STOCHASTIC HOMOGENISATION OF STRONGLY ANISOTROPIC DEGENERATE INTEGRAL FUNCTIONALS

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Abstract. We prove a stochastic homogenisation result for *strongly anisotropic, degenerate* integral functionals under suitable moment conditions. Specifically, we study random vectorial functionals whose integrands exhibit degenerate growth and coercivity of order $p > 1$ governed by two nonnegative, *stationary* weight functions Λ (governing growth) and λ (governing coercivity). We allow the ratio Λ/λ to become unbounded with positive probability, thereby encompassing the case of strongly anisotropic integrands. Our main result shows that when the integrand is convex in the gradient variable and the weights satisfy the moment condition

$$\mathbb{E}[\Lambda^\alpha + \lambda^{-\beta}] < +\infty,$$

for exponents $\alpha \geq 1$, $\beta \geq 1/(p-1)$ such that

$$\frac{1}{\alpha} + \frac{1}{\beta} < \frac{p}{d-1},$$

if $d \geq 3$ (where d is the space-dimension), the functionals *almost surely* homogenise to a non-degenerate limit energy. Furthermore, in the general non-(quasi)convex case, we prove an analogous homogenisation result under the stricter condition

$$\frac{1}{\alpha} + \frac{1}{\beta} < \frac{1}{d-1},$$

which is shown to be optimal for suitable choices of p and d.

Keywords: Stochastic homogenisation, Γ-convergence, integral functionals, degenerate growth, strongly anisotropic materials.
MSC 2020: 49J45, 49J55, 60G10.

1. Introduction

Over the past decade, the homogenisation of elliptic equations and integral functionals featuring *degenerate*, *random* coefficients has attracted growing interest. In particular, quadratic and convex integral functionals with degenerate, random integrands have been the focus of intensive research within stochastic analysis, as they emerge as Dirichlet forms tied to random conductance models. In this context, homogenisation corresponds to an invariance principle for random walks evolving in degenerate random environments. Moreover, energy functionals with degenerate (discrete or continuous) structure naturally arise in variational models of strongly anisotropic or high-contrast materials.

This paper advances the framework of stochastic homogenisation by addressing *nonlinear* models characterised by *highly anisotropic* and *degenerate* coefficients. Namely, we consider a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with a (continuous) measure-preserving group action $\tau = \{\tau_z\}_{z\in\mathbb{R}^d}$ and for every $\omega \in \Omega$ and $\varepsilon > 0$ we focus on integral functionals of the form

$$F_\varepsilon(\omega)(u) = \int_A f\Big(\omega, \frac{x}{\varepsilon}, \nabla u\Big)\, dx, \tag{1.1}$$

where $A \subset \mathbb{R}^d$ is open, bounded with Lipschitz boundary, $u \in W^{1,1}(A;\mathbb{R}^m)$, with $d, m \in \mathbb{N}$, and $d \geq 2$. If we interpret F_ε as the elastic energy of a composite, microstructured material, in (1.1) the parameter $\omega \in \Omega$ represents a realisation of the microstructure, while $\varepsilon > 0$ sets the corresponding micro-scale. The integrand $f\colon \Omega \times \mathbb{R}^d \times \mathbb{R}^{m\times d} \to [0, +\infty)$ is a τ-*stationary* random variable, *i.e.*,

$$f(\tau_z\omega, x, \xi) = f(\omega, x + z, \xi) \quad \text{for every } (z, \omega, x, \xi) \in \mathbb{R}^d \times \Omega \times \mathbb{R}^d \times \mathbb{R}^{m\times d}$$

and obeys *degenerate* growth and coercivity conditions of order $p > 1$:

$$\lambda(\omega, x)|\xi|^p \leq f(\omega, x, \xi) \leq \Lambda(\omega, x)\left(|\xi|^p + 1\right), \tag{1.2}$$

for every $\omega \in \Omega$, $x \in \mathbb{R}^d$, and $\xi \in \mathbb{R}^{m\times d}$. Here $\Lambda, \lambda\colon \Omega \times \mathbb{R}^d \to [0,+\infty)$ are τ-stationary weight functions, with $\lambda \le \Lambda$, which satisfy the following stochastic integrability conditions

$$\Lambda(\cdot,0) \in L^1(\Omega), \qquad \lambda^{-1/(p-1)}(\cdot,0) \in L^1(\Omega). \tag{1.3}$$

In view of the stationarity hypothesis, by the Fubini Theorem, (1.3) implies that, almost surely,

$$\Lambda(\omega,\cdot) \in L^1_{\rm loc}(\mathbb{R}^d)\,, \quad \lambda^{-1/(p-1)}(\omega,\cdot) \in L^1_{\rm loc}(\mathbb{R}^d);$$

in turn, the local integrability assumptions above allow for

$$\sup_{x\in\mathbb{R}^d} \Lambda(\omega,x) = +\infty, \quad \inf_{x\in\mathbb{R}^d} \lambda(\omega,x) = 0$$

to happen with positive probability, thus accommodating a degenerate behaviour of f. Moreover, appealing to the Birkhoff Ergodic Theorem, it can be proven that (1.3) also guarantees that (almost surely) sequences with uniformly bounded energy F_ε are precompact in $W^{1,p}$ with respect to the strong L^1-topology (see [21]).

In the *isotropic* case - that is when $\Lambda \le c\,\lambda$, for some $c \in (0,+\infty)$ - the theory of Γ-convergence and homogenisation of integral functionals with degenerate coefficients is by now well-understood (see, *e.g.*, [7, 10, 11, 15, 20, 13, 21]). Specifically, in the recent work [21] the authors show, among other, that when τ is *ergodic*, under very mild assumptions on f, the integrability conditions (1.3) are in fact enough to prove that the functionals F_ε homogenise, almost surely, to the deterministic functional

$$F_{\rm hom}(u) = \int_A f_{\rm hom}(\nabla u)\,dx,$$

where $f_{\rm hom}$ is given by

$$f_{\rm hom}(\xi) = \lim_{t\to+\infty} \frac{1}{t^d}\mathbb{E}\left[\inf\left\{\int_{Q_t(0)} f(\omega,x,\nabla u+\xi)\,dx\colon\; u \in W^{1,1}_0(Q_t(0);\mathbb{R}^m)\right\}\right]. \tag{1.4}$$

Furthermore, $f_{\rm hom}$ satisfies standard growth and coercivity conditions of order p

$$c_0|\xi|^p \le f_{\rm hom}(\omega,\xi) \le c_1(|\xi|^p+1),$$

with

$$c_0 = \mathbb{E}[\lambda^{-1/(p-1)}(\cdot,0)]^{1-p} \quad \text{and} \quad c_1 = \mathbb{E}[\Lambda(\cdot,0)],$$

so that thanks to (1.3) the domain of $F_{\rm hom}$ coincides with $W^{1,p}(A;\mathbb{R}^m)$. Notably, in [21] it is also shown that the moment conditions in (1.3) are optimal in the sense that if either of the two is violated, examples can be exhibited for which the corresponding $f_{\rm hom}$ degenerates. The proof of the homogenisation result in [21] follows a now-classical strategy which combines the blow-up method [17] with the Subadditive Ergodic Theorem [1], in the spirit of [9]. However, the degenerate growth of f introduces subtle challenges, particularly in establishing the lower-bound inequality. In fact, in order to prove it, one must modify a sequence locally, to match a linear boundary datum, without substantially increasing the energy. In the language of Γ-convergence, this means proving that the functionals F_ε satisfy a so-called *fundamental estimate*, uniformly in ε (see [8]). But in the degenerate setting, sequences with equi-bounded energy converge only in L^1, while the error in the fundamental estimate is controlled by a weighted L^p-norm, where the weight oscillates randomly at the ε scale. To bridge this mismatch, [21] makes use of a vectorial truncation argument (see also [13]), which is then suitably woven with an enhanced version of the Birkhoff Ergodic Theorem. This combination allows one to control the error in the fundamental estimate and to complete the lower-bound proof despite the degeneracy. It is worth mentioning that, up to suitably replacing the integrability assumptions in (1.3), an analogous approach also works for $p=1$, in this case producing a homogenised functional defined in BV, as shown in [23].

In this paper we extend the study in [21] to the strongly anisotropic setting, that is, to the case where the ratio Λ/λ can become unbounded with positive probability. To the best of our knowledge, this is the first paper where a *nonlinear* stochastic homogenisation result which encompasses both strong anisotropy and degeneracy is proven. On the other hand, in the recent paper [2] the authors establish a *quantitative* stochastic homogenisation result for *linear*, non-uniformly elliptic equations in divergence form allowing for strongly anisotropic coefficients.

In Theorem 3.4 we show that if Λ and λ satisfy suitable moment conditions (which are, in general, stronger than (1.3)), then also in the highly anisotropic setting the functionals F_ε homogenise to $F_{\rm hom}$ with probability one.

As is typical when coercivity and growth of f are not comparable, a proof of the lower bound which makes use of the fundamental estimate becomes rather subtle. An effective way to overcome this issue is to assume convexity of f in the gradient variable (see, *e.g.*, [19, 14, 22], where no growth conditions from above are imposed). This allows one to modify a sequence by considering a convex combination with the desired boundary datum by means of a suitable cut-off function. In fact, the convexity enables the cut-off function to be chosen without relying on an *a priori* energy bound, thus sidestepping the challenges posed by the mismatched coercivity/growth structure. However, the convexity alone is not sufficient to conclude since one also needs to control the error due to the modification above. We show that this is possible under additional moment conditions on Λ and λ. Namely, besides the convexity of f, we require that

$$\Lambda(\cdot,0)\in L^\alpha(\Omega),\qquad \lambda^{-1}(\cdot,0)\in L^\beta(\Omega), \tag{1.5}$$

for exponents $\alpha\geq 1$ and $\beta\geq 1/(p-1)$, satisfying

$$\frac{1}{\alpha}+\frac{1}{\beta}<\frac{p}{d-1}, \tag{1.6}$$

when $d\geq 3$. Under these conditions, in Lemma 6.1 we show that in dimension $d-1$ the weighted L^r_Λ-norm is bounded above by the weighted $W^{1,p}_\lambda$-norm for some $r>p$, up to oscillating random coefficients which can be controlled thanks to the Additive Ergodic Theorem. Eventually, in Proposition 6.3 we prove the liminf inequality by pairing the embedding result in dimension $d-1$ with a careful choice of $(d-1)$-dimensional cubes where the gradient of the cut-off function is controlled. The use of similar lower-dimensional arguments to improve the range of the integrability exponents can be also found in [5, 6] and in the more recent [4, 22]. We observe that for $p=2$ the moment conditions (1.5)-(1.6) turn out to be crucial (in fact, optimal) to prove some regularity results for the solutions of linear, non-uniformly elliptic equations in divergence form [3, 4]. Whereas the stochastic homogenisation result in [2] is proven by means of a rigorous renormalisation-group argument under assumptions which are alternative and not directly comparable to (1.6).

In the non(quasi)-convex case a gluing argument based on a simple cut-off construction becomes unfeasible. In fact, in general, such a construction does not provide any control on the energy in the transition region where the sequence is modified in order to match the boundary datum. Here we overcome this issue by imposing the following stronger bound on the integrability exponents:

$$\frac{1}{\alpha}+\frac{1}{\beta}<\frac{1}{d-1}. \tag{1.7}$$

Furthermore, we employ a rather delicate construction based on the use of convolutions with variable kernels to interpolate between the original sequence and the linear boundary datum. Far from the boundary of the transition region, the use of convolutions ensures enough regularity for the modified sequence and allows us to estimate the corresponding energy. Then, the subtle part of the proof is to determine where to let the convolution-radius shrink in order to obtain compatibility between the trace of the original sequence and that of the boundary datum in a controlled way. This is accomplished by selecting the boundary of two suitable concentric cubes via maximal-function estimates applied to Λ^α, $\lambda^{-\beta}$, and $\lambda|\nabla u|^p$. Then, to control the energy of the modified sequence close to the selected cubes' boundaries, the crucial ingredient is a Muckenhoupt-type estimate that stems from (1.7). The construction as above is the core of the proof of Lemma 7.1 which, in turn, is pivotal in the proof of the lower-bound estimate Proposition 7.3.

Eventually, in the general case we show that the bound in (1.7) is essentially optimal, at least for some choices of p, d, and m. Namely, in the periodic case we exhibit an example which shows that if (1.7) is violated, then (1.4) fails to identify the integrand of the homogenised functional. More precisely, for $p=d=m\geq 3$, $\alpha=\infty$ and $\beta<d-1$, we construct a spatially-periodic, polyconvex integrand f for which the corresponding Γ-limit is strictly smaller than one when evaluated at the identity function, while $f_{\rm hom}(I)\geq 1$, $I\in\mathbb{R}^{d\times d}$ being the identity matrix. Since $\beta<d-1$, this is done by exploiting the possibility to construct a sequence which converges to the identity function while having finite $W^{1,p}_\lambda$-norm and arbitrarily small determinant in a subset of the periodic cell with positive volume fraction.

2. Preliminaries

2.1. **Notation.** Throughout we consider $d, m \in \mathbb{N}$ with $d \geq 2$. Given a measurable set $A \subset \mathbb{R}^d$, we denote by $|A|$ its d-dimensional Lebesgue measure; if $|A| > 0$, we define $⨍_A := \frac{1}{|A|}\int_A$. Given $x_0 \in \mathbb{R}^d$ and $\rho > 0$, $B_\rho(x_0)$ denotes the open ball with radius ρ centered at x_0 while we set $Q_\rho(x_0) := x_0 + (-\rho/2, \rho/2)^d$. If $x_0 = 0$ we write B_ρ and Q_ρ, respectively. Moreover, if $\rho = 1$ we simply write Q instead of Q_1. We use the standard notation for L^p-spaces and Sobolev spaces $W^{1,p}$. For $x \in \mathbb{R}^d$ and $\xi \in \mathbb{R}^{m\times d}$ we set $\ell_\xi(x) := \xi x$. We use $\mathcal{L}^d$ for the σ-algebra of Lebesgue-measurable sets in $\mathbb{R}^d$ while the Borel σ-algebra on $\mathbb{R}^{m\times d}$ is denoted by $\mathcal{B}^{m\times d}$. In all that follows $\mathcal{A}$ denotes the collection of all open and bounded subsets of $\mathbb{R}^d$ with Lipschitz boundary. Finally, C stands for a generic positive constant that may vary from line to line, within the same expression.

2.2. **Maximal Function.** We recall here the definition of the Hardy-Littlewood maximal function on an interval together with a property which is relevant for later use.

Definition 2.1. Let $(a, b) \subset \mathbb{R}$ and $f \in L^1(a, b)$; $M(f)$ denotes the maximal function of f on (a, b) and is given by

$$M(f)(x) := \sup_{\rho\in(0,+\infty)} \frac{1}{2\rho}\int_{(x-\rho)\vee a}^{(x+\rho)\wedge b} |f(y)|\, dy, \qquad x \in (a, b).$$

The following result concerns a so-called L^1-$L^{1,\infty}$ estimate for $M(f)$. We state it in a sub-optimal way which is, though, convenient for our purposes. For the properties of the maximal function we refer the reader to [24, Section 1].

Theorem 2.2. *There exists a universal constant $C > 0$ such that for every $f \in L^1(a, b)$ it holds*

$$\mathcal{L}^1\left(\{x \in (a, b) \colon\ M(f)(x) > s\}\right) \leq C\frac{\|f\|_{L^1(a,b)}}{s},$$

for every $s \in (0, +\infty)$.

2.3. **Convolutions with variable kernels.** In this subsection we recall some basic facts concerning convolutions with variable kernels.

The following result holds true.

Lemma 2.3. *Let $A, A' \in \mathcal{A}$ with $A' \Subset A$, $u \in W^{1,1}(A;\mathbb{R}^m)$ and $r_0 > 0$. Let w be defined as*

$$w(x) := ⨍_{Q_{r(x)}(x)} u(y)\, dy \quad \text{if } x \in A', \qquad w(x) := u(x) \quad \text{if } x \in A \setminus A',$$

where $r \colon \mathbb{R}^d \to [0, +\infty)$ is given by

$$r(x) := \frac{\operatorname{dist}(x, \partial A')}{2} \wedge r_0.$$

Then, $w \in W^{1,1}(A;\mathbb{R}^m)$ and

$$|\nabla w(x)| \leq 2⨍_{Q_{r(x)}(x)} |\nabla u(y)|\, dy. \tag{2.1}$$

Proof. Let T_y denote the map $x \mapsto x + r(x)y$; since $DT_y = \mathrm{Id} + y \otimes \nabla r$. Since $|\nabla r| \leq 1/2$, for every $y \in Q$ we have that $\det DT_y \geq 1 - |y|/2 \geq 1/2$. Moreover, we notice that T_y is one-to-one and acts like the identity on $\partial A'$, hence T_y is a bi $W^{1,\infty}$-diffeomorphism form A' to itself. Hence, w has weak derivative on A and we have

$$\begin{aligned}
|\nabla w(x)| &= \left|\nabla\left(⨍_{Q_{r(x)}(x)} u(y)\, dy\right)\right| = \left|\nabla\left(\int_Q u(x + r(x)y)\, dy\right)\right| \\
&\leq \int_Q |\nabla u(x + r(x)y)| + |\nabla u(x + r(x)y) \cdot y\nabla r(x)|\, dy \\
&\leq 2\int_Q |\nabla u(x + r(x)y)|\, dy.
\end{aligned}$$

Moreover, for every nonnegative $f \in L^1(A')$ we have

$$\int_{A'} \fint_{Q_{r(x)}(x)} f(y)\, dydx = \int_{A'} \int_{Q} f(x + r(x)y)\, dydx = \int_{Q} \int_{A'} f(z) \frac{1}{\det DT_y(z)}\, dzdy \le 2\|f\|_{L^1(A')}.$$

This gives that $|w| \in L^1(A')$ and $|\nabla w| \in L^1(A')$ and hence the claim. □

2.4. **Ergodic Theory.** In this subsection we recall some basic concepts from ergodic theory together with the statements of some ergodic theorems which are relevant for our purposes.

In what follows $(\Omega, \mathcal{F}, \mathbb{P})$ denotes a complete probability space.

Definition 2.4 (Measure-preserving group action)**.** A measure-preserving additive group action on $(\Omega, \mathcal{F}, \mathbb{P})$ is a family $\tau := \{\tau_z\}_{z\in\mathbb{R}^d}$ of measurable maps $\tau_z : \Omega \to \Omega$ satisfying the following properties:

(1) the map $(\omega, z) \mapsto \tau_z(\omega)$ is $(\mathcal{F} \otimes \mathcal{L}^d, \mathcal{F})$-measurable;
(2) $\mathbb{P}(\tau_z E) = \mathbb{P}(E)$, for every $E \in \mathcal{F}$ and every $z \in \mathbb{R}^d$;
(3) $\tau_0 = \mathrm{id}_\Omega$ and $\tau_{z_1+z_2} = \tau_{z_2} \circ \tau_{z_1}$ for every $z_1, z_2 \in \mathbb{R}^d$.

If, additionally, every τ-invariant set E has either probability 0 or 1, then τ is called an *ergodic* measure-preserving group action.

Let g is a measurable function on $(\Omega, \mathcal{F}, \mathbb{P})$; we use the standard notation

$$\mathbb{E}[g] := \int_\Omega g(\omega) \mathrm{d}\mathbb{P}$$

for the expected value of g.

Let $g \in L^1(\Omega)$ and $\mathcal{F}' \subset \mathcal{F}$ be a σ-algebra, then $\mathbb{E}[g|\mathcal{F}']$ denotes the conditional expectation of g with respect to $\mathcal{F}'$. That is, $\mathbb{E}[g|\mathcal{F}']$ is the unique $L^1(\Omega)$-function satisfying

$$\int_E \mathbb{E}[g|\mathcal{F}'](\omega) \mathrm{d}\mathbb{P} = \int_E g(\omega) \mathrm{d}\mathbb{P}$$

for every $E \in \mathcal{F}'$.

We also consider the sub-σ-algebra of τ-invariant sets, that is, $\mathcal{F}_\tau := \{E \in \mathcal{F} : \tau E = E\}$.

In what follows we will make use of a variant of the Additive Ergodic Theorem which can be found in [21, Lemma 4.1].

Theorem 2.5 (Additive Ergodic Theorem)**.** *Let $g \in L^1(\Omega)$, let τ be a measure-preserving group-action on $(\Omega, \mathcal{F}, P)$. Then there exists a set $\Omega' \in \mathcal{F}$ with $\mathbb{P}(\Omega') = 1$ such that for every $\omega \in \Omega'$ and for every measurable bounded set $B \subset \mathbb{R}^d$ with $|B| > 0$ there holds*

$$\lim_{t\to+\infty} \fint_{tB} g(\tau_z\omega)\, \mathrm{d}z = \mathbb{E}[g|\mathcal{F}_\tau](\omega), \tag{2.2}$$

where $\mathcal{F}_\tau$ denotes the σ-algebra of τ-invariant sets. Moreover, if τ is ergodic (2.2) becomes

$$\lim_{t\to+\infty} \fint_{tB} g(\tau_z\omega)\, dz = \mathbb{E}[g].$$

For later use we also recall the definition of subadditive process.

Definition 2.6 (Subadditive process)**.** Let τ be a measure-preserving group-action on $(\Omega, \mathcal{F}, P)$. A subadditive process is a function $\mu : \Omega \times \mathcal{A} \to [0, +\infty)$ satisfying the following properties:

(1) (integrability) for every $A \in \mathcal{A}$, $\mu(\cdot, A)$ belongs to $L^1(\Omega)$;
(2) (stationarity) for every $\omega \in \Omega$, $A \in \mathcal{A}$, and $z \in \mathbb{R}^d$
$$\mu(\omega, A + z) = \mu(\tau_z\omega, A);$$
(3) (sub-additivity) for every $\omega \in \Omega$, for every $A \in \mathcal{A}$, and for every *finite* family $(A_i)_{i\in I} \subset \mathcal{A}$ of pairwise disjoint sets such that $A_i \subset A$ for every $i \in I$ and $|A \setminus \cup_{i\in I} A_i| = 0$, there holds
$$\mu(\omega, A) \le \sum_{i\in I} \mu(\omega, A_i).$$

If τ is ergodic then μ is called a subadditive ergodic process.

