

Optimal sources for elliptic PDEs

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Abstract

We investigate optimal control problems governed by the elliptic partial differential equation $-\Delta u = f$ subject to Dirichlet boundary conditions on a given domain Ω . The control variable in this setting is the right-hand side f , and the objective is to minimize a cost functional that depends simultaneously on the control f and on the associated state function u .

We establish the existence of optimal controls and analyze their qualitative properties by deriving necessary conditions for optimality. In particular, when pointwise constraints of the form $\alpha \leq f \leq \beta$ are imposed a priori on the control, we examine situations where a *bang-bang* phenomenon arises, that is where the optimal control f assumes only the extremal values α and β . More precisely, the control takes the form $f = \alpha 1_E + \beta 1_{\Omega \setminus E}$, thereby placing the problem within the framework of shape optimization. Under suitable assumptions, we further establish certain regularity properties for the optimal sets E .

Finally, in the last part of the paper, we present numerical simulations that illustrate our theoretical findings through a selection of representative examples.

Keywords: shape optimization, optimal potentials, regularity, bang-bang property, optimal control problems.

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1 Introduction

In this paper, we study an optimal control problem for a partial differential equation governed by the Laplace operator in a given bounded domain Ω of \mathbb{R}^d , with homogeneous Dirichlet boundary conditions on $\partial\Omega$. The control variable is the right-hand side f , which is required to lie within a suitably chosen admissible class \mathcal{F} . The associated state equation reads

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u \in H_0^1(\Omega). \end{cases} \quad (1.1)$$

and we denote by u_f the unique weak solution corresponding to a given control f .

The cost functional to be minimized is of the form

$$J(f) = \int_{\Omega} j(x, u_f, f) dx, \quad (1.2)$$

where j is a prescribed integrand satisfying appropriate conditions. The optimal control problem can thus be formulated as

$$\min \{ J(f) : f \in \mathcal{F} \}.$$

We focus on the case where the admissible class \mathcal{F} is defined via an integral constraint of the type

$$\mathcal{F} = \left\{ \int_{\Omega} \psi(f) dx \leq m \right\},$$

for some given $m > 0$ and a convex lower semicontinuous function $\psi : \mathbb{R} \rightarrow [0, \infty]$ satisfying the following hypotheses:

$$\begin{cases} \text{int}(D(\psi)) \neq \emptyset \text{ with } D(\psi) = \{s \in \mathbb{R} : \psi(s) < \infty\} \\ \lim_{|s| \rightarrow +\infty} \psi(s) = +\infty. \end{cases}$$

Under these assumptions, the optimization problem we deal with takes the form

$$\min \left\{ \int_{\Omega} j(x, u_f, f) dx : \int_{\Omega} \psi(f) dx \leq m \right\}, \quad (1.3)$$

A particularly interesting case arises when the control f is constrained to lie between two prescribed constants α and β . This constraint can be expressed by taking

$$\psi(s) = +\infty \quad \text{if } s \notin [\alpha, \beta].$$

Under this setting, and for suitable choices of the integrand j in the cost functional, a *bang-bang* phenomenon may occur, meaning that the optimal control f attains only the extreme values α and β . More precisely, the optimal control takes the form

$$f = \beta 1_E + \alpha 1_{\Omega \setminus E}$$

for some measurable subset $E \subset \Omega$. In this regime, the problem naturally transforms into a *shape optimization problem*, where the control variable is the set E itself. We

devote particular attention to this case, discussing several related aspects, including the regularity properties of the optimal sources f and the structural features of the associated optimal sets E .

Finally, in Section 6, we present a series of numerical simulations that illustrate the theoretical phenomena described and provide concrete examples of the optimal configurations.

2 Notation

In this section, for the convenience of the reader, we introduce and summarize the main notation that will be consistently used throughout the paper.

- We denote by Ω a bounded domain in \mathbb{R}^d .
- Let $\psi : \mathbb{R} \rightarrow (-\infty, \infty]$ be a convex lower semicontinuous function. We introduce the following related notions:

- The domain of ψ , denoted by $D(\psi)$, is defined by

$$D(\psi) = \{s \in \mathbb{R} : \psi(s) < \infty\}.$$

- The conjugate function $\psi^* : \mathbb{R} \rightarrow (-\infty, \infty]$ is given by

$$\psi^*(t) := \sup_{s \in D(\psi)} (ts - \psi(s))$$

- The subdifferential of ψ at a point $s \in D(\psi)$, denoted by $\partial\psi(s)$, is defined as

$$\partial\psi(s) = \{\xi \in \mathbb{R}, \quad \psi(r) \geq \psi(s) + \xi(r - s), \quad \forall r \in \mathbb{R}\} = [d_-\psi(s), d_+\psi(s)],$$

where

$$d_-\psi(s) = \lim_{r \nearrow s} \frac{\psi(r) - \psi(s)}{r - s}, \quad d_+\psi(s) = \lim_{r \searrow s} \frac{\psi(r) - \psi(s)}{r - s}$$

denote, respectively, the left and right derivatives of ψ at s .

- The recession limits of ψ , denoted by $c^-(\psi)$ and $c^+(\psi)$, are defined by

$$c^-(\psi) = \lim_{s \rightarrow -\infty} \frac{\psi(s)}{s} \quad c^+(\psi) = \lim_{s \rightarrow +\infty} \frac{\psi(s)}{s}.$$

- For a bounded open set $\Omega \subset \mathbb{R}^d$, we denote by $\mathcal{M}(\Omega)$ the space of bounded Borel measures on Ω .
- Given $f \in \mathcal{M}(\Omega)$, we denote by f^a and f^s the absolutely continuous and singular parts of f in its Radon-Nikodym decomposition:

$$f = f^a dx + f^s.$$

The positive and negative parts of a measure f are denoted by f_- and f_+ respectively. The support of a measure f is denoted by $\text{supp}(f)$.

- For $s, t \in \mathbb{R}$, we denote by $t \wedge s$ and $t \vee s$ the minimum and maximum of s and t , respectively.
- For any $m > 0$, we define the truncation function $T_m : \mathbb{R} \rightarrow [-m, m]$ at height m , by

$$T_m(s) = (m \wedge s) \vee (-m), \quad \forall s \in \mathbb{R}.$$

3 Existence of an optimal source

In this section, we establish the existence of an optimal source term f under suitable mild assumptions. We begin by considering the case where the function ψ is convex and exhibits superlinear growth at infinity, that is

$$\lim_{|s| \rightarrow +\infty} \frac{\psi(s)}{|s|} = +\infty. \quad (3.1)$$

Theorem 3.1. *Suppose that the functional (1.2) is lower semicontinuous with respect to the weak $L^1(\Omega)$ topology, and that the integrand $j(x, s, z)$ satisfies the growth condition*

$$-c|s|^p - a(x) \leq j(x, s, z), \quad \text{with } c > 0, a \in L^1(\Omega), p < d/(d-2). \quad (3.2)$$

If, in addition, the function ψ satisfies the superlinear growth condition (3.1), then the optimization problem (1.3) admits at least one solution $f_{opt} \in L^1(\Omega)$.

Proof. Assuming that ψ grows superlinearly, any minimizing sequence (f_n) for the optimization problem (1.3) is relatively compact in the weak topology of $L^1(\Omega)$. Thus, up to a subsequence, we may suppose that $f_n \rightarrow f$ weakly in $L^1(\Omega)$ for some $f \in L^1(\Omega)$.

Moreover, due to the compact embedding of $L^1(\Omega)$ into $W^{-1,q}(\Omega)$ for every $q < d/(d-1)$, the corresponding solutions u_n to the PDEs (1.1) converge strongly in $W_0^{1,q}(\Omega)$, and hence strongly in $L^p(\Omega)$ for all $p < d/(d-2)$, to the solution u associated with the limit f .

Finally, by the lower semicontinuity of the mappings

$$f \mapsto J(f), \quad \text{and} \quad f \mapsto \int_{\Omega} \psi(f) dx,$$

with respect to the weak $L^1(\Omega)$ topology, it follows that f indeed minimizes the original functional. Consequently, f is an optimal solution. \square

Remark 3.2. A sufficient condition ensuring the weak $L^1(\Omega)$ lower semicontinuity of the functional J defined in (1.2) is that the integrand $j(x, \cdot, \cdot)$ is lower semicontinuous in its arguments for almost every x , and that $j(x, s, \cdot)$ is convex for almost every x and every s . For further details, we refer the reader to [5].

Remark 3.3. If we strengthen the growth assumption on ψ by requiring that there exists $q > 1$ such that

$$c|s|^q - a \leq \psi(s) \quad \text{for some } c > 0, a \in \mathbb{R}, \quad (3.3)$$

then the growth condition (3.2) on the integrand j can be accordingly relaxed and allows for broader classes of nonlinearities and source terms, adapting to the growth properties of ψ . Specifically, we may assume:

$$\begin{cases} -c|s|^p - a(x) \leq j(x, s, z), & \text{with } c > 0, a \in L^1(\Omega), p < \frac{dq}{d-2q} & \text{if } q < d/2 \\ -ce^{|s|^p} - a(x) \leq j(x, s, z), & \text{with } c > 0, a \in L^1(\Omega), p < \frac{d}{d-1} & \text{if } q = d/2 \\ -a_n(x) \leq j(x, s, z) \text{ for } |s| < n, & \text{with } a_n \in L^1(\Omega), \forall n \in \mathbb{N} & \text{if } q > d/2. \end{cases}$$

We now turn our attention to the case when the function ψ exhibits a linear growth, that is,

$$c|s| - a \leq \psi(s) \quad \text{for some constants } c > 0, a \in \mathbb{R}. \quad (3.4)$$

In this setting, the optimal source term may no longer belong to $L^1(\Omega)$, but may instead be represented by a finite Radon measure. Accordingly, the integral $\int_{\Omega} \psi(f)$ must be interpreted in the sense of measures, namely:

$$\int_{\Omega} \psi(f) = \int_{\Omega} \psi(f^a(x)) dx + c^+(\psi) \int df_+^s - c^-(\psi) \int df_-^s. \quad (3.5)$$

It is a classical result that functionals of the form (3.5) are lower semicontinuous with respect to the weak* convergence of measures.

