

# RATE OF CONVERGENCE OF THE VANISHING VISCOSITY METHOD FOR HAMILTON-JACOBI EQUATIONS WITH NEUMANN BOUNDARY CONDITIONS

ALESSANDRO GOFFI

ABSTRACT. We study the quantitative small noise limit in the  $L^\infty$  norm of certain time-dependent Hamilton-Jacobi equations equipped with Neumann boundary conditions, depending on the regularity of the data and the geometric properties of the domain. We first provide a  $\mathcal{O}(\sqrt{\varepsilon})$  rate of convergence for Hamilton-Jacobi equations with locally Lipschitz Hamiltonians posed on convex domains of the Euclidean space. We then enhance this speed of convergence in the case of quadratic Hamiltonians proving one-side rates of order  $\mathcal{O}(\varepsilon)$  and  $\mathcal{O}(\varepsilon^\beta)$ ,  $\beta \in (1/2, 1)$ . The results exploit recent  $L^1$  contraction estimates for Fokker-Planck equations with bounded velocity fields on unbounded domains used to derive differential Harnack estimates for the corresponding Neumann heat flow.

## 1. INTRODUCTION

In this note we study the rate of convergence of the vanishing viscosity approximation for the first-order (backward) Hamilton-Jacobi equation equipped with homogeneous Neumann boundary condition

$$\begin{cases} -\partial_t u + H(Du(x, t)) = f(x, t) & \text{in } \Omega \times (0, T), \\ \partial_\nu u = 0 & \text{on } \partial\Omega \times (0, T), \\ u(x, T) = u_T(x) & \text{in } \Omega, \end{cases}$$

under the main assumption that  $\Omega \subset \mathbb{R}^n$  is *unbounded*. Heuristically, this amounts to study the speed of convergence of  $v^\varepsilon = \partial_\varepsilon u_\varepsilon$  (or, formally, the behavior of  $\frac{u_\varepsilon + \eta - u_\varepsilon}{\eta}$  as  $\eta \rightarrow 0^+$ ), where, denoting by  $\varepsilon > 0$  the (small) viscosity parameter,  $u_\varepsilon$  solves the viscous problem

$$\begin{cases} -\partial_t u_\varepsilon - \varepsilon \Delta u_\varepsilon + H(Du_\varepsilon(x, t)) = f(x, t) & \text{in } \Omega \times (0, T), \\ \partial_\nu u_\varepsilon = 0 & \text{on } \partial\Omega \times (0, T), \\ u_\varepsilon(x, T) = u_T(x) & \text{in } \Omega. \end{cases}$$

We will show that in the case of solutions satisfying a priori Lipschitz bounds (independent of  $\varepsilon$ ) and  $H \in W_{\text{loc}}^{1, \infty}(\mathbb{R}^n)$ , if  $\Omega$  is convex we have

$$\|u_\varepsilon - u\|_{L^\infty(\Omega \times (0, T))} \leq C\sqrt{\varepsilon T}.$$

This bound is of perturbative nature, i.e. it is a consequence of the linear part of the PDE, while the nonlinearity plays a minor role. In addition, this quantitative bound of order  $\mathcal{O}(\sqrt{\varepsilon T})$  is optimal. In fact, it agrees with that of the vanishing viscosity limit of the heat equation. It is easy to see that for  $H = f = 0$  one can write via [9, Lemma II.1.3], see also [20, Lemma 1.14], for  $g(\cdot, t) = u_\varepsilon(\cdot, T - t)$

$$\|g(\cdot) - u_T\|_{L^\infty} = \|e^{t\varepsilon\Delta} u_T - u_T\|_{L^\infty} \leq C\|u_T\|_{W^{1, \infty}} \sqrt{\varepsilon T}.$$

---

*Date:* September 1, 2025.

*2020 Mathematics Subject Classification.* 35F21, 35F31, 41A25.

*Key words and phrases.* Adjoint method, Vanishing viscosity approximation, Neumann boundary condition.

The author is member of the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM), and he was partially supported by the INdAM-GNAMPA project 2025.

If, in addition,  $H$  is quadratic and  $\Delta u_\varepsilon \leq c(t) \in L^1(0, T)$ , we can boost the rate on both sides and get the quantitative estimate

$$-C\varepsilon^\beta \leq u_\varepsilon - u \leq \varepsilon \int_0^T c(t), \quad \beta \in (1/2, 1).$$

The results are based on duality methods and make use of some crucial  $L^1$ -contraction estimates for Fokker-Planck equations discovered recently in [14] within the analysis of geometric estimates for the heat flow. For this reason, we need to add the *geometric* assumption that  $\Omega$  satisfies a (uniform) interior cone property, cf. (1). We mention that our estimates are new even when  $\Omega$  is bounded and convex: in this case we do not need  $\Omega$  to satisfy the interior cone condition, see [14].

