Asymptotic expansions by Γ-convergence

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Abstract

We propose a new framework aimed at constructing approximations of different order to static mechanical models with variational structure. Our starting point is a parameterized family of functionals (a 'theory') and we are interested in approximating the global minimuma of the energy when a secondary small parameter goes to zero. The goal is to develop a set of increasingly accurate asymptotic variational models allowing one to deal with the cases when this secondary parameter is 'small' but finite. At the basis of our approach is the idea of Γ-equivalence, allowing one to divide the given set of 'theories' into classes of asymptotic equivalence with respect to the small parameter. Since Γ-convergence may be nonuniform within a 'theory' we pose a problem of finding a uniform approximation. To achieve this goal we propose a method based on rectifying the singular points in the parameter space by using the blow-up argument and then asymptotically matching the approximations around such points with the regular approximation away from them. We illustrate the main ideas with physically meaningful examples covering broad set of subjects from homogenization and dimension reduction to fracture and phase transitions. The analysis of many of the examples is new and presents an independent interest. In particular, we give considerable attention to the problem of transition from discrete to continuum when the internal and external scales are not well separated and one has to deal with the so called 'size' or 'scale' effects.

Introduction

Most of the models in continuum mechanics describing equilibrium configurations are based on the minimization of functionals which contain a small parameter of either constitutive or geometrical nature. It is then natural to try to use the smallness of the parameter to replace the original model by a simpler one. The well-known examples of the limiting mechanical models which emerge when the small parameter tends to zero include low-dimensional theories of thin-walled structural elements (e.g [25]), the homogenized models of composite materials (e.g. [46]) and the continuum models of crystal lattices (e.g.[10]). In all those cases the limiting models are more tractable than their prototypes because they do not contain the small parameter and enjoy the advantages of reduced dimensionality, homogeneity or continuity.

Often, the simplified model can be constructed by the more or less straightforward dropping of the 'small' terms. While such point-wise limits can be rigorously justified in some situations (e.g. [6]), there are other cases when the limit is nontrivial due to only weak convergence of the minimizers (e.g. [63]). In those case in order to derive the limiting theory one has to use more

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sophisticated methods, in particular, the method of Γ-convergence, which was specially designed to handle such situations [31]. In the last years the construction of Γ-limits proved to be a very useful tool in dealing with otherwise intractable problems [27, 13].

The situations, however, are plentiful when the limiting models derived by the method of Γ-convergence are degenerate and when it is clear that a physically meaningful model must necessarily contain the original parameter even if it is considered small. It is enough to mention the plate theories combining membrane deformations with bending [35], the theories of surface tension [47], and the approaches in fracture mechanics assuming a nonzero toughness [37]. The small parameter must also be preserved when the ratio of the internal to external scales is small but not too small and when the main modeled phenomenon is associated with the 'size' or 'scale' effects [58].

Many asymptotic models containing a small parameter have been suggested in applications as a heuristic way of simplifying the original "microscopic" descriptions. Moreover, it is not uncommon that there exist several different approximate models with apparently overlapping domains of application (e.g., Bernoulli and Timoshenko theories for rods [6, 41, 50], Kirchhoff and Mindlin-Reissner models for plates [3, 6, 41], gradient and nonlocal continuum models of crystal elasticity [47, 39, 55], various regularizations in the models with phase transitions or strain localization [5, 42]). In particular, the present study has been mostly motivated by the desire of the authors to rigorously distinguish the theories of Griffith [37] and Barenblatt [4] in fracture mechanics under the assumption that they represent various asymptotic limits of a lattice model with Lennard-Jones interactions [60, 18].

As we have already mentioned, the asymptotic theories are typically aimed at capturing the global minimum of the energy and in this case the most general approach to formalize the intuitive derivations proposed in applications is to use the techniques of Γ-convergence [27, 13]. Furthermore, to justify the higher order corrections rigorously, it is necessary to extend the idea of the Γ-limit and formulate the concept of Γ-asymptotic expansion. Quite expectedly, this leaves one with a variety of options.

In the current mathematical literature the issue is usually addressed by constructing a Γ-limit in the proper limiting space and then improving it inside the same space through increasingly more accurate approximation of the minimal value (Γ-development, see [2]). The obvious problem with this approach is that the class of minimizers is decided already in the first step and after this class is exhausted, the process of the improvement of the minimizer terminates (locking). In addition, as we show in the paper, the corresponding higher-order Γ-limits may simply ceases to exist (choking). Even more serious problems concern parameterized families of functional where at certain values of parameters the Γ-convergence may not be uniform. In these cases we deal with singular phenomena, associated, for instance, with nucleation, buckling or failure. Around the singular points in the parameter space the precision of the straightforward Γ-development may drop dramatically. Those (locking, chocking and nonuniformity) are probably the reasons why, as a rule, the theories considered best in applications could not be reproduced by this formal method.

In this paper we propose a new methodology aimed at unifying the existing ad hoc approaches and placing them into the formal framework of Γ-convergence. Our starting point is the concept of Γ-asymptotically equivalent functionals of certain order which generalizes the corresponding concept in the classical perturbation methods [34, 38]. We show that, outside the zero order Γ-limit, the class of Γ-equivalent variational approximations of a given minimization problem may be rather large. In particular, the equivalent theories may be very different with respect to the degree of approximation for the minimizers, the general complexity, 'computability' and the ability to capture local minima. We therefore raise the question of the additional criteria securing the

uniqueness of the corresponding asymptotic expansions.

In search of such criteria we move from single functionals to 'theories' interpreting the latter as parameterized sets of functionals, where parameters may characterize the geometry of the domain, the boundary conditions, the bulk 'loading', or the constitutive behavior (see examples in [35, 44, 56, 26]). The presence of parameters raises the issues of continuity and uniformity of the successive Γ-limits. As we show, in the typical cases the Γ-convergence is non-uniform and in this sense the straightforward asymptotic theories based on Γ-development are not universal. The failure of such asymptotic theories is due to the existence of the special points in the space of parameters where the Γ-limit is singular (for instance, discontinuous). To deal with these points we propose particular ways of rectifying the singular behavior by local blow up of the functional and constructing the 'table' of Γ-limit which fully characterizes the singular point. If the 'tables' in all singular points are known, the only remaining problem is to match the boundary layer type Γ-expansions near the singular points with the regular Γ-expansions outside these points.

Despite its mainly theoretical focus, the paper contains a series of illuminating examples of equivalent theories and discusses the multiplicity of ways of generating uniform approximations with respect to classes of boundary conditions. In our selection of examples we intentionally resisted the temptation of dealing with the most general cases and instead limited ourselves to the problems where a one dimensional, scalar version of a particular model could substitute its multi-dimensional, tensorial analog. Some of our examples use arguments that can be derived from the known theories, and in these cases the details are not included or only sketched. The full proofs are given only for the cases that are not present in the literature.

The presentation is organized as follows. To motivate the subsequent definitions we review in Section 1 the existing asymptotic procedures based on Γ-convergence and show that despite their universality, they are not without significant flows. In Section 2 we introduce 'theories' and then show in Section 3 that within a 'theory' the standard Γ-development can be nonuniform with respect to the parameter. To deal with these problems we introduce in Section 4 the concept of Γ-equivalence of functionals and study the main properties of the asymptotic factorization of the set of functionals into equivalence classes. Some systematic methods of generating Γ-equivalent functionals are proposed and discussed in Section 5. In Section 6 we extend the definition of Γ-equivalence to 'theories'. The regular points within the 'theories' are studied in Section 7. The structure of the singular points constitute the subject of Section 8 where we also formally define a 'table of Γ-limits' and present several detailed computations of various 'tables' in the problems of physical interest. In Section 9 we pose the problem of rectifying singular points and produce a rather general recipe for constructing uniformly equivalent theories. Our conclusions are summarized in Section 10.

1 Background

As indicated in the Introduction, often the study of complex minimization problems involving a small parameter ε can be simplified by approximating the original problem with a new one where the dependence on this parameter has been either simplified or completely eliminated. Situations like this are well known in the context of differential equations where the method of geometrical optics and the theory of fluid boundary layers can serve as representative examples. The rigorous theory of perturbation methods for differential equations is a mature field whose origin can be traced to Poincaré (1886).

In the situation where the main object of interest is the global minimum of a functional, the adequate language has been developed by DeGiorgi [31, 29]. His approach is based on the notion of Γ-convergence which we briefly review below to make our presentation self contained (see [27, 13, 14] for more details).

The first requirement to Γ-convergence is that

$$F_\varepsilon \xrightarrow{\Gamma} F^{(0)} \tag{1}$$

implies

$$\min F_\varepsilon \to \min F^{(0)}. \tag{2}$$

The second requirement is that (almost) minimizers of F_ε converge to minimizers of $F^{(0)}$

$$u_\varepsilon \longrightarrow u^{(0)}, \tag{3}$$

even though the meaning of convergence may be very weak. The third requirement concerns the stability of the Γ-limit with respect to the addition of continuous perturbations

$$(F_\varepsilon + G) \xrightarrow{\Gamma} (F^{(0)} + G). \tag{4}$$

In particular, if the condition (4) is satisfied, then, once the Γ-limit $F^{(0)}$ is computed, the result can be used to describe a whole class of problems. This implies that Γ-convergence can deal with some simple 'theories'.

We now proceed with the formal definition:

Definition 1.1 *Let X be a first-countable space (e.g., a metric space) and let $F_\varepsilon : X \to [-\infty, +\infty]$. Then F_ε* Γ-converges *to F_0 as $\varepsilon \to 0$ (and F_0 is the* Γ-limit *of F_ε) if the following two conditions are satisfied for all $x \in X$:*

(i) (lim inf inequality) *for all $x_\varepsilon \to x$ $F_0(x) \le \liminf_{\varepsilon\to 0} F_\varepsilon(x_\varepsilon)$;*
(ii) (existence of a recovery sequence) *there exists $\overline{x}_\varepsilon \to x$ such that $F_0(x) = \lim_{\varepsilon\to 0} F_\varepsilon(\overline{x}_\varepsilon)$.*

From this definition one can see that implication (1) is valid if some *equi-coerciveness* assumptions on F_ε are satisfied (i.e., if we may find converging minimizing sequences) and throughout this paper we suppose that such assumptions indeed hold.

We illustrate the notion of Γ-equivalence by the following examples. They deal with the derivation of a continuum elasticity theory as the asymptotic limit of different lattice models.

Example 1.2 For ε such that $N = \frac{1}{\varepsilon} \in \mathbb{N}$ consider the functional

$$F_\varepsilon(u) = \sum_{i=1}^{N} \varepsilon\, W(u_i) \tag{5}$$

where $u : \{1, \dots, N\} \to \mathbb{R}$ and

$$\sum_{i=1}^{N} \varepsilon\, u_i = 0. \tag{6}$$

The energy (5) describes a chain of springs connected in series. The springs can be viewed as not interacting modulo the 'mean field' interaction with a loading device. Suppose that W is strictly convex. Then one can show that (see [20])

$$F^{(0)}(u) = \int_0^1 W(u)\,dt, \tag{7}$$

with constraint

$$\int_0^1 u(s)\,ds = 0. \tag{8}$$

Example 1.3 To show that the previous result is stable with respect to adding long-range interactions, we introduce a constant $J > 0$ and consider the following 'elastic' Ising model

$$F_\varepsilon(u) = \sum_{i=1}^{N} \varepsilon\Big(W(u_i) - Ju_iu_{i+1}\Big), \tag{9}$$

where we set $u_{N+1} = u_1$ to avoid boundary effects. We again impose the constraint (6). If we now rewrite the energy in the form

$$F_\varepsilon(u) = \sum_{i=1}^{N} \varepsilon\Big(\widetilde{W}(u_i) - 2J\Big|\frac{u_i + u_{i+1}}{2}\Big|^2\Big),$$

it can be interpreted as the model of a chain with the nonlinear interaction of nearest neighbors characterized by the convex potential

$$\widetilde{W}(z) = W(z) + J|z|^2,$$

and an additional linear interaction of the next to nearest neighbors (NNN model). Suppose that $W(z) \geq C(|z|^2 - 1)$ with $C > J$, so that F_ε are equi-coercive. Then, if the function

$$\overline{W}(z) = \widetilde{W}(z) - 2Jz^2 = W(z) - Jz^2.$$

is convex one can show that (see [20] for more details)

$$F^{(0)}(u) = \int_0^1 \overline{W}(u)\,dt.$$

The two examples above represent the simplest cases of periodic convex homogenization where the result could also be obtained by point-wise convergence. The next example shows the simplest case where the notion of Γ-limit is essential.

Example 1.4 Consider again Example 1.2 and suppose now that W is a double-well potential as shown in Fig. 1. We obtain (see [13] Theorem 4.3)

$$F^{(0)}(u) = \int_0^1 W^{**}(u)\,dt, \qquad \int_0^1 u\,dt = 0.$$

Here and often in the sequel we denote by W^{**} the convex envelope of W, whose appearance highlights the formation of microstructure.

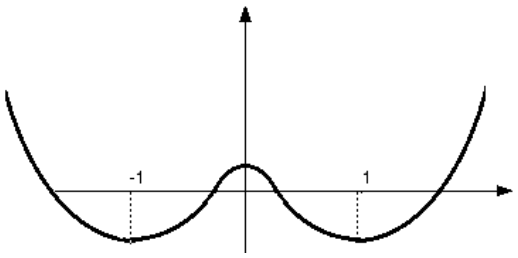

Figure 1: A double-well potential

1.1 Γ-development

If the description given by $F^{(0)}$ is too coarse, further information can be obtained by iteration of the Γ-limit procedure. More specifically if some $\alpha > 0$ exists such that

$$F_\varepsilon^{(\alpha)} := \frac{F_\varepsilon - \min F^{(0)}}{\varepsilon^\alpha} \xrightarrow{\Gamma} F^\alpha, \tag{10}$$

then, using again the fundamental property (2) of Γ-convergence we obtain

$$\min F_\varepsilon^{(\alpha)} \left(= \frac{\min F_\varepsilon - \min F^{(0)}}{\varepsilon^\alpha}\right) \to \min F^{(\alpha)}. \tag{11}$$

In other words, one can write the more accurate development

$$\min F_\varepsilon = \min F^{(0)} + \varepsilon^\alpha \min F^{(\alpha)} + o(\varepsilon^\alpha). \tag{12}$$

Remark 1.5 This convergence of minima can be deduced if we can find a precompact sequence of minimizers of F_ε, or, more in general, if there exists a precompact ε^α-minimizing sequence; i.e., x_ε such that $F_\varepsilon(x_\varepsilon) = \inf F_\varepsilon + o(\varepsilon^\alpha)$, which implies the equi-coerciveness of $F_\varepsilon^{(\alpha)}$.

We say that a Γ-development is *complete* if for all $0 < \gamma < \alpha$ we have

$$F^{(\gamma)}(u) := \Gamma\text{-}\lim_{\varepsilon\to 0} \frac{F_\varepsilon(u) - \min F^{(0)}}{\varepsilon^\gamma} = \begin{cases} 0 & \text{if } u \text{ is a minimizer of } F^{(0)} \\ +\infty & \text{otherwise} \end{cases}$$

Note that if the Γ-development is not complete, i.e., $F^{(\gamma)}$ does not have the form above for some γ, but $\min F^{(\gamma)} = 0$, then such $F^{(\gamma)}$ plays no role in (12).

The process of Γ-*development* [2] (or development by Γ-convergence) is formally resumed in the equality

$$F_\varepsilon \overset{\Gamma}{=} F^{(0)} + \varepsilon^\alpha F^{(\alpha)} + o(\varepsilon^\alpha). \tag{13}$$

The general equality (13) is only formal since the domains of the functionals may be different, and even when they are equal the energy $F^{(0)} + \varepsilon^\alpha F^{(\alpha)}$ is equal to $+\infty$ outside the set of minimizers of $F^{(0)}$; however, if the domains are equal, it suggests that we could expect the energy $F^{(0)} + \varepsilon^\alpha F^{(\alpha)}$ to be used in place of F_ε at scale ε^α.

This example illustrates a remarkable 'stiffness' of the classical elasticity theory.

In the next example we show the first manifestation of a 'size' effect when the the boundary layers, due to lattice incompatibility with the 'shape' of the macroscopic boundary, contribute to higher-order Γ limits.

Example 1.6 To capture the incompatibility effect in the one-dimensional setting we consider the functional almost equivalent to the one in the Example 1.2

$$F_\varepsilon(u) = \sum_{\{i:\varepsilon i\in(0,1)\}} \varepsilon\, W(u_i), \tag{14}$$

again with constraint 6. In this case the identification of a discrete u with a piecewise-constant function is not exact. We obtain

$$F^1_\varepsilon(u) = \int_0^1 W(u)\, dt + c_\varepsilon, \tag{15}$$

with constraint (8), and

$$c_\varepsilon = \varepsilon\Big(\Big[\frac{1}{\varepsilon} - 1\Big] - \frac{1}{\varepsilon}\Big)W(0).$$

Here $[t]$ is the integer part of t.

Remark 1.7 In the computation of the higher-order Γ-limits some non trivial scale analysis must be performed to understand what is the relevant scaling ε^α (or, more general, $f(\varepsilon)$). Note however that, up to scaling, we can always suppose that $F^{(0)}$ is non trivial and, if needed, that the next relevant scale is ε.