Theorem 3.4. *Suppose that the functional (1.2) is weakly* lower semicontinuous in the space $\mathcal{M}(\Omega)$ of finite Radon measures, and that the integrand j satisfies the growth condition*

$$-c|s|^p - a(x) \leq j(x, s, z), \quad \text{for some } c > 0, a \in L^1(\Omega), p < d/(d-2).$$

If, in addition, the function ψ satisfies the linear growth condition (3.4), then the optimization problem (1.3) admits at least one optimal solution f_{opt} , which is a measure with finite total variation.

Proof. The proof proceeds along similar lines as that of Theorem 3.1. Let (f_n) be a minimizing sequence for the optimization problem (1.3). Since (f_n) is bounded in the space of finite Radon measures, by the Banach-Alaoglu theorem, we can extract a subsequence (still denoted by (f_n)) which converges to some measure f in the weak* topology of $\mathcal{M}(\Omega)$.

The corresponding sequence of solutions (u_n) to the PDEs (1.1) then converges strongly in $W_0^{1,q}(\Omega)$ for every $q < d/(d-1)$, and therefore also strongly in $L^p(\Omega)$ for every $p < d/(d-2)$, to the solution u associated with the limit measure f .

Finally, the weak* lower semicontinuity of both terms involved in the optimization problem (1.3),

$$f \mapsto J(f), \quad \text{and} \quad f \mapsto \int_{\Omega} \psi(f),$$

ensures that f is indeed an optimal solution to (1.3). \square

Remark 3.5. A sufficient condition for the lower semicontinuity of the functional J in (1.2) with respect to the weak* convergence of measures is the following (see for example [4]). Suppose the integrand $j(x, s, z)$ admits the decomposition in the form

$$j(x, s, z) = A(x, s) + B(x, z),$$

where the functions A and B satisfy the following properties:

- for almost every $x \in \Omega$ the function $A(x, \cdot)$ is lower semicontinuous;
- there exist constants $c > 0$, $p < d/(d - 2)$ and a function $a \in L^1(\Omega)$ such that

$$A(x, s) \geq -c|s|^p + a(x);$$

- for almost every $x \in \Omega$ the function $B(x, \cdot)$ is convex and lower semicontinuous;
- the associated recession function

$$B^\infty(x, z) = \lim_{t \rightarrow +\infty} \frac{B(x, tz)}{t}$$

is lower semicontinuous with respect to both variables (x, z) ;

- there exist functions $a_0 \in C_0(\Omega)$ and $a_1 \in L^1(\Omega)$ such that

$$B(x, z) \geq a_0(x)z + a_1(x).$$

The assumptions on the function A allow to obtain the lower semicontinuity thanks to the Fatou's lemma, while the assumptions on the function B allow to obtain the lower semicontinuity thanks to the results on functionals defined on measures. For all the details we refer to [4], where more general cases, including the ones where the functional J is not convex, are considered.

4 Necessary conditions of optimality

In this section, we derive some necessary conditions of optimality that any solution f_{opt} must satisfy. These conditions are presented in Theorem 4.1 below. To this end, it is convenient to introduce the resolvent operator \mathcal{R} , which associates to every function f the unique solution u of the partial differential equation (1.1). It is well known that \mathcal{R} is a self-adjoint operator.

Theorem 4.1. *Suppose that the function j appearing in the formulation of the optimal control problem (1.3) satisfies the growth condition*

$$|j(x, s, z)| \leq a(x) + c|s|^p, \quad \text{with } c > 0, a \in L^1(\Omega), p < d/(d - 2).$$

In addition, we assume that one of the following conditions holds.

- (Case of superlinear growth): If ψ satisfies the superlinear growth condition (3.1), then for almost every $x \in \Omega$ and every $(s, z) \in \mathbb{R}^2$, the partial derivatives $\partial_s j(x, s, z)$ and $\partial_z j(x, s, z)$ exist and fulfill

$$\begin{cases} |\partial_s j(x, s, z)| \leq b(x) + \gamma(|s|^\sigma + |z|^\tau) \\ |\partial_z j(x, s, z)| \leq \gamma, \end{cases} \quad (4.1)$$

where $\gamma > 0$, $b \in L^q(\Omega)$ with $q > d/2$, $\sigma < 2/(d-2)$, and $\tau < 2/d$.

- (Case of linear growth): If ψ exhibits a linear growth, meaning $c^+(\psi) - c^-(\psi) > 0$, then $j = j(x, s, z)$ depends only on (x, s) and not on z . In this case, for almost every $x \in \Omega$ and every $s \in \mathbb{R}$, the partial derivative $\partial_s j(x, s)$ exists and satisfies

$$|\partial_s j(x, s)| \leq b(x) + \gamma|s|^\sigma, \quad (4.2)$$

where again $\gamma > 0$, $b \in L^q(\Omega)$ with $q > d/2$, $\sigma < 2/(d-2)$.

Then, if f_{opt} is an optimal solution to the problem (1.3), there exists a non-negative scalar $\lambda \geq 0$ such that

$$\lambda \left(\int_{\Omega} \psi(f_{\text{opt}}) dx - m \right) = 0, \quad (4.3)$$

and, setting

$$w := \mathcal{R}(\partial_s j(x, \mathcal{R}(f_{\text{opt}}), f_{\text{opt}})) + \partial_z j(x, \mathcal{R}(f_{\text{opt}}), f_{\text{opt}}), \quad (4.4)$$

the following alternative holds:

- If $\lambda = 0$, then

$$\begin{cases} w \geq 0 \text{ a.e. in } \Omega \text{ if } \sup(D(\psi)) = +\infty \\ w \leq 0 \text{ a.e. in } \Omega \text{ if } \inf(D(\psi)) = -\infty \\ f_{\text{opt}}^a = \min(D(\psi)) \text{ a.e. in } \{w > 0\} \\ f_{\text{opt}}^a = \max(D(\psi)) \text{ a.e. in } \{w < 0\} \\ \text{supp}(f_{\text{opt}}^s) \subset \{w = 0\}. \end{cases} \quad (4.5)$$

- If $\lambda > 0$, then

$$\begin{cases} \psi(f_{\text{opt}}^a) + \psi^*\left(-\frac{w}{\lambda}\right) = -\frac{w f_{\text{opt}}^a}{\lambda} \text{ a.e. in } \Omega \\ -\lambda c^+(\psi) \leq w \leq -\lambda c^-(\psi) \text{ a.e. in } \Omega \\ \text{supp}(f_{\text{opt},+}^s) \subset \{w + \lambda c^+(\psi) = 0\} \\ \text{supp}(f_{\text{opt},-}^s) \subset \{w + \lambda c^-(\psi) = 0\}. \end{cases} \quad (4.6)$$

Moreover, if the function $j(x, \cdot, \cdot)$ is convex for almost every $x \in \Omega$, then the conditions stated above are not only necessary for optimality but also sufficient.

Proof. Since the function ψ is convex, for any $f \in \mathcal{M}(\Omega)$ satisfying the constraint $\int_{\Omega} \psi(f) dx \leq m$, the mapping

$$\varepsilon \in [0, 1] \mapsto \int_{\Omega} j(x, \mathcal{R}(f_{opt} + \varepsilon(f - f_{opt})), f_{opt} + \varepsilon(f - f_{opt})) dx$$

attains its minimum at $\varepsilon = 0$. Thanks to the regularity assumptions (4.1) or (4.2), combined with the fact that $\mathcal{R}(f_{opt}) \in L^r(\Omega)$ for every $r \in [1, d/(d-2)]$, we can differentiate under the integral sign with respect to ε at $\varepsilon = 0$, leading to

$$\begin{aligned} 0 &\leq \int_{\Omega} \left(\partial_s j(x, \mathcal{R}(f_{opt}), f_{opt}) \mathcal{R}(f - f_{opt}) + \partial_z j(x, \mathcal{R}(f_{opt}), f_{opt})(f - f_{opt}) \right) dx \\ &= \int_{\Omega} \left(\mathcal{R}(\partial_s j(x, \mathcal{R}(f_{opt}), f_{opt})) + \partial_z j(x, \mathcal{R}(f_{opt}), f_{opt}) \right) (f - f_{opt}) dx \\ &= \int_{\Omega} w(f - f_{opt}) dx, \end{aligned}$$

where, recalling (4.4), we have set

$$w = \mathcal{R}(\partial_s j(x, \mathcal{R}(f_{opt}), f_{opt})) + \partial_z j(x, \mathcal{R}(f_{opt}), f_{opt}).$$

Thus, we deduce that f_{opt} solves the following convex minimization problem:

$$\min \left\{ \int_{\Omega} w f dx : \int_{\Omega} \psi(f) \leq m \right\}. \quad (4.7)$$

Applying the Kuhn-Tucker theorem, we infer the existence of a Lagrange multiplier $\lambda \geq 0$ satisfying the complementary condition (4.3), such that f_{opt} is a solution to

$$\begin{cases} \min \left\{ \int_{\Omega} w f dx + \lambda \int_{\Omega} \psi(f) dx : f \in \mathcal{M}(\Omega) \right\} & \text{if } \lambda > 0 \\ \min \left\{ \int_{\Omega} w f dx : f \in \mathcal{M}(\Omega), f^a \in D(\psi) \text{ a.e. in } \Omega \right\} & \text{if } \lambda = 0. \end{cases} \quad (4.8)$$

In particular, this shows that, almost everywhere in Ω , the absolutely continuous part $f_{opt}^a(x)$ solves the following pointwise minimization problem:

$$\begin{cases} \min_{s \in \mathbb{R}} \{ w(x)s + \lambda \psi(s) \} & \text{if } \lambda > 0 \\ \min_{s \in D(\psi)} w(x)s & \text{if } \lambda = 0, \end{cases}$$

thereby establishing the first four conditions in (4.5) and the first condition in (4.6).