The research on quantitative estimates for the vanishing viscosity approximation of Hamilton-Jacobi equations has recently received an increasing interest, mostly in connection with the quantitative study of the convergence problem in Mean Field Control. Classical results provide  $\mathcal{O}(\sqrt{\varepsilon})$  rates for problems posed in  $\mathbb{R}^n$  or  $\mathbb{T}^n$  using maximum principle methods [6], while the more recent [10] provides the same result using integral techniques. This rate can be boosted through estimates in weaker  $L^p$ -norms, cf. [3], via nonlocal approximations [8, 12], when  $H$  is uniformly or strictly convex or under appropriate smallness conditions [5]. The case of problems with boundary conditions is much less studied, and its analysis goes back to [18, 19] for smooth, convex and bounded domains. To our knowledge, quantitative estimates were obtained only in few papers for stationary equations on bounded domains of the Euclidean space, cf. [21, 16]. Our results provide new advances in at least two directions, allowing the possibility of *unbounded convex sets* and the treatment of *time-dependent* problems: for us the result in the stationary case will be a byproduct, as in [22]. This note emphasizes that for problems with boundary conditions the geometry of the domain, together with the regularity of the data and the geometric assumptions of  $H$ , play an important role in the study of the rate of convergence of the vanishing viscosity approximation.

## 2. PRELIMINARY RESULTS ON FOKKER-PLANCK EQUATIONS WITH NEUMANN BOUNDARY CONDITIONS

We assume that  $\Omega \subseteq \mathbb{R}^n$  is a domain of  $\mathbb{R}^n$  with smooth boundary. We also assume that  $\Omega$  is unbounded and satisfies the following condition

- (1)  $\Omega$  satisfies the interior cone condition, i.e. there exists a finite cone such that every point in  $\Omega$  is the vertex of a cone (congruent to the fixed given cone) contained in  $\Omega$ .

This geometric property ensures suitable extension properties of Sobolev spaces and a priori bounds for parabolic equations with divergence-type terms. These properties and estimates are fundamental to deduce a conservation of mass principle, as we describe next. The paper [14] considered the forward Cauchy-Neumann problem solved by the function  $\rho_\varepsilon := \rho_\varepsilon(x, t)$

$$(2) \quad \begin{cases} \partial_t \rho_\varepsilon - \varepsilon \Delta \rho_\varepsilon + \operatorname{div}(b(x, t) \rho_\varepsilon) = 0 & \text{in } Q_\tau = \Omega \times (\tau, T), \\ \varepsilon \partial_\nu \rho - \rho b(x, t) \cdot \nu = 0 & \text{on } \Sigma_\tau := \partial\Omega \times (\tau, T), \\ \rho(x, \tau) = \rho_\tau(x) & \text{in } \Omega. \end{cases}$$

where  $\rho_\tau \in C_0^\infty(\Omega)$  and  $\nu$  is the outward normal to the boundary of  $\Omega$ , under the main assumption that

$$b \in L^\infty(Q_\tau).$$

The regularity condition on  $b$  can be considerably weakened, as discussed in [14].

**Theorem 2.1.** *[Theorem 2.4 in [14]] Assume that  $\Omega$  satisfies (1), and let  $b \in L^\infty(Q_\tau)$ . Then there exists a unique weak energy solution  $\rho \in W := \{f \in L^2(\tau, T; W^{1,2}(\Omega)) \text{ such that } \partial_t f \in L^2(\tau, T; W'_\Omega)\}$  to (2). In addition, if  $\rho_\tau \geq 0$ , then  $\rho(t) \geq 0$  and we have*

$$\int_\Omega \rho(x, t) dx = \int_\Omega \rho_\tau(x) dx$$

for  $t \in (\tau, T]$ .

### 3. RATE OF CONVERGENCE ESTIMATES

In this section we consider domains with smooth boundary satisfying the following additional constraint

$$(3) \quad \Omega \text{ is convex.}$$

We address the vanishing viscosity approximation of the following equation

$$(4) \quad \begin{cases} -\partial_t u + H(Du(x, t)) = f(x, t) & \text{in } Q_T := \Omega \times (0, T), \\ \partial_\nu u = 0 & \text{in } \Sigma_T := \partial\Omega \times (0, T), \\ u(x, T) = u_T(x) & \text{in } \Omega, \end{cases}$$

namely the diffusive Cauchy-Neumann problem

$$(5) \quad \begin{cases} -\partial_t u_\varepsilon - \varepsilon \Delta u_\varepsilon + H(Du_\varepsilon(x, t)) = f(x, t) & \text{in } Q_T, \\ \partial_\nu u_\varepsilon = 0 & \text{in } \Sigma_T, \\ u_\varepsilon(x, T) = u_T(x) & \text{in } \Omega, \end{cases}$$

where  $\Omega$  is either bounded and satisfies (3) or it is unbounded and satisfies (1) and (3). Note that if the domain is bounded the assumption (1) can be dropped, as it is a consequence of the convexity of the ambient space, cf. [14, Section 2.1]. In the case of (4), since  $u$  is not  $C^1$ , the Neumann condition and the notion of solution should be interpreted in the viscosity sense, see [19]. If  $u$  is semi-superharmonic (i.e.  $\Delta u_\varepsilon \leq c$ ), one can also interpret the Neumann condition as in [18, Lemma 8.1]. In this setting, the notion of viscosity solution and the well-posedness results also hold in the case of unbounded domains [19, p. 795 and p. 806], see also [19, Remark p. 804] for additional comments about condition (3). In the sequel we will exploit the notation  $u(t) = u(\cdot, t)$ .