Below we state the pointwise subadditive theorem proven by Akcoglu and Krengel [1, Theorem 2.7].

Theorem 2.7 (Subadditive Ergodic Theorem)**.** *Let $\mu : \Omega \times \mathcal{A} \to [0, +\infty)$ be a subadditive process. Then there exist a $\mathcal{F}$-measurable function $\phi : \Omega \to [0, +\infty)$ and a set $\Omega' \in \mathcal{F}$ with $\mathbb{P}(\Omega') = 1$ such that*

$$\lim_{n\in\mathbb{N}, n\to+\infty} \frac{\mu(\omega, nQ)}{|nQ|} = \phi(\omega),$$

for every $\omega \in \Omega'$ and for every cube $Q := Q_k(x_0)$ with $x_0 \in \mathbb{Z}^d$ and $k \in \mathbb{N}$.

3. Set-up and statement of the main result

In this section we define the random functionals we are going to consider and we state a homogenisation result for these functionals, which holds true under two different sets of assumptions.

We preliminarily need to introduce the class of admissible integrands.

Definition 3.1 (Admissible random integrands)**.** A function $f \colon \Omega\times\mathbb{R}^d\times\mathbb{R}^{m\times d} \to [0, +\infty)$ is an *admissible random integrand* if it satisfies the following assumptions:

(A1) f is $(\mathcal{F}\otimes\mathcal{L}^d\otimes\mathcal{B}^{m\times d})$-measurable and, for every $\omega \in \Omega$ and every $x \in \mathbb{R}^d$, the map $\xi \mapsto f(\omega, x, \xi)$ is lower semicontinuous;

(A2) there exist $p \in (1, \infty)$ and $\Lambda, \lambda \colon \Omega \times \mathbb{R}^d \to [0, +\infty)$ $(\mathcal{F} \otimes \mathcal{L}^d)$-measurable functions with

$$\Lambda(\cdot, 0) \in L^1(\Omega)\,, \quad \lambda^{-1/(p-1)}(\cdot, 0) \in L^1(\Omega), \tag{3.1}$$

such that for every $\omega \in \Omega$, $x \in \mathbb{R}^d$ and $\xi \in \mathbb{R}^{m\times d}$ the following bounds hold

$$\lambda(\omega, x)|\xi|^p \le f(\omega, x, \xi) \le \Lambda(\omega, x)\,(|\xi|^p + 1)\,; \tag{3.2}$$

(A3) the random variables f, Λ, and λ are stationary. That is, there exists a measure-preserving group action $\tau := \{\tau_z\}_{z\in\mathbb{R}^d}$ such that

$$f(\tau_z\omega, x, \xi) = f(\omega, x + z, \xi), \quad \Lambda(\tau_z\omega, x) = \Lambda(\omega, x + z), \quad \lambda(\tau_z\omega, x) = \lambda(\omega, x + z), \tag{3.3}$$

for every $(z, \omega, x, \xi) \in \mathbb{R}^d \times \Omega \times \mathbb{R}^d \times \mathbb{R}^{m\times d}$.

Moreover, we consider the following additional hypotheses on the integrand f and on the weight functions Λ and λ:

(H1) the map $\xi \mapsto f(\omega, x, \xi)$ is convex, for every $\omega \in \Omega$ and $x \in \mathbb{R}^d$ and there holds

$$\Lambda(\cdot, 0) \in L^\alpha(\Omega), \qquad \lambda^{-1}(\cdot, 0) \in L^\beta(\Omega), \tag{3.4}$$

for two exponents $\alpha \ge 1$ and $\beta \ge 1/(p - 1)$. If $d \ge 3$ the exponents α and β additionally satisfy

$$\frac{1}{\alpha} + \frac{1}{\beta} < \frac{p}{d-1}; \tag{3.5}$$

(H2) the integrability assumptions (3.4) hold for exponents $\alpha \ge 1$ and $\beta \ge 1/(p-1)$, which additionally satisfy

$$\frac{1}{\alpha} + \frac{1}{\beta} < \frac{1}{d-1}. \tag{3.6}$$

Remark 3.2. We notice that in view of the stochastic integrability of the weight functions (3.4) and their stationarity (3.3) we can appeal to the Fubini Theorem to deduce that for $\mathbb{P}$-a.e. $\omega \in \Omega$ we have

$$\Lambda(\omega, \cdot) \in L^\alpha_{\rm loc}(\mathbb{R}^d)\,, \quad \lambda^{-1}(\omega, \cdot) \in L^\beta_{\rm loc}(\mathbb{R}^d).$$

Let f be as in Definition 3.1; for every fixed $\varepsilon > 0$ and $\omega \in \Omega$ we consider the integral functional $F_\varepsilon(\omega) \colon L^1_{\rm loc}(\mathbb{R}^d; \mathbb{R}^m) \times \mathcal{A} \longrightarrow [0, +\infty]$ defined as

$$F_\varepsilon(\omega)(u, A) := \begin{cases} \displaystyle\int_A f\left(\omega, \frac{x}{\varepsilon}, \nabla u\right) dx & \text{if } u \in W^{1,1}(A; \mathbb{R}^m), \\ +\infty & \text{otherwise in } L^1_{\rm loc}(\mathbb{R}^d; \mathbb{R}^m). \end{cases} \tag{3.7}$$

Before stating the main result of this paper, we observe that despite the degenerate growth conditions satisfied by f (up to imposing Dirichlet boundary conditions) the domain of the homogenised functional is $W^{1,p}(A; \mathbb{R}^m)$, as the following lemma asserts.

Lemma 3.3 (Equi-coerciveness)**.** *Let f be an admissible random integrand as in Definition 3.1. Let $(u_\varepsilon) \subset W^{1,1}(A;\mathbb{R}^m)$ be a sequence (possibly depending on ω) satisfying*

$$\sup_{\varepsilon>0} F_\varepsilon(\omega)(u_\varepsilon, A) < +\infty.$$

Then, there exists $\Omega' \in \mathcal{F}$ with $\mathbb{P}(\Omega') = 1$ such that (∇u_ε) is relatively weakly compact in $L^1(A;\mathbb{R}^{m\times d})$. Moreover, if (u_ε) is bounded in $L^1(A;\mathbb{R}^m)$ then there exists $u \in W^{1,p}(A;\mathbb{R}^m)$ such that $u_\varepsilon \rightharpoonup u$ weakly in $W^{1,1}(A;\mathbb{R}^m)$.

Proof. Since the proof only rests on the growth condition from below in (3.2), the claim is a direct consequence of [21, Lemma 4.2]. □

We are now in a position to state a *almost sure* homogenisation result for the functionals $F_\varepsilon(\omega)$. This holds true under the two alternative sets of hypotheses (H1) and (H2).

Theorem 3.4 (Homogenisation)**.** *Let f be an admissible random integrand. Assume that, additionally, either (H1) or (H2) holds true. Then, there exists $\Omega' \subset \Omega$ with $\mathbb{P}(\Omega') = 1$ such that for every $\omega \in \Omega'$ and every $A \in \mathcal{A}$ the functionals $F_\varepsilon(\omega)(\cdot, A)$ Γ-converge in $L^1_{\rm loc}(\mathbb{R}^d;\mathbb{R}^m)$ to $F_{\rm hom}(\omega)(\cdot, A)$ with $F_{\rm hom}(\omega)\colon L^1_{\rm loc}(\mathbb{R}^d;\mathbb{R}^m) \times \mathcal{A} \longrightarrow [0, +\infty]$ given by*

$$F_{\rm hom}(\omega)(u, A) := \begin{cases} \displaystyle\int_A f_{\rm hom}(\omega, \nabla u)\, dx & \text{if } u \in W^{1,p}(A;\mathbb{R}^m), \\ +\infty & \text{otherwise in } L^1_{\rm loc}(\mathbb{R}^d;\mathbb{R}^m). \end{cases} \tag{3.8}$$

The random integrand $f_{\rm hom}$ is given by the following asymptotic cell formula

$$f_{\rm hom}(\omega, \xi) := \lim_{t\to+\infty} \frac{1}{t^d} \inf\left\{ \int_{Q_t} f(\omega, x, \nabla u)\, dx \colon\ u - \ell_\xi \in W^{1,1}_0(Q_t;\mathbb{R}^m) \right\}, \tag{3.9}$$

for every $\omega \in \Omega'$ and $\xi \in \mathbb{R}^{m\times d}$.

Moreover, $f_{\rm hom}$ is $(\mathcal{T} \otimes \mathcal{B}^{m\times d})$-measurable, continuous in ξ, and for every $\omega \in \Omega$ it satisfies the following growth conditions

$$c_0(\omega)|\xi|^p \le f_{\rm hom}(\omega, \xi) \le c_1(\omega)(|\xi|^p + 1), \tag{3.10}$$

where

$$c_0(\omega) := \mathbb{E}[\lambda^{-1/(p-1)}(\cdot, 0)|\mathcal{F}_\tau]^{1-p}(\omega) \quad \text{and} \quad c_1(\omega) := \mathbb{E}[\Lambda(\cdot, 0)|\mathcal{F}_\tau](\omega). \tag{3.11}$$

Remark 3.5 (Ergodic case)**.** If (3.3) holds true for a measure-preserving group action which is ergodic, then $f_{\rm hom}$ is deterministic. Therefore, by (3.1), (3.9), (3.10), and the Dominated Convergence Theorem we get

$$f_{\rm hom}(\xi) = \lim_{t\to+\infty} \frac{1}{t^d} \int_\Omega \inf\left\{ \int_{Q_t} f(\omega, x, \nabla u)\, dx \colon\ u - \ell_\xi \in W^{1,1}_0(Q_t;\mathbb{R}^m) \right\} d\mathbb{P},$$

for every $\xi \in \mathbb{R}^{m\times d}$. Moreover, in the ergodic case $f_{\rm hom}$ satisfies the following standard growth conditions

$$c_0|\xi|^p \le f_{\rm hom}(\xi) \le c_1(|\xi|^p + 1),$$

with

$$c_0 := \int_\Omega c_0(\omega)\, d\mathbb{P} = \mathbb{E}[\lambda^{-1/(p-1)}(\cdot, 0)]^{1-p} \quad \text{and} \quad c_1 := \int_\Omega c_1(\omega)\, d\mathbb{P} = \mathbb{E}[\Lambda(\cdot, 0)].$$

The proof of Theorem 3.4 is carried out in three main steps. Under the sole assumption that f is an admissible random integrand, in the first step we establish the existence and spatial homogeneity of $f_{\rm hom}$ (cf. Proposition 4.2). Afterwards, we prove that the upper-bound inequality for F_ε holds true almost surely (cf. Proposition (5.1)). Also in this case the additional hypotheses (H1) and (H2) play no role in the proof. Eventually, the lower-bound inequality for F_ε is the most delicate part of the proof of Theorem 3.4 and it crucially relies on the convexity of f as well as on the choice of the integrability exponents as in (3.5) and (3.6), respectively (cf. Proposition 6.3 and Proposition 7.3).

Remark 3.6 (Convex case, $d = 2$)**.** We observe that if the function $\xi \mapsto f(\omega, x, \xi)$ is convex for every $\omega \in \Omega$ and $x \in \mathbb{R}^d$ and $d = 2$ then Theorem 3.4 holds true under the sole integrability assumptions (3.1). This is due to the fact that for $d = 2$ the estimate in (6.2) (cf. Lemma 6.1) holds true without imposing any constraint on the exponents other than $\alpha \ge 1$ and $\beta \ge 1/(p-1)$. As a consequence, in this case the convexity of f in ξ is enough to prove the lower-bound inequality in Proposition 6.3.

Remark 3.7 (Convex growth)**.** Let f be an admissible random integrand for which there exist two constants $c_2, c_3 > 0$ such that for every $\omega \in \Omega$, $x \in \mathbb{R}^d$, and $\xi \in \mathbb{R}^{m\times d}$ there holds

$$c_2\, g(\omega, x, \xi) \leq f(\omega, x, \xi) \leq c_3\, g(\omega, x, \xi), \tag{3.12}$$

where $g\colon \Omega \times \mathbb{R}^d \times \mathbb{R}^{m\times d} \to [0, +\infty)$ is an admissible random integrand satisfying (H1). Let $F_\varepsilon(\omega)$ be the functionals as in (3.7) with f satisfying (3.12). Then, on account of Proposition 4.2, Proposition 5.1, and by inspecting the proof of Proposition 6.3, it can be easily deduced that the conclusions of Theorem 3.4 hold true for $F_\varepsilon(\omega)$ as well.

4. Existence of the homogenised integrand

In this section we prove the existence of the spatially homogeneous integrand $f_{\rm hom}$. We will follow the classical strategy introduced in [9] based on Theorem 2.7. To this end, for every $\xi \in \mathbb{R}^{m\times d}$, $\omega \in \Omega$, and $A \in \mathcal{A}$ we define the quantity

$$\mu_\xi(\omega, A) := \inf\left\{\int_A f(\omega, x, \nabla v)\, dx\colon\ v \in \ell_\xi + W^{1,1}_0(A;\mathbb{R}^m)\right\}, \tag{4.1}$$

where f is an admissible random integrand in the sense of Definition 3.1. The following lemma states that for every fixed $\xi \in \mathbb{R}^{m\times d}$, the map $\mu_\xi\colon \Omega\times\mathcal{A} \to [0,+\infty)$ defines a subadditive process as in Definition 2.6.

Lemma 4.1. *Let $\xi \in \mathbb{R}^{m\times d}$ be fixed. Then, $\mu_\xi\colon \Omega\times\mathcal{A} \to [0,+\infty)$ defined in (4.1) is a subadditive process.*

Proof. We start by proving that $\omega \mapsto \mu_\xi(\omega, A)$ is integrable for every $A \in \mathcal{A}$. The $\mathcal{F}$-measurability of the map $\omega \mapsto \mu_\xi(\omega, A)$ follows arguing verbatim as in [21, Lemma C.1]. Indeed, the proof in [21, Lemma C.1] is not hinged on having the same growth conditions from below and above. Moreover, we have

$$0 \leq \mu_\xi(\omega, A) \leq \int_A f(\omega, x, \xi)\, dx \leq \int_A \Lambda(\omega, x)(|\xi|^p + 1)\, dx.$$

Then, using Tonelli's Theorem, assumptions (A2) and (A3), and a change of variables in Ω we get

$$\mathbb{E}[\mu_\xi(\cdot, A)] \leq \int_A \mathbb{E}[(|\xi|^p + 1)\Lambda(\cdot, 0)]\, dx = \mathbb{E}[\Lambda(\cdot,0)](|\xi|^p + 1)|A|,$$

thus, $\mu_\xi(\cdot, A) \in L^1(\Omega)$.

To prove the stationarity of μ_ξ we observe that for every $z \in \mathbb{R}^d$ and every $v \in \ell_\xi + W^{1,1}_0(A;\mathbb{R}^m)$, the function $\tilde v(x) := v(x - z) + \ell_\xi(z)$ belongs to $\ell_\xi + W^{1,1}_0(A + z;\mathbb{R}^m)$; therefore appealing to (A3) we get

$$\int_{A+z} f(\omega, x, \nabla\tilde v)\, dx = \int_A f(\omega, x + z, \nabla v)\, dx = \int_A f(\tau_z\omega, x, \nabla v)\, dx.$$

Hence passing to the minimum in $v \in \ell_\xi + W^{1,1}_0(A;\mathbb{R}^m)$ on both sides, we obtain

$$\mu_\xi(\omega, A + z) = \mu_\xi(\tau_z\omega, A),$$

for every $\omega \in \Omega$, $A \in \mathcal{A}$, and $z \in \mathbb{R}^d$.

Finally, we are left prove the subadditivity of μ_ξ as a set function. To this end, fix $\omega\in\Omega$ and $A \in \mathcal{A}$ and let $(A_i)_{i\in I} \subset \mathcal{A}$ be a finite family of pairwise disjoint subsets of A such that $|A \setminus \cup_{i\in I} A_i| = 0$. Let moreover $\eta > 0$, for every $i \in I$ by the definition of μ_ξ we can find $v_i \in \ell_\xi + W^{1,1}_0(A_i;\mathbb{R}^m)$ such that

$$\int_{A_i} f(\omega, x, \nabla v_i)\, dx \leq \mu_\xi(\omega, A_i) + \frac{\eta}{2^i}.$$

Thus, defining $v := \ell_\xi + \sum_{i\in I} v_i\chi_{A_i} \in \ell_\xi + W^{1,1}_0(A;\mathbb{R}^m)$, by additivity and locality we have

$$\mu_\xi(\omega, A) \leq \sum_{i\in I}\int_{A_i} f(\omega, x, \nabla v_i(x))\, dx \leq \sum_{i\in I}\mu_\xi(\omega, A_i) + \eta.$$

Therefore the subadditivity of μ_ξ follows by the arbitrariness of $\eta > 0$. $\square$

On account of Lemma 4.1 we are now in a position to prove the main result of this section.

Proposition 4.2. *Let f be an admissible random integrand. Then, there exists $\Omega' \in \mathcal{F}$ with $\mathbb{P}(\Omega') = 1$ such that for every $\omega \in \Omega'$, $x_0 \in \mathbb{R}^d$, $\rho > 0$, and $\xi \in \mathbb{R}^{m\times d}$, there exists the limit*

$$\lim_{t\to+\infty} \frac{1}{|tQ_\rho(x_0)|} \inf\left\{\int_{tQ_\rho(x_0)} f(\omega, x, \nabla u)\,dx\colon\ u - \ell_\xi \in W^{1,1}_0(tQ_\rho(x_0);\mathbb{R}^m)\right\}$$
$$= \lim_{t\to+\infty} \frac{1}{t^d} \inf\left\{\int_{Q_t(0)} f(\omega, x, \nabla u)\,dx\colon\ u - \ell_\xi \in W^{1,1}_0(Q_t(0);\mathbb{R}^m)\right\} =: f_{\rm hom}(\omega,\xi)$$

and defines a $(\mathcal{F}\otimes\mathcal{B}^{m\times d})$-measurable function on $\Omega\times\mathbb{R}^{m\times d}$.

The function $f_{\rm hom}$ satisfies the following growth conditions

$$c_0(\omega)|\xi|^p \le f_{\rm hom}(\omega,\xi) \le c_1(\omega)(|\xi|^p+1), \tag{4.2}$$

where $c_0(\omega), c_1(\omega)$ are as in (3.11). Moreover, $\xi \mapsto f_{\rm hom}(\omega,\xi)$ is continuous. If $\xi \mapsto f(\omega,x,\xi)$ is convex for every $\omega\in\Omega$ and $x\in\mathbb{R}^d$, then $\xi\mapsto f_{\rm hom}(\omega,\xi)$ is also convex.

If additionally f is stationary with respect to an ergodic measure-preserving group action, then $f_{\rm hom}$ is deterministic and given by

$$f_{\rm hom}(\xi) = \lim_{t\to+\infty}\frac{1}{t^d}\int_\Omega \inf\left\{\int_{Q_t(0)} f(\omega,x,\nabla u)\,dx\colon\ u-\ell_\xi\in W^{1,1}_0(Q_t(0);\mathbb{R}^m)\right\} d\mathbb{P}. \tag{4.3}$$

Proof. Let $\xi\in\mathbb{R}^{m\times d}$ be fixed; by Lemma 4.1 we know that μ_ξ defines a subadditive process. Hence, we can apply Theorem 2.7 to deduce the existence of a set $\Omega_\xi\in\mathcal{F}$ with $\mathbb{P}(\Omega_\xi)=1$ and of an $\mathcal{F}$-measurable function ϕ_ξ such that for every $\omega\in\Omega_\xi$

$$\phi_\xi(\omega) = \lim_{n\in\mathbb{N}, n\to+\infty} \frac{\mu_\xi(\omega, nQ_k(z))}{|nQ_k(z)|}, \tag{4.4}$$

for every $k\in\mathbb{N}$ and $z\in\mathbb{Z}^d$.

We now want to extend the convergence in (4.4) to an arbitrary sequence of positive real numbers $t\to+\infty$ and to every cube $Q_\rho(x)$ with side-length $\rho>0$ and center $x_0\in\mathbb{R}^d$. To this end we closely follow the same method used in the non-degenerate case, now appealing both the fact that $\Lambda(\cdot,0)\in L^1(\Omega)$ and to Theorem 2.5. In what follows, every time we invoke Theorem 2.5 we tacitly assume that it holds for every $\omega\in\Omega_\xi$.

