(U_\varepsilon) < \varepsilon$. Consider the functions $f_\varepsilon \in C(\partial\Omega)$ so that $\mathbf{1}_K \le f_\varepsilon \le 1$ and f_ε is supported in U_ε. Suppose that $\omega^y(K) = \eta > 0$. Then, letting u_ε be a solution of

$$\begin{cases} L_A u_\varepsilon = 0 & \text{in } \Omega \\ u_\varepsilon = f_\varepsilon & \text{on } \partial\Omega \end{cases}$$

it follows from a Harnack chain argument that $u_\varepsilon(y) \le C u_\varepsilon(x)$ for some constant C which depends on x, y, Ω, and the ellipticity of A. On the other hand, the integral representation of u_ε yields

$$\begin{cases} \eta \le u_\varepsilon(y) = \int f_\varepsilon d\omega^y \\ \varepsilon \ge u_\varepsilon(x) = \int f_\varepsilon d\omega^x \end{cases}$$

so that $u_\varepsilon(y) \ge \frac{\eta}{\varepsilon} u_\varepsilon(x)$ reaching a contradiction for $\varepsilon \le C^{-1}\eta$.

Under slightly weaker assumptions than our own, the work of Grüter and Widman [GW82] ensures the existence of a Green function.[3] The book [Ken94] describes the behavior of the elliptic measure and corresponding Green functions. In particular the results proved in [Jon82] for harmonic functions on NTA domains extend to solutions of L_A on NTA domains (or even uniform domains with the CDC) with appropriate adaptations. Throughout the remainder of the preliminaries, we work under the following assumption.

Assumption 2.5. Let $\Lambda_0 > 0$, $n \ge 3$, and $\Omega \subset \mathbb{R}^n$ be an NTA domain. Let $A \in L^\infty(\Omega; \mathbb{R}^{n\times n})$ be symmetric and Λ_0-elliptic matrix-valued function, and let $L_A = -\operatorname{div}(A\nabla\cdot)$. Let $\{\omega^x\}_{x\in\Omega}$ denote the family of elliptic measures associated to L_A in Ω. We refer to ω^x as the *elliptic measure with pole at* x. When Ω is unbounded, we may additionally include the elliptic measure with pole at infinity in this family.

We summarize below the results which will be used later in this paper:

Theorem 2.6 (Green functions, [TZ21, Theorem 2.7])**.** *Let Ω and A be as in Assumption 2.5. There exists a unique $G : \Omega \times \Omega \to [0, \infty]$ such that the following hold:*

(1) for each $y \in \Omega$ and $r > 0$, $G(\cdot, y) \in W^{1,2}(\Omega \setminus B(y, r)) \cap W_0^{1,1}(\Omega)$
(2) for all $\varphi \in C_c^\infty(\Omega)$, $\int \langle A(x)\nabla G(x, y), \nabla\varphi(x)\rangle \, dx = \varphi(y)$
(3) for each $y \in \Omega$, $G(\cdot, y)$, called the Green function of L_A in Ω with pole at y, denoted $G(x) = G(x, y)$ satisfies

$$G \in L^*_{\frac{n}{n-2}}(\Omega) \text{ with } \|G\|_{L^*_{\frac{n}{n-2}}} \le C(n, \Lambda_0).$$

$$\nabla G \in L^*_{\frac{n}{n-1}}(\Omega) \text{ with } \|\nabla G\|_{L^*_{\frac{n}{n-1}}} \le C(n, \Lambda_0).$$

(4) For all $x, y \in \Omega$

$$G(x, y) \le C(n, \Lambda_0)|x - y|^{2-n}$$

$$G(x, y) \ge C(n, \Lambda_0)|x - y|^{2-n}, \quad \text{if } |x - y| \le \delta(y)/2.$$

[3] In fact they do not require the exterior corkscrew condition, and instead use the CDC condition (see e.g. [TZ21, Definition 2.6].

Here, $L_p^*(\Omega)$ denotes the weak L^p space with $p \geq 1$, given by

$$L_p^*(\Omega) = \{f : \Omega \to \mathbb{R} \cup \{\infty\} : f \text{ measurable}, \|f\|_{L_p^*(\Omega)} < \infty\}$$

where

$$\|f\|_{L_p^*(\Omega)} = \sup_{t>0} t|\{x \in \Omega : |f(x)| > t\}|^{\frac{1}{p}}.$$

Note that $\|f\|_{L_p^*(\Omega)} \leq \|f\|_{L^p(\Omega)}$, and for $0 < \sigma \leq p-1$ we have

$$\|f\|_{L^{p-\sigma}(\Omega)} \leq \left(\frac{p}{\sigma}\right)^{\frac{1}{p-\sigma}} |\Omega|^{\frac{\sigma}{p(p-\sigma)}} \|f\|_{L_p^*(\Omega)},$$

where $|\Omega|$ denotes the Lebesgue measure of Ω.

We further recall from [Ken94] some basic results concerning the behavior of L_A-elliptic measures and functions on NTA domains, which are the analogue of properties derived in [Jon82] for $L_A = -\Delta$. Our presentation closely follows [TZ21]. For Ω and A as in Assumption 2.5, we refer to allowable constants as constants depending only on the dimension n, the ellipticity Λ_0, the L^∞-norm of A, and the NTA constants (C_2, R_2) of Ω. Furthermore all of the following results still hold if the exterior corkscrew condition is replaced by the CDC condition.

Lemma 2.7. *Let Ω, A and L_A be as in Assumption 2.5. There exist constants $\beta > 0$ and $C > 0$ depending on allowable constants such that for any $p \in \partial\Omega$, $0 < r < \operatorname{diam}\Omega$, the following holds. If $u \in W^{1,2}(\Omega; [0,\infty))$ with $L_A u = 0$ weakly in $B(p, 2r) \cap \Omega$ and u vanishes continuously[4] on $\Delta(p, 2r) = B(p, 2r) \cap \partial\Omega$, then*

$$u(x) \leq Cr^{-\beta}|x-p|^\beta \|u\|_{C^0(B(p,2r)\cap\Omega)} \qquad \forall x \in \Omega \cap B(p,r).$$

Corollary 2.8. *Let Ω, A, L_A and $\{\omega^x\}_{x\in\Omega}$ be as in Assumption 2.5. There exists $m_0 > 0$ depending on allowable constants such that for any $p \in \partial\Omega$ and $0 < r < R_0$,*

$$\omega^{x_{p,r}}(B(p,r) \cap \partial\Omega) \geq m_0,$$

where $x_{p,r}$ is an interior corkscrew point for $\Delta(p,r)$.

Lemma 2.9. *Let Ω, A and L_A be as in Assumption 2.5. There exists $C > 0$ depending on allowable constants such that for any $p \in \partial\Omega$ and $0 < r < R_0$, the following holds. If $u \in W^{1,2}(\Omega; [0,\infty))$ with $L_A u = 0$ weakly in $B(p,4r) \cap \Omega$ and u vanishes continuously on $\Delta(p,4r)$, then*

$$u(x) \leq Cu(x_{p,r}) \qquad \forall x \in \Omega \cap B(p,r).$$

Lemma 2.10. *Let Ω, A, L_A and $\{\omega^x\}_{x\in\Omega}$ be as in Assumption 2.5. There exists $C > 0$ depending on allowable constants such that for all $p \in \partial\Omega$ and $0 < r < R_0/M$ if $L_A u = 0$ weakly in $B(p,4r) \cap \Omega$ and u vanishes continuously on $\Delta(p,4r)$, then*

$$C^{-1} \leq \frac{\omega^x(\Delta(p,r))}{r^{n-2}u(x_{p,r})} \leq C \qquad \text{for any } x \in \Omega \setminus B(p,4r). \tag{2.3}$$

In particular, this holds when $u(\cdot) = G(\cdot, x)$ is the Green function of L_A in Ω with pole at x as in Theorem 2.6.

Lemma 2.11. *Let Ω, A, L_A and $\{\omega^x\}_{x\in\Omega}$ be as in Assumption 2.5. There exists $C > 0$ depending on allowable constants such that for all $p \in \partial\Omega$ and $0 < r < R_0/2M$, the following holds. If $y \in \Omega \setminus B(p, 2Mr)$, then for $s \in (r/2, r)$,*

$$\omega^y(B(p,r)) \leq C\omega^y(B(p,s)).$$

Proposition 2.12. *Let Ω, A, L_A and $\{\omega^x\}_{x\in\Omega}$ be as in Assumption 2.5. Then for any $y \in \Omega$ and any $\varphi \in C_c^\infty(\mathbb{R}^n)$,*

$$-\int_\Omega \langle A(x)\nabla G(x,y), \nabla\varphi(x)\rangle \, dx = \int_{\partial\Omega} \varphi \, d\omega^y - \varphi(y),$$

where $G(\cdot, y)$ is a Green function for L_A in Ω with pole at y.

---

[4]Since NTA domains are Wiener regular, it is well-known [Wie23, LSW63] that solutions with continuous boundary data are continuous at the boundary. In particular, it is not restrictive to assume that u vanishes continuously on $\Delta(p, 2r)$.

2.4. **Tangent measures and Λ-tangents.** For a compact set $K \subset \mathbb{R}^n$, and two Radon measures μ, ν we define

$$F_K(\mu,\nu) = \sup\left\{ \left| \int f d(\mu-\nu) \right| \mid \mathrm{Lip}(f) \le 1,\ f \in C_c(K) \right\}. \tag{2.4}$$

If $K = B_r$, we simply write $F_r(\cdot,\cdot)$. We recall, see [Mat95, Lemma 14.13] that for a sequence of Radon measures $\{\mu_k\}$ and a Radon measure μ,

$$\mu_k \overset{*}{\rightharpoonup} \mu \iff \lim_{k\to\infty} F_r(\mu_k,\mu) = 0 \quad \forall r > 0. \tag{2.5}$$

It is well-known, see [Pre87, Proposition 1.12], that

$$F(\mu,\nu) := \sum_{\ell=1}^{\infty} 2^{-\ell} \min\{1, F_\ell(\mu,\nu)\}$$

defines a metric on the space of Radon measures. Moreover, F generates the topology of weak-$*$ convergence. We denote $F(\mu) = F(\mu, 0)$.

Throughout the remainder of this section we will suppose that $\Lambda : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ is an invertible matrix-valued function. We denote this by writing $\Lambda : \mathbb{R}^n \to GL(n,\mathbb{R})$.

The notion of tangent measures that we will summarize stems from Preiss' resolution of the density question [Pre87]. The extension of Preiss' machinery to the elliptic setting was introduced and first used in [CGTW25] by Casey, the first author, Toro, and Wilson. The presentation here will follow most similarly to [CGTW25], but we emphasize that the case where $\Lambda \equiv \mathrm{id}$ always reduces to Preiss' work.

Given an invertible matrix-valued function $\Lambda : \mathbb{R}^n \to GL(n,\mathbb{R})$ fixed throughout this section, we consider the ellipse centered at p

$$E_\Lambda(p,r) = p + \Lambda(p)B(0,r),$$

whose eccentricity depends on p. We further consider the rescalings

$$T^\Lambda_{p,r}(y) = \Lambda(p)^{-1}\left(\frac{y-p}{r}\right),$$

and the associated pushforward measures $T^\Lambda_{p,r}[\mu]$. The latter in particular satisfy

$$T^\Lambda_{p,r}[\mu](B_1)) = T_{p,r}[\mu](E_\Lambda(0,1)) = \mu\left(E_\Lambda(p,r)\right).$$

Definition 2.13 (Λ-tangents)**.** If μ is a Radon measure on $\mathbb{R}^n$, $p \in \operatorname{spt}\mu$ and $\Lambda : \mathbb{R}^n \to GL(n,\mathbb{R})$, we define the *class of Λ-tangents to μ at p* as

$$\mathrm{Tan}_\Lambda(\mu,p) := \left\{ \nu \text{ Radon s.t. } \nu = \lim_i c_i T^\Lambda_{p,r_i}[\mu] : c_i > 0,\ r_i \downarrow 0,\ \nu \ne 0 \right\}. \tag{2.6}$$

In the special case where $\Lambda = \mathrm{id}$, the class of Λ-tangents at a given point is simply the class of tangent measures and thus we simply write $\mathrm{Tan}(\mu,p)$; see the discussion in the introduction. Furthermore, we denote by $\mathrm{Tan}_\Lambda[\mu]$ the weak-$*$ closure of the space of measures $\cup_{p\in \operatorname{spt}\mu} \mathrm{Tan}_\Lambda(\mu,p)$. We essentially only consider this space in the case of $\mathrm{Tan}_{\mathrm{id}}[\mu]$ which we will simply denote as $\mathrm{Tan}[\mu]$.

