Γ-CONVERGENCE AND STOCHASTIC HOMOGENIZATION FOR FUNCTIONALS IN THE $\mathcal{A}$-FREE SETTING

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ABSTRACT. We obtain a compactness result for Γ-convergence of integral functionals defined on $\mathcal{A}$-free vector fields. This is used to study homogenization problems for these functionals without periodicity assumptions. More precisely, we prove that the homogenized integrand can be obtained by taking limits of minimum values of suitable minimization problems on large cubes, when the side length of these cubes tends to $+\infty$, assuming that these limit values do not depend on the center of the cube. Under the usual stochastic periodicity assumptions, this result is then used to solve the stochastic homogenization problem by means of the subadditive ergodic theorem.

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CONTENTS

1. INTRODUCTION

Many problems in continuum mechanics and electromagnetism lead to the study of vector fields $u \in L^p(D;\mathbb{R}^d)$ that satisfy a differential constraint of the form

$$\sum_{i=1}^{N} A^i \partial_i u = 0 \quad \text{in } D, \tag{1.1}$$

where $D \subset \mathbb{R}^N$ is a bounded open set and A^i are $l \times d$-matrices that fulfil the constant-rank property (see (2.1)). These vector fields are called $\mathcal{A}$-free. The theory of compensated compactness, developed in [29, 33, 34, 35, 36, 37], provides powerful tools for their analysis and has recently been extended to $\mathcal{A}$-free measures [3, 16].

When f is a Carathéodory function satisfying the usual p-growth condition (see (2.10)), the study of the minimization of the integral functional

$$F(u, D) := \int_D f(x, u(x))\,dx \tag{1.2}$$

among all function $u \in L^p(D;\mathbb{R}^d)$ that satisfy the differential constraint (1.1) leads to the notion of $\mathcal{A}$-quasiconvexity introduced in [20] (see Definition 2.1), and inspired by a slightly different definition found in [10]. This condition is necessary and sufficient for lower semicontinuity of (1.2) with respect to the weak topology of $L^p(D;\mathbb{R}^d)$ under the constraint (1.1). Further results on $\mathcal{A}$-quasiconvex functionals can be found in [5, 4] for the linear growth case, in [9] in the context of partial regularity of minimizers, in [6] in connection with Young measures, in [11, 17] when two operators are present, in [19] with different

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exponents in the bounds of f from below and from above, in [24] in connection with Gårding inequalities, in [25] for the case of boundary $\mathcal{A}$-quasiconvexity, in [30] for an extended-valued function f, in [31] for potentials for $\mathcal{A}$-quasiconvexity, and in [32] for the study of relaxation via $\mathcal{A}$-quasiconvex envelopes.

When $x \mapsto f(x,\xi)$ is periodic in x, the limit behavior as $\varepsilon \to 0^+$ of the minimizers of the functionals

$$F_\varepsilon(u,D) := \int_D f(\tfrac{x}{\varepsilon}, u(x))\,dx \tag{1.3}$$

is studied in [7] using Γ-convergence (see also [28] for the $p=1$ case). More precisely, under an additional p-Lipschitz condition (see (2.13)), the family of functionals $(F_\varepsilon(\cdot,D))_{\varepsilon>0}$, subject to the differential constraint (1.1), Γ-converges as $\varepsilon \to 0^+$, with respect to the weak topology of $L^p(D;\mathbb{R}^d)$, to a homogenized functional of the form

$$F_{\rm hom}(u,D) := \int_D f_{\rm hom}(u(x))\,dx, \tag{1.4}$$

subject to the same differential constraint. The homogenized integrand $f_{\rm hom}$ is obtained from f by solving some auxiliary minimum problems for $F(\cdot,Q_r)$ on cubes Q_r whose side length r tends to infinity. The periodic case was further developed in [14, 15], where the authors established periodic homogenization results for integral energies under periodically oscillating or space-dependent differential constraints.

Aim and main compactness theorem. The aim of this paper is to study, more generally, Γ-convergence of sequences $(F_k)_{k\in\mathbb{N}}$ of functionals of the form

$$F_k(u,D) := \int_D f_k(x,u(x))\,dx \tag{1.5}$$

subject to the differential constraint (1.1). We assume that the integrands f_k satisfy p-growth and p-Lipschitz conditions with constants independent of k, and we prove a compactness result (see Corollary 4.7): there exist a subsequence, which we do not relabel, and a functional F of the form (1.2) such that for every bounded open set $D \subset \mathbb{R}^N$, the sequence of functionals $(F_k(\cdot,D))_{k\in\mathbb{N}}$, subject to the differential constraint (1.1), Γ-converges to $F(\cdot,D)$ with respect to the weak topology of $L^p(D;\mathbb{R}^d)$. Under different hypotheses, Γ-convergence results for functionals of the form (1.5) were studied in [26] in the context of dimension reduction problems.

Strategy of the proof. Following an idea introduced in [2], we prove this result by first studying the Γ-convergence of the sequence of functionals (1.5) without differential constraint but with respect to a topology in $L^p(D;\mathbb{R}^d)$ that takes into account the convergence of $\sum_{i=1}^N A^i\partial_i u$ in $W^{-1,p}(D;\mathbb{R}^l)$ (see Theorem 3.1).

The proof is based on the usual localization technique for Γ-convergence and on a new integral representation result (see Theorem 3.3).

To obtain from this result the Γ-convergence in the $\mathcal{A}$-free setting, we use a modification procedure introduced in [20], which allows us to replace a sequence $(u_k)_{k\in\mathbb{N}}$ with $\sum_{i=1}^N A^i\partial_i u_k \to 0$ in $W^{-1,p}(D;\mathbb{R}^l)$ by a sequence $(v_k)_{k\in\mathbb{N}}$ with the same limit in the weak topology of $L^p(D;\mathbb{R}^d)$ and such that $\sum_{i=1}^N A^i\partial_i v_k = 0$ in D for every k, with a negligible modification of the values of F_k (see Lemmas 4.1 and 4.2).

Characterising the Γ-limit integrand. When $\xi \mapsto f(x,\xi)$ is $\mathcal{A}$-quasiconvex for a.e. $x \in \mathbb{R}^N$, we reconstruct, for a.e. $x \in \mathbb{R}^N$ and every $\xi \in \mathbb{R}^d$, the value of $f(x,\xi)$ via the infima of some auxiliary minimum problems for $F(\cdot,Q_\rho(x))/\rho^N$ on cubes with center x and side length ρ, taking the limit as $\rho \to 0^+$ (see Theorem 5.3). This allows us to characterize the integrand of the Γ-limit of a sequence (1.5) by taking limits, as $k \to \infty$, of the infima of these auxiliary minimum problems for $F_k(\cdot,Q_\rho(x))/\rho^N$ (see Theorem 6.2). We further prove a technical variant of this result (see Theorem 6.9) in which the infima of the auxiliary minimum problems satisfy a subadditivity condition, preparing the ground for stochastic applications.

Homogenisation without periodicity. The preceding characterization of the integrand of the Γ-limit is then used to study the homogenization problem for (1.3) without periodicity assumptions. After a change of variables, the functionals $F_\varepsilon(\cdot,Q_\rho(x))/\rho^N$ are transformed into $F(\cdot,Q_{\rho/\varepsilon}(x/\varepsilon))/(\rho/\varepsilon)^N$. Therefore, the previous results show that the family of functionals $(F_\varepsilon(\cdot,D))_{\varepsilon>0}$, subject to the differential constraint (1.1), Γ-converges as $\varepsilon \to 0^+$, with respect to the weak topology of $L^p(D;\mathbb{R}^d)$, to the functional (1.4) subject to the same differential constraint, assuming only that there exists the limit, as $r \to +\infty$, of the infima of these auxiliary minimum problems for $F(\cdot,Q_r(rx))/r^N$, and that this limit, which defines $f_{\rm hom}$, does not depend on x (see Theorem 7.1). These conditions are satisfied not only when $x \mapsto f(x,\xi)$ is periodic on $\mathbb{R}^N$ for every $\xi \in \mathbb{R}^d$, but also also for suitable perturbations of the periodic case (see Proposition 7.2).

Stochastic homogenisation. When $f = f(\omega, x, \xi)$ depends also on a variable ω running on a probability space, under the standard assumptions of stochastic homogenization (see Definition 8.1), we obtain that the limits that define $f_{\text{hom}}(\omega,\xi)$ exist almost surely; hence, the family of functionals $(F_\varepsilon(\omega,\cdot,D))_{\varepsilon>0}$, subject to the differential constraint (1.1), Γ-converges as $\varepsilon \to 0^+$, with respect to the weak topology of $L^p(D;\mathbb{R}^d)$, to the functional

$$F_{\text{hom}}(\omega,u,D) := \int_D f_{\text{hom}}(\omega,u(x))\,dx$$

subject to the same differential constraint (see Theorem 8.5). Finally, if the stochastic process is ergodic, then it can be shown that the homogenized integrand does not depend on ω.

Overall, we provide a unified Γ-convergence framework for integral functionals with $\mathcal{A}$-free constraints, covering deterministic, non-periodic, and stochastic settings in a single theory.

2. Notation and preliminaries

Throughout this paper, $N, d, l \in \mathbb{N}$ and $p, q \in (1,+\infty)$ are fixed, with $\frac{1}{p} + \frac{1}{q} = 1$. We define $\mathcal{O}(\mathbb{R}^N)$ to be the collection of all bounded open subsets of $\mathbb{R}^N$ and, for $D \in \mathcal{O}(\mathbb{R}^N)$, we define $\mathcal{O}(D) := \{D \cap U : U \in \mathcal{O}(\mathbb{R}^N)\}$ to be the collection of all open subsets of D. For every $x \in \mathbb{R}^N$ and $\rho > 0$, we consider the open cube $Q_\rho(x)$ with center x, side length ρ, and sides parallel to the coordinate axes.

We use the standard notation for Lebesgue and Sobolev spaces. In particular, for $m \in \mathbb{N}$ and $D \in \mathcal{O}(\mathbb{R}^N)$, $W^{-1,p}(D;\mathbb{R}^m)$ is the dual of $W^{1,q}_0(D;\mathbb{R}^m)$. We also consider the dual of $W^{1,q}(D;\mathbb{R}^m)$, which we denote by $\widetilde{W}^{-1,p}(D;\mathbb{R}^m)$. In both cases, the duality product will be denoted by $\langle\cdot,\cdot\rangle$. The norm used in $W^{1,q}_0(D;\mathbb{R}^m)$ is $\|u\|_{W^{1,q}_0(D;\mathbb{R}^m)} := \|\nabla u\|_{L^q(D;\mathbb{R}^{m\times N})}$, and the norm used in $W^{1,q}(D;\mathbb{R}^m)$ is $\|u\|_{W^{1,q}(D;\mathbb{R}^m)} := (\|u\|^q_{L^q(D;\mathbb{R}^m)} + \|\nabla u\|^q_{L^q(D;\mathbb{R}^{m\times N})})^{1/q}$. The norms in $W^{-1,p}(D;\mathbb{R}^m)$ and $\widetilde{W}^{-1,p}(D;\mathbb{R}^m)$ are the corresponding dual norms.

Several of our results involve integral functionals whose fields are subject to linear partial differential constraints with constant coefficients, know as the $\mathcal{A}$-free setting as mentioned in the Introduction, which we now make precise. For $i \in \{1,\dots,N\}$, let $A^i \in \mathbb{R}^{l\times d}$ be $l \times d$ real-valued matrices such that there exists $r \in \mathbb{N}$ satisfying

$$\operatorname{rank}\left(\sum_{i=1}^N A^i w_i\right) = r \tag{2.1}$$

for all $w \in \mathbb{R}^N \setminus \{0\}$. This condition is referred to as the constant-rank property.

For every $u \in L^p_{\text{loc}}(\mathbb{R}^N;\mathbb{R}^d)$, let $\mathcal{A}u$ be the $\mathbb{R}^l$-valued distribution on $\mathbb{R}^N$ defined by

$$\langle \mathcal{A}u, \psi\rangle := -\sum_{i=1}^N \int_{\mathbb{R}^N} (A^i u)\cdot \partial_i \psi\,dx \quad \text{for every } \psi \in C^\infty_c(\mathbb{R}^N;\mathbb{R}^l). \tag{2.2}$$

Given $D \in \mathcal{O}(\mathbb{R}^N)$, we consider the operators

$$\mathcal{A}_D : L^p(D;\mathbb{R}^d) \to W^{-1,p}(D;\mathbb{R}^l) \quad \text{and} \quad \widetilde{\mathcal{A}}_D : L^p(D;\mathbb{R}^d) \to \widetilde{W}^{-1,p}(D;\mathbb{R}^l)$$

that map each $u \in L^p(D;\mathbb{R}^d)$ into the elements $\mathcal{A}_D u$ of $W^{-1,p}(D;\mathbb{R}^l)$ and $\widetilde{\mathcal{A}}_D u$ of $\widetilde{W}^{-1,p}(D;\mathbb{R}^l)$ defined, respectively, by

$$\langle \mathcal{A}_D u, \psi\rangle := -\sum_{i=1}^N \int_D (A^i u)\cdot \partial_i \psi\,dx \quad \text{for every } \psi \in W^{1,q}_0(D;\mathbb{R}^l), \tag{2.3}$$

$$\langle \widetilde{\mathcal{A}}_D u, \psi\rangle := -\sum_{i=1}^N \int_D (A^i u)\cdot \partial_i \psi\,dx \quad \text{for every } \psi \in W^{1,q}(D;\mathbb{R}^l). \tag{2.4}$$

In particular, $\ker \mathcal{A}_D$ is the set of all $u \in L^p(D;\mathbb{R}^d)$ such that

$$\sum_{i=1}^N \int_D (A^i u)\cdot \partial_i \psi\,dx = 0 \quad \text{for every } \psi \in W^{1,q}_0(D;\mathbb{R}^l),$$

in which case we either write $\mathcal{A}_D u = 0$ or $u \in \ker \mathcal{A}_D$. Moreover, we have that

$$\|\mathcal{A}_D u\|_{W^{-1,p}(D;\mathbb{R}^l)} = \|\operatorname{div}(Au)\|_{W^{-1,p}(D;\mathbb{R}^l)}, \tag{2.5}$$

where $Au(x) := (A^1 u(x), ..., A^N u(x)) \in \mathbb{R}^{l\times N}$ for $x \in D$.

The wave cone (or characteristic cone) Λ associated with the operator $\mathcal{A}$ is the subset of $\mathbb{R}^d$ defined by

$$\Lambda := \bigcup_{w\in\mathbb{R}^N\setminus\{0\}} \ker\left(\sum_{i=1}^N A^i w_i\right). \tag{2.6}$$

As extensively studied in [20], the weak lower semicontinuity of functionals defined on $\ker \mathcal{A}_D$ is intimately related to the notion of $\mathcal{A}$-quasiconvexity, which we recall next.

Definition 2.1 ($\mathcal{A}$-quasiconvex functions). *Let $Q \subset \mathbb{R}^N$ be a cube and $g\colon \mathbb{R}^d \to \mathbb{R}$ a locally bounded Borel function. We say that g is $\mathcal{A}$-quasiconvex if*

$$g(\xi) \le \frac{1}{|Q|}\int_Q g(\xi + w(x))\,dx$$

for every $\xi \in \mathbb{R}^d$ and $w \in C^\infty(\mathbb{R}^N;\mathbb{R}^d)$, with w Q-periodic, $\mathcal{A}w = 0$ in $\mathbb{R}^N$, and $\int_Q w(y)dy = 0$.

Remark 2.2. By a change of variables, we see that the previous definition does not depend on the choice of the cube Q. In the unconstrained case, when $\mathcal{A} = 0$, $\mathcal{A}$-quasiconvexity reduces to convexity, while when $\mathcal{A} = \operatorname{curl}$, $\mathcal{A}$-quasiconvexity reduces to quasiconvexity in the sense of Morrey (cf. [20]). By virtue of Jensen's inequality, every convex function is $\mathcal{A}$-quasiconvex.

For every $D \in \mathcal{O}(\mathbb{R}^N)$, let $\|\cdot\|^{\mathcal{A}}_D$ be the norm on $L^p(D;\mathbb{R}^d)$ involving the operator $\mathcal{A}_D$ and defined by

$$\|u\|^{\mathcal{A}}_D := \|u\|_{W^{-1,p}(D;\mathbb{R}^d)} + \|\mathcal{A}_D u\|_{W^{-1,p}(D;\mathbb{R}^l)}.$$

Note that $L^p(D;\mathbb{R}^d)$ endowed with this norm is separable.

Remark 2.3. If g is $\mathcal{A}$-quasiconvex and satisfies the p-growth condition

$$\frac{1}{c_0}|\xi|^p - c_0 \le g(\xi) \le c_0(1 + |\xi|^p) \quad \text{for every } \xi \in \mathbb{R}^d \tag{2.7}$$

for some constant $c_0 \ge 1$, then g is Λ-convex on $\mathbb{R}^d$, i.e.,

$$g(\theta\xi_1 + (1-\theta)\xi_2) \le \theta g(\xi_1) + (1-\theta)g(\xi_2)$$

for every ξ_1, $\xi_2 \in \mathbb{R}^d$ with $\xi_1 - \xi_2 \in \Lambda$ and every $\theta \in [0,1]$ (see, e.g., the proof of [4, Lemma 2.19]). If the vector space $\operatorname{span}(\Lambda)$ generated by the wave cone Λ coincides with $\mathbb{R}^d$, then Λ-convexity and (2.7) imply that there exists a constant $\underline{c}_1 > 0$, depending only on c_1, p, and d, such that the p-Lipschitz condition

$$|g(\xi_1) - g(\xi_2)| \le \underline{c}_1(1 + |\xi_1|^{p-1} + |\xi_2|^{p-1})|\xi_1 - \xi_2| \tag{2.8}$$

holds for every ξ_1, $\xi_2 \in \mathbb{R}^d$ (see, e.g., [23, Lemma 2.3]).

The equality $\operatorname{span}(\Lambda) = \mathbb{R}^d$ is satisfied in many interesting cases, for instance when $\mathcal{A} = \operatorname{curl}$ or $\mathcal{A} = \operatorname{div}$ (see [20, Remark 3.3] for the precise definitions).

Remark 2.4. For functions $g\colon \mathbb{R}^d \to [0,+\infty)$, the p-Lipschitz condition can be expressed in many equivalent ways. Indeed, if g satisfies (2.8), then we can find a constant $\bar{c}_1 > 0$, depending only on $\underline{c}_1$ and p, such that

$$|g(\xi_1) - g(\xi_2)| \le \bar{c}_1\big(1 + (|\xi_1| \wedge |\xi_2|)^{p-1} + |\xi_1 - \xi_2|^{p-1}\big)|\xi_1 - \xi_2|$$

holds for every $\xi_1, \xi_2 \in \mathbb{R}^d$, where $a \wedge b := \min\{a,b\}$. If g satisfies also (2.7), the previous inequality implies that there exists a constant $\hat{c}_1 \ge 1$, depending only on c_0, $\bar{c}_1$, and p, such that

$$|g(\xi_1) - g(\xi_2)| \le \hat{c}_1\big(1 + (g(\xi_1) \wedge g(\xi_2))^{\frac{p-1}{p}} + |\xi_1 - \xi_2|^{p-1}\big)|\xi_1 - \xi_2| \tag{2.9}$$

holds for every $\xi_1, \xi_2 \in \mathbb{R}^d$.

Conversely, if g satisfies (2.7) and (2.9), we can find a constant $\check{c}_1 > 0$, depending only on c_0, $\hat{c}_1$, and p, such that (2.8) holds with $\underline{c}_1$ replaced by $\check{c}_1$.

In the rest of the paper, we prefer to express the p-Lipschitz condition in the rather unusual form (2.9), because in this inequality the constant is stable under Γ-convergence, while this is not true for the more usual form (2.8).

Next, we introduce the class of integrands that we consider in our analysis. Throughout the paper c_0 and c_1 are fixed constants with $c_0 \geq 1$ and $c_1 \geq \hat{c}_1 \geq 1$, where $\hat{c}_1$ is the constant in (2.9) corresponding to the constant $\bar{c}_1$ obtained from $\underline{c}_1$ in (2.8).

Definition 2.5. *Let $\mathcal{F}$ be the collection of all Carathéodory functions $f\colon \mathbb{R}^N \times \mathbb{R}^d \to [0,+\infty)$ satisfying the following p-growth condition:*

$$\frac{1}{c_0}|\xi|^p - c_0 \leq f(x,\xi) \leq c_0(1+|\xi|^p) \tag{2.10}$$

for every $x \in \mathbb{R}^N$ and $\xi \in \mathbb{R}^d$. Let $\mathcal{F}_{\rm Lip}$ be the collection of all $f \in \mathcal{F}$ such that

$$|f(x,\xi_1) - f(x,\xi_2)| \leq c_1\big(1 + (f(x,\xi_1) \wedge f(x,\xi_2))^{\frac{p-1}{p}} + |\xi_1-\xi_2|^{p-1}\big)|\xi_1-\xi_2|, \tag{2.11}$$

for every $x \in \mathbb{R}^N$ and $\xi_1, \xi_2 \in \mathbb{R}^d$. Finally, let $\mathcal{F}_{\rm qc}$ be the collection of all functions $f \in \mathcal{F}$ such that $\xi \mapsto f(x,\xi)$ is $\mathcal{A}$-quasiconvex for every $x \in \mathbb{R}^N$.

We observe that there is no loss of generality in the constraint on the constant c_1 in the preceding definition because if (2.11) holds for a constant c_1, then it also holds for any constant $c \geq c_1$.

Under our assumptions on c_0, c_1, and p, it can be shown that the function $f(x,\xi) := |\xi|^p$ belongs to $\mathcal{F}_{\rm qc}$.

Remark 2.6. By exchanging the roles of ξ_1 and ξ_2, we see that (2.11) is equivalent to

$$f(x,\xi_1) \leq f(x,\xi_2) + c_1\big(1 + f(x,\xi_2)^{\frac{p-1}{p}} + |\xi_1-\xi_2|^{p-1}\big)|\xi_1-\xi_2| \tag{2.12}$$

for every $x \in \mathbb{R}^N$ and every $\xi_1, \xi_2 \in \mathbb{R}^d$.

Remark 2.7. As discussed in Remark 2.4, there exists a constant $\check{c}_1 > 0$, depending only on c_0, c_1, and p, such that (2.10) and (2.11) imply

$$|f(x,\xi_1) - f(x,\xi_2)| \leq \check{c}_1(1 + |\xi_1|^{p-1} + |\xi_2|^{p-1})|\xi_1-\xi_2| \tag{2.13}$$

for every $x \in \mathbb{R}^N$ and every $\xi_1, \xi_2 \in \mathbb{R}^d$.

Remark 2.8. By Remarks 2.3 and 2.4, in the case where the vector space span(Λ) generated by the wave cone Λ coincides with $\mathbb{R}^d$, we have that every $f \in \mathcal{F}_{\rm qc}$ satisfies (2.11) with $\hat{c}_1$ in place of c_1. Since we are assuming throughout that $\hat{c}_1 \leq c_1$, we conclude that $\mathcal{F}_{\rm qc} \subset \mathcal{F}_{\rm Lip}$ in this case.

We now introduce the collection of functionals that correspond to integrands in $\mathcal{F}$.

Definition 2.9. *Let $\{L^p, \mathcal{O}\}$ be the set of pairs (u,D) with $u \in L^p(D,\mathbb{R}^d)$ and $D \in \mathcal{O}(\mathbb{R}^N)$, and let $\mathcal{I}$ be the collection of all functionals $F\colon \{L^p,\mathcal{O}\} \to [0,+\infty)$ satisfying the following properties for every $D \in \mathcal{O}(\mathbb{R}^N)$ and $u \in L^p(D;\mathbb{R}^d)$:*

- (a) $\frac{1}{c_0}\|u\|^p_{L^p(D;\mathbb{R}^d)} - c_0|D| \leq F(u,D) \leq c_0\big(|D| + \|u\|^p_{L^p(D;\mathbb{R}^d)}\big)$,
- (b) *the set function $B \mapsto F(u,B)$ defined for every $B \in \mathcal{O}(D)$ can be extended to a nonnegative measure defined on all Borel subsets of D.*

Here, and henceforth, if $u \in L^p(D;\mathbb{R}^d)$ and $B \in \mathcal{O}(D)$, we simply write $F(u,B)$ instead of $F(u|_B,B)$.