Let $\rho>0$, $x_0\in\mathbb{R}^d$ be fixed; to simplify the notation set $Q:=Q_\rho(x_0)$. For $\delta>0$ let $\rho^\pm_\delta>0$ and $x^\pm_\delta\in\mathbb{Q}^d$ be such that

$$Q^-_\delta \Subset Q \Subset Q^+_\delta \qquad \text{and} \qquad |Q^-_\delta| \ge (1-\delta)|Q^+_\delta|, \tag{4.5}$$

where $Q^\pm_\delta := Q_{\rho^\pm_\delta}(x^\pm_\delta)$. We can find $R:=R_\delta\in\mathbb{Z}^+$ with the property that the cubes $RQ^\pm_\delta$ have integer center and integer vertices. Let $t\gg1$ and set $t_- := \lfloor t\rfloor$. By the subadditivity of μ_ξ and the upper bound in (3.2) we have

$$\frac{1}{|tRQ|}\mu_\xi(\omega,tRQ) \le \frac{1}{|t_-RQ^-_\delta|}\mu_\xi(\omega,t_-RQ^-_\delta) + \frac{|\xi|^p+1}{|tRQ|}\int_{tRQ\setminus t_-RQ^-_\delta}\Lambda(\omega,x)\,dx. \tag{4.6}$$

We observe that by definition $t_-RQ^-_\delta \subset tRQ$, therefore

$$\frac{1}{|tRQ|}\int_{tRQ\setminus t_-RQ^-_\delta}\Lambda(\omega,x)\,dx \le \fint_{tRQ}\Lambda(\omega,x)\,dx - \left(\frac{t_-}{t}\right)^d\frac{|Q^-_\delta|}{|Q|}\fint_{t_-RQ^-_\delta}\Lambda(\omega,x)\,dx.$$

Hence, letting $t\to+\infty$ in (4.6), in view of (4.4), (4.5), and Theorem 2.5 we get

$$\limsup_{t\to+\infty}\frac{1}{|tRQ|}\mu_\xi(\omega,tRQ) \le \phi_\xi(\omega) + \delta(|\xi|^p+1)\mathbb{E}[\Lambda(\cdot,0)|\mathcal{F}_\tau](\omega).$$

In turn, by arbitrariness of $\delta>0$, this yields

$$\limsup_{t\to+\infty}\frac{1}{|tRQ|}\mu_\xi(\omega,tRQ) \le \phi_\xi(\omega). \tag{4.7}$$

Then, choosing Q_δ^+ and $t^+ := \lceil t \rceil$ an analogous argument gives

$$\liminf_{t\to+\infty} \frac{1}{|tRQ|}\mu_\xi(\omega, tRQ) \geq \phi_\xi(\omega). \tag{4.8}$$

Eventually, by combining (4.7) and (4.8) we have that

$$\phi_\xi(\omega) = \lim_{t\to+\infty} \frac{\mu_\xi(\omega, tQ_\rho(x_0))}{|tQ_\rho(x_0)|} = \lim_{t\to+\infty} \frac{\mu_\xi(\omega, Q_t)}{t^d}, \tag{4.9}$$

for every $\omega \in \Omega_\xi$, $\rho > 0$, and $x_0 \in \mathbb{R}^d$, as desired.

Set

$$\Omega' := \bigcap_{\xi\in\mathbb{Q}^{m\times d}} \Omega_\xi;$$

clearly, $\Omega' \in \mathcal{F}$ and $\mathbb{P}(\Omega') = 1$. Moreover, ϕ_ξ is well defined for every $\omega \in \Omega'$, whenever $\xi \in \mathbb{Q}^{m\times d}$. We now claim that the limit in (4.9) is well defined for every $\xi \in \mathbb{R}^{m\times d}$ when $\omega \in \Omega'$. To this end, we introduce the auxiliary functions $\phi_\rho^\pm\colon \Omega' \times \mathbb{R}^d \times \mathbb{R}^{m\times d} \to [0, +\infty)$ defined as

$$\phi_\rho^-(\omega, x, \xi) := \liminf_{t\to+\infty} \frac{\mu_\xi(\omega, Q_{t\rho}(tx))}{t^d\rho^d}, \qquad \phi_\rho^+(\omega, x, \xi) := \limsup_{t\to+\infty} \frac{\mu_\xi(\omega, Q_{t\rho}(tx))}{t^d\rho^d}. \tag{4.10}$$

By definition of Ω' we have that for every $\rho > 0$, $x \in \mathbb{R}^d$, and $\xi \in \mathbb{Q}^{m\times d}$ there holds

$$\phi_\rho^-(\omega, x, \xi) = \phi_\rho^+(\omega, x, \xi) = \phi_\xi(\omega). \tag{4.11}$$

Let $\delta \in (0, 1/2d)$ be fixed; we have

$$Q_{t(1-\delta)\rho}(tx) \Subset Q_{t\rho}(tx) \Subset Q_{t(1+\delta)\rho}(tx).$$

Fix $\xi \in \mathbb{R}^{m\times d}$ and let $(\xi_j) \subset \mathbb{Q}^{m\times d}$ be such that $\xi_j \to \xi$, as $j \to +\infty$. Take $v \in \ell_\xi + W_0^{1,1}(Q_{t\rho}(tx); \mathbb{R}^m)$ and extend it as ℓ_ξ outside $Q_{t\rho}(tx)$. Let $\varphi \in C^\infty(\mathbb{R}^d; [0, 1])$ be a cut-off function with the following properties

$$\varphi \equiv 1 \ \text{ on } Q_{t\rho}(tx), \qquad \varphi \equiv 0 \ \text{ on } \mathbb{R}^d \setminus Q_{t(1+\delta)\rho}(tx), \qquad \|\nabla\varphi\|_{L^\infty(\mathbb{R}^d)} \leq \frac{2}{\delta\rho t}.$$

Set $v_j := \varphi v + (1-\varphi)\ell_{\xi_j} \in \ell_{\xi_j} + W_0^{1,1}(Q_{(t(1+\delta)\rho)}(tx); \mathbb{R}^m)$. By (3.2) we have

$$\begin{aligned}
\mu_{\xi_j}(\omega, Q_{t(1+\delta)\rho}(tx)) &\leq \int_{Q_{t(1+\delta)\rho}(tx)} f(\omega, y, \nabla v_j)\, dy \\
&\leq \int_{Q_{t\rho}(tx)} f(\omega, y, \nabla v)\, dy + \int_{Q_{t(1+\delta)\rho}(tx)\setminus Q_{t\rho}(tx)} \Lambda(\omega, y)\left(|\nabla v_j|^p + 1\right)\, dy \\
&\leq \int_{Q_{t\rho}(tx)} f(\omega, y, \nabla v)\, dy + \int_{Q_{t(1+\delta)\rho}(tx)\setminus Q_{t\rho}(tx)} \Lambda(\omega, y)\left(|\nabla\varphi|^p|\xi-\xi_j|^p|y|^p + |\xi|^p + |\xi_j|^p + 1\right) dy \\
&\leq \int_{Q_{t\rho}(tx)} f(\omega, y, \nabla v) dy + C\left(\left(\frac{\rho+|x|}{\delta\rho}\right)^p |\xi-\xi_j|^p + |\xi|^p + |\xi_j|^p + 1\right) \int_{Q_{t(1+\delta)\rho}(tx)\setminus Q_{t\rho}(tx)} \Lambda(\omega, y) dy.
\end{aligned}$$

Hence, by the arbitrariness of $v \in \ell_\xi + W_0^{1,1}(Q_{(t\rho)}tx; \mathbb{R}^m)$, we get

$$\begin{aligned}
\mu_{\xi_j}(\omega, Q_{t(1+\delta)\rho}(tx)) &\leq \mu_\xi(\omega, Q_{t\rho}(tx)) \\
&\quad + C\left(\left(\frac{\rho+|x|}{\delta\rho}\right)^p |\xi-\xi_j|^p + |\xi|^p + |\xi_j|^p + 1\right) \int_{Q_{t(1+\delta)\rho}(tx)\setminus Q_{t\rho}(tx)} \Lambda(\omega, y) dy.
\end{aligned}$$

Then, dividing the expression above by $(t\rho)^d$, taking the lim inf as $t \to +\infty$, recalling (4.10), (3.1), (3.3), and appealing to Theorem 2.5 we obtain

$$\begin{aligned}
(1+\delta)^d \phi_\rho^-\left(\omega, \frac{x}{1+\delta}, \xi_j\right) &\leq \phi_\rho^-(\omega, x, \xi) \\
&\quad + \left(\left(\frac{\rho+|x|}{\delta\rho}\right)^p |\xi-\xi_j|^p + |\xi|^p + |\xi_j|^p + 1\right)(1-2d\delta)\mathbb{E}[\Lambda(\cdot, 0)|\mathcal{F}_\tau](\omega).
\end{aligned} \tag{4.12}$$

With a similar construction now using the cubes $Q_{t\rho}(tx)$ and $Q_{t(1-\delta)\rho}(tx)$ we can also show that

$$\phi_\rho^+(\omega,x,\xi) \le (1-\delta)^d \phi_\rho^+\left(\omega,\frac{x}{1-\delta},\xi_j\right) + \left(\left(\frac{\rho+|x|}{\delta\rho}\right)^p|\xi-\xi_j|^p + |\xi|^p + |\xi_j|^p + 1\right)(1-2d\delta)\mathbb{E}[\Lambda(\cdot,0)|\mathcal{F}_\tau](\omega). \quad (4.13)$$

On the other hand, since $\xi_j \in \mathbb{Q}^{m\times d}$ we have

$$\phi_\rho^+\left(\omega,\frac{x}{1-\delta},\xi_j\right) = \phi_\rho^-\left(\omega,\frac{x}{1+\delta},\xi_j\right) = \phi_{\xi_j}(\omega), \quad (4.14)$$

for every $\omega\in\Omega'$, $\rho>0$, $x\in\mathbb{R}^d$, and $j\in\mathbb{N}$. Thus, by combining (4.12)–(4.14), passing to the liminf as $j\to+\infty$, and sending $\delta\to 0$ we get

$$\liminf_{j\to+\infty}\phi_{\xi_j}(\omega) \le \phi_\rho^+(\omega,x,\xi) \le \phi_\rho^-(\omega,x,\xi) \le \liminf_{j\to+\infty}\phi_{\xi_j}(\omega).$$

From the latter we infer that ϕ_ρ^+ and ϕ_ρ^- coincide on $\Omega'\times\mathbb{R}^{m\times d}$ and do not depend on $\rho>0$ and $x\in\mathbb{R}^d$. That is, for every $\xi\in\mathbb{R}^{m\times d}$ and $\omega\in\Omega'$ we can set

$$\phi_\xi(\omega) := \phi_\rho^\pm(\omega,x,\xi) = \lim_{t\to+\infty}\frac{1}{|tQ_\rho(x)|}\mu_\xi(\omega,tQ_\rho(x)) = \lim_{t\to+\infty}\frac{1}{t^d}\mu_\xi(\omega,Q_t). \quad (4.15)$$

The function $\omega\mapsto\phi_\xi(\omega)$ is $\mathcal{F}$-measurable on Ω' for every $\xi\in\mathbb{R}^{m\times d}$ and, in view of (4.12), (4.13) and (4.15), it is continuous in ξ for every $\omega\in\Omega'$. Hence ϕ_ξ is $(\mathcal{F}\otimes\mathcal{B}^{m\times d})$-measurable on $\Omega'\times\mathbb{R}^{m\times d}$.

Moreover, we observe that if $\xi\mapsto f(\omega,x,\xi)$ is convex for every $x\in\mathbb{R}^d$ and every $\omega\in\Omega$, then from the definition of μ_ξ it follows that $\xi\mapsto\mu_\xi$ is convex and thus, $\xi\mapsto\phi_\xi(\omega)$ is convex for every $\omega\in\Omega'$.

Now, define $f_{\mathrm{hom}}\colon\Omega\times\mathbb{R}^{m\times d}\to[0,+\infty)$ as

$$f_{\mathrm{hom}}(\omega,\xi) := \begin{cases}\phi_\xi(\omega) & \text{if } \omega\in\Omega',\\ \mathbb{E}[\Lambda(\cdot,0)|\mathcal{F}_\tau](\omega)(|\xi|^p+1) & \text{if } \omega\in\Omega\setminus\Omega',\end{cases}$$

and we observe that by definition f_{hom} is $(\mathcal{F}\otimes\mathcal{B}^{m\times d})$-measurable on $\Omega\times\mathbb{R}^{m\times d}$.

We now show that f_{hom} satisfies (4.2). To this end, it suffices to consider the case $\omega\in\Omega'$. In view of (3.1)-(3.3) and Theorem 2.5, we readily find

$$f_{\mathrm{hom}}(\omega,\xi) = \lim_{t\to+\infty}\frac{\mu_\xi(\omega,Q_t)}{t^d} \le \lim_{t\to+\infty}\fint_{Q_t}\Lambda(\omega,x)(|\xi|^p+1)\,dx = \mathbb{E}[\Lambda(\cdot,0)|\mathcal{F}_\tau](\omega)(|\xi|^p+1)$$

and hence the upper bound.

For the lower bound take $v\in\ell_\xi+W_0^{1,1}(Q;\mathbb{R}^m)$. By the Hölder inequality we have

$$|\xi| = \left|\fint_{Q_t}\nabla v\,dy\right| \le \fint_{Q_t}|\nabla v|\,dy \le \left(\fint_{Q_t}\lambda(\omega,y)|\nabla v|^p\,dy\right)^p\left(\fint_{Q_t}\lambda^{-1/(p-1)}(\omega,y)\,dy\right)^{(p-1)/p}.$$

Therefore, again using (3.2) we have

$$|\xi|^p \le \left(\fint_{Q_t}\lambda^{-1/(p-1)}(\omega,y)\,dy\right)^{p-1}\frac{1}{t^d}\int_{Q_t}f(\omega,x,\nabla v)\,dy$$

and passing to the infimum on $v\in\ell_\xi+W_0^{1,1}(Q;\mathbb{R}^m)$ we infer

$$\left(\fint_{Q_t}\lambda^{-1/(p-1)}(\omega,y)\,dy\right)^{1-p}|\xi|^p \le \frac{1}{t^d}\mu_\xi(\omega,Q_t).$$

Then, passing to the limit as $t\to+\infty$ and appealing to (3.1), (3.3), and to Theorem 2.5 we get

$$\mathbb{E}[\lambda(\cdot,0)^{-1/(p-1)}|\mathcal{F}_\tau]^{p-1}(\omega)|\xi|^p \le f_{\mathrm{hom}}(\omega,\xi)$$

and hence the lower bound.

Eventually, we are left to show that f_{hom} is deterministic if τ is ergodic. By definition of ergodicity, this is equivalent to showing that f_{hom} is invariant under the group action τ; that is,

$$f_{\mathrm{hom}}(\tau_z\omega,\xi) = f_{\mathrm{hom}}(\omega,\xi). \quad (4.16)$$

for every $\omega \in \Omega'$, $z \in \mathbb{R}^d$, and $\xi \in \mathbb{R}^{m\times d}$.

Let $\omega \in \Omega'$, $\xi \in \mathbb{R}^{m\times d}$, and $z \in \mathbb{R}^d$ be fixed. By the stationarity of μ_ξ, for every $t > 0$ we have

$$\mu_\xi(\tau_z\omega, tQ) = \mu_\xi(\omega, t(Q + z/t)), \tag{4.17}$$

where Q is an arbitrary open cube in $\mathbb{R}^d$.

Let $Q' \Subset Q \Subset Q''$, for $t \gg 1$ we also have that $Q' \Subset Q + z/t \Subset Q''$. By the subadditivity of μ_ξ we get

$$\frac{\mu_\xi(\omega, t(Q + z/t))}{|tQ|} \leq \frac{\mu_\xi(\omega, tQ')}{|tQ'|} + \frac{|\xi|^p + 1}{|tQ|}\int_{tQ''\setminus tQ'} \Lambda(\omega, y)\, dy.$$

Recalling (4.17), taking the lim sup as $t \to +\infty$, and invoking Theorem 2.5 we get

$$\limsup_{t\to+\infty} \frac{\mu_\xi(\tau_z\omega, tQ)}{|tQ|} \leq f_{\rm hom}(\omega, \xi) + (|\xi|^p + 1)\frac{|Q''| - |Q'|}{|Q|}\mathbb{E}[\Lambda(\cdot, 0)|\mathcal{F}_\tau](\omega).$$

In turn, since Q', Q'' where arbitrary cubes, this yields

$$\limsup_{t\to+\infty} \frac{\mu_\xi(\tau_z\omega, tQ)}{|tQ|} \leq f_{\rm hom}(\omega, \xi). \tag{4.18}$$

An analogous argument also shows that

$$\liminf_{t\to+\infty} \frac{\mu_\xi(\tau_z\omega, tQ)}{|tQ|} \geq f_{\rm hom}(\omega, \xi). \tag{4.19}$$

Gathering (4.18) and (4.19) gives both that $\tau_z\omega \in \Omega'$ and (4.16), hence the claim is achieved.

Finally, we observe that (4.3) is a direct consequence of (3.1), (4.2), the very definition of $f_{\rm hom}$, and the Dominated Convergence Theorem. □

5. Upper bound

In this section we prove the upper bound inequality. We notice that the proof is the same both in the convex and in the general case.

To prove the existence of a recovery sequence for $u \in W^{1,p}(A;\mathbb{R}^m)$ we argue by density. That is, we first approximate u with a piecewise affine function on A and then, to construct a recovery sequence for u, we focus on a single simplex T where the approximating piecewise affine function has constant gradient.

Proposition 5.1 (Upper bound)**.** *Let f be an admissible random integrand. Then there exists $\Omega' \in \mathcal{F}$ with $\mathbb{P}(\Omega') = 1$ such that for every $\omega \in \Omega'$ and every $u \in L^1_{\rm loc}(\mathbb{R}^d;\mathbb{R}^m)$ there exists a sequence $(u_\varepsilon) \subset L^1_{\rm loc}(\mathbb{R}^d;\mathbb{R}^m)$ with the property that $u_\varepsilon \to u$ in $L^1_{\rm loc}(\mathbb{R}^d;\mathbb{R}^m)$ and*

$$\limsup_{\varepsilon\to 0} F_\varepsilon(\omega)(u_\varepsilon, A) \leq F_{\rm hom}(\omega)(u, A),$$

for every $A \in \mathcal{A}$, where $F_{\rm hom}$ is as in (3.8).

Proof. Let $u \in L^1_{\rm loc}(\mathbb{R}^d;\mathbb{R}^m)$ and $A \in \mathcal{A}$ and assume that $u \in W^{1,p}(A;\mathbb{R}^m)$ otherwise there is nothing to prove.

For this proof, it is convenient to recall the definition of Γ-lim sup of a sequence of functionals. That is $F''(\omega)\colon L^1_{\rm loc}(\mathbb{R}^d;\mathbb{R}^m) \times \mathcal{A} \longrightarrow [0, +\infty]$ is the functional defined as

$$F''(\omega)(u, A) := \inf\left\{\limsup_{\varepsilon\to 0} F_\varepsilon(\omega)(u_\varepsilon, A)\colon u_\varepsilon \to u \text{ in } L^1_{\rm loc}(\mathbb{R}^d;\mathbb{R}^m)\right\}.$$

It is well-known that $u \mapsto F''(\omega)(u, \cdot)$ is lower semicontinuous with respect to the strong $L^1_{\rm loc}(\mathbb{R}^d;\mathbb{R}^m)$-convergence.

With a little abuse of notation let $\Omega' \in \mathcal{F}$ be a set with $\mathbb{P}(\Omega') = 1$ where Lemma 3.3, Theorem 2.5, and Proposition 4.2 hold true. We now claim that, for every $\omega \in \Omega'$, $A \in \mathcal{A}$, and $u \in W^{1,p}(A;\mathbb{R}^m)$ there holds

$$F''(\omega)(u, A) \leq F_{\rm hom}(\omega)(u, A). \tag{5.1}$$

Since A has Lipschitz boundary, without loss of generality we assume that $u \in W^{1,p}(\mathbb{R}^d;\mathbb{R}^m)$. Moreover, we observe that by virtue of the growth conditions from above in (4.2) satisfied by $f_{\rm hom}$ and by its continuity, the functional $F_{\rm hom}$ is continuous with respect to the strong $W^{1,p}$-convergence. Thus, by standard density arguments, it is enough to prove (5.1) for piecewise affine functions u.

To this end, let u be a piecewise affine function on $\mathbb{R}^d$, that is, u is continuous and there exists a locally finite triangulation $\{T_i\}_{i\in\mathbb{N}}$ of $\mathbb{R}^d$ into d-simplices such that $u|_{T_i}$ is affine for every $i \in \mathbb{N}$. We now construct a recovery sequence (u^i_ε) for $u|_{T_i}$ such that $u^i_\varepsilon \in u + W^{1,1}_0(T_i;\mathbb{R}^m)$. In this way, the recovery sequences (u^i_ε) can be glued together to define a global recovery sequence for u on $\mathbb{R}^d$.

