EQUILIBRIA OF AGGREGATION-DIFFUSION MODELS WITH NONLINEAR POTENTIALS

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Abstract. We consider an evolution model with nonlinear diffusion of porous medium type in competition with a nonlocal drift term favoring mass aggregation. The distinguishing trait of the model is the choice of a nonlinear (s,p) Riesz potential for describing the overall aggregation effect. We investigate radial stationary states of the dynamics, showing their relation with extremals of suitable Hardy-Littlewood-Sobolev inequalities. In the case that aggregation does not dominate over diffusion, radial stationary states also relate to global minimizers of a homogeneous free energy functional featuring the (s,p) energy associated to the nonlinear potential. In the limit as the fractional parameter s tends to zero, the nonlocal interaction term becomes a backward diffusion and we describe the asymptotic behavior of the stationary states.

Contents

1. Introduction

We are interested in stationary solutions of aggregation-diffusion models of the form

$$\partial_t \rho = \Delta \rho^m - \chi \operatorname{div}(\rho \nabla \mathcal{S}(\rho)) \qquad \text{in } (0,+\infty)\times\mathbb{R}^N, \tag{1.1}$$

where $\rho = \rho(x,t)$ represents a mass density whose evolution is driven by a porous medium diffusion $(m > 1)$ and a nonlocal interaction modeled by a potential $\mathcal{S}$ that accounts for long range effects. Here, $\chi > 0$ is the sensitivity constant measuring the interaction strength.

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Equations of the form (1.1) typically appear in mathematical biology as macroscopic models of interacting particles/agents [9, 17, 40, 43], such as the Keller-Segel model of chemotaxis [25, 27, 29, 30, 31, 41]. These models usually feature linear potentials in convolution form, i.e., $\mathcal{S}(\rho)$ is the convolution of ρ with some suitable radial convolution kernel accounting for mutual interaction forces.

Among the most relevant modeling examples is the Newtonian or the Riesz (attractive) potential, appearing in the Keller-Segel model and its many variants, which is given by

$$\mathcal{S}(\rho) = K_s * \rho. \tag{1.2}$$

Here, the kernel K_s is defined for $0 < s < N/2$ as

$$K_s(x) := c_{N,s}\,|x|^{2\,s-N}, \qquad c_{N,s} := \pi^{-\frac{N}{2}}\,2^{-2s}\,\frac{\Gamma\left(N/2-s\right)}{\Gamma(s)}, \tag{1.3}$$

and in terms of Fourier transform we have $\hat{K}_s(\xi) = |\xi|^{-2s}$ (with $\hat{u}(\xi) := \int_{\mathbb{R}^N} e^{-ix\cdot\xi}u(x)\,dx$). The particular case of the Newtonian potential corresponds to $s = 1$ if $N \geq 3$. With the choice (1.2), the free energy of the system is

$$\mathcal{F}_s(\rho) = \frac{1}{m-1}\int_{\mathbb{R}^N}\rho^m - \frac{\chi}{2}\int_{\mathbb{R}^N}\rho\,K_s*\rho,$$

featuring the competition among the diffusion term and the total interaction energy associated to the mean field potential. Functional $\mathcal{F}_s$ has to be analyzed among mass densities in the following class (defined for any given $M > 0$)

$$\mathcal{Y}_M = \mathcal{Y}_{M,m} := \left\{\rho \in L^1_+(\mathbb{R}^N)\cap L^m(\mathbb{R}^N)\ :\ \int_{\mathbb{R}^N} x\,\rho(x)\,dx = 0,\ \int_{\mathbb{R}^N}\rho(x)\,dx = M\right\}, \tag{1.4}$$

which naturally arises by taking into account that the evolution problem is formally preserving mass, center of mass and positivity. $\mathcal{F}_s$ is a Lyapunov functional for the dynamics. In fact, (1.1)-(1.2) can be seen as the gradient flow of $\mathcal{F}_s$ with respect to the square Wasserstein distance, see [10, 28]. In the search for stationary solutions to the evolution problem (1.1)-(1.2), it is therefore natural to look for minimizers (if existing) of $\mathcal{F}_s$ over $\mathcal{Y}_M$ and, more generally, for critical points satisfying suitable Euler-Lagrange equations. We also stress a crucial property of functional $\mathcal{F}_s$, which is the homogeneity with respect to the mass invariant dilations

$$\rho^\lambda(x) := \lambda^N\rho(\lambda x), \qquad x\in\mathbb{R}^N,\ \lambda > 0. \tag{1.5}$$

Indeed, we have

$$\mathcal{F}_s(\rho^\lambda) = \frac{\lambda^{N\,(m-1)}}{m-1}\int_{\mathbb{R}^N}\rho^m - \frac{\chi\,\lambda^{N-2s}}{2}\int_{\mathbb{R}^N}\rho\,K_s*\rho.$$

As a consequence, aggregation and diffusion are in balance if $m = 2-2s/N$, which is called the *fair competition regime* [10]. If m is below this threshold, aggregation dominates and concentrating all the mass at a single point (that is, letting $\lambda \to +\infty$) is energetically favorable.

The classical Keller-Segel model [30] of chemotaxis, in its simplest mathematical formulation [4, 8, 24, 29, 48] is a fair competition model, formally obtained by letting $N = 2$, $s = 1$ so that the convolution kernel is the Newtonian kernel (in dimension 2 it is understood that $K_1(x) = -\frac{1}{2\pi}\log|x|$), and by letting the diffusion be linear $m = m_c = 1$ (the diffusion term in the free energy becomes $\int_{\mathbb{R}^N} \rho \log \rho$). It is well known that a critical mass M_c exists in such a model (whose explicit value is $8\pi/\chi$), and that global-in-time solutions for the associated Cauchy problem exist if the mass is not above the critical mass, while blow up in finite time occurs if $M > M_c$. Moreover, stationary states exist only if $M = 8\pi/\chi$, see [5, 9, 14]. The above properties of the classical Keller-Segel model generalize to fair competition models in higher dimension: it has been proven in the Newtonian potential case $s = 1, N \geq 3$ in [6] that a critical mass M_c still appears for $m = 2 - 2/N$, that its value can be written in terms of the best constant of suitable Hardy-Littlewood-Sobolev (HLS) inequalities, and that stationary states exist only if $M = M_c$. The validity of analogous properties for more singular Riesz potentials $0 < s < 1$ has been shown in [10, 11], still in the fair competition regime $m = 2 - 2s/N$. On the other hand, the diffusion dominated regime has been considered in [19], and in such case stationary states exist for every choice of the mass $M > 0$ and can be obtained as minimizers of $\mathcal{F}_s$ over $\mathcal{Y}_M$. In the aggregation dominated regime $m < 2 - 2s/N$ the free energy $\mathcal{F}_s$ is not bounded from below over $\mathcal{Y}_M$ (whatever the choice of $M > 0$), but stationary states of the dynamics can still be obtained, as seen in [20], as solutions to the Euler-Lagrange equation associated with the free energy (see also [3] for the Newtonian case $s = 1$).

2. Main results

2.1. (s, p) **potential, stationary states and HLS inequalities.** In this work we shall investigate the nonlinear potential counterpart of the previous results about stationary states, by considering an interaction described by the nonlinear Riesz potential, which has been introduced in [38], see also [1], [37, Section 4.2], [39, Section 5.4] and the references therein. We let

$$\mathcal{S} = \mathcal{K}_{s,p}, \tag{2.1}$$

where $1 < p < \infty$, $0 < s\,p < N$, and $\mathcal{K}_{s,p}$ stands for the *nonlinear* (s, p) *Riesz potential* given by

$$\mathcal{K}_{s,p}(\rho) := K_{s/2} * (K_{s/2} * \rho)^{p'-1}.$$

Here, p' is the conjugate exponent of p, i.e., $1/p + 1/p' = 1$. The total interaction energy of the mass density ρ associated to the nonlinear potential $\mathcal{K}_{s,p}$ (the (s, p) energy) is given by

$$\mathcal{I}_{s,p}(\rho) = \frac{1}{p'} \int_{\mathbb{R}^N} \rho\, \mathcal{K}_{s,p}(\rho) = \frac{1}{p'} \int_{\mathbb{R}^N} (K_{s/2} * \rho)^{p'},$$

where the second equality is due to Plancherel theorem, which also implies that for $\varepsilon \to 0$

$$\mathcal{I}_{s,p}(\rho + \varepsilon\varphi) = \mathcal{I}_{s,p}(\rho) + \varepsilon \int_{\mathbb{R}^N} (K_{s/2} * \rho)^{p'-1} K_{s/2} * \varphi + o(\varepsilon) = \mathcal{I}_{s,p}(\rho) + \varepsilon \int_{\mathbb{R}^N} \mathcal{K}_{s,p}(\rho)\, \varphi + o(\varepsilon)$$

for every test function φ, showing that indeed $\mathcal{K}_{s,p}$ is the functional derivative of $\mathcal{I}_{s,p}$. The free energy is therefore

$$\mathcal{F}_{s,p}(\rho) = \frac{1}{m-1} \int_{\mathbb{R}^N} \rho^m \, dx - \frac{\chi}{p'} \mathcal{I}_{s,p}(\rho)$$

and the evolution equation (1.1)-(2.1) is formally its Wasserstein gradient flow. The composition property of K_s shows that for $p = 2$ we are reduced to the linear potential case:

$\mathcal{K}_{s,2}(\rho) = K_s * \rho$ and $\mathcal{F}_{s,2}(\rho) = \mathcal{F}_s(\rho)$. For $p \neq 2$ the free energy is still a homogeneous functional, satisfying

$$\mathcal{F}_{s,p}(\rho^\lambda) = \frac{\lambda^{N\,(m-1)}}{m-1} \int_{\mathbb{R}^N} \rho^m - \lambda^{N\,(m_c-1)} \frac{\chi}{p'} \int_{\mathbb{R}^N} (K_{s/2} * \rho)^{p'},$$

where

$$m_c := p' - \frac{s\,p'}{N} \tag{2.2}$$

is the critical exponent. Therefore, we still recognize three regimes according to the value of the diffusion exponent m: we are in the *diffusion dominated regime* if $m > m_c$, in the *fair competition regime* if $m = m_c$, and in the *aggregation dominated regime* if $m < m_c$.

We perform the analysis of stationary states of (1.1)-(2.1). As in the linear potential case, we show that a critical mass appears only if $m = m_c$. Moreover, we show that stationary states are strictly related to optimizers of the following Hardy-Littlewood-Sobolev (HLS) type inequality, stating that if

$$m > (p_s^*)', \qquad \text{where} \qquad p_s^* := \frac{N\,p}{N - s\,p},$$

there exists a constant $H > 0$ such that

$$\|K_{s/2} * h\|_{p'}^{p'} \le H\,\|h\|_1^{p'\,\vartheta_0}\,\|h\|_m^{p'\,(1-\vartheta_0)} \qquad \text{for every } h \in L^1_+(\mathbb{R}^N) \cap L^m(\mathbb{R}^N), \tag{2.3}$$

where $0 < \vartheta_0 < 1$ is given by

$$\vartheta_0 := 1 - \frac{m'}{p_s^*} = \frac{1}{(p_s^*)'}\,\frac{m - (p_s^*)'}{m-1}.$$

We shall prove existence and regularity properties of optimizers of (2.3), which will be shown to be solutions of the nonlocal equation

$$\rho^{m-1} = a\,(\mathcal{K}_{s,p}(\rho) - \mathcal{C})_+ \qquad \text{in } \mathbb{R}^N \tag{2.4}$$

for suitable values of the positive constants $a, \mathcal{C}$. We notice that for $p = 2$, in terms of $u := K_s * \rho$ the above equation becomes the fractional semilinear PDE

$$(-\Delta)^s u = a^{\frac{1}{m-1}}\,(u - \mathcal{C})_+^{\frac{1}{m-1}},$$

which is the fractional plasma equation investigated in [20]. The terminology for such a semilinear equation is due to the fact that the nonlinearity in the right hand side, where $(x)_+ := \max\{x, 0\}$, appears in some classical models of plasma physics [50, 51]. The following is our first main result, which provides the main properties of the HLS optimizers. In the case that $m \ge m_c$, these results can be translated in a statement about minimizers of the free energy $\mathcal{F}_{s,p}$. In this regard, a critical mass appears for $m = m_c$, given by

$$M_c := \left(\frac{p_s^*}{\chi\,H^*_{m_c,s,p}}\right)^{\frac{N}{sp'}}, \tag{2.5}$$

where $H^*_{m,s,p}$ is the best constant in (2.3).

Theorem 2.1. *Let $1 < p < \infty$, $0 < sp < N$, $m > (p_s^*)'$. The best constant in the HLS inequality (2.3) is attained. Each optimizer is radially nonincreasing (up to translation), compactly supported, Hölder regular in $\mathbb{R}^N$ and smooth in the interior of its support. It satisfies (2.4) for suitable values of the constants $a > 0, \mathcal{C} > 0$.*
If $m > m_c$, then for each optimizer h of the HLS inequality (2.3) there exists a unique scaling factor $\lambda > 0$ such that h^λ is (up to translation) a minimizer of functional $\mathcal{F}_{s,p}$ over $\mathcal{Y}_M$, where $M = \|h\|_1$; conversely for every $M > 0$ minimizers of $\mathcal{F}_{s,p}$ over $\mathcal{Y}_M$ exist and are optimizers of (2.3).
If $m = m_c$, then each optimizer h of the HLS inequality (2.3) having mass M_c is (up to translation) a minimizer of $\mathcal{F}_{s,p}$ over $\mathcal{Y}_{M_c}$ and $\mathcal{F}_{s,p}(h) = 0$; conversely minimizers of $\mathcal{F}_{s,p}$ over $\mathcal{Y}_M$ exist if and only if $M = M_c$ and are optimizers of (2.3).