Definition 2.14 (Cones, d-cones, and basis)**.** A collection of nonzero Radon measures $\mathcal{M}$ is called a *cone* if

$$\mu \in \mathcal{M} \implies c\mu \in \mathcal{M} \qquad \text{for all } c > 0\,.$$

A cone of Radon measures is called a *d-cone* or dilation cone if

$$\mu \in \mathcal{M} \implies T_{0,r}[\mu] \in \mathcal{M} \qquad \text{for all } r > 0\,.$$

The *basis* of a d-cone is the collection of $\mu \in \mathcal{M}$ so that $F_1(\mu) = 1$. A d-cone $\mathcal{M}$ is said to have a *closed* (respectively *compact*) *basis* if the basis is closed (respectively compact) with respect to the weak-$*$ topology.

We next introduce a notion of the distance to a d-cone. Let $\mathcal{M}$ be a d-cone and ν a Radon measure in $\mathbb{R}^n$. If $s > 0$ and $0 < F_s(\nu) < \infty$ we define the *distance* between ν and $\mathcal{M}$ at scale s by

$$d_s(\nu, \mathcal{M}) = \inf \left\{ F_s \left(\frac{\nu}{F_s(\nu)}, \mu \right) : \mu \in \mathcal{M} \text{ and } F_s(\mu) = 1 \right\}. \tag{2.7}$$

If $F_s(\nu) \in \{0, \infty\}$, we define $d_s(\nu, \mathcal{M}) = 1$.

Proposition 2.15. [KPT09, Remarks 2.1 & 2.2] *If μ, ν are Radon measures,*

$$F_r(\mu, \nu) = rF_1(T_{0,r}[\mu], T_{0,r}[\nu]). \tag{2.8}$$

If $\mathcal{M}$ is a d-cone and ν a Radon measure,

i) $d_s(\nu, \mathcal{M}) \leq 1$ for all $s > 0$.
ii) $d_s(\nu, \mathcal{M}) = d_1(T_{0,s}[\nu], \mathcal{M})$ for all $s > 0$.
iii) If $\nu_i \overset{*}{\rightharpoonup} \nu$ and $F_s(\nu) > 0$, then $d_s(\nu, \mathcal{M}) = \lim_{i\to\infty} d_s(\nu_i, \mathcal{M})$.

Remark 2.16. Given $\nu \in \mathrm{Tan}_\Lambda(\mu, p)$ with $c_i T^\Lambda_{p,r_i}[\mu] \overset{*}{\rightharpoonup} \nu$ it is easy to check that $cT_{0,r}[\nu] = \lim_i cc_i T^\Lambda_{p,rr_i}[\mu]$ for any $c, r > 0$. In particular $\mathrm{Tan}_\Lambda(\mu, p)$ is a d-cone.

A crucial property of Λ-tangents (and in particular tangent measures) is that tangents to Λ-tangents are Λ-tangents. This is a consequence of the following theorem.

Theorem 2.17. [CGTW25, Theorem 3.3] *Let μ be a Radon measure on $\mathbb{R}^n$ and $\Lambda : \mathbb{R}^n \to GL(n, \mathbb{R})$. Then at μ-a.e. $p \in \mathbb{R}^n$, every $\nu \in \mathrm{Tan}_\Lambda(\mu, p)$ has the following two properties:*

(1) $T_{x,r}[\nu] \in \mathrm{Tan}_\Lambda(\mu, p)$ for all $x \in \operatorname{spt} \nu, r > 0$.
(2) $\mathrm{Tan}(\nu, x) \subset \mathrm{Tan}_\Lambda(\mu, p)$ for all $x \in \operatorname{spt} \nu$.

Λ-tangents were introduced as a tool to transform anisotropic information of a base measure into an isotropic form for the Λ-tangents. It turns out this is equivalent to taking a tangent measure, and then performing a linear rigid change of variables. In this paper, we use the latter approach in order to allow access to all the estimates from Section 2.3 while performing a blow-up without restating Section 2.3 in terms of ellipses (which would also be possible). To formalize this geometric intuition, we recall the Isomorphism Lemma:

Lemma 2.18. [CGTW25, Lemma 3.4] *Let μ be a Radon measure on $\mathbb{R}^n$ and $\Lambda : \mathbb{R}^n \to GL(n, \mathbb{R})$. For a Radon measure ν, the following are equivalent:*

(1) $\nu \in \mathrm{Tan}_\Lambda(\mu, p)$
(2) $\Lambda(p)_\sharp \nu \in \mathrm{Tan}(\mu, p)$
(3) $\nu \in \mathrm{Tan}((\Lambda(p)^{-1})_\sharp \mu, \Lambda(p)^{-1} p)$

Remark 2.19. In Theorem 1.3 (i), one of our goals is to conclude that $\mathrm{Tan}(\mu, p) \subset \mathcal{F}$, the latter being the space of flat measures. Since for any invertible matrix $\Lambda(p)$, $\Lambda(p)_\sharp \sigma \in \mathcal{F}$ for all $\sigma \in \mathcal{F}$, the equivalence of (1) and (2) in Lemma 2.18 immediately shows that this conclusion in Theorem 1.3 can equivalently be stated for $\mathrm{Tan}_\Lambda(\mu, p)$ in place of $\mathrm{Tan}(\mu, p)$.

The next lemma states that $\mathrm{Tan}_\Lambda(\mu, p)$ is connected in a specific sense:

Lemma 2.20. [CGTW25, Corollary 3.6] *Let $\Lambda : \mathbb{R}^n \to GL(n, \mathbb{R})$. Suppose $\mathcal{F} = \cup_{i=1}^\infty \mathcal{F}_i$, each $\mathcal{F}_i$ is a d-cone with compact basis and there exists a d-cone $\mathcal{M}$ with closed basis so that $\mathcal{F} \subset \mathcal{M}$ and for each i the following holds*

$$\begin{cases} \exists \varepsilon_i > 0,\, R_i > 0 \text{ such that } \forall \varepsilon \in (0, \varepsilon_i) \text{ there exists no } \nu \in \mathcal{M} \setminus \cup_{j=1}^{i-1} \mathcal{F}_j \\ \text{satisfying } d_r(\nu, \mathcal{F}_i) \leq \varepsilon \ \forall r \geq R_i > 0 \quad \text{and } d_{R_i}(\nu, \mathcal{F}_i) = \varepsilon. \end{cases} \tag{P_i}$$

Then for any $p \in \mathbb{R}^n$ so that $\mathrm{Tan}_\Lambda(\mu, a) \subset \mathcal{M}$, the following dichotomy holds:

(i) $\mathrm{Tan}_\Lambda(\mu, p) \subset \mathcal{F}$, or
(ii) $\mathrm{Tan}_\Lambda(\mu, p) \cap \mathcal{F} = \emptyset$.

We recall Definition 1.7, where we defined $\mathcal{D}(A^+, A^-)$ to be the class of Radon measures with the following properties:

a) $\mathbb{R}^n \setminus \text{spt } \nu = \Omega^+ \cup \Omega^-$ for unbounded complementary NTA domains $\Omega^\pm$ with fixed NTA constants.
b) There exist non-negative $u^\pm > 0$ that are Green's functions with pole at infinity for the triple $(\Omega^\pm, \nu, L_{A^\pm})$.

3. Reduction to tangents

In this section we will adapt a blow-up argument, pioneered by Kenig and Toro [KT03, KT06] to our setting. Throughout, we will work under the following assumption.

Assumption 3.1. Let $n \geq 0$ and $\Lambda_0 > 0$. Let Ω^+ and $\Omega^- = \mathbb{R}^n \setminus \overline{\Omega^+}$ be complementary NTA domains with $\partial\Omega := \partial\Omega^+ = \partial\Omega^-$, let $A^\pm \in L^\infty(\mathbb{R}^n; \mathbb{R}^{n\times n})$ and let $A^\pm$ be 2-quasicontinuous, symmetric, and Λ_0-elliptic matrix-valued functions, and let $L_{A^\pm} = -\operatorname{div}(A^\pm \nabla \cdot)$ in $\Omega^\pm$. Let $x_0^\pm \in \Omega^\pm$ be fixed poles, and let $u^\pm := G^\pm(\cdot, x_0)$ and $\omega^\pm := \omega^{\pm, x_0^\pm}$ be the respective Green functions and elliptic measures for $L_{A^\pm}$ in $\Omega^\pm$ with poles at $x_0^\pm$.

Let $\Lambda_0, \Omega^\pm, u^\pm, A^\pm$, and $\Omega^\pm$ be as in Assumption 3.1. Given a sequence of positive numbers $\{r_i\}$ with $r_i \downarrow 0$ and $p \in \partial\Omega$, define

$$\Omega_i^\pm = \frac{\Omega^\pm - p}{r_i}, \qquad \partial\Omega_i = \frac{\partial\Omega - p}{r_i} \tag{3.1}$$

together with the functions

$$u_i^\pm(x) = \frac{u^\pm(r_i x + p)}{\omega^\pm(B(p, r_i))} r_i^{n-2} \tag{3.2}$$

and the measures

$$\omega_i^\pm(E) = \frac{\omega^\pm(r_i E + p)}{\omega^\pm(B(p, r_i))} \quad \text{for Borel subsets } E \subset \mathbb{R}^n. \tag{3.3}$$

Note that (3.3) may be rewritten as $\omega_i^\pm = c_i T_{p,r_i}[\omega^\pm]$ where $c_i = \omega^\pm(B(p, r_i))^{-1}$.

Before beginning the proof in earnest, we take note of a few properties of the defined sequences. By Lemma 2.10, for i large enough (so that $r_i \leq R_0/C_0$ and our unnamed pole is outside $B(p, 4r_i)$) it follows that there exists a $C > 0$ depending on the NTA constants, dimension, and ellipticity (but not on i) so that

$$C^{-1} \leq \frac{u^\pm(x^\pm_{p,r_i})}{\omega^\pm(B(p, r_i))} r_i^{n-2} \leq C \tag{3.4}$$

where $x^\pm_{p,r_i} \in \Omega^\pm$ are interior corkscrew points for $\Omega^\pm$ associated to the surface balls $\Delta(p, r_i)$. In addition, the boundary harnack property for NTA domains, Lemma 2.9 yields that for fixed $N > 1$ and all $x \in B(0, N)$, whenever i is large enough depending on N,

$$u^\pm(r_i x + p) \leq C u^\pm(x^\pm_{p,r_i}). \tag{3.5}$$

In particular, combining (3.2), (3.4), and (3.5) we conclude that for all $x \in B(0, N)$:

$$\sup_{i \geq 1} \sup_{x \in B(0,N)} u_i^\pm(x) \leq C < \infty. \tag{3.6}$$

Furthermore, since $\omega^\pm$ are locally doubling (Lemma 2.11), it follows from (3.3) that

$$\sup_{i\geq 1} \omega_i^\pm(B(0, N)) = \sup_{i \geq 1} \frac{\omega^\pm(B(p, r_i N))}{\omega^\pm(B(p, r_i))} \leq C_N < \infty \tag{3.7}$$

where $C_N > 0$ depends on N and allowable constants.