When  $\Omega$  is bounded and smooth, and  $H$  is locally Lipschitz, the convergence problem of the vanishing viscosity method was addressed in [18, 19, 21] for stationary equations, but no results seem available for time-dependent Hamilton-Jacobi equations. We provide for the first time a time-dependent version of the quantitative convergence in [21, Theorem 2]. Our main novelty with respect to [21] is the possibility of allowing unbounded domains and the treatment of evolution problems: the stationary rate of convergence found in [21, Theorem 2] will be a consequence of our next Theorems, see [3, Section 4.3.1]. To our knowledge, only few cases of problems with boundary conditions were addressed in the literature other than [21]. We mention [22] for Dirichlet problems in bounded domains, [15] for problems with state constraints and [16] for general Neumann-type boundary conditions and Hölder continuous solutions.

*Remark 3.1.* All the results in the paper will be stated in the form of a priori estimates. However, one needs some qualitative properties of solutions to perform rigorously the calculations below. When  $\Omega$  is bounded, smooth and convex, solutions of Hamilton-Jacobi equations satisfy the maximal  $L^q$ -regularity property, see [11, 13] (the convexity of  $\Omega$  is not needed when  $H$  has at most natural growth). If the domain is unbounded, we need some extra conditions on the geometry of the domain at infinity to ensure maximal regularity of solutions, see [1, 2] and [14, Section 2.1] for a complete discussion.

**Theorem 3.2.** *Let  $u_\varepsilon$  be a Lipschitz solution to the viscous equation (5) and  $u$  be a Lipschitz viscosity solution to the first-order equation (4), with  $\Omega$  either bounded and satisfying (3) or unbounded and satisfying (1) and (3). Assume that  $H \in W_{\text{loc}}^{1,\infty}(\mathbb{R}^n)$  and  $f \in W^{1,\infty}(Q_T)$ . Then*

$$\|u_\varepsilon - u\|_{L^\infty(Q_T)} \leq M\sqrt{\varepsilon T}.$$

for a positive constant  $M$  depending on  $\|Du\|_{L^\infty(\overline{Q_T})}$ ,  $n$ ,  $\|Df\|_{L^\infty(Q_T)}$ .

*Proof.* We differentiate (5) with respect to  $\varepsilon$  to find the PDE

$$-\partial_t v^\varepsilon - \varepsilon \Delta v^\varepsilon + D_p H(Du_\varepsilon) \cdot Dv^\varepsilon = \Delta u_\varepsilon.$$

equipped with the boundary conditions

$$\partial_\nu v^\varepsilon = 0 \text{ on } \Sigma_T \text{ and } v^\varepsilon(T) = 0 \text{ in } \Omega.$$

This procedure of differentiating with respect to  $\varepsilon$  can be made rigorous via Lemma 2.1 in [5]. We introduce the adjoint problem

$$(6) \quad \begin{cases} \partial_t \rho_\varepsilon - \varepsilon \Delta \rho_\varepsilon - \operatorname{div}(D_p H(Du_\varepsilon) \rho_\varepsilon) = 0 & \text{in } Q_\tau = \Omega \times (\tau, T) \\ \varepsilon \partial_\nu \rho_\varepsilon + \rho_\varepsilon D_p H(Du_\varepsilon) \cdot \nu = 0 & \text{on } \Sigma_\tau, \\ \rho_\varepsilon(x, \tau) = \delta_{x_0} & \text{in } \Omega. \end{cases}$$

Since  $H$  is locally Lipschitz and  $u$  is globally Lipschitz, the velocity field  $b(x, t) = -D_p H(Du_\varepsilon) \in L^\infty(Q_\tau)$ . Thus, (6) admits a unique weak nonnegative solution belonging to  $W$  by Theorem 2.1. By duality, this gives the representation formula

$$(7) \quad \int_\Omega v^\varepsilon(\tau) \rho_\varepsilon(\tau) dx = \int_\Omega v^\varepsilon(T) \rho_\varepsilon(T) dx + \iint_{Q_\tau} \Delta u_\varepsilon \rho_\varepsilon dx dt.$$

We only need to bound the last integral of the right-hand side, since the first one vanishes due to the fact that  $v^\varepsilon(T) = 0$ . Standard computations through the Bochner's identity yield the evolution of  $w_\varepsilon = |Du_\varepsilon|^2$

$$-\partial_t w_\varepsilon - \varepsilon \Delta w_\varepsilon + 2\varepsilon |D^2 u_\varepsilon|^2 + D_p H(Du_\varepsilon) \cdot Dw_\varepsilon = 2Df \cdot Du_\varepsilon.$$