Let us now assume that $c^+(\psi) > 0$ (hence $w \in C^0(\overline{\Omega})$). Suppose by contradiction that there exists $x \in \overline{\Omega}$ such that $w(x) + \lambda c^+(\psi) < 0$. Then, considering the test measure $f = n\delta_x$ with $n > 0$, and letting $n \rightarrow \infty$, we observe that the value of the minimization problem (4.8) would tend to $-\infty$, contradicting the existence of an optimal solution f_{opt} . Consequently, we must have $w(x) + \lambda c^+(\psi) \geq 0$ for all $x \in \overline{\Omega}$.

Furthermore, noting that for any nonnegative singular measure f^s it holds

$$0 \leq \int_{\Omega} (w + \lambda c^+(\psi)) df_+^s,$$

we deduce that the support of the positive part of the singular component satisfies

$$\text{supp}(f_{opt,+}^s) \subset \{w + \lambda c^+(\psi) = 0\}.$$

A similar argument, considering the case $c^-(\psi) < 0$, yields that

$$\begin{cases} w + \lambda c^-(\psi) \leq 0 & \text{in } \overline{\Omega}, \\ \text{supp}(f_{opt,-}^s) \subset \{w + \lambda c^-(\psi) = 0\}. \end{cases}$$

Finally, when $j(x, \cdot, \cdot)$ is convex for almost every $x \in \Omega$, the original optimization problem (1.3) is itself convex. In this case, f_{opt} solves (1.3) if and only if it solves the equivalent convex minimization problem (4.7), and thus if and only if the necessary optimality conditions stated in Theorem 4.1 are satisfied. \square

Remark 4.2. The first condition in (4.6) can equivalently be reformulated in either of the following forms:

$$-w \in \lambda \partial \psi(f_{opt}^a) \text{ a.e. in } \Omega \quad \text{or} \quad f_{opt}^a \in \partial \psi^*\left(-\frac{w}{\lambda}\right) \text{ a.e. in } \Omega. \quad (4.9)$$

The second formulation provides a characterization of the optimal control f_{opt}^a directly in terms of the adjoint variable w .

In the present work, our primary interest is focused on the case where the optimal control f_{opt}^a exhibits a bang-bang structure. According to the second condition in (4.9), such a behavior arises if there exists a point $s \in \text{int}(D(\psi^*))$ where the convex conjugate ψ^* fails to be differentiable. More precisely, under this assumption, we have

$$\partial \psi^*(s) = [d_-\psi^*(s), d_+\psi^*(s)], \quad -\infty < d_-\psi^*(s) < d_+\psi^*(s) < \infty, \quad (4.10)$$

which leads to the following characterization:

$$\begin{cases} f_{opt}^a(x) \geq d_+\psi^*(s) & \text{if } w(x) < -\lambda s \\ f_{opt}^a(x) \leq d_-\psi^*(s) & \text{if } w(x) > -\lambda s. \end{cases} \quad (4.11)$$

It is important to note that if the set $\{x \in \Omega : w(x) = -\lambda s\}$ has a positive Lebesgue measure, then condition (4.11) does not necessarily imply that f_{opt}^a is discontinuous on this set.

Assuming furthermore that the function $j(x, s, z)$ is independent of z , and recalling that the function $w = \mathcal{R}(\partial_s j(x, \mathcal{R}(f_{opt})))$ belongs to $W_{loc}^{2,q}(\Omega)$, it follows that $\Delta w = 0$ almost everywhere in $\{w = s\}$, for every $s \in \mathbb{R}$. Consequently, we obtain:

$$|\{\partial_s j(x, \mathcal{R}(f_{opt})) = 0\}| = 0 \implies |\{w = s\}| = 0, \quad \forall s \in \mathbb{R}. \quad (4.12)$$

A particularly simple sufficient condition to ensure (4.12) is that the map $s \mapsto j(x, s)$ be either strictly increasing or strictly decreasing for each $x \in \Omega$.

On the other hand, it is useful to recall that condition (4.10) is equivalent to the relation

$$\psi(t) = st - \psi^*(s) \quad \forall t \in [d_-\psi^*(s), d_+\psi^*(s)],$$

meaning that ψ must be affine on an interval of positive length. Therefore, a necessary condition on the function ψ for the appearance of bang-bang optimal controls is the existence of a bounded interval with nonempty interior on which ψ is affine, that is, the function ψ must fail to be strictly convex over some nontrivial subinterval.

Remark 4.3. By an argument similar to the one of Remark 3.3, the growth conditions imposed on the function j and its derivatives in Theorem 4.1 can be relaxed when the function ψ satisfies the condition (3.3). Specifically, when $q > d/2$, it suffices to require that, for every $n > 0$,

$$|j(x, s, z)| + |\partial_s j(x, s, z)| \leq a_n(x) + c_n |z|^q \quad \text{for } |s| < n,$$

where $a_n \in L^1(\Omega)$ and $c_n > 0$ are given, and similarly,

$$|\partial_z j(x, s, z)| \leq b_n(x) + \gamma_n |z|^{q-1} \quad \text{for } |s| < n,$$

with $b_n \in L^{q/(q-1)}(\Omega)$ and $\gamma_n > 0$.

We are now ready to illustrate the application of Theorem 4.1 through several important examples of the function ψ .

Example 4.4. Let us now consider the case where $\psi(s) = |s|$. In this setting, problem (1.3), under the assumption that $j(x, s, z)$ is independent of z and satisfies the growth conditions (4.2), can be rewritten as:

$$\min \left\{ \int_{\Omega} j(x, \mathcal{R}(f)) dx : \|f\|_{\mathcal{M}(\Omega)} \leq m \right\}. \quad (4.13)$$

In order to apply Theorem 4.1 together with the characterization provided in Remark 4.2, we first observe the properties of the convex conjugate ψ^* , namely:

$$\psi^*(t) = \begin{cases} 0 & \text{if } t \in [-1, 1] \\ +\infty & \text{otherwise,} \end{cases} \quad \partial\psi^*(t) = \begin{cases} [-\infty, 0] & \text{if } t = -1 \\ 0 & \text{if } t \in (-1, 1) \\ [0, \infty] & \text{if } t = 1. \end{cases}$$

If $\lambda = 0$ in the framework of Theorem 4.1, then, according to condition (4.5) and the fact that $D(\psi) = \mathbb{R}$, the optimality system simply reduces to

$$w = \mathcal{R}(\partial_s j(x, \mathcal{R}(f_{opt}))) = 0 \quad \text{almost everywhere in } \Omega,$$

which is equivalent to the condition:

$$\partial_s j(x, \mathcal{R}(f_{opt})) = 0 \quad \text{a.e. in } \Omega.$$

Let us assume now that we are not in this degenerate case, so that $\lambda > 0$. In this case, Theorem 4.1 combined with the optimality conditions (4.9) yield the following set of properties:

$$\begin{cases} -\lambda \leq w \leq \lambda \text{ a.e. in } \Omega, \\ \text{supp}(f_{opt}) \subset \{|w| = \lambda\}, \\ f_{opt} \geq 0 \text{ in } \{w = -\lambda\}, \\ f_{opt} \leq 0 \text{ in } \{w = \lambda\}, \\ \|f\|_{\mathcal{M}(\Omega)} = m. \end{cases}$$

In particular, let us consider the situation where the function $s \mapsto j(x, s)$ is non-decreasing for almost every $x \in \Omega$. In this case, we have $\partial_s j(x, \cdot) \geq 0$, which, by

the maximum principle applied to w , implies that $w \geq 0$ almost everywhere in Ω . Therefore, w satisfies $0 \leq w \leq \lambda$ a.e. in Ω , and the support of the optimal control is contained in the set $\{w = \lambda\}$, with $f_{opt} \leq 0$.

For instance, if Ω is a ball centered at the origin and $j(x, s) = s$, the solution simplifies further, and the optimal control is given explicitly by:

$$f_{opt} = -m\delta_0.$$

where δ_0 denotes the Dirac mass at the origin.

An entirely similar analysis can be carried out when $j(x, \cdot)$ is non-increasing, leading to the symmetric case.

Example 4.5. In connection with problem (4.13), let us now consider the variational problem

$$\min \left\{ \int_{\Omega} j(x, \mathcal{R}(f)) dx : f \geq 0, \int_{\Omega} f dx \leq m \right\}. \quad (4.14)$$

In this context, the function ψ is given by

$$\psi(s) = \begin{cases} s & \text{if } s \geq 0 \\ +\infty & \text{if } s < 0, \end{cases}$$

and its convex conjugate ψ^* takes the form

$$\psi^*(t) = \begin{cases} 0 & \text{if } t \leq 1 \\ \infty & \text{if } t > 1, \end{cases}$$

with

$$\partial\psi^*(t) = \begin{cases} 0 & \text{if } t < 1 \\ [0, \infty] & \text{if } t = 1. \end{cases}$$

Let f_{opt} be an optimal solution to problem (4.14). Then, by applying the optimality conditions (4.5) and (4.6), we infer the existence of a Lagrange multiplier $\lambda \geq 0$ such that

$$\begin{cases} \lambda \left(\int_{\Omega} \psi(f_{opt}) dx - m \right) = 0 \\ w \geq -\lambda \text{ a.e. in } \Omega \\ \text{supp}(f_{opt}) \subset \{w = -\lambda\}. \end{cases}$$