Existence of optimizers of (2.3) is a standard application of Riesz rearrangement inequalities along with compactness theorems for radially decreasing functions. For an optimizer, the constants $a, \mathcal{C}$ can be explicitly expressed, as well as the optimal dilation factor λ in the case $m > m_c$, as seen through the proof. It is not difficult to check that for every $M > 0$ the infimum of $\mathcal{F}_{s,p}$ over $\mathcal{Y}_M$ equals $-\infty$ if $(p_s^*)' < m < m_c$. However, an optimizer of the HLS inequality is still satisfying (2.4), hence after a suitable mass invariant dilation it satisfies the Euler-Lagrange equation

$$\rho^{m-1} = \frac{m-1}{m}\left(\chi\mathcal{K}_{s,p}(\rho) - \mathcal{Q}\right)_+ \qquad \text{in } \mathbb{R}^N \tag{2.6}$$

associated with functional $\mathcal{F}_{s,p}$, where $\mathcal{Q} > 0$ is a constant playing the role of Lagrange multiplier for the mass constraint. As such, it is (up to translation) a radially nonincreasing stationary state for (1.1)-(2.1) as we discuss in Section 5. About the regularity properties in Theorem 2.1, we mention that boundedness of optimizers has been proved in [19] by a purely variational argument in the case $p = 2$, $m > m_c$, which consists in the construction of a suitable bounded competitor for every unbounded candidate. Such an argument seems not straightforward in the nonlinear potential setting, therefore we prove boundedness by classical bootstrap methods, based on HLS inequalities and on (2.4), that are working for every $m > (p_s^*)'$. We stress that Theorem 2.1 generalizes the previous results in the literature about inequality (2.3): in the case $p = 2$ it is also called the Lane-Emden inequality and has been studied in [10, 15]. Interestingly, other generalizations have been recently investigated in [26], in relation with the Choquard equation, which still leads to radially decreasing compactly supported optimizers for suitable choices of the parameters therein.

2.2. **Asymptotic behavior of stationary states as $s \to 0$.** As observed in [28] by considering that K_s is an approximate identity for small s, the aggregation term can be considered as an approximation of a backward diffusion process, so that the evolution model (1.1)-(2.1) formally becomes the forward-backward diffusion equation

$$\partial_t \rho = \Delta\rho^m - \frac{\chi}{p'}\,\Delta\rho^{p'}.$$

Similarly, the associated free energy $\mathcal{F}_{s,p}$ formally becomes, in the limit $s \to 0$, the following functional featuring the competition of L^m and $L^{p'}$ norms

$$\mathcal{F}_0(\rho) = \frac{1}{m-1}\int_{\mathbb{R}^N}\rho^m - \frac{\chi}{p'}\int_{\mathbb{R}^N}\rho^{p'}. \tag{2.7}$$

Clearly, the minimization problem for functional $\mathcal{F}_0$ in the class $\mathcal{Y}_M$ is strongly influenced by the sign of $m-p'$, which is reflected in the fact that the critical exponent m_c from (2.2) is equal to p' if $s=0$. If $m<p'$, then functional $\mathcal{F}_{s,p}$ does not have minimizers over $\mathcal{Y}_M$ for every small enough s, and moreover $m<(p_s^*)'$ for every small enough s, so that we are not in the range of parameters of Theorem 2.1. Therefore, in our second main result, which is the following, we restrict to $m\geq p'$. The result for $p=2, m>2$ is given in [28].

Theorem 2.2. *Let $1<p<\infty$ and $0<s\,p<N$. Let $m\geq p'$, $M>0$. Let ρ_s be a minimizer of $\mathcal{F}_{s,p}$ over $\mathcal{Y}_M$ for every $s\in(0,N/p)$.*
If $m>p'$, then $\rho_s\to\rho_0$ strongly in $L^q(\mathbb{R}^N)$ for every $1<q<+\infty$ as $s\to 0$, where ρ_0 is the unique radially decreasing minimizer of $\mathcal{F}_0$ over $\mathcal{Y}_M$, which is the characteristic function of a ball.
Else if $m=p'$, we have $\inf_{\mathcal{Y}_M}\mathcal{F}_s\to\inf_{\mathcal{Y}_M}\mathcal{F}_0$ as $s\to 0$. Moreover, $\rho_s\to 0$ uniformly on $\mathbb{R}^N$ if $0<\chi<p$, and $\rho_s\to M\delta_0$ in the sense of measures if $\chi>p$.

Let us conclude this section with a discussion on possible further extensions and open problems. First of all, uniqueness (up to translations) of stationary states of given mass (or of optimizers of the HLS inequality (2.3) up to the natural scaling) would require a further, deep analysis. It has been proved in the case $p=2$ by different methods in [12, 15, 20, 21], and each of them could be suitable for treating the nonlinear potential case as well. The stability result of the HLS inequality in [15] could also be potentially generalized to $p\neq 2$. Second, radiality of *every* stationary solution to (1.1)-(2.1) is not guaranteed. Such a property has been proven in [19] in the linear potential case $p=2$ under some restrictions on m,s (building on the result from [18] for $s=1$). It remains an open problem to extend such result for the case $p\neq 2$. It would prove that all the steady states of the dynamics are actually radially decreasing. Moreover, it would also be interesting to investigate stationary states of the dynamics, meant as solutions to (2.6), in the regime $1<m\leq(p_s^*)'$: in this range radially decreasing solutions are expected to exists only for $\mathcal{Q}=0$ and to be smooth, positive and vanishing at infinity, since this behavior has been proven for $p=2$ in [20].

3. Preliminaries

3.1. Notation and functional framework.

The dimension of the ambient space $\mathbb{R}^N$ will be $N\geq 1$. For $x_0\in\mathbb{R}^N$ and $r>0$, the symbol $B_r(x_0)$ stands for the euclidean $N-$dimensional open ball

$$B_r(x_0)=\left\{x\in\mathbb{R}^N: |x-x_0|<r\right\}.$$

As usual, we will denote with $|\cdot|$ the $N-$dimensional Lebesgue measure. For $1\leq p\leq\infty$, the standard Lebesgue spaces are denoted by $L^p_{\rm loc}$ and L^p, and we will use the shortcut notation $\|\cdot\|_p$ for the $L^p(\mathbb{R}^N)$ norms. For an open set $\Omega\subseteq\mathbb{R}^N$, the notation $W^{1,p}(\Omega)$ and $BV(\Omega)$ stand respectively for the usual Sobolev space and the usual space of bounded variation functions on Ω. We use the following notation for the Hölder spaces

$$C^{0,\alpha}(\Omega):=\left\{u\in C^0(\Omega)\cap L^\infty(\Omega):\ \sup_{\substack{x,y\in\Omega,\\ x\neq y}}\frac{|u(x)-u(y)|}{|x-y|^\alpha}<\infty\right\}.$$

We say that $u\in C^{0,\alpha}_{\rm loc}(\Omega)$ if $u\in C^{0,\alpha}(\Omega')$ for every open set Ω' that is compactly contained in Ω. In particular $C^{0,1}(\Omega)$ is the space of bounded Lipschitz functions on Ω.

For every $\rho \in L^1(\mathbb{R}^N)$ and $\lambda > 0$, the *mass invariant dilation of* ρ *by factor* λ is given by (1.5). Since

$$\|\rho\|_1 = \|\rho^\lambda\|_1,$$

if $\rho \in \mathcal{Y}_M$ then also $\rho^\lambda \in \mathcal{Y}_M$, for every $\lambda > 0$, where $\mathcal{Y}_M$ is defined by (1.4).

With an abuse of notation, we will say that a radially symmetric function $\rho \in L^1(\mathbb{R}^N)$ is nonincreasing if its radial profile is nonincreasing. The *radially symmetric nonincreasing rearrangement* of a function $\rho \in L^1(\mathbb{R}^N)$ will be denoted by ρ^*. For the precise definition and its properties, we refer the reader to [35, Chapter 3]. We recall that the convolution among two nonnegative radially nonincreasing functions on $\mathbb{R}^N$ is still radially nonincreasing on $\mathbb{R}^N$, see [13]. In particular, if f is radially nonincreasing nonnegative, so is $K_s * f$.

The Fourier transform of the Riesz kernel K_s defined in (1.3) is given by (see for example [47, Lemma 1, Chapter V] or also [39, Theorem 2.8] and [49, Proposition 12.10])

$$\hat{K_s}(\xi) = |\xi|^{-2\,s}.$$

Moreover, for the normalization constant $c_{N,s}$ in (1.3) we have the following limiting behavior

$$\lim_{s\to 0} \frac{c_{N,s}}{s} = \frac{\Gamma\left(\frac{N}{2}\right)}{\pi^{\frac{N}{2}}} = \frac{2}{N\,\omega_N}. \tag{3.1}$$

3.2. **Basics on Riesz potentials.** We now recall some facts we will need throughout the whole paper.

Lemma 3.1. *Let $1 \le q < r \le \infty$, $0 < s\,q < N$ and $s\,r > N$. For every $h \in L^q(\mathbb{R}^N) \cap L^r(\mathbb{R}^N)$ we have*

$$\|K_{s/2} * h\|_\infty \le \alpha_s\,\|h\|_q + \beta_s\,\|h\|_r,$$

for some positive constant $\alpha_s = \alpha(N, q, s)$ and $\beta_s = \beta(N, r, s) > 0$. Moreover, we have

$$\lim_{s\to 0} \frac{\alpha_s}{s} = \begin{cases} \pi^{-\frac{N}{2}}\Gamma\left(\frac{N}{2}\right)\left(\omega_N (q-1)\right)^{\frac{q-1}{q}} & \text{if } q > 1,\\[2mm] \pi^{-\frac{N}{2}}\Gamma\left(\frac{N}{2}\right) & \text{if } q = 1,\end{cases} \tag{3.2}$$

and

$$\lim_{s\to 0} \beta_s = 1 \qquad \text{if } r = \infty. \tag{3.3}$$

Proof. Case $q > 1$. Our assumptions imply that

$$q' > \frac{N}{N-s} \qquad \text{and} \qquad r' < \frac{N}{N-s}.$$

Then, for every $x \in \mathbb{R}^N$, Hölder's inequality yields

$$\begin{aligned} \frac{(K_{s/2} * h)(x)}{c_{N,s/2}} &= \int_{\mathbb{R}^N\setminus B_1(x)} \frac{h(y)}{|x-y|^{N-s}}\,dy + \int_{B_1(x)} \frac{h(y)}{|x-y|^{N-s}}\,dy \\ &\le \left(\int_{\mathbb{R}^N\setminus B_1} \frac{dy}{|y|^{(N-s)\,q'}}\right)^{\frac{1}{q'}} \|h\|_q + \left(\int_{B_1} \frac{dy}{|y|^{(N-s)\,r'}}\right)^{\frac{1}{r'}} \|h\|_r \\ &= \left(\frac{N\,\omega_N}{(N-s)\,q'-N}\right)^{\frac{1}{q'}} \|h\|_q + \left(\frac{N\,\omega_N}{N-(N-s)\,r'}\right)^{\frac{1}{r'}} \|h\|_r, \end{aligned} \tag{3.4}$$

which gives the desired conclusion with

$$\alpha_s = \alpha(N, s, q) := c_{N,s/2} \left(\frac{N\omega_N}{(N-s)\, q' - N} \right)^{\frac{1}{q'}},$$

$$\beta_s = \beta(N, s, r) := c_{N,s/2} \left(\frac{N\omega_N}{N - (N-s)\, r'} \right)^{\frac{1}{r'}}.$$

By recalling (3.1), we get the claimed asymptotic behaviors (3.2)-(3.3) for α_s and β_s.

Case $q = 1$. We pass to the limit for $q \searrow 1$ in (3.4) obtaining

$$\frac{(K_{s/2} * h)(x)}{c_{N,s/2}} \le \|h\|_1 + \left(\frac{N\,\omega_N}{N - (N-s)\, r'} \right)^{\frac{1}{r'}} \|h\|_r.$$

This yields our claimed estimate with $\alpha_s = c_{N,s/2}$ and β_s as before. By recalling (3.1), we eventually get the desired asymptotic behaviors. ☐

Next we introduce the *Hardy-Littlewood-Sobolev type inequalities* that are crucial in this work.