Let $\mathcal{I}_{\rm Lip}$ be the collection of all functionals $F \in \mathcal{I}$ such that

- (c) $|F(u,D) - F(v,D)| \leq c_1\big(|D|^{\frac{p-1}{p}} + (F(u,D) \wedge F(v,D))^{\frac{p-1}{p}} + \|u-v\|^{p-1}_{L^p(D;\mathbb{R}^d)}\big)\|u-v\|_{L^p(D;\mathbb{R}^d)}$

for every $D \in \mathcal{O}(\mathbb{R}^N)$ and u, $v \in L^p(D;\mathbb{R}^d)$.

Finally, let $\mathcal{I}_{\rm sc}$ be the collection of all functionals $F \in \mathcal{I}$ such that for every $D \in \mathcal{O}(\mathbb{R}^N)$, the functional $u \mapsto F(u,D)$ is lower semicontinuous for the topology induced on $L^p(D;\mathbb{R}^d)$ by the norm $\|\cdot\|^{\mathcal{A}}_D$.

Remark 2.10. Let $F \in \mathcal{I}$. By the lower bound in (a) of Definition 2.9, if $D \in \mathcal{O}(\mathbb{R}^N)$ and $(u_k)_{k\in\mathbb{N}}$ is a sequence in $L^p(D;\mathbb{R}^d)$ converging to $u \in L^p(D;\mathbb{R}^d)$ in the topology induced by the norm $\|\cdot\|^{\mathcal{A}}_D$ and such that $(F(u_k, D))_{k\in\mathbb{N}}$ is bounded, then $u_k \rightharpoonup u$ weakly in $L^p(D;\mathbb{R}^d)$. Therefore, $F \in \mathcal{I}_{\rm sc}$ if and only if we have for every $D \in \mathcal{O}(\mathbb{R}^N)$ that

$$F(u, D) \le \liminf_{k\to\infty} F(u_k, D)$$

for every $u \in L^p(D;\mathbb{R}^d)$ and every sequence $(u_k)_{k\in\mathbb{N}} \subset L^p(D;\mathbb{R}^d)$ such that $u_k \rightharpoonup u$ weakly in $L^p(D;\mathbb{R}^d)$ and $\mathcal{A}_D u_k \to \mathcal{A}u$ strongly in $W^{-1,p}(D;\mathbb{R}^l)$. In particular, this inequality holds whenever $u_k \to u$ strongly in $L^p(D;\mathbb{R}^d)$. Hence, for every $F \in \mathcal{I}_{\rm sc}$ and $D \in \mathcal{O}(\mathbb{R}^N)$ the functional $F(\cdot, D)$ is lower semicontinuous in the strong topology of $L^p(D;\mathbb{R}^d)$.

Remark 2.11. If $f \in \mathcal{F}$, then the functional F defined by

$$F(u, D) := \int_D f(x, u(x))\,dx, \quad \text{for every } D \in \mathcal{O}(\mathbb{R}^N) \text{ and } u \in L^p(D;\mathbb{R}^d), \tag{2.14}$$

belongs to $\mathcal{I}$. If $f \in \mathcal{F}_{\rm Lip}$, then $F \in \mathcal{I}_{\rm Lip}$. Indeed, condition (c) of Definition 2.9 follows from (2.11) by using Hölder's inequality. If $f \in \mathcal{F}_{\rm qc}$, then $F \in \mathcal{I}_{\rm sc}$ by [20, Theorem 3.7] and Remark 2.10.

Remark 2.12. By analogy with Remark 2.4, there exists a constant $\check{c}_1$, depending only on c_0, c_1, and p, such that if $F \in \mathcal{I}_{\rm Lip}$, then

$$|F(u, D) - F(v, D)| \le \check{c}_1\big(|D|^{\frac{p-1}{p}} + \|u\|^{p-1}_{L^p(D;\mathbb{R}^d)} + \|v\|^{p-1}_{L^p(D;\mathbb{R}^d)}\big)\|u - v\|_{L^p(D;\mathbb{R}^d)} \tag{2.15}$$

for every $D \in \mathcal{O}(\mathbb{R}^N)$ and every $u, v \in L^p(D;\mathbb{R}^d)$. Conversely, if the growth conditions (a) of Definition 2.9 and the Lipschitz condition (2.15) hold, then there exists a constant $\hat{c}_1$, depending only on c_0, $\check{c}_1$, and p, such that

$$|F(u, D) - F(v, D)| \le \hat{c}_1\big(|D|^{\frac{p-1}{p}} + (F(u, D) \wedge F(v, D))^{\frac{p-1}{p}} + \|u - v\|^{p-1}_{L^p(D;\mathbb{R}^d)}\big)\|u - v\|_{L^p(D;\mathbb{R}^d)},$$

which implies that the Lipschitz condition (c) of Definition 2.9 holds with c_1 replaced by $\hat{c}_1$. As before, our preference for condition (c) of Definition 2.9 is due to the fact that, unlike (2.15), it is stable under Γ-convergence.

Remark 2.13. Similarly to Remark 2.6, we see by exchanging the roles of u and v that the Lipschitz condition (c) of Definition 2.9 is equivalent to the inequality

$$F(u, D) \le F(v, D) + c_1\big(|D|^{\frac{p-1}{p}} + F(v, D)^{\frac{p-1}{p}} + \|u - v\|^{p-1}_{L^p(D;\mathbb{R}^d)}\big)\|u - v\|_{L^p(D;\mathbb{R}^d)} \tag{2.16}$$

for every $D \in \mathcal{O}(\mathbb{R}^N)$ and every $u, v \in L^p(D;\mathbb{R}^d)$.

3. Γ-convergence in the unconstrained setting

In this section, we prove a compactness result for the family of functionals $\mathcal{I}$ introduced in Definition 2.9 with respect to Γ-convergence without considering the constraint $\mathcal{A}_D u = 0$.

Theorem 3.1. *Consider the three families of functionals $\mathcal{I}$, $\mathcal{I}_{\rm Lip}$, and $\mathcal{I}_{\rm sc}$ introduced in Definition 2.9. Let $(F_k)_{k\in\mathbb{N}}$ be a sequence in $\mathcal{I}$. Then, there exist a subsequence, which we do not relabel, and a functional $F \in \mathcal{I}_{\rm sc}$ such that for every $D \in \mathcal{O}(\mathbb{R}^N)$, the sequence $(F_k(\cdot, D))_{k\in\mathbb{N}}$ Γ-converges to $F(\cdot, D)$ with respect to the topology induced by $\|\cdot\|^{\mathcal{A}}_D$ on $L^p(D;\mathbb{R}^d)$.*

If, in addition, $F_k \in \mathcal{I}_{\rm Lip}$ for every $k \in \mathbb{N}$, then $F \in \mathcal{I}_{\rm Lip}$.

The proof of this theorem relies on the following technical result, which will be used also in the proof of Proposition 6.8.

Lemma 3.2. *Let $(F_k)_{k\in\mathbb{N}}$ be a sequence in $\mathcal{I}$, let $D_1, D_2, B \in \mathcal{O}(\mathbb{R}^N)$, with $D_1 \subset\subset D_2$, let $u \in L^p(D_2 \cup B;\mathbb{R}^d)$, and let $(u_k)_{k\in\mathbb{N}} \subset L^p(D_2;\mathbb{R}^d)$ and $(v_k)_{k\in\mathbb{N}} \subset L^p(B;\mathbb{R}^d)$ be two sequences such that*

$$\begin{aligned} &\lim_{k\to\infty} \|u_k - u\|^{\mathcal{A}}_{D_2} = 0 \quad \text{and} \quad \limsup_{k\to\infty} F_k(u_k, D_2) < +\infty, \\ &\lim_{k\to\infty} \|v_k - u\|^{\mathcal{A}}_{B} = 0 \quad \text{and} \quad \limsup_{k\to\infty} F_k(v_k, B) < +\infty. \end{aligned} \tag{3.1}$$

Then, setting $D := D_1 \cup B$, there exists a sequence $(w_k)_{k\in\mathbb{N}} \subset L^p(D;\mathbb{R}^d)$ such that

$$w_k = u_k \text{ in } D_1 \quad \text{and} \quad w_k = v_k \text{ in } B \setminus D_2, \tag{3.2}$$

$$w_k \rightharpoonup u \text{ weakly in } L^p(D;\mathbb{R}^d) \quad \text{and} \quad \lim_{k\to\infty} \|w_k - u\|_D^{\mathcal{A}} = 0, \tag{3.3}$$

$$\limsup_{k\to\infty} F_k(w_k, D) \le \limsup_{k\to\infty} \big(F_k(u_k, D_2) + F_k(v_k, B)\big). \tag{3.4}$$

Proof. By (3.1), we may assume without loss of generality that $\sup_k F_k(u_k, D_2) < +\infty$ and $\sup_k F_k(v_k, B) < +\infty$. Consequently, by (a) in Definition 2.9, $(u_k)_{k\in\mathbb{N}}$ is bounded in $L^p(D_2;\mathbb{R}^d)$ and $(v_k)_{k\in\mathbb{N}}$ is bounded in $L^p(B;\mathbb{R}^d)$. In turn, this yields a uniform bound for the total variations of the sequence $(\nu_k)_{k\in\mathbb{N}}$ of Radon measures on $\mathbb{R}^N$ defined by

$$\nu_k(E) := \int_{E\cap D_2\cap B} \big(1 + |u_k(x)|^p + |v_k(x)|^p\big)\, dx \quad \text{for each Borel set } E \subset \mathbb{R}^N.$$

Thus, extracting a subsequence, which we do not relabel, there exists a Radon measure ν on $\mathbb{R}^N$ such that

$$\nu_k \overset{*}{\rightharpoonup} \nu \quad \text{weakly}^* \text{ in the sense of measures on } \mathbb{R}^N. \tag{3.5}$$

Next, we set $\tau := \operatorname{dist}(D_1, \partial D_2) > 0$ and, for $0 < t < \tau$, we define

$$D^t := \big\{x \in \mathbb{R}^N : \operatorname{dist}(x, D_1) < t\big\} \subset\subset D_2.$$

Since ν is a finite measure, we can select $\eta \in (0, \frac{\tau}{2})$ such that and $\nu(\partial D^\eta) = 0$ (see [18, Proposition 1.15]). For every $m \in \mathbb{N}$ with $0 < \frac{1}{m} < \frac{\tau}{2}$, we define the sets

$$L_m := (D^{\eta+\frac{1}{m}} \setminus \overline{D^\eta}) \cap B \subset\subset D_2.$$

We further consider a cut-off function $\theta_m \in C_c^\infty(D^{\eta+\frac{1}{m}};[0,1])$ with $\theta_m = 1$ on $\overline{D^\eta}$, and define

$$w_k^m(x) := \theta_m(x) u_k(x) + (1 - \theta_m(x)) v_k(x) \quad \text{for } x \in D = D_1 \cup B.$$

It is clear that (3.2) holds for any choice of m.

Let $\psi \in W_0^{1,q}(D;\mathbb{R}^l)$. Then, using the fact that $\theta_m\psi$, $\partial_i\theta_m\psi \in W_0^{1,q}(D_2;\mathbb{R}^l)$ and $(1-\theta_m)\psi$, $\partial_i(1-\theta_m)\psi = -\partial_i\theta_m\psi \in W_0^{1,q}(B;\mathbb{R}^l)$, we can find a constant, $c > 0$, depending on $\mathcal{A}$ and m but not on k, such that

$$\begin{aligned}
-\sum_{i=1}^N \int_D A^i(w_k^m - u)\cdot \partial_i\psi\, dx &= -\sum_{i=1}^N \int_D A^i\big(\theta_m(u_k - u) + (1-\theta_m)(v_k - u)\big)\cdot\partial_i\psi\, dx \\
&= -\sum_{i=1}^N \left(\int_{D_2} A^i(u_k - u)\cdot\partial_i(\theta_m\psi)\, dx - \int_{D_2} A^i(u_k - u)\cdot(\partial_i\theta_m\psi)\, dx\right) \\
&\quad - \sum_{i=1}^N \left(\int_B A^i(v_k - u)\cdot\partial_i((1-\theta_m)\psi)\, dx + \int_B A^i(v_k - u)\cdot(\partial_i\theta_m\psi)\, dx\right) \\
&\le c\Big(\|\mathcal{A}_{D_2}(u_k - u)\|_{W^{-1,p}(D;\mathbb{R}^l)} + \|u_k - u\|_{W^{-1,p}(D_2;\mathbb{R}^d)} \\
&\quad + \|\mathcal{A}_B(v_k - u)\|_{W^{-1,p}(D;\mathbb{R}^l)} + \|v_k - u\|_{W^{-1,p}(B;\mathbb{R}^d)}\Big).
\end{aligned}$$

Consequently, we deduce from (3.1) that

$$\lim_{k\to\infty} \|w_k^m - u\|_D^{\mathcal{A}} = 0 \quad \text{for every } m \in \mathbb{N} \text{ with } 0 < \tfrac{1}{m} < \tfrac{\tau}{2}. \tag{3.6}$$

Finally, for every such m, condition (c) and the second inequality in (a) of Definition 2.9 together with (3.1)–(3.5) yield

$$\begin{aligned}
\limsup_{k\to\infty}\ F_k\big(w_k^m, D\big) &\le \limsup_{k\to\infty} \big(F_k(u_k, D_2) + F_k(v_k, B) + F_k(w_k^m, L_m)\big) \\
&\le \limsup_{k\to\infty} \big(F_k(u_k, D_2) + F_k(v_k, B)\big) + c_0 \limsup_{k\to\infty} \int_{L_m} \big(1 + |u_k(x)|^p + |v_k(x)|^p\big)\, dx \\
&\le \limsup_{k\to\infty} \big(F_k(u_k, D_2) + F_k(v_k, B)\big) + c_0\, \nu(\overline{L_m}).
\end{aligned}$$

Together with (3.6), this implies that for every $m \in \mathbb{N}$ with $0 < \frac{1}{m} < \frac{\tau}{2}$ there exists $k_m \in \mathbb{N}$ such that

$$\|w_k^m - u\|_D^{\mathcal{A}} < \tfrac{1}{m} \quad \text{and} \quad F_k(w_k^m, D) < \limsup_{k\to\infty} \big(F_k(u_k, D_2) + F_k(v_k, B)\big) + c_0\, \nu(\overline{L_m}) + \tfrac{1}{m} \tag{3.7}$$

for every $k \ge k_m$. It is not restrictive to assume that $k_m < k_{m+1}$ for every m.

Define $w_k := w_k^m$ for $k_m \le k < k_{m+1}$. Then, (3.7) yields

$$\|w_k - u\|_D^{\mathcal{A}} < \tfrac{1}{m} \quad \text{and} \quad F_k(w_k, D) < \limsup_{k\to\infty} \big(F_k(u_k, D_2) + F_k(v_k, B)\big) + c_0\, \nu(\overline{L_m}) + \tfrac{1}{m}$$

for every $k \ge k_m$. Hence,

$$\limsup_{k\to\infty} \|w_k - u\|_D^{\mathcal{A}} \le \tfrac{1}{m} \quad \text{and} \quad \limsup_{k\to\infty} F_k(w_k, D) \le \limsup_{k\to\infty} \big(F_k(u_k, D_2) + F_k(v_k, B)\big) + c_0\, \nu(\overline{L_m}) + \tfrac{1}{m}.$$

Since $\lim_m \nu(\overline{L_m}) = \nu(\partial D^\eta) = 0$, taking the limit as $m \to \infty$ in the preceding estimates, we obtain the second part of (3.3) and (3.4).

Recalling (a) of Definition 2.9, inequality (3.4) and (3.1) imply that $(w_k)_{k\in\mathbb{N}}$ is bounded in $L^p(D;\mathbb{R}^d)$. Therefore, the first part of (3.3) is a consequence of the second one (see Remark 2.10). $\square$

Proof of Theorem 3.1. For each $D \in \mathcal{O}(\mathbb{R}^N)$, we define $F'(\cdot, D), F''(\cdot, D) : L^p(D;\mathbb{R}^d) \to [0, +\infty]$ by

$$F'(\cdot, D) := \Gamma(\|\cdot\|_D^{\mathcal{A}})\text{-}\liminf_{k\to\infty} F_k(\cdot, D) \quad \text{and} \quad F''(\cdot, D) := \Gamma(\|\cdot\|_D^{\mathcal{A}})\text{-}\limsup_{k\to\infty} F_k(\cdot, D),$$

where the Γ-limits are taken with respect to the topology induced on $L^p(D;\mathbb{R}^d)$ by the norm $\|\cdot\|_D^{\mathcal{A}}$. If $u \in L^p(B;\mathbb{R}^d)$ for some $B \in \mathcal{O}(\mathbb{R}^N)$ containing D, we simply write $F'(u, D)$ and $F''(u, D)$ instead of $F'(u|_D, D)$ and $F''(u|_D, D)$.

We now proceed in several steps.

Step 1 (Monotonicity of F' and F''). We observe that for every $D_1, D_2 \in \mathcal{O}(\mathbb{R}^N)$ with $D_1 \subset D_2$, condition (c) in Definition 2.9 implies that $F_k(u, D_1) \le F_k(u, D_2)$ for every $u \in L^p(D_2;\mathbb{R}^d)$ and every $k \in \mathbb{N}$. Therefore, $F'(u, D_1) \le F'(u, D_2)$ and $F''(u, D_1) \le F''(u, D_2)$ for every $u \in L^p(D_2;\mathbb{R}^d)$.

Step 2 (Upper bound for F''). Let $D \in \mathcal{O}(\mathbb{R}^N)$ and $u \in L^p(D;\mathbb{R}^d)$. Then,

$$F''(u, D) \le c_0(|D| + \|u\|^p_{L^p(D;\mathbb{R}^d)}). \tag{3.8}$$

Indeed, the definition of $F''(\cdot, D)$ and the upper bound in (a) of Definition 2.9 yield

$$F''(u, D) \le \limsup_{k\to\infty} F_k(u, D) \le c_0(|D| + \|u\|^p_{L^p(D;\mathbb{R}^d)}),$$

which proves (3.8).

Step 3 (Nested subadditivity of F''). Let $D_1, D_2, B \in \mathcal{O}(\mathbb{R}^N)$, with $D_1 \subset\subset D_2$, and let $u \in L^p(D_2 \cup B;\mathbb{R}^d)$. We want to prove that

$$F''(u, D_1 \cup B) \le F''(u, D_2) + F''(u, B). \tag{3.9}$$

Let $(u_k)_{k\in\mathbb{N}} \subset L^p(D_2;\mathbb{R}^d)$ and $(v_k)_{k\in\mathbb{N}} \subset L^p(B;\mathbb{R}^d)$ be two sequences such that

$$\lim_{k\to\infty} \|u_k - u\|_{D_2}^{\mathcal{A}} = 0 \quad \text{and} \quad \limsup_{k\to\infty} F_k(u_k, D_2) = F''(u, D_2) < +\infty,$$
$$\lim_{k\to\infty} \|v_k - u\|_{B}^{\mathcal{A}} = 0 \quad \text{and} \quad \limsup_{k\to\infty} F_k(v_k, B) = F''(u, B) < +\infty,$$

which exist by [12, Proposition 8.1]. By Lemma 3.2, there exists a sequence $(w_k)_{k\in\mathbb{N}} \subset L^p(D_1 \cup B;\mathbb{R}^d)$ such that (3.3) and (3.4) hold, and so

$$F''_D(u, D_1 \cup B) \le \limsup_{k\to\infty} F_k(w_k, D_1 \cup B) \le \limsup_{k\to\infty} \big(F_k(u_k, D_2) + F_k(v_k, B)\big) \le F''(u, D_2) + F''(u, B),$$

which proves (3.9).

Step 4 (Compactness property). Let $\mathcal{D}$ be the countable collection of the open sets that are finite unions of open rectangles with rational vertices. Using the compactness of Γ-convergence on separable metric spaces (see [12, Theorem 8.5]) and a diagonal argument, we obtain a subsequence of $(F_k)_{k\in\mathbb{N}}$, which we do not relabel, for which

$$F'(u, D) = F''(u, D) \quad \text{for every } D \in \mathcal{D} \text{ and every } u \in L^p(D;\mathbb{R}^d).$$

For every $D \in \mathcal{O}(\mathbb{R}^N)$ and every $u \in L^p(D;\mathbb{R}^d)$, we define

$$F(u, D) := \sup_{\substack{D' \in \mathcal{D} \\ D' \subset\subset D}} F'(u, D') = \sup_{\substack{D' \in \mathcal{D} \\ D' \subset\subset D}} F''(u, D'). \tag{3.10}$$

From the monotonicity of $F'(u,\cdot)$ and $F''(u,\cdot)$ in Step 1, we deduce for every $D \in \mathcal{O}(\mathbb{R}^N)$ and every $u \in L^p(D;\mathbb{R}^d)$ that

$$F(u,D) = \sup_{\substack{D' \in \mathcal{O}(\mathbb{R}^N)\\ D' \subset\subset D}} F'(u,D') = \sup_{\substack{D' \in \mathcal{O}(\mathbb{R}^N)\\ D' \subset\subset D}} F''(u,D') \tag{3.11}$$

and

$$F(u,D) \le F'(u,D) \le F''(u,D). \tag{3.12}$$

In fact, by (3.10), we clearly have

$$F(u,D) \le \sup_{\substack{\widetilde{D} \in \mathcal{O}(\mathbb{R}^N)\\ \widetilde{D} \subset\subset D}} F'(u,\widetilde{D}). \tag{3.13}$$

Conversely, given $\widetilde{D} \in \mathcal{O}(\mathbb{R}^N)$ with $\widetilde{D} \subset\subset D$, we can find $D' \in \mathcal{D}$ such that $\widetilde{D} \subset D' \subset\subset D$. Thus, by the monotonicity of $F'(u,\cdot)$ and by (3.10), we conclude that

$$F'(u,\widetilde{D}) \le F'(u,D') \le F(u,D).$$

Taking the supremum over all sets $\widetilde{D} \in \mathcal{O}(\mathbb{R}^N)$ with $\widetilde{D} \subset\subset D$ yields the converse inequality of (3.13), which proves the first identity in (3.11). The remaining statements in (3.11) and (3.12) can be proven similarly.

Moreover, by (3.8), we have

$$F(u,D) \le c_0(|D| + \|u\|^p_{L^p(D;\mathbb{R}^d)}) \tag{3.14}$$

for every $D \in \mathcal{O}(\mathbb{R}^N)$ and every $u \in L^p(D;\mathbb{R}^d)$.

Step 5 (Proof of the Γ*-convergence).* By (3.12), we obtain that $(F_k(\cdot,D))_{k\in\mathbb{N}}$ Γ-converges to $F(\cdot,D)$ once we prove that

$$F''(u,D) \le F(u,D) \tag{3.15}$$

for every $D \in \mathcal{O}(\mathbb{R}^N)$ and every $u \in L^p(D;\mathbb{R}^d)$.

Fix any such D and u, and fix $\varepsilon > 0$. Let $K \subset D$ be a compact set such that

$$c_0(|D \setminus K| + \|u\|^p_{L^p(D\setminus K;\mathbb{R}^d)}) < \varepsilon. \tag{3.16}$$

Fix $D_1, D_2 \in \mathcal{O}(\mathbb{R}^N)$, with $K \subset D_1 \subset\subset D_2 \subset\subset D$. By (3.9) with $B := D \setminus K$, (3.8), (3.11), and (3.16), we obtain

$$F''(u,D) \le F''(u,D_2) + F''(u,D\setminus K) \le F(u,D) + c_0(|D\setminus K| + \|u\|^p_{L^p(D\setminus K;\mathbb{R}^d)}) \le F(u,D) + \varepsilon.$$

The arbitrariness of $\varepsilon > 0$ yields (3.15), completing the proof of Γ-convergence. In particular, in view of (3.11), (3.12), and (3.15), we conclude that F is inner regular; that is, we have for every $D \in \mathcal{O}(\mathbb{R}^N)$ and every $u \in L^p(D;\mathbb{R}^d)$ that

$$F(u,D) = \sup_{\substack{D' \in \mathcal{O}(\mathbb{R}^N)\\ D' \subset\subset D}} F(u,D'). \tag{3.17}$$

Moreover, by a general property of Γ-limits, we have for every $D \in \mathcal{O}(\mathbb{R}^N)$ that the functional $u \mapsto F(u,D)$ is lower semicontinuous for the topology induced on $L^p(D;\mathbb{R}^d)$ by the norm $\|\cdot\|^{\mathcal{A}}_D$.