This result admits a more refined characterization under additional assumptions. Suppose that for almost every $x \in \Omega$, the function $s \mapsto j(x, s)$ is strictly concave. In that case, the optimal source f_{opt} must be an extremal point of the admissible set

$$\left\{ f \geq 0 : \int_{\Omega} f dx \leq m \right\}.$$

Consequently, the optimal solution must be a singular measure supported at a point, that is, a multiple of a Dirac delta. Assume furthermore that for almost every $x \in \Omega$,

the function $j(x, \cdot)$ attains its maximum at $s = 0$. Since $f_{opt} \geq 0$, it follows that $\mathcal{R}(f_{opt}) \geq 0$, and hence,

$$\partial_s j(x, \mathcal{R}(f_{opt})) \leq \partial_s j(x, 0) \leq 0 \quad \text{a.e. in } \Omega,$$

implying that the adjoint state $w = \mathcal{R}(\partial_s j(x, \mathcal{R}(f_{opt}))) \leq 0$ almost everywhere in Ω . The case where $w = 0$ a.e. leads to a contradiction, as it would imply $\mathcal{R}(f_{opt}) = 0$ a.e., which would in turn correspond to the maximum, not the minimum, of the functional in (4.14). Thus, we conclude that the Lagrange multiplier λ must be strictly positive, and we obtain the refined optimality condition:

$$-\lambda \leq w \leq 0 \quad \text{a.e. in } \Omega, \quad f_{opt} = m\delta_{x_0} \quad \text{with } w(x_0) = -\lambda. \quad (4.15)$$

As a concrete example, consider the maximization problem

$$\max \left\{ \int_{\Omega} |\mathcal{R}(f)|^p dx : f \geq 0, \quad \int_{\Omega} f dx \leq m \right\}. \quad (4.16)$$

It is readily seen that if $p \geq d/(d-2)$, the functional is unbounded above and the supremum is infinite, hence no optimal solution exists. However, when $p < d/(d-2)$, the problem admits a solution f_{opt} , and it satisfies the structure described in (4.15).

Example 4.6. Let us consider the following optimization problem:

$$\min \left\{ \int_{\Omega} j(x, \mathcal{R}(f), f) dx : \int_{\Omega} f dx \leq m, \quad \alpha \leq f \leq \beta \right\}, \quad (4.17)$$

subject to the bounds

$$\alpha|\Omega| < m \leq \beta|\Omega|, \quad (4.18)$$

where α and β are real constants. Without loss of generality, and to simplify the exposition, we assume $\alpha \geq 0$; the treatment of other cases (e.g., when $\alpha < 0$) follows in a similar way. The admissible set is naturally associated with the function

$$\psi(s) = \begin{cases} s & \text{if } s \in [\alpha, \beta] \\ +\infty & \text{otherwise,} \end{cases}$$

whose convex conjugate is given by

$$\psi^*(t) = \begin{cases} (t-1)\alpha & \text{if } t \leq 1 \\ (t-1)\beta & \text{if } t \geq 1, \end{cases}$$

with

$$\partial\psi^*(t) = \begin{cases} \alpha & \text{if } t < 1 \\ [\alpha, \beta] & \text{if } t = 1 \\ \beta & \text{if } t > 1. \end{cases}$$

By Theorem 4.1, any optimal solution f_{opt} of problem (4.17) must satisfy the point-wise condition

$$f_{opt} = \begin{cases} \beta & \text{if } w < -\lambda \\ \alpha & \text{if } w > -\lambda, \end{cases} \quad (4.19)$$

where w is the adjoint state defined via (4.4), and $\lambda \geq 0$ is a Lagrange multiplier associated with the volume constraint, satisfying the complementary condition

$$\lambda \left(\int_{\Omega} f \, dx - m \right) = 0. \quad (4.20)$$

Since the adjoint variable w is known to vanish on the boundary $\partial\Omega$ due to the properties of \mathcal{R} , the structure of the optimal solution f_{opt} is particularly simple when the following conditions occur:

$$\lambda > 0, \quad |\{w < -\lambda\}| > 0, \quad |\{w = -\lambda\}| = 0.$$

Under these hypotheses, the optimal control f_{opt} is of bang-bang type; that is, it takes only the extremal values α and β almost everywhere in Ω .

Let us now examine how the qualitative nature of f_{opt} depends on the structure of the integrand j . Assume that the function $j(x, s, z)$ is independent of z and is either non-decreasing or non-increasing in the variable s . In the first case, where j is non-decreasing in s , the adjoint state is non-negative:

$$w = \mathcal{R}(\partial_s j(x, \mathcal{R}(f_{opt}))) \geq 0.$$

Then, from (4.19), it follows that $f_{opt} = \alpha$ almost everywhere in Ω .

In contrast, if j is non-increasing in s , then $w \leq 0$ a.e. in Ω . Suppose that the measure of the set $\{w < -\lambda\}$ is zero. Then, again from (4.19), we have $f_{opt} = \alpha$ a.e., and so

$$\int_{\Omega} f_{opt} \, dx = \alpha |\Omega| < m,$$

which implies, by (4.20), that $\lambda = 0$. Consequently, the adjoint state w must vanish identically, and $\partial_s j(x, \mathcal{R}(\alpha)) = 0$ as well. This is only possible if for a.e. $x \in \Omega$ the function $j(x, \cdot)$ is constant in the interval $[\mathcal{R}(\alpha), 0]$ or in the interval $[0, \mathcal{R}(\alpha)]$ (depending on the sign of α).

If this constancy condition is not satisfied, then necessarily $\lambda > 0$, and the volume constraint $\int_{\Omega} f \, dx \leq m$ is saturated. In this situation, the function f_{opt} takes both values α and β , as described by (4.19). In particular, this occurs whenever $j(x, \cdot)$ is strictly decreasing, in which case the condition $|\{w = -\lambda\}| = 0$ is also satisfied, and f_{opt} is indeed a bang-bang control.

Example 4.7. Another interesting example corresponds to

$$\min \left\{ \int_{\Omega} |\mathcal{R}(f) - u_0|^2 \, dx : \int_{\Omega} f \, dx \leq m, \alpha \leq f \leq \beta \right\},$$

with $u_0 \in L^2(\Omega)$ prescribed and m satisfying (4.18). This case has been studied, with $\alpha = 0$ and $\beta = 1$, in [12]. Since this functional is strictly convex, the solution is unique and (4.19), (4.20) are necessary and sufficient conditions for f_{opt} , where now $w = 2\mathcal{R}(\mathcal{R}(f_{opt}) - u_0)$.

Since $f_{opt} \in [\alpha, \beta]$, we have $\mathcal{R}(f_{opt}) \in [\mathcal{R}(\alpha), \mathcal{R}(\beta)]$. If $u_0 \leq \mathcal{R}(\alpha)$ a.e. in Ω , the maximum principle gives $\mathcal{R}(\mathcal{R}(\alpha) - u_0) \geq 0$ in Ω and then $f_{opt} = \alpha$ satisfies (4.19) with $\lambda = 0$. Analogously, if $u_0 \geq \mathcal{R}(\beta)$ a.e. in Ω , then $f_{opt} = \beta$.

Assume $u_0 \in [\mathcal{R}(\alpha), \mathcal{R}(\beta)]$ a.e. in Ω and $u_0 \not\equiv \mathcal{R}(\alpha)$, $u_0 \not\equiv \mathcal{R}(\beta)$. If $f_{opt} = \alpha$ a.e. in Ω , the strong maximum principle gives $w > 0$ in Ω while

$$\int_{\Omega} f \, dx = \alpha|\Omega| < m$$

implies $\lambda = 0$. By (4.19) we conclude that $f_{opt} = \beta$, in contradiction with $f_{opt} = \alpha$. Similarly, if $f_{opt} = \beta$ a.e. in Ω , we get $w > 0$ a.e. in Ω in contradiction with (4.19). Taking into account (4.19) we then deduce that $|\{w = \lambda\}| = 0$ implies that f_{opt} is a bang-bang control.

Another case in which f_{opt} is of bang-bang type, again deduced from the necessary conditions of optimality (4.19), is when $-\Delta u_0 \geq \beta$ a.e. in Ω and $u_0 \geq 0$ on $\partial\Omega$.

Example 4.8. Let us now consider an example where ψ is strictly convex. By Remark 4.2 the optimal controls are not of bang-bang type. We take

$$\min \left\{ \int_{\Omega} j(x, \mathcal{R}(f), f) \, dx : \int_{\Omega} f^2 \, dx \leq m \right\}, \quad m > 0.$$

Now,

$$\psi(s) = s^2, \quad \psi^*(s) = \frac{t^2}{4}, \quad \partial\psi^*(t) = \frac{t}{2}.$$

Therefore, if f is an optimal solution and w is given by (4.4) we have the existence of $\lambda \geq 0$ such that

$$w = 0 \text{ a.e. in } \Omega \quad \text{or} \quad f_{opt} = \frac{\sqrt{m} w^2}{\|w\|_{L^4(\Omega)}^2}.$$

In the second case f_{opt} is a continuous function by the summability assumptions on j and their derivatives.

Example 4.9. Consider the *compliance case*

$$\min \left\{ \int_{\Omega} f \mathcal{R}(f) \, dx : \int_{\Omega} f \, dx \geq m, \alpha \leq f \leq \beta \right\}, \quad (4.21)$$

and assume $0 \leq \alpha < \beta$. To have a nontrivial problem we also assume $\alpha|\Omega| < m < \beta|\Omega|$. Using an integration by parts we have

$$\int_{\Omega} f \mathcal{R}(f) \, dx = -2\mathcal{E}(f)$$

where $\mathcal{E}(f)$ is the energy

$$\mathcal{E}(f) = \min \left\{ \int_{\Omega} \left(\frac{1}{2} |\nabla u|^2 - fu \right) \, dx : u \in H_0^1(\Omega) \right\},$$

and thus the optimization problem can be reformulated as

$$\max \left\{ \mathcal{E}(f) : \int_{\Omega} f \, dx \geq m, \alpha \leq f \leq \beta \right\}.$$

Similarly to example (4.6) we have

$$\psi(s) = \begin{cases} -s & \text{if } s \in [\alpha, \beta] \\ +\infty & \text{otherwise,} \end{cases}$$

$$\psi^*(t) = \begin{cases} (t+1)\alpha & \text{if } t \leq -1 \\ (t+1)\beta & \text{if } t \geq -1, \end{cases} \quad \partial\psi^*(t) = \begin{cases} \alpha & \text{if } t < -1 \\ [\alpha, \beta] & \text{if } t = -1 \\ \beta & \text{if } t > -1, \end{cases}$$

and that m in Theorem 4.1 must be chosen as $-m$.