Lemma 3.2 (Hardy-Littlewood-Sobolev type inequality)**.** *Let* $1 < q < \infty$ *and* $0 < s\,q < N$*. For every* $h \in L^q(\mathbb{R}^N)$*, we have*

$$K_{s/2} * h \in L^{q_s^*}(\mathbb{R}^N), \qquad \textit{where} \;\; q_s^* = \frac{N\,q}{N - s\,q}.$$

More precisely, there exists a sharp constant $\overline{H}_s = \overline{H}(N, s, q) > 0$ *such that*

$$\|K_{s/2} * h\|_{q_s^*} \le \overline{H}_s\, \|h\|_q, \qquad \textit{with} \;\; \limsup_{s \to 0} \overline{H}_s \le 1. \tag{3.5}$$

In particular, for every $h \in L^1(\mathbb{R}^N) \cap L^m(\mathbb{R}^N)$*,* $m > q$*, we have*

$$\|K_{s/2} * h\|_{q_s^*} \le \overline{H}_s\, \|h\|_1^{\vartheta}\, \|h\|_m^{1-\vartheta}, \qquad \textit{where} \;\; \vartheta = \frac{1}{q}\frac{m-q}{m-1}. \tag{3.6}$$

Proof. Inequality (3.5) follows by using in duality the well-known *Hardy-Littlewood-Sobolev Inequality* [35, Theorem 4.3]. Indeed, with the notation therein used, if we plug the following

$$r := q, \quad \lambda := N - s \quad \text{and so} \quad p := (q_s^*)',$$

we get that

$$\int_{\mathbb{R}^N} \varphi\, \left(K_{s/2} * h\right)\, dx = c_{N,s/2} \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{\varphi(x)\, h(y)}{|x-y|^{N-s}}\, dxdy \le C\,(N, q, s)\; c_{N,s/2}\, \|h\|_q,$$

for every $\varphi \in L^{(q_s^*)'}(\mathbb{R}^N)$ with $\|\varphi\|_{(q_s^*)'} = 1$, which allows to conclude. The constant $C\,(N, q, s)$ denotes the sharp constant of [35, Theorem 4.3(1)] and as shown therein we have

$$\begin{aligned}\overline{H}_s = \overline{H}\,(N, q, s) &:= C\,(N, q, s)\; c_{N,s/2} \\ &\le c_{N,s/2}\, \frac{N}{s}\, \omega_N^{(N-s)/N}\, \frac{1}{q\,(q_s^*)'} \left(\left(\frac{(N-s)/N}{1 - 1/(q_s^*)'} \right)^{(N-s)/N} + \left(\frac{(N-s)/N}{1 - 1/q} \right)^{(N-s)/N} \right).\end{aligned}$$

By (3.1), we infer that

$$\limsup_{s \to 0} \overline{H}_s \le 1.$$

Eventually, by the interpolation inequality in L^p-spaces, we also get (3.6). ☐

For our purposes, it will be convenient to rewrite (3.5) and (3.6) with q replaced by $(p_s^*)'$, given that p is the nonlinear Riesz potential exponent appearing in functional $\mathcal{F}_{s,p}$. It reads as follows

Corollary 3.3. *Let $1 < p < \infty$ and $0 < s\,p < N$. We have*

$$\|K_{s/2} * h\|_{p'} \le H_s\, \|h\|_{(p_s^*)'} \quad \text{for every } h \in L^{(p_s^*)'}(\mathbb{R}^N), \tag{3.7}$$

where the sharp constant $H_s = H\,(N, s, p) > 0$ satisfies

$$H_s \le c_{N,s/2}\, \frac{N}{s}\, \omega_N^{(N-s)/N}\, \frac{1}{p\,(p_s^*)'} \left(\left(\frac{(N-s)/N}{1 - 1/p} \right)^{(N-s)/N} + \left(\frac{(N-s)/N}{1 - 1/(p_s^*)'} \right)^{(N-s)/N} \right), \tag{3.8}$$

in particular $\limsup_{s\to 0} H_s \le 1$. Moreover, if $m > (p_s^*)'$ we have

$$\|K_{s/2} * h\|_{p'} \le H_s\, \|h\|_1^{\vartheta_0}\, \|h\|_m^{1-\vartheta_0} \quad \text{for every } h \in L^1(\mathbb{R}^N) \cap L^m(\mathbb{R}^N), \tag{3.9}$$

where $0 < \vartheta_0 < 1$ is given by

$$\vartheta_0 = \vartheta_0(m, N, p, s) := \frac{1}{(p_s^*)'}\, \frac{m - (p_s^*)'}{m - 1} = 1 - \frac{m'}{p_s^*}. \tag{3.10}$$

Proof. We have

$$(p_s^*)' = \frac{N\,p}{N\,(p-1) + s\,p} \in (1,\ N/s)\,,$$

thus the exponent $q := (p_s^*)'$ satisfies the assumptions of Lemma 3.2. Since $q_s^* = p'$, inequality (3.6) can be rewritten as (3.7) where the sharp constant $H_s = H\,(N, s, p) := \overline{H}(N, s, (p_s^*)') > 0$ satisfies (3.8). By the interpolation inequality in L^p-spaces we also get (3.9). □

The following theorem is due to Kurokawa. It will be used in Section 6 to establish convergence results for minimizers of $\mathcal{F}_{s,p}$ as $s \to 0$, see Theorem 6.7 and Proposition 6.8. We recall its elegant proof below.

Theorem 3.4 ([32])**.** *Let $1 < q < p$. For every $h \in L^q(\mathbb{R}^N) \cap L^p(\mathbb{R}^N)$, we have*

$$\lim_{s\to 0} \|K_{s/2} * h - h\|_p = 0.$$

Proof. Since $h \in L^p(\mathbb{R}^N)$, for every $\varepsilon > 0$ we can find $0 < \delta < 1$ such that

$$\int_{\mathbb{R}^N} |h(x-y) - h(x)|^p\, dx < \varepsilon, \quad \text{for } |y| < \delta, \tag{3.11}$$

see for instance [23, Proposition 17.1]. We set

$$K_{s/2}(x) = c_{N,s/2}\, |x|^{s-N}\, 1_{B_\delta}(x) + c_{N,s/2}\, |x|^{s-N}\, 1_{B_\delta^c}(x) =: K_{s/2}^0(x) + K_{s/2}^\infty(x),$$

so we have

$$\|K_{s/2} * h - h\|_p \le \|K_{s/2}^0 * h - h\|_p + \|K_{s/2}^\infty * h\|_p. \tag{3.12}$$

For the first addendum, since

$$\int_{B_\delta} |y|^{s-N}\, dy = N\,\omega_N \int_0^\delta \varrho^{s-1}\, d\varrho = N\,\omega_N\, \frac{\delta^s}{s}, \tag{3.13}$$

by adding and subtracting $c_{N,s/2}\int_{B_\delta}|y|^{s-N}h(x)\,dy$, we get

$$\|K^0_{s/2}*h-h\|_p\le c_{N,s/2}\left(\int_{\mathbb{R}^N}\left|\int_{B_\delta}|y|^{s-N}\,(h(x-y)-h(x))\,dy\right|^p dx\right)^{\frac{1}{p}} + \left(c_{N,s/2}-\frac{s}{N\omega_N\delta^s}\right)\left(\int_{\mathbb{R}^N}\left|\int_{B_\delta}|y|^{s-N}h(x)dy\right|^p dx\right)^{\frac{1}{p}}.$$

By Minkowski's integral inequality, (3.11) and again (3.13) we get

$$\|K^0_{s/2}*h-h\|_p\le N\,\omega_N\,\frac{\delta^s}{s}\,\left[c_{N,s/2}\,\varepsilon+\left(c_{N,s/2}-\frac{s}{N\,\omega_N\,\delta^s}\right)\|h\|_p\right].$$

By (3.1), we then obtain

$$\limsup_{s\to 0}\|K^0_{s/2}*h-h\|_p\le\varepsilon. \tag{3.14}$$

To estimate the second term in the right hand side of (3.12) for small s, we take β, r so that

$$0<\beta<N\left(\frac{1}{q}-\frac{1}{p}\right)\qquad\text{and}\qquad r^*_\beta=p, \tag{3.15}$$

in particular $q<r<p$. Without loss of generality, we can assume that $0<s<\beta$ and observe that

$$|y|^{s-N}=|y|^{(s-\beta)+(\beta-N)}\le\delta^{s-\beta}\,|y|^{\beta-N}\le\delta^{-\beta}\,|y|^{\beta-N},\qquad\text{for } |y|\ge\delta,$$

being $0<\delta<1$. This entails that

$$\|K^\infty_{s/2}*h\|_p=c_{N,s/2}\,\left(\int_{\mathbb{R}^N}\left|\int_{B^c_\delta}|y|^{s-N}\,h(x-y)\,dy\right|^p dx\right)^{\frac{1}{p}}\le c_{N,s/2}\,\delta^{-\beta}\,\left\|K_{\beta/2}*|h|\right\|_p.$$

Since $h\in L^r(\mathbb{R}^N)$, by (3.15) and by the Hardy-Littlewood-Sobolev inequality (3.5) we then obtain that $\left\|K_{\beta/2}*|h|\right\|_p<\infty$ and so

$$\lim_{s\to 0}\|K^\infty_{s/2}*h\|_p=0,$$

by (3.1). By spending this information and (3.14) in (3.12), we get the desired conclusion. □

4. Extremals of the HLS type inequality

4.1. **Existence: the Lieb-Oxford method.** In this section, we discuss the existence of extremals of the Hardy-Littlewood-Sobolev type inequality (3.9).

Let $1<p<\infty$ and $0<s\,p<N$. Let $m>(p^*_s)'$ and $\vartheta_0=1-m'/p^*_s$. The following quantity

$$H^*_{m,s}:=\sup\left\{\frac{\|K_{s/2}*h\|^{p'}_{p'}}{\|h\|_1^{p'\vartheta_0}\|h\|_m^{p'(1-\vartheta_0)}}:h\in L^1(\mathbb{R}^N)\cap L^m(\mathbb{R}^N)\setminus\{0\}\right\}, \tag{4.1}$$

is the sharp constant in the Hardy-Littlewood-Sobolev type inequality (3.9) raised to the power p'. By (3.8) this quantity is finite, since we have $H^*_{m,s}\le H^{p'}_s$.

Remark 4.1. The quotient definining $H^*_{m,s}$ given by

$$h \mapsto \frac{\|K_{s/2} * h\|_{p'}^{p'}}{\|h\|_1^{p'\vartheta_0}\|h\|_m^{p'(1-\vartheta_0)}}, \qquad h \in L^1(\mathbb{R}^N) \cap L^m(\mathbb{R}^N) \setminus \{0\},$$

is *invariant* under the actions of the two families of trasformations

$$\mu \mapsto \mu\, h, \quad \text{for } \mu > 0, \qquad \text{and} \qquad \lambda \mapsto \lambda^N\, h(\lambda\, x), \quad \text{for } \lambda > 0.$$

This property will be crucially exploited in Lemma 4.2 below.

We now infer the existence of extremals for (4.1) by using a classical argument by Lieb and Oxford (see for instance [36, Appendix A] and [34, Theorem 2.5]).

Lemma 4.2 (Existence of extremals)**.** *Let $1 < p < \infty$ and $0 < s\,p < N$. Let $m > (p^*_s)'$. There exists a radially symmetric and nonincreasing function $h_s \in L^1_+(\mathbb{R}^N) \cap L^m(\mathbb{R}^N)$ realizing the supremum in (4.1) and satisfying $\|h_s\|_1 = \|h_s\|_m = 1$.*

Moreover, every function $h_s \in L^1_+(\mathbb{R}^N) \cap L^m(\mathbb{R}^N)$ attaining the supremum in (4.1) is such that $h_s = h^*_s(\cdot - y)$, for some $y \in \mathbb{R}^N$.

Proof. **Step 1: reduction to normalized radially symmetric and nonincreasing functions.** Let $(h_j)_{j\in\mathbb{N}}$ be a maximizing sequence of feasible competitors for $H^*_{m,s}$, i.e.

$$\lim_{j\to\infty} \frac{\|K_{s/2} * h_j\|_{p'}^{p'}}{\|h_j\|_1^{p'\,\vartheta_0}\|h_j\|_m^{p'\,(1-\vartheta_0)}} = H^*_{m,s}, \qquad \text{with } \ h_j \in L^1(\mathbb{R}^N) \cap L^m(\mathbb{R}^N) \setminus \{0\}, \qquad \text{for } \ j \in \mathbb{N}.$$

We can assume that $h_j \geq 0$, since $K_{s/2} * h_j \leq K_{s/2} * |h_j|$ pointwisely. We can further assume that

$$\|h_j\|_1 = \|h_j\|_m = 1, \qquad \text{for } j \in \mathbb{N}. \tag{4.2}$$

This is not restrictive, since we could replace each approximant h_j with a rescaled version given by

$$\widetilde{h_j}(x) = \lambda_j\, h_j(\mu_j\, x), \qquad \text{with } \ \mu_j = \left(\frac{\|h_j\|_1}{\|h_j\|_m}\right)^{\frac{1}{N}\frac{m}{m-1}}, \quad \lambda_j = \frac{\mu_j^N}{\|h_j\|_1}, \qquad \text{for } \ j \in \mathbb{N}.$$

Indeed, we have

$$\|K_{s/2} * \widetilde{h_j}\|_{p'}^{p'} = \int_{\mathbb{R}^N} \left| \int_{\mathbb{R}^N} |x-y|^{s-N}\, \lambda_j\, h(\mu_j\, y) dy \right|^{p'} dx = \frac{\lambda^{p'}}{\mu^{s\,p'+N}} \|K_{s/2} * h_j\|_{p'}^{p'},$$

and

$$\|\widetilde{h_j}\|_1 = \frac{\lambda}{\mu^N}\, \|h_j\|_1, \qquad \|\widetilde{h_j}\|_m = \frac{\lambda}{\mu^{\frac{N}{m}}}\, \|h_j\|_m,$$

for every $j \in \mathbb{N}$. This yields

$$\begin{aligned}\frac{\|K_{s/2} * \widetilde{h_j}\|_{p'}^{p'}}{\|\widetilde{h_j}\|_1^{p'\,\vartheta_0}\|\widetilde{h_j}\|_m^{p'\,(1-\vartheta_0)}} &= \frac{\lambda_j^{p'}}{\lambda_j^{p'\,\vartheta_0}\lambda_j^{p'\,(1-\vartheta_0)}} \frac{\mu_j^{N\,p'\,\vartheta_0 + \frac{N}{m}\,p'\,(1-\vartheta_0)}}{\mu_j^{s\,p'+N}} \frac{\|K_{s/2} * h_j\|_{p'}^{p'}}{\|h_j\|^{p'\,\vartheta_0}\|h_j\|_m^{p'\,(1-\vartheta_0)}} \\ &= \frac{\|K_{s/2} * h_j\|_{p'}^{p'}}{\|h_j\|^{p'\,\vartheta_0}\|h_j\|_m^{p'\,(1-\vartheta_0)}},\end{aligned}$$

for every $j \in \mathbb{N}$.