To this end, fix a single simplex T_i such that $u|_{T_i} = \xi_i x + b_i$ where $\xi_i \in \mathbb{R}^{m\times d}$ and $b_i \in \mathbb{R}^m$. Given $\delta > 0$ small we consider the collection of cubes

$$\mathcal{Q}_\delta(T_i) := \{Q_\delta(\delta z) : \; z \in \mathbb{Z}, \; Q_\delta(\delta z) \subset T_i\}.$$

For convenience, we use the shorthand notation $Q := Q_\delta(\delta z)$ and consider the interior approximation of T_i defined as $T_{i,\delta} := \cup_{Q\in\mathcal{Q}_\delta(T_i)} Q$. We now define a sequence $(u^i_{\varepsilon,\delta})$ on T_i. Recalling (4.1), for every $\varepsilon > 0$ we can find $v^i_{\varepsilon,Q} \in W^{1,1}_0(Q/\varepsilon;\mathbb{R}^m)$ satisfying

$$\int_{Q/\varepsilon} f(\omega, x, \xi_i + \nabla v^i_{\varepsilon,Q}(x))\, dx \le \mu_\xi(\omega, Q/\varepsilon) + \varepsilon. \tag{5.2}$$

Set $u^i_{\varepsilon,\delta} := \xi_i x + b_i + \sum_{Q\in\mathcal{Q}_\delta(T_i)} \varepsilon v^i_{\varepsilon,Q}(x/\varepsilon)\chi_Q(x)$. Since $v^i_{\varepsilon,Q} \in W^{1,1}_0(Q/\varepsilon;\mathbb{R}^m)$ for every $Q \in \mathcal{Q}_\delta(T_i)$ we have that $u^i_{\varepsilon,\delta} \in u + W^{1,1}_0(T_i;\mathbb{R}^m)$. Moreover, we recall that by Proposition 4.2 for every $\omega \in \Omega'$ and every $\xi \in \mathbb{R}^{m\times d}$ there holds

$$\lim_{\varepsilon\to 0} \frac{\mu_\xi(\omega, Q/\varepsilon)}{|Q/\varepsilon|} = f_{\rm hom}(\omega, \xi). \tag{5.3}$$

Moreover, by (3.2) we have

$$\begin{aligned} F_\varepsilon(\omega)(u^i_{\varepsilon,\delta}, T_i) &= \sum_{Q\in\mathcal{Q}_\delta(T_i)} \int_Q f\left(\omega, \frac{x}{\varepsilon}, \xi_i + \nabla v^i_{\varepsilon,Q}\left(\frac{x}{\varepsilon}\right)\right) dx + \int_{T_i\setminus T_{i,\delta}} f\left(\omega, \frac{x}{\varepsilon}, \xi_i\right) dx \\ &\le \sum_{Q\in\mathcal{Q}_\delta(T_i)} \varepsilon^d \int_{Q/\varepsilon} f\left(\omega, y, \xi_i + \nabla v^i_{\varepsilon,Q}\right) dy + \varepsilon^d \int_{(T_i\setminus T_{i,\delta})/\varepsilon} (1 + |\xi_i|^p)\Lambda(\omega, y)\, dy. \end{aligned}$$

Then, passing to the limit as $\varepsilon \to 0$ and recalling (5.2), (5.3) we get

$$\limsup_{\varepsilon\to 0} F_\varepsilon(\omega)(u^i_{\varepsilon,\delta}, T_i) \le \sum_{Q\in\mathcal{Q}_\delta(T_i)} |Q| f_{\rm hom}(\xi_i) + \limsup_{\varepsilon\to 0} \varepsilon^d \int_{(T_i\setminus T_{i,\delta})/\varepsilon} (|\xi_i|^p + 1)\Lambda(\omega, y)\, dy.$$

Hence, by virtue of (3.1), (3.3), Theorem 2.5, and also using that $\nabla u|_{T_i} = \xi_i$, we obtain

$$\limsup_{\varepsilon\to 0} F_\varepsilon(\omega)(u^i_{\varepsilon,\delta}, T_i) \le \int_{T_i} f_{\rm hom}(\nabla u)\, dx + |T_i \setminus T_{i,\delta}|(|\xi_i|^p + 1)\mathbb{E}[\Lambda(\cdot, 0)|\mathcal{F}_\tau](\omega). \tag{5.4}$$

Next set

$$u_{\varepsilon,\delta} := \begin{cases} u^i_{\varepsilon,\delta} & \text{in } \; T_i \quad \text{if } \; T_i \cap A \neq \emptyset, \\ u & \text{otherwise in } \mathbb{R}^d; \end{cases}$$

by (5.4) we have

$$\limsup_{\varepsilon\to 0} F_\varepsilon(\omega)(u_{\varepsilon,\delta}, A) \le \int_A f_{\rm hom}(\nabla u)\, dx + \sum_{i:\, T_i\cap\partial A\neq\emptyset} \int_{T_i} f_{\rm hom}(\nabla u)\, dx + o(1), \tag{5.5}$$

as $\delta \to 0$. We now prove that for ε and δ small enough, $u_{\varepsilon,\delta}$ is L^1-close to u . By combining (5.5), Lemma 3.3, and the Poincaré Inequality, we infer that $(u_{\varepsilon,\delta})$ is bounded in $W^{1,1}(A;\mathbb{R}^m)$ uniformly in ε and that, up to subsequences (not relabelled), $u_{\varepsilon,\delta} \to \tilde{u}_\delta$ as $\varepsilon \to 0$ in $L^1(A;\mathbb{R}^m)$, for some $\tilde{u}_\delta \in W^{1,p}(A;\mathbb{R}^m)$.

Again using the Poincaré inequality on each cube $Q \in \mathcal{Q}_\delta(T_i)$ we have

$$\begin{aligned}\|u-\tilde u_\delta\|_{L^1(T_i;\mathbb{R}^m)} &= \lim_{\varepsilon\to 0}\sum_{Q\in\mathcal{Q}_\delta(T_i)}\int_Q \left|\varepsilon v^i_{\varepsilon,Q}\left(\frac{x}{\varepsilon}\right)\right|\,dx \le C\delta \liminf_{\varepsilon\to 0}\sum_{Q\in\mathcal{Q}_\delta(T_i)}\int_Q \left|\nabla v^i_{\varepsilon,Q}\left(\frac{x}{\varepsilon}\right)\right|\,dx\\ &\le C\delta\bigg(|\xi_i||T_i| + \liminf_{\varepsilon\to 0}\sum_{Q\in\mathcal{Q}_\delta(T_i)}\int_Q\left|\xi_i+\nabla v^i_{\varepsilon,Q}\left(\frac{x}{\varepsilon}\right)\right|\,dx\bigg)\\ &\le C\delta\bigg(|\xi_i||T_i| + \liminf_{\varepsilon\to 0}\int_{T_i}\left|\nabla u^i_{\varepsilon,\delta}(x)\right|\,dx\bigg)\\ &\le C\delta\bigg(|\xi_i||T_i| + \lim_{\varepsilon\to 0}\left(\int_{T_i} f\left(\omega,\frac{x}{\varepsilon},\nabla u^i_\varepsilon\right)\,dx\right)^{1/p}\left(\int_{T_i}\lambda^{-1/(p-1)}\left(\omega,\frac{x}{\varepsilon}\right)\right)^{(p-1)/p}\bigg).\end{aligned}$$

Then, appealing to Theorem 2.5 and (5.4) we deduce that the right hand side of the expression above vanishes as $\delta\to 0$. Hence, $\tilde u_\delta \to u$ in $L^1(T_i;\mathbb{R}^m)$ as $\delta\to 0$ and similarly $\tilde u_\delta\to u$ in $L^1_{\rm loc}(\mathbb{R}^d;\mathbb{R}^m)$. Therefore, $u_{\varepsilon,\delta}\in$) converges to $\tilde u_\delta$ in $L^1(A;\mathbb{R}^m)$ as $\varepsilon\to 0$, which converges in turn to u in $L^1(A;\mathbb{R}^m)$ as $\delta\to 0$.

Now, let $\mathcal{T}$ be the collection of all $T\in(T_i)_{i\in\mathbb{N}}$ such that $T\Subset A$. Using the lower semicontinuity of $u\mapsto F''(\omega)(u,A)$ we infer

$$\begin{aligned}F''(\omega)(u,A) &\le \liminf_{\delta\to 0}F''(\omega)(\tilde u_\delta,A)\le \liminf_{\delta\to 0}\sum_{T\in\mathcal{T}}\limsup_{\varepsilon\to 0}F_\varepsilon(\omega)(u_{\varepsilon,\delta},T)\le\sum_{T\in\mathcal{T}}\int_T f_{\rm hom}(\nabla u)\,dx\\ &\le \int_A f_{\rm hom}(\nabla u)\,dx + c_1(\omega)\,|A\setminus\cup_{T\in\mathcal{T}}T|\,\sup_A|\nabla u|^p.\end{aligned}$$

Since by refining the triangulation associated to u, we can make $|A\setminus\cup_{T\in\mathcal{T}}T|$ arbitrarily small, using a standard diagonal argument, we achieve (5.1) for every u piecewise affine. This completes the proof of the upper-bound inequality. □

6. Lower bound in the convex case

This section is devoted to the proof of the lower bound in the convex case.

6.1. **Some technical lemmas.** We preliminarily need to prove two technical results which will be used in the proof of the liminf inequality.

We start with an embedding result. In the latter the dependence of the weight functions on the random parameter ω is not relevant, therefore, to simplify the notation, below we are going to omit it.

Lemma 6.1. *Let $\Lambda,\lambda\colon\mathbb{R}^d\to[0,+\infty)$ be such that $\Lambda\in L^\alpha_{\rm loc}(\mathbb{R}^d)$, $\lambda^{-1}\in L^\beta_{\rm loc}(\mathbb{R}^d)$, where $\alpha\ge 1$ and $\beta\ge 1/(p-1)$; if $d\ge 3$, assume moreover that*

$$\frac{1}{\alpha}+\frac{1}{\beta}<\frac{p}{d-1}. \tag{6.1}$$

Then, there exist an exponent $r\in(p,\infty)$ and a constant $C=C(p,d,r,\alpha,\beta)>0$ such that

$$\begin{aligned}\frac{1}{\rho^r}\fint_{\partial Q_\rho(x_0)}\Lambda|u|^r\,d\mathcal{H}^{d-1} &\le C\left(\fint_{\partial Q_\rho(x_0)}\Lambda^\alpha\,d\mathcal{H}^{d-1}\right)^{\frac{1}{\alpha}}\left(\fint_{\partial Q_\rho(x_0)}\lambda^{-\beta}\,d\mathcal{H}^{d-1}\right)^{\frac{r}{\beta p}}\left(\fint_{\partial Q_\rho(x_0)}\lambda|\nabla u|^p\,d\mathcal{H}^{d-1}\right)^{\frac{r}{p}}\\ &\quad + C\left(\fint_{\partial Q_\rho(x_0)}\Lambda\,d\mathcal{H}^{d-1}\right)\left(\frac{1}{\rho}\fint_{\partial Q_\rho(x_0)}|u|\,d\mathcal{H}^{d-1}\right)^r,\end{aligned} \tag{6.2}$$

for every $\rho\in(0,1]$, $x_0\in\mathbb{R}^d$, and every $u\in W^{1,1}(\partial Q_\rho(x_0);\mathbb{R}^m)$ such that

$$\int_{\partial Q_\rho(x_0)}|u|\,d\mathcal{H}^{d-1}+\int_{\partial Q_\rho(x_0)}\lambda|\nabla u|^p\,d\mathcal{H}^{d-1}<\infty.$$

Moreover, if $\alpha=\infty$ then (6.2) holds true with $(\fint_{\partial Q_\rho(x_0)}\Lambda^\alpha)^{1/\alpha}$ replaced by $\|\Lambda\|_{L^\infty(\partial Q_\rho(x_0))}$; while if $\beta=\infty$ then (6.2) holds true with $(\fint_{\partial Q_\rho(x_0)}\lambda^{-\beta})^{1/\beta}$ replaced by $\|\lambda^{-1}\|_{L^\infty(\partial Q_\rho(x_0))}$.

Proof. We notice that it is enough to prove that

$$\int_{\partial Q} \Lambda |u|^r \, d\mathcal{H}^{d-1} \leq C \left(\int_{\partial Q} \Lambda^\alpha \, d\mathcal{H}^{d-1} \right)^{\frac{1}{\alpha}} \left(\int_{\partial Q} \lambda^{-\beta} \, d\mathcal{H}^{d-1} \right)^{\frac{r}{\beta p}} \left(\int_{\partial Q} \lambda |\nabla u|^p \, d\mathcal{H}^{d-1} \right)^{\frac{r}{p}} + C \left(\int_{\partial Q} \Lambda \, d\mathcal{H}^{d-1} \right) \left(\int_{\partial Q} |u| \, d\mathcal{H}^{d-1} \right)^r, \tag{6.3}$$

where Q is the unit cube centred at the origin. Indeed, then (6.2) readily follows from (6.3) applied to the scaled and translated functions $u(\rho x + x_0)/\rho$, $\Lambda(\rho x + x_0)$, and $\lambda(\rho x + x_0)$. Furthermore, since $\partial Q = \bigcup_{i=1}^{2d} D_i$, where $(D_i)_i$ are pairwise-disjoint $(d-1)$-dimensional unit cubes, it suffices to show (6.3) for just one of such cubes, which throughout is denoted by D.

We now claim that exist an exponent $r \in (p, \infty)$ and a constant $C = C(p, d, r, \alpha, \beta) > 0$ such that

$$\left(\int_D \Lambda |u - \overline{u}|^r \, d\mathcal{H}^{d-1} \right)^{\frac{1}{r}} \leq C \left(\int_D \Lambda^\alpha \, d\mathcal{H}^{d-1} \right)^{\frac{1}{\alpha r}} \left(\int_D \lambda^{-\beta} \, d\mathcal{H}^{d-1} \right)^{\frac{1}{\beta p}} \left(\int_D \lambda |\nabla u|^p \, d\mathcal{H}^{d-1} \right)^{\frac{1}{p}}, \tag{6.4}$$

where $\overline{u} := \int_D u \, d\mathcal{H}^{d-1}$. Assuming for the moment that (6.4) holds, in view of

$$\int_D \Lambda |\overline{u}|^r \, d\mathcal{H}^{d-1} \leq \left(\int_D \Lambda \, d\mathcal{H}^{d-1} \right) \left(\int_D |u| \, d\mathcal{H}^{d-1} \right)^r,$$

we will readily get (6.3). Therefore, we are left to prove (6.4).

If $d = 2$, we observe that D is a one dimensional closed curve in $\mathbb{R}^2$. Hence in this case (6.4) is a direct consequence of the Sobolev embedding in dimension one and of the Hölder Inequality as follows:

$$\begin{aligned} \left(\int_D \Lambda |u - \overline{u}|^r \, d\mathcal{H}^1 \right)^{\frac{1}{r}} &\leq \left(\int_D \Lambda \, d\mathcal{H}^1 \right)^{\frac{1}{r}} \|u - \overline{u}\|_{L^\infty(D)} \leq C \left(\int_D \Lambda^\alpha \, d\mathcal{H}^1 \right)^{\frac{1}{\alpha r}} \int_D |\nabla u| \, d\mathcal{H}^1 \\ &\leq C \left(\int_D \Lambda^\alpha \, d\mathcal{H}^1 \right)^{\frac{1}{\alpha r}} \left(\int_D \lambda^{-\frac{1}{p-1}} \, d\mathcal{H}^1 \right)^{\frac{p-1}{p}} \left(\int_D \lambda |\nabla u|^p \, d\mathcal{H}^1 \right)^{\frac{1}{p}} \\ &\leq C \left(\int_D \Lambda^\alpha \, d\mathcal{H}^1 \right)^{\frac{1}{\alpha r}} \left(\int_D \lambda^{-\beta} \, d\mathcal{H}^1 \right)^{\frac{1}{\beta p}} \left(\int_D \lambda |\nabla u|^p \, d\mathcal{H}^1 \right)^{\frac{1}{p}}. \end{aligned}$$

We now consider $d \geq 3$. To this end, we need to distinguish between the two cases: $\alpha > 1$ and $\alpha = 1$.

Case 1: $\alpha > 1$. Choose $r \in (p, \infty)$; by (6.1) we immediately get $1/\alpha + 1/\beta < r/(d-1)$. For $\alpha \in (1, \infty)$ by the Hölder Inequality we obtain

$$\left(\int_D \Lambda |u - \overline{u}|^r \, d\mathcal{H}^{d-1} \right)^{\frac{1}{r}} \leq \left(\int_D \Lambda^\alpha \, d\mathcal{H}^{d-1} \right)^{\frac{1}{\alpha r}} \left(\int_D |u - \overline{u}|^{\frac{\alpha r}{\alpha - 1}} \, d\mathcal{H}^{d-1} \right)^{\frac{\alpha - 1}{\alpha r}}, \tag{6.5}$$

whereas for $\alpha = \infty$ we immediately get

$$\left(\int_D \Lambda |u - \overline{u}|^r \, d\mathcal{H}^{d-1} \right)^{\frac{1}{r}} \leq \|\Lambda\|_{L^\infty(D)}^{\frac{1}{r}} \|u - \overline{u}\|_{L^r(D)}.$$

Below we need to consider two further subcases.

Subcase 1.1: assume that

$$\frac{\alpha p}{\alpha - 1} \geq \frac{d-1}{d-2} \tag{6.6}$$

hold. Since $\alpha r/(\alpha - 1) > 1^*$, we can find the exponent $q > 1$ such that $\alpha r/(\alpha - 1)$ is its conjugate Sobolev exponent in dimension $d - 1$; that is

$$q := \left(\frac{\alpha r}{\alpha - 1} \right)_* = \frac{(d-1)\frac{\alpha r}{\alpha - 1}}{\frac{\alpha r}{\alpha - 1} + (d-1)} = \frac{d r \alpha - r \alpha}{r \alpha + d \alpha - d - \alpha + 1}. \tag{6.7}$$

We notice that $q \in (1, r)$. Indeed, the lower bound $q > 1$ follows by the second inequality in (6.6) while the upper bound $q < r$ is a consequence of $\alpha r > d - 1$ given by (6.1) for $\alpha < \infty$. If on the other hand

$\alpha = \infty$ then $q = r_* < r$. For $q = q(r)$ as in (6.7), the Sobolev Inequality provides us with constant $C = C(d, r, \alpha) > 0$ such that

$$\left(\int_D |u - \overline{u}|^{\frac{\alpha r}{\alpha - 1}}\, d\mathcal{H}^{d-1}\right)^{\frac{\alpha-1}{\alpha r}} \leq C \left(\int_D |\nabla u|^q \, d\mathcal{H}^{d-1}\right)^{\frac{1}{q}}. \tag{6.8}$$

Let $q(p)$ be as in (6.7) with r replaced by p. A straightforward computation gives

$$\frac{p - q(p)}{q(p)} = \frac{p}{d-1} - \frac{1}{\alpha},$$

thus appealing to (6.1) we immediately get

$$\frac{p - q(p)}{q(p)} > \frac{1}{\beta}.$$

Therefore, in view of the continuity of $q = q(r)$, up to choosing $r \in (p, p+\sigma)$ for $\sigma > 0$ small enough, the inequality $(p-q)/q > 1/\beta$ holds true for the corresponding values of q as in (6.7). We notice additionally that $1/\beta = 0$ if $\beta = \infty$.

By the Hölder Inequality we obtain

$$\begin{aligned}\left(\int_D |\nabla u|^q \, d\mathcal{H}^{d-1}\right)^{\frac{1}{q}} &= \left(\int_D \lambda^{-\frac{q}{p}}\lambda^{\frac{q}{p}} |\nabla u|^q \, d\mathcal{H}^{d-1}\right)^{\frac{1}{q}} \\ &\leq \left(\int_D \lambda^{-\frac{q}{p-q}} \, d\mathcal{H}^{d-1}\right)^{\frac{p-q}{pq}} \left(\int_D \lambda |\nabla u|^p \, d\mathcal{H}^{d-1}\right)^{\frac{1}{p}}.\end{aligned} \tag{6.9}$$

Since $(p-q)/q > 1/\beta$ appealing to the Hölder Inequality to estimate the first factor on the right hand side of (6.9) we obtain

$$\left(\int_D |\nabla u|^q \, d\mathcal{H}^{d-1}\right)^{\frac{1}{q}} \leq \left(\int_D \lambda^{-\beta} \, d\mathcal{H}^{d-1}\right)^{\frac{1}{\beta p}} \left(\int_D \lambda |\nabla u|^p \, d\mathcal{H}^{d-1}\right)^{\frac{1}{p}}. \tag{6.10}$$

Eventually (6.4) follows by gathering (6.5), (6.8), and (6.10).

Subcase 1.2: assume that

$$\frac{\alpha p}{\alpha - 1} < \frac{d-1}{d-2} \tag{6.11}$$

holds true.

In this case (6.8) holds true with $q = 1$. Indeed, (6.8) holds for those exponents $r \in (p, \infty)$ satisfying

$$\frac{\alpha r}{(\alpha - 1)} < 1^* = \frac{d-1}{d-2},$$

whose existence is guaranteed by (6.11). Furthermore (6.10) holds true with $\beta = 1/(p-1)$. Finally, (6.4) again follows by combining (6.5), (6.8), and (6.10).

Case 2: $\alpha = 1$. In view of (6.1) we immediately deduce that $p > d-1$; moreover, we also get

$$\frac{p - (d-1)}{d-1} = \frac{p}{d-1} - \frac{1}{\alpha} > \frac{1}{\beta}.$$

Hence we can find $\sigma > 0$ such that every $q \in (d-1, d-1+\sigma)$ satisfies $(p-q)/q > 1/\beta$. Then, we can argue as for $\alpha > 1$ to obtain (6.10). The latter ensures in particular that $\nabla u \in L^q(D, \mathcal{H}^{d-1})$, thus since $q > d-1$ we immediately deduce

$$\|u - \overline{u}\|_{L^\infty(D, \mathcal{H}^{d-1})} \leq C \left(\int_D |\nabla u|^q \, d\mathcal{H}^{d-1}\right)^{\frac{1}{q}}, \tag{6.12}$$

for some $C = C(d, q) > 0$. Consequently, for every $r \in (p, \infty)$ the Hölder Inequality together with (6.12) yield

$$\left(\int_D \Lambda|u-\overline{u}|^r \, d\mathcal{H}^{d-1}\right)^{\frac{1}{r}} \le \left(\int_D \Lambda \, d\mathcal{H}^{d-1}\right)^{\frac{1}{r}} \|u-\overline{u}\|_{L^\infty(D,\mathcal{H}^{d-1})}$$
$$\le C \left(\int_D \Lambda \, d\mathcal{H}^{d-1}\right)^{\frac{1}{r}} \left(\int_D |\nabla u|^q \, d\mathcal{H}^{d-1}\right)^{\frac{1}{q}}. \tag{6.13}$$

Eventually, (6.4) follows by combining (6.10) and (6.13). □

We also need a technical result on the limit of the product of weakly converging sequences.