Since $j(x, s, z) = sz$, we have that for a solution f_{opt} of (4.21), the function w defined by (4.4) is given by $w = 2\mathcal{R}(f_{opt})$, where $f_{opt} \in [\alpha, \beta]$ implies $\mathcal{R}(f_{opt})$ strictly positive in Ω . Thus, Theorem 4.1 proves the existence of $\lambda > 0$ such that

$$f_{opt} = \begin{cases} \beta & \text{if } \mathcal{R}(f_{opt}) < \lambda \\ \alpha & \text{if } \mathcal{R}(f_{opt}) > \lambda, \end{cases} \quad \int_{\Omega} f_{opt} ds = m.$$

Moreover, as we saw in Remark 4.2 the set $\{\mathcal{R}(f_{opt}) = \lambda\}$ has null measure. We are then in the bang-bang situation $f_{opt} = \alpha 1_E + \beta 1_{\Omega \setminus E}$ for $E = \{\mathcal{R}(f_{opt}) > \lambda\}$.

5 Regularity of optimal sources

We have seen in Section 4 that if the function ψ in (1.3) is not strictly convex, then the optimal solutions are of bang-bang type, where the interfaces are given by $\{w = s\}$, with w defined by (4.4) and $s \in \mathbb{R}$ (indeed, if this set has positive measure, then the optimal control could be continuous). The question we consider in the present section is to get some regularity results for bang-bang optimal solutions. Since they are discontinuous, we can ask whether they are BV functions, that is, whether the set $\{w = s\}$ has a finite perimeter.

5.1 BV regularity

As a model problem, we can consider the compliance case of Example 4.9:

$$\min \left\{ \int_{\Omega} f \mathcal{R}(f) dx : \int_{\Omega} f dx \geq m, f(x) \in [\alpha, \beta] \right\}, \quad (5.1)$$

with $0 \leq \alpha < \beta$ and $\alpha|\Omega| < m < \beta|\Omega|$. We have seen that the optimal solution f_{opt} is of bang-bang type, that is

$$f_{opt} = \alpha 1_E + \beta 1_{\Omega \setminus E} \quad \text{with } E = \{\mathcal{R}(f_{opt}) < s\},$$

for some positive constant s that has to be chosen such that the integral constraint $\int_{\Omega} f dx \geq m$ is saturated. The function $u = \mathcal{R}(f_{opt})$ thus solves the PDE

$$\begin{cases} -\Delta u = \beta 1_{\{u < s\}} + \alpha 1_{\{u > s\}} & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Theorem 5.1. *The optimal solution f_{opt} of the minimization problem (5.1) is in $BV(\Omega)$, hence the optimal set E above has a finite perimeter*

Proof. It is enough to apply Theorem 3.5 of [6]. □

5.2 A weaker regularity

Similarly to the above example, Theorem 4.1 and Remark 4.2 with $j(x, s, z)$ independent of z prove that for bang-bang optimal controls, the interfaces are of the form $\{u = s\}$ with u the solution of the PDE

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where we set $f = \partial_s j(x, \mathcal{R}(f_{opt}))$. Some results about the regularity of the level sets of the solution of the above problem are simple to obtain. On the one hand, if $f \in L^q(\Omega)$, with $q > d$ then u is in $C^1(\bar{\Omega})$. Thus, the implicit function theorem proves that for every $s \in \mathbb{R}$ the set

$$\{u = s\} \cap \{\nabla u \neq 0\}$$

is a C^1 manifold. On the other hand, for u just in $BV(\Omega)$ the coarea formula ([10], Chapter 5) gives

$$\int_{\Omega} d|\nabla u| = \int_{\mathbb{R}} \|\nabla 1_{\{u>s\}}\|_{\mathcal{M}(\Omega)} ds.$$

Thus, except for s in a subset of \mathbb{R} of null Lebesgue measure we have

$$1_{\{u>s\}} \in BV(\Omega). \quad (5.2)$$

The question is now if, adding some assumptions on f , property (5.2) holds for every $s \in \mathbb{R}$. Since the difficulties appear in the set $\{\nabla u = 0\}$, let us assume that f is positive in Ω (by linearity, if f is negative the argument is similar) in such way that this set has null Lebesgue measure.

The result below is slightly weaker than (5.2). We will only prove that for any $q > 1$, we have

$$\log^{-q} \left(\frac{1}{|\nabla u|} \vee e \right) 1_{\{u>s\}} \in BV(\Omega).$$

Observe that the factor $\log^{-q} (1/|\nabla u| \vee e)$ vanishes on the “bad set” $\{\nabla u = 0\}$ but it goes to zero very slowly with respect to ∇u .

In the following, for a connected bounded open set $\Omega \subset \mathbb{R}^d$, $d \geq 2$, we deal with $\mathcal{R}(f)$ solution of

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \quad (5.3)$$

We start with the following estimates for the solution of (5.3)

Theorem 5.2. *Assume Ω of class $C^{1,1}$; then for every $f \in BV(\Omega)$, there exists $C > 0$, which only depends on Ω such that $u = \mathcal{R}(f)$ satisfies*

$$\int_{\Omega} \frac{1}{|\nabla u|} \left| D^2 u \left(I - \frac{\nabla u \otimes \nabla u}{|\nabla u|^2} \right) \right|^2 dx \leq C \|f\|_{BV(\Omega)}, \quad (5.4)$$

$$\int_{\{|\nabla u| < 1/e\}} \frac{|D^2 u \nabla u|^2}{|\nabla u|^3 \log^q \left(\frac{1}{|\nabla u|} \right)} dx \leq \frac{C}{q-1} \|f\|_{BV(\Omega)}, \quad \forall q > 1. \quad (5.5)$$

Moreover, if f satisfies

$$\exists \alpha > 0 \text{ such that } f \geq \alpha \text{ in } \Omega. \quad (5.6)$$

then, for every $q > 1$ and every $\varepsilon > 0$, we have

$$\int_{\Omega} \frac{1}{|\nabla u| \log^q \left(\frac{1}{|\nabla u|} \vee e \right)} dx \leq C \frac{q^2}{\alpha^2(q-1)} \|f\|_{BV(\Omega)} + \frac{H_{d-1}(\partial\Omega)}{\alpha}, \quad (5.7)$$

$$\frac{1}{\varepsilon} \int_{\{s < u < s + \varepsilon\}} \frac{|\nabla u|}{\log^q \left(\frac{1}{|\nabla u|} \vee e \right)} dx \leq C \frac{q^2}{\alpha(q-1)} \|f\|_{BV(\Omega)} + H_{d-1}(\partial\Omega), \quad (5.8)$$

where H_{d-1} denotes the $(d-1)$ -Hausdorff measure in \mathbb{R}^N .

Proof. It is enough to prove the result for Ω of class $C^{2,\alpha}$, $\alpha > 0$, $f \in C^{2,\alpha}(\overline{\Omega})$, and then $u \in C^{2,\alpha}(\overline{\Omega})$. The general case follows by an approximation argument, recalling that for every $f \in BV(\Omega)$, there exists $f_n \in C^\infty(\overline{\Omega})$ such that

$$f_n \rightarrow f \text{ in } L^{d/(d-1)}(\Omega), \quad \|\nabla f_n\|_{L^1(\Omega)^d} \rightarrow \|\nabla f\|_{\mathcal{M}(\Omega)^d},$$

and that the Calderon-Zygmund theorem implies that u_n satisfies

$$u_n \rightarrow u \text{ in } W^{2,d/(d-1)}(\Omega).$$

In the following we define $\zeta : (0, \infty) \rightarrow \mathbb{R}$ by

$$\zeta(s) = \begin{cases} 0 & \text{if } s = 0 \\ \frac{1}{\log(\frac{1}{s} \vee e)} & \text{if } 0 < s. \end{cases}$$

Let us prove (5.4), (5.5). We use that the derivatives of u satisfy (see [7])

$$\begin{cases} -\Delta \partial_i u = \partial_i f & \text{in } \Omega, \quad 1 \leq i \leq d \\ \nabla u = -|\nabla u| \nu & \text{on } \partial\Omega \\ -D^2 u \nu \cdot \nu = f + h \cdot \nabla u & \text{on } \partial\Omega, \end{cases}$$

where $\nu = -\nabla u / |\nabla u|$ is the unitary outside normal to Ω , and h is a function in $L^\infty(\partial\Omega)^d$, depending only on Ω . For $\delta > 0$ small enough, we take

$$\frac{\partial_i u}{|\nabla u| + \delta} \zeta(|\nabla u| + \delta)^{q-1}$$

as test function in the equation for $\partial_i u$. Summing with respect to the index i and integrating by parts, we get

$$\begin{aligned} & \int_{\Omega} \frac{|D^2 u|^2}{|\nabla u| + \delta} \zeta(|\nabla u| + \delta)^{q-1} dx - \int_{\Omega} \frac{|D^2 u \nabla u|^2}{|\nabla u| (|\nabla u| + \delta)^2} \zeta(|\nabla u| + \delta)^{q-1} dx \\ & \quad + (q-1) \int_{\{|\nabla u| + \delta < 1/e\}} \frac{|D^2 u \nabla u|^2}{|\nabla u| (|\nabla u| + \delta)^2} \zeta(|\nabla u| + \delta)^q dx \\ & = \int_{\partial\Omega} \frac{|\nabla u| (f + h \cdot \nabla u)}{|\nabla u| + \delta} \zeta(|\nabla u| + \delta)^{q-1} dH_{d-1}(x) \\ & \quad + \int_{\Omega} \frac{\nabla f \cdot \nabla u}{|\nabla u| + \delta} \zeta(|\nabla u| + \delta)^{q-1} dx. \end{aligned} \quad (5.9)$$