Once this first development is computed, the analysis at successively lower scales

$$1 >> f_1(\varepsilon) >> \ldots >> f_m(\varepsilon)$$

can be performed by iteration (in these notations, in the development above we have taken $f_1(\varepsilon) = f(\varepsilon)$, or ε^α). Then we obtain a development

$$F_\varepsilon \overset{\Gamma}{=} F^{(0)} + f_1(\varepsilon)F^{(1)} + \cdots + f_m(\varepsilon)F^{(m)} + o(f_m(\varepsilon)), \tag{16}$$

where (with a little abuse of notation with respect to (13)) we have set

$$F^{(j)} = \Gamma\text{-}\lim_{\varepsilon\to 0^+} F^{(j)}_\varepsilon, \text{ where } F^{(j)}_\varepsilon(u) := \frac{F_\varepsilon(u) - \sum_{i<j} f_i(\varepsilon)\, m^{(i)}}{f_j(\varepsilon)}, \tag{17}$$

and $m^{(i)} = \min F^{(i)}$.

We now discuss some limitations of the straightforward Γ-development.

1.2 Locking of minimizers

A rather unfortunate consequence of (11) is that this approximate energy is infinite outside the set of minimizers of $F^{(0)}$. The latter may reduce to a trivial set (e.g. a single point) that cannot be refined by the successive Γ-limits. In what follows we shall be referring to this property of the straightforward Γ-development as the *locking* of the minimizers.

Remark 1.8 If $F^{(0)}$ has a unique minimum point u_0 then the computation of $F^{(\alpha)}$ reduces to that of $F^{(\alpha)}(u_0)$. If F_ε are equicoercive, then this amounts to computing

$$F^{(\alpha)}(u_0)\ \Big(=\min F^{(\alpha)}\Big)=\lim_{\varepsilon\to 0}\frac{\min F_\varepsilon-\min F^{(0)}}{\varepsilon^\alpha}.$$

This observation will be frequently used in the sequel.

Example 1.9 To illustrate the locking phenomenon with a simplest example, consider

$$F_\varepsilon(u)=\int_0^1 (|u'|^2+\varepsilon|u|^2)\,dt,\qquad u(0)=0,\ u(1)=1. \tag{18}$$

We can compute the Γ-development with respect to the strong L^2 topology (or equivalently with respect to the weak H^1 topology), and obtain $I_\varepsilon=F^{(0)}+\varepsilon F^{(1)}+o(\varepsilon)$, where

$$F^{(0)}(u)=\int_0^1 |u'|^2\,dt,\qquad u(0)=0,\ u(1)=1. \tag{19}$$

and

$$F^{(1)}(u)=\begin{cases}\frac{1}{3} & \text{if } u=t\\ +\infty & \text{otherwise.}\end{cases} \tag{20}$$

The computation of the last Γ-limit is trivial since the first one admits the only minimum point

$$u(t)=t.$$

This expression should be compared to the actual minimizer u_ε of the original problem, whose formal asymptotic expansion goes as follows

$$u_\varepsilon(t)=t+\varepsilon\frac{1}{6}(t^3-t)-\varepsilon^3\frac{1}{18}(t-t^3)+o(\varepsilon^2).$$

One can see that the successive Γ-development locks the minimizer and does not allow to improve its quality beyond what have been found in the first approximation even though finer and finer information about the minimizer is needed to compute the higher-order Γ-limits.

The next example shows that the locking of the minimizer may not happen at the level of the first approximation but may instead occur during the subsequent higher-order development.

Example 1.10 Let $W:\mathbb{R}\to\mathbb{R}$ be a continuous double-well potential with wells (absolute minima) in ±1 and growing more than linearly at ∞ (e.g., $W(s)=\min\{(s+1)^2,(s-1)^2\}$). Consider

$$F_\varepsilon(u)=\int_\Omega (W(u)+C\varepsilon^2|\nabla u|^2)\,dx,\qquad u\in H^1(\Omega),\ \int_\Omega u\,dx=0.$$

If we use the weak L^1-topology, suggested by the superlinear growth conditions of W at infinity, then the first Γ-limit is (see, e.g., [14, 13])

$$F^{(0)}(u) = \int_\Omega W^{**}(u)\,dx, \quad u \in L^1(0,1), \ \int_\Omega u\,dx = 0. \tag{21}$$

Note that the minimizer is piecewise constant $u = \pm 1$ and due to the degeneracy of the energy (21) it is not unique. In fact only the measures of the sets where $u = 1$ and $u = -1$ are known at this stage. In particular the location of the interfaces (internal boundary layers) between the sates with $u = 1$ and $u = -1$ and even their number remain unspecified. This information, however, can be recovered in the next step of Γ-development, which finally locks the minimizer. Indeed, we obtain (see, e.g., [14, 13])

$$F^{(1)}(u) = c_W \mathcal{H}^{n-1}(S(u)), \qquad |u| = 1 \tag{22}$$

Here $u \in \{\pm 1\}$ is piecewise constant, $\int_\Omega u\,dx = 0$ and

$$c_W = 2\sqrt{C} \int_{-1}^{1} \sqrt{W(s) - \min W}\,ds,$$

where $S(u)$ denotes the interface between the phases $\{u = \pm 1\}$ and $\mathcal{H}^{n-1}$ the $(n-1)$-dimensional (surface) measure.

The minimization of the functional (22) allows one to fix the location of the interface which locks the minimizer. In the subsequent approximations only the minimal value is changing. Interestingly, the relevant scale successive to ε, is of exponential type $\varepsilon e^{-c_1/\varepsilon}$ and not of the form ε^α [22, 66]. Indeed, we recall that in the one-dimensional case with $\Omega = (0,1)$ the set of 'locked' minimizers is $\{u_0, -u_0\}$, where

$$u_0(t) = \begin{cases} -1 & \text{if } t < 1/2, \\ 1 & \text{if } t > 1/2. \end{cases}$$

The development reads as

$$F_\varepsilon \overset{\Gamma}{=} F^{(0)} + \varepsilon F^{(1)} + \varepsilon e^{-1/2\varepsilon} F^\infty + o(\varepsilon e^{-1/2\varepsilon}),$$

where

$$F^\infty(u) = \begin{cases} C^\infty & \text{if } u = \pm u_0 \\ +\infty & \text{otherwise,} \end{cases}$$

and the constant C^∞ may be computed in terms of the limit of minimum problems as in Remark 1.8.

One can see that the first approximation locks the phase fractions, the second fixes the geometry of the interface, while the higher-order approximations describe exponentially weak corrections due to interaction of the interface with the external boundary (size effect). The latter will be the subject of a detailed analysis in Example 8.4, where we consider the case when the size effect becomes dominant.

Remark 1.11 Since the weak-L^1 equi-coerciveness improves to *strong*-L^1 coerciveness at scale ε, then we may (a posteriori) choose to compute the first Γ-limit $F^{(0)}$ with respect to the strong L^1-topology, obtaining

$$F^{(0)}(u) = \int_\Omega W(u)\,dx,$$

while $F^{(1)}$ remains unchanged. This shows that sometimes the Γ-limit may look superficially as a point-wise limit even if the argumentation behind its derivation is entirely different.

1.3 'Choking' of Γ-development

Locking of the minimizer is not the only problem which one may encounter while constructing the Γ-development. Thus, the whole process may simply 'choke' if one of the higher-order limits does not exist. We illustrate this phenomenon by the following examples.

Example 1.12 Consider a non-constant strictly positive and bounded one-periodic function $a : \mathbb{R} \to \mathbb{R}$, and the functionals

$$F_\varepsilon(u) = \int_0^1 a\Big(\frac{t}{\varepsilon}\Big)|u'|^2\,dt \tag{23}$$

defined on functions subject to the boundary conditions

$$u(0) = 0, \qquad u(1) = 1.$$

A standard argument shows that for all strictly positive and bounded $f : [0,1] \to \mathbb{R}$ we have

$$\min\Big\{\int_0^1 f(t)|u'|^2\,dt :\ u(0) = 0,\ u(1) = 1\Big\} = \underline{f},\ \text{where } \frac{1}{\underline{f}} = \int_0^1 \frac{1}{f(s)}\,ds; \tag{24}$$

i.e., $\underline{f}$ is the *harmonic mean* of f on $[0,1]$. We can now apply this computation to $f(s) = a_\varepsilon(s) = a(s/\varepsilon)$, and recalling that $a_\varepsilon^{-1} \rightharpoonup (\underline{a})^{-1}$, obtain the well know fact that the Γ-limit of F_ε is (see , for instance, [19])

$$F^{(0)}(u) = \underline{a}\int_0^1 |u'|^2\,dt, \qquad u(0) = 0,\ u(1) = 1. \tag{25}$$

The unique minimum point of $F^{(0)}$ is $\overline{u}(t) = t$, and in order to compute any further development it suffices to compute it at this function (locking). The next meaningful order is ε. The Γ-limit can be computed for sequences $\varepsilon_j \to 0$ if there exists the limit (see Remark 1.8)

$$\begin{aligned} &\lim_j \frac{1}{\varepsilon_j}\Big(\min\Big\{\int_0^1 a\Big(\frac{t}{\varepsilon_j}\Big)|u'|^2\,dt :\ u(0) = 0,\ u(1) = 1\Big\} - \underline{a}\Big) \\ = \ &\lim_j \frac{1}{\varepsilon_j}\left(\Big(\int_0^1 \frac{1}{a(s/\varepsilon_j)}\,ds\Big)^{-1} - \underline{a}\right) = \lim_j \underline{a}^2 \int_{[1/\varepsilon_j]}^{1/\varepsilon_j}\Big(\frac{1}{\underline{a}} - \frac{1}{a(s)}\Big)\,ds. \end{aligned} \tag{26}$$

The value of the limit (26), which must be equal to $F^{(1)}(\overline{u})$, depends on the sequence (ε_j) which means that the development at order ε does not exist. This obviously terminates the process preventing one from improving upon the first approximation.

Here we encounter another example of the interaction between the boundary of a body and the homogenization procedure. In the case when the scales are well separated $\varepsilon \ll 1$ (i.e., we consider only the development at order 1) the energy associated with the boundary layers is negligible. As the external and internal scale get closer to each other, we enter the domain of a size effect where the oscillatory structure of the corrections to the homogenized theory, preventing the minimal value of the approximate functional from converging, becomes more and more noticeable.

The next example shows that the Γ-development may not terminate till sufficiently high order of the approximation.

Example 1.13 Consider again Example 1.4 where the functional F_ε (see 5) can be thought as the discretization of the continuum energy $\int_0^1 W(u)\,dt$. We assume that the discretization is perfectly compatible with the 'shape' of the body, meaning that we set $\varepsilon = 1/N$. Suppose that W is a double-well potential. In Example 1.4 we have obtained the following result

$$F^{(0)}(u) = \int_0^1 W^{**}(u)\,dt, \qquad \int_0^1 u\,dt = 0.$$

One can also show that F_ε is equivalent to $F^{(0)}$ also at order ε, as in the convex case but not at order ε^2. To check this we may refer to the study of the parameterized minimum problems

$$m_\varepsilon(\lambda) = \min\{F_\varepsilon(u) :\ \textstyle\sum_i \varepsilon u_i = \lambda\}.$$

An analysis of the exact solution of the discrete problem (e.g. [33, 53]) pictures the dependence on λ as in Fig. 2 for the case $W(s) = (s-1)^2 \wedge (s=1)^2$, from which we see that $\varepsilon^{-2} m_\varepsilon(0)$ does not converge to $0 = \min F^{(0)}$.

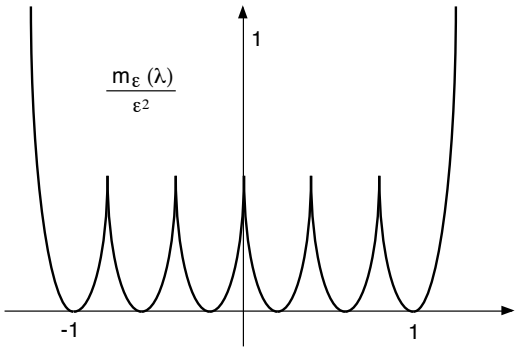

Figure 2: Minimal energy of a chain of bi-stable springs.

It is easy to see that the minimal value at, say, $\lambda = 0$ fluctuates as the small parameter goes to zero, and therefore the limit does not exist. This example shows that for non-convex energies homogenization methods must take into account the scale of the approximation and that different theories may have to be used at different scales (contrary to what we have observed in the convex case).

Now, observe that if parameter λ in Example 1.13 is allowed to vary, the fact of the non-existence of the second order Γ-limit remains true for the whole interval $(-1, 1)$ of parameters λ. In the next Section we discuss more systematically some other typical problems which arise when the Γ-development is applied to parameterized family of functionals.

2 Theories

We recall that Γ-convergence has been designed to handle automatically the functionals parameterized by 'lower-order terms'. In this case the parametrization does not affect the first order Γ-limit in the essential way, with the corresponding terms being either continuous perturbation or in some way 'compatible' with Γ-convergence. However the higher-order Γ-limits do not enjoy the same 'invariance' property with respect to the same 'continuous' extensions. Moreover, in the typical problems of interest the parameter enters the functional in a variety of ways that are not at all 'compatible' with Γ-convergence.

In what follows we shall refer to a class of minimization problems originating from a parameterized family of functionals as a 'theory'. This terminology comes from applications where one encounters, for instance, a multiplicity of theories of beams, plates and shells, theories of low or high frequency vibrations, quasi-continuum theories of crystals and cohesive theories of cracks. We adopt the following formal definition

Definition 2.1 *Let $\mathcal{E}$ be a set of positive real numbers with $0 \in \overline{\mathcal{E}}$, and let Λ be a subset of a topological space. Then a family of functionals F_ε^λ is called a* parameterized family *on Λ (the* space of parameters*) or a 'theory'.*

We begin the analysis of the 'theories' by listing a series of examples.

1. *Van der Waals's theory of phase transitions* ([65]). Suppose that W is a double-well potential as in Example 1.10. Take $\Lambda = \mathbb{R}$ or $\Lambda = [-1, 1]$ and define

$$F_\varepsilon^\lambda(u) = \int_0^1 \Big(W(u) + \varepsilon^2 |u'|^2\Big)\, dt, \qquad \int_0^1 u(t)\, dt = \lambda. \tag{27}$$

In this case λ represents an imposed integral constraint representing, for instance, average strain in a bar if $v' = u$. A typical problem in this theory is to determine the function

$$m(\lambda, \varepsilon) = m_\varepsilon(\lambda) = \min\Big\{ \int_0^1 (W(u) + \varepsilon^2 |u'|^2)\, dt : \int_0^1 u\, dt = \lambda \Big\}. \tag{28}$$

whose first derivative in λ gives the effective stress-strain relation. When $\varepsilon = 0$, we obtain the famous Maxwell 'common-tangent' construction (e.g.[6]), which does not handle nucleation appropriately even under the assumption of global minimization. Indeed, at finite ε the first nucleus of the new phase appears with finite 'size' and this effect is lost in the limit $\varepsilon \to 0$. It is then of interest to construct an approximate theory corresponding to the case when ε is small but finite which captures the above effect.

Different theories of the type (27) have recently been unified under the general title of a 'phase field' model (e.g. [23]). The main goal of the phase field theory is numerical capturing of the sharp discontinuities. In this framework our task is to construct an intermediate theory which avoids the drawbacks of the sharp interface limit but does not have to resolve the higher derivative terms where it is not absolutely necessary.

2. *1D Lattice theory of fracture* ([60]). Let $J : [0, +\infty) \to [0, +\infty)$ be a Lennard-Jones interatomic potential with minimum in 1 (see Fig. 3), and consider the scaled energy

$$F_\varepsilon^\lambda(u) = \sum_{i=1}^{N} \varepsilon\, J\Big(\frac{u_i - u_{i-1}}{\varepsilon}\Big) \tag{29}$$

with boundary conditions $u_0 = 0$, $u_N = \lambda$. In this case λ represents an imposed displacement in a hard device. A problem of interest is again to compute

$$m(\lambda, \varepsilon) = ..$$

whose first derivative defines the effective stress-strain relation. When $\varepsilon = 0$, we obtain a material that does not support tension and breaks at infinitesimal tension ([60, 18]). Real cracks, on the contrary, appear only at finite tension. The challenge is then to capture this effect when ε is small but finite.

While the theory of fracture in this example looks superficially very different from the theory of phase transitions discussed above, we show in what follows that the two theories are in fact remarkably similar.

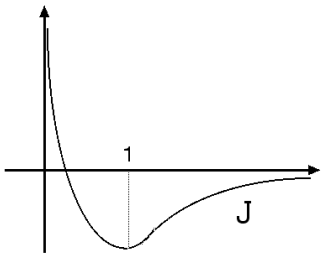

Figure 3: a Lennard-Jones potential.