Lemma 6.2. *Let $U \subset \mathbb{R}^d$ be measurable with $|U| < +\infty$. Let $(u_n), (v_n)$, and (z_n) be sequences of positive measurable functions on U satisfying the following properties:*

(i) $u_n \to u$ a.e in U;
(ii) $(v_n) \subset L^1(U)$, $v_n \rightharpoonup v$ weakly in $L^1(U)$;
(iii) $\sup_{n\in\mathbb{N}} \|z_n\|_{L^\infty(U)} < +\infty$

for some measurable functions u, v with $v \in L^1(U)$. If, additionally, $uv \in L^1(U)$ and

$$\sup_{n\in\mathbb{N}} \int_U u_n^q v_n z_n \, dx < +\infty, \tag{6.14}$$

for some $q > 1$, then

$$\limsup_{n\to+\infty} \int_U u_n v_n z_n \, dx \le \sup_{n\in\mathbb{N}} \|z_n\|_{L^\infty(U)} \int_U uv \, dx.$$

Proof. Let $M \ge 1$ and consider the truncated functions $u_n^M := u_n \wedge M$; by assumption u_n^M converges to $u^M := u \wedge M$ a.e. on U. By [16, Proposition 2.61], for every $M \ge 1$ the sequence $(u_n^M v_n)$ converges weakly in $L^1(U)$ to $u^M v$. Moreover, by (6.14) there exists $C > 0$ such that for every $n \in \mathbb{N}$ we have

$$M^{q-1} \int_{\{u_n \ge M\}} u_n v_n z_n \, dx \le \int_U u_n^q v_n z_n \, dx \le C.$$

Thus, for every $M \ge 1$ there holds

$$\limsup_{n\to+\infty} \left| \int_U u_n v_n z_n \, dx - \int_U u_n^M v_n z_n \, dx \right| \le \limsup_{n\to+\infty} \int_{\{u_n \ge M\}} u_n v_n z_n \, dx \le \frac{C}{M^{q-1}}.$$

Therefore,

$$\limsup_{n\to+\infty} \int_U u_n v_n z_n \, dx \le \limsup_{n\to+\infty} \int_U u_n^M v_n z_n \, dx + \frac{C}{M^{q-1}}$$
$$\le \sup_{n\in\mathbb{N}} \|z_n\|_{L^\infty(U)} \lim_{n\to+\infty} \int_U u_n^M v_n \, dx + \frac{C}{M^{q-1}}$$
$$\le \sup_{n\in\mathbb{N}} \|z_n\|_{L^\infty(U)} \int_U u^M v \, dx + \frac{C}{M^{q-1}}.$$

Since $u^M v \to uv$ a.e. in U for $M \to +\infty$ and $uv \in L^1(U)$, we conclude by letting $M \to +\infty$ and invoking the Dominated Convergence Theorem. □

We are now in a position to prove the lower bound inequality for the Γ-limit in the convex case, under the stochastic integrability assumptions (3.4)-(3.5) on the weight functions.

Proposition 6.3 (Lower bound)**.** *Let f be an admissible random integrand and assume that (H1) holds. Then, there exists $\Omega' \in \mathcal{F}$ with $\mathbb{P}(\Omega') = 1$ such that for every $\omega \in \Omega'$, every $(u_\varepsilon) \subset L^1_{\mathrm{loc}}(\mathbb{R}^d; \mathbb{R}^m)$, and $u \in L^1_{\mathrm{loc}}(\mathbb{R}^d; \mathbb{R}^m)$ with $u_\varepsilon \to u$ in $L^1_{\mathrm{loc}}(\mathbb{R}^d; \mathbb{R}^m)$ there holds*

$$F_{\mathrm{hom}}(\omega)(u, A) \le \liminf_{\varepsilon\to 0} F_\varepsilon(\omega)(u_\varepsilon, A), \tag{6.15}$$

for every $A \in \mathcal{A}$, where F_{hom} is as in (3.8).

Proof. With a little abuse of notation let $\Omega' \in \mathcal{F}$ denote a set with $\mathbb{P}(\Omega') = 1$ where both Proposition 4.2 and Lemma 3.3 hold true. Let $\omega \in \Omega'$ and $A \in \mathcal{A}$ be fixed.

Without loss of generality we can assume that the right-hand side in (6.15) is finite and, passing to a (not relabelled) subsequence, that the liminf is actually a limit. Then, by Lemma 3.3 we infer that $u \in W^{1,p}(A;\mathbb{R}^m)$. Let $\varepsilon > 0$ be fixed, on $\mathcal{B}(A)$ we consider the finite Radon measure defined as

$$\nu_\varepsilon(\omega, B) := \int_B f\left(\omega, \frac{x}{\varepsilon}, \nabla u_\varepsilon\right)\, dx.$$

By assumption (ν_ε) is an equibounded sequence, hence, up to a subsequence, $\nu_\varepsilon \overset{*}{\rightharpoonup} \nu$ for some nonnegative finite Radon measure ν, possibly depending on ω. Appealing to the Lebesgue Decomposition Theorem we can write $\nu = \tilde{f}dx + \nu_s$ where $\tilde{f}$ is a nonnegative integrable function and $\nu_s \perp dx$. Since A is open, the convergence $\nu_\varepsilon \overset{*}{\rightharpoonup} \nu$ implies that

$$\lim_{\varepsilon\to 0} F_\varepsilon(\omega)(u_\varepsilon, A) = \lim_{\varepsilon\to 0} \nu_\varepsilon(\omega, A) \geq \nu(\omega, A) \geq \int_A \tilde{f}(\omega, x)\, dx.$$

Thus, to prove (6.15) it suffices to show that for a.e. $x_0 \in A$ the following inequality holds

$$\tilde{f}(\omega, x_0) \geq f_{\rm hom}(\omega, \nabla u(x_0)), \tag{6.16}$$

where $f_{\rm hom}$ is as in (3.9).

For $x \in A$ let $\rho_x > 0$ be such that $Q_\rho(x) \subset A$ for every $0 < \rho < \rho_x$. Since ν is a finite Radon measure on A, we have that $\nu(\omega, \partial Q_\rho(x)) = 0$ for $\mathcal{L}^1$-a.e. $\rho \in (0, \rho_x)$.

In view of the Besicovitch Differentiation Theorem and the Portmanteau Theorem, we infer that for a.e. $x_0 \in A$

$$\tilde{f}(\omega, x_0) = \lim_{\rho\to 0} \frac{\nu(\omega, Q_\rho(x_0))}{\rho^d} = \lim_{\rho\to 0}\lim_{\varepsilon\to 0} \frac{\nu_\varepsilon(\omega, Q_\rho(x_0))}{\rho^d}.$$

Therefore, to get (6.16) we are going to show that for a.e. $x_0 \in A$ we have

$$\liminf_{\rho\to 0}\liminf_{\varepsilon\to 0} \fint_{Q_\rho(x_0)} f\left(\omega, \frac{x}{\varepsilon}, \nabla u_\varepsilon\right)\, dx \geq f_{\rm hom}(\omega, \nabla u(x_0)). \tag{6.17}$$

Assume now that $x_0 \in A$ is a Lebesgue point both for u and for ∇u and define the linearisation of u in x_0 as

$$L_u(x) := u(x_0) + \nabla u(x_0)(x - x_0)$$

where for simplicity we drop the explicit dependence on x_0. Since $u \in W^{1,p}(A;\mathbb{R}^m)$ we can additionally assume that

$$\lim_{\rho\to 0} \frac{1}{\rho^p} \fint_{Q_\rho(x_0)} |u - L_u|^p\, dx = 0. \tag{6.18}$$

We now modify u_ε in a neighbourhood of $\partial Q_\rho(x_0)$ so that it attains a boundary datum which is approximately L_u. Moreover, we show that this modification does not essentially increase the energy. To this end, let $\delta > 0$ be sufficiently small and let $\varphi \in C_c^\infty(Q_\rho(x_0); [0,1])$ be a cut-off function with $\varphi = 1$ in $Q_{(1-2\delta)\rho}(x_0)$ and $\varphi = 0$ in $Q_\rho(x_0) \setminus Q_{(1-\delta)\rho}(x_0)$. Then, $\mathrm{supp}(\nabla\varphi) \subset C_{\delta,\rho}(x_0)$, where $C_{\delta,\rho}(x_0) := Q_{(1-\delta)\rho}(x_0) \setminus Q_{(1-2\delta)\rho}(x_0)$. Let $\tau \in (0,1)$ and set

$$w_\varepsilon := \tau\varphi u_\varepsilon + \tau(1-\varphi)L_u.$$

By definition $(w_\varepsilon) \subset W^{1,p}(A;\mathbb{R}^m)$ and $w_\varepsilon = \tau L_u$ in a neighbourhood of $\partial Q_\rho(x_0)$.

We now estimate F_ε along the sequence (w_ε). By virtue of the convexity of f and of (3.2) we get

$$\begin{aligned}
\frac{1}{\rho^d}F_\varepsilon(\omega)(w_\varepsilon, Q_\rho(x_0)) &= \fint_{Q_\rho(x_0)} f\left(\omega, \frac{x}{\varepsilon}, \tau\varphi\nabla u_\varepsilon + \tau(1-\varphi)\nabla u(x_0) + \tau\frac{1-\tau}{1-\tau}\nabla\varphi\otimes(u_\varepsilon - L_u)\right) dx \\
&\leq \fint_{Q_\rho(x_0)} \tau\varphi f\left(\omega, \frac{x}{\varepsilon}, \nabla u_\varepsilon\right) dx + \fint_{Q_\rho(x_0)} \tau(1-\varphi)f(\omega, \frac{x}{\varepsilon}, \nabla u(x_0))\, dx \\
&+ (1-\tau)\fint_{Q_\rho(x_0)} \Lambda\left(\omega, \frac{x}{\varepsilon}\right)\left(\frac{\tau^p}{(1-\tau)^p}|\nabla\varphi|^p|u_\varepsilon - L_u|^p + 1\right) dx \\
&\leq \frac{\tau}{\rho^d}F_\varepsilon(\omega)(u_\varepsilon, Q_{(1-\delta)\rho}(x_0)) + \frac{\tau}{\rho^d}\int_{Q_\rho(x_0)\setminus Q_{(1-2\delta)\rho}(x_0)} \Lambda\left(\omega, \frac{x}{\varepsilon}\right)(|\nabla u(x_0)|^p + 1)\, dx \\
&+ (1-\tau)\fint_{Q_\rho(x_0)} \Lambda\left(\omega, \frac{x}{\varepsilon}\right) dx + \frac{\tau^p}{(1-\tau)^{p-1}}\fint_{Q_\rho(x_0)} \Lambda\left(\omega, \frac{x}{\varepsilon}\right)|\nabla\varphi|^p|u_\varepsilon - L_u|^p\, dx.
\end{aligned} \tag{6.19}$$

Let $\delta \in (0, 1/4)$; we now claim that for every $\varepsilon > 0$ we can find a cut-off function φ_ε (also depending on δ) satisfying $|\nabla\varphi_\varepsilon| \leq C/(\delta\rho)$ as well as

$$\limsup_{\varepsilon\to 0}\fint_{Q_\rho(x_0)} \Lambda\left(\omega, \frac{x}{\varepsilon}\right)|\nabla\varphi_\varepsilon|^p|u_\varepsilon - L_u|^p\, dx \leq \frac{Cc_1(\omega)}{\delta^p\rho^p}\fint_{Q_\rho(x_0)} |u - L_u|^p\, dx, \tag{6.20}$$

where $c_1(\omega)$ is as in (3.11). To this end we consider the interval $I_1^\varepsilon \subset (1-2\delta, 1-\delta)$ defined as

$$I_1^\varepsilon := \left\{\sigma \in (1-2\delta, 1-\delta)\colon\ \fint_{\partial Q_{\sigma\rho}(x_0)} \Lambda^\alpha\left(\omega, \frac{x}{\varepsilon}\right) d\mathcal{H}^{d-1} \leq \frac{5}{d\delta}\frac{2^{d-2}}{\rho^d}\int_{C_{\delta,\rho}(x_0)} \Lambda^\alpha\left(\omega, \frac{x}{\varepsilon}\right) dx\right\}.$$

We notice that $\mathcal{L}^1(I_1^\varepsilon) \geq 4\delta/5$. Indeed, setting

$$H_\Lambda^\varepsilon(\sigma) := \int_{\partial Q_{\sigma\ell}(x_0)} \Lambda^\alpha\left(\omega, \frac{x}{\varepsilon}\right) d\mathcal{H}^{d-1},$$

recalling that $\delta < 1/4$, we have

$$\mathcal{H}^{d-1}(\partial Q_{\sigma\rho}(x_0)) = 2d(\sigma\rho)^{d-1} \geq 2d\rho^{d-1}/2^{d-1},$$

hence

$$I_1^\varepsilon \supset \left\{\sigma \in (1-2\delta, 1-\delta)\colon\ H_\Lambda^\varepsilon(\sigma) \leq \frac{5}{\delta\rho}\int_{C_{\delta,\rho}(x_0)} \Lambda^\alpha\left(\omega, \frac{x}{\varepsilon}\right) dx\right\}.$$

On the other hand, an application of the Fubini Theorem gives

$$\begin{aligned}
\int_{1-2\delta}^{1-\delta} H_\Lambda^\varepsilon\, d\sigma &= \int_{1-2\delta}^{1-\delta}\int_{\partial Q_{\sigma\rho}(x_0)} \Lambda^\alpha\left(\omega, \frac{x}{\varepsilon}\right) d\mathcal{H}^{d-1}\, d\sigma \\
&= \frac{1}{\rho}\int_{(1-2\delta)\rho}^{(1-\delta)\rho}\int_{\partial Q_\sigma(x_0)} \Lambda^\alpha\left(\omega, \frac{x}{\varepsilon}\right) d\mathcal{H}^{d-1}\, d\sigma = \frac{1}{\rho}\int_{C_{\delta,\rho}(x_0)} \Lambda^\alpha\left(\omega, \frac{x}{\varepsilon}\right) dx,
\end{aligned}$$

thus, appealing to the Chebychev Inequality gives

$$\mathcal{L}^1\left(\left\{\sigma \in (1-2\delta, 1-\delta)\colon H_\Lambda^\varepsilon(\sigma) \geq \frac{5}{\delta\rho}\int_{C_{\delta,\rho}(x_0)} \Lambda^\alpha\left(\omega, \frac{x}{\varepsilon}\right) dx\right\}\right) \leq \frac{\delta}{5}.$$

Analogously, define the sets

$$I_2^\varepsilon := \left\{\sigma \in (1-2\delta, 1-\delta)\colon \fint_{\partial Q_{\sigma\rho}(x_0)} \lambda^{-\beta}\left(\omega, \frac{x}{\varepsilon}\right) d\mathcal{H}^{d-1} \leq \frac{5}{d\delta}\frac{2^{d-2}}{\rho^d}\int_{C_{\delta,\rho}(x_0)} \lambda^{-\beta}\left(\omega, \frac{x}{\varepsilon}\right) dx\right\},$$

$$I_3^\varepsilon := \left\{\sigma \in (1-2\delta, 1-\delta)\colon \fint_{\partial Q_{\sigma\rho}(x_0)} |u_\varepsilon|\, d\mathcal{H}^{d-1} \leq \frac{5}{d\delta}\frac{2^{d-2}}{\rho^d}\int_{C_{\delta,\rho}(x_0)} |u_\varepsilon|\, dx\right\},$$

$$I_4^\varepsilon := \left\{\sigma \in (1-2\delta, 1-\delta)\colon \fint_{\partial Q_{\sigma\rho}(x_0)} \lambda^{-\beta}\left(\omega, \frac{x}{\varepsilon}\right)|\nabla u_\varepsilon|^p\, d\mathcal{H}^{d-1} \leq \frac{5}{d\delta}\frac{2^{d-2}}{\rho^d}\int_{C_{\delta,\rho}(x_0)} \lambda^{-\beta}\left(\omega, \frac{x}{\varepsilon}\right)|\nabla u_\varepsilon|^p\, dx\right\};$$

then, arguing exactly as above we can infer that $\mathcal{L}^1(I_2^\varepsilon) \geq 4\delta/5$, $\mathcal{L}^1(I_3^\varepsilon) \geq 4\delta/5$ and $\mathcal{L}^1(I_4^\varepsilon) \geq 4\delta/5$. Hence, defining $I^\varepsilon \subset (1-2\delta, 1-\delta)$ as $I^\varepsilon := I_1^\varepsilon \cap I_2^\varepsilon \cap I_3^\varepsilon \cap I_4^\varepsilon$, we get that $\mathcal{L}^1(I^\varepsilon) \geq \delta/5$.

We now consider the cut-off function $\varphi_\varepsilon \in C_c^\infty(Q_\rho(x_0);[0,1])$ with $\varphi_\varepsilon = 1$ in $Q_{(1-2\delta)\rho}(x_0)$, $\varphi_\varepsilon = 0$ in $Q_\rho(x_0) \setminus Q_{(1-\delta)\rho}(x_0)$ and

$$\operatorname{supp}(\nabla\varphi_\varepsilon) \subset \bigcup_{\sigma\in I^\varepsilon} \partial Q_{\sigma\rho}(x_0), \quad \|\nabla\varphi_\varepsilon\|_{L^\infty} \leq \frac{5}{\delta\rho}, \tag{6.21}$$

which is possible since $\mathcal{L}^1(I^\varepsilon) \geq \delta/5$.

We now claim that there exists $r \in (p,\infty)$ such that

$$\sup_{\varepsilon>0} \fint_{Q_\rho(x_0)} \Lambda\left(\omega, \frac{x}{\varepsilon}\right) |\nabla\varphi_\varepsilon|^r |u_\varepsilon|^r \, dx \leq \widehat{C} \tag{6.22}$$

where $\widehat{C} = \widehat{C}\,(\rho,\delta,r,p,d,\alpha,\beta,\Lambda,\lambda,\omega) > 0$.

Let $r \in (p,\infty)$ be the integrability exponent as in Lemma 6.1. In view of (6.21) and (6.2) we deduce the existence of constant $\widehat{C} > 0$ (possibly depending on all the parameters but ε) such that

$$\begin{aligned}
&\fint_{Q_\rho(x_0)} \Lambda\left(\omega,\frac{x}{\varepsilon}\right)|\nabla\varphi_\varepsilon|^r|u_\varepsilon|^r\,dx \leq \widehat{C}\int_{I^\varepsilon}\frac{1}{(\sigma\rho)^r}\fint_{\partial Q_{\sigma\rho}(x_0)}\Lambda\left(\omega,\frac{x}{\varepsilon}\right)|u_\varepsilon|^r\,d\mathcal{H}^{d-1}\,d\sigma\\
&\leq \widehat{C}\int_{I^\varepsilon}\left(\fint_{\partial Q_{\sigma\rho}(x_0)}\Lambda^\alpha\left(\omega,\frac{x}{\varepsilon}\right)d\mathcal{H}^{d-1}\right)^{\frac{1}{\alpha}}\left(\fint_{\partial Q_{\sigma\rho}(x_0)}\lambda^{-\beta}\left(\omega,\frac{x}{\varepsilon}\right)d\mathcal{H}^{d-1}\right)^{\frac{r}{\beta p}}\left(\fint_{\partial Q_{\sigma\rho}(x_0)}\lambda\left(\omega,\frac{x}{\varepsilon}\right)|\nabla u_\varepsilon|^p d\mathcal{H}^{d-1}\right)^{\frac{r}{p}}d\sigma\\
&\quad+\widehat{C}\int_{I^\varepsilon}\left(\fint_{\partial Q_{\sigma\rho}(x_0)}\Lambda\left(\omega,\frac{x}{\varepsilon}\right)d\mathcal{H}^{d-1}\right)\left(\frac{1}{\sigma\rho}\fint_{\partial Q_{\sigma\rho}(x_0)}|u_\varepsilon|\,d\mathcal{H}^{d-1}\right)^r d\sigma\\
&\leq \widehat{C}\left(\fint_{Q_\rho(x_0)}\Lambda^\alpha\left(\omega,\frac{x}{\varepsilon}\right)dx\right)^{\frac{1}{\alpha}}\left(\fint_{Q_\rho(x_0)}\lambda^{-\beta}\left(\omega,\frac{x}{\varepsilon}\right)dx\right)^{\frac{r}{\beta p}}\left(\fint_{Q_\rho(x_0)}\lambda\left(\omega,\frac{x}{\varepsilon}\right)|\nabla u_\varepsilon|^p\,dx\right)^{\frac{r}{p}}\mathcal{L}^1(I^\varepsilon)\\
&\quad+\widehat{C}\left(\fint_{Q_\rho(x_0)}\Lambda^\alpha\left(\omega,\frac{x}{\varepsilon}\right)dx\right)^{\frac{1}{\alpha}}\left(\frac{1}{\rho}\fint_{Q_\rho(x_0)}|u_\varepsilon|\,dx\right)^r\mathcal{L}^1(I^\varepsilon),
\end{aligned}\tag{6.23}$$

where to establish the last estimate we also used the Hölder Inequality.

We observe that by Theorem 2.5 we have

$$\begin{aligned}
&\lim_{\varepsilon\to0}\left(\fint_{Q_\rho(x_0)}\Lambda^\alpha\left(\omega,\frac{x}{\varepsilon}\right)dx\right)^{\frac{1}{\alpha}} = \mathbb{E}[\Lambda^\alpha(\cdot,0)|\mathcal{F}_\tau]^{\frac{1}{\alpha}}(\omega),\\
&\lim_{\varepsilon\to0}\left(\fint_{Q_\rho(x_0)}\lambda^{-\beta}\left(\omega,\frac{x}{\varepsilon}\right)dx\right)^{\frac{1}{\beta}} = \mathbb{E}[\lambda^{-\beta}(\cdot,0)|\mathcal{F}_\tau]^{\frac{1}{\beta}}(\omega),
\end{aligned}\tag{6.24}$$

for $\mathbb{P}$-a.e. $\omega \in \Omega$. With a little abuse of notation we use the notation Ω' for the $\mathcal{F}$-measurable set of probability 1 where also (6.24) holds.