The two first terms in the left-hand side can be written as

$$\int_{\Omega} \frac{\zeta(|\nabla u| + \delta)^{q-1}}{|\nabla u| + \delta} \left(\left| D^2 u \left(I - \frac{\nabla u \otimes \nabla u}{|\nabla u|^2} \right) \right|^2 + \frac{\delta |D^2 u \nabla u|^2}{|\nabla u|^2 (|\nabla u| + \delta)^2} \right) dx,$$

where the integrand is non-negative. Thus, we can use the monotone convergence theorem to pass to the limit as $\delta \rightarrow 0$ in (5.9) to get

$$\begin{aligned} & \int_{\Omega} \frac{\zeta(|\nabla u|)^{q-1}}{|\nabla u|} \left| D^2 u \left(I - \frac{\nabla u \otimes \nabla u}{|\nabla u|^2} \right) \right|^2 dx + (q-1) \int_{\{|\nabla u| < 1/e\}} \frac{|D^2 u \nabla u|^2}{|\nabla u|^3} \zeta(|\nabla u|)^q dx \\ &= \int_{\partial\Omega} (f + h \cdot \nabla u) \zeta(|\nabla u|)^{q-1} dH_{d-1}(x) + \int_{\Omega} \frac{\nabla f \cdot \nabla u}{|\nabla u|} \zeta(|\nabla u|)^{q-1} dx. \end{aligned}$$

Using $\zeta \leq 1$ and that $u \in W^{2,d/(d-1)}(\Omega)$ implies $\nabla u \in L^1(\partial\Omega)^d$ we deduce (5.5). Inequality (5.4) follows letting $q \rightarrow 1$ in the above equality.

Let us now prove (5.7), (5.8). We multiply (5.3) by

$$\frac{\zeta^q(|\nabla u| + \delta)}{|\nabla u| + \delta},$$

with $\delta > 0$ and then we integrate in $\{u < t\}$, for $t > 0$, such that $\{u = t\}$ is a C^1 manifold (this holds for ever t outside a subset of $(0, \infty)$ with null measure). We get

$$\begin{aligned} & \int_{\{u < t\}} \frac{D^2 u \nabla u \cdot \nabla u}{|\nabla u| (|\nabla u| + \delta)^2} \zeta^q(|\nabla u| + \delta) \left(-1 + q \zeta(|\nabla u| + \delta) 1_{\{|\nabla u| + \delta < \frac{1}{e}\}} \right) dx \\ & \quad + \int_{\partial\Omega} \frac{|\nabla u| \zeta^q(|\nabla u| + \delta)}{|\nabla u| + \delta} dH_{d-1}(x) \\ &= \int_{\{u=t\}} \frac{|\nabla u| \zeta^q(|\nabla u| + \delta)}{|\nabla u| + \delta} dH_{d-1}(x) + \int_{\{u < t\}} f \frac{\zeta^q(|\nabla u| + \delta)}{|\nabla u| + \delta} dx. \end{aligned}$$

Using (5.6) in the last term and Young's inequality in the first one, this gives

$$\begin{aligned} & \int_{\{u=t\}} \frac{|\nabla u| \zeta^q(|\nabla u| + \delta)}{|\nabla u| + \delta} dH_{d-1}(x) + \frac{\alpha}{2} \int_{\{u < t\}} \frac{\zeta^q(|\nabla u| + \delta)}{|\nabla u| + \delta} dx \\ & \leq \frac{1}{2\alpha} \int_{\{u < t\}} \frac{|D^2 u \nabla u|^2}{(|\nabla u| + \delta)^3} \zeta^q(|\nabla u| + \delta) \left(-1 + q \zeta(|\nabla u| + \delta) 1_{\{|\nabla u| + \delta < \frac{1}{e}\}} \right)^2 dx \\ & \quad + \int_{\partial\Omega} \frac{|\nabla u| \zeta^q(|\nabla u| + \delta)}{|\nabla u| + \delta} dH_{d-1}(x). \end{aligned}$$

Thanks to (5.5) we deduce that this inequality holds for every $t > 0$. Moreover, it allows us to pass to the limit when $\delta \rightarrow 0$ using the Lebesgue's dominated convergence theorem in right-hand side and the monotone convergence theorem in the left-hand side. Thus, we get

$$\begin{aligned} & \int_{\{u=t\}} \zeta^q(|\nabla u|) dH_{d-1}(x) + \frac{\alpha}{2} \int_{\{u < t\}} \frac{\zeta^q(|\nabla u|)}{|\nabla u|} dx \\ & \leq \frac{1}{2\alpha} \int_{\{u < t\}} \frac{|D^2 u \nabla u|^2}{|\nabla u|^3} \zeta^q(|\nabla u|) \left(-1 + q \zeta(|\nabla u|) 1_{\{|\nabla u| < \frac{1}{e}\}} \right)^2 dx \\ & \quad + \int_{\partial\Omega} \zeta^q(|\nabla u|) dH_{d-1}(x), \end{aligned}$$

and then, by (5.5) and $0 \leq \zeta \leq 1$, that there exists $C > 0$ satisfying

$$\begin{aligned} \int_{\{u=t\}} \zeta^q(|\nabla u|) dH_{d-1}(x) + \frac{\alpha}{2} \int_{\{u<t\}} \frac{\zeta^q(|\nabla u|)}{|\nabla u|} dx \\ \leq \frac{Cq^2}{\alpha(q-1)} \|f\|_{BV(\Omega)} + H_{d-1}(\partial\Omega). \end{aligned} \quad (5.10)$$

Estimate (5.7) just follows from this inequality taking t tending to infinity.

To get estimate (5.7) we recall the coarea formula for Lipschitz functions ([10], Chapter 5) which establishes

$$\int_{\mathbb{R}} g|\nabla u| dx = \int_{\mathbb{R}} \int_{\{u=t\}} g dH_{d-1}(x) dt, \quad \forall g \in L^1(\Omega). \quad (5.11)$$

Using (5.11) with $g = \zeta^q(|\nabla u|)1_{\{s<u<s+\varepsilon\}}$ we get (5.8) from (5.10). \square

Corollary 5.3. *For $\Omega \in C^{1,1}$ and $f \in BV(\Omega)$, satisfying (5.6), the function $u = \mathcal{R}(f)$ is such that*

$$z := \frac{1}{\log^q\left(\frac{1}{|\nabla u|} \vee e\right)}$$

belongs to $W^{1,1}(\Omega)$, for every $q > 0$, and there exists $C > 0$ depending only on Ω such that

$$\|\nabla z\|_{L^1(\Omega)^N} \leq C \left(\frac{q+1}{\alpha q} \|f\|_{BV(\Omega)} + H_{d-1}(\partial\Omega) \right). \quad (5.12)$$

Proof. Taking into account

$$\begin{aligned} |\nabla z| &= \frac{|D^2u \nabla u|}{|\nabla u|^2 \log^{q+1}\left(\frac{1}{|\nabla u|} \vee e\right)} 1_{\{|\nabla u| < 1/e\}} \\ &= \frac{|D^2u \nabla u|}{|\nabla u|^{\frac{3}{2}} \log^{\frac{q+1}{2}}\left(\frac{1}{|\nabla u|} \vee e\right)} \frac{1}{|\nabla u|^{\frac{1}{2}} \log^{\frac{q+1}{2}}\left(\frac{1}{|\nabla u|} \vee e\right)} 1_{\{|\nabla u| < 1/e\}}, \end{aligned}$$

Using the Cauchy-Schwarz inequality the result follows from (5.5) and (5.7) with q replaced by $q-1$. \square

Our main result about the regularity of the function $1_{\{u>s\}}$ is given by

Theorem 5.4. *Assume Ω of class $C^{1,1}$ and let $f \in BV(\Omega)$ satisfying (5.6). Then the function $u = \mathcal{R}(f)$ satisfies for every $s > 0$ and every $q > 1$*

$$\frac{1}{\log^q\left(\frac{1}{|\nabla u|} \vee e\right)} 1_{\{u>s\}} \in BV(\Omega). \quad (5.13)$$

Moreover

$$\frac{1}{\log^q\left(\frac{1}{|\nabla u|} \vee e\right)} \nabla 1_{\{u>s\}} \in \mathcal{M}(\Omega), \quad (5.14)$$

and there exists $C > 0$ only depending on Ω such that

$$\left\| \frac{1}{\log^q\left(\frac{1}{|\nabla u|} \vee e\right)} \nabla 1_{\{u>s\}} \right\|_{\mathcal{M}(\Omega)^d} \leq C \frac{q^2}{\alpha(q-1)} \|f\|_{BV(\Omega)} + H_{d-1}(\partial\Omega). \quad (5.15)$$

Proof. We fix $s > 0$ and $q > 1$, then, for $\varepsilon > 0$, we define

$$v_\varepsilon := \frac{T_\varepsilon(u-s)^+}{\varepsilon \log^q\left(\frac{1}{|\nabla u|} \vee e\right)}.$$

By the Lebesgue dominated convergence theorem, we have

$$v_\varepsilon \rightarrow \frac{1}{\log^q\left(\frac{1}{|\nabla u|} \vee e\right)} 1_{\{u>s\}} \text{ in } L^p(\Omega), \forall p \in [1, \infty).$$

Moreover,

$$\nabla v_\varepsilon = \frac{\nabla u}{\varepsilon} 1_{\{s<|u|<s+\varepsilon\}} \frac{1}{\log^q\left(\frac{1}{|\nabla u|} \vee e\right)} + \frac{T_\varepsilon(u-s)^+}{\varepsilon} \nabla \left(\frac{1}{\log^q\left(\frac{1}{|\nabla u|} \vee e\right)} \right), \quad (5.16)$$

where the right-hand side is bounded in $L^1(\Omega)^N$ by (5.8) and (5.12). This proves (5.13).