3. *Homogenization theory (e.g.* [46]). Take $\Lambda = (0, +\infty)$ and consider a functional

$$F_\varepsilon^\lambda(u) = \frac{1}{\lambda}\int_0^\lambda a\Big(\frac{t}{\varepsilon}\Big)|u'|^2\,dt \qquad u(0) = 0,\ u(\lambda) = \lambda. \tag{30}$$

In this case λ is a geometrical parameter (the length of a bar). The typical problem in this theory is to find the minimum $m(\lambda, \varepsilon)$ of F_ε^λ and then compute effective elastic modulus $\bar{a} = 2m(\lambda, \varepsilon)/\lambda^2$. The problem has a classical solution when $\varepsilon = 0$. The computation of a correction to this result at $\varepsilon \sim \lambda$ constitutes the main task of the theory of 'size' effect in homogenization.

4. *Theory of finite scale micro-structures (e.g.* [54]*).* This title refers to the broad class of models with competing interactions where certain factors drive the coarsening of the microstructure while the other factors facilitate its refinement. Here we consider the simplest model of this type [61, 1]. Suppose that $\Lambda = [0, +\infty)$ and define

$$F_\varepsilon^\lambda(u) = \int_0^1 \Big(W(u') + \varepsilon^2|u''|^2 + \lambda u^2\Big)\,dt, \qquad u(0) = u(1) = 0 \tag{31}$$

In this case λ represents a combination of material and geometrical parameters and characterizes the 'anti-ferromagnetic' component of the interactions which drives the system towards the refinement of the microstructure. This interaction competes with a 'ferro-magnetic' contribution due to ε term in the energy, which drives the system towards coarsening. The problem is to characterize the scale of the microstructure in the limit when $\varepsilon \to 0$ and due to almost periodic arrangement of

the optimal microstructure, the adequate parameter is the density of interfaces for the minimizer $N(\lambda, \varepsilon)$. If λ is finite, then $N(\lambda, 0) = \infty$. An interesting question is to predict the value of $N(\lambda, \varepsilon)$ when $\varepsilon \sim \lambda$ and λ is small but finite.

In the next Section we further elaborate on the first two of these examples in order to illustrate the typical problems encountered by the straightforward Γ-development applied to 'theories'. The last two examples will be studied in the subsequent Sections.

3 Non-uniformity of Γ-developments

We begin with the theory of phase transitions.

Example 3.1 Consider the family of functionals(27)). It is well known (see [48, 14]) that the volume constraint $\int u\,dt = \lambda$ is compatible with the Γ-limit procedure within the gradient theory of phase transitions. Let $m_\varepsilon(\lambda)$ be defined by (28). If $|\lambda| \geq 1$ then the unique minimizer of F_ε^λ is the constant state $u = \lambda$ for all ε, the Γ-development consist only of the first term, and $m_\varepsilon(\lambda) = m^{(0)}(\lambda) = W^{**}(\lambda)$. For all $\lambda \in (-1, 1)$ the development of the minimum values is given by $m^{(0)}(\lambda) + \varepsilon m^{(1)}(\lambda) + o(\varepsilon)$, where

$$m^{(1)}(\lambda) = c_W \min\Big\{\#(S(u)) : u \in \{\pm 1\}, \int_0^1 u\,dt = \lambda\Big\} = c_W.$$

The plot of the function $m^{(0)}(\lambda) + \varepsilon m^{(1)}(\lambda)$ giving the Γ-development for the minimum values of the functional (27)) is given in Fig. 4 together with the corresponding effective stress-strain relation. The horizontal region on the stress-strain curve corresponds to Maxwell construction. Observe that the nucleation (annihilation) takes place at points 1 and -1 and the newly forming nucleus has infinitesimal 'size'. This result can be compared with the exact solution of the problem at finite ε (e.g. [61, 64]) showing that the nucleation starts at a finite distance from the points 1 and -1 and that the first nucleus is finite. More precisely, our approximate theory of order ε, needs to be corrected near these points at the same order ε.

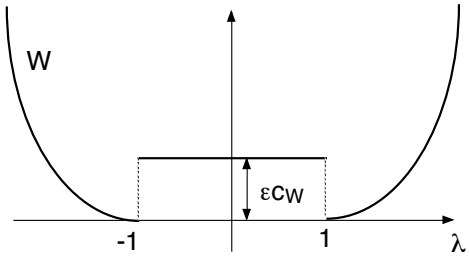

Figure 4: Approximate minimum values by Γ-development .

In the more formal language, we can reformulate the above observations as follows. We first recall that the value $m_\varepsilon(\lambda)$ is continuous with respect to λ. Then one can write

$$\lim_{\lambda \to 1^-} (m_\varepsilon(\lambda) - m^{(0)}(\lambda) - \varepsilon m^{(1)}(\lambda)) = \varepsilon c_W,$$

from which we argue that the description given by the Γ-development at scale ε is not accurate close to the point -1. Similar result can be obtained for point 1.

The key point is that near both limits $\lambda = \pm 1$ the external and internal length scales are no longer well separated. The external length scale here is given by the distance between the location of the interface and the exterior boundary of the body: during nucleation both scales become comparable. One encounters similar non-uniformity in the multi-dimensional setting when the two interfaces get sufficiently closer to each other or when the curvature of one of the interface increases considerably (say, in the neighborhood of a topological transition).

Despite its rather different formal appearance, the next example is very similar to the previous one. Here instead of the new phase we deal with the nucleation of a crack.

Example 3.2 Consider a lattice theory of fracture with the energy (29). After identifying discrete functions with their piecewise-affine interpolations the Γ-limit at order 1 can be computed as in the other cases involving the passage from discrete to continuous and is simply

$$F^0_\lambda(u) = \int_0^1 J^{**}(u')\,dt$$

with boundary conditions $u(0) = 0$, $u(1) = \lambda$, defined on all increasing functions $u : [0, 1] \to \mathbb{R}$ such that $u(0) = 0$ and $u(1) = \lambda$ (see e.g. [60, 13]; see also Example 6.5). Note that such functions may be discontinuous, but the derivative u' is defined almost everywhere. The integrand J^{**} is constant and equal to $\min J = J(1)$ on $[1, +\infty)$, so that

$$\min F^0_\lambda = J^{**}(\lambda) = \begin{cases} J(\lambda) & \text{if } \lambda \le 1 \\ J(1) & \text{if } \lambda > 1. \end{cases}$$

Indeed by Jensen's inequality $\min F^0_\lambda \ge J^{**}(\lambda)$. If $\lambda \le 1$ then $u_\lambda(t) = \lambda t$ is the only test function for which we have equality. If $\lambda > 1$ then all increasing functions satisfying the boundary conditions and with $u' \ge 1$ are minimizers; in particular the function

$$\widehat{u}_\lambda(t) = \begin{cases} t & \text{if } 0 \le t < 1 \\ \lambda & \text{if } t = 1, \end{cases}$$

which satisfies the boundary conditions, jumps at $t = 1$, but has $u' = 1$ almost everywhere. One can see that in this approximation the effective material does not support any tension.

The next scale is ε, for which we have

$$F^1_\lambda(u) = \begin{cases} 0 & \text{if } u = u_\lambda \\ +\infty & \text{otherwise} \end{cases}$$

if $\lambda \le 1$, and

$$F^1_\lambda(u) = \begin{cases} -J(1)\#(S(u)) & \text{if } u \text{ is piecewise affine and } u' = 1 \text{ a.e.} \\ +\infty & \text{otherwise} \end{cases}$$

for $\lambda > 1$. Here it is understood that u is increasing and satisfies the boundary values (see [13] for more details). In particular we see that $\widehat{u}_\lambda$ above is a minimizer for F^1_λ and $\min F^1_\lambda = -J(1)$.

Again we can compute the approximation of $m_\varepsilon(\lambda) = \min F^\lambda_\varepsilon$ given by the development by Γ-convergence $m^{(0)}(\lambda) + \varepsilon m^{(1)}(\lambda)$, and we get

$$m^{(0)}(\lambda) + \varepsilon m^{(1)}(\lambda) = \begin{cases} J(\lambda) & \text{if } \lambda \le 1 \\ J(1) - \varepsilon J(1) & \text{if } \lambda > 1. \end{cases}$$

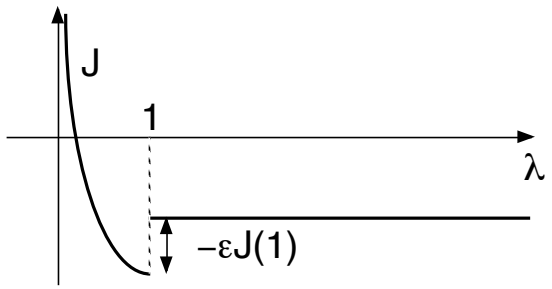

Figure 5: Approximate minimum values by Γ-development.

The plot of the approximate minimum values obtained by Γ-development is given in Fig. 5 together with the effective stress-strain relation. One can see that the first order refined theory again delivers the same physically absurd result that the fracture occurs at zero tension. More formally, as in the previous example we can write

$$\lim_{\lambda \to 1^+} (m_\varepsilon(\lambda) - m^{(0)}(\lambda) - \varepsilon m^{(1)}(\lambda)) = -\varepsilon J(1),$$

from which one can argue that the description given by the Γ-development at scale ε is not accurate close to the point 1. Here again, we deal with the phenomenon of nucleation, this time of a crack. For the newly formed micro-crack the opening is comparable to the small parameter ε, which breaks scale separation legitimizing the Γ-development.

It is again instructive to compare the approximation of the minimal value delivered by the Γ-development and shown in Fig. 5 with the exact solution of the fracture problem at finite ε (see [60, 18] for more details). The local picture near the singular point $\lambda = 1$ is exactly the same as in the case of phase transition problem near the points $\lambda = \pm 1$. Thus, again, at small but finite ε, one can always get sufficiently close to the nucleation point in order to find that the approximation which is supposed to capture the terms of the order ε makes an error which is at least of the same order. The resulting non uniformity of the Γ-development is illustrated in Fig. 6, where we show the ε dependence of the function $m_\varepsilon(\lambda)$ in the case of fracture as $\lambda \to 1$. This picture can obviously be repeated for both singular points $\lambda = \pm 1$ appearing in the case of a phase transition.

Remark 3.3 Recall that the breakdown of uniform convergence (non uniformity) in conventional asymptotic expansions for the functions $f_\varepsilon(x)$ is often due to the formation of the boundary layers in x space. As our examples show, in the case of functionals, the non-uniformity of the Γ-development can also present itself through the formation of boundary layers, but now for the function describing the distribution of the minimal values of the functional (function $m_\varepsilon(\lambda)$) in the space of parameters (λ space).

Based on the analysis of the examples in the previous sections, we can conclude that despite the universal nature of the Γ-development method, it does not offer a fully satisfactory solution to the problem of the asymptotic approximation for variational problems encountered in applications.

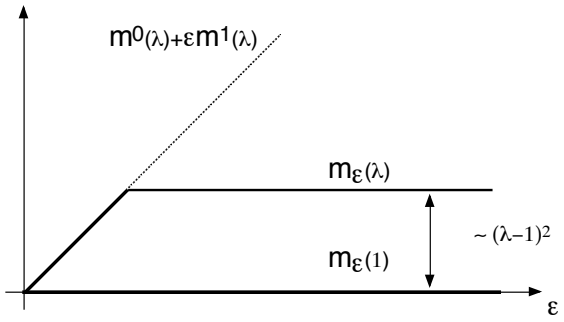

Figure 6: Non-uniformity of the Γ-development associated with the phenomenon of nucleation.

Most disturbing, if the method is applied to 'theories' it may generate an approximation that is not uniform with respect to the parameter leading to rather poor quality of approximation in certain regimes even if it is good in the others. This may be one of the main reasons, why the general paradigm of Γ-development is often in conflict with the approximate methods used by the practitioners.

Our main goal in the rest of the paper is to develop an adequate *vocabulary* aimed at overcoming the above drawbacks of the straightforward Γ-convergence and to find the way of reinterpreting rigorously the 'good' approximate theories used in applications.

4 Γ-equivalence

Our first observation is that equality of Γ-limits gives an equivalence relation between families of energies; i.e., if $\Gamma\text{-}\lim F_\varepsilon = \Gamma\text{-}\lim G_\varepsilon$ then we may say that F_ε is *equivalent* to G_ε. In this way the concept of Γ-limit can be replaced by that of an equivalence class. Note that the domain of equivalent F_ε and G_ε may be completely different.

Remark 4.1 In order not to make the extraction of a Γ-converging sequence a loss of generality, from now on we will tacitly assume that our Γ-limits are computed with respect to a separable metrizable convergence (which is usually the case in applications).

Definition 4.2 *F_ε and G_ε are* equivalent at order ε^α *if there exist translations m_ε such that for all sequences ε_j for which the limits exist we have*

$$\Gamma\text{-}\lim_j \frac{F_{\varepsilon_j} - m_{\varepsilon_j}}{\varepsilon_j^\alpha} = \Gamma\text{-}\lim_j \frac{G_{\varepsilon_j} - m_{\varepsilon_j}}{\varepsilon_j^\alpha},$$

and these limits are non trivial (i.e., they do not take the value $-\infty$ and are not identically $+\infty$).

Observe, that in the spirit of (13) we may write

$$F_\varepsilon \overset{\Gamma}{=} G_\varepsilon + o(\varepsilon^\alpha),$$

even when no Γ-development of either functional exists. If $\alpha = 0$ within this definition, F_ε is equivalent to itself, even when (F_ε) does not converge, and two sequences converging to $+\infty$ are not always equivalent. Note moreover that in the definition above we may always choose $m_\varepsilon = \min F_\varepsilon$.

While we may also define 'equivalence at order $f(\varepsilon)$', where $f(\varepsilon)$ is any function of ε, we limit ourselves to the scaling ε^α for the sake of simplicity.

Theorem 4.3 *Let $H_\varepsilon, H'_\varepsilon$ converge continuously to H (i.e, $H_\varepsilon(x_\varepsilon) \to H(x)$ if $x_\varepsilon \to x$; e.g., $H_\varepsilon = H$ a continuous function) and let F_ε and G_ε be equivalent at order ε^α; then $F_\varepsilon + \varepsilon^\alpha H_\varepsilon$ and $G_\varepsilon + \varepsilon^\alpha H'_\varepsilon$ are equivalent at order ε^α*

Proof. This follows immediately from the definition, and reduces to the compatibility of the Γ-limit with respect to continuous perturbations if $\alpha = 0$.

Theorem 4.4 *Let (F_ε) and (G_ε) be equi-coercive and equivalent at order ε^α; then we have*

$$\inf F_\varepsilon = \inf G_\varepsilon + o(\varepsilon^\alpha).$$

Proof. The functionals

$$F^\alpha_\varepsilon = \frac{F_\varepsilon - m_\varepsilon}{\varepsilon^\alpha}, \qquad G^\alpha_\varepsilon = \frac{G_\varepsilon - m_\varepsilon}{\varepsilon^\alpha}$$

are equicoercive. Given (ε_j) converging to 0, upon extraction of a subsequence, by the compactness of Γ-convergence, $F^\alpha_{\varepsilon_j} \to H$ and $G^\alpha_{\varepsilon_j} \to H$ for some coercive H and hence we have

$$\begin{aligned} \lim_j \frac{\inf F_{\varepsilon_j} - \inf G_{\varepsilon_j}}{\varepsilon^\alpha_j} &= \lim_j \Big(\frac{\inf F_{\varepsilon_j} - m_{\varepsilon_j}}{\varepsilon^\alpha_j} - \frac{\inf G_{\varepsilon_j} - m_{\varepsilon_j}}{\varepsilon^\alpha_j} \Big) \\ &= \min H - \min H = 0. \end{aligned}$$

Hence the thesis follows by the arbitrariness of (ε_j).

Note that, since we do not require the existence of the Γ- limits , minimizers may not converge. However, arguing by subsequences, we still deduce that the cluster points of minimizers of F_ε are the same as those of G_ε. In the particular case when the functional H in the proof above has a unique minimizer, then we may conclude that minimizers of converging subsequences of F_ε and G_ε indexed by the same (ε_j), have the same limits, and in this sense are close.

Below we present several examples of equivalent functionals of different order.

Example 4.5 The first example illustrates the fact that already in the linear case a multiplicity of equivalent functionals can always be found without modifying the principal structure of the original problem. Thus, an equivalent energy at order ε to F_ε in Example 1.9 can be searched among quadratic functionals of the form

$$G_\varepsilon(u) = \int_0^1 (a_\varepsilon |u'|^2 + b_\varepsilon |u|^2)\, dt, \qquad u(0) = 0,\ u(1) = 1, \tag{32}$$

The condition of equivalence at order 1 gives $a_\varepsilon = 1 + o(1)$ and $b_\varepsilon = o(1)$ and the condition of equivalence at order ε gives

$$3a_\varepsilon + b_\varepsilon = 3 + \varepsilon + o(\varepsilon).$$

One can choose for example either

$$G_\varepsilon(u) = \Big(1 + \frac{\varepsilon}{3}\Big) \int_0^1 |u'|^2 \, dt, \qquad u(0) = 0, \ u(1) = 1, \tag{33}$$

or

$$G_\varepsilon(u) = \int_0^1 ((1+\varepsilon)|u'|^2 - 2\varepsilon|u|^2) \, dt, \qquad u(0) = 0, \ u(1) = 1. \tag{34}$$

Example 4.6 The next example shows that also in nonlinear case, equivalent problems may have the same general form and differ only in details. For instance, let

$$F_\varepsilon(u) = \int_\Omega \Big(W(u) + C\varepsilon^2 |\nabla u|^2\Big) \, dx$$

and

$$\widetilde{F}_\varepsilon(u) = \int_\Omega \Big(\widetilde{W}(u) + \widetilde{C}\varepsilon^2 |\nabla u|^2\Big) \, dx$$

be two energies as in Example 1.10 with W and $\widetilde{W}$ two double-well potentials with wells in ± 1. Then $\widetilde{F}_\varepsilon$ and F_ε are equivalent at order ε if and only if $\min W = \min \widetilde{W}$ and

$$\sqrt{C} \int_{-1}^1 \sqrt{W(s) - \min W} \, ds = \sqrt{\widetilde{C}} \int_{-1}^1 \sqrt{\widetilde{W}(s) - \min \widetilde{W}} \, ds.$$

In this case, by Example 1.10 they are both equivalent to

$$\mathcal{F}(u) = |\Omega| \min W + \varepsilon F^{(1)}(u),$$

with $F^{(1)}$ given by (22). The conditions of equivalence at order 1 are different if we take the weak or the strong L^1-topology. In the first case the condition is $W^{**} = (\widetilde{W})^{**}$; in the second one $W = \widetilde{W}$.