Finally, to state the main theorem of this section, we decompose the boundary $\partial\Omega$ of $\partial\Omega^\pm$.

$$F_1 = \left\{ p \in \partial\Omega : 0 < h(p) := \frac{d\omega^-}{d\omega^+}(p) = \lim_{r \to 0} \frac{\omega^-(B(p, r))}{\omega^+(B(p, r))} < \infty \right\}, \tag{3.8}$$

$$F_2 = \left\{ p \in \partial\Omega : \frac{d\omega^-}{d\omega^+}(p) = \infty \right\} \tag{3.9}$$

$$F_3 = \left\{ p \in \partial\Omega : \frac{d\omega^-}{d\omega^+}(p) = 0 \right\} \tag{3.10}$$

$$F_4 = \left\{ p \in \partial\Omega : \frac{d\omega^-}{d\omega^+}(p) \text{ does not exist} \right\}. \tag{3.11}$$

Given a Radon measure μ and a μ-measurable set E, we denote by $\Theta(\mu, p, E)$ the density

$$\theta(\mu, p, E) := \lim_{r \downarrow 0} \frac{\mu(B(p,r) \cap E)}{\mu(B(p,r))},$$

whenever such a limit exists. In addition, given a μ-measurable function f, we say that x is a point of μ-approximate continuity for f if

$$f(x) = \lim_{r\to 0} \fint_{B(x,r)} f\, d\mu \quad \text{and} \quad \lim_{r\to 0} \fint_{B(x,r)} |f(x) - f(y)| d\mu(y) = 0.$$

We denote the set of μ-approximate continuity points for f by $\mathcal{C}(\mu, f)$, and define

$$F_0 = \left\{ p \in F_1 \cap \mathcal{C}\Big(\omega^+, \frac{d\omega^-}{d\omega^+}\Big) \cap \mathcal{C}(\omega^\pm, A^\pm) : \theta(\omega^\pm, p, F_1) \text{ exists and equals } 1 \right\}. \tag{3.12}$$

Remark 3.2. By the Lebesgue-Besicovitch-Federer differentiation theorem, we know that:

- $\partial\Omega = F_1 \sqcup F_2 \sqcup F_3 \sqcup F_4$;
- $\omega^+(F_2) = 0 = \omega^-(F_3) = \omega^\pm(F_4)$;
- $\omega^+ \llcorner F_1$ and $\omega^- \llcorner F_1$ are mutually absolutely continuous;
- $\frac{d\omega^-}{d\omega^+} \in L^1_{\text{loc}}(\omega^+)$ and $\frac{d\omega^+}{d\omega^-} = 1/\left(\frac{d\omega^-}{d\omega^+}\right) \in L^1_{\text{loc}}(\omega^-)$;
- $\omega^\pm(F_1 \setminus F_0) = 0$.

Note that the last property additionally uses the fact that $A^\pm$ are 2-quasicontinuous and [HKM06, Theorem 11.14].

We introduce a final piece of notation which is subtle, but crucial to succinctly talk about the results of this section. We recall that we often have two matrix-valued functions $A^\pm$, and the two corresponding operators

$$L_{A^\pm} u = -\text{div}(A^\pm(\cdot)\nabla u(\cdot)) = -\text{div}(A^\pm \nabla u)$$

where $(\cdot)$ should be filled with the spatial variable. In contrast, letting

$$A(u, x) := \begin{cases} A^+(x) & u > 0 \\ A^-(x) & u \leq 0, \end{cases}$$

we define operator

$$L^A u := -\text{div}(A(u, \cdot)\nabla u(\cdot)) = -\text{div}(A(u)\nabla u). \tag{3.13}$$

We are now ready to state the main result concerning blow-ups of our two-phase problem.

Theorem 3.3. *Let $\Omega^\pm$, $\partial\Omega$, $A^\pm$, $L_{A^\pm}$, $u^\pm$ and $\omega^\pm$ be as in Assumption 3.1. Using the notation above, for any $p \in F_0$, there exists a subsequence (which we do not relabel) so that as $i \to \infty$ the following converge in Hausdorff distance uniformly on compact sets:*

$$\Omega_i^\pm \to \Omega_\infty^\pm \quad \textit{and} \quad \partial\Omega_i \to \partial\Omega_\infty, \tag{3.14}$$

where $\Omega^\pm_\infty$ are unbounded NTA domains with the same NTA constants as $\Omega^\pm$ and satisfy $\partial\Omega^+_\infty = \partial\Omega^-_\infty = \partial\Omega_\infty$. Moreover, there exist $u^\pm_\infty \in C(\mathbb{R}^n)$ (extended by zero) such that

$$u_i^\pm \to u_\infty^\pm \ \textit{uniformly on compact sets}, \tag{3.15}$$

and $u := u^+_\infty - u^-_\infty$ is a weak $W^{1,2}_{\text{loc}}(\mathbb{R}^n)$-solution to

$$L^{A_p} u = 0 \quad \textit{on } \mathbb{R}^n, \tag{3.16}$$

where $A_p(u) := A^+(p)\mathbf{1}_{\{u>0\}} + A^-(p)\mathbf{1}_{\{u\leq 0\}}$[5] *and L^{A_p} is as in (3.13).*

Furthermore, there exist Radon measures $\omega^\pm_\infty$ so that

$$\omega_i^\pm \overset{*}{\rightharpoonup} \omega_\infty^\pm, \tag{3.17}$$

[5]Note that $A^\pm(p)$ are well-defined since $p \in F_0$.

and $\omega_\infty^\pm$ *are the elliptic measures of* $\Omega_\infty^\pm$ *(with pole at infinity) corresponding to the operators* $L_{A^\pm(p)}$*. Moreover,* $\omega_\infty^+ = \omega_\infty^- =: \omega_\infty$*. That is, for all* $\varphi \in C_c^\infty(\mathbb{R}^n)$*,*

$$\int_{\Omega_\infty^\pm} A^\pm(p)\nabla u_\infty^\pm \cdot \nabla\varphi = \int_{\partial\Omega_\infty} \varphi d\omega_\infty. \tag{3.18}$$

Finally, we note that there exists $C > 0$ *depending on the NTA constants so that*

$$0 \in \operatorname{spt}\omega_\infty \quad \textit{and} \quad C^{-1} \le \omega_\infty(B_1) \le 1. \tag{3.19}$$

Notice that Theorem 3.3 is a more detailed re-statement of Theorem 1.8; in other words, we are concluding that w*-lim $\omega_i^\pm = \omega^\infty \in \mathcal{D}(A^+(p), A^-(p))$ and (3.18) holds.

Proof. Thanks to pre-compactness following from (3.6) and (3.7), the fact that there exists a subsequence and limiting objects so that (3.14), (3.15), and (3.17) all hold follows as in [KT03, Section 4]. The fact that $0 \in \operatorname{spt}\omega_\infty$ is a generic fact about tangent measures. That $\omega_\infty(B_1) \le 1$ follows from (3.3) and the characterization of weak-$*$ convergence in terms of open sets. The fact that $C^{-1} \le \omega_\infty(B_1)$ follows from the fact that the sequence ω_i is uniformly doubling (Lemma 2.11) with $\omega_i(B_1) = 1$ implying $\omega_i(\overline{B_{3/4}}) \ge C^{-1}$ for all i, and then using the characterization of weak-$*$ convergence in terms of compact sets.

Moreover, by [AM19, Lemma 3.11 (f) and (3-11)] (see also [KT03, Theorem 4.1]) and the $\omega^\pm$-approximate continuity of $A^\pm$ at p, we know that

$$\int_{\Omega_\infty^\pm} A^\pm(p)\nabla u_\infty^\pm \cdot \nabla\varphi = \int_{\partial\Omega_\infty} \varphi d\omega_\infty^\pm, \tag{3.20}$$

and

$$\begin{cases} L_{A^\pm(p)} u_\infty^\pm = 0 & \text{on } \Omega_\infty^\pm \\ u_\infty^\pm = 0 & \text{on } \mathbb{R}^n \setminus \Omega_\infty^\pm \\ u_\infty^\pm > 0 & \text{on } \Omega_\infty^\pm. \end{cases} \tag{3.21}$$

While similar statements are known in the literature, see in particular [KPT09, KT03, KT06, AM19], we prove that under the assumption that $p \in F_0$ both (3.16) and (3.18) hold for our pair of operators with a discontinuity in the coefficients across the boundary. Verifying (3.18) from (3.20) and (3.21) it suffices to prove that $\omega_\infty^+ = \omega_\infty^-$.

To confirm $\omega_\infty^+ = \omega_\infty^-$ we first observe that by duality, our rescaled measures (3.3) satisfy

$$\int_{\partial\Omega_i} \varphi\, d\omega_i^\pm = \frac{1}{\omega^\pm(B(p,r_i))}\int_{\partial\Omega} \varphi\left(\frac{q-p}{r_i}\right) d\omega^\pm(q) \qquad \forall\varphi \in C_c^\infty(\mathbb{R}^n). \tag{3.22}$$

In particular,

$$\begin{aligned} \int_{\partial\Omega_i} \varphi\, d\omega_i^- &= \frac{1}{\omega^-(B(p,r_i))}\int_{F_1} \varphi\left(\frac{q-p}{r_i}\right) \frac{d\omega^-}{d\omega^+}(q) d\omega^+(q) \\ &\quad + \frac{1}{\omega^-(B(p,r_i))}\int_{\partial\Omega\setminus F_1} \varphi\left(\frac{q-p}{r_i}\right) d\omega^-(q) \\ &=: (I) + (II). \end{aligned}$$

We note that as $i \to \infty$ the size of (II) goes to zero. Indeed, combine the doubling of ω^- (3.7) with the compact support of φ and the fact that $p \in F_0$ requires $\Theta(\omega^-, p, \partial\Omega \setminus F_1) = 0$. Denoting $h = \frac{d\omega^-}{d\omega^+}$, we next write

$$\begin{aligned} (I) &= \frac{\omega^+(B(p,r_i))}{\omega^-(B(p,r_i))}\frac{1}{\omega^+(B(p,r_i))}\int_{F_1} \varphi\left(\frac{q-p}{r_i}\right) h(q) d\omega^+(q) \\ &= h(p)\frac{\omega^+(B(p,r_i))}{\omega^-(B(p,r_i))}\frac{1}{\omega^+(B(p,r_i))}\int_{F_1} \varphi\left(\frac{q-p}{r_i}\right) d\omega^+(q) \\ &+ \frac{\omega^+(B(p,r_i))}{\omega^-(B(p,r_i))}\frac{1}{\omega^+(B(p,r_i))}\int_{F_1} \varphi\left(\frac{q-p}{r_i}\right) (h(q)-h(p)) d\omega^+(q) \\ &=: (III) + (IV). \end{aligned}$$

Since p is a point of approximate continuity for h, (3.7) ensures that (IV) goes to zero as $i \to \infty$. Finally, we see that

$$\lim_{r\to 0} \frac{\omega^+(B(p,r_i))}{\omega^-(B(p,r_i))} = \frac{1}{h(p)}$$

so that, when combined with (3.17) and (3.22), (III) converges to $\int_{\partial\Omega^+_\infty} \varphi d\omega^+_\infty$. In summary, we have shown

$$\int_{\partial\Omega^-_\infty} \varphi d\omega^-_\infty = \lim_{i\to\infty} (II) + (III) + (IV) = \lim_{i\to\infty} (III) = \int_{\partial\Omega^+_\infty} \varphi d\omega^+_\infty \qquad \forall \varphi \in C^\infty_c(\mathbb{R}^n)$$

verifying that $\omega^+_\infty = \omega^-_\infty = \omega_\infty$ and consequently (3.18).

We now turn to (3.16). We deduce from Theorem 2.6 that $u \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^n)$ since it is a difference of two Green functions with pole at infinity. Thus, from (3.21), for $\varphi \in C^\infty_c(\mathbb{R}^n)$:

$$\begin{aligned}\int_{\mathbb{R}^n} \langle A_p(u)\nabla u, \nabla\varphi\rangle &= \int_{\Omega^+_\infty} \langle A^+(p)\nabla u, \nabla\varphi\rangle - \int_{\Omega^-_\infty} \langle A^-(p)\nabla u, \nabla\varphi\rangle \\ &= \int_{\partial\Omega_\infty} \varphi d\omega^+_\infty - \int_{\partial\Omega_\infty} \varphi d\omega^-_\infty = 0\end{aligned}$$

verifying (3.16). □

4. Free boundary interpretation of the blow-up domains

We are now in the position to study the free boundary problem (3.16). We begin by using ([AM19, Lemma 3.8, 3.9]), which we state in a simplified version that is sufficient for our needs:

Lemma 4.1 ([AM19, Lemma 3.8, Lemma 3.9])**.** *Let Ω, A, L_A and $\{\omega^x\}_{x\in\Omega}$ be as in Assumption 2.5. Let $p \in \partial\Omega$ and let $\Lambda(p) \in \mathbb{R}^{n\times n}$ be a Λ_0-elliptic matrix. Let $u = G(\cdot, x)$ be a Green function for L_A on Ω with pole at $x \in \Omega$, and let $T(x) = \Lambda(p)^{-1}(x - p)$. Then $u \circ T^{-1}, T[\omega^x]$, are respectively a Green function and elliptic measure with pole at $T(x)$ for $L_{\tilde{A}}$ on $T(\Omega)$, where $\tilde{A}(y) = \Lambda(p)^{-1}A(\Lambda(p)y + p)\Lambda(p)^{-1}$.*

We note that in the more general setting of [AM19], the above lemma contains a factor of $|\det(\Lambda(p))|$ in the definition of $\tilde{A}$. However, since this is a constant in our setting, it can be removed for aesthetic purposes. As an immediate corollary, we obtain the following.