Eventually, we claim that it is not restrictive to assume that h_j is radially nonincreasing, for every $j \in \mathbb{N}$. Indeed, take $\varphi_j \in L^p(\mathbb{R}^N)$ with $\|\varphi_j\|_p = 1$ such that

$$\|K_{s/2} * h_j\|_{p'} = \int_{\mathbb{R}^N} \varphi_j \, \left(K_{s/2} * h_j\right) \, dx.$$

We denote with φ_j^* and h_j^* the radially symmetric nonincreasing rearrangement of φ_j and h_j, respectively. By using the Riesz's rearrangement inequality [35, Theorem 3.7], we get

$$\|K_{s/2} * h_j\|_{p'} = c_{N,s/2} \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{\varphi_j(x)\, h_j(y)}{|x-y|^{N-s}} dxdy \leq c_{N,s/2} \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{\varphi_j^*(x)\, h_j^*(y)}{|x-y|^{N-s}} dxdy.$$

Since $\|\varphi_j^*\|_p = 1$, passing to the supremum on the right hand side we obtain

$$\|K_{s/2} * h_j\|_{p'} \leq \|K_{s/2} * h_j^*\|_{p'},$$

which proves our claim.

Step 2: the supremum is achieved. Thanks to *Step 1* we can assume that h_j is a nonnegative radially symmetric nonincreasing function with $\|h_j\|_1 = \|h_j\|_m = 1$, for every $j \in \mathbb{N}$. By using spherical coordinates, by the monotonicity of h_j, we can infer that for every $R > 0$ we have

$$1 = \|h_j\|_1 \geq N\,\omega_N \int_0^R \xi_j(r)\, r^{N-1}\, dr \geq N\,\omega_N\, \xi_j(R) \int_0^R r^{N-1}\, dr = \omega_N\, \xi_j(R)\, R^N,$$

where $\xi_j(|x|) := h_j(x)$ is the one-dimensional radial profile of h_j, for every $j \in \mathbb{N}$. Similarly, we have

$$1 = \|h_j\|_m^m \geq \omega_N\, \xi_j(R)^m\, R^N,$$

that is

$$\sup_{(R,\infty)} |\xi_j| \leq \frac{1}{\omega_N} \, \min\left\{\frac{1}{R^N}, \frac{1}{R^{N/m}}\right\} =: \omega(R), \qquad \text{for every } R > 0,\ j \in \mathbb{N}. \tag{4.3}$$

Lebesgue's differentiation theorem for monotone functions (see [2, Corollary 3.29] for instance) entails that

$$\int_R^\infty |\xi_j'(r)|\, dr \leq \omega(R), \qquad \text{for every } R > 0,\ j \in \mathbb{N}.$$

By means of a diagonal argument and *Helly's Selection Theorem* (see [23, Proposition 19.1c]), we can extract a subsequence (not relabeled) of nonincreasing functions $(\xi_j)_j$ converging everywhere to a nonincreasing function ξ in (R, ∞), for every rational number $R > 0$. This implies that

$$\lim_{j\to\infty} h_j(x) = \lim_{j\to\infty} \xi_j(|x|) = \xi(|x|) =: h_s(x), \qquad \text{for every } x \in \mathbb{R}^N \setminus \{0\}. \tag{4.4}$$

By collecting the previous information, we infer that h_s is a radially nonincreasing function satisfying

$$0 \leq h_s(x) \leq \omega(|x|) =: v(x), \qquad \text{for every } x \in \mathbb{R}^N \setminus \{0\}. \tag{4.5}$$

Observe that

$$v \in L^q(\mathbb{R}^N), \qquad \text{for every } 1 < q < m. \tag{4.6}$$

By using (4.2) and (4.4), Fatou's Lemma entails that

$$\|h_s\|_1 \leq 1 \qquad \text{and} \qquad \|h_s\|_m \leq 1, \tag{4.7}$$

thus in particular $h_s \in L^1(\mathbb{R}^N) \cap L^m(\mathbb{R}^N)$. Moreover, we have

$$\left(K_{s/2} * h_s\right)(x) = \lim_{j\to\infty} \left(K_{s/2} * h_j\right)(x) \leq \left(K_{s/2} * v\right)(x), \qquad \text{a.e. } x \in \mathbb{R}^N. \tag{4.8}$$

Indeed, from (4.3) we get the inequality in (4.8) and

$$\left(K_{s/2} * h_j\right)(x) = c_{N,s} \int_{\mathbb{R}^N} \frac{h_j(y)}{|x-y|^{N-s}} dy \leq c_{N,s} \int_{\mathbb{R}^N} \frac{v(y)}{|x-y|^{N-s}} dy = \left(K_{s/2} * v\right)(x).$$

From (4.6) and the properties of the Riesz potential (see [47, Theorem 1, Chapter V])

$$\left(K_{s/2} * v\right)(x) < \infty, \qquad \text{a.e. } x \in \mathbb{R}^N.$$

Then, by using (4.4) and Lebesgue's Dominated Convergence Theorem, we obtain the equality in (4.8). Observe that, by using (4.6) and Corollary 3.3, we have $K_{s/2} * v \in L^{p'}(\mathbb{R}^N)$. By using Lebegue's Dominated Convergence Theorem, by (4.8) and by recalling (4.2), we infer that

$$H^*_{m,s} = \lim_{j\to\infty} \|K_{s/2} * h_j\|_{p'}^{p'} = \|K_{s/2} * h_s\|_{p'}^{p'} \leq \frac{\|K_{s/2} * h_s\|_{p'}^{p'}}{\|h_s\|_1^{p'\vartheta_0} \|h_s\|_m^{p'(1-\vartheta_0)}},$$

where in the last inequality we also used (4.7). Then, the maximality of $H^*_{m,s}$ among functions in $L^1(\mathbb{R}^N) \cap L^m(\mathbb{R}^N)$ entails that

$$\|h_s\|_1^{p'\vartheta_0} \|h_s\|_m^{p'(1-\vartheta_0)} = 1.$$

This combined with (4.7) gives that $\|h_s\|_1 = \|h_s\|_m = 1$.

To complete the proof, we are only left out to prove that every other nonnegative extremal of (4.1) must be radially nonincreasing up to translations. Assume that $h_s \in L^1_+(\mathbb{R}^N) \cap L^m(\mathbb{R}^N) \setminus \{0\}$ satisfies equality in (4.1). In light of Corollary 3.3 we know that $K_{s/2} * h_s \in L^{p'}(\mathbb{R}^N)$, thus we take $\varphi \in L^p(\mathbb{R}^N)$ such that

$$\|K_{s/2} * h_s\|_{p'} = \int_{\mathbb{R}^N} \varphi \left(K_{s/2} * h_s\right) dx.$$

By using the Riesz's rearrangement inequality in strict form [35, Theorem 3.9], we get

$$\|K_{s/2} * h_s\|_{p'} = c_{N,s/2} \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{\varphi(x)\, h_s(y)}{|x-y|^{N-s}}\, dxdy \leq \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{\varphi^*(x)\, h_s^*(y)}{|x-y|^{N-s}} dxdy$$

with equality holding only if $\varphi = \varphi^*(\cdot - y)$ and $h_s = h_s^*(\cdot - y)$, for some $y \in \mathbb{R}^N$. This proves our claim and ends the proof. ☐

Remark 4.3 (Extremals in $\mathcal{Y}_M$)**.** Under the assumptions of Lemma 4.2, we can infer that for every *prescribed mass* $M > 0$ there exists a nonnegative radially symmetric nonincreasing function $h_{s,M} \in \mathcal{Y}_M$ realizing the supremum in (4.1). More precisely, it is obtained as

$$h_{s,M} := M\, h_s,$$

where h_s is an extremal of (3.9) provided by Lemma 4.2 (thus satisfying $\|h_s\|_1 = 1$) with barycenter at the origin.

4.2. **Euler-Lagrange equation.** The Euler-Lagrange equation satisfied by nonnegative extremals of (3.9) is derived below. We share arguments from [16, Theorem 3.1].

Lemma 4.4. *Let $1 < p < \infty$ and $0 < s\,p < N$. Let $m > (p_s^*)'$. For every extremal $h_0 \in L^1_+(\mathbb{R}^N) \cap L^m(\mathbb{R}^N)$ of the HLS (3.9), we have*

$$\mathcal{A}_s\, h_0^{m-1} = \Big(\mathcal{K}_{s,p}(h_0) - \mathcal{C}_s\Big)_+ \qquad \text{in } \mathbb{R}^N, \tag{4.9}$$

where

$$\mathcal{A}_s = \frac{m'}{p_s^*}\frac{\|K_{s/2} * h_0\|_{p'}^{p'}}{\|h_0\|_m^m}, \qquad \mathcal{C}_s = \left(1 - \frac{m'}{p_s^*}\right)\frac{\|K_{s/2} * h_0\|_{p'}^{p'}}{\|h_0\|_1} > 0. \tag{4.10}$$

Proof. We set

$$\mathcal{G}_s(h) := H^*_{m,s}\, \|h\|_1^{p'\,\vartheta_0}\, \|h\|_m^{p'\,(1-\vartheta_0)} - \|K_{s/2} * h\|_{p'}^{p'}, \qquad \text{for } h \in L^1_+(\mathbb{R}^N) \cap L^m(\mathbb{R}^N) \setminus \{0\},$$

where $H^*_{m,s}$ and ϑ_0 are respectively given by (4.1) and (3.10). Let h_0 be as in the statement. We shall make perturbations of h_0 that preserve positivity. We take $\varphi \in C_0^\infty(\mathbb{R}^N)$ and set

$$\psi := \varphi\, h_0, \qquad \varepsilon_0 := \frac{1}{2\|\varphi\|_\infty}, \qquad h_\varepsilon := h_0 + \varepsilon\,\psi \geq 0, \qquad \text{for } 0 \leq |\varepsilon| < \varepsilon_0. \tag{4.11}$$

By the optimality of h_0, we have

$$\mathcal{G}_s(h_\varepsilon) \geq \mathcal{G}(h_0) = 0, \qquad \text{for every } 0 \leq |\varepsilon| < \varepsilon_0,$$

this entails that

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\bigg|_{\varepsilon=0} \mathcal{G}_s(h_\varepsilon) = 0. \tag{4.12}$$

We have

$$\mathcal{G}_s(h_\varepsilon) = H^*_{m,s}\, \|h_0 + \varepsilon\,\psi\|_1^{p'\,\vartheta_0}\, \|h_0 + \varepsilon\,\psi\|_m^{p'\,(1-\vartheta_0)} - \|K_{s/2} * (h_0 + \varepsilon\,\psi)\|_{p'}^{p'}. \tag{4.13}$$

We expand to the first order, with respect to the variable ε, the three integral terms appearing in the rightmost term. For the first one, it is clear that

$$\int_{\mathbb{R}^N} (h_0 + \varepsilon\psi)\, dx = \int_{\mathbb{R}^N} h_0\, dx + \varepsilon \int_{\mathbb{R}^N} \psi\, dx, \qquad \text{for } 0 < |\varepsilon| < \varepsilon_0.$$

For the second integral term, we have

$$(h_0 + \varepsilon\,\psi)^m - h_0^m = \varepsilon \int_0^1 m\,(h_0 + \varepsilon\,t\,\psi)^{m-1}\,\psi\, dt, \qquad \text{a.e. in } \mathbb{R}^N, \quad \text{for } 0 < |\varepsilon| < \varepsilon_0.$$

By integrating over $\mathbb{R}^N$, dividing by ε and using Fubini theorem we obtain

$$\int_{\mathbb{R}^N} \frac{(h_0 + \varepsilon\,\psi)^m - h_0^m}{\varepsilon}\, dx = \int_0^1 \mathscr{H}_\varepsilon(t)\, dt, \tag{4.14}$$

where we set

$$\mathscr{H}_\varepsilon(t) = m \int_{\mathbb{R}^N} (h_0 + \varepsilon\,t\,\psi)^{m-1}\,\psi\, dx, \qquad \text{for } t \in [0,1].$$

By Hölder's inequality and (4.11), we infer that

$$|\mathscr{H}_\varepsilon(t)| \leq m\, \|h_0 + \varepsilon\,t\,\psi\|_m^{\frac{m}{m'}}\, \|\psi\|_m \leq m\, (\|\psi\|_m + \varepsilon_0\, \|\psi\|_m)^{\frac{m}{m'}}\, \|\psi\|_m, \qquad \text{for } t \in [0,1],$$

for every $0 < |\varepsilon| < \varepsilon_0$. By using Lebesgue's Dominated Convergence Theorem in (4.14)

$$\int_{\mathbb{R}^N} (h_0 + \varepsilon\,\psi)^m\;dx = \int_{\mathbb{R}^N} h_0^m\,dx + \varepsilon \int_{\mathbb{R}^N} m\,h_0^{m-1}\psi\,dx + o(\varepsilon), \qquad \text{as } \varepsilon \to 0.$$