The next two examples illustrate the fact that the equivalent theories may also have a rather different structure.

Example 4.7 Consider the discrete model (9) with the double-well energy

$$W(z) = \min\{(z-1)^2, (z+1)^2\},$$

and $J < 1$, so that F_ε are equi-coercive. It is easy to see that $\overline{W}$ from Example 1.3 is itself a double-well potential, with symmetric wells that we denote by $\pm a$. Let $m^0 = \min F^{(0)} = \min \overline{W} = \overline{W}(a)$.

Remark 4.8 In the case of a general W the 'effective' potential $\overline{W}$ is given by the more complex formula

$$\overline{W}(z) = \frac{1}{2} \inf\{W(z_1) + W(z_2) : z_1 + z_2 = 2z\} - 2Jz^2$$

(see [20, 13]) highlighting oscillations at microscopic scale. Some equivalent energies in this case can be deduced from the analysis in [15].

We may apply to F_ε the first-order analysis of [15] obtaining that the next meaningful scale is ε, and that the next term in the development is

$$F^{(1)}(u) = K_W \#(S(u)), \qquad u \in \{\pm a\},$$

where

$$\begin{aligned} K_W \;=\; \inf\Big\{ \sum_{i=-\infty}^{+\infty} \Big(\frac{1}{2}\big(W(v_i) + W(v_{i-1}) - 2Jv_iv_{i+1}\big) - m^0\Big) : \\ v : \mathbb{Z} \to \mathbb{R},\ v_i = -a \text{ if } i \le -N,\ v_i = a \text{ if } i \ge N,\ N \in \mathbb{N}\Big\} \end{aligned}$$

The value K_W represents the energy of an interface by means of a 'discrete optimal-profile problem' between the two constant (minimal) states $\pm a$. Note that for fixed N the terms in the sum in the minimum problem are 0 for $i \ge N$ and $i < -N-1$; moreover the function $v_i = a \operatorname{sign} i$ is an admissible test function for all $N \ge 1$, from which we obtain $K_W \le 4Ja^2$.

This development gives the equivalent energy at scale ε

$$G_\varepsilon(u) = \begin{cases} m^0 + \varepsilon K_W \#(S(u)) & \text{if } u \in \{\pm a\} \\ +\infty & \text{otherwise.} \end{cases}$$

A comparison with the gradient theory of phase transitions shows that the functional

$$\widetilde{G}_\varepsilon(u) = \int_0^1 (\overline{W}(u) + \varepsilon^2 C|u'|^2)\, dt, \tag{35}$$

is also equivalent to F_ε at order ε, provided that C is chosen such that

$$K_W = 2\sqrt{C} \int_{-a}^{a} \sqrt{\overline{W}(s) - \min \overline{W}}\, ds.$$

The minimizers of the approximate energy (35) agree with the exact solution of the original problem which is known in the explicit form for finite ε [62]. It is interesting that another approximation at order ε with the same structure as in (35) but different $\overline{W}(u)$ and C can be formally obtained by a point-wise limit [62], however it appears to be working only in the limited domain of parameters J and its rigorous status remains to be clarified (see [9] for more general results in this direction).

The next example shows that even the number of variables in equivalent theories with otherwise similar structure may be different.

Example 4.9 Consider a functional which one encounters in the Timoshenko theory of beams

$$F_\varepsilon(u, \phi) = \int_0^l \Big(E|\phi'|^2 + \frac{H}{\varepsilon^2}(\phi - u')^2\Big)\, dt.$$

Below we prove (see also [6]) that the corresponding minimization problem is Γ-equivalent at order 1 to the more conventional Euler-Bernoulli bending problem characterized by the functional

$$G(u) = E \int_0^l |u''|^2\, dt.$$

Here we have assumed the identification of G with

$$G(u,\phi) = \begin{cases} E\displaystyle\int_0^l |u''|^2\,dt & \text{if } \phi = u' \\ +\infty & \text{otherwise.} \end{cases}$$

To justify the claim we have to show that $\Gamma\text{-}\lim_{\varepsilon\to 0^+} F_\varepsilon = G$. It suffices to prove that if $u_\varepsilon, \phi_\varepsilon$ are such that $\sup F_\varepsilon(u_\varepsilon,\phi_\varepsilon) < +\infty$, then, up to subsequences and translations by constants (for ϕ_ε) and affine functions (for u_ε), we have $\phi_\varepsilon \rightharpoonup \phi$ and $u_\varepsilon \rightharpoonup u$ weakly in $H^1(0,l)$, with $u \in H^2(0,l)$ and $u' = \phi$, and

$$\liminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon,\phi_\varepsilon) \geq E\int_0^l |u''|^2\,dt.$$

Now, from $\sup \int_0^l |\phi_\varepsilon'|^2\,dt < +\infty$ we deduce $\phi_\varepsilon \rightharpoonup \phi$, while from

$$\int_0^l |u_\varepsilon'|^2\,dt \leq C\int_0^l (|\phi_\varepsilon|^2 + |u_\varepsilon' - \phi_\varepsilon|^2)\,dt \leq C(1+\varepsilon^2)$$

we deduce that $u_\varepsilon \rightharpoonup u$. By the lower semi-continuity of the norm then

$$\int_0^l |u' - \phi|^2\,dt \leq \liminf_{\varepsilon\to 0}\int_0^l |u_\varepsilon' - \phi_\varepsilon|^2\,dt = 0,$$

so that $u' = \phi$ and $u \in H^2(0,l)$. Finally,

$$E\int_0^l |u''|^2\,dt = E\int_0^l |\phi'|^2\,dt \leq \liminf_{\varepsilon\to 0} E\int_0^l |\phi'|^2\,dt \leq \liminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon,\phi_\varepsilon).$$

The obtained result is stable with respect to the addition of the boundary conditions, prescribing, for instance, displacements (hinging) and rotations (clamping) at the end points.

5 Systematic methods

Although a sufficiently general method of generating the whole class of Γ- equivalent functional does not exist, we discuss in this Section three rather systematic approaches of producing equivalent functionals.

5.1 'Taylor' expansion

First we observe that if a Γ-development exists then it is easy to construct an equivalent family as follows.

Theorem 5.1 *Let F_ε and m_ε^α be such that the limit*

$$F^{(\alpha)} = \Gamma\text{-}\lim_{\varepsilon\to 0^+} \frac{F_\varepsilon - m_\varepsilon^\alpha}{\varepsilon^\alpha}$$

exists and is not trivial. Then (F_ε) is equivalent to

$$G_\varepsilon(u) := m_\varepsilon^\alpha + \varepsilon^\alpha F^{(\alpha)}(u)$$

at order ε^α. In particular, if a Γ-development $F^{(0)} + \varepsilon^{\beta_1} F^{(1)} + \cdots + \varepsilon^{\beta_M} F^{(M)} + \varepsilon^\alpha F^{(\alpha)}$ exists, with $0 = \beta_0 < \cdots < \beta_M < \alpha$, then we may take

$$G_\varepsilon(u) := \sum_{k=0}^{M} \varepsilon^{\beta_k} m^{(k)} + \varepsilon^\alpha F^{(\alpha)}(u),$$

with $m^{(k)} = \min F^{(k)}$.

Proof. It suffices to apply Definition 4.2 above, with $m_\varepsilon^\alpha = \sum_{k=0}^{M} \varepsilon^{\beta_k} m^{(k)}$ in the case of a Γ-development.

Remark 5.2 It must be noted that in the case of a Γ-development only the values $m^{(k)} = \min F^{(k)}$ take part in the definition of G_ε and not the actual form of each $F^{(k)}$. In particular, energies with different developments may be equivalent at scale ε^α.

Example 5.3 An equivalent energy at order ε to F_ε in Example 1.9 can be obtained directly from the development as

$$G_\varepsilon(u) = \begin{cases} 1 + \frac{\varepsilon}{3} & \text{if } u = t \\ +\infty & \text{otherwise.} \end{cases} \tag{36}$$

Example 5.4 An equivalent energy at order 1 and ε in Example 1.12 is given by

$$G_\varepsilon(u) = \Big(\underline{a} + \varepsilon \underline{a}^2 b\Big(\frac{1}{\varepsilon}\Big)\Big) \int_0^1 |u'|^2 \, dt, \qquad u(0) = 0,\ u(1) = 1, \tag{37}$$

where

$$b(t) = \int_0^t \Big(\frac{1}{\underline{a}} - \frac{1}{a(s)}\Big)\, ds \tag{38}$$

(note that b is 1-periodic, so that we may remove the integer part in the lower extreme of the integral in (26)). Also, note that $\Big(\int_0^1 \frac{1}{a(s/\varepsilon_j)}\, ds\Big)^{-1}$ can be developed in a power series in terms of ε, $\underline{a}$ and b, obtaining equivalent energies up to order ε^k for all $k \in \mathbb{N}$ of the form

$$G_\varepsilon(u) = \underline{a}\Big(1 + \varepsilon \underline{a} b\Big(\frac{1}{\varepsilon}\Big) + \cdots + \varepsilon^k \underline{a}^k \Big(b\Big(\frac{1}{\varepsilon}\Big)\Big)^k\Big) \int_0^1 |u'|^2 \, dt, \qquad u(0) = 0,\ u(1) = 1.$$

We reiterate here that despite its systematic nature, the straightforward Γ-development suffers from the rather poor representation of the minimizers and potential non-uniformity in the representation of the minimal values.

5.2 'Stretched' variables and Ansätze

Another scheme of constructing equivalent energies, usually much more relevant for applications, is by the computation of the Γ-development of functionals obtained through a special change of variables (ansatz) which anticipates the structure of the minimizer.

To illustrate the subsequent formal development, we first consider the simplest case where one seeks to construct the lowest order of approximation to a family of functionals $F_\varepsilon(u)$. Suppose that the Γ-limit of $F_\varepsilon(u)$ is either trivial or does not exist. Suppose further that one can find a new variable v, such that

$$u = \frac{1}{\phi(\varepsilon)}\Phi(v) \tag{39}$$

and that a re-scaled functional

$$\Psi_\varepsilon(v) = \psi(\varepsilon)F_\varepsilon\Big(\frac{\Phi(v)}{\phi(\varepsilon)}\Big) \tag{40}$$

has a nontrivial Γ-limit. Then the functional Ψ_ε expressed in terms of u will deliver the desired equivalent theory. The nontrivial limits of the type (40) reveal the so called self-similar structure of the singularity at $\varepsilon = 0$.

A more general formal procedure goes as follows:

1. Find a bijective change of variables $\Phi_\varepsilon : V \to X$ and define $H_\varepsilon(v) = F_\varepsilon(\Phi_\varepsilon(v))$;
2. Compute the Γ-limit $H : V \to X$ of H_ε;
3. Define $G_\varepsilon(u) = H(\Phi_\varepsilon^{-1}(u))$ and prove that G_ε is equivalent to F_ε.

We remark that sometimes the last passage is not straightforward since the domain of H may be different from that of H_ε. It becomes feasible, however, if other invertible changes of variables exist $\Theta_\varepsilon : V \to V$ carrying the domain of H into the domain of H_ε. In this case we need to put $G_\varepsilon(u) = H(\Theta_\varepsilon^{-1}(\Phi_\varepsilon^{-1}(u)))$.

In many cases, a possible change of variable is of the type $u = u_0 + \varepsilon^\gamma v$ where u_0 is a minimizer for $F^{(0)}$; i.e., $\Phi_\varepsilon(v) = u_0 + \varepsilon^\gamma v$ in the remark above. It is clear that finding the relevant scaling (stretching) requires a deep understanding of the solution to the original problem and can not be fully formalized even though one can of course try to make an exhaustive search through the particular classes of ansatzes. The well know examples of the use of stretched variables can be found in various derivations for the theories of plates and rods (see, for instance, [6, 41, 52, 35]). Another nontrivial applications of the method can be found in the higher-order approximations for composites [7, 57].

We illustrate this approach by the following simple examples.

Example 5.5 Consider again Example 1.9. Our goal is to obtain an equivalent energy at scale ε^2 to F_ε. In this case the locked minimizer is $u_0(t) = t$, so we may choose a change of variables of the form $\Phi_\varepsilon(v) = u_0 + \varepsilon^\gamma v$ with $v(0) = v(1) = 0$, and compute the Γ-limit of

$$H_\varepsilon(v) = \frac{1}{\varepsilon^2}\Big(F_\varepsilon(\Phi_\varepsilon(v)) - 1 - \frac{1}{3}\varepsilon\Big). \tag{41}$$

If $\gamma < 1$ this limit is identically 0, while if $\gamma > 1$ it is trivial. We then choose and the change of variables $u = u_0 + \varepsilon v$, for which the Γ-limit of H_ε is

$$H(v) = \int_0^1 (|v'|^2 + 2tv)\,dt.$$

By inverting the change of variables $v = (u-t)/\varepsilon$ we get the integral $\frac{1}{\varepsilon^2}\int_0^1 (|1-u'|^2 + 2\varepsilon t(u-t))\,dt$, and, after integrating by parts using the boundary condition, an equivalent energy at scale ε^2 to F_ε of the form

$$G_\varepsilon(u) = \int_0^1 (|u'|^2 + \varepsilon(|u|^2 - |u-t|^2)\,dt.$$

Example 5.6 The same scheme can be applied for the linearization of finite elasticity, where the starting energy is of the form

$$F_\varepsilon(u) = \int_\Omega f(\nabla u)\,dx, \qquad u(x) = x + \varepsilon\phi(x) \text{ on } \partial\Omega$$

with f a hyperelastic energy density with its minimum on $SO(3)$. The Γ-limits of higher order are locked on the identity $u_0(x) = x$. A change of variables $\Phi(v) = u_0 + \varepsilon v$ allows to express a functional equivalent to F_ε at order ε^2 in terms of the functionals of linearized elasticity. For details we refer to [28].

A nontrivial application of the method will be given in the Example 8.5.

5.3 Matched expansions

Another rather general approximation scheme assumes that the original functional is restricted to the part of the domain (say, around the singularities in the original problem) while the truncated Γ-development of a finite order is operative in its complement (say, far away from the singularities). In this case the full description corresponding to finite ε is preserved in the domain which is shrinking as $\varepsilon \to 0$, while an approximate description is used in the domain which is enlarging as the small parameter diminishes. The matching of the two pictures is achieved automatically with the variational principle delivering the 'natural' matching conditions . This approach is often used for the the fully discrete resolution of the cores of the defects within continuum elasticity (quasi-continuum method, [59])

This method is illustrated by the following simplest examples. The first example deals with the boundary layers in homogenization.

Example 5.7 Consider again Example 1.12. It is not difficult to see that an equivalent theory of arbitrary order can be obtained if we take

$$\widetilde{G}_\varepsilon(u) = \int_0^1 a_\varepsilon(t)|u'|^2\,dt, \qquad u(0) = 0,\ u(1) = 1$$

where

$$a_\varepsilon(t) = \begin{cases} \underline{a} & \text{if } t'_\varepsilon \le t \le t''_\varepsilon \\ a\Big(\dfrac{t}{\varepsilon}\Big) & \text{otherwise,} \end{cases}$$

where $0 \le t'_\varepsilon \le t''_\varepsilon \le 1$ and $t''_\varepsilon - t'_\varepsilon \in \varepsilon\mathbb{N}$. To prove this we need to observe that under the imposed conditions the minimum values of the approximate and original functional simply coincide.

The second example, instead of external boundary layers in a linear setting is dealing with internal boundary layers emerging due to the non-convexity of the energy.

Example 5.8 Consider the energy in Example 4.7 with conditions $\sum_i u_i = 0$ and set for simplicity $u_{N+1} = u_N$. In this case a minimizer of the Γ-limit at order ε has a unique phase transition at the point $1/2$. To resolve the fine features of the transition we can use the discrete formulation close to $1/2$, while using the continuous description 'far' from $1/2$. This can be done by considering the equivalent functional

$$G_\varepsilon(u) = \int_{(0,1)\setminus I_\varepsilon} (\overline{W}(u) + C\varepsilon^2|u'|^2)\,dt + \sum_{i=-N_\varepsilon}^{N_\varepsilon-1} \varepsilon(W(u_i) - Ju_iu_{i+1}),$$

defined on H^1 functions coinciding in I_ε with their interpolations on the lattice $\varepsilon\mathbb{Z}$, where we have chosen $N_\varepsilon \in \mathbb{N}$ with $N_\varepsilon \to +\infty$ and $\varepsilon N_\varepsilon \to 0$, and set $I_\varepsilon = \{|t - 1/2| \le \varepsilon N_\varepsilon\}$.