Corollary 4.2. *Suppose that μ is a Radon measure on $\mathbb{R}^n$, and let Ω, A be as in Assumption 2.5. Then for every $p \in \partial\Omega$, the following holds. If $\mathrm{Tan}(\mu, p) \subset \mathcal{D}(A^+(p), A^-(p))$, $\Lambda(p) = \sqrt{A^+(p)}$ and $M_p = \Lambda(p)^{-1}A^-(p)\Lambda(p)^{-1}$ then $\mathrm{Tan}_\Lambda(\mu, p) \subset \mathcal{D}(\mathrm{id}, M_p)$.*

The corollary follows directly from the equivalence of (1) and (2) in Lemma 2.18 and Lemma 4.1. Therefore, by working with $\mathrm{Tan}_\Lambda(\omega^\pm, p)$ instead of $\mathrm{Tan}(\omega^\pm, p)$, we reduce to working under the following assumption throughout this section:

Assumption 4.3. Let $n \geq 1$ and $M \in \mathbb{R}^{n\times n}$ be a Λ_0-elliptic matrix. Let $u \in W^{1,2}_{\mathrm{loc}} \cap C(\mathbb{R}^n)$ be a weak solution to

$$L^M u = -\operatorname{div}(M(u)\nabla u) = 0 \qquad \text{on } \mathbb{R}^n\,, \tag{4.1}$$

where

$$M(u) := \mathrm{id}\,\mathbf{1}_{\{u>0\}} + M\mathbf{1}_{\{u\leq 0\}}\,.$$

The main result of this section can be loosely summarized as follows: if $n \geq 3$, $\omega \in \mathcal{D}(\mathrm{id}, M)$, and ω is sufficiently close to flat at infinity, then in fact $\omega \in \mathcal{F}$. This will follow by showing a Liouville-type theorem for the difference of Green's functions $u^\pm$ with poles at infinity for $(\Omega^\pm(u), \omega, L^M)$, which we will see form a viscosity solution of an appropriate two-phase free-boundary problem; see Corollary 4.20 for a precise statement of the result.

4.1. **Weak solutions are viscosity solutions.**

Definition 4.4. We say that a continuous function u is a viscosity sub-solution to our free boundary problem $L^M(\cdot) = 0$ if:

(1) $\Delta u \geq 0$ in $\Omega^+(u) = \{u > 0\}$ and $L_M u \geq 0$ in $\Omega^-(u) = \{u < 0\}$.
(2) If $B(y, \rho) \subset \Omega^+(u)$ and $x_0 \in \partial B(y, \rho) \cap \partial\Omega^+(u)$, then for the inward pointing normal ν of $\partial B(y, \rho)$ at x_0,
$$u(x) \geq \alpha(\nu \cdot (x - x_0))^+ - \beta(\nu(x - x_0))^- + o(|x|)$$
for some $\beta \geq 0$ and $\alpha \geq \sqrt{\sum_{i,j=1}^n \nu_i \nu_j M^{ij}}\beta$ where M^{ij} are the coefficients of M.

A super-solution is defined similarly, with all the inequalities above reversed and the ball $B(y, \rho) \subset \Omega^-(u)$.

A solution is any continuous function that is both a sub- and a super-solution.

We will frequently use the notion of *two-plane solutions*:

Definition 4.5. We refer to a function $P(x) = \alpha(\nu \cdot x)^+ - \beta(\nu \cdot x)^-$ for some choice of $\nu \in \mathbb{S}^{n-1}$, $\alpha, \beta \geq 0$ as a two-plane solution. Given M as in Assumption 4.3, if α, β satisfy the property (2) of Definition 4.4 for this choice of M, we refer to P as a two-plane sub-solution of $L^M(\cdot) = 0$. We analogously define two-plane super-solutions of $L^M(\cdot) = 0$, and we refer to P as a two-plane solution of $L^M(\cdot) = 0$ if it is both a two-plane sub- and super-solution. Note that, without specifying the coefficients for which P is a two-plane solution, we are assuming no relationship between α and β.

We begin with the following lemma.

Lemma 4.6. *Suppose that M and u are as in Assumption 4.3. If $B(y, \rho) \subset \Omega^-(u)$ or $B(y, \rho) \subset \Omega^+(u)$ with $x_0 \in \partial B(y, \rho) \cap \{u = 0\}$, then there exists a unique two-plane solution $P = P_{x_0}$ so that*
$$u(x) - P(x - x_0) = o(|x - x_0|). \tag{4.2}$$
In particular, the rescalings
$$u_r(\cdot) := \frac{u(r \cdot + x_0)}{r}$$
converge locally uniformly to P. Moreover, the following three properties hold:

(1) For all $N > 0$,
$$\frac{|(\Omega^\pm(u)\Delta\Omega^\pm(P)) \cap B(x_0, rN)|}{|B(x_0, rN)|} \xrightarrow{r \to 0} 0. \tag{4.3}$$
(2) $u_r \to P$ in $L^2_{\text{loc}}(\mathbb{R}^n)$ and $\nabla u_r \to \nabla P$ weakly in $L^2_{\text{loc}}(\mathbb{R}^n)$
(3) P satisfies $L^M P = 0$ in a weak sense.

Proof. (4.2) is the conclusion of [AM13, Lemma 3.6]. This immediately implies the rescalings u_r converge locally uniformly to P. To prove (4.3) we first suppose without loss of generality that $x_0 = 0$ and observe that $\Omega^\pm(P)$ are half spaces and hence invariant under scaling. Then for any fixed $N > 0$ we have
$$\frac{|(\Omega^\pm(u)\Delta\Omega^\pm(P)) \cap B_{rN}|}{|B_{rN}|} = \frac{|(\Omega^\pm(u_{rN})\Delta\Omega^\pm(P)) \cap B_1|}{|B_1|}$$
$$\leq \int_{B_1} |\mathbf{1}_{\Omega^+(u_{rN})} - \mathbf{1}_{\Omega^+(P)}| + |\mathbf{1}_{\Omega^-(u_{rN})} - \mathbf{1}_{\Omega^-(P)}|,$$
which is seen to go to zero by the dominated convergence theorem and the fact that $u_r \to P$ locally uniformly, so in particular pointwise, which in turn implies that $\Omega^\pm(u_r)$ converges pointwise to $\Omega^\pm(P)$.

To prove (2) we must first show that for any $N > 1$ and all r sufficiently small, $\|u_r\|_{W^{1,2}(B_N)} \leq C < \infty$ for some constant C. Indeed, this suffices since then for any sequence $\{u_{r_i}\}$ there exists a subsequence $\{u_{r_{i_j}}\}$ and a function $g \in W^{1,2}(B_N)$ (which may a priori depend on the subsequence) so that $u_{r_{i_j}} \xrightarrow{L^2} g$ and $\nabla u_{r_{i_j}} \xrightarrow{L^2} \nabla g$. However, by (4.2) we then deduce that $g = P$. Since N is arbitrary, this will prove (2).

To verify the $W^{1,2}(B_N)$ bound, we first use the Caccioppoli inequality with a function $\varphi \in W_0^{1,\infty}(B_{N+1})$ so that $\varphi \equiv 1$ on B_N and $\|\nabla\varphi\|_{L^\infty(B_{N+1})} \lesssim 1$ to deduce

$$\int_{B_N} |\nabla u_r|^2 \leq \int |\nabla u_r|^2 \varphi^2 \lesssim_{\Lambda_0} \int |\nabla \varphi|^2 u^2 \lesssim \int_{B_{N+1}} u_r^2.$$

By (4.2), it follows that for r small enough that on B_{N+1} we have $|u_r| \lesssim N+1$ with suppressed constants independent of N so that

$$\int_{B_N} |\nabla u_r|^2 \lesssim (N+1)^2 |B_{N+1}|.$$

In particular again using (4.2) implies that

$$\|u_r\|^2_{W^{1,2}(B_N)} = \|u_r\|^2_{L^2(B_N)} + \|\nabla u_r\|^2_{L^2(B_N)} \lesssim N^2 + (N+1)^2 < \infty$$

for all r sufficiently small, completing the proof of (2).

We next claim that (3) follows from (1) and (2). Indeed, for $\varphi \in W_0^{1,2}(B_N)$, (2) implies

$$\int \langle M(P)\nabla P, \nabla\varphi\rangle \overset{(2)}{=} \lim_{r\to 0} \int \langle M(P)\nabla u_r, \nabla\varphi\rangle\,. \tag{4.4}$$

On the other hand,

$$\begin{aligned}
\limsup_{r\to 0} &\left| \int \langle M(P)\nabla u_r, \nabla\varphi\rangle - \int \langle M(u_r)\nabla u_r, \nabla\varphi\rangle \right| \\
&\leq \limsup_{r\to 0} \int_{(\Omega^+(u_r)\Delta\Omega^+(P))\cap B_N} |M(u_r) - M(P)||\nabla u_r||\nabla\varphi| \\
&\lesssim_{\Lambda_0} \limsup_{r\to 0} \left(\int_{(\Omega^+(u_r)\Delta\Omega^+(P))\cap B_N} |\nabla\varphi|^2 \right)^{1/2} \|\nabla u_r\|_{L^2(B_N)} = 0
\end{aligned}$$

where the final equality uses (1), (2), Cauchy-Schwarz, and the fact that $\nabla\varphi \in L^2$, so that in particular $|\nabla\varphi|^2\,dx$ defines a measure which is mutually absolute continuous with respect to the Lebesgue measure. Combining this with (4.4) verifies (3). □

We note that uniqueness of the two-plane solution from [AM13, Lemma 3.6] is guaranteed because the touching ball prevents the zero-set of the blow-up from rotating. Indeed, in general the uniqueness may fail, see [AK20].

Lemma 4.7. *Let M and u be as in Assumption 4.3. Then u satisfies $L^M u = 0$ in a viscosity sense.*

In [AM13] it is claimed without proof that (4.2) implies weak solutions are viscosity solutions. While the fact that the first condition in Definition 4.4 is satisfied is classical, we fill in the details to prove the second condition in Definition 4.4 also follows from Lemma 4.6 as claimed.

Proof of Lemma 4.7. By (4.2) from Lemma 4.6, we know that if there exists a touching ball $B(y,\rho)$ contained in either Ω^+ or Ω^- and $\partial B(y,\rho) \cap \{u = 0\} \ni x_0$ then if ν is the inward pointing normal direction to $B(y,\rho)$ at x_0, there exists α, β so that

$$u(x) = \alpha((x-x_0)\cdot\nu)^+ + \beta((x-x_0)\cdot\nu)^- + o(|x-x_0|) =: P(x-x_0) + o(|x-x_0|).$$

Without loss of generality, suppose $x_0 = 0$. It remains to check that the relationship between α and β. Let $v_1(x) = \alpha(x\cdot\nu)^+$ and $v_2(x) = \beta(x\cdot M\nu)^-$ so that $M(P)\nabla P = \mathbf{1}_{P>0}\nabla v_1 + \mathbf{1}_{P\leq 0}\nabla v_2$. Fix $0 \leq \varphi \in W_0^{1,2}(B_r)$. Then, because P satisfies $L^M P = 0$ in a weak sense, by Lemma 4.6 (3) we have

$$\begin{aligned}
0 &\leq \int_{B_r} M(P)\nabla P \cdot \nabla\varphi\,dx \\
&= \int_{B_r\cap\Omega^+(P)} \nabla v_1 \cdot \nabla\varphi\,dx + \int_{B_r\cap\Omega^-(P)} \nabla v_2 \cdot \nabla\varphi\,dx
\end{aligned}$$

$$
\begin{aligned}
&= -\int_{B_r\cap\Omega^+(P)} \varphi\Delta v_1\,dx - \int_{B_r\cap\Omega^-(P)} \varphi\Delta v_2\,dx \\
&+ \int_{\partial(B_r\cap\Omega^+(P))} \varphi\partial_\nu v_1 d\mathcal{H}^{n-1} + \int_{\partial(B_r\cap\Omega^-(P))} \varphi\partial_\nu v_2 d\mathcal{H}^{n-1} \\
&= \int_{\{P=0\}\cap B_r} \varphi\,(\alpha\nu - M\beta\nu)\cdot\nu d\mathcal{H}^{n-1}
\end{aligned}
$$

Where the final equality uses $\varphi|_{\partial B_r} = 0$ (in the Sobolev trace sense) and $\Delta v_1 = \Delta v_2 = 0$. Since $\varphi \geq 0$ was arbitrary in $W_0^{1,2}(B_r)$, this in turn implies $\alpha = \alpha\nu^t\nu \geq \beta\nu^t M\nu$ confirming the desired inequality. Repeating the argument with a touching ball from the other side, and using that P is a weak super-solution of $L^M(\cdot) = 0$ confirms $\alpha \leq \beta\nu^t M\nu$ verifying that P is a solution to $L^M(\cdot) = 0$ in the viscosity sense. □

4.2. **ε-monotonicity for blow-ups.** In this section we will work under the following assumption, which we know is satisfied by the Λ-tangents in F^* thanks to Theorem 3.3, Corollary 4.2, and Lemma 4.7.