For the third integral term, we have a.e. in $\mathbb{R}^N$ the following identity

$$\big(K_{s/2} * (h_0 + \varepsilon\,\psi)\big)^{p'} - \big(K_{s/2} * h_0\big)^{p'} = \varepsilon\,p' \int_0^1 \big(K_{s/2} * (h_0 + \varepsilon\,t\psi)\big)^{p'-1} \big(K_{s/2} * \psi\big)\;dt,$$

for $0 < |\varepsilon| < \varepsilon_0$. By integrating over $\mathbb{R}^N$, dividing by ε and by Fubini's theorem

$$\int_{\mathbb{R}^N} \frac{1}{\varepsilon}\Big[\big(K_{s/2} * (h_0 + \varepsilon\,\psi)\big)^{p'} - \big(K_{s/2} * h_0\big)^{p'}\Big]\,dx = \int_0^1 \mathscr{K}_\varepsilon(t)dt, \tag{4.15}$$

where we set

$$\mathscr{K}_\varepsilon(t) = p' \int_{\mathbb{R}^N} \big(K_{s/2} * (h_0 + \varepsilon\,t\,\psi)\big)^{p'-1} \big(K_{s/2} * \psi\big)\;dx, \qquad \text{for } t \in [0,1].$$

By Hölder's inequality and the HLS-type inequality (3.9), we get

$$\begin{aligned}
|\mathscr{K}_\varepsilon(t)| &\le p'\,\|K_{s/2} * (h_0 + \varepsilon\,t\,\psi)\|_{p'}^{\frac{p'}{p}}\,\|K_{s/2} * \psi\|_{p'} \\
&\le p'\,H^*_{m,s}\,\|h_0 + \varepsilon\,t\,\psi\|_1^{\frac{p'}{p}\,\vartheta_0}\,\|h_0 + \varepsilon\,t\,\psi\|_m^{\frac{p}{p'}\,(1-\vartheta_0)}\,\|\psi\|_1^{\vartheta_0}\|\psi\|_m^{1-\vartheta_0},
\end{aligned}$$

where $H^*_{m,s}$ and ϑ_0 are respectively given by (4.1) and (3.10). By Minkowski's inequality, we further have

$$|\mathscr{K}_\varepsilon(t)| \le p'\,H^*_{m,s}\,\left(\|h_0\|_1 + \varepsilon_0\,\|\psi\|_1\right)^{\frac{p'}{p}\,\vartheta_0}\,\left(\|h_0\|_m + \varepsilon_0\,\|\psi\|_m\right)^{\frac{p'}{p}\,(1-\vartheta_0)}\,\|\psi\|_1^{\vartheta_0}\,\|\psi\|_m^{1-\vartheta_0},$$

for $t \in [0,1]$, for $0 < |\varepsilon| < \varepsilon_0$. Thus we can use Lebesgue's Dominated Convergence Theorem in (4.15), obtaining

$$\begin{aligned}
\int_{\mathbb{R}^N} \big(K_{s/2} * (h_0 + \varepsilon\,\psi)\big)^{p'}\,dx &= \int_{\mathbb{R}^N} \big(K_{s/2} * h_0\big)^{p'}\,dx + \varepsilon\,p' \int_{\mathbb{R}^N} \big(K_{s/2} * h_0\big)^{p'-1} \big(K_{s/2} * \psi\big)\,dx + o(\varepsilon) \\
&= \int_{\mathbb{R}^N} \big(K_{s/2} * h_0\big)^{p'}\,dx + \varepsilon\,p' \int_{\mathbb{R}^N} K_{s/2} * \big(K_{s/2} * h_0\big)^{p'-1}\;\psi\,dx + o(\varepsilon),
\end{aligned}$$

as $\varepsilon \to 0$, where the last identity follows from Plancherel's theorem. By collecting the previous asymptotic expansions and by using that

$$H^*_{m,s} = \frac{\|K_{s/2} * h_0\|_{p'}^{p'}}{\|h_0\|_1^{p'\,\vartheta_0}\,\|h_0\|_m^{p'\,(1-\vartheta_0)}},$$

from (4.13) we get

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\bigg|_{\varepsilon=0} \mathcal{G}_s(h_\varepsilon) =& p'\,\vartheta_0\,\frac{\|K_{s/2} * h_0\|_{p'}^{p'}}{\|h_0\|_1} \int_{\mathbb{R}^N} \psi\,dx + p'\,(1-\vartheta_0)\,\frac{\|K_{s/2} * h_0\|_{p'}^{p'}}{\|h_0\|_m^m} \int_{\mathbb{R}^N} h_0^{m-1}\,\psi\,dx \\
&- p' \int_{\mathbb{R}^N} K_{s/2} * \big(K_{s/2} * h_0\big)^{p'-1}\;\psi\,dx.
\end{aligned}$$

By recalling (4.11) and (4.12), this entails that

$$\int_{\mathbb{R}^N} \left(p'\vartheta_0 \frac{\|K_{s/2} * h_0\|_{p'}^{p'}}{\|h_0\|_1} + p'(1-\vartheta_0)\frac{\|K_{s/2} * h_0\|_{p'}^{p'}}{\|h_0\|_m^m} h_0^{m-1} - p' K_{s/2} * (K_{s/2} * h_0)^{p'-1}\right) \varphi\,h_0\,dx = 0,$$

for every $\varphi \in C_0^\infty(\mathbb{R}^N)$. By the positivity of h_0 on its support (which is either a ball or $\mathbb{R}^N$, as a consequence of Lemma 4.2) and by recalling the expression of ϑ_0 given by (3.10), we obtain

$$\frac{m'}{p_s^*}\,\frac{\|K_{s/2}*h_0\|_{p'}^{p'}}{\|h_0\|_m^m}\,h_0^{m-1} = \mathcal{K}_{s,p}(h_0) - \left(1-\frac{m'}{p_s^*}\right)\frac{\|K_{s/2}*h_0\|_{p'}^{p'}}{\|h_0\|_1} \qquad \text{in } \mathrm{supp}(h_0). \tag{4.16}$$

In order to deduce a condition outside the support of h_0, we take any nonnegative function $\varphi \in C_0^\infty(\mathbb{R}^N)\setminus\{0\}$ and set

$$h_\varepsilon := h_0 + \varepsilon\,\varphi, \qquad \text{for } \varepsilon \ge 0.$$

Since $\varphi \ge 0$ and $\varepsilon \ge 0$, we have $h_\varepsilon \ge 0$ thus by minimality of h_0 for $\mathcal{G}$ we infer

$$\lim_{\varepsilon\to 0^+}\frac{\mathcal{G}(h_\varepsilon)-\mathcal{G}(h_0)}{\varepsilon} \ge 0.$$

By arguing as before, we get

$$\int_{\mathbb{R}^N}\left(p'\,\vartheta_0\,\frac{\|K_{s/2}*h_0\|_{p'}^{p'}}{\|h_0\|_1} + p'\,(1-\vartheta_0)\,\frac{\|K_{s/2}*h_0\|_{p'}^{p'}}{\|h_0\|_m^m}\,h_0^{m-1} - p'\,K_{s/2}*(K_{s/2}*h_0)^{p'-1}\right)\varphi\,dx \ge 0,$$

for every nonnegative function $\varphi \in C_0^\infty(\mathbb{R}^N)$. This entails that

$$p'\,\vartheta_0\,\frac{\|K_{s/2}*h_0\|_{p'}^{p'}}{\|h_0\|_1} - p' K_{s/2}*\left(K_{s/2}*h_0\right)^{p'-1}(x) \ge 0 \qquad \text{in } \mathrm{supp}(h_0)^c. \tag{4.17}$$

The proof is thereby complete, in light of (4.16) and (4.17). □

Remark 4.5. We can express the constants $\mathcal{A}_s$ and $\mathcal{C}_s$ in (4.10) in terms of $H^*_{m,s}$: we have

$$\mathcal{A}_s = \frac{m'}{p_s^*}\,H^*_{m,s}\,\|h\|_m^{p'-m}\left(\frac{\|h\|_m}{\|h\|_1}\right)^{p'\left(\frac{m'}{p_s^*}-1\right)}, \qquad \mathcal{C}_s = \left(1-\frac{m'}{p_s^*}\right)H^*_{m,s}\,\|h\|_1^{p'-1}\left(\frac{\|h\|_m}{\|h\|_1}\right)^{\frac{p'\,m'}{p_s^*}}.$$

4.3. **Regularity properties.** Next we show that any extremal of the HLS inequality (3.9) has compact support and it is bounded. We rely on a *bootstrap argument* based on the combination of the HLS inequality (3.5) and Lemma 3.1.

Lemma 4.6 ($L^\infty-$bound and compactness of the support)**.** *Let $1<p<\infty$ and $0<s\,p<N$. Let $m>(p_s^*)'$. For every extremal $h_s \in L^1_+(\mathbb{R}^N)\cap L^m(\mathbb{R}^N)$ of the HLS inequality (3.9), we have that $h_s \in L^\infty(\mathbb{R}^N)$. Moreover, the support of h_s is compact.*

Proof. We start by proving that $\mathrm{supp}(h_s)$ is compact. Recall that by Lemma 4.2, the support of h_s is either a ball or $\mathbb{R}^N$. By contradiction, assume that $\mathrm{supp}(h_s)=\mathbb{R}^N$. Our assumptions entail that $h_s \in L^{(p_s^*)'}(\mathbb{R}^N)$. By using twice the HLS inequality (3.5), we infer that $\mathcal{K}_{s,p}(h_s) \in L^{p_s^*}(\mathbb{R}^N)$, so in particular it vanishes at infinity. By using (4.9) and by recalling that $\mathcal{C}_s>0$ from (4.10), we get a contradiction.

We now prove that $h_s \in L^\infty(\mathbb{R}^N)$. We set $m_1 := m$ and distinguish two cases according to whether $s\,m_1 \ge N$ or $s\,m_1 < N$.

Case 1: $s\,m_1 \ge N$. Since $h_s \in L^1(\mathbb{R}^N)\cap L^{m_1}(\mathbb{R}^N)$, in particular $h_s \in L^r(\mathbb{R}^N)$ for every $1<r<N/s$. By Corollary 3.3, we infer that

$$K_{s/2}*h_s \in L^t(\mathbb{R}^N), \qquad \text{for every } N/(N-s)<t<\infty,$$

and so

$$(K_{s/2} * h_s)^{p'-1} \in L^{t\,(p-1)}(\mathbb{R}^N), \qquad \text{for every } N/(N-s) < t < \infty.$$

Since $s\,p < N$, we have

$$\lim_{t \to \frac{N}{N-s}} t\,(p-1) < N/s.$$

These considerations entail that

$$(K_{s/2} * h_s)^{p'-1} \in L^{t_1}(\mathbb{R}^N) \cap L^{t_2}(\mathbb{R}^N), \qquad \text{for some } t_1 < N/s \text{ and } t_2 > N/s.$$

By Lemma 3.1, we then obtain $K_{s/2} * \big(K_{s/2} * h_s\big)^{p'-1} \in L^\infty(\mathbb{R}^N)$. In turn, from the Euler-Lagrange equation Lemma 4.4, we conclude that $h_s \in L^\infty(\mathbb{R}^N)$.

Case 2: $s\,m_1 < N$. Since $h_s \in L^1(\mathbb{R}^N) \cap L^{m_1}(\mathbb{R}^N)$, from the HLS inequality (3.5), we infer that $K_{s/2} * h_s \in L^r(\mathbb{R}^N) \cap L^{(m_1)^*_s}(\mathbb{R}^N)$, for every $N/(N-s) < r < (m_1)^*_s$. This entails that

$$(K_{s/2} * h_s)^{p'-1} \in L^{r(p-1)}(\mathbb{R}^N) \cap L^{(m_1)^*_s(p-1)}(\mathbb{R}^N), \qquad \text{for every } N/(N-s) < r < (m_1)^*_s.$$

If $(m_1)^*_s\,(p-1) \geq N/s$, by arguing as in *Case 1*, we conclude that $h_s \in L^\infty(\mathbb{R}^N)$ and we stop. If otherwise $(m_1)^*_s\,(p-1) < N/s$, by the HLS inequality (3.5) we infer

$$K_{s/2} * (K_{s/2} * h_s)^{p'-1} \in L^{((m_1)^*_s\,(p-1))^*_s}(\mathbb{R}^N).$$

In turn, from the Euler-Lagrange equation, this implies that

$$h_s \in L^{m_2}(\mathbb{R}^N),$$

where we have set

$$m_2 := (m-1)\,((m_1)^*_s\,(p-1))^*_s = \frac{N\,(m-1)\,(p-1)\,m_1}{N - s\,p\,m_1},$$

and observe that $m_2 > m_1$, being this condition equivalent to $m > (p^*_s)'$. In general, let $k \in \mathbb{N} \setminus \{0\}$ and assume that $m_i < N/s$ for every $1 \leq i \leq k$. We define

$$m_{k+1} := (m-1)\,((m_k)^*_s\,(p-1))^*_s = \frac{N\,(m-1)\,(p-1)\,m_k}{N - s\,p\,m_k}. \tag{4.18}$$

We want to prove by induction that $m_{k+1} > m_k$. Our inductive assumption reads as

$$m_{i+1} > m_i, \qquad \text{for every } 1 \leq i \leq k-1. \tag{4.19}$$

Since $m > (p^*_s)'$, we have

$$m > \frac{(N - s\,m)\,p}{N\,(p-1)} = \frac{(N - s\,m_1)\,p}{N\,(p-1)} > \frac{(N - s\,m_k)\,p}{N\,(p-1)},$$

where in the last inequality we used (4.19). In particular

$$m > \frac{(N - s\,m_k)\,p}{N\,(p-1)} \quad \Longleftrightarrow \quad m_{k+1} > m_k,$$

as we can infer by recalling (4.18). We now claim that

$$\lim_{k \to \infty} m_k = +\infty. \tag{4.20}$$

This would entail that for some $\overline{k}$ we must have $m_{\overline{k}} \geq N/s$, thus, by arguing as in *Case 1*, this would also end the proof. Since $m_k \geq m$, we have

$$\frac{m_{k+1}}{m_k} = \frac{N\,(m-1)\,(p-1)}{N - s\,p\,m_k} \geq \frac{N\,(m-1)\,(p-1)}{N - s\,p\,m} = \frac{m_2}{m_1}, \qquad \text{for every } k \geq 1.$$

Moreover, as we have already observed

$$\frac{m_2}{m_1} > 1 \quad \iff \quad m > (p_s^*)'.$$

The last two facts entail that

$$\liminf_{k\to\infty} \frac{m_{k+1}}{m_k} > 1,$$

and so our claim (4.20). □

In the next lemma, we can readily adapt the argument of [19, Theorem 8] to infer Hölder regularity for extremals of (3.9).