The main difficulty in applying this method is that the exact location of the domain where the full description should be used, is unknown a priori. This poses an additional 'localization' problem. An interesting example of the matching method involving first the localization of the fully resolved domain in the case of 1D fracture can be found in [8].

6 Uniform Γ-equivalence

In this Section we present a more systematic analysis of the parameterized families of functionals ('theories'). We begin with a definition of equivalence for parameterized functionals.

Definition 6.1 *Two families of functionals F_ε^λ and G_ε^λ are* equivalent *at order ε^α at $\lambda_0 \in \Lambda$ if $F_\varepsilon^{\lambda_0}$ and $G_\varepsilon^{\lambda_0}$ are* equivalent *at order ε^α*

The definition is illustrated by an example dealing with a size effect in composites.

Example 6.2 Let $\lambda > 0$ and consider the one-dimensional homogenization problem of the form

$$\min\Big\{\frac{1}{\lambda}\int_0^\lambda a\Big(\frac{x}{\varepsilon}\Big)|u'|^2\,dx : u(0) = 0, u(\lambda) = \lambda\Big\}, \tag{42}$$

where the function a is 1-periodic, bounded and strictly positive. It is convenient to rewrite these problems as minimum problems with the energies

$$F_\varepsilon^\lambda(u) = \int_0^1 a\Big(\frac{\lambda x}{\varepsilon}\Big)|u'|^2\,dx, \qquad u(0) = 0,\ u(1) = 1. \tag{43}$$

As shown in Example 1.12 the Γ-limit of F_ε^λ is

$$F_\lambda^0(u) = \underline{a}\int_0^1 |u'|^2\,dx, \qquad u(0) = 0,\ u(1) = 1, \tag{44}$$

independent of λ, while the Γ-development does not exist at scale ε. An equivalent parameterized functional for all $\lambda > 0$ at scale 1 and ε is

$$G_\varepsilon^\lambda(u) = \Big(\underline{a} + \varepsilon\underline{a}^2 b\Big(\frac{\lambda}{\varepsilon}\Big)\Big)\int_0^1 |u'|^2\,dt, \qquad u(0) = 0,\ u(1) = 1. \tag{45}$$

In unscaled variables we obtain

$$F_\lambda^0(u) = \frac{\underline{a}}{\lambda}\int_0^\lambda |u'|^2\,dx, \qquad u(0)=0,\ u(\lambda)=\lambda, \tag{46}$$

and

$$G_\varepsilon^\lambda(u) = \Big(\frac{\underline{a}}{\lambda} + \varepsilon\frac{\underline{a}^2}{\lambda} b\Big(\frac{\lambda}{\varepsilon}\Big)\Big)\int_0^\lambda |u'|^2\,dt, \qquad u(0)=0,\ u(\lambda)=\lambda. \tag{47}$$

Note that for all ε fixed we have

$$\begin{aligned}\lim_{\lambda\to 0^+}\min F_\varepsilon^\lambda &= \min\Big\{\int_0^1 a(0)|u'|^2\,dt : u(0)=0, u(1)=1\Big\}\\ &= a(0) \neq \underline{a} = \lim_{\lambda\to 0^+}\min F_\lambda^0,\end{aligned}$$

and

$$\lim_{\varepsilon\to 0}\limsup_{\lambda\to 0}\big|\underline{a} - \min G_\varepsilon^\lambda\big| = 0.$$

As we see the equivalence is not uniform because in the case of sufficiently small bodies homogenization starts to interfere with the boundaries and in the limit $\lambda\to 0$ the boundary layers at the external boundaries dominate the effective response of the body. This phenomenon in the case of discrete lattices was studied in more detail in [24, 21].

In view of the previous example it is of interest to upgrade Definition 6.1 to a uniform equivalence of parameterized functionals.

Definition 6.3 *Two families of functionals F_ε^λ and G_ε^λ are* (uniformly) equivalent *at order ε^α at $\lambda_0\in\Lambda$ if there exist translations m_ε^λ such that for all $\varepsilon_j\to 0$, $\lambda_0\in\Lambda$ and all $\lambda_j\to\lambda_0$ we have, upon extraction of subsequences,*

$$\Gamma\text{-}\lim_{\varepsilon\to 0}\frac{F_{\varepsilon_j}^{\lambda_j} - m_{\varepsilon_j}^{\lambda_j}}{\varepsilon^\alpha} = \Gamma\text{-}\lim_{\varepsilon\to 0}\frac{G_{\varepsilon_j}^{\lambda_j} - m_{\varepsilon_j}^{\lambda_j}}{\varepsilon^\alpha}, \tag{48}$$

and these Γ-limits are non trivial.

We say that F_ε^λ and G_ε^λ are uniformly equivalent at order ε^α on Λ if they are uniformly equivalent at all $\lambda_0\in\Lambda$.

Note again that the space on which (uniformly) equivalent functionals may be different, and may vary with ε and λ.

Theorem 6.4 *Let Λ be sequentially compact, and let F_ε^λ and G_ε^λ be uniformly coercive and uniformly equivalent at order ε^α on Λ. Then*

$$\sup_{\lambda\in\Lambda}\big|\inf G_\varepsilon^\lambda - \inf F_\varepsilon^\lambda\big| = o(\varepsilon^\alpha). \tag{49}$$

Proof. By contradiction, suppose that a sequence (λ_j) exists such that

$$|\inf G_{\varepsilon_j}^{\lambda_j} - \inf F_{\varepsilon_j}^{\lambda_j}| \geq C{\varepsilon_j}^{\alpha}.$$

By the compactness of Λ we may suppose that $\lambda_j \to \lambda_0$, and by the definition above that (48) holds. By coerciveness we then obtain that

$$\lim_{\varepsilon\to 0} \frac{\inf F_{\varepsilon_j}^{\lambda_j} - m_{\varepsilon_j}^{\lambda_j}}{\varepsilon^{\alpha}} = \lim_{\varepsilon\to 0} \frac{\inf G_{\varepsilon_j}^{\lambda_j} - m_{\varepsilon_j}^{\lambda_j}}{\varepsilon^{\alpha}},$$

and that the limit is finite. From this we immediately obtain a contradiction.

To give another illustration of the notion of uniform Γ-equivalence we revisit the problem of discretization for non-convex energies (see our Example 1.2).

Example 6.5 For ε such that $N = \frac{1}{\varepsilon} \in \mathbb{N}$, and for $\lambda \in \mathbb{R}$ we consider again the parameterized functional

$$F_{\varepsilon}^{\lambda}(u) = \sum_{i=1}^{N} \varepsilon\, W(u_i) \qquad \sum_{i=0}^{N} \varepsilon\, u_i = \lambda \tag{50}$$

where $u : \{1, \dots, N\} \to \mathbb{R}$. We recall that this problem can be viewed as the 'naive' discretization of $\int_0^1 W(u)\, dt$ with the same boundary conditions, after the identification of u with a piecewise-affine function on $(0, 1)$ and $u_i = u(i/N)$.

If W is strictly convex then 'naive' approach works and $F_{\varepsilon}^{\lambda}$ is uniformly equivalent at all orders to the parameterized continuous functional

$$G_{\varepsilon}^{\lambda}(u) = G^{\lambda}(u) = \int_0^1 W(u)\, dt \qquad \int_0^1 u(s)\, ds = \lambda \tag{51}$$

(independent of ε).

If W is not convex then the situation is different and $F_{\varepsilon}^{\lambda}$ and $G_{\varepsilon}^{\lambda}$ are not equivalent at order ε^2, as we have shown in Example 1.13 for $\lambda = 0$. In fact none of the points in the interval $\lambda \in (-1, 1)$ is a point of uniformity. To construct a uniformly equivalent functional at order ε^2 in the case $W(z) = \min\{(z-1)^2, (z+1)^2\}$ we can, for instance, take

$$W_{\varepsilon}(z) = \min\Big\{\Big(z - 1 + \frac{2i}{N}\Big)^2 : \ i = 0, \dots, N\Big\}, \tag{52}$$

and define

$$G_{\varepsilon}^{\lambda}(u) = \int_0^1 \Big(W_{\varepsilon}(u) + |u'|^2\Big)\, dt \qquad \int_0^1 u(s)\, ds = \lambda. \tag{53}$$

This construction is in some sense trivial, because it presupposes the complete knowledge of the minimizer for the original discrete problem at finite ε. The development of a nontrivial continuum model with the same degree of approximation poses considerable challenge because in the interval $\lambda \in (-1, 1)$ the system behaves as 'strongly discrete'. Indeed in this regime the individual elements transform independently, one after another, so even the weakest forms of the Cauchy-Born rule [32], which is usually the basis of a continuum approximation, can not be hoped to be true. In the more mathematical language, the difficulty here arises from the fact that we are dealing with a case when there is a whole interval of singular points.

To summarize, as we have seen in the preceding examples, for parameterized families of minimization problems one can distinguish (in the space of parameters) the regular points, where the approximation is uniform, and the singular points, where it is not. Below we introduce the formal definitions and treat the regular and singular cases separately.

7 Regular points

We begin with the formal definition of a regular point in the $\lambda - \varepsilon$ space.

Definition 7.1 *Let (F_ε^λ) be a family of parameterized functionals. A point $\lambda_0 \in \Lambda$ is a* regular point *for (F_ε^λ) at scale ε^α if for all $\varepsilon_j \to 0$ and sequences m_j, $\lambda_j \to \lambda_0, \lambda_j' \to \lambda_0$ we have, upon extraction of a subsequence, that*

$$\Gamma\text{-}\lim_j \frac{F_{\varepsilon_j}^{\lambda_j} - m_j}{\varepsilon_j^\alpha} = \Gamma\text{-}\lim_j \frac{F_{\varepsilon_j}^{\lambda_j'} - m_j}{\varepsilon_j^\alpha}. \tag{54}$$

Remark 7.2 We may take $m_j = \inf F_{\varepsilon_j}^{\lambda_j}$ so that for this sequence (m_j) the first limit in (54) is non trivial.

Note that we may take $\lambda_j' = \lambda_0$ so that for all $\lambda_j \to \lambda_0$

$$\Gamma\text{-}\lim_j \frac{F_{\varepsilon_j}^{\lambda_j} - m_j}{\varepsilon_j^\alpha} = \Gamma\text{-}\lim_j \frac{F_{\varepsilon_j}^{\lambda_0} - m_j}{\varepsilon_j^\alpha}. \tag{55}$$

Theorem 7.3 *Let (F_ε^λ) be an equi-coercive parameterized family such that all λ are regular points at scale ε^α, and let*

$$F_\lambda^\alpha = \Gamma\text{-}\lim_{\varepsilon \to 0} \frac{F_\varepsilon^\lambda - m_\varepsilon^\lambda}{\varepsilon^\alpha} \tag{56}$$

exist and be non trivial, where

$$m_\varepsilon^\lambda = \inf F_\varepsilon^\lambda.$$

Then $\lambda \mapsto F_\lambda^\alpha$ is continuous with respect to Γ-convergence

Proof. Since we suppose that our energies are equi- coercive, the topology of Γ-convergence is metrizable and compact (see [27]). Let $\lambda_k \to \lambda_0$ be such that $F_{\lambda_k}^\alpha$ Γ-converge to some F. By a diagonal argument we can find a sequence ε_k such that

$$F = \Gamma\text{-}\lim_{\varepsilon \to 0} \frac{F_{\varepsilon_k}^{\lambda_k} - m_{\varepsilon_k}^{\lambda_k}}{{\varepsilon_k}^\alpha}. \tag{57}$$

We then have

$$\begin{aligned} F &= \Gamma\text{-}\lim_k \frac{F_{\varepsilon_k}^{\lambda_k} - m_{\varepsilon_k}^{\lambda_k}}{{\varepsilon_k}^\alpha} \\ &= \Gamma\text{-}\lim_k \frac{F_{\varepsilon_k}^{\lambda} - m_{\varepsilon_k}^{\lambda_k}}{{\varepsilon_k}^\alpha} = F_{\lambda_0}^\alpha + \lim_k \frac{m_{\varepsilon_k}^{\lambda} - m_{\varepsilon_k}^{\lambda_k}}{{\varepsilon_k}^\alpha}. \end{aligned}$$

Hence, F and $F^\alpha_{\lambda_0}$ differ by a constant. Note however that by the property of convergence of minima and the renormalization by m^λ_ε we have $\min F = \min F^\alpha_{\lambda_0} = 0$, so that $F = F^\alpha_{\lambda_0}$.

Theorem 7.4 *Let (F^λ_ε) be a equi-coercive parameterized family such that all λ are regular points at scale ε^α. If for all λ fixed a development of the form*

$$F^\lambda_\varepsilon = \varepsilon^{\beta_0} F^{(0)}_\lambda + \varepsilon^{\beta_1} F^{(1)}_\lambda + \cdots + \varepsilon^{\beta_M} F^{(M)}_\lambda + \varepsilon^\alpha F^{(\alpha)}_\lambda + o(\varepsilon^\alpha) \tag{58}$$

exists with $\beta_0 < \ldots < \beta_M < \alpha$, and we set

$$m^\alpha_\varepsilon(\lambda) = \sum_{k=0}^{M} \varepsilon^{\beta_k} \min F^{(k)}_\lambda, \tag{59}$$

then F^λ_ε is uniformly equivalent to the family

$$G^\lambda_\varepsilon(u) = m^\alpha_\varepsilon(\lambda) + \varepsilon^\alpha F^{(\alpha)}_\lambda(u) \tag{60}$$

at scale ε^α.

Proof. This immediately follows from Theorem 7.3 above, setting $m^\lambda_\varepsilon = m^\alpha_\varepsilon(\lambda)$.

Our next example illustrates the effect of the *regular* interference between the size of the support of a distributed force and the scale of homogenization.

Example 7.5 Consider the one-dimensional homogenization problem

$$\min\Big\{ \int_{-1}^{1} a\Big(\frac{t}{\varepsilon}\Big)\, dt + \frac{1}{2\lambda} \int_{-\lambda}^{\lambda} u\, dt : u(\pm 1) = 0 \Big\}.$$

Here $\lambda \geq 0$ represents the size of the region where forces are applied; in the singular case $\lambda = 0$, which will be considered in more detailed in the next Section, the second integral is replaced by $u(0)$. Consider the related family of parameterized functionals

$$F^\lambda_\varepsilon(u) = \int_{-1}^{1} a\Big(\frac{t}{\varepsilon}\Big) |u'|^2\, dt + \frac{1}{2\lambda} \int_{-\lambda}^{\lambda} u\, dt, \qquad u(\pm 1) = 0, \tag{61}$$

if $\lambda > 0$, and

$$F^0_\varepsilon(u) = \int_{-1}^{1} a\Big(\frac{t}{\varepsilon}\Big) |u'|^2\, dt + u(0), \qquad u(\pm 1) = 0, \tag{62}$$

if $\lambda = 0$. Since the second terms are continuous perturbations, the corresponding Γ-limits are simply

$$F^0_\lambda(u) = \underline{a} \int_{-1}^{1} |u'|^2\, dt + \frac{1}{2\lambda} \int_{-\lambda}^{\lambda} u\, dt, \qquad u(\pm 1) = 0, \tag{63}$$

if $\lambda > 0$, and

$$F^0_0(u) = \underline{a} \int_{-1}^{1} |u'|^2\, dt + u(0), \qquad u(\pm 1) = 0, \tag{64}$$

if $\lambda = 0$. Note that F^0_λ has a unique minimizer for all $\lambda \geq 0$, which we denote by u^λ_0, giving a minimum value $m^{(0)}(\lambda)$. A straightforward calculation shows that

$$m^{(0)}(\lambda) = \frac{1}{8\underline{a}}\Big(-1 + \frac{2}{3}\lambda\Big)$$

Suppose that a is even, so that by a reflection argument also the minimizer u^λ_ε of F^λ_ε is even. In this case the computation are easily carried over. We have $(u^\lambda_\varepsilon)'(0) = 0$, while u^λ_ε is characterized by the Euler-Lagrange equation

$$\begin{cases} a\Big(\dfrac{t}{\varepsilon}\Big)u' = \dfrac{t}{4\lambda} & \text{for } 0 < t < \lambda \\ a\Big(\dfrac{t}{\varepsilon}\Big)u' = \dfrac{1}{4} & \text{for } \lambda < t < 1, \end{cases}$$

and the only boundary condition $u^\lambda_\varepsilon(1) = 0$. After computing this solution, we obtain

$$m^\varepsilon(\lambda) := \min F^\lambda_\varepsilon = -\frac{1}{8\lambda^2}\int_0^1 \frac{\min\{s^2, \lambda^2\}}{a(s/\varepsilon)}\, ds.$$

At scale ε the Γ-limit is

$$F^1_\lambda(u) = \Gamma\text{-}\lim_j \frac{F^\lambda_{\varepsilon_j}(u) - m^{(0)}(\lambda)}{\varepsilon_j} = \begin{cases} K & \text{if } u = u^\lambda_0 \\ +\infty & \text{otherwise,} \end{cases}$$

where $K = -\frac{1}{8}\lim_j b(1/\varepsilon_j)$ (b as in (38)). Since we want to analyze the dependence on λ we fix a sequence ε_j such that this limit K exists. In this case all points $\lambda > 0$ are easily seen to be regular.