Assumption 4.8. Assume $n \geq 3$ and that $\omega \in \mathcal{D}(\mathrm{id}, M)$ for some Λ_0-elliptic matrix M. In particular, $0 \in \operatorname{spt}\omega$ and $L^M u = 0$ on $\mathbb{R}^n$ in a viscosity sense, where $u = u^+ - u^-$, and $u^\pm$ are Green's functions with poles at infinity for $(\Omega^\pm(u), \omega, L^M)$.

We begin by showing that closeness to flat in the sense of measures $\omega \in \mathcal{D}(\mathrm{id}, M)$ guarantees closeness to two-plane solutions for the associated difference of Green's functions.

Lemma 4.9. *Let $\Lambda_0, r_0 > 0$ be fixed. For any $\eta > 0$ there exists $\delta = \delta(n, \Lambda_0, \eta, r_0) > 0$ such that the following holds. Suppose M, ω and u are as in Assumption 4.8. If for all $r \geq r_0$,*

$$d_r(\omega, \mathcal{F}) \leq \delta$$

then

$$\inf_{P \text{ two-plane solutions}} \|u - P\|_{L^\infty(B_1)} \leq \eta\,\omega(B_1)\,,$$

where we are taking two-plane solutions P of $L^M(\cdot) = 0$.

Proof. Suppose that the conclusion of the lemma fails. Then, there exists a sequence of measures $\widetilde{\omega}_k \in \mathcal{D}(\mathrm{id}, M)$, which we immediately replace with the measures $\omega_k = \widetilde{\omega_k}(B_1))^{-1}\widetilde{\omega_k}$, so that

$$d_r(\omega_k, \mathcal{F}) \leq 2^{-k} \qquad \forall r \geq r_0\,, \tag{4.5}$$

but

$$\inf_{P \text{ two plane solutions}} \|u_k - P\|_{L^\infty(B_1)} \geq \eta_0 \tag{4.6}$$

where $u_k^\pm$ are the Green's functions with pole at infinity guaranteed in Definition 1.7. In particular, (4.6) implies that for every two plane solution P there exists a sequence of points $x_{P,k}$ such that $u_k(x_{P,k}) \geq \eta_0$.

Since $\omega_k(B_1) = 1$ for each k, Lemma 2.11 in turn implies

$$\omega_k(B_{2^N}) \lesssim C^N \tag{4.7}$$

where C is the doubling constant from Lemma 2.11 and the suppressed constants are independent of k. Thus, there exists a convergent subsequence (not relabeled) with $\omega_k \overset{*}{\rightharpoonup} \omega_\infty$. Since $d_r(\cdot, \mathcal{F})$ is continuous with respect to weak-$*$ convergence, it follows from (4.5) that $d_r(\omega_\infty, \mathcal{F}) = 0$ for all $r \geq r_0$. This guarantees that $\omega_\infty \in \mathcal{F}$. On the other hand, since (4.7) holds for all $k \in \mathbb{N}$, Lemma 2.10 implies $|u_k^\pm(x_{0,N})| \lesssim_N 1$ for all k. By Harnack chains this implies $\|u_k^\pm\|_{L^2(B(0,N))} \lesssim_N 1$ for all k and all N. In particular, De Giorgi-Nash-Moser implies $\{u_k\}$ are locally uniformly equicontinuous and we assumed $u_k(0) = 0$. So, Arzela-Ascoli's theorem produces some function u_∞ that is the locally uniform limit of u_k, up to another subsequence that we do not relabel.

As in Theorem 3.3, the relationship between u_k and ω_k passes to limit, that is we also know

$$\int \varphi\,d\omega_\infty = \int \langle M(u_\infty)\nabla u_\infty, \nabla\varphi\rangle \qquad \forall \varphi \in C_c^\infty(\mathbb{R}^n)$$

and in particular that $L^M u_\infty = 0$. Writing $u_\infty = u_\infty^+ - u_\infty^-$. Since $\omega_\infty \in \mathcal{F}$ we have that u_∞^+ is a positive harmonic function on a half-space, vanishing at the boundary. This implies $u_\infty^+ = \alpha(x \cdot \nu)^+$, where ν is the inward pointing normal to $\Omega^+(u)$ (see e.g. [BB88]). Similarly, one confirms that u_∞^- is of the form $\beta(x \cdot \nu)^-$ by a rigid change of variables. In particular, there is a two-plane solution P so that $u_\infty = P$. But since u_k converges to u_∞ uniformly on B_1, this contradicts (4.6). □

Recalling that we aim to use the techniques of [Caf87, Caf89, Caf88] (see also [AM13, Fel97]), we are now in a position to derive ε-monotonicity of solutions to our limiting free boundary problem.

Definition 4.10 (ε-monotonicity)**.** Let $\varepsilon > 0$ and $U \subset \mathbb{R}^n$ open. We say that $u \in C(U)$ is ε-monotone in direction $v \in \mathbb{S}^{n-1}$ on U if for every $\tilde{\varepsilon} \geq \varepsilon$ we have $u(x - \tilde{\varepsilon} v) \leq u(x)$ for every $x \in U$.

We say that u is ε-monotone in the cone $\Gamma(\theta, e) := \{v \in \mathbb{S}^{n-1} : v \cdot e \geq \cos\theta\}$ if u is ε-monotone in direction v for every $v \in \Gamma(\theta, e)$.

Lemma 4.11. *Let $\Lambda_0, r_0 > 0$ and let ω, u, M be as in Assumption 4.8. For every $\varepsilon \in (0,1)$ and $0 < \eta < \frac{\varepsilon}{4C}$ for a constant C depending only on the NTA constants the following holds:*

There exists $0 < \delta_0 = \delta_0(n, \Lambda_0, C_2, R_2, \eta)$ given by the conclusion of Lemma 4.9 such that if $d_r(\omega, \mathcal{F}) \leq \delta_0$ for every $r \geq r_0$, then there is a direction ν so that u is ε-monotone in $B_{1/2}$ in the cone $\Gamma(\theta, \nu)$ for any θ so that $\cos(\theta) \geq \frac{4\eta C}{\varepsilon}$.

Remark 4.12. In order to derive ε-monotonicity, we require uniform nondegeneracy of u. Notice that in [AM13, Lemma 6.3], this comes from considering a rescaling of u for a sufficiently small scale around a point in $p \in \{u = 0\}$ where the notion of "sufficiently small" depends on p. Because we are striving for uniform global estimates on $\{u = 0\}$, we require additional assumptions like Assumption 4.8 or Assumption 4.3 plus NTA.

Proof. Fix $\varepsilon \in (0,1)$ and $0 < \eta < \frac{\varepsilon}{4C}$. By Lemma 4.9, there exists $\delta_0 = \delta_0(n, \Lambda_0, \eta, C_2, R_2) > 0$ such that if $d_r(\omega, \mathcal{F}) \leq \delta_0$ for all $r \geq 1$ then there exists a two-plane solution $P = \alpha(x \cdot \nu)^+ - \beta(x \cdot \nu)^-$ so that

$$\|u - P\|_{L^\infty(B_1)} \leq \eta\, \omega(B_1). \tag{4.8}$$

We claim this implies u is ε-monotone in $\Gamma(\theta, \nu)$ for θ such that $\cos(\theta) \geq \frac{4\eta C}{\varepsilon}$. Note that such θ exists since $\eta < \frac{\varepsilon}{4C}$.

Indeed, if $x_{0,1}^{\pm}$ denote corkscrew points for $\Delta(0,1) \subset \partial\Omega(u)$, then Lemma 2.10 implies that $u^\pm(x_{0,1}^\pm) \geq C^{-1}\omega(B_1)$ for some constant $C > 0$ depending only on n and the NTA constants C_2, R_2. Since in particular $\eta < C^{-1}/2$, we deduce from (4.8), the fact that $x_{0,1}^\pm \in B_1$, and the fact that P is a two plane solution with $\{P = 0\} = \nu^\perp$ that

$$|P(\pm\nu)| \geq \left|P\big(x_{0,1}^\pm\big)\right| \geq \left|u^\pm\big(x_{0,1}^\pm\big)\right| - \left|u^\pm\big(x_{0,1}^\pm\big) - P\big(\pm x_{0,1}^\pm\big)\right| \geq C^{-1}\omega(B_1)/2 =: m_0.$$

In particular, since P is a two-plane solution this implies that

$$P(x + t\nu) \geq m_0 t + P(x) \qquad \forall x \in \mathbb{R}^n \quad \forall t > 0. \tag{4.9}$$

Combining (4.8) with (4.9), it follows if $x \in B_{1/2}$,

$$u(x) \leq P(x) + \eta\omega(B_1) \leq P(x + t\nu) - m_0 t + \eta\omega(B_1) \leq u(x + t\nu) - m_0 t + 2\eta\omega(B_1).$$

so that $u(x + t\nu) \geq u(x)$ if $t \geq \frac{2\eta\omega(B_1)}{m_0} = 4\eta C$. Since $\varepsilon \geq 4\eta C$, this demonstrates that u is ε-monotone in direction ν. Because P is a two-plane solution, one could re-write (4.9) as saying that

$$P(x + te) \geq m_0 \langle e, \nu \rangle t + P(x)\,,$$

so an analogous computation verifies ε-monotonicity for $\varepsilon \geq \frac{4\eta}{C\cos(\theta)}$ in $\Gamma(\theta, \nu)$. Recalling the definition of m_0, the proof is complete. □

4.3. **ε-monotonicity implies Lipschitz boundary.** Armed with Lemma 4.11, we are now in a position to follow the arguments of [Caf89, Wan02, AM13] to obtain (in Lemma 4.14) local Lipschitz regularity of our free boundary, which will in turn imply (in Theorem 4.19) local $C^{1,\alpha}$ regularity, see [Caf87, Fel97]. This will then allow us to obtain a global Liouville-type result (see Corollary 4.20). The reader may observe that although the results in [Caf89] are stated for a rather specific two-phase free boundary problem where the operator on each side is the Laplacian, with a transmission condition across the boundary, the results hold for more general free-boundary problems, in particular the one of the setting herein, provided that

(i) One is considering viscosity solutions to a two-phase free-boundary problem with a transmission condition at the boundary that satisfies a suitable monotonicity and regularity condition; see e.g. [Caf89, Theorem 1].
(ii) The operator for the interior constraint in the free-boundary problem satisfies a suitable comparison principle, including at the boundary (boundary Harnack). Note that one does not require the operator to be linear in general, see e.g. [Wan02].
(iii) The solution to our free boundary problem is ε-monotone in some initial cone of directions.

We refer the reader to [CS05, Chapters 4 & 5] for a good presentation of the ideas in such arguments.

Remark 4.13. Although the more recent and flexible methods of De Silva [DS11] are now commonly used in place of Caffarelli's original techniques to establish local $C^{1,\alpha}$ regularity for sufficiently flat free boundaries, it seems that the methods of Caffarelli are more well-suited to the kind of global rigidity statement we are seeking herein, due to the fact that the argument passses through an intermediate Lipschitz bound.