Step 6 (Proof of condition (a) of Definition 2.9 for F). Fix $D \in \mathcal{O}(\mathbb{R}^N)$ and $u \in L^p(D;\mathbb{R}^d)$. By Step 5, there exists a sequence $(u_k)_{k\in\mathbb{N}} \subset L^p(D;\mathbb{R}^d)$ such that

$$\lim_{k\to\infty} \|u_k - u\|^{\mathcal{A}}_D = 0 \quad \text{and} \quad \limsup_{k\to\infty} F_k(u_k,D) = F(u,D).$$

Then, using the lower bound in condition (a) of Definition 2.9 for F_k, we have for all sufficiently large $k \in \mathbb{N}$ that

$$\frac{1}{c_0}\|u_k\|^p_{L^p(D;\mathbb{R}^d)} - c_0|D| \le F_k(u_k,D) \le F(u,D) + 1.$$

The preceding estimate and (3.14) yield that $(u_k)_{k\in\mathbb{N}}$ is a bounded sequence in $L^p(D;\mathbb{R}^d)$. Hence, $u_k \rightharpoonup u$ weakly in $L^p(D;\mathbb{R}^d)$, and so

$$F(u,D) = \limsup_{k\to\infty} F_k(u_k,D) \ge \limsup_{k\to\infty}\Big(\frac{1}{c_0}\|u_k\|^p_{L^p(D;\mathbb{R}^d)} - c_0|D|\Big) \ge \frac{1}{c_0}\|u\|^p_{L^p(D;\mathbb{R}^d)} - c_0|D|. \tag{3.18}$$

Finally, (3.14) and (3.18) show that condition (a) of Definition 2.9 holds for F.

Step 7 (Proof of condition (b) of Definition 2.9 for F). Fix $D \in \mathcal{O}(\mathbb{R}^N)$ and $u \in L^p(D;\mathbb{R}^d)$, and let $\alpha\colon \mathcal{O}(D) \to [0,+\infty)$ be the (increasing) set function defined by setting $\alpha(B) := F(u,B)$ for every $B \in \mathcal{O}(D)$. Invoking [12, Theorem 14.23], condition (b) of Definition 2.9 is satisfied for F provided that α is subadditive, superadditive, and inner regular in $\mathcal{O}(D)$. The inner regularity holds by (3.17), while the simple proof of the superadditivity can be obtained as in [12, Proposition 16.12]. We are then left to show that α is superadditive in $\mathcal{O}(D)$, which amounts to proving that

$$\alpha(B_1 \cup B_2) \le \alpha(B_1) + \alpha(B_2) \quad \text{for all } B_1, B_2 \in \mathcal{O}(D). \tag{3.19}$$

Let $B_1, B_2 \in \mathcal{O}(D)$, and fix $\delta > 0$. By (3.17), we can find $B' \subset\subset B_1 \cup B_2$ such that

$$\alpha(B_1 \cup B_2) - \delta < F''(u,B').$$

Let $B_1', B_1'', B_2' \in \mathcal{O}(D)$ be such that $B_1' \subset\subset B_1'' \subset\subset B_1$, $B_2' \subset\subset B_2$, and $B' \subset\subset B_1' \cup B_2'$ (see [12, Lemma 14.20] for instance). Then, using (3.9), (3.17), and the monotonicity of $F''(u,\cdot)$ proved in Step 1, we obtain

$$\alpha(B_1 \cup B_2) - \delta < F''(u,B') \le F''(u,B_1' \cup B_2') \le F''(u,B_1'') + F''(u,B_2') \le \alpha(B_1) + \alpha(B_2),$$

from which (3.19) follows by letting $\delta \to 0$.

Step 8 (Proof of condition (c) of Definition 2.9 for F). Assume that $F_k \in \mathcal{I}_{Lip}$ for every $k \in \mathbb{N}$ and fix $D \in \mathcal{O}(\mathbb{R}^N)$. By (2.16) for F_k, we have

$$F_k(u,D) \le F_k(v,D) + c_1\big(|D|^{\frac{p-1}{p}} + F_k(v,D)^{\frac{p-1}{p}} + \|u-v\|^{p-1}_{L^p(D;\mathbb{R}^d)}\big)\|u-v\|_{L^p(D;\mathbb{R}^d)} \tag{3.20}$$

for every $u, v \in L^p(D;\mathbb{R}^d)$. We claim that this inequality passes to the Γ-limit. Indeed, given $u, v \in L^p(D;\mathbb{R}^d)$, we can find a sequence $(v_k)_{k\in\mathbb{N}}$ in $L^p(D;\mathbb{R}^d)$ converging to v in the topology induced on $L^p(D;\mathbb{R}^d)$ by the norm $\|\cdot\|^{\mathcal{A}}_D$ and such that

$$\lim_{k\to\infty} F_k(v_k,D) = F(v,D). \tag{3.21}$$

For every $k \in \mathbb{N}$, let $u_k := v_k + u - v$. By (3.20), we have

$$F_k(u_k,D) \le F_k(v_k,D) + c_1\big(|D|^{\frac{p-1}{p}} + F_k(v_k,D)^{\frac{p-1}{p}} + \|u-v\|^{p-1}_{L^p(D;\mathbb{R}^d)}\big)\|u-v\|_{L^p(D;\mathbb{R}^d)}. \tag{3.22}$$

On the other hand, since $(u_k)_{k\in\mathbb{N}}$ converges to u in the topology induced on $L^p(D;\mathbb{R}^d)$ by the norm $\|\cdot\|^{\mathcal{A}}_D$, we have by Γ-convergence that

$$F(u,D) \le \liminf_{k\to\infty} F_k(u_k,D).$$

This inequality, together with (3.21) and (3.22), leads to (2.16) for F. As observed in Remark 2.13, we then conclude that condition (c) of Definition 2.9 holds for F. □

We now prove that every functional in $\mathcal{I}_{\mathrm{Lip}}$ can be represented by an integral whose integrand belongs to $\mathcal{F}_{\mathrm{Lip}}$.

Theorem 3.3. *Let $F \in \mathcal{I}_{\mathrm{Lip}}$. For every $x \in \mathbb{R}^N$ and $\xi \in \mathbb{R}^d$, we set*

$$f(x,\xi) := \limsup_{\rho\to 0^+} \frac{F(\xi, Q_\rho(x))}{\rho^N}. \tag{3.23}$$

Then, $f \in \mathcal{F}_{\mathrm{Lip}}$ and

$$F(u,D) = \int_D f(x,u(x))\,dx \tag{3.24}$$

for every $D \in \mathcal{O}(\mathbb{R}^N)$ and every $u \in L^p(D,\mathbb{R}^d)$. If, in addition, $F \in \mathcal{I}_{\mathrm{Lip}} \cap \mathcal{I}_{\mathrm{sc}}$, then the function $\xi \mapsto f(x,\xi)$ is $\mathcal{A}$-quasiconvex for every $x \in \mathbb{R}^N$; that is, $f \in \mathcal{F}_{\mathrm{Lip}} \cap \mathcal{F}_{\mathrm{qc}}$.

Proof. By condition (a) of Definition 2.9 and (3.23), the function f satisfies (2.10). Since F satisfies (2.16) (see Remark 2.13), we deduce that (2.12) holds for f, which is equivalent to (2.11) (see Remark 2.6).

Let us fix $D \in \mathcal{O}(\mathbb{R}^N)$. By condition (b) of Definition 2.9 the set function $B \mapsto F(\xi,B)$ defined on $\mathcal{O}(D)$ can be extended to a measure defined on all Borel subsets of D. By condition (a) of Definition 2.9, this measure is absolutely continuous with respect to the Lebesgue measure. By (3.23) and by the Lebesgue Differentiation Theorem, the function $x \mapsto f(x,\xi)$ is measurable on $\mathbb{R}^N$ for every $\xi \in \mathbb{R}^d$, and

$$F(\xi,B) = \int_B f(x,\xi)\,dx \quad \text{for every Borel set } B \subset D. \tag{3.25}$$

The measurability of $x \mapsto f(x,\xi)$, together with (2.10) and (2.11), which encode the continuity of $\xi \mapsto f(x,\xi)$, implies that $f \in \mathcal{F}$.

Let $u\colon D \to \mathbb{R}^d$ be a piecewise constant function, that is, there exists a finite family $(B_i)_{i\in I}$ of pairwise disjoint sets in $\mathcal{O}(D)$, covering almost all of D, and a finite family $(\xi_i)_{i\in I}$ in $\mathbb{R}^d$ such that for every $i \in I$ we have $u(x) = \xi_i$ for every $x \in B_i$. By conditions (a) and (b) of Definition 2.9 the set function $B \mapsto F(u,B)$ is a measure that is absolutely continuous with respect to the Lebesgue measure. By applying (3.25) to ξ_i and B_i, we obtain

$$F(u,D) = \sum_{i\in I} F(\xi_i,B_i) = \sum_{i\in I} \int_{B_i} f(x,\xi_i)\,dx = \int_D f(x,u(x))\,dx.$$

Consider now an arbitrary function $u \in L^p(D,\mathbb{R}^d)$. There exists a sequence $(u_k)_{k\in\mathbb{N}}$ of piecewise constant functions converging to u in $L^p(D,\mathbb{R}^d)$. By the previous step, we have

$$F(u_k,D) = \int_D f(x,u_k(x))\,dx$$

for every $k \in \mathbb{N}$. By condition (a) of Definition 2.9, using (2.10), (2.13), and (2.15) we can pass to the limit in both terms as $k \to \infty$ and we obtain (3.24).

If $F \in \mathcal{I}_{\mathrm{sc}}$, then for a.e. $x \in \mathbb{R}^N$ the function $\xi \mapsto f(x,\xi)$ is $\mathcal{A}$-quasiconvex by [20, Theorem 3.6]. Fix $x \in \mathbb{R}^N$ and $\rho > 0$, and let $g\colon \mathbb{R}^N \to \mathbb{R}$ be the function defined by

$$g(\xi) := \frac{1}{\rho^N} \int_{Q_\rho(x)} f(y,\xi)\,dy. \tag{3.26}$$

Let $Q \subset \mathbb{R}^N$ be a cube, and let $w \in C^\infty(\mathbb{R}^N;\mathbb{R}^d)$ be a Q-periodic function, with $\mathcal{A}w = 0$ in $\mathbb{R}^N$ and $\int_Q w(y)\,dy = 0$. By $\mathcal{A}$-quasiconvexity of f, we have for a.e. $y \in \mathbb{R}^N$ that

$$f(y,\xi) \le \frac{1}{|Q|} \int_Q f(y,\xi + w(z))\,dz.$$

Integrating with respect to y and using Fubini's theorem, we get

$$\int_{Q_\rho(x)} f(y,\xi)\,dy \le \frac{1}{|Q|} \int_Q \Big(\int_{Q_\rho(x)} f(y,\xi + w(z))\,dy \Big)\,dz.$$

In view of (3.26), this gives

$$g(\xi) \le \frac{1}{|Q|} \int_Q g(\xi + w(z))\,dz.$$

Hence, g is $\mathcal{A}$-quasiconvex. By (3.24) this is equivalent to saying that the function

$$\xi \mapsto \frac{F(\xi, Q_\rho(x))}{\rho^N}$$

is $\mathcal{A}$-quasiconvex. Moreover, by Fatou's lemma for bounded sequences (see [18, Lemma 1.83 (ii)]), it can be similarly checked that the lim sup of locally equi-bounded functions preserves $\mathcal{A}$-quasiconvexity. We then deduce from (3.23) that $\xi \mapsto f(x,\xi)$ is $\mathcal{A}$-quasiconvex for every $x \in \mathbb{R}^N$. □

We are now in a position to prove a compactness result for the collection of integrands $\mathcal{F}$.

Corollary 3.4. *Let $(f_k)_{k\in\mathbb{N}}$ be a sequence in $\mathcal{F}_{\mathrm{Lip}}$ and let $(F_k)_{k\in\mathbb{N}}$ be the corresponding sequence of functionals in $\mathcal{I}$ defined by (2.14). Then, there exist a subsequence, which we do not relabel, and a function $f \in \mathcal{F}_{\mathrm{Lip}} \cap \mathcal{F}_{\mathrm{qc}}$ such that for every $D \in \mathcal{O}(\mathbb{R}^N)$, the sequence $(F_k(\cdot,D))_{k\in\mathbb{N}}$ Γ-converges to $F(\cdot,D)$ defined by (2.14) with respect to the topology induced by $\|\cdot\|_D^{\mathcal{A}}$ on $L^p(D;\mathbb{R}^d)$.*

Proof. The result follows from Remark 2.11 and Theorems 3.1 and 3.3. □

The following theorem, which was communicated to us by Jean-François Babadjian, shows that the hypothesis of p-Lipschitz continuity can be omitted under a condition on the wave cone Λ defined by (2.6).

Theorem 3.5. *Assume that the vector space* $\mathrm{span}(\Lambda)$ *generated by the wave cone Λ coincides with $\mathbb{R}^d$. Then, Theorem 3.3 is still satisfied if we replace $\mathcal{I}_{\mathrm{Lip}}$ by $\mathcal{I}_{\mathrm{sc}}$ and $\mathcal{F}_{\mathrm{Lip}}$ by $\mathcal{F}_{\mathrm{Lip}} \cap \mathcal{F}_{\mathrm{qc}}$. Moreover, the inclusion $\mathcal{I}_{\mathrm{sc}} \subset \mathcal{I}_{\mathrm{Lip}}$ holds. Finally, Corollary 3.4 remains valid if we assume only that $(f_k)_{k\in\mathbb{N}}$ is a sequence in $\mathcal{F}$.*

Proof. Assume that $F \in \mathcal{I}_{\rm sc}$ and let f be defined by (3.23). Arguing as in Theorem 3.3, we show that the function $x \mapsto f(x,\xi)$ is measurable on $\mathbb{R}^N$ for every $\xi \in \mathbb{R}^d$, that f satisfies the p-growth condition (2.10), and that (3.24) holds for every $D \in \mathcal{O}(\mathbb{R}^N)$ and every piecewise constant function $u\colon D \to \mathbb{R}^d$.

We claim that for every $x \in \mathbb{R}^N$ the function $\xi \mapsto f(x,\xi)$ is Λ-convex on $\mathbb{R}^d$, i.e.,

$$f(x, \theta\xi_1 + (1-\theta)\xi_2) \le \theta f(x,\xi_1) + (1-\theta) f(x,\xi_2) \tag{3.27}$$

for every $\xi_1, \xi_2 \in \mathbb{R}^d$ with $\xi_1 - \xi_2 \in \Lambda$ and every $\theta \in [0,1]$. To this aim, we fix ξ_1, ξ_2, and θ as required and we consider the periodic extension $\chi\colon \mathbb{R} \to [0,1]$ of the characteristic function of $[0,\theta]$. By the definition of wave cone (see (2.6)), there exists $w \in \mathbb{R}^N \setminus \{0\}$ such that

$$\xi_1 - \xi_2 \in \ker\left(\sum_{i=1}^N A^i w_i\right). \tag{3.28}$$

We consider the piecewise constant function $u_\varepsilon \in L^p_{\rm loc}(\mathbb{R}^N;\mathbb{R}^d)$ defined by

$$u_\varepsilon(x) = \chi(x\cdot w/\varepsilon)\xi_1 + (1-\chi(x\cdot w/\varepsilon))\xi_2.$$

By the Riemann–Lebesgue Lemma, $u_\varepsilon \rightharpoonup \theta\xi_1 + (1-\theta)\xi_2$ weakly in $L^p(D;\mathbb{R}^d)$ for every $D \in \mathcal{O}(\mathbb{R}^N)$. Moreover,

$$\mathcal{A}u_\varepsilon = \frac{1}{\varepsilon}\left(\sum_{i=1}^N A^i w_i\right)(\xi_1-\xi_2)\sum_{j\in\mathbb{Z}}(\mathcal{H}^{N-1} \llcorner \{x\cdot w = j\} - \mathcal{H}^{N-1} \llcorner \{x\cdot w = \theta + j\}) \tag{3.29}$$

in the sense of distributions on $\mathbb{R}^N$, where $\mathcal{H}^{N-1}$ is the $(N-1)$-dimensional Hausdorff measure and $(H^{N-1} \llcorner E)(B) = H^{N-1}(E \cap B)$ for every pair of Borel sets E, $B \subset \mathbb{R}^N$. By (3.28), the right-hand side of (3.29) equals 0, hence $\mathcal{A}u_\varepsilon = 0$ in the sense of distributions on $\mathbb{R}^N$.

Since $F \in \mathcal{I}_{\rm sc}$, for every $D \in \mathcal{O}(\mathbb{R}^N)$ the functional $F(\cdot, D)$ is lower semicontinuous with respect to the norm $\|\cdot\|^{\mathcal{A}}_D$ on $L^p(D;\mathbb{R}^d)$. Thus, by Remark 2.10,

$$F(\theta\xi_1 + (1-\theta)\xi_2, D) \le \liminf_{\varepsilon\to 0+} F(u_\varepsilon, D).$$

Using the integral representation of F on piecewise constant functions, from the previous inequality we obtain

$$\begin{aligned} F(\theta\xi_1 + (1-\theta)\xi_2, D) &\le \liminf_{\varepsilon\to 0+} \int_D f(x, u_\varepsilon(x))\,dx \\ &= \lim_{\varepsilon\to 0+}\int_D \big(\chi(x\cdot w/\varepsilon) f(x,\xi_1) + (1-\chi(x\cdot w/\varepsilon)) f(x,\xi_2)\big)\,dx \\ &= \theta\int_D f(x,\xi_1)\,dx + (1-\theta)\int_D f(x,\xi_2)\,dx = \theta F(\xi_1,D) + (1-\theta)F(\xi_2,D), \end{aligned}$$

where the second equality follows again from the Riemann–Lebesgue Lemma.

Given $x \in \mathbb{R}^N$ and $\rho > 0$, we take $D = Q_\rho(x)$ in the previous inequality. Dividing by ρ^N and using (3.23), we obtain (3.27), which shows that the function $\xi \mapsto f(x,\xi)$ is Λ-convex on $\mathbb{R}^d$ for every $x \in \mathbb{R}^N$.

By (2.10), we can apply [23, Lemma 2.3]) (see also [21, Lemma 4.6]) to obtain that there exists $\check{c}_1 > 0$, depending only on c_0, p, and d, such that (2.13) holds for every $x \in \mathbb{R}^N$ and every $\xi_1, \xi_2 \in \mathbb{R}^d$. Since f is measurable with respect to x and satisfies the growth conditions (2.10), we conclude that $f \in \mathcal{F}$.

Thanks to (2.13), the integral representation on piecewise constant functions leads to the following inequality

$$F(u,D) \le \int_D f(x,u(x))\,dx \quad \text{for all } D \in \mathcal{O}(\mathbb{R}^N) \text{ and } u \in L^p(D;\mathbb{R}^d)$$

by the lower semicontinuity of $F(\cdot, D)$ in the strong topology of $L^p(D;\mathbb{R}^d)$ (see Remark 2.10) and the continuity of $u \mapsto \int_D f(x,u(x))\,dx$ in the strong topology of $L^p(D;\mathbb{R}^d)$ due to (2.13).

To prove the equality, we fix $u \in L^p_{\rm loc}(\mathbb{R}^N;\mathbb{R}^d)$ and we use the translation argument introduced in the proof of [8, Lemma 4.1] (see also [12, Theorem 21.1]). We define

$$G(v,D) = F(u+v, D) \quad \text{for all } D \in \mathcal{O}(\mathbb{R}^N) \text{ and } v \in L^p(D;\mathbb{R}^d).$$

We observe that G satisfies the growth conditions

(a′) $\dfrac{1}{C}\|v\|^p_{L^p(D;\mathbb{R}^d)} - C\big(|D| + \|u\|^p_{L^p(D;\mathbb{R}^d)}\big) \le G(v,D) \le C\big(|D| + \|u\|^p_{L^p(D;\mathbb{R}^d)}\big) + C\|v\|^p_{L^p(D;\mathbb{R}^d)}$, for suitable a constant $C > 1$. This is a slight variant of condition (a) of Definition 2.9.

Arguing as before, we obtain a Carathéodory function $g\colon \mathbb{R}^N \times \mathbb{R}^d \to \mathbb{R}$ with p-growth and p-Lipschitz with respect to the second variable such that for every $D \in \mathcal{O}(\mathbb{R}^N)$ we have

$$G(v, D) = \int_D g(x, v(x))\, dx \quad \text{if } v \text{ is piecewise constant on } D,$$

from which it follows that

$$G(v, D) \leq \int_D g(x, v(x))\, dx \quad \text{for all } v \in L^p(D; \mathbb{R}^d).$$

Let $D \in \mathcal{O}(\mathbb{R}^N)$ and let $(u_k)_{k\in\mathbb{N}}$ be a sequence of piecewise constant functions such that $u_k \to u$ strongly in $L^p(D; \mathbb{R}^d)$. Since $v \mapsto \int_D f(x, v(x))\, dx$ and $v \mapsto \int_D g(x, v(x))\, dx$ are continuous in the strong topology of $L^p(D; \mathbb{R}^d)$, the equalities and inequalities satisfied by $F(v, D)$, $G(v, D)$, $\int_D f(x, v(x))\, dx$ and $\int_D g(x, v(x))\, dx$ give

$$\begin{aligned} F(u, D) &\leq \int_D f(x, u(x))\, dx = \lim_{k\to\infty} \int_D f(x, u_k(x))\, dx = \lim_{k\to\infty} F(u_k, D) = \lim_{k\to\infty} G(u_k - u, D) \\ &\leq \lim_{k\to\infty} \int_D g(x, u_k(x) - u(x))\, dx = \int_D g(x, 0)\, dx = G(0, D) = F(u, D), \end{aligned}$$

and we obtain

$$F(u, D) = \int_D f(x, u(x))\, dx. \tag{3.30}$$

Since every $u \in L^p(D; \mathbb{R}^d)$ can be extended to a function of $L^p_{\rm loc}(D; \mathbb{R}^d)$, (3.24) holds for every $D \in \mathcal{O}(\mathbb{R}^N)$ and $u \in L^p(D; \mathbb{R}^d)$.

Since $f \in \mathcal{F}$ and $F \in \mathcal{I}_{\rm sc}$, using [20, Theorem 3.6] we deduce from (3.30) that the function $\xi \mapsto f(x, \xi)$ is $\mathcal{A}$-quasiconvex for a.e. $x \in \mathbb{R}^N$. Arguing as in the last part of the proof of Theorem 3.3, we obtain that $\xi \mapsto f(x, \xi)$ is $\mathcal{A}$-quasiconvex for every $x \in \mathbb{R}^N$, hence $f \in \mathcal{F}_{\rm qc}$. By Remark 2.8 ,we have $\mathcal{F}_{\rm qc} \subset \mathcal{F}_{\rm Lip}$, hence $f \in \mathcal{F}_{\rm Lip} \cap \mathcal{F}_{\rm qc}$. This concludes the proof of the modified version of Theorem 3.3.

To prove the inclusion $\mathcal{I}_{\rm sc} \subset \mathcal{I}_{\rm Lip}$, we observe that, by the modified version of Theorem 3.3 for every $F \in \mathcal{I}_{\rm sc}$ there exists $f \in \mathcal{F}_{\rm Lip} \cap \mathcal{F}_{\rm qc}$ such that (3.24) holds for every $D \in \mathcal{O}(\mathbb{R}^N)$ and every $u \in L^p(D, \mathbb{R}^d)$. By Remark 2.11, this implies that $F \in \mathcal{I}_{\rm Lip}$.

The modified version of Corollary 3.4 follows easily from this modified version of Theorem 3.3. □

4. Γ-convergence in the $\mathcal{A}$-free setting

In this section, we study Γ-convergence in the $\mathcal{A}$-free setting, i.e., with the constraint $\mathcal{A}u = 0$. We begin with some preliminary lemmas. The following result has been established in [20, Lemma 2.15].