Lemma 4.7 (Hölder regularity)**.** *Let $1 < p < \infty$ and $0 < s\,p < N$. Let $m > (p_s^*)'$. For every extremal $h_s \in L^1_+(\mathbb{R}^N) \cap L^m(\mathbb{R}^N)$ of the HLS inequality (3.9), we have*

- *for $0 < s < 1/2$*

$$\begin{cases} h_s \in C^{0,1}(\mathbb{R}^N), & \text{if } m \le 2, \\ h_s \in C^{0,\frac{1}{m-1}}(\mathbb{R}^N), & \text{if } 2 < m < m^*, \\ h_s \in C^{0,\gamma}(\mathbb{R}^N), & \text{if } m^* \le m, \end{cases}$$

where $m^* = \dfrac{2-2s}{1-2s}$ and $\gamma \in \left(0, \dfrac{2\,s}{m-2}\right)$.

- *for $s \ge 1/2$*

$$h_s \in C^{0,\gamma}(\mathbb{R}^N), \qquad \text{where } \gamma = \min\left\{1, \frac{1}{m-1}\right\}.$$

Moreover, h_s has $C^\infty-$regularity in the interior of its support.

Proof. First, we assume $0 < s < 1/2$. We will take advantage of the embeddings between Bessel potential spaces, fractional Sobolev spaces and Hölder spaces. We briefly recall that for $1 \le q < \infty$ the fractional Sobolev space $W^{s,q}(\mathbb{R}^N)$ is given by

$$W^{s,q}(\mathbb{R}^N) = \left\{u \in L^q(\mathbb{R}^N) : [u]_{W^{s,q}(\mathbb{R}^N)} < \infty\right\},$$

where $[\,\cdot\,]_{W^{s,q}(\mathbb{R}^N)}$ denotes the Gagliardo-Slobodeckiĭ seminorm

$$[u]_{W^{s,q}(\mathbb{R}^N)} := \left(\iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|u(x)-u(y)|^q}{|x-y|^{N+s\,q}}\,dxdy\right)^{\frac{1}{q}}.$$

The Bessel potential spaces $\mathcal{L}^{s,q}(\mathbb{R}^N)$, where $1 < q < \infty$, are defined through the Fourier transform, see for instance [47, Section V.3], [52, Section 2.2.2], [45, Section 27.3]. They can be characterized as

$$\mathcal{L}^{s,q}(\mathbb{R}^N) = \{u \in L^q(\mathbb{R}^N) : u = K_{s/2} * h, \text{ for some } h \in L^q(\mathbb{R}^N)\},$$

see for example [46, Theorem 2] (or also [45, Theorem 26.8, Theorem 27.3]). By recalling [47, Theorem 5, pag. 155] and [22, Theorem 4.47], the following continuous embeddings holds true:

$$\mathcal{L}^{s,q}(\mathbb{R}^N) \hookrightarrow W^{s,q}(\mathbb{R}^N), \qquad \text{for } q \ge 2, \tag{4.21}$$

and

$$W^{s,q}(\mathbb{R}^N) \hookrightarrow C^{0,\gamma}(\mathbb{R}^N), \qquad \text{where } \gamma = s - N/q, \text{ for } q > N/s. \tag{4.22}$$

Let h_s be as in the statement. By Lemma 4.2, we can assume it is radially nonincreasing. By Lemma 3.1, Lemma 4.4 and Lemma 4.6, we have that

$$h_s \in L^1(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N) \quad \text{and} \quad K_{s/2}*h_s \in L^q(\mathbb{R}^N), \quad \text{for every } N/(N-s) < q \leq \infty. \tag{4.23}$$

In particular, since $N/(N-s) < N/s$ for every $0 < s < 1/2$, we have that

$$K_{s/2} * h_s \in \mathcal{L}^{s,q}(\mathbb{R}^N), \qquad \text{for every } q > N/s.$$

In view of the embeddings (4.21) and (4.22), this entails that

$$K_{s/2} * h_s \in C^{0,\gamma}(\mathbb{R}^N), \qquad \text{where } \gamma = s - N/q, \text{ for } q > N/s.$$

Since h_s is radially nonincreasing, so is of $K_{s/2} * h_s$, see [13]. Moreover, $K_{s/2} * h_s$ is clearly positive, bounded and vanishing at infinity. In particular it is bounded away from zero on compact sets. For these reasons we get $(K_{s/2}*h_s)^{p'-1} \in C^{0,\gamma}_{\text{loc}}(\mathbb{R}^N)$. Therefore for every $R > 0$, by using (4.23), Lemma 3.1 and the identity

$$(-\Delta)^{s/2}\left(K_{s/2} * (K_{s/2} * h_s)^{p'-1}\right) = (K_{s/2} * h_s)^{p'-1} \qquad \text{in } B_{2R},$$

from [44, Corollary 3.5] we can infer that

$$\begin{aligned} &\|K_{s/2} * (K_{s/2}*h_s)^{p'-1}\|_{C^{0,\gamma+s}(B_R)} \leq \\ &\qquad \leq c\left(\|K_{s/2} * (K_{s/2} * h_s)^{p'-1}\|_{L^\infty(\mathbb{R}^N)} + \|(K_{s/2} * h_s)^{p'-1}\|_{C^{0,\gamma}(B_{2R})}\right), \end{aligned} \tag{4.24}$$

for some $c = c(N, s, R) > 0$ (notice that since $s < 1/2$, then $\gamma + s$ is not an integer as $\gamma = s - N/q$) and so

$$K_{s/2} * (K_{s/2} * h_s)^{p'-1} \in C^{0,\gamma+s}(B_R), \qquad \text{where } \gamma = s - N/q, \text{ for } q > N/s.$$

From the Euler-Lagrange equation provided by Lemma 4.4, this entails that $h_s^{m-1} \in C^{0,\gamma+s}(B_R)$, for $R > 0$. By using the fact that h_s has compact support, we infer

$$h_s \in C^{0,(\gamma+s)\alpha}(\mathbb{R}^N), \quad \text{where } \alpha = \min\left\{1, \frac{1}{m-1}\right\} \text{ and } \gamma = s - N/q, \text{ for } q > N/s. \tag{4.25}$$

Now we distinguish three cases.

Case 1: $(p_s^*)' < m \leq 2$. By using (4.23) and (4.25), from [44, Corollary 3.5] we have that

$$K_{s/2} * h_s \in C^{0,\gamma+2s}_{\text{loc}}(\mathbb{R}^N),$$

if $\gamma+2s$ is not an integer. Since $K_{s/2}*h_s$ is bounded and bounded away from zero on compact sets, as before we deduce

$$(K_{s/2} * h_s)^{p'-1} \in C^{0,\gamma+2s}_{\text{loc}}(\mathbb{R}^N).$$

If $\gamma+2s > 1$, we get $(K_{s/2}*h_s)^{p'-1} \in C^{0,1}_{\text{loc}}(\mathbb{R}^N)$ and so also $K_{s/2}*(K_{s/2}*h_s)^{p'-1} \in C^{0,1}_{\text{loc}}(\mathbb{R}^N)$. Thus, in light of Lemma 4.4, using that $m \leq 2$ and the compactness of the support of h_s, we get $h_s \in C^{0,1}(\mathbb{R}^N)$ as desired. On the other hand, if $\gamma + 2s < 1$ we newly apply (4.24) and [44, Corollary 3.5] obtaining that

$$K_{s/2} * (K_{s/2} * h_s)^{p'-1} \in C^{0,\gamma+3s}_{\text{loc}}(\mathbb{R}^N),$$

if $\gamma + 3\,s$ is not an integer. Observe that we gained $2s$ derivatives starting from (4.25), and the gain in regularity depends therefore on s but not on p. In other words, the regularity gain provided by the nonlinear potential $\mathcal{K}_{s,p}$ does not depend on p and it is the same of the linear potential $\mathcal{K}_{s,2}$. Thus, the proof gets reduced to the case $p = 2$ which is given in [19, Theorem 8]. For this reason, we just sketch the conclusion of the argument, omitting some details. We take an integer $j \geq 1$ such that

$$\frac{1}{2\,(j+1)} < s < \frac{1}{2\,j}, \tag{4.26}$$

and set $\gamma_j := \gamma + (j-1)\,2\,s = 2\,s\,j - N/q$, where $q > N/s$ is choosen large enough so that $1 - 2\,s < \gamma_j < 1$. This is a feasible choice thanks to (4.26). By iterating the previous argument j times starting from (4.25), we get $K_{s/2} * (K_{s/2} * h_s)^{p'-1} \in C^{0,\gamma_j+2s}_{\rm loc}(\mathbb{R}^N) \subseteq C^{0,1}_{\rm loc}(\mathbb{R}^N)$, being $\gamma_j + 2s > 1$ by construction. By using the Euler-Lagrange equation provided by Lemma 4.4 and the compactness of $\mathrm{supp}(h_s)$ from Lemma 4.6 and since $m \leq 2$, we conclude that $h_s \in C^{0,1}(\mathbb{R}^N)$.

Case 2: $2 < m < m^*$. Starting from (4.25), we can improve the Hölder regularity of h_s by a bootstrap argument, as in the previous case. We give the details of one iteration, in order to clarify that the same argument used in the proof of [19, Theorem 8] still holds. In light of (4.23) and (4.25), we can apply [44, Corollary 3.5] to infer that

$$(K_{s/2} * h_s)^{p'-1} \in C^{0,\frac{\gamma+s}{m-1}+s}_{\rm loc}(\mathbb{R}^N), \qquad \text{where } \gamma = s - N/q, \text{ for } q > N/s,$$

if $(\gamma + s)/(m-1) + s$ is not an integer. By newly applying [44, Corollary 3.5], we obtain

$$K_{s/2} * (K_{s/2} * h_s)^{p'-1} \in C^{0,\frac{\gamma+s}{m-1}+2\,s}_{\rm loc}(\mathbb{R}^N),$$

if $(\gamma+s)/(m-1)+2\,s$ is not an integer. By reasoning as in the previous case, if $\gamma+2\,s/(m-1) + 2\,s > 1$, we have $h_s^{m-1} \in C^{0,1}(\mathbb{R}^N)$ and so $h_s \in C^{0,\frac{1}{m-1}}(\mathbb{R}^N)$. On the other hand, if $\gamma + 2\,s/(m-1) + 2\,s < 1$, by always using the Euler-Lagrange equation provided by Lemma 4.4 and the fact that h_s has compact support, we obtain $h_s^{m-1} \in C^{0,\frac{\gamma+s}{m-1}+2\,s}(\mathbb{R}^N)$, which entails that

$$h_s \in C^{0,\gamma_1}(\mathbb{R}^N), \qquad \text{where } \gamma_1 = \frac{\gamma+s}{(m-1)^2} + \frac{2\,s}{m-1},$$

if $(\gamma + s)/(m-1) + 2\,s$ is not an integer, where $\gamma = s - N/q$, for $q > N/s$. In general, by iterating this argument following [19, Theorem 8], we can improve the Hölder regularity of h_s to infer that $h_s^{m-1} \in C^{0,1}(\mathbb{R}^N)$, which yields $h_s \in C^{0,\frac{1}{m-1}}(\mathbb{R}^N)$, as desired.

Case 3: $m \geq m^*$. We can proceed with the same bootstrap argument, however without reaching Lipschitz regularity of h_s^{m-1}. We observe that [19, Remark 2], to which we refer, still holds and gives the desired result.

In order to conclude, we observe that the case $1/2 \leq s$ is simpler than the case $0 < s < 1/2$ and can be treated in the same way, up to some minor modifications, as done in [19, Theorem 8]. Eventually, by using the same argument of [19, Theorem 10], from Lemma 4.4 and Lemma 4.6, we obtain the $C^\infty-$regularity of h_s in the interior of its support. □

We end this section by remarking that the extremals of (4.1) are always in $W^{1,1}(\mathbb{R}^N)$:

Corollary 4.8. *Let $1 < p < \infty$ and $0 < s\,p < N$. Let $m > (p_s^*)'$. For every function $h_s \in L^1_+(\mathbb{R}^N) \cap L^m(\mathbb{R}^N)$ attaining the supremum in (4.1), we have $h_s \in W^{1,1}(\mathbb{R}^N)$.*

Proof. The desired conclusion follows from Lemma 4.2, Lemma 4.4 and Lemma 4.6 by arguing as in [28, Proposition 2.10]. □

5. Minimizers of the energy functional

In this section, we analyze minimizers of functional $\mathcal{F}_{s,p}$ over $\mathcal{Y}_M$ thus concluding the proof of Theorem 2.1. We start by considering the *diffusion dominated regime*, that is the case $m > m_c$.