Assertion (5.14) also comes from (5.16), which gives

$$\begin{aligned} \frac{\nabla u}{\varepsilon} 1_{\{s<|u|<s+\varepsilon\}} \frac{1}{\log^q\left(\frac{1}{|\nabla u|} \vee e\right)} &= \nabla v_\varepsilon - \frac{T_\varepsilon(u-s)^+}{\varepsilon} \nabla \left(\frac{1}{\log^q\left(\frac{1}{|\nabla u|} \vee e\right)} \right) \\ &\stackrel{*}{=} \nabla \left(\frac{1}{\log^q\left(\frac{1}{|\nabla u|} \vee e\right)} 1_{\{u>s\}} \right) - 1_{\{u>s\}} \nabla \left(\frac{1}{\log^q\left(\frac{1}{|\nabla u|} \vee e\right)} \right) \\ &= \frac{1}{\log^q\left(\frac{1}{|\nabla u|} \vee e\right)} \nabla 1_{\{u>s\}} \text{ in } \mathcal{M}(\Omega)^N, \end{aligned}$$

taking into account that the left-hand side is bounded in $L^1(\Omega)$ by (5.8). Inequality (5.15) is also a consequence of the estimate of the left-hand side by (5.8). \square

Remark 5.5. As we said at the beginning of subsection 5.2, assumption (5.6) implies that the set $\{\nabla \mathcal{R}(f) = 0\}$ has null measure. A further result is given by (5.7) which proves that $(|\nabla u| \log^q(1/|\nabla u|))^{-1}$ is integrable for $q > 1$ and ∇u close to zero. Observe that this result does not extend to $q = 1$. For example, taking $f = 1$ and Ω the annulus $B(0, 2) \setminus \overline{B}(0, 1)$ we have

$$\nabla u = \begin{cases} \frac{1}{2} \left(-|x| + \frac{3}{2 \log 2 |x|} \right) \frac{x}{|x|} & \text{if } d = 2 \\ \frac{1}{d} \left(-|x| + \frac{3(d-2)2^{d-2}}{2(2^{d-2}-1)|x|^{d-1}} \right) \frac{x}{|x|} & \text{if } d > 2. \end{cases}$$

Thus, using that ∇u vanishes on $\{|x| = r\}$ for some $r \in (1, 2)$, we easily get

$$\int_{\Omega} \frac{1}{|\nabla u| \log^q\left(\frac{1}{|\nabla u|} \vee e\right)} dx < \infty \iff q > 1.$$

Estimate (5.7) allows us to prove that for every $f \in W^{1,p}(\Omega)$, $p > d$, which satisfies (5.6), the Hausdorff dimension of the set $\{\nabla \mathcal{R}(f) = 0\}$ is at most $d - 1$. However, we are not able to prove $H_{d-1}(\{\nabla u = 0\}) < \infty$ as in the example in Remark 5.5. In order to give a more accurate result, we introduce the following refinement of the usual H_{d-1} -measure.

Definition 5.6. For $q \geq 0$ and $A \subset \mathbb{R}^d$, we define

$$H_{d-1,q}^\delta(A) = \inf \left\{ \sum_{i=1}^n \frac{r_i^{d-1}}{\log^q\left(\frac{1}{r_i}\right)} : A \subset \bigcup_{i=1}^n B(x_i, r_i), r_i < \delta \right\}, \quad 0 < \delta < 1,$$

and

$$H_{d-1,q}(A) = \lim_{\delta \rightarrow 0} H_{d-1,q}^\delta(A) = \sup_{\delta > 0} H_{d-1,q}^\delta(A).$$

Remark 5.7. Clearly, $H_{d-1,q}$ is an outer measure. It agrees with the usual $(d-1)$ -Hausdorff measure for $q = 0$ and satisfies

$$H_{d-1,q}(A) = 0 \text{ for some } q \geq 0 \implies H_s(A) = 0, \quad \forall s > d-1,$$

with H_s the s -Hausdorff measure. Thus, every set A with $H_{d-1,q}(A) = 0$ for some $q \geq 0$ has Hausdorff dimension at most $d-1$.

Theorem 5.8. Assume $\Omega \in C^{1,1}$ and $f \in W^{1,p}(\Omega)$, with $p > d$ such that (5.6) is satisfied. Then, for every $q > 1$ the solution u of (5.3) satisfies

$$H_{d-1,q}(\{\nabla u = 0\}) = 0. \quad (5.17)$$

Proof. We take $A := \{\nabla u = 0\}$. By (5.7) and $|A| = 0$, for every $\varepsilon > 0$ there exists an open set $U \subset \Omega$ with $A \subset U$ and

$$\int_U \frac{1}{|\nabla u| \log^q\left(\frac{1}{|\nabla u|} \vee e\right)} dx < \varepsilon.$$

Let $\delta \in (0, 1)$ be. Using that A is compact, we can find $x_i \in A$ and $0 < r_i < \delta$, $1 \leq i \leq m$, such that

$$A \subset \bigcup_{i=1}^m B(x_i, r_i), \quad \overline{B}(x_i, r_i) \subset U, \quad 1 \leq i \leq m. \quad (5.18)$$

By the Vitali's covering theorem we can now extract n balls $B(x_{i_j}, r_{i_j})$, $1 \leq j \leq n$, which are disjoint and satisfy

$$\bigcup_{i=1}^m \overline{B}(x_i, r_i) \subset \bigcup_{j=1}^n \overline{B}(x_{i_j}, 5r_{i_j}). \quad (5.19)$$

On the other hand, since $f \in W^{1,p}(\Omega)$, $p > d$ implies that ∇u is Lipschitz and $\nabla u(x_{i_j}) = 0$, there exists $L > 0$ such that

$$|\nabla u(x)| \leq L|x - x_{i_j}|, \quad \forall x \in \Omega, \quad 1 \leq j \leq n.$$

Then, for a certain constant $c > 0$, we have

$$\int_{B(x_{i_j}, r_{i_j})} \frac{1}{|\nabla u| \log^q\left(\frac{1}{|\nabla u|} \vee e\right)} dx \geq c \int_0^{r_{i_j}} \frac{r^{d-2}}{\log^q(1/r)} dr,$$

where an integration by parts gives

$$\int_0^{r_{i_j}} \frac{r^{d-2}}{\log^q(1/r)} dr = \frac{r_{i_j}^{d-1}}{(d-1)\log^q(1/r_{i_j})} + \frac{1}{d-1} \int_0^{r_{i_j}} \frac{r^{d-2}}{\log^{q+1}(1/r)} dr,$$

and then, assuming δ small enough and recalling that $r_{i_j} < \delta$, we get

$$\int_0^{r_{i_j}} \frac{r^{d-2}}{\log^q(1/r)} dr \geq \frac{r_{i_j}^{d-1}}{2(d-1)\log^q(1/r_{i_j})}.$$

Using that

$$c \sum_{j=1}^n \int_0^{r_{i_j}} \frac{r^{d-2}}{\log^q(1/r)} dr \leq \int_U \frac{1}{|\nabla u| \log^q\left(\frac{1}{|\nabla u|} \vee e\right)} dx < \varepsilon,$$

we deduce

$$\sum_{j=1}^n \frac{r_{i_j}^{d-1}}{\log^q(1/r_{i_j})} \leq \frac{2(d-1)}{c} \varepsilon.$$

By (5.18) and (5.19) we then have

$$H_{d-1,q}^\delta(A) \leq \sum_{j=1}^n \frac{(5r_{i_j})^{d-1}}{\log^q(1/(5r_{i_j}))} \leq \frac{5^{d-1}2(d-1)}{c} \varepsilon,$$

which by the arbitrariness of ε proves (5.17). \square

5.3 The case Ω convex

When the domain Ω is convex, in some cases we can obtain a better regularity for the optimal right-hand side f_{opt} . Let us return to the compliance case

$$\min \left\{ \int_{\Omega} f \mathcal{R}(f) dx : \int_{\Omega} f dx \geq m, 0 \leq f \leq 1 \right\}$$

with $0 < m < |\Omega|$, and assume Ω convex. We have seen in Example 4.9 that the optimal right-hand side f_{opt} is of bang-bang type: $f_{opt} = 1_E$ with $E = \{w < s\}$ for a suitable s such that $|E| = m$, where w is the solution of the PDE

$$\begin{cases} -\Delta w = 1_{\{w < s\}} & \text{in } \Omega \\ w = 0 & \text{on } \partial\Omega. \end{cases} \quad (5.20)$$

Lemma 5.9. *The set $E = \{w < s\}$ above is convex.*

Proof. It is enough to apply Theorem 1.2 of [3]. In fact this theorem applies to solutions v of

$$\begin{cases} -\Delta v = \phi(v) & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega \end{cases}$$

with ϕ Hölder continuous such that

- (i) $\sqrt{\Phi}$ is concave,
- (ii) Φ/ϕ is convex on $]0, M[$,

where Φ is the primitive of ϕ with $\Phi(0) = 0$, and

$$M = \inf \{t > 0 : \phi(t) = 0\}.$$

By approximating our function $\phi = 1_{[0,s]}$ by

$$\phi_n(t) = \begin{cases} 1 - (t/s)^n & \text{if } t \leq s \\ 0 & \text{if } t > s \end{cases}$$

we see that ϕ_n satisfies conditions (i) and (ii), hence the level sets of the functions v_n solutions of the PDE

$$\begin{cases} -\Delta v = \phi_n(v) & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega \end{cases}$$

are convex. Passing to the limit as $n \rightarrow \infty$, we have that the level sets of the solution v are convex too. \square

Proposition 5.10. *The set E is of class C^1 .*

Proof. By Lemma 5.9 the set E is convex; assume by contradiction that it has a corner. The solution w of (5.20) satisfies the PDE

$$\begin{cases} -\Delta w = 1 & \text{in } \Omega \setminus E \\ w = 0 & \text{on } \partial\Omega, \quad w = s & \text{on } \partial E; \end{cases}$$

in addition, by (5.20) we have that w is $W^{2,p}$ regular near the corner for every p , which is impossible by the well-known theory of elliptic PDEs in domains with re-entrant corners. \square

6 Numerical simulations

In this section we show some numerical examples, in the two-dimensional case, for problem (1.3). We consider three cases:

- Problem (4.16) relative to the maximization of the L^p norm of $\mathcal{R}(f)$ when f is non-negative and has a bounded mass;
- The minimization problem (4.17) in Example 4.6 in the case of a linear cost $j(x, s) = g(x) s$ for some suitable function g ;
- The minimization problem (4.17) in Example 4.6 in the quadratic case $j(x, s) = |s - u_0(x)|^2$ for some suitable function u_0 .