Lemma 4.1. *Let $D \in \mathcal{O}(\mathbb{R}^N)$, let $u \in L^p(D; \mathbb{R}^d)$, and let $(u_k)_{k\in\mathbb{N}} \subset L^p(D; \mathbb{R}^d)$ be a sequence such that*

$$u_k \rightharpoonup u \text{ weakly in } L^p(D; \mathbb{R}^d) \quad \text{and} \quad \mathcal{A}_D u_k \to 0 \text{ in } W^{-1,p}(D; \mathbb{R}^l).$$

Then, there exists a p-equi-integrable sequence $(v_k)_{k\in\mathbb{N}} \subset L^p(D; \mathbb{R}^d)$ satisfying

$$\begin{gathered} v_k \rightharpoonup u \text{ weakly in } L^p(D; \mathbb{R}^d), \quad \mathcal{A}_D v_k = 0, \quad \int_D v_k\, dx = \int_D u\, dx, \\ \lim_{k\to\infty} \|u_k - v_k\|_{L^r(D;\mathbb{R}^d)} = 0 \quad \text{for all } 1 \leq r < p. \end{gathered} \tag{4.1}$$

If D is a cube Q, $u \in L^p_{\rm loc}(\mathbb{R}^N; \mathbb{R}^d)$ is Q-periodic, and $\mathcal{A}u = 0$ in $\mathbb{R}^N$, then, in addition to the previous properties, we can obtain that $v_k \in L^p_{\rm loc}(\mathbb{R}^N; \mathbb{R}^d)$ is Q-periodic and satisfies $\mathcal{A}v_k = 0$ in $\mathbb{R}^N$.

The following result will be used to deduce the Γ-convergence in the $\mathcal{A}$-free setting from the Γ-convergence woth respect to the topology induced by $\|\cdot\|^{\mathcal{A}}_D$.

Lemma 4.2. *Let D, u, $(u_k)_{k\in\mathbb{N}}$, and $(v_k)_{k\in\mathbb{N}}$ be as in Lemma 4.1, and let $(F_k)_{k\in\mathbb{N}}$ be a sequence in $\mathcal{I}_{\rm Lip}$. Then,*

$$\limsup_{k\to\infty} \big(F_k(v_k, D) - F_k(u_k, D)\big) \leq 0. \tag{4.2}$$

In particular,

$$\liminf_{k\to\infty} F_k(v_k, D) \le \liminf_{k\to\infty} F_k(u_k, D) \quad \text{and} \quad \limsup_{k\to\infty} F_k(v_k, D) \le \limsup_{k\to\infty} F_k(u_k, D). \tag{4.3}$$

Proof. This result was established in [7] when either F_k are independent of k or F_k are the functionals associated to functions f_k with $f_k(x,\xi) = f(kx,\xi)$ for some $f \in \mathcal{F}$ periodic in the first variable. To prove that (4.2) also holds in our setting, we apply Theorem 3.3 to obtain a sequence $(f_k)_{k\in\mathbb{N}}$ in $\mathcal{F}_{\rm Lip}$ such that

$$F_k(v, D) = \int_D f_k(x, v(x))\, dx \quad \text{for every } v \in L^p(D;\mathbb{R}^d).$$

We define

$$w_k(x) := f_k(x, u_k(x)) \quad \text{and} \quad z_k(x) := f_k(x, v_k(x)), \tag{4.4}$$

and show that

$$w_k - z_k \to 0 \text{ in measure.} \tag{4.5}$$

Indeed, introducing

$$\alpha := \frac{q}{q-1} \in (0,1), \quad s := \frac{1}{\alpha} \in (1,+\infty), \quad \text{and} \quad t \in (1,+\infty) \quad \text{such that } \frac{1}{s} + \frac{1}{t} = 1,$$

we have $s\alpha = 1$ and $\alpha t = q = \frac{p}{p-1}$; hence, using (2.13) and Hölder's inequality, it follows, for some constant c independent of k, that

$$\begin{aligned}\int_D |w_k - z_k|^\alpha\, dx &\le c_2^\alpha \int_D (1 + |u_k|^{p-1} + |v_k|^{p-1})^\alpha |u_k - v_k|^\alpha\, dx \\ &\le c_2^\alpha \Big(\int_D (1 + |u_k|^{p-1} + |v_k|^{p-1})^{\alpha t}\, dx\Big)^{\frac{1}{t}} \Big(\int_D |u_k - v_k|^{\alpha s}\, dx\Big)^{\frac{1}{s}} \\ &\le c\Big(\int_D (1 + |u_k|^p + |v_k|^p)\, dx\Big)^{\frac{\alpha}{q}} \Big(\int_D |u_k - v_k|\, dx\Big)^\alpha \to 0\end{aligned}$$

by (4.1) and the boundedness of the sequences $(u_k)_{k\in\mathbb{N}}$ and $(v_k)_{k\in\mathbb{N}}$ in $L^p(D;\mathbb{R}^d)$. Thus, (4.5) holds.

To conclude, we fix $\delta > 0$ and, setting $w^+ := \max\{0, w\}$, we observe that

$$\begin{aligned}0 \le \int_D (z_k - w_k)^+\, dx &= \int_{\{x\in D:\ (z_k - w_k)^+(x) > \delta\}} (z_k - w_k)\, dx + \int_{\{x\in D:\ (z_k-w_k)^+ \le \delta\}} (z_k - w_k)^+\, dx \\ &\le \int_{\{x\in D:\ |z_k(x) - w_k(x)| > \delta\}} z_k\, dx + \delta \mathcal{L}^N(D),\end{aligned}$$

where we used the inequalities $-w_k \le 0$ (by the nonnegativity of f_k) and $(z_k - w_k)^+ \le |z_k - w_k|$. By (2.10) the p-equi-integrability of $(v_k)_{k\in\mathbb{N}}$ implies the equi-integrability of $(z_k)_{k\in\mathbb{N}}$. Therefore, the preceding estimate and the convergence in measure in (4.5) yield

$$\lim_{k\to\infty} \int_D (z_k - w_k)^+\, dx = 0.$$

Since $z_k - w_k \le (z_k - w_k)^+$, we obtain

$$\limsup_{k\to\infty} \int_D (z_k - w_k)\, dx \le 0\,,$$

which is equivalent to (4.2) by (4.4). The implication (4.2)⇒(4.3) is trivial. □

In the proof of Proposition 6.8, we need the following technical result, which can be obtained from the previous lemmas.

Corollary 4.3. *For every $\varepsilon > 0$ and $C > 0$ there exists $\eta > 0$ with the following property: for every open cube $Q \subset \mathbb{R}^N$ with side length less than or equal to 1 and every $u \in L^p(Q;\mathbb{R}^d)$, with $\|u\|^p_{L^p(Q;\mathbb{R}^d)} < C|Q|$, $\operatorname{supp} u \subset\subset Q$, and $\|\widetilde{\mathcal{A}}_Q u\|^p_{\widetilde{W}^{-1,p}(Q;\mathbb{R}^l)} < \eta|Q|$, there exists $v \in L^p_{\rm per}(Q;\mathbb{R}^d)$, with $\|v - u\|^p_{W^{-1,p}(Q;\mathbb{R}^d)} < \varepsilon|Q|$, $\mathcal{A}v = 0$ in $\mathbb{R}^N$, and $\int_Q v\, dx = \int_Q u\, dx$, such that*

$$F(v, Q) < F(u, Q) + \varepsilon|Q| \tag{4.6}$$

for every $F \in \mathcal{I}_{\rm Lip}$.

Proof. It is clear that the result does not depend on the center of the cube. We claim that is enough to prove it for the cube $Q := Q_1(0)$ with center 0 and side length 1. Indeed, if $Q_\rho = Q_\rho(0)$ is the cube with center 0 and side length ρ and $u_\rho \in L^p(Q_\rho;\mathbb{R}^d)$ is a function with $\operatorname{supp} u_\rho \subset\subset Q_\rho$, we consider the rescaled function $u \in L^p(Q;\mathbb{R}^d)$ defined by

$$u(x) := u_\rho(\rho x) \quad \text{for every } x \in Q.$$

Then, $\operatorname{supp} u \subset\subset Q$. Moreover, by a change of variables in the integrals, we see that

$$\|u\|^p_{L^p(Q;\mathbb{R}^d)} < C|Q| = C \quad \Longleftrightarrow \quad \|u_\rho\|^p_{L^p(Q_\rho;\mathbb{R}^d)} < C|Q_\rho| = C\rho^N.$$

Furthermore, as we prove next, if

$$\|\widetilde{\mathcal{A}}_{Q_\rho} u_\rho\|^p_{\widetilde{W}^{-1,p}(Q_\rho;\mathbb{R}^l)} < \eta|Q_\rho| = \eta\rho^N, \tag{4.7}$$

then

$$\|\widetilde{\mathcal{A}}_Q u\|^p_{\widetilde{W}^{-1,p}(Q;\mathbb{R}^l)} < \eta = \eta|Q|. \tag{4.8}$$

In fact, for every $\psi \in W^{1,q}(Q;\mathbb{R}^l)$, we consider the function $\psi_\rho \in W^{1,q}(Q_\rho;\mathbb{R}^l)$ defined by

$$\psi_\rho(x) := \psi(\tfrac{x}{\rho}) \quad \text{for every } x \in Q_\rho.$$

We first observe that

$$\begin{aligned}
\|\psi\|^q_{W^{1,q}(Q;\mathbb{R}^l)} &= \int_Q |\psi(x)|^q\,dx + \int_Q |\nabla\psi(x)|^q\,dx = \int_Q |\psi_\rho(\rho x)|^q\,dx + \rho^q \int_Q |\nabla\psi_\rho(\rho x)|^q\,dx \\
&= \rho^{-N}\int_{Q_\rho} |\psi_\rho(x)|^q\,dx + \rho^{q-N}\int_{Q_\rho} |\nabla\psi_\rho(x)|^q\,dx \geq \rho^{q-N}\|\psi_\rho\|^q_{W^{1,q}(Q_\rho;\mathbb{R}^l)},
\end{aligned} \tag{4.9}$$

where we used the fact that $0 < \rho \leq 1$ in the last inequality. Moreover,

$$\begin{aligned}
\langle \widetilde{\mathcal{A}}_Q u, \psi\rangle &= -\sum_{i=1}^N \int_Q (A^i u(x)) \cdot \partial_i\psi(x)\,dx = -\rho\sum_{i=1}^N \int_Q (A^i u_\rho(\rho x)) \cdot \partial_i\psi_\rho(\rho x)\,dx \\
&= -\rho^{1-N}\sum_{i=1}^N \int_{Q_\rho} (A^i u_\rho(x)) \cdot \partial_i\psi_\rho(x)\,dx = \rho^{1-N}\langle \widetilde{\mathcal{A}}_{Q_\rho} u_\rho, \psi_\rho\rangle.
\end{aligned} \tag{4.10}$$

Thus, (4.9) and (4.10) yield

$$\frac{\langle \widetilde{\mathcal{A}}_Q u, \psi\rangle}{\|\psi\|_{W^{1,q}(Q;\mathbb{R}^l)}} \leq \frac{\rho^{1-N}\langle \widetilde{\mathcal{A}}_{Q_\rho} u_\rho, \psi_\rho\rangle}{\rho^{1-\frac{N}{q}}\|\psi_\rho\|_{W^{1,q}(Q_\rho;\mathbb{R}^l)}} = \frac{\langle \widetilde{\mathcal{A}}_{Q_\rho} u_\rho, \psi_\rho\rangle}{\rho^{\frac{N}{p}}\|\psi_\rho\|_{W^{1,q}(Q_\rho;\mathbb{R}^l)}},$$

from which we conclude that

$$\|\widetilde{\mathcal{A}}_Q u\|^p_{\widetilde{W}^{-1,p}(Q,\mathbb{R}^l)} \leq \frac{1}{\rho^N}\|\widetilde{\mathcal{A}}_{Q_\rho} u\|^p_{\widetilde{W}^{-1,p}(Q_\rho,\mathbb{R}^l)}.$$

Therefore, (4.7) implies (4.8). Consequently, u fulfils all hypotheses required on Q.

Assuming that the result is proved for Q, let $\rho \in (0,1]$, Q_ρ, and u_ρ be given with $u_\rho \in L^p(Q_\rho;\mathbb{R}^d)$, $\|u_\rho\|^p_{L^p(Q_\rho;\mathbb{R}^d)} < C|Q_\rho|$, $\operatorname{supp} u_\rho \subset\subset Q_\rho$, and $\|\widetilde{\mathcal{A}}_{Q_\rho} u_\rho\|^p_{\widetilde{W}^{-1,p}(Q_\rho;\mathbb{R}^l)} < \eta|Q_\rho|$. As above, set $u(x) := u_\rho(\rho x)$ for $x \in Q$. Then, there exists $v \in L^p_{\mathrm{per}}(Q;\mathbb{R}^d)$ satisfying the properties considered in the statement for Q relative to u. Let $v_\rho \in L^p_{\mathrm{per}}(Q_\rho;\mathbb{R}^d)$ be defined by

$$v_\rho(x) := v(\tfrac{x}{\rho}) \quad \text{for every } x \in Q_\rho.$$

Given $\psi_\rho \in W^{1,q}_0(Q_\rho;\mathbb{R}^d)$, we consider $\psi \in W^{1,q}(Q;\mathbb{R}^d)$ defined by $\psi(x) := \psi_\rho(\rho x)$, $x \in Q$, and observe that

$$\begin{aligned}
\|\psi\|^q_{W^{1,q}_0(Q;\mathbb{R}^d)} &= \int_Q |\nabla\psi(x)|^q\,dx = \rho^q\int_Q |\nabla\psi_\rho(\rho x)|^q\,dx \\
&= \rho^{q-N}\int_{Q_\rho} |\nabla\psi_\rho(x)|^q\,dx = \rho^{q-N}\|\psi_\rho\|^q_{W^{1,q}(Q_\rho;\mathbb{R}^d)}.
\end{aligned}$$

Moreover,

$$\begin{aligned}\langle v-u,\psi\rangle &= \int_Q (v(x)-u(x))\cdot\psi(x)\,dx = \int_Q (v_\rho(\rho x)-u_\rho(\rho x))\cdot\psi_\rho(\rho x)\,dx\\ &= \rho^{-N}\int_{Q_\rho}(v_\rho(x)-u_\rho(x))\cdot\psi_\rho(x)\,dx = \rho^{-N}\langle v_\rho-u_\rho,\psi_\rho\rangle.\end{aligned}$$

From the two preceding chain of equalities, we conclude that

$$\frac{\langle v-u,\psi\rangle}{\|\psi\|_{W_0^{1,q}(Q;\mathbb{R}^d)}} = \frac{\rho^{-N}\langle v_\rho-u_\rho,\psi_\rho\rangle}{\rho^{1-\frac{N}{q}}\|\psi_\rho\|_{W_0^{1,q}(Q_\rho;\mathbb{R}^d)}} = \frac{\langle v_\rho-u_\rho,\psi_\rho\rangle}{\rho^{1+\frac{N}{p}}\|\psi_\rho\|_{W_0^{1,q}(Q_\rho;\mathbb{R}^d)}}.$$

Thus,

$$\|v_\rho-u_\rho\|^p_{W^{-1,p}(Q_\rho;\mathbb{R}^d)} \le \rho^{p+N}\|v-u\|^p_{W^{-1,p}(Q;\mathbb{R}^d)}.$$

Because $\|v-u\|^p_{W^{-1,p}(Q;\mathbb{R}^d)} < \varepsilon|Q| = \varepsilon$, we obtain for $0<\rho<1$ that

$$\|v_\rho-u_\rho\|^p_{W^{-1,p}(Q_\rho;\mathbb{R}^d)} < \varepsilon\rho^{p+N} < \varepsilon\rho^N = \varepsilon|Q_\rho|.$$

The equalities $\mathcal{A}v_\rho = 0$ in $\mathbb{R}^N$ and $\int_{Q_\rho} v_\rho\,dx = \int_{Q_\rho} u_\rho\,dx$ can be obtained from the corresponding properties of v and u by a change of variables. As for (4.6) for v_ρ and u_ρ, given $F_\rho\in\mathcal{I}_{\rm Lip}$, let $f_\rho\in\mathcal{F}_{\rm Lip}$ be the corresponding integrand (see Theorem 3.3). Let $f\in\mathcal{F}_{\rm Lip}$ be the function defined by

$$f(x,\xi) := f_\rho(\rho x,\xi) \quad \text{for every } x\in\mathbb{R}^N \text{ and } \xi\in\mathbb{R}^d,$$

and denote by $F\in\mathcal{I}_{\rm Lip}$ the corresponding functional (see Remark 2.11). Then, by (4.6) for Q, we have

$$F(v,Q) < F(u,Q)+\varepsilon|Q| = F(u,Q)+\varepsilon. \tag{4.11}$$

On the other hand,

$$\begin{aligned}F_\rho(v_\rho,Q_\rho) &= \int_{Q_\rho} f_\rho(x,v_\rho(x))\,dx = \rho^N\int_Q f_\rho(\rho x,v_\rho(\rho x))\,dx = \rho^N\int_Q f(x,v(x))\,dx = \rho^N F(v,Q),\\ F_\rho(u_\rho,Q_\rho) &= \int_{Q_\rho} f_\rho(x,u_\rho(x))\,dx = \rho^N\int_Q f_\rho(\rho x,u_\rho(\rho x))\,dx = \rho^N\int_Q f(x,u(x))\,dx = \rho^N F(u,Q),\end{aligned}$$

which, together with (4.11), yield

$$F_\rho(v_\rho,Q_\rho) < F_\rho(u_\rho,Q_\rho)+\varepsilon\rho^N = F_\rho(u_\rho,Q_\rho)+\varepsilon|Q_\rho|.$$

Since F_ρ is an arbitrary element of $\mathcal{I}_{\rm Lip}$, we obtain (4.6) for v_ρ and u_ρ. This concludes the proof of the claim that it is enough to prove the corollary for $Q := Q_1(0)$, which we establish next.

We argue by contradiction. Assume that the statement for Q is false. Then, there exist $\varepsilon>0$ and $C>0$ such that for every $k\in\mathbb{N}$ there exists $u_k\in L^p(Q;\mathbb{R}^d)$, with $\|u_k\|^p_{L^p(Q;\mathbb{R}^d)} < C$, $\operatorname{supp} u_k \subset\subset Q$, and $\|\widetilde{\mathcal{A}}_Q u_k\|^p_{\widetilde{W}^{-1,p}(Q;\mathbb{R}^l)} < 1/k$, such that for every $v\in L^p_{\rm per}(Q;\mathbb{R}^d)$, with $\|v-u_k\|^p_{W^{-1,p}(Q;\mathbb{R}^d)}<\varepsilon$, $\mathcal{A}v=0$ in $\mathbb{R}^N$, and $\int_Q v\,dx = \int_Q u_k\,dx$, there exists $F_{k,v}\in\mathcal{I}_{\rm Lip}$ such that

$$F_{v,k}(v,Q) \ge F_{v,k}(u_k,Q)+\varepsilon. \tag{4.12}$$

Since $(u_k)_{k\in\mathbb{N}}$ is bounded in $L^p(Q;\mathbb{R}^d)$, a subsequence of $(u_k)_{k\in\mathbb{N}}$, not relabeled, converges weakly in $L^p(Q;\mathbb{R}^d)$ to a function $u\in L^p(Q;\mathbb{R}^d)$. We extend each u_k to a Q-periodic function, still denoted u_k. Then, for every $D\in\mathcal{O}(\mathbb{R}^N)$, u_k converges weakly in $L^p(D;\mathbb{R}^d)$ to the periodic extension of u, still denoted by u. To prove that $\mathcal{A}u=0$ in $\mathbb{R}^N$, we start by setting $Q_m := Q_m(0)$ for every $m\in\mathbb{N}$, and we show that

$$\|\widetilde{\mathcal{A}}_{Q_m} u_k\|_{\widetilde{W}^{-1,p}(Q_m;\mathbb{R}^l)} < \frac{m^N}{k^{1/p}}. \tag{4.13}$$

To prove this inequality, we define $A_m := \{1,\dots,m\}^N$ and for every $\alpha=(\alpha_1,\dots,\alpha_N)\in A_m$, we set

$$x(\alpha) := \Big(-\frac{m}{2}-\frac{1}{2}+\alpha_1,\dots,-\frac{m}{2}-\frac{1}{2}+\alpha_N\Big).$$

We observe that

$$\overline{Q}_m = \bigcup_{\alpha\in A_m}\overline{Q}_1(x(\alpha)).$$

Let $\psi \in W^{1,q}(Q_m;\mathbb{R}^l)$. For every $\alpha \in A_m$, we define $\psi_\alpha \in W^{1,q}(Q;\mathbb{R}^l)$ by $\psi_\alpha(x) := \psi(x + x(\alpha))$. Using the Q-periodicity of u_k, we get from (2.4) that

$$\langle \widetilde{\mathcal{A}}_{Q_m} u_k, \psi\rangle = -\sum_{i=1}^N \int_{Q_m} (A^i u_k)\cdot \partial_i\psi \, dx = -\sum_{i=1}^N \sum_{\alpha\in A_m} \int_Q (A^i u_k)\cdot \partial_i\psi_\alpha \, dx = \sum_{\alpha\in A_m} \langle \widetilde{\mathcal{A}}_Q u_k, \psi_\alpha\rangle.$$

Since $\|\widetilde{\mathcal{A}}_Q u_k\|^p_{\widetilde{W}^{-1,p}(Q;\mathbb{R}^l)} < 1/k$, we deduce that

$$\big|\langle \widetilde{\mathcal{A}}_{Q_m} u_k, \psi\rangle\big| \le \frac{1}{k^{1/p}} \sum_{\alpha\in A_m} \|\psi_\alpha\|_{W^{1,q}(Q;\mathbb{R}^l)} \le \frac{m^N}{k^{1/p}} \|\psi\|_{W^{1,q}(Q_m;\mathbb{R}^l)},$$

which concludes the proof of (4.13). Because u_k converges to u weakly in $L^p(Q_m;\mathbb{R}^d)$, we obtain from (4.13) that $\widetilde{\mathcal{A}}_{Q_m} u = 0$ for every $m \in \mathbb{N}$. We then conclude that $\mathcal{A}u = 0$ in $\mathbb{R}^N$.

By Lemma 4.1, there exists a sequence $(v_k)_{k\in\mathbb{N}}$ in $L^p_{\rm per}(Q;\mathbb{R}^d)$ satisfying (4.1) with $D = Q$. Since the embedding of $L^p(Q;\mathbb{R}^d)$ into $W^{-1,p}(Q;\mathbb{R}^d)$ is compact, the sequences $(v_k)_{k\in\mathbb{N}}$ and $(u_k)_{k\in\mathbb{N}}$ converge to u strongly in $W^{-1,p}(Q;\mathbb{R}^d)$, and so $\|v_k - u_k\|^p_{W^{-1,p}(Q;\mathbb{R}^d)} < \varepsilon$ for k large enough.

For every k, let $F_k := F_{v_k,k}$. By Lemma 4.2, we have

$$\limsup_{k\to\infty} \big(F_k(v_k, Q) - F_k(u_k, Q)\big) \le 0,$$

while (4.12) gives $F_k(v_k, Q) - F_k(u_k, Q) \ge \varepsilon$ for k large enough. This contradiction concludes the proof. ☐

Definition 4.4. *For every $F \in \mathcal{I}$ and every $D \in \mathcal{O}(\mathbb{R}^N)$, let $F^{\mathcal{A}}(\cdot, D)$ be the restriction of $F(\cdot, D)$ to* $\ker \mathcal{A}_D$.