Proposition 5.1 (Diffusion dominated regime)**.** *Let $1 < p < \infty$ and $0 < s\,p < N$. Let $m > m_c$ and let $\chi, M > 0$. Define the functional*

$$\mathcal{Y}_M \ni \rho \mapsto \Lambda(\rho) := \left(\frac{\|K_{s/2} * \rho\|_{p'}^{p'\,(m-1)}}{\|\rho\|_m^{m\,(m_c-1)}} \right)^{\frac{1}{m-m_c}},$$

which is invariant by mass-invariant dilations. Let moreover

$$\kappa := \left(\frac{\chi}{p'}\right)^{\frac{m-1}{m-m_c}} \left(\frac{p'}{p_s^*}\right)^{\frac{m_c-1}{m-m_c}} \left(\frac{m_c - m}{m-1}\right).$$

For every $\rho \in \mathcal{Y}_M$, there exists a unique positive number $\lambda_*(\rho)$, called the optimal dilation factor of ρ, such that

$$\mathcal{F}_{s,p}(\rho^\lambda) \geq \mathcal{F}_{s,p}(\rho^{\lambda_*(\rho)}) = \kappa\,\Lambda(\rho) \quad \text{for every } \lambda > 0, \qquad \text{with equality only if } \lambda = \lambda_*(\rho).$$

It is expressed as

$$\lambda_*(\rho) = \left(\frac{\chi}{p_s^*}\, \frac{\|K_{s/2} * \rho\|_{p'}^{p'}}{\|\rho\|_m^m} \right)^{\frac{1}{N\,(m-m_c)}}. \tag{5.1}$$

Proof. Let $\rho \in \mathcal{Y}_M$. For $\lambda > 0$, we consider the function given by

$$\lambda \mapsto f_\rho(\lambda) := \mathcal{F}_{s,p}(\rho^\lambda) = \frac{\lambda^{N\,(m-1)}}{m-1}\,\|\rho\|_m^m - \lambda^{N\,(m_c-1)}\,\frac{\chi}{p'}\,\|K_{s/2} * \rho\|_{p'}^{p'}. \tag{5.2}$$

Recall that, since $1 < p < N/s$, we have $N/(N-s) < p' < \infty$, thus $m_c = p'(1 - s/N) > 1$. By optimizing with respect to λ, we get

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\mathcal{F}(\rho^\lambda) = N\,\lambda^{N\,(m-1)-1}\,\|\rho\|_m^m - \frac{N\,(m_c-1)\,\chi}{p'}\lambda^{N\,(m_c-1)-1}\,\|K_{s/2} * \rho\|_{p'}^{p'} = 0. \tag{5.3}$$

The unique extremal is given by (5.1). Clearly, at $\lambda_*(\rho)$, the function given by (5.2) attains a global minimum, and notice also that we have

$$\lim_{\lambda \to +\infty} \mathcal{F}_{s,p}(\rho^\lambda) = +\infty \qquad \text{and} \qquad \lim_{\lambda \to 0} \mathcal{F}_{s,p}(\rho^\lambda) = 0.$$

Furthermore, by using (5.1) we can write

$$\begin{aligned}\mathcal{F}_{s,p}(\rho^{\lambda_*(\rho)}) =& \frac{1}{m-1}\left(\frac{\chi}{p_s^*}\frac{\|K_{s/2} * \rho\|_{p'}^{p'}}{\|\rho\|_m^m} \right)^{\frac{m-1}{m-m_c}} \|\rho\|_m^m - \frac{\chi}{p'}\left(\frac{\chi}{p_s^*}\frac{\|K_{s/2} * \rho\|_{p'}^{p'}}{\|\rho\|_m^m} \right)^{\frac{m_c-1}{m-m_c}} \|K_{s/2} * \rho\|_{p'}^{p'} \\ =& \kappa\,\Lambda(\rho) < 0,\end{aligned}$$

where Λ and κ are defined as in the statement. □

For the *fair competition regime*, that is $m = m_c$, we have the following two sided-estimate for the energy, which extends [6, Proposition 3.4].

Proposition 5.2 (Fair competition regime)**.** *Let $1 < p < \infty$ and $0 < s\,p < N$. Let $\chi, M > 0$. For every $\rho \in \mathcal{Y}_M$, we have*

$$\frac{\chi}{p'}\, H^*_{m_c,s} \left(M_c^{p' \frac{s}{N}} - M^{p' \frac{s}{N}} \right) \|\rho\|_{m_c}^{m_c} \le \mathcal{F}_{s,p}(\rho) \le \frac{\chi}{p'}\, H^*_{m_c,s} \left(M_c^{p' \frac{s}{N}} + M^{p' \frac{s}{N}} \right) \|\rho\|_{m_c}^{m_c},$$

where $H^*_{m_c,s}$ is given by (4.1) and M_c is given by (2.5).

Proof. For every $\rho \in \mathcal{Y}_M$, by recalling (4.1), we get

$$\begin{aligned}\mathcal{F}_{s,p}(\rho) = \frac{\|\rho\|_{m_c}^{m_c}}{m_c - 1} - \frac{\chi}{p'} \|K_{s/2} * \rho\|_{p'}^{p'} &\ge \left(\frac{1}{m_c - 1} - \frac{\chi}{p'}\, H^*_{m_c,s}\, M^{p' \frac{s}{N}} \right) \|\rho\|_{m_c}^{m_c} \\ &= \frac{\chi}{p'}\, H^*_{m_c,s} \left(M_c^{p' \frac{s}{N}} - M^{p' \frac{s}{N}} \right) \|\rho\|_{m_c}^{m_c},\end{aligned}$$

where M_c is given by (2.5). On the other hand, by recalling (2.2), we also have

$$\mathcal{F}_{s,p}(\rho) \le \frac{\chi}{p'}\, H^*_{m_c,s} \left(M_c^{p' \frac{s}{N}} + M^{p' \frac{s}{N}} \right) \|\rho\|_{m_c}^{m_c},$$

which yields the claimed estimate. □

Proposition 5.3 (Infimum of $\mathcal{F}_{s,p}$)**.** *Let $1 < p < \infty$, $0 < s\,p < N$ and let $\chi > 0$. For every $M > 0$, we have*

$$\inf_{\rho \in \mathcal{Y}_M} \mathcal{F}_{s,p}(\rho) = \begin{cases} -\infty, & \text{if } 1 < m < m_c, \\ \nu_s, & \text{if } m = m_c, \\ \mu_s, & \text{if } m > m_c, \end{cases}$$

for some $\mu_s = \mu\,(N, p, s, \chi, m, M) < 0$, where $\nu_s = \nu_s\,(N, p, s, \chi, M)$ is given by

$$\nu_s = \begin{cases} 0, & \text{if } 0 < M \le M_c, \\ -\infty, & \text{if } M > M_c, \end{cases}$$

being M_c the critical mass introduced in (2.5).

Proof. As in the beginning of the proof of Proposition 5.1, we take $\rho \in \mathcal{Y}_M$ and consider the function $f_\rho(\lambda) := \mathcal{F}_{s,p}(\rho^\lambda)$, $\lambda > 0$, whose expression is given by (5.2) for every $m > 1$.

If $1 < m < m_c$, by sending $\lambda \nearrow \infty$ we infer that, for every $M > 0$,

$$\inf_{\rho \in \mathcal{Y}_M} \mathcal{F}_{s,p}(\rho) = \lim_{\lambda \to \infty} f_\rho(\lambda) = -\infty.$$

If $m > m_c$, from Proposition, 5.1 and by recalling (2.2) and (3.10), we get

$$\Lambda(\rho)^{\frac{m - m_c}{m-1}} = \frac{\|K_{s/2} * \rho\|_{p'}^{p'}}{\|\rho\|_m^{\frac{m}{m-1} \frac{p'}{p_s^*}}} = \frac{\|K_{s/2} * \rho\|_{p'}^{p'}}{\|\rho\|_m^{p' (1 - \vartheta_0)}} = M^{p' \vartheta_0} \left(\frac{\|K_{s/2} * \rho\|_{p'}^{p'}}{M^{p' \vartheta_0}\, \|\rho\|_m^{p' (1 - \vartheta_0)}} \right). \tag{5.4}$$

Since $m > m_c > (p_s^*)'$, the Hardy-Littlewood-Sobolev type inequality (3.9) entails therefore that

$$\sup_{\rho\in\mathcal{Y}_M} \Lambda(\rho) \in (0,+\infty)$$

for every $M > 0$. In turn, by Proposition 5.1

$$\inf_{\rho\in\mathcal{Y}_M} \mathcal{F}_{s,p}(\rho) = \inf_{\rho\in\mathcal{Y}_M} (\kappa\,\Lambda(\rho)) = \kappa \sup_{\rho\in\mathcal{Y}_M} \Lambda(\rho)$$

since $\kappa < 0$, thus

$$\mu_s = \mu(N,p,s,\chi,m,M) := \inf_{\rho\in\mathcal{Y}_M} \mathcal{F}_{s,p}(\rho) \in (-\infty,0).$$

If $m = m_c$, we need to distinguish two cases.

Case $0 < M \le M_c$. We take $\rho \in \mathcal{Y}_M$ and test the energy $\mathcal{F}_{s,p}$ with its mass invariant dilations ρ^λ given by (1.5). By the change of variable formula, we have

$$\|\rho^\lambda\|_{m_c}^{m_c} = \lambda^{N(m_c-1)}\|\rho\|_{m_c}^{m_c}, \qquad \text{for} \;\; \lambda > 0.$$

Since $m_c > 1$, using Proposition 5.2 and sending $\lambda \searrow 0$ we get

$$\inf_{\rho\in\mathcal{Y}_M} \mathcal{F}_{s,p}(\rho) = \lim_{\lambda\to 0} \mathcal{F}_{s,p}(\rho^\lambda) = 0. \tag{5.5}$$

Case $M > M_c$. Let h_s be the unit mass extremal (with barycenter at the origin) of the HLS-type inequality (3.9), provided by Lemma 4.2. We set for every $\lambda > 0$

$$\rho_\lambda(x) := M\,\lambda^N\,h_s(\lambda\,x) \in \mathcal{Y}_M.$$

By recalling (2.2) and (2.5), we have

$$\begin{aligned}\mathcal{F}_{s,p}(\rho_\lambda) &= \frac{\|\rho_\lambda\|_{m_c}^{m_c}}{m_c-1} - \frac{\chi}{p'}\,\|K_{s/2}*\rho_\lambda\|_{p'}^{p'} = \lambda^{2\,N\,m_c-N}\,\frac{M^{m_c}}{m_c-1} - \frac{\chi}{p'}\,\lambda^{(N-s)\,p'-N}\,M^{p'}\,H^*_{m_c,s}\\ &= \lambda^{p'\,(N-s)}\,\frac{p_s^*}{p'}\,M^{p'}\left[\frac{1}{M^{\frac{s\,p'}{N}}} - \frac{1}{M_c^{\frac{s\,p'}{N}}}\right] < 0,\end{aligned}$$

from which, by sending $\lambda \to \infty$, we get the desired conclusion. □

Remark 5.4. By the previous proposition, we infer that there are no minimizers in $\mathcal{Y}_M$ of the energy functional $\mathcal{F}_{s,p}$ in the aggregation dominated regime $m \in (1,m_c)$, whatever the value of the mass $M > 0$. In the fair competition regime, $m = m_c$, still there are no minimizers in $\mathcal{Y}_M$ for prescribed mass $M \in (M_c,\infty)$. Also for values of the mass $M \in (0,M_c)$, there are no minimizers of $\mathcal{F}_{s,p}$ in $\mathcal{Y}_M$, in light of the leftmost inequality in Proposition 5.2.

Remark 5.5. By inspecting the proof of Proposition 5.1 we can infer that, for every $m > 1$, any critical point ρ of $\mathcal{F}_{s,p}$ necessarily satisfies the relevant identity

$$\frac{\chi}{p_s^*}\|K_{s/2}*\rho\|_{p'}^{p'} = \|\rho\|_m^m. \tag{5.6}$$

Indeed, (5.2) holds for every $m > 1$. Therefore imposing criticality of ρ only with respect to mass invariant dilations, i.e. imposing (5.3), yields (5.6). Notice that if $1 < m < m_c$ then ρ is a maximum, and not a minumum, in the family $\{\rho^\lambda\}_{\lambda>0}$. Notice also that, in the case $m = m_c$, (5.6) is equivalent to $\mathcal{F}_{s,p}(\rho) = 0$.

Next, we discuss the relation between extremals of (3.9) and minimizers of $\mathcal{F}_{s,p}$.

Corollary 5.6 (Extremals HLS vs minimizers of $\mathcal{F}_{s,p}$)**.** *Let $1<p<\infty$ and $0<s\,p<N$.*

Let $m\ge m_c$ and let $M>0$. If $\rho_s\in\mathcal{Y}_M$ attains the infimum of the energy functional $\mathcal{F}_{s,p}$, among functions in $\mathcal{Y}_M$, then ρ_s is an extremal of the Hardy-Littlewood-Sobolev type inequality (3.9), that is ρ_s attains the supremum in (4.1).