The convexity of  $\Omega$ , instead, provides the inequality  $\partial_\nu w_\varepsilon \leq 0$  on  $\Sigma_\tau$ , cf. [14, Lemma 2.1]. By duality with the adjoint problem and using the conservation of mass property in Theorem 2.1 we have the following bound

$$(8) \quad |Du_\varepsilon(x, \tau)| + 2\varepsilon \int_\tau^T \int_\Omega |D^2 u_\varepsilon|^2 \rho dx dt \leq C_L$$

for a constant  $C_L$  depending on  $\|Du_T\|_{L^\infty(\Omega)}, \|Df\|_{L^\infty(Q_\tau)}$ . Note that this step requires the convexity of the ambient space, since we use that  $\partial_\nu w_\varepsilon \leq 0$ ,  $\partial_\nu u_\varepsilon = \partial_\nu \rho_\varepsilon = 0$  on  $\Sigma_T$ , along with the nonnegativity of  $\rho_\varepsilon$  on  $Q_\tau$ . Therefore, we can use (8) and Theorem 2.1 to bound by the Hölder's inequality

$$\iint_{Q_\tau} \Delta u_\varepsilon \rho_\varepsilon dx dt \leq \sqrt{n} \left( \iint_{Q_\tau} |D^2 u_\varepsilon|^2 \rho \right)^{\frac{1}{2}} \left( \iint_{Q_\tau} \rho \right)^{\frac{1}{2}} \leq \frac{\sqrt{C_L n T}}{\sqrt{\varepsilon}}.$$

Integrating in  $\varepsilon$  and noting that the right-hand side is integrable near  $\varepsilon = 0$ , we get for  $\varepsilon_1 \geq \varepsilon_2 > 0$

$$\|(u^{\varepsilon_1} - u^{\varepsilon_2})(\tau)\|_{L^\infty(\Omega)} \leq M \sqrt{T} (\sqrt{\varepsilon_1} - \sqrt{\varepsilon_2}),$$

where  $M = 2\sqrt{n C_L}$ . We conclude by letting  $\varepsilon_2 \rightarrow 0$ .  $\square$

*Remark 3.3.* In the case of bounded domains, the same result can be achieved using the standard maximum principle for parabolic equations applied to the auxiliary function

$$F_\pm(x, t) = \sqrt{\varepsilon} \partial_\varepsilon u_\varepsilon(x, t) \pm |Du_\varepsilon(x, t)|^2.$$

Our proof provides a global approach to this problem that avoids the use maximum principles on unbounded spaces.

*Remark 3.4.* The result can be made unconditional using Lipschitz estimates independent of the viscosity  $\varepsilon$  for the Cauchy-Neumann problem of Hamilton-Jacobi equations. These are well-known for bounded convex domains, see e.g. [18, 19], and require additional coercivity conditions on  $H$ . Examples of Lipschitz bounds in the unbounded case can be found in [14] for quadratic Hamilton-Jacobi equations. At this stage, we do not know how to remove the convexity condition on  $\Omega$  to obtain the second order integral bound in (8). However, Lipschitz estimates are available under weaker geometric assumptions, see e.g. [19, p. 807].

We now investigate an improvement of the convergence rate in Theorem 3.2. The authors in [21, p. 18] highlighted that for a certain one-dimensional, quadratic, Hamilton-Jacobi equation with Neumann conditions, one should expect a rate of convergence better than the order  $\mathcal{O}(\sqrt{\varepsilon})$ . Our next result confirms the observations of [21], as it provides an improvement when the solutions are semi-superharmonic and  $H$  is quadratic, under additional conditions on  $f$ , cf. (9). In particular, it gives a quantitative statement of [18, Theorem 8.1]. If  $f$  is semi-superharmonic and  $H$  is uniformly

convex, this upgrade is known under certain conditions of the state space: the paper [5] provides two-sides  $\mathcal{O}(\varepsilon |\log \varepsilon|)$  rates of convergence in the periodic setting, see also [4] for the whole space case. Both of them exploit entropy bounds of solutions to the adjoint problem (6). The rate can be also improved under smallness conditions (e.g. short time horizons) or under geometric requirements on the solutions, at least for domains without boundary, cf. Remark 3.3 in [5].