Remark 4.5. Let $(F_k)_{k\in\mathbb{N}}$ be a sequence of functionals in $\mathcal{I}$, let $F \in \mathcal{I}$, and let $D \in \mathcal{O}(\mathbb{R}^N)$. If we extend $F^{\mathcal{A}}(\cdot, D)$ to $L^p(D;\mathbb{R}^d)$ by setting $F^{\mathcal{A}}(u, D) = +\infty$ for every $u \in L^p(D;\mathbb{R}^d) \setminus \ker \mathcal{A}_D$, by the lower bound in (a) of Definition 2.9 we can apply [12, Proposition 8.16] to conclude that the sequence $(F_k^{\mathcal{A}}(\cdot, D))_{k\in\mathbb{N}}$ Γ-converges to $F^{\mathcal{A}}(\cdot, D)$ in $\ker \mathcal{A}_D$ with respect to the weak topology of $L^p(D;\mathbb{R}^d)$ if and only if

- (LI) for every $u \in \ker \mathcal{A}_D$ and every sequence $(u_k)_{k\in\mathbb{N}}$ in $\ker \mathcal{A}_D$ converging to u weakly in $L^p(D;\mathbb{R}^d)$, we have $F^{\mathcal{A}}(u, D) \le \liminf_{k\to\infty} F_k^{\mathcal{A}}(u_k, D)$,
- (LS) for every $u \in \ker \mathcal{A}_D$ there exists a sequence $(v_k)_{k\in\mathbb{N}}$ in $\ker \mathcal{A}_D$ converging to u weakly in $L^p(D;\mathbb{R}^d)$ such that $F^{\mathcal{A}}(v, D) \ge \limsup_{k\to\infty} F_k^{\mathcal{A}}(v_k, D)$.

We now prove the equivalence between Γ-convergence with respect to the topology induced by $\|\cdot\|^{\mathcal{A}}_D$ and Γ-convergence in the $\mathcal{A}$-free setting.

Theorem 4.6. *Let $(F_k)_{k\in\mathbb{N}}$ be a sequence of functionals in $\mathcal{I}_{\rm Lip}$, let $F \in \mathcal{I}$, and let $F_k^{\mathcal{A}}$ and $F^{\mathcal{A}}$ be as in Definition 4.4. The the following conditions are equivalent:*

- (a) *For every $D \in \mathcal{O}(\mathbb{R}^N)$, the sequence $F_k(\cdot, D)$ Γ-converges to $F(\cdot, D)$ with respect to the topology induced by $\|\cdot\|^{\mathcal{A}}_D$ on $L^p(D;\mathbb{R}^d)$;*
- (b) *for every $D \in \mathcal{O}(\mathbb{R}^N)$, the sequence $(F_k^{\mathcal{A}}(\cdot, D))_{k\in\mathbb{N}}$ Γ-converges to $F^{\mathcal{A}}(\cdot, D)$ in $\ker \mathcal{A}_D$ with respect to the weak topology of $L^p(D;\mathbb{R}^d)$.*

Proof. Assume (a). Let $u \in \ker \mathcal{A}_D$ and let $(u_k)_{k\in\mathbb{N}}$ be a sequence in $\ker \mathcal{A}_D$ converging to u weakly in $L^p(D;\mathbb{R}^d)$. Since the embedding of $L^p(D;\mathbb{R}^d)$ into $W^{-1,p}(D;\mathbb{R}^d)$ is compact and $\mathcal{A}u_k = \mathcal{A}u = 0$, the sequence $(u_k)_{k\in\mathbb{N}}$ converges to u strongly in the topology induced by the norm $\|\cdot\|^{\mathcal{A}}_D$ on $L^p(D;\mathbb{R}^d)$. By (a), this implies that

$$F^{\mathcal{A}}(u, D) \le \liminf_{k\to\infty} F_k^{\mathcal{A}}(u_k, D).$$

Hence, condition (LI) of Remark 4.5 is satisfied.

To prove (LS), we fix $u \in \ker \mathcal{A}_D$ and a sequence $(u_k)_{k\in\mathbb{N}} \subset L^p(D;\mathbb{R}^d)$ such that

$$\lim_{k\to\infty} \|u_k - u\|^{\mathcal{A}}_D = 0 \quad \text{and} \quad \limsup_{k\to\infty} F_k(u_k, D) = F(u, D) = F^{\mathcal{A}}(u, D) < +\infty,$$

which exists by [12, Proposition 8.1]. We observe further that $u_k \rightharpoonup u$ weakly in $L^p(D;\mathbb{R}^d)$ (see Remark 2.10). Let $(v_k)_{k\in\mathbb{N}}$ be the sequence in $\ker\mathcal{A}_D$ provided by Lemma 4.1. By Lemma 4.2, we have

$$\limsup_{k\to\infty} F_k^{\mathcal{A}}(v_k,D) = \limsup_{k\to\infty} F_k(v_k,D) \le \limsup_{k\to\infty} F_k(u_k,D) = F^{\mathcal{A}}(u,D),$$

which proves condition (LS) of Remark 4.5. This concludes the proof of (b).

Assume (b). By the Theorem 3.1, there exist a subsequence, which we do not relabel, and a functional $\hat F \in \mathcal{I}_{\rm sc}$ such that for every $D \in \mathcal{O}(\mathbb{R}^N)$, the sequence $(F_k(\cdot,D))_{k\in\mathbb{N}}$ Γ-converges to $\hat F(\cdot,D)$ with respect to the topology induced by $\|\cdot\|_D^{\mathcal{A}}$ on $L^p(D;\mathbb{R}^d)$. Since (a) implies (b), we have $\hat F(u,D) = F(u,D)$ for every $D \in \mathcal{O}(\mathbb{R}^N)$ and every $u \in \ker\mathcal{A}_D$. Let $\hat f$ and f be the integrands corresponding to $\hat F$ and F, respectively, defined by (3.23). Since every constant function belongs to $\ker\mathcal{A}_D$, we deduce that $\hat f = f$; by Theorems 3.3 and 3.5 this implies $\hat F = F$. Since the Γ-limit does not depend on the subsequence, we obtain (a) by the Urysohn property of Γ-convergence (see [12, Proposition 8.3]). □

From Theorems 3.1 and 4.6, we deduce the following compactness result in the $\mathcal{A}$-free setting.

Corollary 4.7. *Let $(F_k)_{k\in\mathbb{N}}$ be a sequence of functionals in $\mathcal{I}_{\rm Lip}$ and let $F_k^{\mathcal{A}}$ be as in Definition 4.4. Then, there exist a subsequence, which we do not relabel, and a functional $F \in \mathcal{I}_{\rm sc}$ such that for every $D \in \mathcal{O}(\mathbb{R}^N)$, the sequence $(F_k^{\mathcal{A}}(\cdot,D))_{k\in\mathbb{N}}$ Γ-converges to $F^{\mathcal{A}}(\cdot,D)$ in $\ker\mathcal{A}_D$ with respect to the weak topology of $L^p(D;\mathbb{R}^d)$.*

5. The integrand obtained from minimum values on small cubes

In this section, given an integrand $f \in \mathcal{F}_{\rm qc}$, we reconstruct its values $f(x,\xi)$ at a pair $(x,\xi) \in \mathbb{R}^N\times\mathbb{R}^d$ by taking the limit, as $\rho \to 0^+$, of the infima of some minumum problems related to f and ξ in the cubes $Q_\rho(x)$.

Definition 5.1. *For every cube $Q \subset \mathbb{R}^N$, we consider the sets*

$$\mathcal{U}(Q) := \Big\{ u \in L^p_{\rm loc}(\mathbb{R}^N;\mathbb{R}^d) \colon u \text{ is } Q\text{-periodic}, \int_Q u\,dx = 0, \text{ and } \mathcal{A}u = 0 \text{ in } \mathbb{R}^N \text{ in the sense of (2.2)} \Big\},$$

$$\mathcal{U}_c(Q) := \Big\{ u \in L^p(Q;\mathbb{R}^d) \colon \operatorname{supp} u \subset\subset Q, \int_Q u\,dx = 0, \text{ and } \mathcal{A}_Q u = 0 \text{ in } Q \text{ in the sense of (2.3)} \Big\}.$$

For every $f \in \mathcal{F}$ and $\xi \in \mathbb{R}^d$, we set

$$\begin{aligned} M(f,\xi,Q) &:= \inf\big\{F(\xi+u,Q) : u \in \mathcal{U}(Q)\big\},\\ M_c(f,\xi,Q) &:= \inf\big\{F(\xi+u,Q) : u \in \mathcal{U}_c(Q)\big\}, \end{aligned} \tag{5.1}$$

where F is defined by (2.14).

Remark 5.2. If $u \in \mathcal{U}_c(Q)$, then its Q-periodic extension belongs to $\mathcal{U}(Q)$. In fact, denoting by $\tilde u$ the Q-periodic extension of u, let us prove that $\mathcal{A}\tilde u = 0$ in $\mathbb{R}^N$. Given $\psi \in C_c^\infty(\mathbb{R}^N;\mathbb{R}^l)$, we can find a finite family of mutually disjoint translations of Q, $\{Q_j\}_{j=1}^M$, such that

$$\operatorname{supp}\psi \subset\subset \operatorname{int}\Big(\bigcup_{j=1}^M \overline{Q_j}\Big).$$

Moreover, setting $K := \operatorname{supp} u \subset\subset Q$, let $K_j \subset\subset Q_j$ be the corresponding translations of K. In other words, $K_j = \operatorname{supp} \tilde u_{|Q_j}$. Finally, let $\theta_j \in C_c^\infty(Q_j;[0,1])$ be a cut-off function with $\theta_j = 1$ on K_j. Then,

$$\begin{aligned} \langle \mathcal{A}\tilde u, \psi\rangle &= -\sum_{i=1}^N\sum_{j=1}^M \int_{Q_j} (A^i\tilde u)\cdot\partial_i\psi\,dx = -\sum_{i=1}^N\sum_{j=1}^M \int_{Q_j} (A^i\tilde u)\cdot\partial_i(\theta_j\psi)\,dx\\ &= -\sum_{i=1}^N\sum_{j=1}^M \int_Q (A^i u)\cdot\partial_i\tilde\psi_j\,dx = 0, \end{aligned}$$

where $\tilde\psi_j \in C_c^\infty(Q;\mathbb{R}^l)$ is the translation of $(\theta_j\psi)_{|Q_j}$ from Q_j to Q, and where we used the fact that $\mathcal{A}_Q u = 0$ in the last equality.

Hence, $M(f,\xi,Q) \le M_c(f,\xi,Q)$ for every cube $Q \subset \mathbb{R}^N$, every $f \in \mathcal{F}$, and every $\xi \in \mathbb{R}^d$.

We now prove that the value $f(x,\xi)$ of an integrand $f \in \mathcal{F}_{\rm qc}$ can be reconstructed using the minimum values introduced in (5.1) on cubes shrinking to x.

Theorem 5.3. *Let $f \in \mathcal{F}_{\rm Lip} \cap \mathcal{F}_{\rm qc}$. Then, for a.e. $x \in \mathbb{R}^N$ and every $\xi \in \mathbb{R}^d$, we have that*

$$f(x,\xi) = \lim_{\rho\to 0^+} \frac{M(f,\xi,Q_\rho(x))}{\rho^N} = \lim_{\rho\to 0^+} \frac{M_c(f,\xi,Q_\rho(x))}{\rho^N}.$$

Proof. For $\xi \in \mathbb{R}^d$ fixed, we have for every Lebesgue point for $f(\cdot,\xi)$ that

$$f(x,\xi) = \lim_{\rho\to 0^+} \frac{1}{\rho^N} \int_{Q_\rho(x)} f(y,\xi)\, dy. \tag{5.2}$$

Thus, for a.e. $x \in \mathbb{R}^N$ and for all $\xi \in \mathbb{Q}^d$, (5.2) holds. Consequently, using the Lipschitz continuity in (2.11), we conclude that (5.2) holds for a.e. $x \in \mathbb{R}^N$ and for all $\xi \in \mathbb{R}^d$. Then, for a.e. $x \in \mathbb{R}^N$ and for all $\xi \in \mathbb{R}^d$, we have that

$$f(x,\xi) = \lim_{\rho\to 0^+} \frac{1}{\rho^N} \int_{Q_\rho(x)} f(y,\xi)\, dy \geq \limsup_{\rho\to 0^+} \frac{M_c(f,\xi,Q_\rho(x))}{\rho^N}.$$

By Remark 5.2, it remains to prove that

$$f(x,\xi) \leq \liminf_{\rho\to 0^+} \frac{M(f,\xi,Q_\rho(x))}{\rho^N} \tag{5.3}$$

holds for a.e. $x \in \mathbb{R}^N$ and for all $\xi \in \mathbb{R}^d$.

Fix $x \in \mathbb{R}^N$ and $\xi \in \mathbb{R}^d$. Given $\rho > 0$, (5.1) yields a function $u_\rho \in \mathcal{U}(Q_\rho(x))$ satisfying

$$\int_{Q_\rho(x)} f(y, \xi + u_\rho(y))\, dy < M(f,\xi,Q_\rho(x)) + \rho^{N+1}.$$

Then, to prove (5.3), it suffices to show that

$$f(x,\xi) \leq \liminf_{\rho\to 0^+} \frac{1}{\rho^N} \int_{Q_\rho(x)} f(y,\xi + u_\rho(y))\, dy. \tag{5.4}$$

To prove this inequality, we have to compare the values of f at two different points, x, $y \in \mathbb{R}^N$. For this reason, for $m \in \mathbb{N}$ and for $x, y \in \mathbb{R}^N$, we define

$$\omega_m(x,y) := \sup_{|\eta|\leq m} |f(y,\eta) - f(x,\eta)|. \tag{5.5}$$

We claim that there exists a set $E \subset \mathbb{R}^N$ with measure 0 such that for every $x \in \mathbb{R}^N \setminus E$ and every $m \in \mathbb{N}$, we have

$$\lim_{\rho\to 0^+} \frac{1}{\rho^N} \int_{Q_\rho(x)} \omega_m(x,y)\, dy = 0. \tag{5.6}$$

To prove this claim, we fix $m \in \mathbb{N}$ and $k \in \mathbb{N}$ and find $n_{m,k} \in \mathbb{N}$ and a finite family $(\eta_i^{m,k})_{i=1}^{n_{m,k}}$ in $B_m(0)$ such that

$$\overline{B_m(0)} \subset \bigcup_{i=1}^{n_{m,k}} B_{\frac{1}{k}}(\eta_i^{m,k}).$$

Then, for $\eta \in B_{\frac{1}{k}}(\eta_i^{m,k})$, we have

$$\begin{aligned} |f(y,\eta) - f(x,\eta)| &\leq |f(y,\eta_i^{m,k}) - f(x,\eta_i^{m,k})| + |f(y,\eta) - f(y,\eta_i^{m,k})| + |f(x,\eta) - f(x,\eta_i^{m,k})| \\ &\leq |f(y,\eta_i^{m,k}) - f(x,\eta_i^{m,k})| + C(1 + m^{p-1})\tfrac{1}{k}, \end{aligned}$$

where we used (2.10) and (2.11), and C depends on c_0 and c_1. Consequently,

$$\omega_m(x,y) \leq \sup_{1\leq i\leq n_{m,k}} |f(y,\eta_i^{m,k}) - f(x,\eta_i^{m,k})| + C(1+m^{p-1})\tfrac{1}{k},$$

which gives

$$\frac{1}{\rho^N} \int_{Q_\rho(x)} \omega_m(x,y)\, dy \leq \sum_{i=1}^{n_{m,k}} \frac{1}{\rho^N} \int_{Q_\rho(x)} |f(y,\eta_i^{m,k}) - f(x,\eta_i^{m,k})|\, dy + C(1+m^{p-1})\frac{1}{k}.$$

Hence, considering the Lebesgue points x for all functions $f(\cdot,\eta_i^{m,k})$ with $m\in\mathbb{N}$, $k\in\mathbb{N}$, and $i\in\{1,...,n_{m,k}\}$, we find a set $E\subset\mathbb{R}^N$ with measure 0 such that for every $x\in\mathbb{R}^N\setminus E$, $m\in\mathbb{N}$, and $k\in\mathbb{N}$ we have

$$\lim_{\rho\to0^+}\frac{1}{\rho^N}\int_{Q_\rho(x)}\omega_m(x,y)\,dy\le C(1+m^{p-1})\frac{1}{k},$$

from which (5.6) follows by taking the limit $k\to\infty$. Since $f\in\mathcal{F}_{\rm qc}$ for every $x\in\mathbb{R}^N\setminus E$ the function $\xi\mapsto f(x,\xi)$ is $\mathcal{A}$-quasiconvex.

We now fix $x\in\mathbb{R}^N\setminus E$, $\xi\in\mathbb{R}^d$, and a sequence $\rho_j\to0^+$ such that

$$\lim_{j\to\infty}\frac{1}{\rho_j^N}\int_{Q_{\rho_j}(x)}f(y,\xi+u_{\rho_j}(y))\,dy=\liminf_{\rho\to0^+}\frac{1}{\rho^N}\int_{Q_\rho(x)}f(y,\xi+u_\rho(y))\,dy. \tag{5.7}$$

By (2.10),

$$\begin{aligned}\int_{Q_{\rho_j}(x)}\Big(\frac{1}{c_0}|\xi+u_{\rho_j}(y)|^p-c_0\Big)\,dy\le\int_{Q_{\rho_j}(x)}f(y,\xi+u_{\rho_j}(y))\,dy<M(f,\xi,Q_{\rho_j}(x))+\rho_j^{N+1}\\ \le c_0(1+|\xi|^p)\rho_j^N+\rho_j^{N+1},\end{aligned}$$

which gives

$$\frac{1}{\rho_j^N}\int_{Q_{\rho_j}(x)}|\xi+u_{\rho_j}(y)|^p\,dy\le C_\xi \tag{5.8}$$

for some positive constant C_ξ, independent of j. Then, setting

$$v_j(z):=u_{\rho_j}(x+\rho_jz)\qquad\text{for } z\in Q,$$

with Q the unit cube centered at the origin, we have that $v_j\in\mathcal{U}(Q)$ with $\int_Q|\xi+v_j(z)|^p\,dz\le C_\xi$ by (5.8). Then, we can use [7, Lemma 3.1] to find a p-equi-integrable sequence $(\tilde v_j)_{j\in\mathbb{N}}\subset\mathcal{U}(Q)$ such that

$$\limsup_{j\to\infty}\int_Q f(x+\rho_jz,\xi+\tilde v_j(z))\,dz\le\limsup_{j\to\infty}\int_Q f(x+\rho_jz,\xi+v_j(z))\,dz.$$

Next, we set $\tilde u_j(y):=\tilde v_j(\frac{y-x}{\rho_j})$, and we observe that the preceding inequality becomes

$$\limsup_{j\to\infty}\frac{1}{\rho_j^N}\int_{Q_{\rho_j}(x)}f(y,\xi+\tilde u_j(y))dy\le\lim_{j\to\infty}\frac{1}{\rho_j^N}\int_{Q_{\rho_j}(x)}f(y,\xi+u_{\rho_j}(y))dy.$$

This inequality, together with (5.7), implies that (5.4) is a consequence of the inequality

$$f(x,\xi)\le\liminf_{j\to\infty}\frac{1}{\rho_j^N}\int_{Q_{\rho_j}(x)}f(y,\xi+\tilde u_j(y))\,dy, \tag{5.9}$$

which we establish next. Observing that $(\tilde u_j)_{j\in\mathbb{N}}\subset\mathcal{U}(Q_{\rho_j}(x))$, we have by the definition of $\mathcal{A}$-quasiconvexity (Definition 2.1) that

$$f(x,\xi)\le\frac{1}{\rho_j^N}\int_{Q_{\rho_j}(x)}f(x,\xi+\tilde u_j(y))\,dy\qquad\text{for all } j\in\mathbb{N}. \tag{5.10}$$

To prove (5.9), we compare the integrals

$$\frac{1}{\rho_j^N}\int_{Q_{\rho_j}(x)}f(x,\xi+\tilde u_j(y))\,dy\qquad\text{and}\qquad\frac{1}{\rho_j^N}\int_{Q_{\rho_j}(x)}f(y,\xi+\tilde u_j(y))\,dy.$$

To this aim, for $m\in\mathbb{N}$, we set

$$\hat Q_j^m(x):=\{y\in Q_{\rho_j}(x):|\xi+\tilde u_j(y)|\le m\}\qquad\text{and}\qquad\check Q_j^m(x):=\{y\in Q_{\rho_j}(x):|\xi+\tilde u_j(y)|>m\},$$

and observe that (5.10) and (5.5) yield

$$\begin{aligned}f(x,\xi)&\le\frac{1}{\rho_j^N}\int_{\hat Q_j^m(x)}f(x,\xi+\tilde u_j(y))\,dy+\frac{1}{\rho_j^N}\int_{\check Q_j^m(x)}f(x,\xi+\tilde u_j(y))\,dy\\&\le\frac{1}{\rho_j^N}\int_{Q_{\rho_j}(x)}f(y,\xi+\tilde u_j(y))\,dy+\frac{1}{\rho_j^N}\int_{Q_{\rho_j}(x)}\omega_m(x,y)\,dy+\frac{1}{\rho_j^N}\int_{\check Q_j^m(x)}f(x,\xi+\tilde u_j(y))\,dy.\end{aligned}$$

Passing to the limit as $j\to\infty$, we obtain from (5.6) that

$$f(x,\xi)\le\liminf_{j\to\infty}\frac{1}{\rho_j^N}\int_{Q_j(x)}f(y,\xi+\tilde u_j(y))\,dy+\limsup_{j\to\infty}\frac{1}{\rho_j^N}\int_{\check Q_j^m(x)}f(x,\xi+\tilde u_j(y))\,dy.$$

To conclude the proof of (5.9), it is enough to show that

$$\limsup_{j\to\infty}\frac{1}{\rho_j^N}\int_{\check{Q}_j^m(x)} f(x,\xi+\tilde{u}_j(y))\,dy \le \lambda_m \quad \text{with} \lim_{m\to\infty}\lambda_m = 0.$$

The preceding estimate is a consequence of the p-equi-integrability of the sequence $(\tilde{v}_j)_{j\in\mathbb{N}}$ in Q together with the inequality

$$\frac{1}{\rho_j^N}\int_{\check{Q}_j^m(x)} f(x,\xi+\tilde{u}_j(y))\,dy \le \frac{c_0}{\rho_j^N}\int_{\check{Q}_j^m(x)} (1+|\xi+\tilde{u}_j(y)|^p)\,dy = c_0\int_{\check{Q}_j^m}\big(1+|\xi+\tilde{v}_j(z)|^p\big)\,dz,$$

where $\check{Q}_j^m := \{z\in Q : |\tilde{v}_j(z)|>m\}$, with $|\check{Q}_j^m|\to 0$ as $m\to\infty$. □

6. The Γ-limit obtained from minimum values on small cubes

In this section, we prove that the integrand of the Γ-limit of a sequence $(F_k)_{k\in\mathbb{N}}$ of functionals in $\mathcal{I}$ can be obtained by talking the limit, first as $k\to\infty$ and then as $\rho\to 0^+$, of the infima of some minimum problems for F_k on the cubes $Q_\rho(x)$, see (6.6).

We begin by proving that if $F_k(\cdot,D)$ Γ-converges to $F(\cdot,D)$, then the corresponding infima introduced in (5.1) satisfy some inequalities.

Proposition 6.1. *Let $(f_k)_{k\in\mathbb{N}}$ be a sequence in $\mathcal{F}_{\rm Lip}$, let $f\in\mathcal{F}$, and let F_k and F be the corresponding functionals in $\mathcal{I}$ defined by (2.14). Assume that for every $D\in\mathcal{O}(\mathbb{R}^N)$, the sequence $(F_k(\cdot,D))_{k\in\mathbb{N}}$ Γ-converges to $F(\cdot,D)$ with respect to the topology induced by $\|\cdot\|_D^{\mathcal{A}}$ on $L^p(D;\mathbb{R}^d)$. Let $Q\subset\mathbb{R}^N$ be an open cube and let $\xi\in\mathbb{R}^d$. Then,*

$$\limsup_{k\to\infty} M(f_k,\xi,Q)\le M_c(f,\xi,Q), \tag{6.1}$$

$$\liminf_{k\to\infty} M(f_k,\xi,Q)\ge M(f,\xi,Q). \tag{6.2}$$

Proof. Let $\delta>0$. By (5.1), there exists $u\in L^p(Q;\mathbb{R}^N)$, with $\operatorname{supp} u\subset\subset Q$, $\int_Q u\,dx=0$, and $\mathcal{A}_Q u=0$, such that

$$F(\xi+u,Q)<M_c(f,\xi,Q)+\delta.$$

By Γ-convergence, there exists a sequence $(u_k)_{k\in\mathbb{N}}$ in $L^p(Q,\mathbb{R}^N)$ such that $u_k\to u$ in $W^{-1,p}(Q,\mathbb{R}^N)$, $\mathcal{A}_Q u_k\to\mathcal{A}_Q u=0$ in $W^{-1,p}(Q,\mathbb{R}^l)$, and

$$\lim_{k\to\infty}F_k(\xi+u_k,Q)=F(\xi+u,Q)<M_c(f,\xi,Q)+\delta<+\infty. \tag{6.3}$$

By (2.10), this implies that $(u_k)_{k\in\mathbb{N}}$ is bounded in $L^p(Q,\mathbb{R}^N)$; hence, $u_k\rightharpoonup u$ weakly in $L^p(Q,\mathbb{R}^N)$. Using Lemmas 4.1 and 4.2, we can find a p-equi-integrable sequence $(v_k)_{k\in\mathbb{N}}\subset L^p_{per}(Q;\mathbb{R}^d)$ satisfying

$$v_k\rightharpoonup u \text{ in } L^p(Q;\mathbb{R}^d),\quad \mathcal{A}v_k=0 \text{ in } \mathbb{R}^N,\quad \int_Q v_k\,dx=\int_Q u\,dx=0,$$

and

$$\limsup_{k\to\infty}F_k(\xi+v_k,Q)\le\limsup_{k\to\infty}F_k(\xi+u_k,Q). \tag{6.4}$$

By (5.1), we have $M(f_k,\xi,Q)\le F_k(\xi+v_k,Q)$. This inequality, together with (6.3) and (6.4), yields

$$\limsup_{k\to\infty}M(f_k,\xi,Q)\le M_c(f,\xi,Q)+\delta.$$

Since $\delta>0$ is arbitrary, we obtain (6.1).