As we show below the point $\lambda = 0$, on the contrary, is not regular and its appropriate neighborhood represents the domain of the size effect. This singular case will be treated in Example 7.5.

Now, we formulate a simple necessary condition of regularity.

Theorem 7.6 *If (F^λ_ε) is a equi-coercive parameterized family which for each λ admits a Γ-development of the form (58), is regular for all scales ε^β with $\beta < \alpha$ and is regular at λ_0 at scale ε^α, then $\lambda \mapsto \min F^{(M)}_\lambda$ is continuous at λ_0.*

Proof. The proof can be performed by induction. It suffices to check the case $M = 0$ and $\beta_0 = 0$, in which case the thesis is that $\lambda \mapsto \min F_\lambda$ is continuous at λ_0, where $F_\lambda = \Gamma\text{-}\lim_{\varepsilon\to 0^+} F^\lambda_\varepsilon$. Indeed, if $\lambda_k \to \lambda_0$ then $\min F_{\lambda_k} = \lim_{\varepsilon\to 0^+} \min F^{\lambda_k}_\varepsilon$. By a diagonal argument we may find $\varepsilon_k \to 0$ such that $\lim_k \min F_{\lambda_k} = \lim_k \min F^{\lambda_k}_{\varepsilon_k}$. By the regularity at λ_0 we have that the Γ-limit of $F^{\lambda_k}_{\varepsilon_k}$ is the same as that of $F^\lambda_{\varepsilon_k}$; i.e., F_λ. By the convergence of minima we then obtain $\min F_\lambda = \lim_k \min F_{\lambda_k}$.

In the next example we list several cases studied above where the necessary condition of regularity suggested by Theorem 7.6 fails.

Example 7.7 If F^λ_ε is as in Example 3.1 then all λ different from ± 1 are regular points. From the study of minimum problems for the limit as summarized in Fig. 5, by Theorem 7.6 we deduce that ± 1 are not regular points for F^λ_ε at scale ε. Similarly, for Example 3.2 we deduce that 1 is not a regular point, and for Example 6.2 that 0 is not a regular point.

8 Singular points and 'tables' of Γ-limits

We are now in the position to give the formal definition of a singular point.

Definition 8.1 *Let F_ε^λ be a family of parameterized functionals, with $\lambda \in \Lambda$. We say that λ_0 is a* singular point at scale ε^α *if it is not regular; i.e., if there exist m_ε, $\lambda'_\varepsilon \to \lambda_0$ and $\lambda''_\varepsilon \to \lambda_0$ such that (up to subsequences)*

$$\Gamma\text{-}\lim_{\varepsilon\to 0} \frac{F_\varepsilon^{\lambda'_\varepsilon} - m_\varepsilon}{\varepsilon^\alpha} \neq \Gamma\text{-}\lim_{\varepsilon\to 0} \frac{F_\varepsilon^{\lambda''_\varepsilon} - m_\varepsilon}{\varepsilon^\alpha}. \tag{65}$$

Example 8.2 In Example 3.1 the points ± 1 are singular at scale ε; in Example 3.2 the point 1 is singular at scale ε. In both cases the theorem does not hold even though the limit $F_\lambda^{(1)}$ exists, and we have (taking e.g. Λ a compact set containing a neighbourhood of 1)

$$\sup_\Lambda \left| \min F^{\lambda_\varepsilon} - m^{(0)}(\lambda) - \varepsilon \min F_\lambda^{(1)} \right| \geq C\varepsilon;$$

At a singular point λ_0 the computation of the Γ-limit or Γ-development with fixed λ_0 is not sufficient to accurately describe the behavior of minimum problems. We have then to look at the possibility of different limits along different paths $\lambda_\varepsilon \to \lambda_0$. To simplify the bookkeeping of various distinct limits around the singular point we introduce the notion of a 'table' of Γ-limits. Below we limit ourselves to the analysis at scale 1 and ε; the general case requiring only a more complex notation.

Definition 8.3 *The* table of Γ-limits at scale 1 *for F_ε^λ at λ_0 are all sequences $(\varepsilon_j, \lambda_j)$, and functionals $F^{(0)}_{(\varepsilon_j,\lambda_j)}$ with $\varepsilon_j \to 0$, $\lambda_j \to \lambda_0$, and*

$$F^{(0)}_{(\varepsilon_j,\lambda_j)} = \Gamma\text{-}\lim_j F_{\varepsilon_j}^{\lambda_j}.$$

The table of Γ-limits at scale ε *for F_ε^λ at λ_0 are all sequences $(\varepsilon_j, \lambda_j)$, and functionals $F^{(1)}_{(\varepsilon_j,\lambda_j)}$ with $\varepsilon_j \to 0$, $\lambda_j \to \lambda_0$, and*

$$F^{(1)}_{(\varepsilon_j,\lambda_j)} = \Gamma\text{-}\lim_j \frac{F_{\varepsilon_j}^{\lambda_j} - \min F^{(0)}_{(\varepsilon_j,\lambda_j)}}{\varepsilon_j},$$

etc.

Note that if λ_0 is regular then

$$F^{(1)}_{(\varepsilon_j,\lambda_j)} = \Gamma\text{-}\lim_j \frac{F_{\varepsilon_j}^{\lambda_j} - \min F^{(0)}_{\lambda_0}}{\varepsilon_j},$$

Below we give several examples of singular points which can be fully rectified. We begin with the situations when the boundaries between different entries in the table are sharp as in the case of nucleation (see below), fracture (see below) or buckling (not discussed here, see [49, 40, 36]), when the table is nontrivial due to the existence of several drastically different regimes in the behavior of the system.

The first example summarizes what we have learned about the phenomenon of nucleation in the case of gradient theory of phase transitions.

Example 8.4 Consider again the energy (27). Suppose that W is of class C^2, with minimum 0 and that $W^{**}(z) = W(z)$ if $|z| \geq 1$. We focus on the singular point $\lambda_0 = 1$. Note that the functionals $F^{(1)}_{(\varepsilon_j,\lambda_j)}$ are finite only at the constant function $u = 1$, so that it suffices to compute the limit

$$\begin{aligned} F^{(1)}_{(\varepsilon_j,\lambda_j)}(1) &= \lim_j \min\Big\{\int_0^1 \Big(\frac{W(v)}{\varepsilon_j} + \varepsilon_j|v'|^2\Big)\,dt : \int_0^1 u\,dt = \lambda_j\Big\} \\ &= \lim_j \min\Big\{\alpha\frac{(1-\lambda_j)^2}{\varepsilon_j}, \beta\Big\}, \end{aligned} \tag{66}$$

where

$$\alpha = \frac{1}{2}W''(1), \qquad \beta = c_W. \tag{67}$$

In this case the existence of the Γ-limit $F^{(1)}_{(\varepsilon_j,\lambda_j)}$ is equivalent to the existence of the last limit depending on the ratio $(1-\lambda_j)^2/\varepsilon_j$. The proof of the second inequality in (66), which is a bit more technical, can be found in the Appendix A1.

We can summarize our findings in a form of the following table:

1. If $(1-\lambda)^2 = C\varepsilon$, where $C \leq \frac{\beta}{\alpha}$, then $m_1 = \alpha C$
2. If $(1-\lambda)^2 \geq \frac{\beta}{\alpha}$ then $m_1 = \beta$

The behavior of the system close to the point $\varepsilon = 0$ and $\lambda = 1$ can be pictured in the ε–λ plane, where the line $\varepsilon = \frac{\alpha}{\beta}(1-\lambda)^2$ (for $\lambda < 1$) (*nucleation threshold*) separates the zone with phase mixture from the one where the stable configuration is homogeneous.

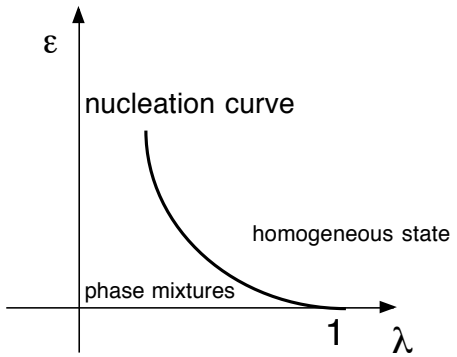

Figure 7: Nucleation curve in the ε–λ space

The next example concerns the nucleation of cracks in the discrete lattices with Lennard-Jones interactions.

Example 8.5 We consider the theory (29) and focus as in the Example 3.2 on the singular point $\lambda_0 = 1$. Observe that the functionals $F^{(1)}_{(\varepsilon_j,\lambda_j)}$ are finite only at the affine function $\overline{u}(t) = t$. Then to find the limit along arbitrary sequence in the parameter space it suffices to compute

$$F^{(1)}_{(\varepsilon_j,\lambda_j)}(\overline{u}) = \lim_j \min\Big\{\sum_i \Big(J\Big(\frac{u_i - u_{i-1}}{\varepsilon_j}\Big) - J(1)\Big) : u_0 = 0,\ u_N = \lambda_j\Big\}. \tag{68}$$

In this case, following the method of Section 5.2 we may change variables around $\overline{u}$, setting

$$v_i = \frac{u_i - \varepsilon_j i}{\sqrt{\varepsilon}_j}, \qquad \psi_j(z) = \frac{1}{\varepsilon_j}(J(1+\sqrt{\varepsilon}_j z) - J(1)),$$

so that

$$F^{(1)}_{(\varepsilon_j,\lambda_j)}(\overline{u}) \;=\; \lim_j \min\Big\{\sum_i \varepsilon_j \psi_j\Big(\frac{v_i - v_{i-1}}{\varepsilon_j}\Big) :\; v_0 = 0,\; v_N = \frac{\lambda_j - 1}{\sqrt{\varepsilon}_j}\Big\}. \tag{69}$$

After this change of variables we are in the position of describing the behavior of these minimum problems via the computation of the Γ-limit of the corresponding functionals in the variable v. This Γ-limit has been computed in [21] (see also [20]), from which we then get

$$\begin{aligned} F^{(1)}_{(\varepsilon_j,\lambda_j)}(\overline{u}) \;&=\; \lim_j \min\Big\{\alpha \int_0^1 |v'|^2\, dt + \beta \#(S(v)) : v(0) = 0, v(1) = \frac{\lambda_j - 1}{\sqrt{\varepsilon}_j}\Big\} \\ &=\; \lim_j \min\Big\{\frac{\alpha(\lambda_j - 1)^2}{\varepsilon_j}, \beta\Big\}, \end{aligned}$$

where

$$\alpha = \frac{1}{2}J''(1), \qquad \beta = -J(1). \tag{70}$$

Note the similarity of this case with the previous example even though our methodologies of finding the final result have been (superficially) different. Indeed in the case of phase transitions we used the direct method, while in the case of fracture we used the method of 'stretched' variables.

We can now summarize our results concerning fracture in discrete lattice in a form of the following 'table' :

1. If $(1-\lambda)^2 = C\varepsilon$, and $C \le \frac{\beta}{\alpha}$, then $m_1 = \alpha C$
2. If $(1-\lambda)^2 \ge \frac{\beta}{\alpha}\varepsilon$ then $m_1 = \beta$

The behavior of the system close to the point $\varepsilon = 0$ and $\lambda = 1$ can be again pictured in the ε–λ plane, where the line $\varepsilon = \frac{\alpha}{\beta}(1-\lambda)^2$ (for $\lambda < 1$) (*fracture threshold*) separates a zone where there is a crack from one where the stable configuration is homogeneous.

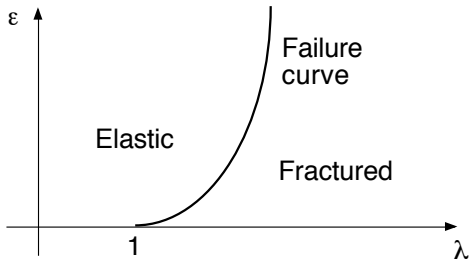

Figure 8: Failure curve in the ε–λ space

The next example shows that there may be an infinity of distinct entries characterizing a 'table' of a particular singular point. This example deals with the theory of finite-scale micro-structures introduced in Section (2), which we simplify here a little bit to avoid inessential technical difficulties.

Example 8.6 Taking the gradient theory of phase transitions into account, we may simplify the functional in (31) (the equivalence in some regimes of these functionals can be deduced from [51, 1]) and analyze the simplified functional

$$F_\varepsilon^\lambda(u) = 2\varepsilon \#(S(u')) + \lambda \int_0^1 u^2\,dt, \qquad u(0) = u(1) = 0,\ |u'| = 1,$$

where the first two terms of the integral in (31) have been replaced by (twice) the number of jump points of the derivative and the additional constraint that u is piecewise affine with gradient $u' \in \{\pm 1\}$.

The Γ-limit of F_ε^λ is

$$F_\lambda(u) = \lambda \int_0^1 u^2\,dt,\ u(0) = u(1) = 0,\ |u'| \le 1,$$

with minimum equal to 0. If $\lambda > 0$ the unique minimizer is $u(t) = 0$, while the limit is identically 0 on all admissible functions if $\lambda = 0$.

We may explicitly compute the minimum $m_\varepsilon(\lambda) = \min F_\varepsilon^\lambda$, which is obtained on the function u_N with N creases in $(0,1)$, $2/N$=periodic, odd, and equal to $(|2Nt-1|-1)/2N$ on $[0,1/N]$, for which we have

$$F_\varepsilon^\lambda(u_N) = 2\varepsilon N + \frac{\lambda}{12N^2}.$$

The optimal $N = N(\varepsilon,\lambda)$ is obtained by minimizing this quantity for $N \in \mathbb{N}$, $N \ge 1$, obtaining

$$N(\varepsilon,\lambda) \in \left\{ \left[\sqrt[3]{\frac{\lambda}{12\varepsilon}}\right] \vee 1, \left[\sqrt[3]{\frac{\lambda}{12\varepsilon}}\right] + 1 \right\},$$

and

$$\min F_\varepsilon^\lambda = \min\left\{ 2\varepsilon + \frac{\lambda}{12}, 2\varepsilon\Big[\sqrt[3]{\frac{\lambda}{12\varepsilon}}\Big] + \frac{\lambda}{12\Big[\sqrt[3]{\frac{\lambda}{12\varepsilon}}\Big]^2}, 2\varepsilon\Big(\Big[\sqrt[3]{\frac{\lambda}{12\varepsilon}}\Big] + 1\Big) + \frac{\lambda}{12\Big(\Big[\sqrt[3]{\frac{\lambda}{12\varepsilon}}\Big] + 1\Big)^2} \right\}. \quad (71)$$

For fixed λ we then have that

$$N(\varepsilon,\lambda) \sim \sqrt[3]{\frac{\lambda}{12\varepsilon}} \qquad \text{and} \qquad \min F_\varepsilon^\lambda \sim \varepsilon^{2/3}\lambda^{1/3}\sqrt[3]{\frac{2}{3}}.$$

We can examine the behavior at $\lambda = 0$. The next meaningful scale here is ε. The scaled energy is then simply

$$\frac{1}{\varepsilon}F_\varepsilon^\lambda(u) = 2\#(S(u')) + \frac{\lambda}{\varepsilon}\int_0^1 u^2\,dt, \qquad u(0) = u(1) = 0,\ |u'| = 1, \quad (72)$$

whose Γ-limit depends indeed on the ratio λ/ε. If $\lambda_\varepsilon/\varepsilon \to p \in [0,+\infty)$ then the corresponding Γ-limit is

$$F_p^1(u) = 2\#(S(u')) + p\int_0^1 u^2\,dt, \qquad |u'| = 1.$$

This shows that $\lambda = 0$ is a singular point at scale ε and gives the table of Γ-limits.

We can now summarize our results concerning fracture in discrete lattice in a form of the following 'table':

$$\text{if } C_{k-1}\varepsilon \le \lambda \le C_k\varepsilon, \text{ then } N = k,$$

where C_k is an increasing sequence tending to $+\infty$ with $C_0 = 0$, whose values can be computed from (71). The behavior of $F^1_{\lambda/\varepsilon}$ is pictured in Fig. 9 through the number of interfaces of the minimizer. Note that the behavior of this functional differs from that of functional (31) close to $p = 0$, where the original functional admits homogeneous minimizers (e.g. [61]) that are nor allowed by the simplified one.

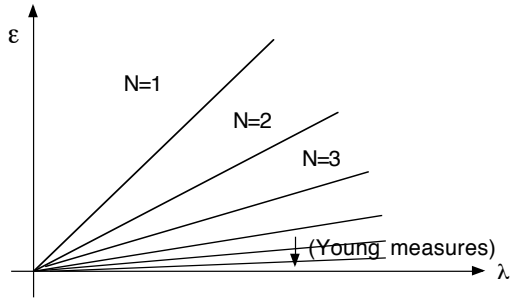

Figure 9: Boundaries in the parameter space separating microstructures with different number of interfaces

The next two examples illustrate the case when the boundary between different entries in a given table is not sharp but diffuse. The first example deals with size-effects in homogenization.