We apply a gradient descent method derived from an appropriate use of the optimality conditions given by Theorem 4.1. We refer to [1, 2, 8] for other algorithms related to similar problems. The algorithm is as follows.

- Initialization: choose an admissible function $f_0 \in L^1(\Omega)$.
- For $n \geq 0$, iterate until stop condition as follows.
 - Compute w_n as in (4.4) for $f_{opt} = f_n$.
 - Compute \hat{f}_n descent direction associated as:

* Example 4.5

$$\hat{f}_n(x) = m\delta_{x_n}$$

with x_n the point where the minimum of w_n is attained.

* Example 4.6

$$\hat{f}_n(x) = \begin{cases} \beta & \text{if } w_n(x) < -\lambda_n, \\ \alpha & \text{in other case,} \end{cases}$$

where λ_n is the Lagrange multiplier associated to the volume constraint.

- For $\varepsilon_n \in [0, 1)$ small enough, update the function f_n :

$$f_{n+1} = f_n + \varepsilon_n(\hat{f}_n - f_n).$$

- Stop if $\frac{|I_n - I_{n-1}|}{|I_0|} < tol$, for $tol > 0$ small, with

$$I_n = \int_{\Omega} (j(x, \mathcal{R}(f_n)) + \psi(f_n)) dx, \quad n \geq 0.$$

The computation has been carried out using the free software FreeFem++ v4.5 ([11], available in <http://www.freefem.org>). The picture of figures are made in Paraview 5.10.1 (available at <https://www.kitware.com/open-source/#paraview>), which is free too, except Figure 3 which is made with MATLAB. We use P1-Lagrange finite element approximations for the control function f , the state $\mathcal{R}(f)$ and costate w . For all simulations of the Example 4.6 where the parameters α and β appear, we consider the normalized values $\alpha = 0$ and $\beta = 1$.

Example 6.1. We consider the maximization problem

$$\max \left\{ \int_{\Omega} |\mathcal{R}(f)|^p dx : f \geq 0, \quad \int_{\Omega} f dx \leq m \right\}$$

in dimension two, with $p = 4$ and volume constraint $m = 10$. The domain Ω is a ball with a non-centered hole and a sharp mesh with 87806 triangles, see Figure 1. According to the analysis of optimality condition made in Example 4.5, the optimal right-hand side $f_{opt} = m\delta_{x_0}$ is a Dirac mass where the point x_0 is explicitly computed by $(-0.429729, 0.212863)$, see Figure 3. In Figure 2 we can observe the decreasing cost evolution for the minimization algorithm.

THE MESH

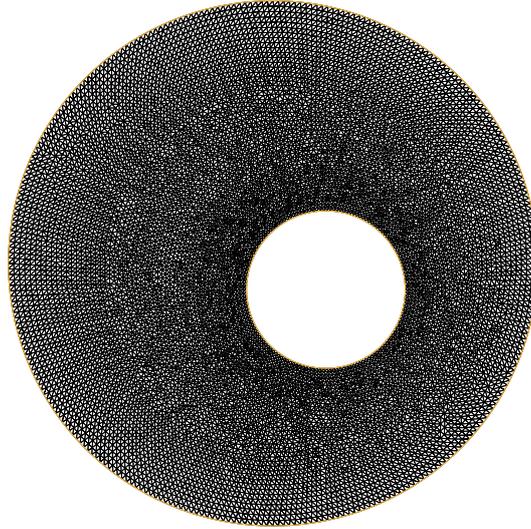


Figure 1: First numerical simulation: the mesh.

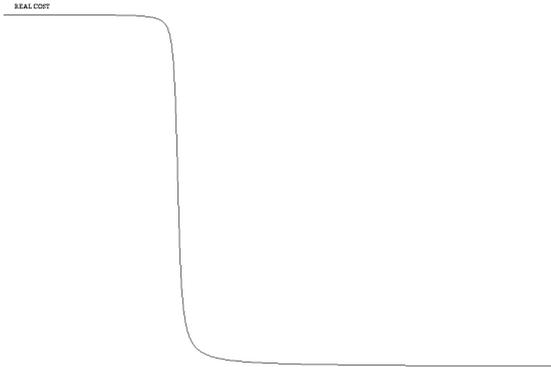


Figure 2: First numerical simulation: cost evolution.

Example 6.2. We solve numerically the problem (4.17) for Ω the unit ball of \mathbb{R}^2 and the linear cost given by $j(x, s, z) = g(x) s$ with $g(x, y) = x^2 - y^2$. We take $m = 1.25$ corresponding to use, approximately, a maximum of 40% of β . We can observe the computed optimal right-hand side f_{opt} in Figure 4.

Example 6.3. In this last example we solve numerically also, the problem (4.17) for Ω the unit ball of \mathbb{R}^2 and $m = 1.25$ as in the previous case, but we consider $j(x, s, z) = |s - u_0|^2$ taking a constant function $u_0 = 0.1$. As we can expect f_{opt} is a bang-bang control, see Figure 5.

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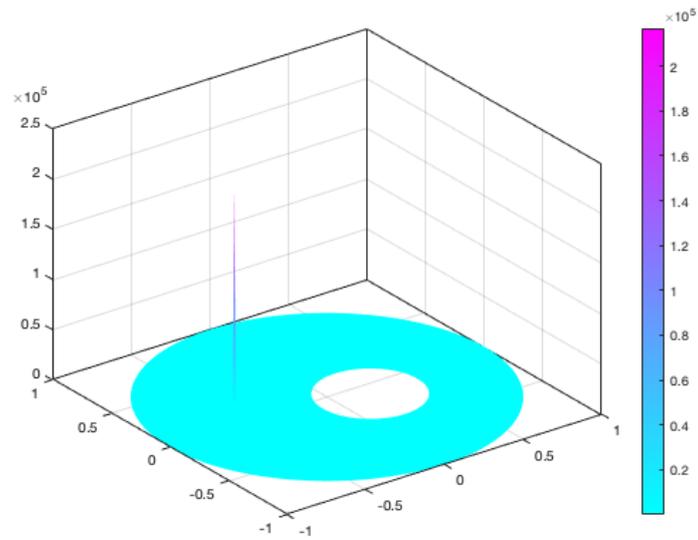


Figure 3: First numerical simulation: the optimal right-hand side $f_{opt} = m\delta_{x_0}$.

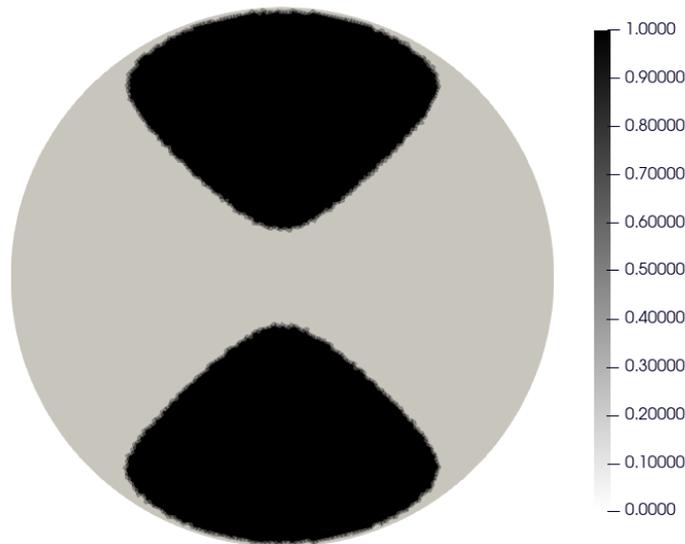


Figure 4: Second numerical simulation: the optimal right-hand side f_{opt} .

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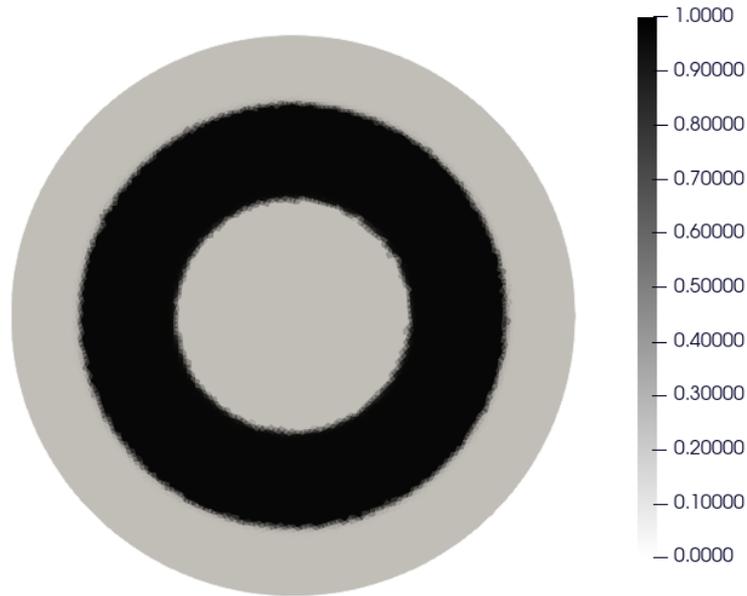


Figure 5: Third numerical simulation: f_{opt} is bang-bang.

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