Moreover, for ε small enough we have

$$\frac{1}{\rho}\fint_{Q_\rho(x_0)}|u_\varepsilon|\,dx \leq \frac{2}{\rho}\fint_{Q_\rho(x_0)}|u|\,dx, \qquad \fint_{Q_\rho(x_0)}\lambda\left(\omega,\frac{x}{\varepsilon}\right)|\nabla u_\varepsilon|^p\,dx \leq \frac{1}{\rho^d}F_\varepsilon(\omega)(u_\varepsilon,Q_\rho(x_0)) \leq \frac{C}{\rho^d}. \tag{6.25}$$

Therefore gathering (6.23)-(6.25) yields (6.22). In turn, (6.22) and (6.21) allow us to infer that

$$\sup_{\varepsilon>0}\fint_{Q_\ell}\Lambda\left(\omega,\frac{x}{\varepsilon}\right)|\nabla\varphi_\varepsilon|^p|u_\varepsilon - L_u|^r dx \leq \widehat{C}, \tag{6.26}$$

with $\widehat{C} := \widehat{C}(\rho,\delta,r,p,d,\alpha,\beta,\Lambda,\lambda,\omega,x_0,u)$ positive and finite.

By virtue of (6.26), Theorem 2.5, and (6.21) we can now apply Lemma 6.2 with $q = r/p > 1$ to the sequences

$$(|u_\varepsilon - L_u|^p)_\varepsilon, \quad \Big(\Lambda\big(\omega,\cdot/\varepsilon\big)\Big)_\varepsilon, \quad (|\nabla\varphi_\varepsilon|^p)_\varepsilon,$$

to get (6.20), as claimed.

Therefore by combing the convexity estimate (6.19) together with (6.20) we deduce that

$$\liminf_{\varepsilon\to 0}\frac{1}{\rho^d}F_\varepsilon(\omega)(w_\varepsilon,Q_\rho(x_0))\leq \liminf_{\varepsilon\to 0}\frac{1}{\rho^d}F_\varepsilon(\omega)(u_\varepsilon,Q_\rho(x_0))+\delta Cc_1(\omega)(|\nabla u(x_0)|^p+1)$$
$$+(1-\tau)c_1(\omega)+\frac{Cc_1(\omega)}{\delta^p(1-\tau)^{p-1}}\fint_{Q_\rho(x_0)}\frac{|u-L_u|^p}{\rho^p}\,dx, \tag{6.27}$$

where the constant $C>0$ depends only on d and p. On the other hand, taking into account that F_ε is invariant under vertical translations, by the definition of w_ε a change of variables gives

$$\lim_{\varepsilon\to 0}\frac{\varepsilon^d}{\rho^d}\mu_{\tau\nabla u(x_0)}\big(\omega,Q_{\frac{\rho}{\varepsilon}}(x_0/\varepsilon)\big)\leq \liminf_{\varepsilon\to 0}\frac{1}{\rho^d}F_\varepsilon(\omega)(w_\varepsilon,Q_\rho(x_0)), \tag{6.28}$$

where $\mu_{\tau\nabla u(x_0)}$ is as in (4.1) with $\xi=\tau\nabla u(x_0)$.

Then, gathering (6.27), (6.28), invoking Proposition 4.2, and (6.18) we obtain

$$f_{\rm hom}(\omega,\tau\nabla u(x_0))\leq \liminf_{\rho\to 0}\liminf_{\varepsilon\to 0}\frac{1}{\rho^d}F_\varepsilon(\omega)(u_\varepsilon,Q_\rho(x_0))$$
$$+\delta Cc_1(\omega)(|\nabla u(x_0)|^p+1)+(1-\tau)c_1(\omega),$$

for every $\delta\in(0,1/4)$ and $\tau\in(0,1)$. Eventually, letting $\delta\to 0^+$, $\tau\to 1^-$, thanks to the continuity of $\xi\mapsto f_{\rm hom}(\omega,\xi)$ we get (6.17) and hence the lower bound. ☐

7. Lower bound in the non-(quasi)convex case

The following technical lemma will be pivotal in the proof of the lower bound inequality for non (quasi)convex integrands. To simplify the notation, below we omit the dependence of Λ and λ on ω.

Lemma 7.1. *Let $\Lambda,\lambda\colon\mathbb{R}^d\to[0,+\infty)$ be such that*

$$\Lambda\in L^\alpha_{\rm loc}(\mathbb{R}^d),\quad \lambda^{-1}\in L^\beta_{\rm loc}(\mathbb{R}^d) \tag{7.1}$$

where $\alpha\geq 1$ and $\beta\geq 1/(p-1)$ satisfy

$$\frac{1}{\alpha}+\frac{1}{\beta}<\frac{1}{d-1}. \tag{7.2}$$

Let $x_0\in\mathbb{R}^d$, $\rho\in(0,1]$, and let $Q_\rho(x_0)$ be the open cube with side length $\rho>0$ centred at x_0. Let $s,t\in(0,1)$ be such that $1/2<s<t\leq 1-(t-s)$ and consider the open cubes $Q_{s\rho}(x_0)\subset Q_{t\rho}(x_0)\subset Q_\rho(x_0)$. Set $\delta:=(t-s)/2$, so that $\mathrm{dist}(\partial Q_{t\rho}(x_0),Q_{s\rho}(x_0))=\delta\rho$.

Let moreover $u,v\in W^{1,1}(Q_\rho(x_0);\mathbb{R}^m)$ be such that

$$\int_{Q_{t\rho}(x_0)\setminus Q_{s\rho}(x_0)}\lambda|\nabla u|^p\,dx+\int_{Q_{t\rho}(x_0)\setminus Q_{s\rho}(x_0)}\lambda|\nabla v|^p\,dx<+\infty.$$

Then, for every $\eta>0$ there exist a function $w\in W^{1,1}(Q_\rho(x_0);\mathbb{R}^m)$, a constant $C:=C(\eta,\delta,d,p,\alpha,\beta,\Lambda,\lambda)$, and an open interval $J\Subset(s,t)$ with $\mathcal{L}^1(J)\geq c\delta$, $c:=c(\eta)>0$, such that $w=u$ in $Q_{s\rho}(x_0)$, $w=v$ in $Q_\rho(x_0)\setminus Q_{t\rho}(x_0)$ and

$$\frac{1}{\rho^d}\int_{Q_{t\rho}(x_0)\setminus Q_{s\rho}(x_0)}\Lambda|\nabla w|^p\,dx\leq\frac{\eta}{\rho^d}\int_{Q_{t\rho}(x_0)\setminus Q_{s\rho}(x_0)}(\lambda|\nabla u|^p+\lambda|\nabla v|^p)\,dx+C\left(\frac{1}{\rho^d}\int_{\partial Q_{\ell\rho}(x_0)}|u-v|\,d\mathcal{H}^{d-1}\right)^p, \tag{7.3}$$

for $\mathcal{L}^1$-a.e. every $\ell\in J$.

Proof. In the proof we will show that (7.3) holds for a constant $C>0$ depending on Λ and λ exactly through $\fint_{Q_\rho(x_0)}\Lambda^\alpha$ and $\fint_{Q_\rho(x_0)}\lambda^{-\beta}$.

We divide the proof into a number of intermediate steps.

Step 1: scaling and translation argument. We observe that it is enough to prove the lemma for $x_0=0$ and $\rho=1$. Indeed, assume that claim holds for $x_0=0$ and $\rho=1$; for $x\in Q$ set

$$u^\rho(x):=\frac{u(\rho x+x_0)}{\rho},\quad v^\rho(x):=\frac{v(\rho x+x_0)}{\rho},\quad \Lambda^\rho(x):=\Lambda(\rho x+x_0),\quad \lambda^\rho(x):=\lambda(\rho x+x_0).$$

Then, applying the result to $u^\rho, v^\rho, \Lambda^\rho, \lambda^\rho$ provides us with a function $w^\rho \in W^{1,1}(Q;\mathbb{R}^m)$ satisfying

$$\int_{Q_t\setminus Q_s} \Lambda^\rho |\nabla w^\rho|^p dx \le \eta \int_{Q_t\setminus Q_s} (\lambda^\rho|\nabla u^\rho|^p + \lambda^\rho|\nabla v^\rho|^p)\,dx + C\left(\int_{\partial Q_\ell} |u^\rho - v^\rho|\,d\mathcal{H}^{d-1}\right)^p,$$

for $\mathcal{L}^1$-a.e. every $\ell \in J$. Hence, by a change of variables we readily see that $w(x) := \rho w^\rho((x-x_0)/\rho)$ belongs to $W^{1,1}(Q_\rho(x_0);\mathbb{R}^m)$ and satisfies (7.3) with the same η and the same constant C and therefore

$$C = C\Big(\eta,\delta,d,p,\alpha,\beta,\fint_Q (\Lambda^\rho)^\alpha, \fint_Q (\lambda^\rho)^{-\beta}\Big) = C\Big(\eta,\delta,d,p,\alpha,\beta,\fint_{Q_\rho(x_0)} \Lambda^\alpha, \fint_{Q_\rho(x_0)} \lambda^{-\beta}\Big).$$

Step 2: maximal functions estimates. Let $N \in \mathbb{N}$ to be chosen later and consider $s', t' \in (0,1)$ with $s < s' < t' < t$ and $t' - s' = 2\delta/N$, such that

$$\begin{aligned}
&\int_{Q_{t'}\setminus Q_{s'}} (\lambda|\nabla u|^p + \lambda|\nabla v|^p)\,dx \le \frac{3}{N}\int_{Q_t\setminus Q_s} (\lambda|\nabla u|^p + \lambda|\nabla v|^p)\,dx,\\
&\int_{Q_{t'}\setminus Q_{s'}} \Lambda^\alpha\,dx \le \frac{3}{N}\int_{Q_t\setminus Q_s} \Lambda^\alpha\,dx\\
&\int_{Q_{t'}\setminus Q_{s'}} \lambda^{-\beta}\,dx \le \frac{3}{N}\int_{Q_t\setminus Q_s} \lambda^{-\beta}\,dx.
\end{aligned} \tag{7.4}$$

Set

$$H_\Lambda(\sigma) := \int_{\partial Q_\sigma} \Lambda^\alpha\,d\mathcal{H}^{d-1}(x), \quad H_\lambda(\sigma) := \int_{\partial Q_\sigma} \lambda^{-\beta}\,d\mathcal{H}^{d-1}(x).$$

Recalling the definition of maximal function in (Definition 2.1), we claim that we can find $s'' \in [s', s' + (t'-s')/3]$ and $t'' \in [t' - (t'-s')/3, t']$ such that the following inequalities hold for a universal constant $C > 0$ and for every $\sigma \in (s'', t']$ and $\tau \in [s', t'')$:

$$\int_{Q_\sigma\setminus Q_{s''}} \lambda|\nabla u|^p dx \le C\frac{\sigma - s''}{t'-s'}\int_{Q_{t'}\setminus Q_{s'}} \lambda|\nabla u|^p dx, \tag{7.5}$$

$$\int_{Q_{t''}\setminus Q_\tau} \lambda|\nabla v|^p dx \le C\frac{t'' - \tau}{t'-s'}\int_{Q_{t'}\setminus Q_{s'}} \lambda|\nabla v|^p\,dx, \tag{7.6}$$

$$M(H_\Lambda)(s'') \le \frac{C}{t'-s'}\int_{Q_{t'}\setminus Q_{s'}} \Lambda^\alpha\,dx, \qquad M(H_\lambda)(s'') \le \frac{C}{t'-s'}\int_{Q_{t'}\setminus Q_{s'}} \lambda^{-\beta}\,dx, \tag{7.7}$$

$$M(H_\Lambda)(t'') \le \frac{C}{t'-s'}\int_{Q_{t'}\setminus Q_{s'}} \Lambda^\alpha\,dx, \qquad M(H_\lambda)(t'') \le \frac{C}{t'-s'}\int_{Q_{t'}\setminus Q_{s'}} \lambda^{-\beta}\,dx. \tag{7.8}$$

Below we only prove (7.5) and (7.7), the proof of (7.6) and (7.8) being analogous.

To this end we start observing that the functions H_Λ, H_λ, and

$$H(\sigma) := \int_{\partial Q_\sigma} \lambda|\nabla u|^p\,d\mathcal{H}^{d-1}$$

belong to $L^1(s', t')$. Hence, in view of Theorem 2.2, we can find a universal constant $C > 0$ such that

$$\mathcal{L}^1\left(\left\{\sigma \in (s',t'):\ M(H)(\sigma) > 12C\frac{\|H\|_{L^1(s',t')}}{t'-s'}\right\}\right) \le \frac{t'-s'}{12}, \tag{7.9}$$

and analogously

$$\begin{aligned}
&\mathcal{L}^1\left(\left\{\sigma \in (s',t'):\ M(H_\Lambda)(\sigma) > 12C\frac{\|H_\Lambda\|_{L^1(s',t')}}{t'-s'}\right\}\right) \le \frac{t'-s'}{12},\\
&\mathcal{L}^1\left(\left\{\sigma \in (s',t'):\ M(H_\lambda)(\sigma) > 12C\frac{\|H_\lambda\|_{L^1(s',t')}}{t'-s'}\right\}\right) \le \frac{t'-s'}{12}.
\end{aligned} \tag{7.10}$$

In turn, (7.9) and (7.10) ensure the existence of a subset I of $(s', s' + (t'-s')/3)$ with positive measure and a universal constant $C > 0$ such that for every $\sigma \in I$ the following inequalities simultaneously hold true

$$
\begin{aligned}
&M(H)(\sigma) \le \frac{C}{t'-s'} \int_{Q_{t'} \setminus Q_{s'}} \lambda |\nabla u|^p dx, \\
&M(H_\Lambda)(\sigma) \le \frac{C}{t'-s'} \int_{Q_{t'} \setminus Q_{s'}} \Lambda^\alpha dx, \quad M(H_\lambda)(\sigma) \le \frac{C}{t'-s'} \int_{Q_{t'} \setminus Q_{s'}} \lambda^{-\beta} dx.
\end{aligned}
$$

Now let $s'' \in I$; by the Fubini Theorem it is straightforward to conclude that (7.7) holds. As for (7.5), by Definition 2.1 for every $\sigma \in (s'', t']$ we have

$$
\begin{aligned}
\frac{1}{2(\sigma - s'')} \int_{Q_\sigma \setminus Q_{s''}} \lambda |\nabla u|^p \, dx &= \frac{1}{2(\sigma - s'')} \int_{s''}^{\sigma} H(\theta) \, d\theta \le M(H)(s'') \\
&\le C \frac{1}{t'-s'} \int_{Q_{t'} \setminus Q_{s'}} \lambda |\nabla u|^p \, dx.
\end{aligned}
$$

Eventually, set

$$
J := \left(s'' + 2\frac{t''-s''}{5}, t'' - 2\frac{t''-s''}{5} \right) \qquad \text{and} \qquad r_0 := \frac{t''-s''}{16} \ge \frac{\delta}{24N}. \tag{7.11}
$$

Step 3: Λ and λ satisfy a Muckenhoupt-type condition. We now claim that by virtue of (7.2) the following bounds hold true for every $x \in \partial Q_{s''} \cup \partial Q_{t''}$ and every $r \le \delta/N$. Namely, we have

$$
\left(\fint_{Q_r(x)} \Lambda \, dy \right) \left(\fint_{Q_r(x)} \lambda^{-\frac{1}{p-1}} dy \right)^{p-1} \le \frac{C_d}{\delta^{\left(\frac{1}{\alpha}+\frac{1}{\beta}\right)}} r^{-(d-1)\left(\frac{1}{\alpha}+\frac{1}{\beta}\right)} \left(\int_Q \Lambda^\alpha dy \right)^{\frac{1}{\alpha}} \left(\int_Q \lambda^{-\beta} dy \right)^{\frac{1}{\beta}}, \tag{7.12}
$$

where $C_d > 0$ is a dimensional constant.

To this end, let $x \in \partial Q_{s''}$ and $r \le \delta/N$; appealing to the Fubini Theorem, the Hölder Inequality, and the Jensen Inequality we get

$$
\begin{aligned}
\fint_{Q_r(x)} \Lambda \, dy &= \fint_{s''-r/2}^{s''+r/2} \frac{1}{r^{d-1}} \int_{Q_r(x) \cap \partial Q_\sigma} \Lambda \, d\mathcal{H}^{d-1}(y) \, d\sigma \\
&\le \fint_{s''-r/2}^{s''+r/2} \frac{1}{r^{d-1}} \mathcal{H}^{d-1} \left(Q_r(x) \cap \partial Q_\sigma \right)^{\frac{\alpha-1}{\alpha}} \left(\int_{Q_r(x) \cap \partial Q_\sigma} \Lambda^\alpha \, d\mathcal{H}^{d-1}(y) \right)^{\frac{1}{\alpha}} d\sigma \\
&\le \fint_{s''-r/2}^{s''+r/2} \frac{(dr^{d-1})^{\frac{\alpha-1}{\alpha}}}{r^{d-1}} \left(\int_{\partial Q_\sigma} \Lambda^\alpha \, d\mathcal{H}^{d-1}(y) \right)^{\frac{1}{\alpha}} d\sigma \\
&\le d^{\frac{\alpha-1}{\alpha}} r^{-(d-1)\frac{1}{\alpha}} \left(\fint_{s''-r/2}^{s''+r/2} H_\Lambda(\sigma) \, d\sigma \right)^{\frac{1}{\alpha}},
\end{aligned}
$$

where in the last inequality we used the definition of H_Λ. In turn, by the definition of maximal function, also invoking (7.7) and (7.4), we obtain

$$
\begin{aligned}
\fint_{Q_r(x)} \Lambda \, dy &\le d^{\frac{\alpha-1}{\alpha}} r^{-(d-1)\frac{1}{\alpha}} \left(M H_\Lambda(s'') \right)^{\frac{1}{\alpha}} \le d^{\frac{\alpha-1}{\alpha}} r^{-(d-1)\frac{1}{\alpha}} \left(\frac{C}{t'-s'} \int_{Q_{t'} \setminus Q_{s'}} \Lambda^\alpha \, dx \right)^{\frac{1}{\alpha}} \\
&\le C d^{\frac{\alpha-1}{\alpha}} r^{-(d-1)\frac{1}{\alpha}} \left(\frac{N}{2\delta} \frac{3}{N} \int_Q \Lambda^\alpha \, dx \right)^{\frac{1}{\alpha}} \le C_d \delta^{-\frac{1}{\alpha}} r^{-(d-1)\frac{1}{\alpha}} \left(\int_Q \Lambda^\alpha \, dx \right)^{\frac{1}{\alpha}}.
\end{aligned} \tag{7.13}
$$

Similarly, by the Fubini Theorem, the Hölder Inequality with exponent $\beta(p-1) \geq 1$, and the Jensen Inequality we have

$$\begin{aligned}
\fint_{Q_r(x)} \lambda^{-\frac{1}{p-1}}\, dy &= \fint_{s''-r/2}^{s''+r/2} \frac{1}{r^{d-1}} \int_{Q_r(x)\cap\partial Q_\sigma} \lambda^{-\frac{1}{p-1}}\, d\mathcal{H}^{d-1}(y)\, d\sigma \\
&\leq \fint_{s''-r/2}^{s''+r/2} \frac{1}{r^{d-1}} \mathcal{H}^{d-1}\left(Q_r(x)\cap\partial Q_\sigma\right)^{1-\frac{1}{(p-1)\beta}} \left(\int_{Q_r(x)\cap\partial Q_\sigma} \lambda^{-\beta}\, d\mathcal{H}^{d-1}(y)\right)^{\frac{1}{(p-1)\beta}} d\sigma \\
&\leq \fint_{s''-r/2}^{s''+r/2} \frac{(dr^{d-1})^{1-\frac{1}{(p-1)\beta}}}{r^{d-1}} \left(\int_{\partial Q_\sigma} \lambda^{-\beta}\, d\mathcal{H}^{d-1}(y)\right)^{\frac{1}{(p-1)\beta}} d\sigma \\
&\leq C_d r^{-(d-1)\frac{1}{(p-1)\beta}} \left(\fint_{s''-r/2}^{s''+r/2} H_\lambda(\sigma)\, d\sigma\right)^{\frac{1}{(p-1)\beta}}.
\end{aligned}$$

Then, by the definition of maximal function, invoking (7.7) and (7.4), we obtain

$$\begin{aligned}
\left(\fint_{Q_r(x)} \lambda^{-\frac{1}{p-1}}\, dy\right)^{p-1} &\leq C_d r^{-(d-1)\frac{1}{\beta}} \left(MH_\lambda(s'')\right)^{\frac{1}{\beta}} \leq C_d r^{-(d-1)\frac{1}{\beta}} \left(\frac{C}{t'-s'} \int_{Q_{t'}\setminus Q_{s'}} \lambda^\beta\, dx\right)^{\frac{1}{\beta}} \\
&\leq C_d \delta^{-\frac{1}{\beta}} r^{-(d-1)\frac{1}{\beta}} \left(\int_Q \lambda^\beta\, dx\right)^{\frac{1}{\beta}}. \qquad (7.14)
\end{aligned}$$

Moreover, for $x \in \partial Q_{t''}$ we can argue as above now replacing (7.7) with (7.8). Finally, (7.12) follows by gathering (7.13) and (7.14).

Step 4: definition of w and energy estimate. We preliminarily observe that since $u, v \in W^{1,1}(Q;\mathbb{R}^m)$, then for $\mathcal{L}^1$-a.e. $\ell \in J$ we have

$$\int_{\partial Q_\ell} |u-v|\, d\mathcal{H}^{d-1} < +\infty. \qquad (7.15)$$

Now choose $\ell \in J$ so that (7.15) holds, then the function defined as

$$\tilde{w} := \begin{cases} u & \text{in } Q_\ell, \\ v & \text{in } Q\setminus Q_\ell, \end{cases}$$

belongs to $SBV(Q;\mathbb{R}^m)$. Moreover, the the total variation of $D\tilde{w}$ clearly satisfies

$$|D\tilde{w}| \leq |\nabla u|\mathcal{L}^d \lfloor_{Q_\ell} + |\nabla v|\mathcal{L}^d \lfloor_{Q\setminus Q_\ell} + |u-v|\mathcal{H}^{d-1}\lfloor_{\partial Q_\ell}.$$

Now, let $r\colon Q \to [0,+\infty)$ be the variable radius defined as

$$r(x) := \begin{cases} 0 & \text{if } x \in Q\setminus(Q_{t''}\setminus Q_{s''}) \\ \dfrac{\operatorname{dist}(x, \partial Q_{s''}\cup\partial Q_{t''})}{2} \wedge r_0 & \text{otherwise,} \end{cases}$$

where r_0 is as in (7.11). For every $x \in Q$ we now define the function w as

$$w(x) := \fint_{Q_{r(x)}(x)} \tilde{w}(y)\, dy, \qquad (7.16)$$

with the understanding that $w(x) = \tilde{w}(x)$ if $r(x) = 0$.