**Theorem 3.5.** *Let  $u_\varepsilon \in W_x^{1,\infty}$  be a semi-superharmonic solution (i.e. satisfying the one-side bound  $\Delta u_\varepsilon \leq c(t)$ ,  $c \in L^1(0, T)$  independent of  $\varepsilon$ ) to the viscous equation (5) and  $u \in W_x^{1,\infty}$  be a semi-superharmonic solution to the first-order equation (4), with  $\Omega$  as above. Assume that  $H(p) = |p|^2$  and*

$$(9) \quad \Delta f \leq c_f \text{ in } Q_T, \text{ and } \partial_\nu f \geq 0 \text{ on } \Sigma_T.$$

*Then, for all  $\beta \in (1/2, 1)$ , there exists  $C = C(\beta)$  depending in addition on  $n, c_f, T, \|Du\|_{L^\infty(\bar{Q}_T)}$ ,  $\|Df\|_{L^\infty(Q_T)}$ ,  $\|(\Delta u_T)^+\|_{L^\infty(\Omega)}$  such that*

$$-C\varepsilon^\beta \leq u_\varepsilon - u \leq \varepsilon \int_0^T c(t) dt, \quad \beta \in (1/2, 1).$$

*Remark 3.6.* Some observations about (9) are in order. While the one-side condition  $\Delta f \leq c_f$  is natural to derive unilateral second order bounds, the second condition is in general necessary. We refer to [18, Remark 8.2] for some explicit examples in bounded and unbounded convex domains showing that this condition cannot be dropped.

*Proof.* We start again with the equation solved by  $v^\varepsilon$ , namely

$$-\partial_t v^\varepsilon - \varepsilon \Delta v^\varepsilon + D_p H(Du_\varepsilon) \cdot Dv^\varepsilon = \Delta u_\varepsilon$$

equipped with the boundary conditions

$$\partial_\nu v^\varepsilon = 0 \text{ on } \Sigma_T \text{ and } v^\varepsilon(T) = 0 \text{ in } \Omega.$$

By duality, we have the representation formula

$$\int_\Omega v^\varepsilon(\tau) \rho_\varepsilon(\tau) dx = \int_\Omega v^\varepsilon(T) \rho_\varepsilon(T) dx + \iint_{Q_\tau} \Delta u_\varepsilon \rho_\varepsilon dx dt.$$

Since  $\Delta u_\varepsilon \leq c(t)$  and  $v^\varepsilon(T) = 0$ , by the conservation of mass for the adjoint problem in Theorem 2.1 we have

$$\iint_{Q_\tau} \Delta u_\varepsilon \rho_\varepsilon dx dt \leq \int_\tau^T \|(\Delta u_\varepsilon(t))^+\|_{L^\infty(\Omega)} \int_\Omega \rho_\varepsilon dx dt \leq \int_0^T c(t) dt.$$

Integrating with respect to  $\varepsilon$  we conclude

$$\|(u_\varepsilon - u)^+(\tau)\|_{L^\infty(\Omega)} \leq \varepsilon \int_0^T c(t) dt.$$

We now prove the leftmost estimate. We first show that for  $\alpha \in (1, 2)$  we have the bound

$$(10) \quad \int_\tau^T \int_\Omega (t - \tau)^\alpha |D^2 u_\varepsilon|^2 \rho_\varepsilon dx dt \leq K,$$

where  $\rho_\varepsilon$  solves (6) with  $b(x, t) = -D_p H(Du_\varepsilon) = -2Du_\varepsilon$  and  $K$  depends on  $n, \alpha, T, u_T, f$ . To this aim, we exploit the uniform convexity of the Hamiltonian. We first find, by differentiating twice the equation for  $u_\varepsilon$  and using the Bochner's identity  $\Delta |Du_\varepsilon|^2 = 2|D^2 u_\varepsilon|^2 + 2Du_\varepsilon \cdot D\Delta u_\varepsilon$ , the following inequality solved by the function  $z_\varepsilon(x, t) = (t - \tau)^\alpha \Delta u_\varepsilon(x, t)$

$$-\partial_t z_\varepsilon - \varepsilon \Delta z_\varepsilon + 2(t - \tau)^\alpha |D^2 u_\varepsilon|^2 + 2Du_\varepsilon \cdot Dz_\varepsilon = -\alpha(t - \tau)^{\alpha-1} \Delta u_\varepsilon + (t - \tau)^\alpha \Delta f \quad \text{in } Q_T.$$

Note that since  $u_\varepsilon$  solves the Hamilton-Jacobi equation,  $z$  satisfies the boundary condition

$$\partial_\nu z_\varepsilon = (t - \tau)^\alpha \partial_\nu \Delta u_\varepsilon(x, t) = (t - \tau)^\alpha \partial_\nu \left\{ \frac{1}{\varepsilon} (-\partial_t u_\varepsilon + |Du_\varepsilon|^2 - f) \right\} \leq 0 \text{ on } \Sigma_T.$$

To see the last inequality, use the identity  $\partial_\nu(\partial_t u_\varepsilon) = 0$ ,  $\partial_\nu |Du_\varepsilon|^2 \leq 0$  because  $\partial_\nu u_\varepsilon = 0$  and  $\Omega$  is convex, and also (9). By duality and integrating in  $\Omega \times (\tau, T)$  we have

$$\begin{aligned} \underbrace{\int_{\Omega} z_\varepsilon(\tau) \rho_\varepsilon(\tau) dx}_{=0} + 2 \int_{\tau}^T \int_{\Omega} (t-\tau)^\alpha |D^2 u_\varepsilon|^2 \rho_\varepsilon dx dt &\leq \underbrace{\int_{\Omega} z_\varepsilon(T) \rho_\varepsilon(T) dx}_{\leq (T-\tau)^\alpha \|(\Delta u_T)^+\|_{L^\infty(\Omega)}} \\ &\quad - \alpha \int_{\tau}^T \int_{\Omega} (t-\tau)^{\alpha-1} \Delta u_\varepsilon \rho_\varepsilon dx dt + \int_{\tau}^T \int_{\mathbb{T}^n} (t-\tau)^\alpha \Delta f \rho_\varepsilon dx dt. \end{aligned}$$