To prove (6.2), we choose $u_k\in L^p_{\rm loc}(\mathbb{R}^N;\mathbb{R}^d)$ for every k, with u_k Q-periodic, $\int_Q u_k\,dx=0$, and $\mathcal{A}u_k=0$ in $\mathbb{R}^N$, such that

$$F_k(\xi+u_k,Q)<M(f_k,\xi,Q)+\tfrac{1}{k}. \tag{6.5}$$

By (6.1), the right-hand side of the previous inequality is bounded; hence, a subsequence of $(u_k)_{k\in\mathbb{N}}$, not relabeled, converges to some function u weakly in $L^p(Q;\mathbb{R}^d)$.

Since all functions u_k are Q-periodic, the function u can be extended to a Q-periodic function, still denoted by u. Because the embedding of $L^p(Q;\mathbb{R}^d)$ into $W^{-1,p}(Q;\mathbb{R}^d)$ is compact and $\mathcal{A}u_k=0$ for every k, we deduce that $(u_k)_{k\in\mathbb{N}}$ converges to u in the topology induced by $\|\cdot\|_D^{\mathcal{A}}$ on $L^p(D;\mathbb{R}^d)$ and that $\mathcal{A}u=0$ in $\mathbb{R}^N$. Therefore, by Γ-convergence and by (5.1), we have

$$M(f,\xi,Q)\le F(\xi+u,Q)\le\liminf_{k\to\infty}F_k(\xi+u_k,Q).$$

Together with (6.5), the preceding estimate gives (6.2). □

Theorem 6.2. *Let $(f_k)_{k\in\mathbb{N}}$ be a sequence in $\mathcal{F}_{\rm Lip}$, let $(F_k)_{k\in\mathbb{N}}$ be the corresponding sequence of functionals in $\mathcal{I}$ defined by (2.14), and let $(F_k^{\mathcal{A}})_{k\in\mathbb{N}}$ be the sequence obtained as in Definition 4.4. Suppose that*

(a) *there exists $f\colon \mathbb{R}^N\times\mathbb{R}^d\to[0,+\infty)$ such that*

$$f(x,\xi)=\limsup_{\rho\to0^+}\liminf_{k\to\infty}\frac{M(f_k,\xi,Q_\rho(x))}{\rho^N}=\limsup_{\rho\to0^+}\limsup_{k\to\infty}\frac{M(f_k,\xi,Q_\rho(x))}{\rho^N}\tag{6.6}$$

for a.e. $x\in\mathbb{R}^N$ and every $\xi\in\mathbb{R}^d$.

Then, there exists $\hat f\in\mathcal{F}_{\rm Lip}\cap\mathcal{F}_{\rm qc}$ such that $f(x,\xi)=\hat f(x,\xi)$ for a.e. $x\in\mathbb{R}^N$ and every $\xi\in\mathbb{R}^d$, and the functionals F and $F^{\mathcal{A}}$ introduced in (2.14) and Definition 4.4 satisfy the following properties:

(b) *for every $D\in\mathcal{O}(\mathbb{R}^N)$, the sequence $(F_k(\cdot,D))_{k\in\mathbb{N}}$ Γ-converges to $F(\cdot,D)$ with respect to the topology induced by $\|\cdot\|_D^{\mathcal{A}}$ on $L^p(D;\mathbb{R}^d)$;*

(c) *for every $D\in\mathcal{O}(\mathbb{R}^N)$, the sequence $(F_k^{\mathcal{A}}(\cdot,D))_{k\in\mathbb{N}}$ Γ-converges to $F^{\mathcal{A}}(\cdot,D)$ in $\ker\mathcal{A}_D$ with respect to the weak topology of $L^p(D;\mathbb{R}^d)$.*

Conversely, if $f\in\mathcal{F}$ and the functionals F and $F^{\mathcal{A}}$ introduced in (2.14) and Definition 4.4 satisfy (b) or (c), then f satisfies (a).

Proof. Since $F_k\in\mathcal{F}_{\rm Lip}$ for every $k\in\mathbb{N}$ by Remark 2.11, the equivalence between (b) and (c) is proved in Theorem 4.6. If these conditions are satisfied, then by Corollary 3.4 there exists $\hat f\in\mathcal{F}_{\rm Lip}\cap\mathcal{F}_{\rm qc}$ such that

$$F(u,D):=\int_D\hat f(x,u(x))\,dx,\quad \text{for every } D\in\mathcal{O}(\mathbb{R}^N)\text{ and } u\in L^p(D;\mathbb{R}^d),$$

which implies $f(x,\xi)=\hat f(x,\xi)$ for a.e. $x\in\mathbb{R}^N$ and every $\xi\in\mathbb{R}^d$.

Next, we assume that (b) holds, and we show that $\hat f$ satisfies (6.6). Fix $x\in\mathbb{R}^N$ and $\xi\in\mathbb{R}^d$. By Proposition 6.1 for every $\rho>0$ we have that

$$\limsup_{k\to\infty}M(f_k,\xi,Q_\rho(x))\le M_c(f,\xi,Q_\rho(x)),$$
$$M(f,\xi,Q_\rho(x))\le\liminf_{k\to\infty}M(f_k,\xi,Q_\rho(x)).$$

By Theorem 5.3, we obtain (6.6) with $\hat f$ for a.e. $x\in\mathbb{R}^N$ and every $\xi\in\mathbb{R}^d$, concluding the proof of (a).

Finally, we assume that (a) holds, and we prove that (b) is also satisfied. By Corollary 3.4, there exists a subsequence $(f_{k_j})_{j\in\mathbb{N}}$ and a function $\hat f\in\mathcal{F}_{\rm Lip}\cap\mathcal{F}_{\rm qc}$ such that for every $D\in\mathcal{O}(\mathbb{R}^N)$ the sequence $(F_{k_j}(\cdot,D))_{j\in\mathbb{N}}$ Γ-converges to $\widehat F(\cdot,D)$ with respect to the topology induced by $\|\cdot\|_D^{\mathcal{A}}$ on $L^p(D;\mathbb{R}^d)$, where $\widehat F$ is the functional associated with $\hat f$ as in (2.14). Since (b)⇒(a) by the preceding part of the proof, we have for a.e. $x\in\mathbb{R}^N$ and every $\xi\in\mathbb{R}^d$ that

$$\hat f(x,\xi)=\limsup_{\rho\to0^+}\liminf_{j\to\infty}\frac{M(f_{k_j},\xi,Q_\rho(x))}{\rho^N}=\limsup_{\rho\to0^+}\limsup_{j\to\infty}\frac{M(f_{k_j},\xi,Q_\rho(x))}{\rho^N}.$$

By (6.6), this implies that $\hat f(x,\xi)=f(x,\xi)$ for a.e. $x\in\mathbb{R}^N$ and every $\xi\in\mathbb{R}^d$; hence, $\widehat F=F$. Since the Γ-limit does not depend on the subsequence, we obtain (b) by the Urysohn property of Γ-convergence (see [12, Proposition 8.3]). □

The result of the previous theorem cannot be applied directly to the study of stochastic homogenization by means of the subadditive ergodic theorem [1, Theorem 2.7] (see also [13] and [27]) because the term $M(f,\xi,Q)$ is not subadditive; that is, we do not know if

$$M(f,\xi,Q)\le\sum_{i\in I}M(f,\xi,Q_i)\tag{6.7}$$

when $(Q_i)_{i\in I}$ is a finite decomposition of Q into disjoint cubes. We now introduce a variant of $M(f,\xi,Q)$ that satisfies this property. The idea is to relax the constraint $\mathcal{A}u=0$ that was used in the definition of $M(f,\xi,Q)$. We begin with a technical result that will be useful to impose a constraint on the norm $\|\mathcal{A}_Du\|_{W^{-1,p}(D;\mathbb{R}^l)}$ depending additively on D.

Remark 6.3. For every $D \in \mathcal{O}(\mathbb{R}^N)$ and $u \in L^p(D;\mathbb{R}^d)$, there exists $V \in L^p(D;\mathbb{R}^{l\times N})$ such that

$$\langle \widetilde{\mathcal{A}}_D u, \psi\rangle = \int_D V\cdot\nabla\psi\,dx \quad \text{for every } \psi \in W^{1,q}(D;\mathbb{R}^l), \tag{6.8}$$

where $\cdot$ denotes the Euclidean scalar product between matrices. By (2.4), this equality is satisfied, for instance, when for every $x \in D$ and $i = 1,\dots,N$, the i-th column of the matrix $V(x)$ is given by the vector $A^i u(x)$. If (6.8) holds, then

$$\|\widetilde{\mathcal{A}}_D u\|_{\widetilde{W}^{-1,p}(D;\mathbb{R}^l)} \le \|V\|_{L^p(D;\mathbb{R}^{l\times N})}. \tag{6.9}$$

The following lemma provides a partial converse to inequality (6.9), which will be used later in the proof of Proposition 6.8.

Lemma 6.4. *Let $D \subset \mathbb{R}^N$ be a bounded connected open set with Lipschitz boundary, and let $K \subset D$ be a compact set. Then, there exists a constant $C_{K,D} > 0$ such that for every $U \in L^p(D;\mathbb{R}^{l\times N})$, with* $\operatorname{supp} U \subset K$*, there exists $V \in L^p(D;\mathbb{R}^{l\times N})$ satisfying*

$$\int_D V\cdot\nabla\psi\,dx = \int_D U\cdot\nabla\psi\,dx \quad \textit{for every } \psi \in W^{1,q}(D,\mathbb{R}^l), \tag{6.10}$$

$$\|V\|_{L^p(D;\mathbb{R}^{l\times N})} \le C_{K,D}\|\operatorname{div} U\|_{W^{-1,p}(D;\mathbb{R}^l)}. \tag{6.11}$$

Proof. Fix $D \subset \mathbb{R}^N$, $K \subset D$, and $U \in L^p(D;\mathbb{R}^{l\times N})$ as in the statement. Let $W^{1,q}_m(D;\mathbb{R}^l) := \{v \in W^{1,q}(D;\mathbb{R}^l)\colon \int_D v\,dx = 0\}$, with the norm induced by $W^{1,q}(D;\mathbb{R}^l)$, let $\widetilde{W}^{1,p}_m(D;\mathbb{R}^l)$ be the dual space of $W^{1,q}_m(D;\mathbb{R}^l)$, endowed with the dual norm, let $T\colon W^{1,q}_m(D;\mathbb{R}^l) \to \widetilde{W}^{1,p}_m(D;\mathbb{R}^l)$ be the monotone operator defined by

$$\langle T(v),\psi\rangle := \int_D |\nabla v|^{q-2}\nabla v\cdot\nabla\psi\,dx \quad \text{for every } v,\psi \in W^{1,q}_m(D;\mathbb{R}^l),$$

and let $\mathcal{G} \in \widetilde{W}^{1,p}_m(D;\mathbb{R}^l)$ be defined by

$$\langle \mathcal{G},\psi\rangle := \int_D U\cdot\nabla\psi\,dx \quad \text{for every } \psi \in W^{1,q}_m(D;\mathbb{R}^l).$$

By the Hartman–Stampacchia Theorem [22, Lemma 3.1], there exists a unique function $v \in W^{1,q}_m(D;\mathbb{R}^l)$ such that $T(v) = \mathcal{G}$; i.e.,

$$\int_D |\nabla v|^{q-2}\nabla v\cdot\nabla\psi\,dx = \int_D U\cdot\nabla\psi\,dx \quad \text{for every } \psi \in W^{1,q}(D;\mathbb{R}^l). \tag{6.12}$$

Taking $\psi = v$ in (6.12), we obtain

$$\int_D |\nabla v|^q\,dx = \int_D U\cdot\nabla v\,dx. \tag{6.13}$$

Let $\omega \in C^\infty_c(D)$ with $\omega = 1$ on K. Since $\operatorname{supp} U \subset K$, we get from (6.13) that

$$\int_D |\nabla v|^q\,dx = \int_D U\cdot\nabla(\omega v)\,dx = -\langle \operatorname{div} U,\omega v\rangle \le \|\operatorname{div} U\|_{W^{-1,p}(D;\mathbb{R}^l)}\|\omega v\|_{W^{1,q}_0(D;\mathbb{R}^l)}. \tag{6.14}$$

Recalling the definition of $\|\cdot\|_{W^{1,q}_0(D;\mathbb{R}^l)}$ given at the beginning of Section 2, the Poincaré–Wirtinger Inequality yields a constant $C_{D,\omega} > 0$ such that

$$\|\omega v\|_{W^{1,q}_0(D;\mathbb{R}^l)} \le \|\omega\|_{L^\infty(D)}\|\nabla v\|_{L^q(D;\mathbb{R}^{l\times N})} + \|\nabla\omega\|_{L^\infty(D;\mathbb{R}^N)}\|v\|_{L^q(D;\mathbb{R}^l)} \le C_{D,\omega}\|\nabla v\|_{L^q(D;\mathbb{R}^{l\times N})}.$$

From this inequality and from (6.14), we get

$$\|\nabla v\|^q_{L^q(D;\mathbb{R}^{l\times N})} = \int_D |\nabla v|^q dx \le C_{D,\omega}\|\operatorname{div} U\|_{W^{-1,p}(D;\mathbb{R}^l)}\|\nabla v\|_{L^q(D;\mathbb{R}^{l\times N})}.$$

Hence,

$$\|\nabla v\|^{q-1}_{L^q(D;\mathbb{R}^{l\times N})} \le C_{D,\omega}\|\operatorname{div} U\|_{W^{-1,p}(D;\mathbb{R}^l)}. \tag{6.15}$$

Let $V := |\nabla v|^{q-2}\nabla v$. Equality (6.10) follows from (6.12). Since $|V|^p = |\nabla v|^{p(q-1)} = |\nabla v|^q$, we have $\|V\|_{L^p(D;\mathbb{R}^{l\times N})} = \big(\int_D |\nabla v|^q dx\big)^{1/p} = \|\nabla v\|^{q/p}_{L^q(D;\mathbb{R}^{l\times N})} = \|\nabla v\|^{q-1}_{L^q(D;\mathbb{R}^{l\times N})}$. By (6.15), this gives (6.11). ◻

Definition 6.5. *Given $D \in \mathcal{O}(\mathbb{R}^N)$ and $\eta > 0$, we set*

$$\mathcal{V}^\eta(D) := \big\{V \in L^p(D;\mathbb{R}^{l\times N})\colon \|V\|^p_{L^p(D;\mathbb{R}^{l\times N})} < \eta|D|\big\},$$

$$\mathcal{U}^\eta_c(D) := \Big\{u \in L^p(D;\mathbb{R}^d)\colon \operatorname{supp} u \subset\subset D, \int_D u\,dx = 0, \text{ and (6.8) holds for some } V \in \mathcal{V}^\eta(D)\Big\}.$$

For every $f \in \mathcal{F}$ and every $\xi \in \mathbb{R}^d$, we set

$$M^\eta_c(f,\xi,D) := \inf\big\{F(\xi+u,D)\colon u \in \mathcal{U}^\eta_c(D)\big\}, \tag{6.16}$$

where F is defined by (2.14).

We will see in Lemma 8.4 that M^η_c satisfies the subadditivity property mentioned in (6.7).

Remark 6.6. If $R \subset \mathbb{R}^N$ is an open rectangle, $u \in \mathcal{U}^\eta_c(R)$, and $U \in \mathcal{V}^\eta(R)$ satisfies (6.8), we can extend u and U by R-periodicity, which extensions we do not relabel. Then, using the fact that $\operatorname{supp} u \subset\subset R$, we obtain that $\mathcal{A}u = -\operatorname{div} U$ in $\mathbb{R}^N$ in the sense of distributions.

Remark 6.7. For every $D \in \mathcal{O}(\mathbb{R}^N)$ and every $\eta > 0$, we observe that $0 \in \mathcal{U}^\eta_c(D)$. Therefore, we have $M^\eta_c(f,\xi,D) \le F(\xi,D)$ for every $\xi \in \mathbb{R}^d$. By (2.10), this implies

$$M^\eta_c(f,\xi,D) \le c_0(1+|\xi|^p)|D|. \tag{6.17}$$

If $f \in \mathcal{F}_{\mathrm{Lip}}$, it follows immediately from the definition of M^η_c and from (2.12) that for every ξ_1 and $\xi_2 \in \mathbb{R}^d$, we have

$$\frac{M^\eta_c(f,\xi_1,D)}{|D|} \le \frac{M^\eta_c(f,\xi_2,D)}{|D|} + c_1\Big(1+\Big(\frac{M^\eta_c(f,\xi_2,D)}{|D|}\Big)^{\frac{p-1}{p}} + |\xi_1-\xi_2|^{p-1}\Big)|\xi_2-\xi_1|.$$

Exchanging the roles of ξ_1 and ξ_2 and using (6.17), we obtain

$$\Big|\frac{M^\eta_c(f,\xi_1,D)}{|D|} - \frac{M^\eta_c(f,\xi_2,D)}{|D|}\Big| \le c_5\big(1+|\xi_1|+|\xi_2|\big)^{p-1}|\xi_2-\xi_1| \tag{6.18}$$

for a suitable constant c_5 depending only on c_1 and p.

The following result concerns the behavior of $M^\eta_c(f_k,\xi,Q)$ on a cube Q when the functionals corresponding to f_k Γ-converge.

Proposition 6.8. *Let $(f_k)_{k\in\mathbb{N}}$ be a sequence in $\mathcal{F}_{\mathrm{Lip}}$, let $f \in \mathcal{F}$, and let F_k and F be the corresponding functionals in $\mathcal{I}$ defined by (2.14). Assume that for every $D \in \mathcal{O}(\mathbb{R}^N)$, the sequence $(F_k(\cdot,D))_{k\in\mathbb{N}}$ Γ-converges to $F(\cdot,D)$ with respect to the topology induced by $\|\cdot\|^{\mathcal{A}}_D$ on $L^p(D;\mathbb{R}^d)$. Then, for every $\eta > 0$, every cube $Q \subset \mathbb{R}^N$, and every $\xi \in \mathbb{R}^d$, we have*

$$\limsup_{k\to\infty} M^\eta_c(f_k,\xi,Q) \le M_c(f,\xi,Q). \tag{6.19}$$

Moreover, for every $\varepsilon > 0$, there exists $\eta > 0$ such that for every cube $Q \subset \mathbb{R}^N$ with side length less than or equal to 1 and every $\xi \in \mathbb{R}^d$, we have

$$M(f,\xi,Q) \le \liminf_{k\to\infty} M^\eta_c(f_k,\xi,Q) + \varepsilon|Q|. \tag{6.20}$$

Consequently,

$$\sup_{\eta>0}\limsup_{k\to\infty} M^\eta_c(f_k,\xi,Q) \le M_c(f,\xi,Q), \tag{6.21}$$

$$M(f,\xi,Q) \le \sup_{\eta>0}\liminf_{k\to\infty} M^\eta_c(f_k,\xi,Q), \tag{6.22}$$

for every cube $Q \subset \mathbb{R}^N$ with side length less than or equal to 1 and every $\xi \in \mathbb{R}^d$.

Proof. Fix $\xi \in \mathbb{R}^d$ and $\delta > 0$. By (5.1), there exists $u \in L^p(Q;\mathbb{R}^d)$, with $\operatorname{supp} u \subset\subset Q$, $\int_Q u\,dx = 0$, and $\mathcal{A}_Q u = 0$, such that

$$F(\xi+u,Q) < M_c(f,\xi,Q) + \delta.$$

By Γ-convergence, there exists a sequence $(u_k)_{k\in\mathbb{N}}$ in $L^p(Q;\mathbb{R}^d)$ such that $u_k \to u$ in $W^{-1,p}(Q;\mathbb{R}^d)$, $\mathcal{A}_Q u_k \to \mathcal{A}_Q u$ in $W^{-1,p}(Q;\mathbb{R}^l)$, and

$$\lim_{k\to\infty} F_k(\xi+u_k,Q) = F(\xi+u,Q) < M_c(f,\xi,Q)+\delta < +\infty. \tag{6.23}$$

By (2.10), this inequality implies that $(u_k)_{k\in\mathbb{N}}$ is bounded in $L^p(Q;\mathbb{R}^d)$; hence, $u_k \rightharpoonup u$ weakly in $L^p(Q;\mathbb{R}^d)$.

Fix a compact set K, with $\operatorname{supp} u \subset K \subset\subset Q$, such that $c_0(1+|\xi|^p)|Q\setminus K| < \delta$, and let $D_1, D_2 \in \mathcal{O}(\mathbb{R}^N)$ be such that $K \subset D_1 \subset\subset D_2 \subset\subset Q$. We apply Lemma 3.2 with $v_k := u$ for every $k\in\mathbb{N}$ and $B := Q\setminus K$ to obtain a sequence $w_k \in L^p(Q;\mathbb{R}^N)$ such that

$$w_k = u = 0 \ \text{ in } Q\setminus D_2, \quad w_k \rightharpoonup u \quad \text{weakly in } L^p(Q;\mathbb{R}^d), \quad w_k \to u \text{ in } W^{-1,p}(Q;\mathbb{R}^d), \tag{6.24}$$

$$\mathcal{A}_Q w_k \to \mathcal{A}_Q u = 0 \ \text{ in } W^{-1,p}(Q;\mathbb{R}^l),$$

$$\limsup_{k\to\infty} F_k(\xi+w_k,Q) \le \limsup_{k\to\infty} \big(F_k(\xi+u_k,D_2) + F_k(\xi,Q\setminus K)\big), \tag{6.25}$$

where we used in (6.25) the fact that

$$\limsup_{k\to\infty} \big(F_k(\xi+u_k,D_2) + F_k(\xi+u,Q\setminus K)\big) = \limsup_{k\to\infty} \big(F_k(\xi+u_k,D_2) + F_k(\xi,Q\setminus K)\big)$$

because $\operatorname{supp} u \subset K$. By (2.10) and by our choice of K, we have $F_k(\xi,Q\setminus K) \le c_0(1+|\xi|^p)|Q\setminus K| < \delta$, and so (6.25) gives

$$\limsup_{k\to\infty} F_k(\xi+w_k,Q) \le \lim_{k\to\infty} F_k(\xi+u_k,Q)+\delta. \tag{6.26}$$

By (6.24), we have $\int_Q w_k\,dx \to \int_Q u\,dx = 0$. Fix $\varphi \in C_c^\infty(Q)$, with $\int_Q \varphi\,dx = 1$ and $\operatorname{supp}\varphi \subset\subset D_2$, and set $z_k := w_k - \varphi\int_Q w_k\,dx$. By the first formula in (6.24), we have

$$z_k = 0 \ \text{ in } Q\setminus D_2.$$

Moreover, $\mathcal{A}_Q z_k \to 0$ in $W^{-1,p}(Q;\mathbb{R}^l)$, and we have by (2.11) that

$$\limsup_{k\to\infty} F_k(\xi+z_k,Q) = \limsup_{k\to\infty} F_k(\xi+w_k,Q).$$

This inequality, together with (6.23) and (6.26), gives

$$\limsup_{k\to\infty} F_k(\xi+z_k,Q) < M_c(f,\xi,Q)+2\delta. \tag{6.27}$$

Since the supports of the functions z_k are contained in $\overline{D}_2$, recalling (2.5), Remark 6.3 and Lemma 6.4 yield a constant $C = C_{\overline{D}_2,Q} > 0$ such that for every $k\in\mathbb{N}$, there exists $V_k \in L^p(Q;\mathbb{R}^{l\times N})$ satisfying

$$\int_Q V_k\cdot\nabla\psi\,dx = \langle\widetilde{\mathcal{A}}_Q z_k,\psi\rangle \quad \text{for every } \psi\in W^{1,q}(Q;\mathbb{R}^l),$$

$$\|V_k\|_{L^p(Q;\mathbb{R}^{l\times N})} \le C\|\mathcal{A}_Q z_k\|_{W^{-1,p}(Q;\mathbb{R}^l)}. \tag{6.28}$$

Fix $\eta > 0$. Since $\mathcal{A}_Q z_k \to 0$ in $W^{-1,p}(Q;\mathbb{R}^l)$, we obtain from (6.28) that, for k large enough, the functions V_k belong to the set $\mathcal{V}^\eta(Q)$ introduced in Definition 6.5. Moreover, since $\operatorname{supp} z_k \subset\subset Q$ and $\int_Q z_k\,dx = 0$, we have for k large enough that the functions z_k belong to the set $\mathcal{U}_c^\eta(Q)$ introduced in Definition 6.5. By (6.16), this implies that $M_c^\eta(f_k,\xi,Q) \le F_k(\xi+z_k,Q)$. Together with (6.27), this yields

$$\limsup_{k\to\infty} M_c^\eta(f_k,\xi,Q) < M_c(f,\xi,Q)+2\delta.$$

Given the arbitrariness of $\delta > 0$, we obtain (6.19), from which (6.21) follows.