We notice that $w \in W^{1,1}(Q;\mathbb{R}^m)$. Indeed, "far from ∂Q_ℓ" we can apply the Lemma 2.3 since by definition $\tilde{w} \in W^{1,1}(Q_{s''+4r_0}\setminus Q_{s''};\mathbb{R}^m) \cap W^{1,1}(Q_{t''}\setminus Q_{s''+12r_0};\mathbb{R}^m)$. On the other hand, "close to ∂Q_ℓ", that is, for $x \in Q_{s''+12r_0}\setminus Q_{s''+4r_0}$, we have $r(x) = r_0$, hence $w \in W^{1,\infty}(Q_{s''+12r_0}\setminus Q_{s''+4r_0};\mathbb{R}^m)$ by the standard theory of mollification for SBV functions.

We have

$$\int_{Q_{t''}\setminus Q_{s''}} \Lambda|\nabla w|^p\, dx \leq \int_{Q_{s''+4r_0}\setminus Q_{s''}} \Lambda|\nabla w|^p\, dx + \int_{Q_{s''+12r_0}\setminus Q_{s''+4r_0}} \Lambda|\nabla w|^p\, dx + \int_{Q_{t''}\setminus Q_{s''+12r_0}} \Lambda|\nabla w|^p\, dx.$$

Then, by (2.1) in Lemma 2.3 applied in $Q_{s''+4r_0} \setminus Q_{s''}$ and in $Q_{t''} \setminus Q_{s''+12r_0}$, we deduce the existence of a constant $C = C(d,p) > 0$ such that

$$\begin{aligned}
\int_{Q_{t''}\setminus Q_{s''}} \Lambda |\nabla w|^p\,dx &\leq C \int_{Q_{s''+4r_0}\setminus Q_{s''}} \Lambda \left| \fint_{Q_{r(x)}(x)} \nabla\tilde{w}\,dy \right|^p dx + C \int_{Q_{s''+12r_0}\setminus Q_{s''+4r_0}} \Lambda \left| \fint_{Q_{r_0}(x)} dD\tilde{w} \right|^p dx \\
&\quad + C \int_{Q_{t''}\setminus Q_{s''+12r_0}} \Lambda \left| \fint_{Q_{r(x)}(x)} \nabla\tilde{w}\,dy \right|^p dx \\
&\leq C \int_{Q_{s''+4r_0}\setminus Q_{s''}} \Lambda \left(\fint_{Q_{r(x)}(x)} |\nabla u|\,dy \right)^p dx + C \int_{Q_{s''+12r_0}\setminus Q_{s''+4r_0}} \Lambda \left(\fint_{Q_{r_0}(x)} |\nabla u|\,dy \right)^p dx \\
&\quad + C \int_{Q_{s''+12r_0}\setminus Q_{s''+4r_0}} \Lambda \left(\fint_{Q_{r_0}(x)} |\nabla v|\,dy \right)^p dx \\
&\quad + C \int_{Q_{s''+12r_0}\setminus Q_{s''+4r_0}} \Lambda \left(\frac{1}{r_0^d} \int_{Q_{r_0}(x)\cap\partial Q_\ell} |u-v|\,d\mathcal{H}^{d-1} \right)^p dx, \\
&\quad + C \int_{Q_{t''}\setminus Q_{s''+12r_0}} \Lambda \left(\fint_{Q_{r(x)}(x)} |\nabla v|\,dy \right)^p dx,
\end{aligned} \tag{7.17}$$

where to establish the last inequality we used the fact that by definition of $r(x)$ and of $\ell \in J$ we have that $Q_{r(x)}(x) \cap \partial Q_\ell = \emptyset$ for every $x \in (Q_{s''+4r_0} \setminus Q_{s''}) \cup (Q_{t''} \setminus Q_{s''+12r_0})$.

We now estimate the second term on the right hand side of (7.17), and notice third term can be treated similarly. We preliminarily observe that for every $x \in Q_{s''+12r_0} \setminus Q_{s''+4r_0}$ and since $2r_0 \leq \operatorname{dist}(x, \partial Q_{s''} \cup \partial Q_{t''}) \leq 4r_0$, there holds

$$\left(\fint_{Q_{2r_0}(x)} \Lambda\,dy \right) \left(\fint_{Q_{2r_0}(x)} \lambda^{-1/(p-1)}\,dy \right)^{p-1} \leq C(\delta,d,\Lambda,\lambda) r_0^{-(d-1)\left(\frac{1}{\alpha}+\frac{1}{\beta}\right)}. \tag{7.18}$$

In fact, this is a consequence of (7.12) applied in a cube centred at the closest point to x in $\partial Q_{s''} \cup \partial Q_{t''}$ and with side-length $16r_0$. We now cover the set $Q_{s''+12r_0} \setminus Q_{s''+4r_0}$ with $n_d \in \mathbb{N}$ cubes of side-length r_0, where n_d depends only on the dimension. Hence, thanks to the Hölder Inequality, and (7.18) we infer

$$\begin{aligned}
\int_{Q_{s''+12r_0}\setminus Q_{s''+4r_0}} \Lambda \left(\fint_{Q_{r_0}(x)} |\nabla u|\,dy \right)^p dx &\leq \sum_{i=1}^{n_d} \int_{Q_{r_0}(x_i)} \Lambda \left(\fint_{Q_{r_0}(x)} |\nabla u|\,dy \right)^p dx \\
&\leq 2^{pd} \sum_{i=1}^{n_d} \left(\fint_{Q_{2r_0}(x_i)} |\nabla u|\,dx \right)^p \int_{Q_{r_0}(x_i)} \Lambda\,dx \\
&\leq 2^{pd} \sum_{i=1}^{n_d} \int_{Q_{2r_0}(x_i)} \lambda |\nabla u|^p\,dx \left(\fint_{Q_{2r_0}(x_i)} \lambda^{-1/(p-1)}\,dx \right)^{p-1} \fint_{Q_{2r_0}(x_i)} \Lambda\,dx \\
&\leq C(\delta,d,p,\Lambda,\lambda) r_0^{-(d-1)\left(\frac{1}{\alpha}+\frac{1}{\beta}\right)} \sum_{i=1}^{n_d} \fint_{Q_{2r_0}(x_i)} \lambda |\nabla u|^p\,dx \\
&\leq C(\delta,d,p,\Lambda,\lambda) N^{(d-1)\left(\frac{1}{\alpha}+\frac{1}{\beta}\right)} \int_{Q_{t''}\setminus Q_{s''}} \lambda |\nabla u|^p\,dx,
\end{aligned}$$

where in the last inequality we have used the fact that $r_0 \geq \delta/(24N)$.

Eventually, invoking (7.4) we deduce that

$$\int_{Q_{s''+12r_0}\setminus Q_{s''+4r_0}} \Lambda \left(\fint_{Q_{r(x)}(x)} |\nabla u|\,dy \right)^p dx \leq C(\delta,d,p,\Lambda,\lambda) N^{(d-1)\left(\frac{1}{\alpha}+\frac{1}{\beta}\right)} \frac{1}{N} \int_{Q_t\setminus Q_s} \lambda |\nabla u|^p dx. \tag{7.19}$$

We now estimate the first term in (7.17) and we observe that an estimate on the last term in (7.17) can be proven using analogous arguments. To this end we resort to a "dyadic" decomposition of the set

$Q_{s''+4r_0} \setminus Q_{s''}$, namely we write

$$\int_{Q_{s''+4r_0}\setminus Q_{s''}} \Lambda \left(\fint_{Q_{r(x)}(x)} |\nabla u|\, dy \right)^p dx = \sum_{k=-2}^{+\infty} \int_{Q_{s''+r_0/2^k}\setminus Q_{s''+r_0/2^{k+1}}} \Lambda \left(\fint_{Q_{r(x)}(x)} |\nabla u|\, dy \right)^p dx. \tag{7.20}$$

We notice that $r_0/2^{k+2} \le r(x) \le r_0/2^{k+1}$ for every $x \in Q_{s''+r_0/2^k} \setminus Q_{s''+r_0/2^{k+1}}$. Moreover, for every $x \in Q_{s''+r_0/2^k} \setminus Q_{s''+r_0/2^{k+1}}$ and also taking into account that $r_0/2^{k+1} \le \operatorname{dist}(x, \partial Q_{s''}) \le r_0/2^k$, there holds

$$\left(\fint_{Q_{r_0/2^{k+1}}(x)} \Lambda\, dy \right) \left(\fint_{Q_{r_0/2^{k+1}}(x)} \lambda^{-1/(p-1)}\, dy \right)^{p-1} \le C(\delta, d, \Lambda, \lambda) \left(\frac{r_0}{2^k} \right)^{-(d-1)\left(\frac{1}{\alpha}+\frac{1}{\beta}\right)}. \tag{7.21}$$

In fact, (7.21) follows from (7.12) applied to the cube centred at the closest point to x belonging to $\partial Q_{s''}$ and with side-length $r_0/2^{k-1}$.

Now let $k \ge -2$ be fixed; we decompose the set $Q_{s''+r_0/2^k} \setminus Q_{s''+r_0/2^{k+1}}$ into $n_{d,k} \in \mathbb{N}$ cubes of side-length $r_0/2^{k+2}$. We observe that though $n_{d,k}$ depends on the dimension and on k, the number of cubes of this decomposition which may overlap depends on the dimension d only. Hence, for every $k \ge -2$, by the Hölder Inequality, and (7.21) we get

$$\begin{aligned}
\int_{Q_{s''+r_0/2^k}\setminus Q_{s''+r_0/2^{k+1}}} \Lambda \left(\fint_{Q_{r(x)}(x)} |\nabla u|\, dy \right)^p dx &\le 2^{pd} \int_{Q_{s''+r_0/2^k}\setminus Q_{s''+r_0/2^{k+1}}} \Lambda \left(\fint_{Q_{r_0/2^{k+1}}(x)} |\nabla u|\, dy \right)^p dx \\
&\le 2^{pd} \sum_{i=1}^{n_{d,k}} \int_{Q_{r_0/2^{k+1}}(x_i)} \Lambda \left(\fint_{Q_{r_0/2^{k+1}}(x)} |\nabla u|\, dy \right)^p dx \\
&\le 4^{pd} \sum_{i=1}^{n_{d,k}} \left(\int_{Q_{r_0/2^k}(x_i)} \Lambda\, dx \right) \left(\fint_{Q_{r_0/2^k}(x_i)} |\nabla u|\, dx \right)^p \\
&\le 4^{pd} \sum_{i=1}^{n_{d,k}} \left(\fint_{Q_{r_0/2^k}(x_i)} \Lambda\, dx \right) \left(\fint_{Q_{r_0/2^k}(x_i)} \lambda^{-1/(p-1)}\, dx \right)^{p-1} \int_{Q_{r_0/2^k}(x_i)} \lambda |\nabla u|^p\, dx \\
&\le C(\delta, d, p, \Lambda, \lambda) \left(\frac{r_0}{2^{k-1}} \right)^{-(d-1)\left(\frac{1}{\alpha}+\frac{1}{\beta}\right)} \sum_{i=1}^{n_{d,k}} \int_{Q_{r_0/2^k}(x_i)} \lambda |\nabla u|^p\, dx \\
&\le C(\delta, d, p, \Lambda, \lambda) \left(\frac{r_0}{2^{k-1}} \right)^{-(d-1)\left(\frac{1}{\alpha}+\frac{1}{\beta}\right)} \int_{Q_{s''+r_0/2^{k-1}}\setminus Q_{s''}} \lambda |\nabla u|^p\, dx.
\end{aligned}$$

In turn, by (7.4), (7.5) and recalling that $r_0 \le \delta/(24N)$, we deduce

$$\begin{aligned}
&\int_{Q_{s''+r_0/2^k}\setminus Q_{s''+r_0/2^{k+1}}} \Lambda \left(\fint_{Q_{r(x)}(x)} |\nabla u|\, dy \right)^p dx \\
&\qquad \le C(\delta, d, p, \Lambda, \lambda) \left(\frac{r_0}{2^{k-1}} \right)^{-(d-1)\left(\frac{1}{\alpha}+\frac{1}{\beta}\right)} \frac{r_0}{2^{k-1}} \frac{1}{t'-s'} \int_{Q_{t'}\setminus Q_{s'}} \lambda |\nabla u|^p\, dx \\
&\qquad \le C(\delta, d, p, \Lambda, \lambda) \left(\frac{1}{N} \right)^{1-(d-1)\left(\frac{1}{\alpha}+\frac{1}{\beta}\right)} (2^k)^{(d-1)\left(\frac{1}{\alpha}+\frac{1}{\beta}\right)-1} \int_{Q_t\setminus Q_s} \lambda |\nabla u|^p\, dx.
\end{aligned}$$

Therefore, by (7.20), summing over k by virtue of (7.2) we get

$$\int_{Q_{s''+4r_0}\setminus Q_{s''}} \Lambda \left(\fint_{Q_{r(x)}(x)} |\nabla u|\, dy \right)^p dx \le C(\delta, d, p, \Lambda, \lambda) N^{(d-1)\left(\frac{1}{\alpha}+\frac{1}{\beta}\right)-1} \int_{Q_t\setminus Q_s} \lambda |\nabla u|^p\, dx. \tag{7.22}$$

Now let $\eta > 0$ be arbitrary and fixed; again appealing to (7.2) we can choose N so large that such that

$$C(\delta, d, p, \Lambda, \lambda) N^{(d-1)\left(\frac{1}{\alpha}+\frac{1}{\beta}\right)-1} \le \eta.$$

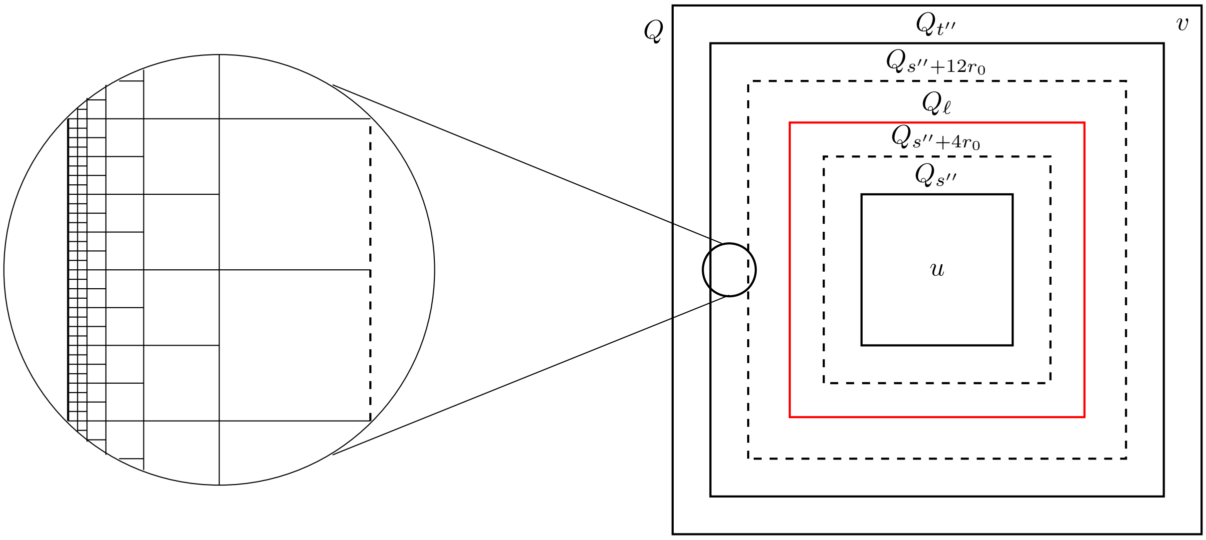

FIGURE 1. The dyadic decomposition in cubes employed in the estimate.

Hence, (7.22) becomes

$$\int_{Q_{s''+12r_0}\setminus Q_{s''}} \Lambda \left(\fint_{Q_{r(x)}(x)} |\nabla u|\, dy \right)^p dx \le \eta \int_{Q_t \setminus Q_s} \lambda |\nabla u|^p\, dx, \tag{7.23}$$

We estimate the one but last term in (7.17). Since $r_0 \geq \delta/(24N)$ we have

$$\begin{aligned} &\int_{Q_{s''+12r_0}\setminus Q_{s''+4r_0}} \Lambda \left(\frac{1}{r_0^d} \int_{Q_{r_0}(x)\cap \partial Q_\ell} |u-v|\, d\mathcal{H}^{d-1} \right)^p dx \\ &\quad \le \left(\left(\frac{24N}{\delta}\right)^d \int_{\partial Q_\ell} |u-v|\, d\mathcal{H}^{d-1} \right)^p \int_{Q_{s''+12r_0}\setminus Q_{s''+4r_0}} \Lambda\, dx \le C \left(\int_{\partial Q_\ell} |u-v|\, d\mathcal{H}^{d-1} \right)^p . \end{aligned}$$

where $C = C(N, \delta, d, p, \Lambda) > 0$. That is, since we choose N depending on η, δ, d, p, Λ and λ,

$$\int_{Q_{s''+12r_0}\setminus Q_{s''+4r_0}} \Lambda \left(\frac{1}{r_0^d} \int_{Q_{r_0}(x)\cap \partial Q_\ell} |u-v|\, d\mathcal{H}^{d-1} \right)^p dx \le C(\eta,\delta,d,p,\Lambda,\lambda) \left(\int_{\partial Q_\ell} |u-v|\, d\mathcal{H}^{d-1} \right)^p, \tag{7.24}$$

where $C(\eta, \delta, d, p, \Lambda, \lambda)$ blows up as $\eta \to 0^+$.

Finally, since all the estimates above are independent of the choice of $\ell \in J$, we conclude by gathering (7.23)-(7.24). □

Remark 7.2. We observe that Lemma 7.1 holds true under the following alternative set of assumptions on the weight functions Λ and λ: There exists a nonincreasing function $\psi\colon [0,1] \to [0,+\infty)$ with $\int_0^1 \psi\, dt < +\infty$ such that for every $x \in \mathbb{R}^d$ and $r \in (0,1)$ there holds

$$\left(\fint_{Q_r(x)} \Lambda(y)\, dy \right) \left(\fint_{Q_r(x)} \lambda^{-1/(p-1)}(y)\, dy \right)^{p-1} \le \psi(r). \tag{7.25}$$

In fact, in this case we can argue as in the proof of Lemma 7.1 now choosing s'' and t'' so that only (7.5) and (7.6) hold true. Moreover, we can skip Step 3 since a Muckenhoupt-like condition is satisfied by assumption, despite with a different right hand-side than in (7.12). Finally, in the dyadic decomposition employed in Step 4, the convergence of the corresponding series is now guaranteed by the fact that

$$\sum_{k=0}^{+\infty} \frac{1}{2^k} \psi\left(\frac{1}{2^k}\right) \le 2 \int_0^1 \psi\, dt.$$

We are now ready to prove the lower bound in the non-(quasi)convex case.

Proposition 7.3. *Let f be an admissible random integrand and assume that (H2) holds. Then, there exists $\Omega' \in \mathcal{F}$ with $\mathbb{P}(\Omega') = 1$ such that for every $\omega \in \Omega'$, every $(u_\varepsilon) \subset L^1_{\rm loc}(\mathbb{R}^d;\mathbb{R}^m)$, and $u \in L^1_{\rm loc}(\mathbb{R}^d;\mathbb{R}^m)$ with $u_\varepsilon \to u$ in $L^1_{\rm loc}(\mathbb{R}^d;\mathbb{R}^m)$ there holds*

$$F_{\rm hom}(\omega)(u, A) \le \liminf_{\varepsilon\to 0} F_\varepsilon(\omega)(u_\varepsilon, A), \tag{7.26}$$

for every $A \in \mathcal{A}$, where $F_{\rm hom}$ is as in (3.8).

Proof. Up to (6.18), the proof is exactly as that of Proposition 6.3. Therefore, here we entirely omit this common part and directly refer to the proof of Proposition 6.3 for it and for the corresponding notation.

Now let $\delta > 0$ be sufficiently small, set $C_{\delta,\rho}(x_0) := Q_{(1-\delta)\rho}(x_0) \setminus Q_{(1-2\delta)\rho}(x_0)$. By applying Lemma 7.1 with $s = 1-2\delta$, $t = 1-\delta$, $u = u_\varepsilon$ and $v = L_u$, for every $\eta > 0$ we find a function $w_\varepsilon \in W^{1,1}(Q_\rho(x_0);\mathbb{R}^m)$, a constant $C = C(\eta, \delta, d, p, \Lambda, \lambda, \omega) > 0$, and an open interval $J_\varepsilon \Subset (1-2\delta, 1-\delta)$ with measure $\mathcal{L}^1(J_\varepsilon) \ge c\delta$ where $c = c(\eta) > 0$, such that

$$w_\varepsilon = u_\varepsilon \quad \text{in } Q_{(1-2\delta)\rho}(x_0), \qquad w_\varepsilon = L_u \quad \text{in } Q_\rho(x_0) \setminus Q_{(1-\delta)\rho}(x_0)$$

and

$$\begin{aligned} \frac{1}{\rho^d}\int_{C_{\delta,\rho}(x_0)} \Lambda\left(\omega, \frac{x}{\varepsilon}\right)|\nabla w_\varepsilon|^p\,dx \le{}& \frac{\eta}{\rho^d}\int_{C_{\delta,\rho}(x_0)} \lambda\left(\omega, \frac{x}{\varepsilon}\right)(|\nabla u_\varepsilon|^p + |\nabla u(x_0)|^p)\,dx \\ &+ C\left(\frac{1}{\rho^d}\int_{\partial Q_{\ell\rho}(x_0)} |u_\varepsilon - L_u|\,d\mathcal{H}^{d-1}\right)^p, \end{aligned} \tag{7.27}$$

for $\mathcal{L}^1$-a.e. every $\ell \in J_\varepsilon$. We recall that the constant C in (7.27) can be taken independent of ε. In fact, the proof con Lemma 7.1 shows that C depends on ε only through the quantities

$$⨍_{Q_\rho(x_0)} \Lambda^\alpha\left(\omega, \frac{x}{\varepsilon}\right)dx, \qquad ⨍_{Q_\rho(x_0)} \lambda^{-\beta}\left(\omega, \frac{x}{\varepsilon}\right)dx.$$

Hence, invoking Theorem 2.5 we obtain the almost sure convergence of the sequences as above to

$$\mathbb{E}[\Lambda^\alpha(\cdot, 0)|\mathcal{F}_\tau], \qquad \mathbb{E}[\lambda^{-\beta}(\cdot, 0)|\mathcal{F}_\tau],$$

respectively, thus C can be chosen independently of ε as claimed.