Note that to integrate by parts we used the boundary conditions  $\partial_\nu z_\varepsilon \leq 0$ ,  $\partial_\nu \rho_\varepsilon = \partial_\nu u_\varepsilon = 0$  on  $\Sigma_T$ , together with the fact that  $\rho_\varepsilon \geq 0$  by Theorem 2.1. We now use the Young's inequality and Theorem 2.1 as follows

$$\begin{aligned} &-\alpha \int_{\tau}^T \int_{\Omega} (t-\tau)^{\alpha-1} \Delta u_\varepsilon \rho_\varepsilon dx dt \\ &\leq \frac{1}{n} \int_{\tau}^T \int_{\Omega} (t-\tau)^\alpha |\Delta u_\varepsilon|^2 \rho_\varepsilon dx dt + \frac{n\alpha^2}{4} \int_{\tau}^T \int_{\Omega} (t-\tau)^{\alpha-2} \rho_\varepsilon dx dt \\ &\leq \int_{\tau}^T \int_{\Omega} (t-\tau)^\alpha |D^2 u_\varepsilon|^2 \rho_\varepsilon dx dt + \frac{n\alpha^2}{4} \int_{\tau}^T (t-\tau)^{\alpha-2} dt \\ &\leq \int_{\tau}^T \int_{\Omega} (t-\tau)^\alpha |D^2 u_\varepsilon|^2 \rho_\varepsilon dx dt + \frac{n\alpha^2}{4(\alpha-1)} T^{\alpha-1}. \end{aligned}$$

We then obtain

$$\int_{\tau}^T \int_{\Omega} (t-\tau)^\alpha |D^2 u_\varepsilon|^2 \rho_\varepsilon dx dt \leq \frac{n\alpha^2}{4(\alpha-1)} T^{\alpha-1} + T^\alpha \|(\Delta u_T)^+\|_{L^\infty(\Omega)} + \frac{T^{\alpha+1}}{\alpha+1} c_f =: K.$$

We now conclude the estimate. Assume first that  $\tau + \varepsilon < T$ . Then, back to the representation formula (7), we have

$$\begin{aligned} \left| \int_{\tau}^T \int_{\Omega} \Delta u_\varepsilon \rho_\varepsilon dx dt \right| &\leq \sqrt{n} \left( \int_{\tau+\varepsilon}^T \int_{\Omega} (t-\tau)^{\alpha/2} |D^2 u_\varepsilon| \rho_\varepsilon (t-\tau)^{-\alpha/2} dx dt + \int_{\tau}^{\tau+\varepsilon} \int_{\Omega} |D^2 u_\varepsilon| \rho_\varepsilon dx dt \right) \\ &\leq \sqrt{n} \left( \int_{\tau+\varepsilon}^T \int_{\Omega} (t-\tau)^\alpha |D^2 u_\varepsilon|^2 \rho_\varepsilon dx dt \right)^{\frac{1}{2}} \left( \int_{\tau+\varepsilon}^T \int_{\Omega} (t-\tau)^{-\alpha} \rho_\varepsilon dx dt \right)^{\frac{1}{2}} \\ &\quad + \sqrt{n} \left( \int_{\tau}^{\tau+\varepsilon} \int_{\Omega} |D^2 u_\varepsilon|^2 \rho_\varepsilon dx dt \right)^{\frac{1}{2}} \left( \int_{\tau}^{\tau+\varepsilon} \int_{\Omega} \rho_\varepsilon dx dt \right)^{\frac{1}{2}} \\ &\leq \sqrt{\frac{nK\varepsilon^{1-\alpha}}{\alpha-1}} + \sqrt{nC_L}, \end{aligned}$$

where we used the estimate (3), the conservation of mass of the adjoint problem and the Lipschitz bound in [5, Lemma 2.3] (note that the Lipschitz bound also holds for convex domains, as noted in (8)). If  $\tau \geq T - \varepsilon$ , then the same estimate holds, without the first constant  $\sqrt{\frac{nK\varepsilon^{1-\alpha}}{\alpha-1}}$ , since in this case there is no need to split the first integral. We conclude that

$$v^\varepsilon(x_0, \tau) \geq -\sqrt{\frac{nK}{\alpha-1}} \varepsilon^{\frac{1-\alpha}{2}} - \sqrt{nC_L}.$$