The main result of this section is the following, which is the conclusion of [AM13, Proposition 6.11, Theorem 6.12] (see also [CS05, Lemma 5.7]).

Theorem 4.14. *Suppose that u and M satisfy Assumption 4.3 with $u(0) = 0$, let $\frac{\pi}{4} < \theta_0 \leq \theta \leq \frac{\pi}{2}$, and let $\nu \in \mathbb{S}^{n-1}$. There exist $\varepsilon_1 = \varepsilon_1(n, \Lambda_0, \theta_0) > 0$, $\lambda = \lambda(n, \Lambda_0, \theta_0) \in (0, 1)$, and $c_0 = c_0(n, \Lambda_0, \theta_0) > 0$ then for any for $\varepsilon \in (0, \varepsilon_1)$ if u is ε-monotone in $B_{1/2}$ in the cone $\Gamma(\theta, \nu)$, then u is $\lambda\varepsilon$-monotone in the cone $\Gamma(\theta - c_0\varepsilon^{1/4}, \nu)$.*

In particular, if u, ω, M satisfy Assumption 4.8 and $r_0 > 0$ is fixed, there exists $\delta_1 = \delta_1(n, \Lambda_0, C_2, R_2) > 0$ such that if $d_r(\omega, \mathcal{F}) \leq \delta_1$ for every $r \geq r_0$, then u is fully monotone in $\mathcal{C}_{1/4} := (B_{1/4} \cap \nu^\perp) \times (-\frac{1}{4}, \frac{1}{4})$ in the cone $\Gamma(\theta_1, \nu)$ for some $\theta_1(\theta_0, \varepsilon)$. Moreover, $\{u = 0\} \cap \mathcal{C}_{1/4}$ is a Lipschitz graph with Lipschitz constant $\kappa = \kappa(n, \Lambda_0, C_2, R_2) > 0$, over $\nu^\perp$. Here, $\nu^\perp$ denotes the $(n-1)$-dimensional linear subspace of $\mathbb{R}^n$ orthogonal to ν.

We will not provide all of the details of the proof of Theorem 4.14, since they are contained in [AM13, Section 6]. Nevertheless, for the purpose of clarity, we will provide a breakdown of the key intermediate results leading towards its conclusion, and emphasize the parts for which the arguments differ to their classical counterparts in [Caf89].

The starting point is the following lemma, which allows one to improve ε-monotonicity to full monotonicity for a barrier solution associated to u in balls with radius given by a function, in a cone whose angle is determined by the gradient of this function.

Lemma 4.15. [Caf89, Lemma 2] *Let $\varphi \in C^2(B_{1/2}; (0, \rho_0])$ with $\rho_0 < \frac{1}{2}$ and suppose that $u \in C(B_{1/2})$ is ε-monotone in a cone $\Gamma(\theta, \nu)$. For $s \leq \frac{1}{2} - \|\varphi\|_{L^\infty}$, consider the function v on B_s given by*

$$v(x) := \sup_{B(x, \varphi(x))} u \,. \tag{4.10}$$

Assume in addition that $\tilde{\theta}$ satisfies

$$\sin \tilde{\theta} \leq \frac{1}{1 + |\nabla \varphi|} \left(\sin \theta - \frac{\varepsilon}{2\varphi} \cos^2 \theta - |\nabla \varphi| \right) .$$

Then v is monotone in the cone $\Gamma(\tilde{\theta}, \nu)$. In particular, in this case the level sets of v are Lipschitz graphs with Lipschitz constant $\bar{L} \leq \cot \tilde{\theta}$ over hyperplanes orthogonal to ν in $\mathbb{R}^n$.

Note that Lemma 4.15 does not require any assumptions on u other than ε-monotonicity.

An additional key observation about the function v in (4.10), as demonstrated in [Fel97], is that it is a subsolution of our free boundary problem away from $\{v = 0\}$.

Lemma 4.16. [Fel97, Lemma 7] *Let u, M, and L^M be as in Assumption 4.3. There exists $C = C_{4.16}(n, \Lambda_0) > 0$ large enough such that the following holds. Suppose that $\varphi \in C^2(B_{1/2}; (0, (0, \frac{1}{2}))$ satisfies*

$$\varphi L^M \varphi \geq C|\nabla\varphi|^2 .$$

Then v as in (4.10) for the function u satisfies $L^M v \leq 0$ in $B_{1/2-\|\varphi\|_{L^\infty}} \setminus \{v = 0\}$.

Note that although in [Fel97], v is defined as a supremum over $\partial B(x, \varphi(x))$ in place of $B(x, \varphi(x))$, in Lemma 4.16 this is equivalent to our definition in light of the maximum principle. The next step is to establish existence of a preliminary 1-parameter family of functions $\{\varphi_t\}_t$ that will satisfy suitable estimates in order for us to use them in Lemma 4.15. This will provide us with something that will be almost a sub-solution to our free-boundary problem.

Lemma 4.17. *Let $\pi \subset \mathbb{R}^n$ be a hyperplane. Let G be the graph $\{(x', f(x') : x' \in \pi \cap B_{1/2})$ of a Lipschitz function f with $f(0) = 0$ and Lipschitz constant $\bar{L}$, and let $\mathcal{C} = (\pi \cap B_{1/2}) \times [-2\bar{L}, 2\bar{L}]$. Then there exists a constant $C = C_{4.17}(n, \Lambda_0) \geq C_{4.16}$ such that for every $\delta \in (0, \frac{1}{2})$, there exists a family $\{\varphi_t\}_{t\in[0,1]} \subset C^2(B_{1/2})$ satisfying*

(i) $1 \leq \varphi_t \leq 1 + t$;
(ii) $\varphi_t L^M \varphi_t \geq C|\nabla\varphi_t|^2$;
(iii) $\varphi_t \simeq 1$ *on* $A_\delta := \{\mathrm{dist}(\cdot, A \cap \partial\mathcal{C}) < \delta)\}$;
(iv) $\varphi_t(x) \geq 1 + t\left[1 - C\delta\,\mathrm{dist}(x, \partial\mathcal{C})^{-2}\right]$ *on* $\{\mathrm{dist}(\cdot, \partial\mathcal{C}) > \delta\}$;
(v) $|\nabla\varphi_t| \leq \frac{Ct}{\delta}$.

The proof of Lemma 4.17 can be found in [Wan02, Lemma 3], combined with the observation that $L^M\varphi_t \geq \mathcal{M}^-(D^2\varphi_t, \Lambda_0^{-1}, \Lambda_0)$, where $\mathcal{M}^-(D^2\varphi_t, \Lambda_0^{-1}, \Lambda_0)$ denotes the Pucci minimal operator as defined in e.g. [CC95, Section 2.2].

For u as in Lemma 4.15 and $\{\varphi_t\}$ as in Lemma 4.17, the function

$$v_t(x) := \sup_{B(x,\sigma\varphi_t(x))} u\,, \tag{4.11}$$

with $\frac{\varepsilon}{2} < \sigma < 2\varepsilon$ is well-defined in $\mathcal{C}_{1/2-4\varepsilon} := (\nu^\perp \cap B_{1/2-4\varepsilon}) \times (-1 - 8\varepsilon, 1 + 8\varepsilon)$. Moreover, Lemma 4.16 guarantees that v is a subsolution of $L^M(\cdot) = 0$ away from $\{v = 0\}$. It remains to correct v_t to ensure that it is a subsolution at its free boundary. This is done via a small perturbation by a solution of $L^M(\cdot) = 0$, which, if done in $\Omega^+(v_t)$, may be chosen to be a harmonic function due to our definition of M.

First, however, recall that if u is ε-monotone in $B_{1/2}$ in the direction $e \in \Gamma(\theta_0, \nu)$ for $\theta_0 > \frac{\pi}{4}$, then there exists a Lipschitz graph G with Lipschitz constant $\bar{L} < 1$ over some hyperplane $\pi \subset \mathbb{R}^n$ such that $\{u = 0\}$ is contained in the neighborhood $N_\varepsilon(G) := \{\mathrm{dist}(\cdot, G) < \varepsilon\}$, see for instance [CS05, Proposition 11.14]. Thus, Lemma 4.17 applies. Furthermore, u is fully monotone in $B_{1/2} \setminus N_{\kappa\varepsilon}(G)$ for some $\kappa > 0$ (independent of ε).

Lemma 4.18. *Suppose that u and M are as in Assumption 4.8 and suppose that u is ε-monotone in $B_{1/2}$ in a cone $\Gamma(\theta_0, \nu)$ with $\theta_0 \geq \frac{\pi}{4}$. Consider*

(a) *the family of functions $\{\varphi_t\}$ as in Lemma 4.17 and the corresponding v_t from (4.11);*
(b) *a harmonic function w_t with boundary data*

$$\begin{cases} u & \text{on } \partial N_{C\kappa\varepsilon}(A) \cap \Omega^+(v_t) \\ 0 & \text{otherwise}\,, \end{cases}$$

for some large constant C, extended by zero;

(c) *the function $\bar{v}_t = v_t + \eta w_t$ for $\eta > 0$, defined in $\mathcal{C}_{1/2-4\varepsilon}$.*

Then, for δ as in Lemma 4.17 and σ as above, if $\eta \geq \frac{C\sigma}{\delta}$ and $\frac{\sigma}{\delta}$ is sufficiently small, $\bar{v}_t$ is a viscosity sub-solution to $L^M(\cdot) = 0$ in $\mathcal{C}_{1/2-4\varepsilon}$.

See [AM13, Lemma 6.9], [CS05, Lemma 5.5] for the proof of Lemma 4.18. Note that we take w_t to be harmonic due to the fact that we are defining it in $\Omega^+(v_t)$, where the coefficient M is the identity for our operator L^M. Combining these Lemmas, one may follow the reasoning of [CS05, Lemma 5.7] and [AM13, Proposition 6.11] in order to conclude the validity of Theorem 4.14. Note that in this argument, a key step is the boundary Harnack principle for v_t and w_t. Furthermore, one requires the knowledge that for a viscosity solution u and a viscosity subsolution $\bar{v}_t$ of our free boundary problem (as in Lemma 4.18), $\{u = 0\}$ and $\{\bar{v}_t = 0\}$ cannot touch. Indeed this is a consequence of [Fel97, Lemma 6] and [AM13, Lemma 6.7] (see also [CS05, Lemma 4.9]).

4.4. **Global Lipschitz boundaries are flat.** With Theorem 4.14 at hand, we may now combine with the results of [Fel97] and [AM13, Section 7], which rely on the techniques of [Caf87]. The strategy is similar to that from Section 4.3. Namely, one must construct a family of subsolutions, that, in this setting, provides an improvement of monotonicity from the initial cone given by the conclusion of Theorem 4.14, to a cone with larger angle, which amounts to improving the Lipschitz constant of the graph of $\{u = 0\}$, in a slightly smaller cylinder in the domain.

We omit the details here, and simply refer the reader to the aforementioned references, providing merely the final conclusion in the form that we wish to use globally.

Theorem 4.19. *Suppose that u and M satisfy Assumption 4.3 and that $\{u = 0\} \cap \mathcal{C}_{1/4}$ is the graph of a Lipschitz function with Lipschitz constant $\kappa > 0$. Then $\{u = 0\} \cap \mathcal{C}_{1/8}$ is $C^{1,\alpha}$ for some $\alpha(n, \kappa, \Lambda_0) > 0$.*

We are now in a position to conclude the key rigidity theorem for globally flat free boundaries (cf. [DSFS14, Lemma 6.2]).

Corollary 4.20. *Suppose that u, ω, M satisfy Assumption 4.8 and let $r_0 > 0$ be fixed. For $\delta_1 = \delta_1(n, \Lambda_0, C_2, R_2) > 0$ given by Theorem 4.14, if $d_r(\omega, \mathcal{F}) \leq \delta_1$ for every $r \geq r_0$, then u is a two-plane solution of $L^M u = 0$. In particular $\omega \in \mathcal{F}$ and $\{u = 0\} \in G(n-1, n)$.*

Remark 4.21. In light of Theorem 4.14 and Theorem 4.19, as in other two-phase free boundary problems, it is natural to ask about the existence of singularities of $\{u = 0\}$, the structure of said singularities, and the behavior of u at such points. In the classical case of $M = \mathrm{id}$, the monotonicity formula of Alt-Caffarelli-Friedman may be used to detect flat portions of the free boundary, characterizing the blow-ups of u as two-plane solutions at such points. Thus singularities are contained in the set where the ACF density is vanishing, thus characterizing the order of vanishing of u as being superlinear at singular points. An analogue of the ACF monotonicity formula was established in the multi-operator setting by Terracini-Soave [ST23], but this monotonicity formula detects an order of vanishing that is necessarily *sublinear* in the case where the coefficients of the operators have a discontinuity at the boundary. This is supplemented by the existence of an example in $\mathbb{R}^3$ of a pair of complementary solutions to the multi-operator elliptic problem that have the same sublinear decay to zero at the free boundary. However, in that example the zero set has positive Lebesgue measure. It therefore remains an open question whether such an example can exist under Assumption 4.8. Due to Theorems 4.14 and 4.19, such an example, if it exists, must have a zero set that is sufficiently far from flat.