Example 8.7 We consider F_ε^λ as in (43). Note that the definition can be extended by continuity to $\lambda = 0$ setting

$$F_\varepsilon^0(u) = \int_0^1 a(0)|u'|^2\,dt, \qquad u(0) = 0,\ u(1) = 1,$$

independent of ε.

The point $\lambda = 0$ is the only singular point at scale 1. The table of Γ-limits as 0 is obtained by looking at the Γ-limits of functionals

$$F_\varepsilon^{\lambda_\varepsilon}(u) = \int_0^1 a\Big(\frac{\lambda_\varepsilon t}{\varepsilon}\Big)|u'|^2\,dt, \qquad u(0) = 0,\ u(1) = 1.$$

Essentially, we have here two cases which can be presented in a form of a table:

1) If $\lambda_\varepsilon/\varepsilon \to p \in [0,+\infty)$, then $F^0(u) = \int_0^1 a(pt)|u'|^2\,dt$ with $u(0)=0,\ u(1)=1$;

2) If $\lambda_\varepsilon >> \varepsilon$, then $F^0(u) = \underline{a}\int_0^1 |u'|^2\,dt$ with $u(0)=0,\ u(1)=1$.

Notice that the two approximations may overlap and therefore the boundary between different asymptotic regimes is 'diffuse'.

Finally we observe that at scale ε all $\lambda > 0$ are singular points, and the Γ-limits of the table at order ε are characterized by the existence (upon subsequences) of the limit

$$K_0 = \lim_{\varepsilon\to 0} b\Big(\frac{\lambda_\varepsilon}{\varepsilon}\Big)$$

($\lambda_\varepsilon \to \lambda$) and equal to

$$F^1_\lambda(u) = \begin{cases} \underline{a}^2 K_0 & \text{if } u(t) = t \\ +\infty & \text{otherwise.} \end{cases} \tag{73}$$

The next example, illustrating the same effect, deals with a concentrated force in a composite.

Example 8.8 We consider Example 7.5 and construct the table of Γ-limits at the singular point 0 for the sequence $\varepsilon_j = \frac{1}{j}$, for which $b(1/\varepsilon_j) = 0 = K$. First, we observe that the Γ-limit

$$F^1 = \Gamma\text{-}\lim_{\varepsilon\to 0^+} \frac{F^{\lambda_j}_{\varepsilon_j} - m^{(0)}(\lambda_j)}{\varepsilon_j}$$

is finite only at u^0_0. The Γ-limit exists if the limit

$$K_1 = \lim_j \frac{\varepsilon_j^2}{8\lambda_j^2}\int_0^{\lambda_j/\varepsilon_j} \Big(t^2 - \Big(\frac{\lambda_j}{\varepsilon_j}\Big)^2\Big)\Big(\frac{1}{\underline{a}} - \frac{1}{a(t)}\Big)\,dt$$

exists, in which case we have the two regimes. They may be presented in the form of a table:

1) If $\lambda_j/\varepsilon_j \to p \in [0,+\infty)$, then

$$F^1(u^0_0) = \begin{cases} 0 & \text{if } p = 0 \\ \dfrac{1}{8p^2}\displaystyle\int_0^p \big(t^2 - p^2\big)\Big(\frac{1}{\underline{a}} - \frac{1}{a(t)}\Big)\,dt & \text{if } p > 0; \end{cases}$$

2) If $\lambda_j >> \varepsilon_j$, then $F^1(u^0_0) = 0$.

We observe that again, since we are dealing with a singular point,

$$\sup_{\lambda\geq 0}\Big|\min F^\lambda_\varepsilon - m^{(0)}(\lambda) - \varepsilon m^{(1)}(\lambda)\Big| \geq c\varepsilon,$$

which means that the straightforward Γ-development can not be used. The general division of the parameter space into domains of applicability of different asymptotic theories remains in the present case basically the same as in the previous example.

Another set of examples, showing overlapping domains of validity of different asymptotic theories in the parameter space, can be found in the problems involving dimension reduction. In this case the role of λ is played by applied loads which have to be scaled with ε if one wants to get a nontrivial Γ-limit. There may be several such directions and in [35, 50], one can find some partially filled 'tables', which are based on the rigorous justification of the semi-empirical ansatzes first proposed by engineers (Kirchoff, von Karman, etc.) but also contain some new entries.

9 Uniform approximations

To construct a uniform approximation for a given theory one has to know the location of the singular points because they have to be treated differently than the regular points. This general observation is illustrated by the following example.

Example 9.1 We may construct equivalent theories at scale 1 to F_ε^λ in (43) in Example 6.2, removing the singular behavior at $\lambda = 0$ by setting

$$G_\varepsilon^\lambda(u) = \int_0^1 a_\varepsilon\Big(\frac{\lambda}{\varepsilon}x\Big)|u'|^2\,dx,$$

where

$$a_\varepsilon(y) = \begin{cases} a(y) & \text{if } |y| \le \rho_\varepsilon \\ \underline{a} & \text{if } |y| > \rho_\varepsilon \end{cases}$$

and $\rho_\varepsilon \to +\infty$ are such that $\varepsilon\rho_\varepsilon \to 0$. In this case the homogenized description with the modulus $\underline{a}$ is used only at a sufficiently large scale, while we are resolving all the microscopic details at the small scale.

In most cases, one needs to know not only the location of the singular points but also their structure. We have seen how the behavior of parameterized energies at singular points may be sometimes analyzed in terms of curves in the ε–λ space, along which a regular Γ-development exists. Although this is not the general case (for example, when we have oscillating behaviors as in Example 1.13 at scale ε^2, or in Example 8.7 at scale ε), it is frequent in applications. Now, if all singular points are rectifiable in this sense, then there exists a specific way to construct a uniform approximation.

To formulate the method we would need the following definition of a blown-up (or rectified) functional:

Definition 9.2 *Let λ_0 be a singular point for F_ε^λ at scale 1. We say that F_ε^λ admits a* blow up at λ_0 *at order 1 if energies H_ε^p exist and a continuous function $p = p(\lambda, \varepsilon)$ such that*

(i) *H_ε^p Γ-converge to H^p, and all p are regular points;*

(ii) *$F_\varepsilon^\lambda = H_\varepsilon^{p(\lambda,\varepsilon)}$ for (λ, ε) in a neighbourhood of $(0, \lambda_0)$.*

The definition can be easily extended to the scales ε^α.

We have already seen several examples of singular points where the singular behavior in original variables can be replaced by a regular behavior of a 'blow up' functional. It will be convenient to have the rectified functionals ready in the following two cases.

Example 9.3 In Example 8.4 we may 'blow up' the functionals F_ε^λ at the point $\lambda_0 = 1$ at order ε. This is equivalent to blowing up the functionals $\frac{1}{\varepsilon}(F_\varepsilon^\lambda - \min F_\lambda^{(0)})$ at order 1. We may then take $p = (1-\lambda)^2/\varepsilon$, and define

$$H_\varepsilon^p(u) = \int_0^1 \Big(\frac{W(v)}{\varepsilon} + \varepsilon|v'|^2\Big)\,dt, \qquad \int_0^1 u\,dt = 1 - \sqrt{\varepsilon p}.$$

The rectified functional then takes the form

$$H^p(u) = \begin{cases} \min\{\alpha p, \beta\} & \text{if } u(t) = 1 \\ +\infty & \text{otherwise} \end{cases}$$

where parameters α and β have been defined in (67).

Example 9.4 As above, in Example 8.5 at $\lambda = 1$ we can take $p = (1-\lambda)^2/\varepsilon$ and define

$$H^p_\varepsilon(u) = \sum_i \Big(J\Big(\frac{u_i - u_{i-1}}{\varepsilon}\Big) - J(1)\Big), \qquad u_0 = 0,\ u_N = 1 + \sqrt{\varepsilon p}.$$

The rectified functional can then be written in the form

$$H^p(u) = \begin{cases} \min\{\alpha p, \beta\} & \text{if } u(t) = t \\ +\infty & \text{otherwise} \end{cases}$$

where parameters α and β have been defined in (70).

The following theorem states that for F^λ_ε which admits a blow up a simple uniformly-equivalent family is given by H^p computed for $p = p(\lambda, \varepsilon)$.

Theorem 9.5 *Let F^λ_ε admit a blow up at λ_0 by means of energies H^p_ε with $p \in \Pi$ and Π compact; then F^λ_ε is uniformly equivalent to $G^\lambda_\varepsilon = H^{p(\lambda,\varepsilon)}$ at λ_0*

Proof. Let $\lambda_\varepsilon \to \lambda_0$; up to subsequences we may suppose there exists the limits $\lim_{\varepsilon\to 0^+} p(\lambda_\varepsilon, \varepsilon) = p_0$ and

$$\Gamma\text{-}\lim_{\varepsilon\to 0^+} F^{\lambda_\varepsilon}_\varepsilon = \Gamma\text{-}\lim_{\varepsilon\to 0^+} H^{p(\lambda_\varepsilon,\varepsilon)}_\varepsilon = H^{p_0} = \Gamma\text{-}\lim_{\varepsilon\to 0^+} H^{p(\lambda_\varepsilon,\varepsilon)}$$

by the regularity of p_0.

Now it is clear that in some cases a uniform approximation can be constructed by asymptotic matching of the rectified structures of the functional around the isolated singular points with the standard Γ-development around the regular points. Here by matching we mean construction of the energies that are equivalent to the Γ-limit (or Γ-development) far from singular points, and to the 'rectified' energies close to singular points. We may apply this method to a generic theory F^λ_ε by using the following algorithm:

1. Compute the table of Γ-limits of F^λ_ε at every point λ. This actually often subdivides into two steps

1a) identify regular points and compute Γ-developments;
1b) identify singular points and compute the complete table;

2. Choose the classes of theories which are compatible with the tables of Γ-limits; i.e., such that in those classes we may find parameterized

3. Tune the parameters in the class of theories chosen in the previous step to obtain an equivalent G^λ_ε of the desired form. In practice this is often done separately on regular and singular points. The method is applicable if the corresponding (locally) equivalent theories can be matched.

Of course, the choice of the functionals at Step 2 is not unique, and additional criteria (simplicity, computability, closeness to well-known theories, ability to describe local minimizers, etc) can drive the final selection. In general those functionals will range from 'locked' energies which only bear information about limit minimizers, or little more detail, to theories as complex as the original functionals F^λ_ε, so that we may either enlarge the domain of 'locked' equivalent theories; or choose

parameterized functionals on a large class that we know have the same Γ-limits as (or include) the class of F_ε^λ;

We conclude with two examples showing how the above algorithm can be actually implemented. The first example deals with the lattice model of fracture.

Example 9.6 In the case of a 1D lattice with Lennard-Jones interactions (see 2) the interesting interval of boundary conditions is $\lambda > 1$ because for $\lambda \leq 1$ the Γ-limit already gives a uniformly-equivalent theory. Indeed one can write a class of uniformly equivalent theories at $\lambda \leq 1$ at all orders in the form

$$G_\varepsilon^\lambda(u) = \int_0^1 \psi(u')\,dt, \qquad u(0) = 0,\ u(1) = \lambda, \tag{74}$$

provided that $\psi(z) = J(z)$ and $\psi^{**}(z) \geq J^{**}(z)$ for $z \leq 1$.

A singular point is located at $\lambda = 1$ and we know that to rectify the behavior at this point one has to look at the Γ-limits of the scaled functionals in (68) in the variable $v(t) = (u(t) - t)/\sqrt{\varepsilon}$. In the limit we obtain the Mumford-Shah functional

$$\alpha \int_0^1 |v'|^2\,dt + \beta \#(S(v))$$

with the additional constraint of 'increasing jumps' $v^+ > v^-$ on $S(v)$ (see [21]). Now, we follow our algorithm and formally pull back the variable $u = t + \sqrt{\varepsilon}v$ in the limiting functional. We obtain

$$\varepsilon\Big(\alpha \int_0^1 |v'|^2\,dt + \beta \#(S(v))\Big) = \alpha \int_0^1 |u' - 1|^2\,dt + \varepsilon\beta \#(S(u)).$$

This approximation of the singular behavior has to be matched with our approximation in the regular points $\lambda \leq 1$ (see (74)) and another regular approximation at $\lambda > 1$

Having in mind the blown-up energies at $\lambda = 1$ computed above, a class of equivalent theories at $\lambda > 1$ at scale ε (and uniformly equivalent on all compact sets of $(0, +\infty)$) can be chosen in the same general form. Thus, if we consider the class of energies

$$G_\varepsilon^\lambda(u) = \int_0 1\psi_\varepsilon(u')\,dt + \varepsilon \sum_{S(u)} g_\varepsilon(u^+ - u^-), \qquad u(0) = 0,\ u(1) = \lambda,$$

there are two conditions of equivalence: $\psi_\varepsilon = \psi$ with a unique minimum in 1 of value $J(1)$ and $\lim_{\varepsilon\to 0} g_\varepsilon(z) = \beta$. In our case these conditions are satisfied by the same energy densities as in the blown-up functional; i.e., $\psi(z) = J(1) + (z-1)^2$ and $g_\varepsilon = \beta$. This gives us a uniformly-equivalent theory for $\lambda \geq 1$ in the form

$$G_\varepsilon^\lambda(u) = J(1) + \alpha \int_0 1|u' - 1|^2\,dt + \varepsilon\,\beta \#(S(u)), \qquad u(0) = 0,\, u(1) = \lambda \tag{75}$$

with the condition $u^+ > u^-$ on $S(u)$. For $\lambda \leq 1$ we may directly use (74) with $\psi(z) = J(z)$.

One can see that we obtained the Griffith's theory of brittle fracture with a unilateral condition on the opening. We reiterate, that this theory contains an internal parameter ε and therefore cannot be obtained as a Γ-limit. The dependence of the minimum values of G_ε^λ on λ and the corresponding stress-strain relation are shown in Fig. 10. We see that fracture in this approximate theory does not

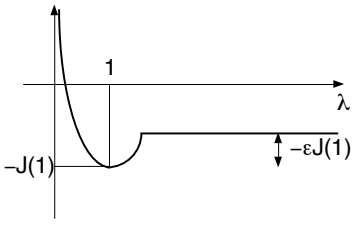

Figure 10: Approximate minimum values for G_ε^λ.

take place at infinitesimal tension, as in the straightforward Γ-development (see our Fig. 5), and that the nucleated crack has a finite opening. These features can be easily verified by computing the exact solution of the discrete problem at finite ε (see [60]).

Once the structure of one uniformly equivalent theory is established, other fracture energies may be constructed that belong to the same equivalence class but may have additional beneficial features. The corresponding development is usually not systematic and requires additional knowledge about the structure of the minimizers in the original problem at finite ε. For instance, one may consider the *cohesive zone* theories of the form

$$G_\varepsilon^\lambda(u) = J(1) + \alpha \int_0^1 |u' - 1|^2 \, dt + \varepsilon \, \beta \sum_{t \in S(u)} g\Big(\frac{u^+ - u^-}{\varepsilon}\Big), \qquad u(0) = 0, \, u(1) = \lambda.$$

Within the class of functions $g \geq 0$, which are also concave and non-decreasing, the conditions of equivalence are: $g'(0) > 0$ and $\lim_{z \to +\infty} g(z) = \beta = -J(1)$. In the case (75) we had $g(z) = 0, z = 0$, and $g(z) = \beta, z > 0$. By modifying appropriately the singular behavior of the function $g(z)$ near $z = 0$ one can hope to avoid some known limitations of the Griffith's theory regarding the poor resolution of the nucleation threshold and prediction of the singularity near the tip of the crack [45, 43]. For example one can take take $g(z) = \min\{z, 1\}$ as in the *Dugdale's* theory of fracture. The Dugdale's theory does a much better job in approximating the local minimizers than the Griffith's theory [43]. It can be further improved if $g(z)$ is taken to coincide with a particular rescaling (and translation) of a concave branch of the function $J(z)$, as shown in ([60]). In this case the resulting uniformly equivalent theory, which now adequately describes the fine structure of the bifurcation of a solution with a crack from a homogeneous (Cauchy-Born) one ([60, 18]), coincides with the Barenblatt's model. The latter was originally obtained from heuristic considerations (see [4]) and its asymptotic status has been obscure.

Our second example concerns the gradient theory of phase transitions (see Section 2) and here the goal of a uniformly equivalent theory is to describe interfaces that are not well separated from each other and/or from the exterior boundaries. Such approximation should be able to take into account that an interface adjusts its internal structure while approaching an obstacle. Due to the mentioned similarity between the fracture problem and the phase transition problem, we will use below the insights obtained from the formal matching procedure described in the preceding example.

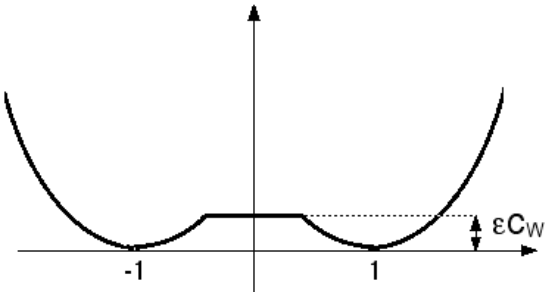

Figure 11: approximate minimum values for G^λ_ε.