Since the previous estimate holds for all  $x_0, \tau$ , we obtain, by setting  $\frac{3-\alpha}{2} = \beta \in (1/2, 1)$ ,

$$u_\varepsilon - u \geq -\frac{1}{\beta} \sqrt{\frac{nK}{2(1-\beta)}} \varepsilon^\beta - \sqrt{nC_L} \varepsilon.$$

□

Theorem 3.5 can be made unconditional using semi-superharmonic bounds for strictly convex Hamilton-Jacobi equations. We recall that under the uniform convexity condition on  $H$ , solutions are known to be semiconcave when  $\Omega = \mathbb{R}^n$  or  $\mathbb{T}^n$ . However, this property remains an open question for Neumann problems posed on bounded convex domains. We show however that second order bounds on pure derivatives independent of the viscosity can be achieved for Hamilton-Jacobi Neumann boundary-value problems. The proof uses an integral approach based again on Theorem 2.1. Earlier results in this direction appeared in [18] via maximum principle methods on bounded convex domains.

**Theorem 3.7.** *Let  $\Omega$  be an unbounded domain satisfying (1)-(3) (or bounded and satisfying (3)), and assume that  $\|(\Delta u_T)^+\|_{L^\infty(\Omega)} \leq M_0$ , with  $H$  satisfying*

$$H(p) = |p|^\gamma, \quad \gamma > 1.$$

Assume also

$$\Delta f \leq c_f(t) \in L^1(0, T) \text{ in } Q_T \text{ and } \partial_\nu f \geq 0 \text{ on } \Sigma_T.$$

Then, any Lipschitz solution of (5) satisfies

$$\|(\Delta u)^+(t)\|_{L^\infty(\Omega)} \leq M_0 + \int_0^T c_f(t) dt.$$

*Sketch of the proof.* We follow the proof of the Li-Yau estimate in [14]. When  $H(p) = |p|^2$ , it is enough to observe that  $\tilde{z}(x, t) = \Delta u_\varepsilon$  satisfies the inequality

$$-\partial_t \tilde{z}_\varepsilon - \varepsilon \Delta \tilde{z}_\varepsilon + 2|D^2 u_\varepsilon|^2 + D_p H(Du_\varepsilon) \cdot D\tilde{z}_\varepsilon = \Delta f \text{ in } Q_T$$

with the boundary condition  $\partial_\nu \tilde{z} \leq 0$  on  $\Sigma_T$ , as in Theorem 3.5. A similar calculation in the more general case  $H(Du_\varepsilon) = |Du_\varepsilon|^\gamma$ , or its smooth approximation  $H_\delta(Du_\varepsilon) = (\delta + |Du_\varepsilon|^2)^{\frac{\gamma}{2}}$ , leads to

$$-\partial_t \tilde{z}_\varepsilon - \varepsilon \Delta \tilde{z}_\varepsilon + \gamma |Du_\varepsilon|^{\gamma-2} Du_\varepsilon \cdot D\tilde{z}_\varepsilon \leq \Delta f.$$

In this case  $\partial_\nu \tilde{z}_\varepsilon \leq 0$  on  $\partial\Omega$ , since  $\partial_\nu |Du_\varepsilon|^\gamma \leq 0$  on  $\partial\Omega$  because  $\Omega$  is convex. By duality with the adjoint problem (6) and using its conservation of mass, it is immediate to obtain the desired estimates, which are independent of the viscosity.  $\square$

*Remark 3.8.* It would be worth investigating the improvement of the rate of convergence in weaker  $L^1$  norms, as in [3, 17]. In this case, starting from the identity

$$\int_\Omega v^\varepsilon(\tau) \rho_\varepsilon(\tau) dx = \int_\Omega v^\varepsilon(T) \rho_\varepsilon(T) dx + \iint_{Q_\tau} \Delta u_\varepsilon \rho_\varepsilon dx dt,$$

we can estimate

$$\iint_{Q_\tau} \Delta u_\varepsilon \rho_\varepsilon dx dt \leq \|\Delta u_\varepsilon\|_{L^1(Q_\tau)} \|\rho_\varepsilon\|_{L^\infty(Q_\tau)}.$$

If the right-hand side does not depend on  $\varepsilon$  we would obtain a  $\mathcal{O}(\varepsilon)$  rate of convergence. The estimate  $\|\Delta u_\varepsilon\|_{L^1(Q_\tau)}$  can be obtained on bounded convex domains [18], while one needs maximum principle estimates for the adjoint problem independent of the viscosity, under the assumption that  $[\operatorname{div}(b)]^- < \infty$ . A study in this direction was carried out in [7] on unbounded domains when  $\operatorname{div}(b) \in L^\infty$ .

*Remark 3.9* (Towards the  $\mathcal{O}(\varepsilon |\log \varepsilon|)$  rate of convergence). In the case of domains without boundary, recent results provided a better  $\mathcal{O}(\varepsilon |\log \varepsilon|)$  speed of convergence, cf. [4, 5]. These are based on entropy-type estimates for solutions to (6) which are unknown in the context of unbounded Neumann problems. We do not know whether in the setting of this manuscript the speed of convergence can be improved.