To prove (6.20), we fix $\varepsilon > 0$ and set $C := 2^{p-1}(c_0+1)|\xi|^p + 2^{p-1}c_0(1+2c_0)$. Let $\eta > 0$ be as in Corollary 4.3. Using the definition of M_c^η (see (6.16)) and (6.9), we choose $u_k \in L^p(Q;\mathbb{R}^d)$ for every $k\in\mathbb{N}$, with $\operatorname{supp} u_k \subset\subset Q$, $\int_Q u_k\,dx = 0$, and $\|\widetilde{\mathcal{A}} u_k\|_{\widetilde{W}^{-1,p}(Q;\mathbb{R}^l)} < \eta|Q|$, such that

$$\frac{1}{c_0}\int_Q |\xi+u_k|^p\,dx - c_0|Q| \le F_k(\xi+u_k,Q) < M_c^\eta(f_k,\xi,Q) + \tfrac{1}{k}|Q| \le \Big(c_0(1+|\xi|^p)+1\Big)|Q|, \tag{6.29}$$

where the first and last inequality follow from (a) in Definition 2.9. These inequalities imply that $\|u_k\|^p_{L^p(Q;\mathbb{R}^d)} < C|Q|$ for every $k\in\mathbb{N}$, which allows us to extract a subsequence of $(u_k)_{k\in\mathbb{N}}$, not relabeled, that converges to some function u weakly in $L^p(Q;\mathbb{R}^d)$.

We extend each u_k to a Q periodic function, still denoted u_k. Then, for every $D \in \mathcal{O}(\mathbb{R}^N)$, the sequence $(u_k)_{k\in\mathbb{N}}$ converges weakly in $L^p(D;\mathbb{R}^d)$ to the periodic extension of u, still denoted by u. Since the embedding of $L^p(Q;\mathbb{R}^d)$ into $W^{-1,p}(Q;\mathbb{R}^d)$ is compact, $(u_k)_{k\in\mathbb{N}}$ converges to u in $W^{-1,p}(Q;\mathbb{R}^l)$; hence, $\|u_k - u\|_{W^{-1,p}(Q;\mathbb{R}^d)} \le \varepsilon|Q|$ for k large enough.

Therefore, by Corollary 4.3, there exists $v_k \in L^p_{\rm per}(Q;\mathbb{R}^d)$, with $\|v_k - u_k\|^p_{W^{-1,p}(Q;\mathbb{R}^d)} < \varepsilon|Q|$, $\mathcal{A}v_k = 0$ in $\mathbb{R}^N$, and $\int_Q v_k\,dx = \int_Q u_k\,dx = 0$, such that

$$F_k(\xi + v_k, Q) < F_k(\xi + u_k, Q) + \varepsilon|Q|. \tag{6.30}$$

Since the right-hand side of (6.29) is bounded, the previous inequality and (2.10) imply that $(v_k)_{k\in\mathbb{N}}$ is bounded in $L^p(Q;\mathbb{R}^d)$; hence, a subsequence of $(v_k)_{k\in\mathbb{N}}$, not relabeled, converges to some function v weakly in $L^p(Q;\mathbb{R}^d)$.

By periodicity, we have for every $D \in \mathcal{O}(\mathbb{R}^N)$ that the sequence $(v_k)_{k\in\mathbb{N}}$ converges weakly in $L^p(D;\mathbb{R}^d)$ to the periodic extension of v, still denoted by v. Since the embedding of $L^p(D;\mathbb{R}^d)$ into $W^{-1,p}(D;\mathbb{R}^d)$ is compact and $\mathcal{A}v_k = 0$ in $\mathbb{R}^N$ for every k, we deduce that $\mathcal{A}v = 0$ in $\mathbb{R}^N$ and that $(v_k)_{k\in\mathbb{N}}$ converges to v in the topology induced by $\|\cdot\|^{\mathcal{A}}_D$ on $L^p(D;\mathbb{R}^d)$.

Moreover, since $\int_Q v_k\,dx = 0$ for every k, we have also $\int_Q v\,dx = 0$. Therefore, by (5.1) and by Γ-convergence, we have

$$M(f,\xi,Q) \le F(\xi + v, Q) \le \liminf_{k\to\infty} F_k(\xi + v_k, Q).$$

Together with (6.29) and (6.30), the preceding estimate gives (6.20). Since $\varepsilon > 0$ is arbitrary, we obtain (6.22) from (6.20). □

By analogy with Theorem 6.2, we are now ready to present the characterization of the Γ-convergence of the functionals associated with f_k by means of the behavior of $M^\eta_c(f_k,\xi,Q)$ on small cubes Q.

Theorem 6.9. *Let $(f_k)_{k\in\mathbb{N}}$ be a sequence in $\mathcal{F}_{\rm Lip}$, let $(F_k)_{k\in\mathbb{N}}$ be the corresponding sequence of functionals in $\mathcal{I}$ defined by (2.14), and let $(F^{\mathcal{A}}_k)_{k\in\mathbb{N}}$ be the sequence obtained as in Definition 4.4. Suppose that*

(a) *there exists $f\colon \mathbb{R}^N \times \mathbb{R}^d \to [0,+\infty)$ such that*

$$f(x,\xi) = \limsup_{\rho\to 0^+} \sup_{\eta>0} \liminf_{k\to\infty} \frac{M^\eta_c(f_k,\xi,Q_\rho(x))}{\rho^N} = \limsup_{\rho\to 0^+} \sup_{\eta>0} \limsup_{k\to\infty} \frac{M^\eta_c(f_k,\xi,Q_\rho(x))}{\rho^N} \tag{6.31}$$

for a.e. $x \in \mathbb{R}^N$ and every $\xi \in \mathbb{R}^d$.

Then, there exists $\hat f \in \mathcal{F}_{\rm Lip} \cap \mathcal{F}_{\rm qc}$ such that $f(x,\xi) = \hat f(x,\xi)$ for a.e. $x \in \mathbb{R}^N$ and every $\xi \in \mathbb{R}^d$, and the functionals F and $F^{\mathcal{A}}$ introduced in (2.14) and Definition 4.4 satisfy the following properties:

(b) *for every $D \in \mathcal{O}(\mathbb{R}^N)$, the sequence $(F_k(\cdot,D))_{k\in\mathbb{N}}$ Γ-converges to $F(\cdot,D)$ with respect to the topology induced by $\|\cdot\|^{\mathcal{A}}_D$ on $L^p(D;\mathbb{R}^d)$;*
(c) *for every $D \in \mathcal{O}(\mathbb{R}^N)$, the sequence $(F^{\mathcal{A}}_k(\cdot,D))_{k\in\mathbb{N}}$ Γ-converges to $F^{\mathcal{A}}(\cdot,D)$ in $\ker\mathcal{A}_D$ with respect to the weak topology of $L^p(D;\mathbb{R}^d)$.*

Conversely, if $f \in \mathcal{F}$ and the functionals F and $F^{\mathcal{A}}$ introduced in (2.14) and Definition 4.4 satisfy (b) or (c), then f satisfies (a).

Proof. Since $F_k \in \mathcal{F}_{\rm Lip}$ for every $k \in \mathbb{N}$ by Remark 2.11, the equivalence between (b) and (c) is proved in Theorem 4.6. If these conditions are satisfied, then by Corollary 3.4 there exists $\hat f \in \mathcal{F}_{\rm Lip} \cap \mathcal{F}_{\rm qc}$ such that

$$F(u,D) := \int_D \hat f(x,u(x))\,dx, \quad \text{for every } D \in \mathcal{O}(\mathbb{R}^N) \text{ and } u \in L^p(D;\mathbb{R}^d),$$

which implies that $f(x,\xi) = \hat f(x,\xi)$ for a.e. $x \in \mathbb{R}^N$ and every $\xi \in \mathbb{R}^d$.

Assume (b). Fix $x \in \mathbb{R}^N$ and $\xi \in \mathbb{R}^d$. By Proposition 6.8, we have for every $0 < \rho \le 1$ that

$$\sup_{\eta>0} \limsup_{k\to\infty} M^\eta_c(f_k,\xi,Q_\rho(x)) \le M_c(f,\xi,Q_\rho(x)), \tag{6.32}$$

$$M(f,\xi,Q_\rho(x)) \le \sup_{\eta>0} \liminf_{k\to\infty} M^\eta_c(f_k,\xi,Q_\rho(x)). \tag{6.33}$$

Using Theorem 5.3, (6.32), and (6.33), we conclude that

$$\hat{f}(x,\xi) = \lim_{\rho\to 0^+} \frac{M(\hat{f},\xi,Q_\rho(x))}{\rho^N} = \lim_{\rho\to 0^+} \frac{M(f,\xi,Q_\rho(x))}{\rho^N} \le \limsup_{\rho\to 0^+} \sup_{\eta>0} \liminf_{k\to\infty} \frac{M_c^\eta(f_k,\xi,Q_\rho(x))}{\rho^N}$$

$$\le \limsup_{\rho\to 0^+} \sup_{\eta>0} \limsup_{k\to\infty} \frac{M_c^\eta(f_k,\xi,Q_\rho(x))}{\rho^N} \le \limsup_{\rho\to 0^+} \frac{M_c(f,\xi,Q_\rho(x))}{\rho^N} = \limsup_{\rho\to 0^+} \frac{M_c(\hat{f},\xi,Q_\rho(x))}{\rho^N} = \hat{f}(x,\xi).$$

Thus, (6.31) holds for a.e. $x \in \mathbb{R}^N$ and every $\xi \in \mathbb{R}^d$, concluding the proof of (a).

Assume (a). By Corollary 3.4, there exists a subsequence $(f_{k_j})_{j\in\mathbb{N}}$ and a function $\hat{f} \in \mathcal{F}_{\rm Lip} \cap \mathcal{F}_{\rm qc}$ such that for every $D \in \mathcal{O}(\mathbb{R}^N)$, the sequence $(F_{k_j}(\cdot,D))_{j\in\mathbb{N}}$ Γ-converges to $\widehat{F}(\cdot,D)$ with respect to the topology induced by $\|\cdot\|_D^{\mathcal{A}}$ on $L^p(D;\mathbb{R}^d)$, where $\widehat{F}$ is the functional associated with $\hat{f}$ by (2.14). Since (b)⇒(a), as proved above, we have for a.e. $x \in \mathbb{R}^N$ and every $\xi \in \mathbb{R}^d$ that

$$\hat{f}(x,\xi) = \limsup_{\rho\to 0^+} \sup_{\eta>0} \liminf_{j\to\infty} \frac{M_c^\eta(f_{k_j},\xi,Q_\rho(x))}{\rho^N} = \limsup_{\rho\to 0^+} \sup_{\eta>0} \limsup_{j\to\infty} \frac{M_c^\eta(f_{k_j},\xi,Q_\rho(x))}{\rho^N}.$$

By (6.31), this implies that $\hat{f}(x,\xi) = f(x,\xi)$ for a.e. $x \in \mathbb{R}^N$ and every $\xi \in \mathbb{R}^d$; hence, $f \in \mathcal{F}_{\rm qc}$ and $\widehat{F} = F$. Since the Γ-limit does not depend on the subsequence, we obtain (b) from the Urysohn property of Γ-convergence (see [12, Proposition 8.3]). □

7. Homogenization without periodicity assumptions

Throughout this section, we fix a function $f \in \mathcal{F}_{\rm Lip}$. We observe that if the vector space span(Λ) generated by the wave cone Λ coincides with $\mathbb{R}^d$ then every $f \in \mathcal{F}_{\rm qc}$ belongs to $\mathcal{F}_{\rm Lip}$ by Remark 2.8. For every $\varepsilon > 0$, we consider the functions $f_\varepsilon \in \mathcal{F}$ defined by

$$f_\varepsilon(x,\xi) := f(\tfrac{x}{\varepsilon},\xi)$$

for every $x \in \mathbb{R}^N$ and $\xi \in \mathbb{R}^d$, the functionals $F_\varepsilon \in \mathcal{I}$ associated with f_ε by (2.14), and the corresponding $\mathcal{A}$-free functionals $F_\varepsilon^{\mathcal{A}}$ introduced in Definition 4.4. Note that $f_\varepsilon \in \mathcal{F}_{\rm Lip}$ for every $\varepsilon > 0$.

The following theorem provides very general conditions on the function f which ensure that there exists a function $f_{\rm hom} \in \mathcal{F}$, independent of x, such that for every $D \in \mathcal{O}(\mathbb{R}^N)$, the family of functionals $(F_\varepsilon(\cdot,D))_{\varepsilon>0}$ Γ-converges as $\varepsilon \to 0^+$ to the functional $F_{\rm hom}(\cdot,D)$ corresponding to $f_{\rm hom}$. By this we mean that, for every sequence $(\varepsilon_k)_{k\in\mathbb{N}}$ of positive numbers converging to 0, the sequence $(F_{\varepsilon_k}(\cdot,D))_{k\in\mathbb{N}}$ Γ-converges to $F_{\rm hom}(\cdot,D)$.

Theorem 7.1. *Suppose that for every $x \in \mathbb{R}^N$, $\xi \in \mathbb{Q}^d$, and $k \in \mathbb{N}$, the limit*

$$f^k_{\rm hom}(\xi) := \lim_{r\to+\infty} \frac{M_c^{1/k}(f,\xi,Q_r(rx))}{r^N} \tag{7.1}$$

exists and is independent of x (see Definition 6.5). Then, $f^k_{\rm hom}$ can be extended as a continuous function on $\mathbb{R}^d$, which we still denote by $f^k_{\rm hom}$, and (7.1) holds for every $\xi \in \mathbb{R}^N$. Let $f_{\rm hom}\colon \mathbb{R}^d \to [0,+\infty)$ be the function defined by

$$f_{\rm hom}(\xi) := \sup_{k\in\mathbb{N}} f^k_{\rm hom}(\xi) = \lim_{k\to\infty} f^k_{\rm hom}(\xi) \tag{7.2}$$

for every $\xi \in \mathbb{R}^d$. Then, $f_{\rm hom} \in \mathcal{F}_{\rm Lip} \cap \mathcal{F}_{\rm qc}$ and the following properties hold:

(a) *for every $D \in \mathcal{O}(\mathbb{R}^N)$, the family $(F_\varepsilon(\cdot,D))_{\varepsilon>0}$ Γ-converges as $\varepsilon \to 0^+$ to $F_{\rm hom}(\cdot,D)$ with respect to the topology induced by $\|\cdot\|_D^{\mathcal{A}}$ on $L^p(D;\mathbb{R}^d)$;*

(b) *for every $D \in \mathcal{O}(\mathbb{R}^N)$, the family $(F_\varepsilon^{\mathcal{A}}(\cdot,D))_{\varepsilon>0}$ Γ-converges as $\varepsilon \to 0^+$ to $F^{\mathcal{A}}_{\rm hom}(\cdot,D)$ in $\ker \mathcal{A}_D$ with respect to the weak topology of $L^p(D;\mathbb{R}^d)$.*

We will see in Section 8 that (7.1) is satisfied almost surely under the standard hypotheses of stochastic homogenization. In particular, it is satisfied when $x \mapsto f(x,\xi)$ is $Q_1(0)$-periodic for every $\xi \in \mathbb{R}^d$. The following proposition examines another simple case in which (7.1) holds: f is the sum of a periodic function with respect to x and a function whose support has compact projection onto $\mathbb{R}^N$.

Proposition 7.2. *Assume that f can be written as*

$$f = f_{per} + f_{comp}, \tag{7.3}$$

where f_{per}, $f_{comp} \in \mathcal{F}_{\rm Lip}$ satisfy the two following properties:

(a) $x \mapsto f_{per}(x,\xi)$ *is* $Q_1(0)$*-periodic for every* $\xi \in \mathbb{R}^d$; (7.4)

(b) *there exists* $R > 0$ *such that* $f_{comp}(x,\xi) = 0$ *for every* $x \in \mathbb{R}^N \setminus Q_R(0)$ *and every* $\xi \in \mathbb{R}^d$. (7.5)

Then, for every $x \in \mathbb{R}^N$, $\xi \in \mathbb{Q}^d$, *and* $k \in \mathbb{N}$, *the limit*

$$f^k_{\rm hom}(\xi) := \lim_{r\to+\infty} \frac{M_c^{1/k}(f,\xi,Q_r(rx))}{r^N} \tag{7.6}$$

exists and is independent of x.

Proof. By (7.4), we can apply Theorem 8.5 to f_{per}, considered as a stochastically periodic random integrand (independent of ω), and we obtain for every $k \in \mathbb{N}$, $x \in \mathbb{R}^N$, and $\xi \in \mathbb{Q}^d$ that the limit

$$f^k_{\rm hom}(\xi) := \lim_{r\to+\infty} \frac{M_c^{1/k}(f_{\rm per},\xi,Q_r(rx))}{r^N} \tag{7.7}$$

exists and is independent of x. We remark that since probability is not involved here, this result can be obtained directly by adapting some arguments of [7].

We claim that

$$f^k_{\rm hom}(\xi) = \lim_{r\to+\infty} \frac{M_c^{1/k}(f,\xi,Q_r(rx))}{r^N} \tag{7.8}$$

for every $k \in \mathbb{N}$, $x \in \mathbb{R}^N$, and $\xi \in \mathbb{Q}^d$. Since $f_{comp} \in \mathcal{F}$, we have $0 \le f_{comp}$ by Definition 2.5. Hence, $f_{per} \le f$ by (7.3). This implies that $M_c^{1/k}(f_{per},\xi,Q_r(rx)) \le M_c^{1/k}(f,\xi,Q_r(rx))$, which, together with (7.7), yields

$$f^k_{\rm hom}(\xi) \le \liminf_{r\to+\infty} \frac{M_c^{1/k}(f,\xi,Q_r(rx))}{r^N}. \tag{7.9}$$

In order to prove that

$$\limsup_{r\to+\infty} \frac{M_c^{1/k}(f,\xi,Q_r(rx))}{r^N} \le f^k_{\rm hom}(\xi), \tag{7.10}$$

we fix $k \in \mathbb{N}$, $x = (x_1,\dots,x_N) \in \mathbb{R}^N$, and $\xi \in \mathbb{Q}^d$. Similarly to the proof of Corollary 4.3, given $m \in \mathbb{N}$, we set $A_m := \{1,\dots,m\}^N$ and for every $\alpha = (\alpha_1,\dots,\alpha_N) \in A_m$, we consider the point

$$x(\alpha) := (x_1,\dots,x_N) + \Big(-\frac12 - \frac{1}{2m} + \frac{\alpha_1}{m},\dots,-\frac12 - \frac{1}{2m} + \frac{\alpha_N}{m}\Big). \tag{7.11}$$

We observe that

$$\overline{Q_1(x)} = \bigcup_{\alpha\in A_m} \overline{Q_{1/m}(x(\alpha))} \quad \text{and} \quad Q_{1/m}(x(\alpha)) \cap Q_{1/m}(x(\beta)) = \emptyset \text{ for } \alpha \ne \beta.$$

Hence, for every $r > 0$,

$$\overline{Q_r(rx)} = \bigcup_{\alpha\in A_m} \overline{Q_{r/m}(rx(\alpha))} \quad \text{and} \quad Q_{r/m}(rx(\alpha)) \cap Q_{r/m}(r(x(\beta)) = \emptyset \text{ for } \alpha \ne \beta. \tag{7.12}$$

Given $r \ge mR$, we set

$$A_m^{r,R} := \big\{\alpha \in A_m \colon Q_{r/m}(rx(\alpha)) \cap Q_R(0) \ne \emptyset\big\}. \tag{7.13}$$

From (7.11), we deduce that $\alpha \in A_m^{r,R}$ if and only if for every $j = 1,\dots,N$, we have

$$\Big(rx_j - \frac{r}{2} - \frac{r}{m} + \frac{r\alpha_j}{m}, rx_j - \frac{r}{2} + \frac{r\alpha_j}{m}\Big) \cap \Big(-\frac{R}{2},\frac{R}{2}\Big) \ne \emptyset. \tag{7.14}$$

Since the intervals depending on α_j in the previous formula are pairwise disjoint and their length is $\frac{r}{m} \ge R$, for every $j = 1,\dots,N$, there are at most two elements $\alpha_j \in \{1,\dots,m\}$ such that (7.14) holds. This implies that the number $\#A_m^{r,R}$ of elements of $A_m^{r,R}$ satisfies

$$\#A_m^{r,R} \le 2^N. \tag{7.15}$$

Given $\delta > 0$, we use Definition 6.5 to find for every $\alpha \in A_m$ a function $u_\alpha \in \mathcal{U}_c^{1/k}(Q_{r/m}(rx(\alpha)))$ such that

$$\int_{Q_{r/m}(rx(\alpha))} f_{per}(y,\xi + u_\alpha(y))\,dy < M_c^{1/k}(f_{\rm per},\xi,Q_{r/m}(rx(\alpha))) + \delta\frac{r^N}{m^N}. \tag{7.16}$$

By (7.12), we can define $u\colon Q_r(rx) \to \mathbb{R}^d$ by setting $u(y) := 0$ for every $\alpha \in A_m^{r,R}$ and $u(y) := u_\alpha(y)$ for every $\alpha \in A_m \setminus A_m^{r,R}$. Recalling Definition 6.5 and (7.12), we see that $u \in \mathcal{U}_c^{1/k}(Q_r(rx))$, and so

$$M_c^{1/k}(f, \xi, Q_r(rx)) \leq \int_{Q_r(rx)} f(y, \xi + u(y))\, dy.$$

By the definition of u and by (2.10), (7.3), (7.5), (7.13), (7.15), and (7.16), we have

$$\begin{aligned} M_c^{1/k}(f, \xi, Q_r(rx)) &\leq \sum_{\alpha \in A_m^{r,R}} \int_{Q_{r/m}(rx(\alpha))} f(y, \xi)\, dy + \sum_{\alpha \in A_m \setminus A_m^{r,R}} \int_{Q_{r/m}(rx(\alpha))} f_{per}(y, \xi + u_\alpha(y))\, dy \\ &\leq 2^N c_0 (1 + |\xi|^p) \frac{r^N}{m^N} + \delta r^N + \sum_{\alpha \in A_m} M_c^{1/k}(f_{\mathrm{per}}, \xi, Q_{r/m}(rx(\alpha))). \end{aligned} \tag{7.17}$$

Since $Q_{r/m}(rx(\alpha)) = Q_{r/m}((r/m)(mx(\alpha))$, the equality

$$M_c^{1/k}(f_{\mathrm{per}}, \xi, Q_{r/m}(rx(\alpha))) = M_c^{1/k}(f_{\mathrm{per}}, \xi, Q_{r/m}((r/m)(mx(\alpha))),$$

together with (7.7), leads to

$$\lim_{r \to +\infty} \frac{M_c^{1/k}(f_{\mathrm{per}}, \xi, Q_{r/m}(rx(\alpha)))}{r^N} = \frac{1}{m^N} f_{hom}^{1/k}(\xi).$$

From this equality and from (7.17), we get

$$\limsup_{r \to +\infty} \frac{M_c^{1/k}(f, \xi, Q_r(rx)}{r^N} \leq c_0 (1 + |\xi|^p) \frac{2^N}{m^N} + \delta + f_{hom}^{1/k}(\xi).$$

Taking the limit as $m \to +\infty$ and $\delta \to 0$, we obtain (7.10), which, together with (7.9), yields (7.8), concluding the proof of (7.6). □

Proof of Theorem 7.1. By (6.18), we have for every ξ_1, $\xi_2 \in \mathbb{R}^d$ and every $k \in \mathbb{N}$ that

$$\Big| \frac{M_c^{1/k}(f, \xi_1, Q_r(rx))}{r^N} - \frac{M_c^{1/k}(f, \xi_2, Q_r(rx))}{r^N} \Big| \leq c_5 \big(1 + |\xi_1| + |\xi_2|\big)^{p-1} |\xi_2 - \xi_1|.$$

Hence,

$$|f_{\mathrm{hom}}^k(\xi_1) - f_{\mathrm{hom}}^k(\xi_2)| \leq c_5 \big(1 + |\xi_1| + |\xi_2|\big)^{p-1} |\xi_2 - \xi_1|$$

for every ξ_1 and $\xi_2 \in \mathbb{Q}^d$. This implies that f_{hom}^k can be extended as a continuous function on $\mathbb{R}^d$, which we still denote by f_{hom}^k, and that (7.1) holds for every $\xi \in \mathbb{R}^N$.