Proof of Corollary 4.20. By a combination of Theorem 4.14 and Theorem 4.19, for δ_1 as in Theorem 4.19, up to a rotation of coordinates, we deduce that

$$\{u = 0\} \cap \mathcal{C}_{1/4} = \{(x', x_n) : x_n = g(x')\},$$

where g is Lipschitz with constant $\kappa > 0$, and thus $\{u = 0\} \cap \mathcal{C}_{1/8}$ is $C^{1,\alpha}$ for $\alpha > 0$ depending only on n, κ and Λ_0. The latter in particular yields the estimate

$$|g(x') - g(0) - \nabla g(0) \cdot x'| \leq C|x'|^{1+\alpha} \qquad \forall x' \in B_{1/8} \cap e_n^\perp, \tag{4.12}$$

where $C = C(n, \kappa, \Lambda_0) > 0$ and e_n is a unit normal to the hyperplane $\{x_n = 0\}$. On the other hand, observe that for any $R > 0$, the rescaling $g_R(x') := R^{-1} g(Rx')$ still satisfies the hypotheses of Theorem 4.19. The estimate 4.12 for g_R becomes

$$|g(y') - g(0) - \nabla g(0) \cdot y'| \leq CR^{-\alpha}|y'|^{1+\alpha} \qquad \forall x' \in B_{R/8} \cap e_n^{\perp}\,. \tag{4.13}$$

Taking $R \to +\infty$ in (4.13) implies g is affine. This in turn implies u is a two-plane solution. Since u is a two-plane solution $\{u = 0\} =: \pi \in G(n-1, n)$. Since $\omega \in \mathcal{D}(\mathrm{id}, M)$, $\operatorname{spt}\omega = \{u = 0\} = \pi$. As a two-plane solution u is translation invariant for $x \in \pi$. This in turn ensures ω has constant density on π, proving $\omega \in \mathcal{F}$. ☐

5. Proof of Theorem 1.3

Recalling the strategy from Section 1.3, we wish to apply Lemma 2.20 to the dilation cone $\mathcal{D}(A_1, A_2)$. With this in mind, we first require the following lemma.

Proposition 5.1. *Let $A^{\pm} \in \mathbb{R}^{n\times n}$ be a pair of elliptic matrices. The dilation cone $\mathcal{D}(A^+, A^-)$ is compact.*

Proof. Suppose $\{\omega_k\}$ is a sequence contained in the basis of $\mathcal{D}(A^+, A^-)$, see Definition 2.14. Then by definition of the basis of a d-cone, $F_1(\omega_k) = 1$ for all k. In particular, for all k:

$$\frac{1}{2}\omega_k(B_{1/2}) \leq \int_{B_1} (1 - |x|)d\omega_k = F_1(\omega_k) = 1.$$

So, by Lemma 2.11, $\{\omega_k\}$ is uniformly locally bounded and hence pre-compact with respect to the weak-$*$ topology. So, there exists some Radon measure ω and a subsequence (which we do not relabel) so that $\omega_k \overset{*}{\rightharpoonup} \omega$ and $F_1(\omega) = 1$. We need to show that $\omega \in \mathcal{D}(A^+, A^-)$.

Moreover, for each k, there is are Green's functions $u_k^{\pm}$ with pole at infinity associated to $(\Omega_k^{\pm}, \omega_k, L_{A^{\pm}})$ and $\Omega_k^{\pm}$ are NTA with the same constants, independently of k. By Theorem 3.3, we know that there exist NTA domains $\Omega^{\pm}$ with the same NTA constants and a subsequence which we do not relabel so that $\Omega_k^{\pm} \to \Omega$ and $\partial\Omega_k \to \partial\Omega$ locally in Hausdorff distance. Moreover, $\operatorname{spt}\omega = \partial\Omega$.

Now, almost exactly as in the proof of Lemma 4.6, we can show that the sequence of functions $\{u_k\}$ for $u_k = u_k^+ - u_k^-$ are uniformly bounded in $W^{1,2}(B_N)$ for all $N \in \mathbb{N}$. The only difference to that proof, is that we use the uniform local boundedness of ω_k and Lemma 2.10 to prove that $\|u_k\|_{L^\infty(B_N)}$ is bounded independently of k, instead of using the proximity to a fixed two-plane solution. This in turn bounds the L^2-norms uniformly and allows us to use the Caccioppoli inequality to deduce that the gradients are also uniformly bounded in L^2. In particular, there is a $W^{1,2}_{\mathrm{loc}}(\mathbb{R}^n)$ function u so that (up to a subsequence that we do not relabel) $u_k \to u$ in $L^2_{\mathrm{loc}}(\mathbb{R}^n)$ and $\nabla u_k \rightharpoonup \nabla u$ weakly in $L^2_{\mathrm{loc}}(\mathbb{R}^n)$. The weak convergence of the gradients combined with the local convergence in Hausdorff distance of the $\Omega_k^{\pm}$ and $\partial\Omega_k$ guarantees that u is a Green's function with pole at infinity for $(\Omega^{\pm}, \omega, L_{A^{\pm}})$. Indeed, for all $\varphi \in C_c^\infty(\mathbb{R}^n)$, writing $u = u^+ - u^-$:

$$\begin{aligned}\int_{\partial\Omega} \varphi\, d\omega &= \lim_{k\to\infty} \int_{\partial\Omega_k} \varphi\, d\omega_k = \lim_{k\to\infty} \int_{\Omega_k^{\pm}} \left\langle A^{\pm}\nabla u_k^{\pm}, \nabla\varphi\right\rangle dx \\ &= \int_{\Omega^{\pm}} \left\langle A^{\pm}\nabla u, \nabla\varphi\right\rangle dx.\end{aligned}$$

That is, $u^{\pm}$ are Green's functions with pole at infinity for $(\Omega^{\pm}, \omega, L_{A^{\pm}})$ and Ω is NTA with the same NTA constants, that is $\omega \in \mathcal{D}(A^+, A^-)$, confirming that $\mathcal{D}(A^+, A^-)$ is a compact dilation cone. ☐

We are now in a position to prove Theorem 1.3.

Proof of Theorem 1.3. Recall the sets $F_0, \ldots, F_4$ from (3.8) - (3.12). We prove the theorem holds when F^* is the subset of F_0 where the "tangents to tangents are tangents" property holds, i.e., the conclusion of Theorem 2.17 holds and with $S = F_2 \cup F_3$, and $N = F_4 \cup (F_1 \setminus F^*)$. In particular, by Remark 3.2 it follows that $\partial\Omega = F^* \sqcup S \sqcup N$.

Similarly, (ii) holds by Remark 3.2 and (iii) holds by combining Remark 3.2 with Theorem 2.17. It remains to check (i). The claim of mutual absolute continuity in (i) follows again by

Remark 3.2. To this end, we now show that if $p \in F^*$ then $\mathrm{Tan}_\Lambda(\omega^\pm, p) \subset \mathcal{F}$, which by Lemma 2.18 in turn implies $\mathrm{Tan}(\omega^\pm, p) \subset \mathcal{F}$.

For $p \in F^*$, Corollary 4.2 ensures

$$\mathrm{Tan}_\Lambda(\omega^\pm, p) \subset \mathcal{D}(\mathrm{id}, M_p)\,,$$

where $M_p = \Lambda(p)^{-1}A^-(p)\Lambda(p)^{-1}$ for $\Lambda(p) = \sqrt{A^+(p)}$. From Proposition 5.1, we know that $\mathcal{M} = \mathcal{D}(\mathrm{id}, M_p)$ is a compact dilation cone. Thus combining Corollary 4.20 and Lemma 2.20, we deduce that either $\mathrm{Tan}_\Lambda(\omega^\pm, p) \subset \mathcal{F}$ or $\mathrm{Tan}_\Lambda(\omega^\pm, p) \cap \mathcal{F} = \emptyset$, where $\mathcal{F}$ is as in (1.1).

It therefore remains to rule out the possibility that $\mathrm{Tan}_\Lambda(\omega^\pm, p) \cap \mathcal{F} = \emptyset$. Indeed, for any $\omega \in \mathcal{D}(\mathrm{id}, M_p)$ we know that there exists a ball $B \subset \Omega^+(u)$ with $\partial B \cap \partial\Omega = \{p_0\}$ at some point $p_0 \in \{u = 0\}$. Then, by Lemma 4.6, $\mathrm{Tan}(\omega, p_0) \subset \mathcal{F}$. On the other hand, since $p \in F^*$, Theorem 2.17 ensures that $\mathrm{Tan}(\omega, p_0) \subset \mathrm{Tan}_\Lambda(\omega^\pm, p)$, thus implying $\mathrm{Tan}_\Lambda(\omega^\pm, p) \cap \mathcal{F} \neq \emptyset$ as desired.

Finally, we show $\dim_{\mathcal{H}}(F^*) \le n - 1$. We wish to use Lemma 2.4. Recalling the definition of $\Theta_{\partial\Omega}$ and β_{F^*} from (2.1) and (2.2), we claim that $\lim_{r\to 0} \Theta_{\partial\Omega}(p, r) = 0$ for all $p \in F^*$. Indeed, suppose not. Then there exists $p \in F^*$, $\delta > 0$, and a sequence $r_i \to 0$ so that

$$\Theta_{\partial\Omega}(p, r_i) \ge \delta \qquad \forall i. \tag{5.1}$$

By Theorem 3.3 we know there exists a subsequence so that $\partial\Omega_i$ as in (3.1) and $\omega_i^\pm$ as in (3.3) converge simultaneously to some $\partial\Omega_\infty$ and ω_∞ respectively (in local Hausdorff distance and in the weak-$*$ topology respectively). Moreover, we know that $\operatorname{spt}\omega_\infty = \partial\Omega_\infty$. We already showed that since $p \in F^*$, it follows $\omega_\infty \in \mathcal{F}$ so that $\operatorname{spt}\omega_\infty = \partial\Omega_\infty \in G(n-1, n)$. But starting from (5.1) this implies

$$\delta \le \Theta_{\partial\Omega}(p, r_i) = \Theta_{\partial\Omega_i}(0, 1) \xrightarrow{i\to\infty} \Theta_{\partial\Omega_\infty}(0, 1) = 0,$$

yielding a contradiction. In particular, since $F^* \subset \partial\Omega$, it follows that for all $p \in F^*$ we have

$$0 \le \limsup_{r\to 0} \beta_{F^*}(p, r) \le \lim_{r\to 0} \Theta_{\partial\Omega}(p, r) = 0.$$

Now Lemma 2.4 implies $\dim_{\mathcal{H}}(F^*) \le n - 1$, verifying the final piece of (i). □

Appendix A. Preiss argument for sets

In Section 5, we used Preiss' tangent measures and a connectedness argument for the space of tangent measures to prove that if under one tangent measure at a point p in a full measure subset of F_0 is flat, then all tangent measures at p are flat. There, our notion of flatness was in terms of measures, which may be unnatural when discussing the free-boundary problem in other settings. In this appendix, we prove that this machinery can be extended to a set-theoretic approach, which we believe may be useful in other PDE contexts, particularly those where one does not have a monotonicity formula that allows one to analyze singularities.

More specifically, we define a set-theoretic notion of space of tangents for which the analogue the key connectedness lemma, Lemma 2.20, holds; see Theorem A.5.