By Fubini's Theorem we have

$$\begin{aligned} \int_{J_\varepsilon}\int_{\partial Q_{\ell\rho}(x_0)} |u_\varepsilon - L_u|\;d\mathcal{H}^{d-1}\,d\ell &= \frac{1}{\rho}\int_{\rho J_\varepsilon}\int_{\partial Q_\ell(x_0)} |u_\varepsilon - L_u|\;d\mathcal{H}^{d-1}\,d\ell \\ &\le \frac{1}{\rho}\int_{C_{\delta,\rho}(x_0)} |u_\varepsilon - L_u|\;dx; \end{aligned}$$

therefore, since $\mathcal{L}^1(J) \ge c\delta$ we can find $\ell = \ell(\varepsilon) \in J_\varepsilon$ such that

$$\int_{\partial Q_{\ell\rho}} |u_\varepsilon - L_u|\;d\mathcal{H}^{d-1} \le \frac{C(\delta)}{\rho}\int_{C_{\delta,\rho}} |u_\varepsilon - L_u|\;dx. \tag{7.28}$$

Then, by combining (7.27), (7.28) we infer the existence of a constant $C = C(\eta, \delta) > 0$ such that for every $\varepsilon > 0$ sufficiently small

$$\begin{aligned} \frac{1}{\rho^d}\int_{C_{\delta,\rho}(x_0)} \Lambda\left(\omega, \frac{x}{\varepsilon}\right)|\nabla w_\varepsilon|^p\,dx \le{}& \frac{\eta}{\rho^d}\int_{C_{\delta,\ell}(x_0)} \lambda\left(\omega, \frac{x}{\varepsilon}\right)(|\nabla u_\varepsilon|^p + |\nabla u(x_0)|^p)\,dx \\ &+ C\left(\frac{1}{\rho^{d+1}}\int_{C_{\delta,\rho}(x_0)} |u_\varepsilon - L_u|\,dx\right)^p. \end{aligned} \tag{7.29}$$

Hence, invoking (3.2), by (7.29) we get

$$\begin{aligned}\frac{1}{\rho^d}F_\varepsilon(\omega)(w_\varepsilon,Q_\rho(x_0)) &\le \frac{1}{\rho^d}F_\varepsilon(\omega)(u_\varepsilon,Q_{(1-2\delta)\rho}(x_0)) + \frac{1}{\rho^d}\int_{Q_\rho(x_0)\setminus Q_{(1-\delta)\rho}(x_0)} \Lambda\left(\omega,\frac{x}{\varepsilon}\right)(|\nabla u(x_0)|^p+1)\\ &+\frac{1}{\rho^d}\int_{C_{\delta,\rho}(x_0)}\Lambda\left(\omega,\frac{x}{\varepsilon}\right)(|\nabla w_\varepsilon|^p+1)\,dx\\ &\le \frac{1+\eta}{\rho^d}F_\varepsilon(\omega)(u_\varepsilon,Q_\rho(x_0)) + \frac{1+\eta}{\rho^d}\int_{Q_\rho(x_0)\setminus Q_{(1-2\delta)\rho}(x_0)}\Lambda\left(\omega,\frac{x}{\varepsilon}\right)(|\nabla u(x_0)|^p+1)\\ &+C\left(\frac{1}{\rho}\fint_{Q_\rho(x_0)}|u_\varepsilon-L_u|\,dx\right)^p,\end{aligned}$$

for every $\varepsilon>0$ small enough and every $\rho\in(0,1)$.

Then, passing to the liminf as $\varepsilon\to 0$ and recalling that $u_\varepsilon\to u$ in $L^1_{\mathrm{loc}}(\mathbb{R}^d;\mathbb{R}^m)$, we obtain

$$\begin{aligned}\liminf_{\varepsilon\to 0}\frac{1}{\rho^d}F_\varepsilon(\omega)(w_\varepsilon,Q_\rho(x_0)) &\le \liminf_{\varepsilon\to 0}(1+\eta)\frac{1}{\rho^d}F_\varepsilon(\omega)(u_\varepsilon,Q_\rho(x_0))\\ &+C(d)\delta c_1(\omega)(|\nabla u(x_0)|^p+1)+C\left(\frac{1}{\rho}\fint_{Q_\rho(x_0)}|u-L_u|\,dx\right)^p,\end{aligned}\tag{7.30}$$

where $C(d)>0$ is a dimensional constant.

Taking into account that F_ε is invariant under vertical translations, by the definition of w_ε a change of variables gives

$$\lim_{\varepsilon\to 0}\frac{\varepsilon^d}{\rho^d}\mu_{\nabla u(x_0)}\big(\omega,Q_{\frac{\rho}{\varepsilon}}(x_0/\varepsilon)\big)\le\liminf_{\varepsilon\to 0}\frac{1}{\rho^d}F_\varepsilon(\omega)(w_\varepsilon,Q_\rho(x_0)),\tag{7.31}$$

where $\mu_{\nabla u(x_0)}$ is as in (4.1) with $\xi=\nabla u(x_0)$. Moreover, recalling that $u\in W^{1,p}(A;\mathbb{R}^m)$, and that x_0 is a Lebesgue point both for u and ∇u, the L^1-differentiability of u yields

$$\lim_{\rho\to 0}\frac{1}{\rho}\fint_{Q_\rho(x_0)}|u-L_u|\,dx=0.\tag{7.32}$$

Hence, gathering (7.30), (7.31), (7.32), and invoking Proposition 4.2 we infer that for every $\delta>0$ small we have

$$f_{\mathrm{hom}}(\omega,\nabla u(x_0))\le(1+\eta)\liminf_{\rho\to 0}\liminf_{\varepsilon\to 0}\fint_{Q_\rho(x_0)}f\left(\omega,\frac{x}{\varepsilon},\nabla u_\varepsilon\right)dx+C(d)\delta c_1(\omega)\left(|\nabla u(x_0)|^p+1\right).$$

Eventually, the arbitrariness of $\delta>0$ and $\eta>0$ entails the desired lower bound (cf. (6.17)). □

8. On the optimality of the integrability assumptions in the general case

In this section we show that when $p=d=m\ge 3$ the integrability assumptions (3.6) on the weight functions are optimal. Namely, in the deterministic periodic setting we exhibit an admissible periodic integrand f for which the lower bound inequality established in Proposition 7.3 does not hold true. The example we are going to discuss is obtained by suitably modifying a construction proposed in [18].

In order to construct the counterexample, we preliminarily need to introduce some notation.

For $x\in\mathbb{R}^d$ set $x':=(x_1,\dots,x_{d-1})\in\mathbb{R}^{d-1}$. Given $x'\in\mathbb{R}^{d-1}$ and $r>0$ we consider the $(d-1)$-dimensional ball $B'_r(x')$ of center x' and radius r and define the cylinder in $\mathbb{R}^d$

$$\mathbf{C}(x',r):=B'_r(x')\times(-1/2,1/2).$$

Similarly, if $Q'_r(x')$ denotes the $(d-1)$-dimensional cube with center x' and side-length $r>0$, we define

$$\mathbf{D}(x',r):=Q'_r(x')\times(-1/2,1/2).$$

Let $\sigma>0$ be such that $0<\sigma<d^2-2d$. We define the continuous function $\lambda\colon Q\to[0,1]$ as

$$\lambda(x):=\begin{cases}2^{1+\sigma}|x'|^{1+\sigma} & \text{if } x\in\mathbf{C}(0,\frac12),\\ 1 & \text{otherwise in } Q.\end{cases}\tag{8.1}$$

We extend the function as above by Q-periodicity to the whole $\mathbb{R}^d$ and with a little abuse of notation we still denote with λ this extension. We notice that, by construction, λ is continuous on $\mathbb{R}^d$ and $0 \le \lambda \le 1$. Moreover, $\lambda^{-1} \in L^\beta_{\rm loc}(\mathbb{R}^d)$ for every $\beta < \frac{d-1}{1+\sigma}$; additionally, $\beta > \frac{1}{d-1}$ thanks to our choice of σ.

Now, let $\eta > 0$ be fixed and arbitrary and let $f\colon \mathbb{R}^d \times \mathbb{R}^{d\times d} \to [0,\infty)$ be defined as

$$f(x,\xi) := \eta\lambda(x)|\xi|^d + |\det \xi|. \tag{8.2}$$

Therefore, for every $x \in \mathbb{R}^d$ and $\xi \in \mathbb{R}^{d\times d}$ we have

$$\eta\lambda(x)|\xi|^d \le f(x,\xi) \le (1+\eta)|\xi|^d,$$

so that (3.2) is satisfied with $\Lambda \equiv 1+\eta$ and $p = d$. We observe that f is an admissible (periodic) integrand in the sense of Definition 3.1.

We now show that the lower-bound inequality in Proposition 7.3 is violated if λ and f are as in (8.1) and (8.2), respectively. To do this, we need to prove the following lemma.

Lemma 8.1. *Let $\varepsilon \in (0,1)$ and $r > 0$. Then, there exists a one-to-one map $u \in W^{1,\infty}(\mathbf{D}(0,2r);\mathbf{D}(0,2r))$ such that $u(x) = x$ in $\mathbf{D}(0,2r)\setminus\mathbf{C}(0,r)$, $|u(x)-x| \le r$ in $\mathbf{C}(0,r)\setminus\mathbf{C}(0,\varepsilon r)$. Moreover, the following estimates hold*

$$\int_{\mathbf{D}(0,2r)} \lambda\left(\frac{x}{2r}\right)|\nabla u|^d\,dx \le C_0 r^{d-1}, \qquad \int_{\mathbf{C}(0,r)} |\det \nabla u|\,dx \le \varepsilon r^{d-1}, \tag{8.3}$$

for some $C_0 = C_0(d,\sigma) > 0$.

Proof. Let $\delta, \gamma \in (0,1)$ be two parameters to be chosen later. We consider the functions $h, g \in W^{1,\infty}([0,1])$ defined as

$$h(t) := \begin{cases} \dfrac{1-\gamma}{\delta-\gamma\delta^2} & 0 \le t \le \delta, \\ \dfrac{1-\gamma}{t-\gamma t^2} & \delta < t \le 1; \end{cases} \qquad g(t) := \begin{cases} \gamma & 0 \le t \le \delta, \\ \dfrac{1-\gamma}{\delta}t + 2\gamma - 1 & \delta < t < 2\delta, \\ 1 & 2\delta \le t \le 1. \end{cases}$$

We start observing that for every $t \in (0,1)$ we have

$$h(t) \le \min\{1/\delta, 1/t\}, \quad th'(t) \le 1/t, \quad g(t) \le 1, \quad g'(t) \le 1/\delta. \tag{8.4}$$

Moreover, a straightforward computation shows that for $t \in (\delta,1)$ there holds

$$0 < h(t) + th'(t) = \frac{\gamma t^2(1-\gamma)}{(1-\gamma t^2)^2} \le 2\gamma(1-\gamma) \le 2\gamma. \tag{8.5}$$

For $x \in \mathbf{C}(0,r)$, we define $u := (u_1,\dots,u_d)$ componentwise as

$$u_i := h\left(\frac{|x'|}{r}\right)x_i \quad \text{for } i = 1,\dots,d-1, \qquad u_d := g\left(\frac{|x'|}{r}\right)x_d,$$

while for $x \in \mathbf{D}(0,2r)\setminus\mathbf{C}(0,r)$ set $u(x) := x$.

Then, for $x \in \mathbf{C}(0,r)$ we get

$$\nabla u(x) = \begin{pmatrix} h\left(\frac{|x'|}{r}\right)I_{d-1} + \dfrac{h'\left(\frac{|x'|}{r}\right)}{|x'|r}x'(x')^T & 0 \\ \dfrac{g'\left(\frac{|x'|}{r}\right)}{|x'|r}x_d(x')^T & g\left(\frac{|x'|}{r}\right) \end{pmatrix}, \tag{8.6}$$

where I_{d-1} denotes the identity matrix in $\mathbb{R}^{(d-1)\times(d-1)}$.

We now estimate $\det \nabla u$ on $\mathbf{C}(0,r)$. By (8.6), for every $x \in \mathbf{C}(0,r)$ we have

$$\begin{aligned} 0 < \det \nabla u(x) &= h\left(\frac{|x'|}{r}\right)^{d-2} g\left(\frac{|x'|}{r}\right)\left(h\left(\frac{|x'|}{r}\right) + \frac{|x'|}{r}h'\left(\frac{|x'|}{r}\right)\right) \\ &\le \begin{cases} \delta^{1-d}\gamma & 0 < |x'| \le \delta r, \\ \delta^{2-d}2\gamma & \delta r < |x'| < r, \end{cases} \end{aligned} \tag{8.7}$$

where to establish the last inequality we used the definition of h and g together with (8.5). In turn, from (8.7) we deuce that

$$\int_{\mathbf{C}(0,r)} |\det \nabla u|\, dx = \int_{\mathbf{C}(0,r)} \det \nabla u \, dx \leq C\gamma\delta^{2-d} r^{d-1}. \tag{8.8}$$

where $C > 0$ is a dimensional constant.

On the other hand, in view of (8.4) and (8.6) we find

$$|\nabla u(x)| \leq C \begin{cases} \left(\dfrac{1}{\delta r} + \dfrac{r}{|x'|}\right) & 0 < |x'| < 2\delta r, \\ \dfrac{r}{|x'|} & 2\delta r \leq |x'| < 1. \end{cases} \tag{8.9}$$

Therefore, by (8.9) we get

$$\begin{aligned} \int_{\mathbf{C}(0,r)} \lambda\left(\frac{x}{2r}\right) |\nabla u|^d \, dx &\leq C \int_{\mathbf{C}(0,2\delta r)} \frac{|x'|^{1+\sigma}}{r^{1+\sigma}} \left(\frac{1}{\delta r} + \frac{r}{|x'|}\right)^d dx + C \int_{\mathbf{C}(0,r)\setminus\mathbf{C}(0,2\delta r)} \frac{|x'|^{1+\sigma}}{r^{1+\sigma}} \frac{r^d}{|x'|^d}\, dx \\ &\leq C \int_0^{2\delta r} \rho^{d-1+\sigma} r^{-1-\sigma} \left(\delta^{-d} r^{-d} + r^d \rho^{-d}\right) d\rho + C r^{d-1-\sigma} \int_{2\delta r}^{r} \rho^{d-2} \rho^{1+\sigma-d} d\rho \\ &\leq C\delta^{-\delta} r^{-1-\sigma-d} \int_0^{2\delta r} \rho^{d-1-\sigma}\, d\rho + C r^{d-1-\sigma} \int_0^r \rho^{\sigma-1}\, d\rho \\ &\leq C r^{d-1} r^{-d} \delta^{\sigma} + \frac{C}{\sigma} r^{d-1}. \end{aligned} \tag{8.10}$$

For $\varepsilon \in (0,1)$ fixed, we now need to suitably choose $\delta, \gamma \in (0,1)$. First, we impose that $\delta < \varepsilon/2$ so that by definition of u the inequality $|u(x) - x| \leq r$ is satisfied in $\mathbf{C}(0,r) \setminus \mathbf{C}(0,\varepsilon r)$. In order for (8.3) to be satisfied we additionally impose that $r^{-d}\delta^{\sigma} \leq 1$. Moreover, in view of (8.8), we need to choose γ so that $\gamma\delta^{2-d} \leq \varepsilon$. Eventually, since $\nabla u = I$ on $\mathbf{D}(0,2r) \setminus \mathbf{C}(0,r)$, the first estimate in (8.3) follows by combining (8.10) and (8.1). □

Example 8.2 (Counterexample to Proposition 7.3)**.** In this example we show the existence of a sequence $(u_n) \subset W^{1,\infty}(Q;\mathbb{R}^d)$ such that $u_n \rightharpoonup \mathrm{id}$ in $W^{1,1}(Q;\mathbb{R}^d)$ and

$$\liminf_{n\to+\infty} \int_Q f(nx, \nabla u_n)\, dx < \int_Q f_{\hom}(I)\, dx, \tag{8.11}$$

for $\eta > 0$ sufficiently small.

We preliminarily observe that thanks to Proposition 4.2 we have

$$f_{\hom}(I) = \lim_{t\to+\infty} \frac{1}{t^d} \inf\left\{\int_{Q_t} \left(\eta\lambda(x)|\nabla v|^d + |\det \nabla v|\right) dx \colon v \in \ell_I + W_0^{1,1}(Q_t;\mathbb{R}^d)\right\}.$$

Then, using the fact that the determinant is a null-Lagrangian, for every $v \in \ell_I + W_0^{1,1}(Q_t;\mathbb{R}^d)$ we get

$$⨍_{Q_t} \left(\eta\lambda(x)|\nabla v|^d + |\det \nabla v|\right) dx \geq ⨍_{Q_t} |\det \nabla v|\, dx \geq \left| ⨍_{Q_t} \det \nabla v \, dx \right| = 1.$$

Hence, taking the infimum in v readily gives

$$f_{\hom}(I) \geq 1. \tag{8.12}$$

We now use Lemma 8.1 to construct a sequence $(u_n) \subset W^{1,\infty}(Q;\mathbb{R}^d)$ satisfying $u_n \rightharpoonup \mathrm{id}$ in $W^{1,1}(Q;\mathbb{R}^d)$ and

$$\liminf_{n\to+\infty} \int_Q f(nx, \nabla u_n)\, dx < 1,$$

for $\eta > 0$ small enough.

To this end, let $n \in 2\mathbb{N}$ and divide the cube Q into n^{d-1} translated copies of $\mathbf{D}(0,1/n)$. On each $\mathbf{D}(x',1/n)$ we apply Lemma 8.1 with $\varepsilon = 1/n$ to construct functions $u_n^{(j)} \colon \mathbf{D}(x',1/n) \to \mathbf{D}(x',1/n)$ for $j = 1, \ldots, n^{d-1}$. Then, we have

$$|u_n^{(j)}(y) - y| \leq 1/n \quad \text{in} \quad \mathbf{C}(x',1/(2n)) \setminus \mathbf{C}(x',1/(2n^2)), \quad u_n^{(j)} = \mathrm{id} \quad \text{in} \quad \mathbf{D}(x',1/n) \setminus \mathbf{C}(x',1/(2n)),$$

for every $j = 1, \dots, n^{d-1}$. Moreover, there exists $C_0 = C_0(d, \sigma) > 0$ with the following property

$$\int_{\mathbf{D}(x',1/n)} \lambda(ny)|\nabla u_n^{(j)}|^d \, dy \leq \frac{C_0}{n^{d-1}}, \qquad \int_{\mathbf{C}(x',1/(2n))} |\det \nabla u_n^{(j)}| \, dy \leq \frac{1}{n}\frac{1}{n^{d-1}}, \tag{8.13}$$

for every $j = 1, \dots, n^{d-1}$.

We now consider the sequence $(u_n) \subset W^{1,\infty}(Q; \mathbb{R}^d)$ obtained by gluing together the functions $u_n^{(j)}$ on Q. Thus, by (8.13) we deduce

$$\int_Q \lambda(nx)|\nabla u_n|^d \, dx \leq \sum_{j=1}^{n^{d-1}} \frac{C_0}{n^{d-1}} = C_0,$$

as well as

$$\int_Q |\det \nabla u_n| \, dx \leq 1 - \frac{\omega_{d-1}}{2^{d-1}} + \sum_{j=1}^{n^{d-1}} \frac{1}{n^d} = 1 - \frac{\omega_{d-1}}{2^{d-1}} - \frac{1}{n},$$

where ω_{d-1} denotes the Lebesgue measure of the unit ball in $\mathbb{R}^{d-1}$.

Therefore, for $\eta > 0$ small enough we have

$$\liminf_{n \to +\infty} \int_Q f(nx, \nabla u_n) \, dx \leq \eta C_0 + 1 - \frac{\omega_{d-1}}{2^{d-1}} < 1. \tag{8.14}$$

Then, (8.11) follows by gathering (8.14) and (8.12). Moreover, by definition (u_n) converges in measure to the identity on Q and is uniformly bounded in $L^\infty(Q; \mathbb{R}^d)$. Then, by virtue of (8.14) we can appeal to Lemma 3.3 to deduce that $u_n \rightharpoonup \mathrm{id}$ weakly in $W^{1,1}(Q; \mathbb{R}^d)$, as $n \to +\infty$. Thus the claim is achieved.

Statements and declarations

Conflict of interest. The authors have no conflicts of interest to declare.

Data availability. The authors do not analyse or generate any datasets.

Acknowledgments

This work was supported by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) under the Germany Excellence Strategy EXC 2044-390685587, Mathematics Münster: Dynamics–Geometry–Structure. D. R. is a member of the Italian "Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni" (GNAMPA-INdAM).

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