As we show next, by Definition 6.5 and a change of variables, we have for every $x \in \mathbb{R}^N$, $\xi \in \mathbb{R}^d$, $\rho > 0$, and $k \in \mathbb{N}$, that

$$M_c^{1/k}\big(f_\varepsilon, \xi, Q_\rho(x)\big) = \varepsilon^N M_c^{1/k}\big(f, \xi, Q_{\rho/\varepsilon}(\tfrac{x}{\varepsilon})\big) = \rho^N \Big(\frac{\varepsilon}{\rho}\Big)^N M_c^{1/k}\big(f, \xi, Q_{\rho/\varepsilon}\big(\tfrac{\rho}{\varepsilon}(\tfrac{x}{\rho})\big)\big). \tag{7.18}$$

In fact, let $u_\varepsilon \in \mathcal{U}_c^{1/k}(Q_{\rho/\varepsilon}(\frac{x}{\varepsilon}))$ and $V_\varepsilon \in \mathcal{V}^{1/k}(Q_{\rho/\varepsilon}(\frac{x}{\varepsilon}))$ be such that

$$\begin{aligned} &\int_{Q_{\rho/\varepsilon}(\frac{x}{\varepsilon})} |V_\varepsilon(y)|^p\, dy \leq \tfrac{1}{k} |Q_{\rho/\varepsilon}(\tfrac{x}{\varepsilon})| = \tfrac{1}{k} (\tfrac{\rho}{\varepsilon})^N, \\ &-\sum_{i=1}^N \int_{Q_{\rho/\varepsilon}(\frac{x}{\varepsilon})} A^i u_\varepsilon(y) \cdot \partial_i \psi(y)\, dy = \int_{Q_{\rho/\varepsilon}(\frac{x}{\varepsilon})} V_\varepsilon(y) \cdot \nabla \psi(y)\, dy \quad \text{for every } \psi \in W^{1,q}(Q_{\rho/\varepsilon}(\tfrac{x}{\varepsilon}); \mathbb{R}^l). \end{aligned}$$

Define

$$w_\varepsilon(z) := u_\varepsilon(\tfrac{z}{\varepsilon}) \quad \text{and} \quad W_\varepsilon(z) := V_\varepsilon(\tfrac{z}{\varepsilon}) \quad \text{for } z \in Q_\rho(x).$$

Then, using a change of variables and the definition of $\mathcal{U}_c^{1/k}(\cdot)$ and $\mathcal{V}^{1/k}(\cdot)$, it can be checked that $w_\varepsilon \in L^p(Q_\rho(x);\mathbb{R}^d)$, $\operatorname{supp} w_\varepsilon \subset\subset Q_\rho(x)$, and $W_\varepsilon \in L^p(Q_\rho(x);\mathbb{R}^{l\times N})$. Moreover,

$$\int_{Q_\rho(x)} w_\varepsilon(z)\,dz = \int_{Q_\rho(x)} u_\varepsilon(\tfrac{z}{\varepsilon})\,dz = \varepsilon^N \int_{Q_{\rho/\varepsilon}(\frac{x}{\varepsilon})} u_\varepsilon(y)\,dy = 0,$$
$$\int_{Q_\rho(x)} |W_\varepsilon(z)|^p\,dz = \int_{Q_\rho(x)} |V_\varepsilon(\tfrac{z}{\varepsilon})|^p\,dz = \varepsilon^N \int_{Q_{\rho/\varepsilon}(\frac{x}{\varepsilon})} |V_\varepsilon(y)|^p\;dy \le \tfrac{1}{k}\rho^N = \tfrac{1}{k}|Q_\rho(x)|,$$
$$\int_{Q_\rho(x)} f(\tfrac{z}{\varepsilon},\xi + w_\varepsilon(z))\,dz = \int_{Q_\rho(x)} f(\tfrac{z}{\varepsilon},\xi + u_\varepsilon(\tfrac{z}{\varepsilon}))\,dz = \varepsilon^N \int_{Q_{\rho/\varepsilon}(\frac{x}{\varepsilon})} f(y,\xi + u_\varepsilon(y)\,dy.$$

Furthermore, given $\theta \in W^{1,q}(Q_\rho(x);\mathbb{R}^l)$, we define $\psi \in W^{1,q}(Q_{\rho/\varepsilon}(\frac{x}{\varepsilon});\mathbb{R}^l)$ by setting $\psi(y) := \frac{1}{\varepsilon}\theta(\varepsilon y)$ for $y \in Q_{\rho/\varepsilon}(\frac{x}{\varepsilon})$, and observe that

$$\begin{aligned}
-\sum_{i=1}^N \int_{Q_\rho(x)} A^i w_\varepsilon(z)\cdot\partial_i\theta(z)\,dz &= -\sum_{i=1}^N \int_{Q_\rho(x)} A^i u_\varepsilon(\tfrac{z}{\varepsilon})\cdot\partial_i\theta(z)\,dz = -\varepsilon^N \sum_{i=1}^N \int_{Q_{\rho/\varepsilon}(\frac{x}{\varepsilon})} A^i u_\varepsilon(y)\cdot\partial_i\theta(\varepsilon y)\,dz\\
&= -\varepsilon^N \sum_{i=1}^N \int_{Q_{\rho/\varepsilon}(\frac{x}{\varepsilon})} A^i u_\varepsilon(y)\cdot\partial_i\psi(y) = \varepsilon^N \int_{Q_{\rho/\varepsilon}(\frac{x}{\varepsilon})} V_\varepsilon(y)\cdot\nabla\psi(y)\,dy\\
&= \int_{Q_\rho(x)} V_\varepsilon(\tfrac{z}{\varepsilon})\cdot\nabla\theta(z)\,dz = \int_{Q_\rho(x)} W_\varepsilon(z)\cdot\nabla\theta(z)\,dz.
\end{aligned}$$

Hence, recalling Definition 6.5, we conclude that $M_c^{1/k}\big(f_\varepsilon,\xi,Q_\rho(x)\big) \le \varepsilon^N M_c^{1/k}\big(f,\xi,Q_{\rho/\varepsilon}(\frac{x}{\varepsilon})\big)$. The converse inequality can be proved similarly, from which (7.18) follows.

Combining (7.1) and (7.18), recalling that the limit in the former is assumed to exist and to be independent of x, we get for every $\xi \in \mathbb{R}^d$ and $\rho > 0$ that

$$f^k_{\mathrm{hom}}(\xi) = \lim_{\varepsilon\to0^+} \frac{M_c^{1/k}\big(f,\xi,Q_{\rho/\varepsilon}(\frac{\rho}{\varepsilon}(\frac{x}{\rho}))\big)}{(\rho/\varepsilon)^N} = \lim_{\varepsilon\to0^+} \frac{M_c^{1/k}\big(f_\varepsilon,\xi,Q_\rho(x)\big)}{\rho^N}.$$

Thus, (7.2) gives for every $\rho > 0$ that

$$f_{\mathrm{hom}}(\xi) = \sup_{k\in\mathbb{N}} f^k_{\mathrm{hom}}(\xi) = \sup_{k\in\mathbb{N}}\lim_{\varepsilon\to0^+} \frac{M_c^{1/k}\big(f_\varepsilon,\xi,Q_\rho(x)\big)}{\rho^N} = \sup_{\eta>0}\lim_{\varepsilon\to0^+} \frac{M_c^{\eta}\big(f_\varepsilon,\xi,Q_\rho(x)\big)}{\rho^N},$$

where in the last equality we used the monotonicity of M_c^η with respect to η. Then, by Theorem 6.9, the function f_{hom} belongs to $\mathcal{F}_{\mathrm{Lip}} \cap \mathcal{F}_{\mathrm{qc}}$ and (a) and (b) are satisfied. □

8. Stochastic homogenization

In this section, we study stochastic homogenization problems in the $\mathcal{A}$-free setting. To this aim, we fix a probability space $(\Omega,\mathcal{T},P)$ and a group $(\tau_z)_{z\in\mathbb{Z}^N}$ of P-preserving transformations on $(\Omega,\mathcal{T},P)$, i.e., a family $(\tau_z)_{z\in\mathbb{Z}^N}$ of $\mathcal{T}$-measurable bijective maps $\tau_z\colon \Omega\to\Omega$, with $P(\tau_z^{-1}(E)) = P(E)$ for every $E\in\mathcal{T}$ and every $z\in\mathbb{Z}^N$, satisfying the group property: $\tau_0 = \mathrm{id}_\Omega$ (the identity map on Ω) and $\tau_{z+z'} = \tau_z\circ\tau_{z'}$ for every $z,z'\in\mathbb{Z}^N$. We recall that the group is called ergodic if every set $E\in\mathcal{T}$ with $\tau_z(E) = E$ for every $z\in\mathbb{Z}^N$ has probability 0 or 1.

We introduce now the classes of random integrands that we are going to consider.

Definition 8.1 (Stochastically periodic random integrands). *Let $\mathcal{F}^{\mathrm{st}}_{\mathrm{Lip}}$ be the collection of functions $f\colon\Omega\times\mathbb{R}^N\times\mathbb{R}^d\to[0,+\infty)$ satisfying the following properties:*

(a) *f is $\mathcal{T}\times\mathcal{L}\times\mathcal{B}$-measurable, where $\mathcal{L}$ is the σ-algebra of Lebesgue measurable subsets of $\mathbb{R}^N$ and $\mathcal{B}$ is the Borel σ-algebra of $\mathbb{R}^d$;*
(b) *setting $f(\omega) := f(\omega,\cdot,\cdot)$, we have $f(\omega)\in\mathcal{F}_{\mathrm{Lip}}$ for every $\omega\in\Omega$;*
(c) *f is stochastically periodic with respect to $(\tau_z)_{z\in\mathbb{Z}^N}$, i.e.,*

$$f(\omega,x+z,\xi) = f(\tau_z(\omega),x,\xi)$$

for every $\omega\in\Omega$, $x\in\mathbb{R}^N$, $z\in\mathbb{Z}^N$, and $\xi\in\mathbb{R}^d$.

Finally, let $\mathcal{F}^{\rm st}_{\rm qc}$ be the collection of all functions $f \in \mathcal{F}^{\rm st}$ such that $f(\omega) \in \mathcal{F}_{\rm qc}$ for P-a.e. $\omega \in \Omega$.

We recall that if the vector space $\mathrm{span}(\Lambda)$ *generated by the wave cone Λ coincides with $\mathbb{R}^d$ then $\mathcal{F}^{\rm st}_{\rm qc} \subset \mathcal{F}^{\rm st}_{\rm Lip}$ by Remark 2.8 .*

We now introduce the notion of subadditive process. Let $\mathcal{R}$ be the collection of all rectangles of the form

$$[a, b) := \{x \in \mathbb{R}^N : a_i \leq x_i < b_i \text{ for } i = 1, \cdots, d\} \quad \text{with } a, b \in \mathbb{R}^N.$$

Definition 8.2 (Covariant subadditive process). *A covariant subadditive process with respect to $(\tau_z)_{z\in\mathbb{Z}^N}$ is a function $\Phi\colon \Omega \times \mathcal{R} \to [0, +\infty)$ with the following properties:*

(a) *(measurability) for every $R \in \mathcal{R}$, the function $\omega \mapsto \Phi(\omega, R)$ is $\mathcal{T}$-measurable on Ω;*
(b) *(covariance) $\Phi(\omega, R + z) = \Phi(\tau_z(\omega), R)$ for every $\omega \in \Omega$, $R \in \mathcal{R}$, and $z \in \mathbb{Z}^N$;*
(c) *(subadditivity) if $R \in \mathcal{R}$ and $(R_i)_{i\in I} \subset \mathcal{R}$ is a finite partition of R, then*

$$\Phi(\omega, R) \leq \sum_{i\in I} \Phi(\omega, R_i) \quad \textit{for every } \omega \in \Omega\,;$$

(d) *(boundedness) there exists $c > 0$ such that $0 \leq \Phi(\omega, R) \leq c|R|$ for every $\omega \in \Omega$ and $R \in \mathcal{R}$.*

We will use the following variant of the Subadditive Ergodic Theorem [1, Theorem 2.7], see also [13] and [27].

Theorem 8.3. *Let Φ be a covariant subadditive process with respect to $(\tau_z)_{z\in\mathbb{Z}^N}$. Then, there exist a $\mathcal{T}$-measurable function $\varphi\colon \Omega \to [0, +\infty)$ and a set $\Omega' \in \mathcal{T}$, with $P(\Omega') = 1$, such that*

$$\lim_{r\to+\infty} \frac{\Phi(\omega, Q(rx, r))}{r^N} = \varphi(\omega)$$

for every $x \in \mathbb{R}^N$ and every $\omega \in \Omega'$. If, in addition, $(\tau_z)_{z\in\mathbb{Z}^N}$ is ergodic, then φ is constant P-a.e. in Ω.

Lemma 8.4. *Let $f \in \mathcal{F}^{\rm st}$, $\xi \in \mathbb{R}^d$, and $\eta > 0$. Then, the function $\Phi^\eta_\xi\colon \Omega \times \mathcal{R} \to [0, +\infty)$ defined by*

$$\Phi^\eta_\xi(\omega, R) := M^\eta_c(f(\omega), \xi, \mathring{R})$$

is a covariant subadditive process.

Proof. Using (2.11) and Remark 6.3, we see that the infimum in (6.16) defining $M^\eta_c(f(\omega), \xi, \mathring{R})$ can be obtained using a suitable dense sequence of functions u. Since for every $u \in L^p(R; \mathbb{R}^d)$, the function $\omega \mapsto \int_R f(\omega, x, \xi + u(x))\,dx$ is $\mathcal{T}$-measurable by (a) of Definition 8.1, we conclude that $\omega \mapsto \Phi^\eta_\xi(\omega, R)$ is $\mathcal{T}$-measurable. The boundedness and the covariance property are clear.

We now prove subadditivity. Fix $\omega \in \Omega$, $R \in \mathcal{R}$, a finite partition $(R_i)_{i\in I}$ of R, and $\delta > 0$. For every $i \in I$, there exist $u_i \in L^p(R_i; \mathbb{R}^d)$ and $V_i \in L^p(R_i; \mathbb{R}^{l\times N})$ such that $\mathrm{supp}\, u_i \subset\subset \mathring{R}_i$, $\int_{R_i} u_i\,dx = 0$,

$$\begin{aligned}
&\langle \widetilde{\mathcal{A}}_{\mathring{R}_i} u_i, \psi\rangle = \int_{R_{1,i}} V_i \cdot \nabla\psi\,dx \quad \text{for every } \psi \in W^{1,q}(\mathring{R}_i; \mathbb{R}^l),\\
&\int_{R_i} |V_i|^p dx < \eta|R_i|,\\
&\int_{R_i} f(\omega, x, \xi + u_i(x))\,dx < M^\eta_c(f(\omega), \xi, \mathring{R}_i) + \delta|R_i|.
\end{aligned}$$

We define $u \in L^p(R; \mathbb{R}^d)$ and $V \in L^p(R; \mathbb{R}^{l\times N})$ by setting $u := u_i$ and $V := V_i$ on R_i for every $i \in I$. Then,

$$\langle \widetilde{\mathcal{A}}_{\mathring{R}} u, \psi\rangle = \int_R V \cdot \nabla\psi\,dx \quad \text{for every } \psi \in W^{1,q}(\mathring{R}; \mathbb{R}^l),$$

Moreover, by additivity, we have

$$\begin{aligned}
&\int_R u\,dx = 0, \quad \int_R |V|^p dx < \eta|R|, \quad \text{and}\\
&\int_R f(\omega, x, \xi + u(x))\,dx < \sum_{i\in I} M^\eta_c(f(\omega), \xi, \mathring{R}_i) + \delta|R|.
\end{aligned}$$

By (6.16), we obtain $M_c^\eta(f(\omega),\xi,\mathring{R}) \leq \sum_{i\in I} M_c^\eta(f(\omega),\xi,\mathring{R}_i) + \delta|R|$. Hence, due to the arbitrariness of $\delta > 0$, we conclude that $M_c^\eta(f(\omega),\xi,\mathring{R}) \leq \sum_{i\in I} M_c^\eta(f(\omega),\xi,\mathring{R}_i)$, which proves the subadditivity of $\Phi_\xi^\eta(\omega,R)$. □

We are now in a position to state our main result on stochastic homogenization of $\mathcal{A}$-free integral functionals. Given a stochastic integrand $f \in \mathcal{F}_{\rm Lip}^{\rm st}$ and $\varepsilon > 0$, we consider the stochastic integrands $f_\varepsilon\colon \Omega \times \mathbb{R}^N \times \mathbb{R}^d \to [0,+\infty)$ defined by

$$f_\varepsilon(\omega,x,\xi) := f(\omega,x/\varepsilon,\xi)$$

for every $x \in \mathbb{R}^N$ and $\xi \in \mathbb{R}^d$. Setting $f_\varepsilon(\omega) := f_\varepsilon(\omega,\cdot,\cdot)$, we have that $f_\varepsilon(\omega) \in \mathcal{F}_{\rm Lip}$ for every $\omega \in \Omega$, so we can consider the functionals $F_\varepsilon(\omega) \in \mathcal{I}_{\rm Lip}$ associated with $f_\varepsilon(\omega)$ by (2.14), and the corresponding $\mathcal{A}$-free functionals $F_\varepsilon^{\mathcal{A}}(\omega)$ introduced in Definition 4.4.

Theorem 8.5. *Let $f \in \mathcal{F}_{\rm Lip}^{\rm st}$ and, for every $\varepsilon > 0$ and for every $\omega \in \Omega$, let $F_\varepsilon(\omega) \in \mathcal{I}_{\rm Lip}$ and $F_\varepsilon^{\mathcal{A}}(\omega)$ be the corresponding functionals. Then, there exists a set $\Omega' \in \mathcal{T}$, with $P(\Omega') = 1$, such that for every $k \in \mathbb{N}$, $\omega \in \Omega'$, and $\xi \in \mathbb{Q}^d$, the limit*

$$f_{\rm hom}^k(\omega,\xi) := \lim_{r\to+\infty} \frac{M_c^{1/k}(f(\omega),\xi,Q_r(rx))}{r^N} \tag{8.1}$$

exists and is independent of x. Moreover, for every $\omega \in \Omega'$, the function $f_{\rm hom}^k(\omega,\cdot)$ can be extended as a continuous function on $\mathbb{R}^d$, which we still denote by $f_{\rm hom}^k(\omega,\cdot)$, and (8.1) holds for every $\xi \in \mathbb{R}^d$.

Let $f_{\rm hom}\colon \Omega \times \mathbb{R}^d \to [0,+\infty)$ be the function defined for every $\xi \in \mathbb{R}^d$ by

$$f_{\rm hom}(\omega,\xi) := \begin{cases} \sup_{k\in\mathbb{N}} f_{\rm hom}^k(\omega,\xi) = \lim_{k\to\infty} f_{\rm hom}^k(\omega,\xi) & \text{if } \omega \in \Omega' \text{ and } \xi \in \mathbb{R}^d, \\ \int_{\Omega'} f_{\rm hom}(\omega',\xi)dP(\omega') & \text{if } \omega \in \Omega\setminus\Omega' \text{ and } \xi \in \mathbb{R}^d. \end{cases}$$

Then, $f_{\rm hom} \in \mathcal{F}_{\rm Lip}^{\rm st} \cap \mathcal{F}_{\rm qc}^{\rm st}$ and, setting $f_{\rm hom}(\omega) := f_{\rm hom}(\omega,\cdot)$ for every $\omega \in \Omega$, the corresponding functionals $F_{\rm hom}(\omega) \in \mathcal{I}_{\rm Lip} \cap \mathcal{I}_{\rm qc}$ defined by (2.14) and $F_{\rm hom}^{\mathcal{A}}(\omega)$ introduced in Definition 4.4 satisfy the following properties:

(a) *for every $\omega \in \Omega'$ and every $D \in \mathcal{O}(\mathbb{R}^N)$, the family $(F_\varepsilon(\omega,\cdot,D))_{\varepsilon>0}$ Γ-converges as $\varepsilon \to 0^+$ to $F_{\rm hom}(\omega,\cdot,D)$ with respect to the topology induced by $\|\cdot\|_D^{\mathcal{A}}$ on $L^p(D;\mathbb{R}^d)$;*

(b) *for every for every $\omega \in \Omega'$ and $D \in \mathcal{O}(\mathbb{R}^N)$, the family $(F_\varepsilon^{\mathcal{A}}(\omega,\cdot,D))_{\varepsilon>0}$ Γ-converges as $\varepsilon \to 0^+$ to $F_{\rm hom}^{\mathcal{A}}(\omega,\cdot,D)$ in $\ker \mathcal{A}_D$ with respect to the weak topology of $L^p(D;\mathbb{R}^d)$.*

If, in addition, $(\tau_z)_{z\in\mathbb{Z}^N}$ is ergodic, then we can select Ω' so that $f_{\rm hom}^k$ and $f_{\rm hom}$ do not depend on ω.

Proof. The existence of $\Omega' \in \mathcal{T}$, with $P(\Omega') = 1$, such that (8.1) holds follows from Theorem 8.3 and Lemma 8.4. The properties of $f_{\rm hom}^k(\omega,\xi)$ and $f_{\rm hom}(\omega,\xi)$ for $\omega \in \Omega'$ are given by Theorem 7.1, which implies also that (a) and (b) hold and that $f_{\rm hom}(\omega) \in \mathcal{F}_{\rm Lip} \cap \mathcal{F}_{\rm qc}$ for every $\omega \in \Omega$; hence, $f_{\rm hom} \in \mathcal{F}_{\rm Lip}^{\rm st} \cap \mathcal{F}_{\rm qc}^{\rm st}$.

If $(\tau_z)_{z\in\mathbb{Z}^N}$ is ergodic, then Theorem 8.3 implies that for every $\xi \in \mathbb{Q}^N$, the function $\omega \mapsto f_{\rm hom}^k(\omega,\xi)$ is constant P-a.e. in Ω. Therefore, we can select Ω' so that $f_{\rm hom}^k$ does not depend on ω for $\xi \in \mathbb{Q}^N$, and so does its continuous extension to $\mathbb{R}^N$ and its limit $f_{\rm hom}$. □

Raiţă

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