The argument that close to flat in a set-theoretic sense implies close to a two plane solution, and in turn ε-monotonicity, and finally the disconnectedness of the dilation cone of tangent boundaries we do not repeat. This is because for the multi-operator problem studied here, these results follow by analogous reasoning to that in Section 4. But, in other contexts, certain hypotheses must be replaced with other suitable ones. Two such instances are the use of the estimates from Lemma 2.11 and Harnack chains for Lemma 4.9, and the use of Lemma 2.10 for Lemma 4.11. These must be replaced with appropriate hypotheses to guarantee suitably strong compactness for the contradiction sequence u_k in the proof of Lemma 4.9, as well as uniform nondegeneracy of u in Lemma 4.11. Such hypotheses are dependent on the nature of the PDE problem at hand, but in certain circumstances can indeed be guaranteed. In fact, in the latter case, one can also treat certain degenerate cases (cf. [CS05, Section 5.6]).

Another difficulty of the set theoretic approach is phrasing an analogue of "Tangents to Tangents are Tangents" (henceforth, TTT, cf. Theorem 2.17), because without an underlying measure, what does it mean for such a property to hold almost-everywhere? In the setting herein, one can use Theorem 2.17 to prove that TTT holds for set theoretic tangents almost everywhere with respect to the elliptic measures (and in particular is reliant upon our setting and not a generic fact like Theorem 2.17). This is precisely because you can show that TTT holds for sets whenever it holds for the elliptic measures; see Lemma A.6. On the other hand, in [AM13] the authors used the Laplacian measure Δu^+ as a suitable choice of underlying measure when showing TTT holds in their setting, almost everywhere. Thus, it seems that a set-theoretic version of TTT requires an underlying measure that is naturally associated to the set in question so that the weak-$*$ convergence will be compatible with the local Hausdorff convergence of the sets.

We begin by defining set theoretic space of tangents of a closed set F.

$$\mathrm{Tan}(F,p) = \{\text{non-empty and closed } \Sigma \subset \mathbb{R}^n : \exists r_i \downarrow 0 \; s.t. \; \Sigma = \lim_{i\to\infty} \frac{F-p}{r_i}\}$$

where the limit is taken with respect to the local Hausdorff convergence.

Proposition A.1. *Suppose F is a closed set and $\Sigma \in \mathrm{Tan}(F,p)$. Then the following hold:*

1. *If $\Sigma \in \mathrm{Tan}(F,p)$ then for all $\lambda > 0$, we know $\lambda\Sigma \in \mathrm{Tan}(F,p)$.*
2. *$0 \in \Sigma$.*

Definition A.2. We say that $\mathcal{S}$ is a dilation cone of closed sets if $\Sigma \in \mathcal{S}$ implies $\lambda\Sigma = \{\lambda x : x \in \Sigma\} \in \mathcal{S}$ for all $\lambda > 0$. We say that $\mathcal{S}$ is a closed dilation cone if it is closed in with respect to convergence in the local Hausdorff distance.

We note that Proposition A.1(1) says precisely that $\mathrm{Tan}(\partial\Omega,p)$ is a dilation cone and (2) is a consequence of (1). One can prove (1) because if $\Sigma = \lim_{i\to\infty} \frac{\partial\Omega - p}{r_i}$ then $\lambda\Sigma = \frac{\partial\Omega-p}{r_i/\lambda}$.

We define the following distance between between a closed set and a dilation cone at scale R:

$$D_R(K,\mathcal{S}) = \inf_{\Sigma\in\mathcal{S}} \tau_r(K,\Sigma)$$

where

$$\tau_r(K,\Sigma) = \frac{1}{r}\,\mathrm{dist}_{\mathcal{H}}[K\cap B_r; \Sigma\cap B_r]\,.$$

Remark A.3. We note that for $\lambda > 0$,

$$\begin{aligned} D_R(\lambda K,\mathcal{S}) &= \inf_{\Sigma\in\mathcal{S}} \tau_R(\lambda K,\lambda\Sigma) = \inf_{\Sigma\in\mathcal{S}} \frac{1}{R}\,\mathrm{dist}_{\mathcal{H}}[\lambda K\cap B_R; \lambda\Sigma\cap B_R] \\ &= \inf_{\Sigma\in\mathcal{S}} \frac{\lambda}{R}\,\mathrm{dist}_{\mathcal{H}}[K\cap B_{R/\lambda}; \Sigma\cap B_{R/\lambda}] \\ &= \inf_{\Sigma\in\mathcal{S}} \tau_{R/\lambda}(K,\Sigma) = D_{R/\lambda}(K,\mathcal{S}) \end{aligned}$$

In other words:

$$D_R(K,\mathcal{S}) = D_{R/\lambda}(\lambda^{-1}K,\mathcal{S}) \tag{A.1}$$

Definition A.4. We say that a closed dilation cone of sets $\mathcal{S}$ has property (P) with respect to a closed dilation cone $\mathcal{K} \subset \mathcal{S}$ if there exists some $r_0 > 0$ and $\varepsilon_0 > 0$ so that for any $\varepsilon \in (0,\varepsilon_0)$ there exists no $\Sigma \in \mathcal{S}$ with

$$D_r(\Sigma,\mathcal{K}) \le \varepsilon \qquad \forall r > r_0$$

and

$$D_{r_0}(\Sigma,\mathcal{K}) = \varepsilon.$$

Theorem A.5 (Connectedness of space of tangents)**.** *If $\mathrm{Tan}(\partial\Omega,p) \subset \mathcal{S}$ for some closed dilation cone $\mathcal{S}$ that has property (P) with respect to $\mathcal{K}$, then either:*

- *$\mathrm{Tan}(\partial\Omega,p) \subset \mathcal{F}$,*
- *or $\mathrm{Tan}(\partial\Omega,p) \cap \mathcal{F} = \emptyset$.*

Before we begin we remark that the proof of Theorem A.5 is strictly easier than the proof of Lemma 2.20 because of the fact that every dilation cone of sets is pre-compact with respect to local Hausdorff convergence. In the measure theoretic setting, the proof is more complicated because one doesn't immediately know that (the analogue of) $\{\Gamma(\rho_i)\}$ has a convergent subsequence in that setting.

Proof. Suppose that $\mathrm{Tan}(\partial\Omega, p) \subset \mathcal{M}$. We proceed by contradiction. That is, assume there exists $\Sigma_1 \in \mathcal{F} \cap \mathrm{Tan}(\partial\Omega, p)$ and $\Sigma_2 \in \mathrm{Tan}(\partial\Omega, p) \setminus \mathcal{F}$.

Then, there exists sequences $s_i \downarrow 0$ and $r_i \downarrow 0$ so that

$$\lim_{i\to\infty} \frac{\partial\Omega - p}{r_i} = \Sigma_1 \quad \text{and} \quad \lim_{i\to\infty} \frac{\partial\Omega - p}{s_i} = \Sigma_2.$$

For any $t > 0$, let $\Gamma(t) = \frac{\partial\Omega - p}{t}$.

Note that by continuity of $D_1(\cdot, \mathcal{M})$ with respect to the local weak Hausdorff convergence which I'm only assuming holds true for now, but seems quite obvious and since $\mathcal{F}$ is closed, there exists $\varepsilon_1 \in (0, \frac{1}{2}\min\{\varepsilon_0, 1\})$, where ε_0 comes from Property (P) so that:

$$\lim_{i\to\infty} D_1(\Gamma(r_i), \mathcal{F}) = D_1(\Sigma_1, \mathcal{F}) = 0 \quad \text{and} \quad \liminf_{i\to\infty} D_1(\Gamma(s_i), \mathcal{F}) = D_1(\Sigma_2, \mathcal{F}) > 2\varepsilon_1 > 0.$$

Suppose without loss of generality that for all i,

$$D_1(\Gamma(r_i), \mathcal{F}) < \varepsilon_1 \quad \text{and} \quad D_1(\Gamma(s_i), \mathcal{F}) > \varepsilon_1 \tag{A.2}$$

and that $s_i < r_i$. Let $\tau_i \in \left(\frac{s_i}{r_i}, 1\right)$ be the largest number τ such that $\tau r_i \in (s_i, r_i)$ satisfies

$$D_1(\Gamma(\tau r_i), \mathcal{F}) \geq \varepsilon_1 \,.$$

In particular, for $\tau_i r_i =: \rho_i$ we must have

$$D_1(\Gamma(\rho_i), \mathcal{F}) = \varepsilon_1 \,. \tag{A.3}$$

The existence of such a ρ_i follows from the continuity of $t \mapsto D_1(\Gamma(t), \mathcal{F})$ and (A.2).

Then, for all $\alpha \in (\tau_i, 1)$, by the definition of τ_i and (A.1), we have

$$D_{\alpha/\tau_i}(\Gamma(\rho_i), \mathcal{F}) = D_1(\Gamma(\alpha r_i), \mathcal{F}) < \varepsilon_1. \tag{A.4}$$

We next claim that $\tau_i \to 0$ as $i \to \infty$. Otherwise, there exists $\tau_{i_k} \to \tau \in (0, 1]$ and

$$\varepsilon_1 = D_1(\Gamma(\tau_{i_k} r_i, \mathcal{F})) = D_{\tau_{i_k}}(\Gamma(r_i), \mathcal{F}) \xrightarrow{i\to\infty} D_\tau(\Sigma_2, \mathcal{F}) = 0$$

a contradiction.

Thus, combining (A.4) with (A.3) we determine

$$\lim_{i\to\infty} D_1(\Gamma(\rho_i)), \mathcal{F}) = \varepsilon_1$$

and, due to the fact that $\tau_i \to 0$, for all $r > 1$ (which we identify with $\frac{\alpha}{\tau_i}$ above) we have

$$\limsup_{i\to\infty} D_r(\Gamma(\rho_i)), \mathcal{F}) < \varepsilon_1.$$

But then, there exists a subsequence i_k so that $\lim_{k\to\infty} \Gamma(\rho_{i_k}) = \Gamma_\infty \in \mathrm{Tan}(\partial\Omega, p) \subset \mathcal{M}$ and

$$D_1(\Gamma_\infty, \mathcal{F}) = \varepsilon_1 \geq D_r(\Gamma_\infty, \mathcal{F}) \qquad \forall r > 1,$$

contradicting that $\mathcal{M}$ has property (P). □

Finally, we confirm that tangents to tangents are tangents holds ω-a.e. $p \in \partial\Omega$ for an NTA domain Ω.

Lemma A.6. *Suppose $p \in \partial\Omega$ is such that $\mathrm{Tan}[\nu] \subset \mathrm{Tan}(\omega, p)$ for all $\nu \in \mathrm{Tan}(\omega, p)$. Then for any $\Sigma \in \mathrm{Tan}(\partial\Omega, p)$, all $x \in \Sigma$ and all $r > 0$, we have*

$$\frac{\Sigma - x}{r} \in \mathrm{Tan}(\partial\Omega, p).$$

Proof. Fix $\Sigma \in \mathrm{Tan}(\partial\Omega, p)$. Then there exists $r_i \to 0$ so that

$$\Sigma = \lim_{i\to\infty} \frac{\partial\Omega - p}{r_i}$$

By Theorem 3.3, we know that by defining $\omega_i = \frac{T_{p,r_i}[\omega]}{\omega(B(p,r_i))}$ there exists a converging subsequence we neglect to relabel and a limiting measure ω_∞ so that $\omega_i \overset{*}{\rightharpoonup} \omega_\infty$. Moreover, we know that $\Sigma = \mathrm{spt}\,\omega$.

We now fix $x \in \Sigma = \mathrm{spt}\,\omega$. Because $\mathrm{Tan}[\omega_\infty] \subset \mathrm{Tan}(\omega, p)$ it follows that $T_{x,r}[\omega] \subset \mathrm{Tan}(\omega, p)$. Moreover,

$$\mathrm{spt}\, T_{x,r}[\omega_\infty] = \frac{\Sigma - x}{r}$$

But then, since $T_{x,r}[\omega_\infty] \in \mathrm{Tan}(\omega, p)$ there exists a sequence $\rho_i \to 0$ and $c_i > 0$ so that $c_i T_{p,\rho_i}[\omega] \overset{*}{\rightharpoonup} T_{x,r}[\omega_\infty]$. Again, by Theorem 3.3, we know that

$$\frac{\partial\Omega - p}{\rho_i} \to \mathrm{spt}\, T_{x,r}[\omega_\infty] = \frac{\Sigma - x}{r},$$

proving $\frac{\Sigma - x}{r} \in \mathrm{Tan}(\partial\Omega, p)$ as desired. □

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