Example 9.7 Consider again the 'theory' given by (27). To construct the simplest uniformly equivalent theory one can try to modify the straightforward development of F^λ_ε at the singular points at $\lambda = \pm 1$. For simplicity we suppose $\min W = 0$ and define the following parameterized family of functionals

$$G^\lambda_\varepsilon(u) = \begin{cases} \begin{cases} W(\lambda) & \text{if } u = \lambda \\ +\infty & \text{otherwise} \end{cases} & \text{if } |\lambda| \geq 1 \\ \begin{cases} \min\Big\{\varepsilon c_W, \frac{1}{2}\Big(\big(W''(1)(\lambda-1)^2\big) \wedge \big(W''(-1)(\lambda+1)^2\big)\Big)\Big\}\#(S(u)) & \text{if } |u| = 1 \text{ a.e.} \\ +\infty & \text{otherwise} \end{cases} & \text{if } |\lambda| < 1. \end{cases} \tag{76}$$

Observe that we have matched the regular approximation at $|\lambda| \geq 1$ at order ε given by

$$G^\lambda_\varepsilon(u) = \begin{cases} W(\lambda) & \text{if } u = \lambda \\ +\infty & \text{otherwise} \end{cases}$$

with the one at $|\lambda| < 1$, given by

$$G^\lambda_\varepsilon(u) = \begin{cases} \varepsilon c_W \#(S(u)) & \text{if } |u| = 1 \text{ a.e.} \\ +\infty & \text{otherwise} \end{cases}$$

and, finally, with the rectified singular approximation at $\lambda = 1$ (and similarly at -1) given by

$$G^\lambda_\varepsilon(u) = \begin{cases} \min\Big\{\varepsilon c_W, \frac{1}{2}\big(W''(1)(\lambda-1)2\big)\Big\}\#(S(u)) & \text{if } |u| = 1 \text{ a.e.} \\ +\infty & \text{otherwise.} \end{cases}$$

(see Example 9.4) The uniform equivalence of the resulting theory (76) and the original theory (27) is immediately proven by Remark 1.8. The dependence of the minimum values of G^λ_ε on λ and the corresponding stress-strain relation are shown in Fig. 9.7. We see that the phase transition in this approximate theory takes place at finite stress, contrary to what has been suggested by the straightforward Γ-development (see our Fig. 4), and that the nucleus has a finite size. These features can be easily verified by computing the exact solution of the discrete problem at finite ε (see [61, 64]).

The main defect of the constructed approximate 'theory' is its rigid structure adopted to the given boundary conditions. Such theory can not be easily generalized to cover other types of

boundary conditions, describe higher dimensional cases or deal with local minimizers. The origin of the problem is that the equivalent energy, which we have chosen for the singular point, is excessively simple. To improve the situation we can choose a broader class of energies with the correct structure of the singularities which would then generates a different uniformly equivalent theory. This choice should be driven by the computation of the energy singularity at ± 1 and in our case it implicitly suggests enlarging the functional space to piecewise-constant functions or to SBV functions.

Indeed, we can follow the pattern of the fracture theory and take

$$G_\varepsilon^\lambda(u) = \int_0^1 (W(u) + \overline{C}\varepsilon^2|u'|^2)\,dx + \varepsilon\beta\#(S(u)), \qquad \int_0^1 u\,dx = \lambda \tag{77}$$

defined on $SBV(0,1)$.

Moreover, since only equivalence at scale ε is required, the potential W may be replaced by its piece-wise quadratic analog

$$W_0(z) = \frac{1}{2}\min\{W''(-1)(z+1)^2, W''(1)(z-1)^2\},$$

provided that $\overline{C}$ is chosen such that

$$2\sqrt{\overline{C}}\int_{-1}^1 \sqrt{W_0(s)}\,ds \geq c_W$$

(see Appendix A.2(c)). The analysis of the resulting expression shows that the convex component of W is now represented by the bulk term in (77) while the concave component (the 'spinodal region) is described by a (constant) surface energy. We emphasize once again that the resulting theory contains a small parameter ε and therefore cannot be obtained as a Γ-limit.

Now, as in fracture mechanics we may try to modify our surface energy further in order, for instance, to capture the local minimizers. We begin by characterizing the classes of eligible surface energies in the general case of n-dimensions. By enlarging the space of competing functions from piecewise-constant functions with values ± 1 to all piecewise-constant functions we obtain the class of functionals

$$G_\varepsilon(u) = \int_\Omega W(u)\,dx + \varepsilon\int_{S(u)} g(u^+,u^-)d\mathcal{H}^{n-1}, \tag{78}$$

where $u^\pm$ are the traces on both sides of the set $S(u)$ of discontinuity points of u, and $g \geq 0$ is a subadditive function. Clearly, G_ε is equivalent to F_ε at scale 1. Sufficient conditions on g for G_ε to be equivalent to F_ε at scale ε are

$$g(u,v) \geq 2\sqrt{C}\left|\int_u^v \sqrt{W(s)}\,ds\right| \quad \text{for all } u,v,$$

and $g(1,-1) = g(-1,1) = c_W$; for example, $g(u,v) \equiv c_W$ (see Appendix A.2(a)). An even larger space allowing for discontinuities is the space $SBV(\Omega)$ (see [30, 12]), where we may take

$$G_\varepsilon(u) = \int_\Omega (W(u) + \overline{C}\varepsilon^2|\nabla u|^2)\,dx + \varepsilon\int_{S(u)} g(u^+,u^-)\,d\mathcal{H}^{n-1} \tag{79}$$

for $u \in SBV(\Omega)$, with g as above and $\overline{C} \geq C$ (see Appendix A.2(b)). If $\overline{C} > C$ then sharp phase transitions are favored. Inside this class a uniformly equivalent theory can be obtained if we take

$\{u \in SBV(0,1) : |u| \geq T\}$ for some $0 < T < 1$ together with the constraint that $u^- \leq -T$ and $u^+ \geq T$ (or the converse) on $S(u)$ and write a functional

$$G^\lambda_\varepsilon(u) = \int_{\Omega_+(u)} \Big(\frac{1}{2}W''(1)(u-1)^2 + \varepsilon^2 C|u'|^2\Big)\,dt + \int_{\Omega_-(u)} \Big(\frac{1}{2}W''(-1)(u+1)^2 + \varepsilon^2 C|u'|^2\Big)\,dt + \varepsilon \sum_{S(u)} g(u^-, u^+),$$

where $\Omega_\pm(u) = \{\pm u \geq T\}$. In this case some more technical conditions must be imposed on T, C and g , involving the notion of subadditive envelope (see Appendix A.2(d)). As we have already seen those conditions are satisfied by taking g to be equal to constant c_W, however,the example of fracture shows, that one can do better to capture the local minimizers. For instance, if we identify g with appropriately re-scaled and translated spinodal component of the function W we obtain an approximate theory with a much broader reach .

10 Conclusions

The goal of this paper is twofold. First, we wanted to show that the previous attempts to extend the idea of Γ-convergence beyond the first Γ-limit have not been fully satisfactory. Second, we wanted to find the way of constructing a rigorous asymptotic Γ-expansion which is devoid of the detected flaws by extending to functionals the corresponding machinery developed by Poincare for solutions of differential equations. This has placed the main focus of the paper on *definitions* rather than theorems.

Along the way, we have found it necessary to extended to functionals the concept of an asymptotic equivalence at a particular order. This allowed us to represent the whole set of functionals depending on a small parameter as a union of the classes of equivalence. The realization of the rich structure of each of these classes makes the task of constructing a unique approximation hopeless, unless additional criteria have been specified. While one can propose many different criteria (analytical simplicity, computability, ability to capture local minimizers, etc.), we have selected the one which emphasizes that in application the particular functionals usually appear as the representatives of the parameterized families (e.g., Von-Karman theory of plates, lubrication theory, the theory of incompressible elastic solids, etc.) To deal with such 'theories', we had to extended to parameterized families of functionals the concept of Γ-equivalence. Since the so defined equivalence may not be uniform with respect to parameters, we have been naturally led to the important distinction between the regular and singular values of parameters. Behind the concept of a singular point is a realization that in theories one generically encounters the values of parameters where the conventional Γ-limits ceases to exist. Such points in the parameter space often correspond to interesting physical phenomena (buckling, nucleation, size effect, etc.)

Following the corresponding methodology developed for functions, we have shown that even if a limit does not exist, one can often reconstruct the structure of the singularity by the blow up procedure and give the complete characterization of the singular structure of the asymptotic expansion in terms of the corresponding 'table' of limits. We have then showed that the knowledge of the full table for all critical point allows one to construct matched asymptotic Γ-expansions delivering globally uniform approximation of a given order. Although the uniformity principle still

does not provide us with a unique approximation of a given order, it considerably narrows the class of admissible approximate theories.

We have applied the proposed methodology to problems of practical interest and produced several interesting approximate theories. In some cases our formal development has given a rigorous justification for the existing semi-empirical procedures used by practitioners (e.g. cohesive models of fracture, quasi-continuum models in elasticity, etc.). In the other cases, the entirely new approximate theories have been advanced (e.g. a theory of phase transitions with surface energy combined with a possibility of a generic discontinuity in the gradients).

In conclusion we mention several limitations of our approach. The main problem is that Γ-convergence deals with the global minima while in applications the situations are plentiful when the energy landscape is rather rugged and it is the knowledge of the local minima which is crucial. Then, the proposed methodology is not universal and its implementation depends on the detailed knowledge of the minimizers of the original functional, which is seldom readily available. Finally, even if successful, the method delivers only one particular approximate theory which may well coexist in applications with other equivalent theories. In those cases one needs to find additional criteria which would justify the use of a particular theory in the physical problem of interest.

Appendix

A.1 We prove the second inequality in (66), namely that

$$\lim_j \min\Big\{\int_0^1 \Big(\frac{W(v)}{\varepsilon_j} + \varepsilon_j |v'|^2\Big)\,dt : \int_0^1 u\,dt = \lambda_j\Big\} = \lim_j \min\Big\{c_W, \frac{1}{2}W''(1)\frac{(1-\lambda_j)^2}{\varepsilon_j}\Big\}.$$

We first note that

$$\begin{aligned} m_j &:= \min\Big\{\int_0^1 \Big(\frac{W(v)}{\varepsilon_j} + \varepsilon_j |v'|^2\Big)\,dt : \int_0^1 u\,dt = \lambda_j\Big\} \\ &\leq \frac{W(\lambda_j)}{\varepsilon_j} = \frac{1}{2}W''(1)\frac{(1-\lambda_j)^2}{\varepsilon_j} + o\Big(\frac{(1-\lambda_j)^2}{\varepsilon_j}\Big) \end{aligned} \tag{80}$$

by testing with $v = \lambda_j$, This shows that if $(1-\lambda_j)^2 << \varepsilon_j$ then $\lim_j m_j = 0$ as desired. Conversely, let

$$(1-\lambda_j)^2 \geq C\varepsilon_j \tag{81}$$

for some $C > 0$. Note that in this case we can construct a sequence of the form

$$\widehat{v}_j(t) = \overline{u}\Big(\frac{t-t_j}{\varepsilon_j}\Big),$$

where $\overline{u}$ is a solution of the optimal profile problem

$$\min\Big\{\int_{-\infty}^{+\infty} (W(u) + |u'|^2)\,dt : u(\pm\infty) = \pm 1\Big\} = c_W,$$

and t_j are suitable translations in order to match the integral constraint, such that $\lim_j \frac{1}{\varepsilon_j} F_{\varepsilon_j}^{\lambda_j}(\widehat{v}_j) = c_W$. By this computation and (80) we then get

$$\lim_j m_j \leq \lim_j \min\Big\{c_W, \frac{1}{2}W''(1)\frac{(1-\lambda_j)^2}{\varepsilon_j}\Big\}.$$

To prove the converse inequality, consider v_j a minimizer for m_j. If $\liminf_j v_j \le -1$ then there exit points x_j^-, x_j^+ such that $\lim_j v_j(x_j^\pm) = \pm 1$, and then

$$\liminf_j m_j \ge \liminf_j \Big| \int_{x_j^-}^{x_j^+} \Big(\frac{W(v_j)}{\varepsilon_j} + \varepsilon_j |v_j'|^2 \Big) \, dt \Big| \ge \liminf_j \Big| \int_{x_j^-}^{x_j^+} \sqrt{W(v_j)} |v_j'| \, dt \Big| = c_W.$$

If otherwise, $\liminf_j v_j \ge C > -1$ then for fixed $\eta > 0$ we have

$$|\{v_j < 1 - \eta\}| \le \frac{\varepsilon}{C_\eta} \int_0^1 \Big(\frac{W(v_j)}{\varepsilon_j} + \varepsilon_j |v_j'|^2 \Big) \, dt \le \varepsilon \frac{c_W}{C_\eta}, \tag{82}$$

where $C_\eta = \min\{W(s) : C \le s \le 1 - \eta\}$. Note moreover that it is not restrictive to suppose that $v_j < 1 + \eta$. Let $c_\eta = o(1)$ as $\eta \to 0$ be such that $W(z) \ge \frac{1}{2}(W''(1) - c_\eta)(1 - z)^2$ on $[1 - \eta, 1 + \eta]$; then, by Jensen's inequality and (81)

$$\begin{aligned} \liminf_j m_j &\ge \liminf_j \int_{\{v_j > 1-\eta\}} \frac{W(v_j)}{\varepsilon_j} \, dt \ge \liminf_j \frac{1}{2}(W''(1) - c_\eta) \frac{1}{\varepsilon_j} (1 - \lambda_j + O(\varepsilon))^2 \\ &\ge \liminf_j \frac{1}{2}(W''(1) - c_\eta) \frac{1}{\varepsilon_j} (1 - \lambda_j)^2. \end{aligned}$$

Letting $\eta \to 0$ we have the desired inequality.

A.2 We sketch here the proofs for the equivalence statements in Example 9.7 .

(a) After noting that the domain of the Γ-limit at order 1 consists of functions in $BV(\Omega; \{-1, 1\})$ we remark that

$$\Gamma\text{-}\lim_{\varepsilon \to 0} \frac{G_\varepsilon - \min W}{\varepsilon} \ge \overline{H}, \tag{83}$$

where $\overline{H}$ is the lower-semicontinuous envelope of

$$H(u) = \int_{S(u)} g(u^+, u^-) d\mathcal{H}^{n-1}$$

defined on piecewise-constant functions. From the relaxation theory for those functionals (see, e.g., [11, 17, 16]) we deduce that the conditions on g imply that $\overline{H}(u) = H(u) = c_W \mathcal{H}^{n-1}(S(u))$ if $u \in BV(\Omega; \{-1, 1\})$, which gives the liminf inequality. Finally, we note that a recovery sequence for such u is simply given by $u_\varepsilon = u$.

(b) As for (a) we can use the same relaxation argument, taking now

$$H(u) = 2\sqrt{\overline{C}} \int_\Omega \sqrt{W(u)} |Du| \, dx \int_{S(u)} g(u^+, u^-) d\mathcal{H}^{n-1}.$$

(c) We can follow the argument in A.1 above. The condition of equivalence at scale ε is then that

$$c_W = \min\Big\{ \int_{-\infty}^{+\infty} (W_0(u) + \overline{C}|u'|^2) \, dt + \sum_{S(u)} g(u^+, u^-) : u(\pm\infty) = \pm 1 \text{ or } \mp 1 \Big\},$$

which is implied by the conditions assumed.

(d) As in (c) we may use the argument in A.1, provided that

$$\begin{aligned} c_W &= \min\Big\{\int_{R_+(u)}\Big(\frac{1}{2}W''(1)(u-1)^2+\varepsilon^2C|u'|^2\Big)\,dt \\ &\quad +\int_{R_-(u)}\Big(\frac{1}{2}W''(-1)(u+1)^2+\varepsilon^2C|u'|^2\Big)\,dt+\sum_{S(u)}g(u^+,u^-) : u(\pm\infty)=\pm1 \text{ or } \mp1\Big\}, \end{aligned}$$

where the infimum is taken over all $u \in SBV(\mathbb{R})$ such that $|u| \geq T$ and $u^- \leq -T$ and $u^+ \geq T$ (or the converse) on $S(u)$, and $R_\pm(u) = \{\pm u \geq T\}$.

Note that if we set

$$g^{T,C}(u,v) = \begin{cases} g(u,v) & \text{if } u \leq -T \text{ and } v \geq T \text{ or the converse} \\ 2\sqrt{C}\Big|\displaystyle\int_u^v |s-1|ds\Big| & \text{if } u,v \geq T \\ 2\sqrt{C}\Big|\displaystyle\int_u^v |s+1|ds\Big| & \text{if } u,v \leq T \\ +\infty & \text{otherwise,} \end{cases}$$

then the above condition can be equivalently expressed as

$$c_W = \min\Big\{\sum_{i=1}^N g^{T,C}(z_i,z_{i-1}) : z_0=\pm1,\ z_N=\mp1,\ N\in\mathbb{N}\Big\};$$

i.e., that the subadditive envelope of $g^{T,C}$ computed in $(-1,1)$ and $(1,-1)$ is c_W (see [13]).

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