## REFERENCES

- [1] H. Amann. Existence and regularity for semilinear parabolic evolution equations. *Ann. Sc. Norm. Super. Pisa, Cl. Sci., IV. Ser.*, 11:593–676, 1984.
- [2] F. E. Browder. A-priori estimates for solutions of elliptic boundary-value problems. I. *Nederl. Akad. Wet., Proc., Ser. A*, 63:145–159, 160–169, 1960.
- [3] F. Camilli, A. Goffi, and C. Mendico. Quantitative and qualitative properties for hamilton-jacobi pdes via the nonlinear adjoint method, to appear in *Ann. Sc. Norm. Super. Pisa, Cl. Sci.*, 2023.
- [4] L.-P. Chaintron and S. Daudin. Optimal rate of convergence in the vanishing viscosity for uniformly convex Hamilton-Jacobi equations. Preprint, arXiv:2506.13255 [math.AP], 2025.
- [5] M. Cirant and A. Goffi. Convergence rates for the vanishing viscosity approximation of Hamilton-Jacobi equations: the convex case. Preprint, arXiv:2502.15495 [math.AP], to appear in *Indiana Univ. Math. J.*, 2025.
- [6] M. G. Crandall and P.-L. Lions. Two approximations of solutions of Hamilton-Jacobi equations. *Math. Comp.*, 43(167):1–19, 1984.
- [7] G. Crippa, C. Donadello, and L. Spinolo. A note on the initial-boundary value problem for continuity equations with rough coefficients. *Hyperbolic problems: theory, numerics and applications*, pp. 957-966, AIMS Series on Appl. Math., 8, 2014.
- [8] J. Droniou and C. Imbert. Fractal first-order partial differential equations. *Arch. Ration. Mech. Anal.*, 182(2):299–331, 2006.
- [9] K.-J. Engel and R. Nagel. *One-parameter semigroups for linear evolution equations*, volume 194 of *Grad. Texts Math.* Berlin: Springer, 2000.
- [10] L. C. Evans. Adjoint and compensated compactness methods for Hamilton-Jacobi PDE. *Arch. Ration. Mech. Anal.*, 197(3):1053–1088, 2010.
- [11] A. Goffi. On the optimal  $L^q$ -regularity for viscous Hamilton-Jacobi equations with subquadratic growth in the gradient. *Commun. Contemp. Math.*, 26(6):31, 2024. Id/No 2350019.
- [12] A. Goffi. Remarks on the rate of convergence of the vanishing viscosity process of Hamilton-Jacobi equations. Preprint, arXiv:2412.15651 [math.AP], 2024.
- [13] A. Goffi and F. Pediconi. Sobolev regularity for nonlinear Poisson equations with Neumann boundary conditions on Riemannian manifolds. *Forum Math.*, 35(2):431–456, 2023.
- [14] A. Goffi and G. Tralli. Global geometric estimates for the heat equation via duality methods. Preprint, arXiv:2409.15456 [math.AP], 2024.
- [15] Y. Han and S. N. T. Tu. Remarks on the vanishing viscosity process of state-constraint Hamilton-Jacobi equations. *Appl. Math. Optim.*, 86(1):42, 2022. Id/No 3.
- [16] E. R. Jakobsen and C. A. Georgelin. Continuous dependence results for non-linear Neumann type boundary value problems. *J. Differ. Equations*, 245(9):2368–2396, 2008.
- [17] C.-T. Lin and E. Tadmor.  $L^1$ -stability and error estimates for approximate Hamilton-Jacobi solutions. *Numer. Math.*, 87(4):701–735, 2001.
- [18] P.-L. Lions. *Generalized solutions of Hamilton-Jacobi equations*, volume 69 of *Research Notes in Mathematics*. Pitman (Advanced Publishing Program), Boston, Mass.-London, 1982.
- [19] P.-L. Lions. Neumann type boundary conditions for Hamilton-Jacobi equations. *Duke Math. J.*, 52:793–820, 1985.
- [20] A. J. Majda and A. L. Bertozzi. *Vorticity and incompressible flow*. Camb. Texts Appl. Math. Cambridge: Cambridge University Press, 2002.
- [21] B. Perthame and R. Sanders. The Neumann problem for nonlinear second order singular perturbation problems. *SIAM J. Math. Anal.*, 19(2):295–311, 1988.
- [22] H. V. Tran. Adjoint methods for static Hamilton-Jacobi equations. *Calc. Var. Partial Differential Equations*, 41(3-4):301–319, 2011.

DIPARTIMENTO DI MATEMATICA E INFORMATICA “ULISSE DINI”, UNIVERSITÀ DEGLI STUDI DI FIRENZE, VIALE G. MORGAGNI 67/A, 50134 FIRENZE (ITALY)

Email address: [alessandro.goffi@unifi.it](mailto:alessandro.goffi@unifi.it)