Viceversa, for $m>m_c$, if $M>0$ and $\rho_s\in\mathcal{Y}_M$ is an extremal of the Hardy-Littlewood-Sobolev type inequality (3.9), then its optimal dilation, in the sense of Proposition 5.1*, attains the infimum of $\mathcal{F}_{s,p}$ over $\mathcal{Y}_M$. If $m=m_c$ and $\rho_s\in\mathcal{Y}_{M_c}$ is an extremal of the Hardy-Littlewood-Sobolev type inequality (3.9), then ρ_s is a minimizer of $\mathcal{F}_{s,p}$ over $\mathcal{Y}_{M_c}$.*

Proof. Assume first that $m>m_c$, $M>0$. Let $\rho_s\in\mathcal{Y}_M$ be a minimizer of $\mathcal{F}_{s,p}$ over $\mathcal{Y}_M$. By Proposition 5.1, we infer that ρ_s maximizes Λ among functions in $\mathcal{Y}_M$, where Λ is the functional defined therein. In particular, by Remark 4.1 and by (5.4), ρ_s is an extremal of the Hardy-Littlewood-Sobolev type inequality (3.9). On the other hand, let $\rho_s\in\mathcal{Y}_M$ be an extremal of the Hardy-Littlewood-Sobolev inequality (3.9) (existence of an extremal in $\mathcal{Y}_M$ is guaranteed by Remark 4.3). Therefore it also satisfies

$$\Lambda(\rho_s)\ge\Lambda(\rho)\qquad\text{for every }\rho\in\mathcal{Y}_M,$$

in view of (5.4) and Remark 4.1. Its optimal dilation in the sense of Proposition 5.1, which is still an extremal of (3.9) in light of Remark 4.1, is given by

$$\widetilde{\rho_s}:=\rho_s^{\lambda_*(\rho_s)},$$

for $\lambda_*(\rho_s)$ as in (5.1). We claim that $\widetilde{\rho_s}$ minimizes the energy functional $\mathcal{F}_{s,p}$ among all functions in $\mathcal{Y}_M$. Indeed, for every $\rho\in\mathcal{Y}_M$, by using the maximality of ρ_s, the invariance by dilations property of Λ and Proposition 5.1, we have

$$\mathcal{F}_{s,p}(\widetilde{\rho_s})=\mathcal{F}_{s,p}(\rho_s^{\lambda_*(\rho_s)})=\kappa\,\Lambda(\rho_s)\le\kappa\,\Lambda(\rho)\le\mathcal{F}_{s,p}(\rho)$$

where the first inequality comes by recalling that κ is negative. This proves the claim.

If $m=m_c$, take any minimizer $\rho_s\in\mathcal{Y}_{M_c}$ of $\mathcal{F}_{s,p}$ over $\mathcal{Y}_{M_c}$. By Proposition 5.3 we have $\mathcal{F}_{s,p}(\rho_s)=0$, that is

$$\frac{1}{m_c-1}\,\|\rho_s\|_{m_c}^{m_c}=\frac{\chi}{p'}\,\|K_{s/2}*\rho_s\|_{p'}^{p'}. \tag{5.7}$$

By recalling (2.5) and (4.1), this entails that

$$\frac{\|K_{s/2}*\rho_s\|_{p'}^{p'}}{M_c^{\frac{p'\,s}{N}}\,\|\rho_s\|_{m_c}^{m_c}}=H^*_{m_c,s}, \tag{5.8}$$

as desired. On the other hand, let $\rho_s\in\mathcal{Y}_{M_c}$ be an extremal of (3.9). Then, always by recalling (2.5), (5.8) implies (5.7) that is $\mathcal{F}_{s,p}(\rho_s)=0$, proving that ρ_s minimizes $\mathcal{F}_{s,p}$ over $\mathcal{Y}_{M_c}$ in view of Proposition 5.3. □

The equation satisfied by the global minimizers of $\mathcal{F}_{s,p}$, provided by Corollary 5.6, reads as follows.

Lemma 5.7. *Let $1<p<\infty$ and $0<s\,p<N$, and let $\chi,M>0$. For $m>m_c$, if $\rho_s\in\mathcal{Y}_M$ is a minimizer of $\mathcal{F}_{s,p}$ over $\mathcal{Y}_M$ then it solves*

$$\frac{m}{m-1}\rho_s^{m-1}=\Big(\chi\,\mathcal{K}_{s,p}(\rho_s)-\mathcal{D}_s\Big)_+\qquad\text{in }\mathbb{R}^N, \tag{5.9}$$

with

$$\mathcal{D}_s := \left(\frac{p_s^* - m'}{M}\right) \|\rho_s\|_m^m = \left(\frac{p_s^* - m'}{M}\right) \frac{\chi}{p_s^*} \|K_{s/2} * \rho_s\|_{p'}^{p'}. \tag{5.10}$$

In particular, it is bounded with compact support, radially nonincreasing, $W^{1,1}(\mathbb{R}^N)$ *and satisfies the Hölder regularity properties of* Lemma 4.7*. The same holds for* $m = m_c$*, by taking* $M = M_c$*.*

Proof. Let $\rho_s \in \mathcal{Y}_M$ be as in the statement. By Corollary 5.6, we have that ρ_s is an extremal of the Hardy-Littlewood-Sobolev type inequality (3.9). In light of Lemma 4.4, it satisfies (4.9). If $m > m_c$, Proposition 5.1 implies that ρ_s coincides with its optimal dilation, i.e., $\lambda_*(\rho_s) = 1$, where λ_* is given by (5.1), so that (5.6) holds. If $m = m_c$, then Remark 5.4 and Lemma 5.3 imply $M = M_c$ and $\mathcal{F}_{s,p}(\rho_s) = 0$ which in turn directly yields (5.6). By inserting (5.6) in (4.10), we see that (4.9) becomes (5.9)-(5.10). By Corollary 4.8, Lemma 4.6 and Lemma 4.7 we then infer the desired conclusions. □

Remark 5.8. For future purposes we record the following identities, valid for $m > m_c$, involving the constant $\mathcal{D}_s$ appearing in the Euler-Lagrange equation (5.9) satisfied by a minimizer of $\mathcal{F}_{s,p}$ over $\mathcal{Y}_M$:

$$\mathcal{D}_s = \left(\frac{p_s^* - m'}{M}\right) \|\rho_s\|_m^m = \left(\frac{p_s^* - m'}{M}\right) \frac{\chi}{p_s^*} \|K_{s/2} * \rho_s\|_{p'}^{p'} = \left(\frac{p_s^* - m'}{M}\right) \frac{(m_c - 1)(m-1)}{m_c - m} \mathcal{F}_{s,p}(\rho_s),$$

which follows from (5.10) and the definition of $\mathcal{F}_{s,p}$.

Remark 5.9. The conclusion of Lemma 5.7 holds also for continuous radially decreasing critical points of $\mathcal{F}_{s,p}$ over $\mathcal{Y}_M$ in the case $(p_s^*)' < m < m_c$. These are defined as continuous radially decreasing solutions to (5.9), with $\mathcal{D}_s$ still expressed by (5.10). Indeed, a first variation argument along the line of Lemma 4.4 proves that the Euler-Lagrange equation that is necessarily satisfied by a continuous radially decreasing critical point ρ of $\mathcal{F}_{s,p}$, constrained to $\mathcal{Y}_M$, is of the form (2.6), for some suitable constant $\mathcal{Q}$ having the role of Lagrange multiplier for the mass constraint. Moreover $\mathcal{Q}$ necessarily coincides with $\mathcal{D}_s$ from (5.10) in view of the criticality condition (5.6). Existence of such critical points, for every mass $M > 0$, is deduced from the existence of radially decreasing extremals of the HLS inequality (3.9) having mass M and satisfying (5.6), which is guaranteed by Lemma 4.2 and Remark 4.1: notice indeed that an extremal of mass M satisfies (5.6) after taking a mass invariant dilation (thus preserving extremality), see Remark 5.5. A HLS extremal having mass M and satisfying (5.6) does satisfy (5.9)-(5.10), thanks to Lemma 4.4, as seen by plugging (5.6) in (4.9)-(4.10). The further regularity of such critical points is then deduced in the same way starting from the Euler-Lagrange equation, see Lemma 4.6, Lemma 4.7 and Corollary 4.8.

Proof of Theorem 2.1. The first part of Theorem 2.1 follows by Lemma 4.2, Lemma 4.4, Lemma 4.6 and Lemma 4.7. The second part follows by Corollary 5.6 and Lemma 5.7. □

6. The limit $s \to 0$

This section is devoted to study the asymptotic behavior of the minimizers ρ_s, provided by Corollary 5.6, as s tends to 0. Since the critical exponent m_c given by (2.2) tends to p', as s goes to zero, we need discuss separately three cases according to whether $m < p'$, $m = p'$ or $m > p'$.

If $m < p'$, we have $m < (p_s^*)'$ for s small enough. By Proposition 5.3, we get $\inf_{\rho \in \mathcal{Y}_M} \mathcal{F}_{s,p} = -\infty$ thus there are no minimizers of $\mathcal{F}_{s,p}$, and not even stationary states obtained from extremals of the HLS inequality according to Remark 5.9. The case $m = p'$ (that we call the limiting fair competition regime) will be discussed in Section 6.3. The limiting diffusion dominated regime $m > p'$ is the most interesting one and it will be treated in the first part of this section. The main property is contained in Theorem 6.7 below: we will prove that if $m > p'$, any family of minimizers $(\rho_s)_{s \in (0,N/p)}$ of the free energy functional $\mathcal{F}_{s,p}$ provided by Corollary 5.6 strongly converges as $s \searrow 0$ to the unique minimizer in $\mathcal{Y}_M$ of a limit functional $\mathcal{F}_0$. In addition, $\mathcal{F}_{s,p}$ $\Gamma-$converges to $\mathcal{F}_0$ on $\mathcal{Y}_M$, with respect to the strong convergence in $L^{p'}(\mathbb{R}^N)$, see Proposition 6.8.

6.1. **The limit functional.** Concerning the limit functional $\mathcal{F}_0$ defined by (2.7) we have the following

Proposition 6.1. *Let $1 < p < \infty$ and $m > p'$. There exists a unique radially symmetric and nonincreasing minimizer of $\mathcal{F}_0$ in $\mathcal{Y}_M$, given by*

$$\rho_0(x) = \left(\frac{\chi}{p}\right)^{\frac{1}{m-p'}} 1_{B_{R_0}}(x), \qquad \textit{where} \;\; R_0 = \left(\frac{M}{\omega_N}\left(\frac{p}{\chi}\right)^{\frac{1}{m-p'}}\right)^{\frac{1}{N}}. \tag{6.1}$$

Proof. Let $\rho \in \mathcal{Y}_M$ and let ρ_λ be its mass invariant dilation given by (1.5). We have

$$\mathcal{F}_0(\rho^\lambda) = \frac{\lambda^{N(m-1)}}{m-1}\|\rho\|_m^m - \frac{\chi}{p'}\lambda^{N(p'-1)}\|\rho\|_{p'}^{p'},$$

and

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\mathcal{F}_0(\rho^\lambda) = N\lambda^{N(m-1)-1}\|\rho\|_m^m - \frac{\chi}{p}N\lambda^{N(p'-1)-1}\|\rho\|_{p'}^{p'} = N\,\lambda^{N(p'-1)-1}\left[\lambda^{N(m-p')}\|\rho\|_m^m - \frac{\chi}{p}\|\rho\|_{p'}^{p'}\right],$$

for $\lambda > 0$. By optimizing in λ, we find that

$$\lambda_*(\rho) := \left(\frac{\chi}{p}\frac{\|\rho\|_{p'}^{p'}}{\|\rho\|_m^m}\right)^{\frac{1}{N(m-p')}}, \tag{6.2}$$

is the unique global minimum of $\lambda \mapsto \mathcal{F}_0(\rho^\lambda)$, for $\lambda > 0$. We then have

$$\mathcal{F}_0(\rho^{\lambda_*(\rho)}) = \kappa\,\Lambda(\rho),$$

where

$$\kappa = -\left(\frac{\chi}{p}\right)^{\frac{m-1}{m-p'}} \qquad \text{and} \qquad \Lambda(\rho) = \frac{\|\rho\|_{p'}^{p'\,\frac{m-1}{m-p'}}}{\|\rho\|_m^{m\,\frac{p'-1}{m-p'}}}.$$

In order to minimize the functional $\mathcal{F}_0$ on $\mathcal{Y}_M$ we can equivalently maximize Λ. Moreover, by symmetrization, we can look for maximizer of Λ in the restricted class of

$$\widetilde{\mathcal{Y}}_M := \{\rho \in \mathcal{Y}_M \,:\, \rho \text{ is radially symmetric and nonincreasing}\}.$$

By using Hölder's inequality, we get that for $\rho \in \widetilde{\mathcal{Y}}_M$

$$\|\rho\|_{p'}^{p'} \le M^{\frac{m-p'}{m-1}} \, \|\rho\|_m^{m\,\frac{p'-1}{m-1}} \quad \iff \quad \Lambda(\rho) \le M,$$

and the equality is satisfied by a function ρ_0 if and only if $\rho_0(x) = c\,1_F(x)$, for some measurable set $F \subseteq \mathbb{R}^N$, see [35, Theorem 2.3 (ii.b)] for example. Since $\rho_0 \in \widetilde{\mathcal{Y}}_M$ and by using that $\|\rho_0\|_1 = M$, we infer that $F = B_R$, for some $R > 0$, and $c = M/|B_R|$. Moreover, since ρ_0 minimizes $\mathcal{F}_0$ it must coincide with its optimal dilation given by (6.2), i.e. $\lambda_*(\rho_0) = 1$. This entails that

$$M = \Lambda(\rho_0) = \left(\frac{p}{\chi}\right)^{\frac{m-1}{m-p'}} \frac{M^m}{|B_R|^{m-1}},$$

from which we infer (6.1). □

In the next result we infer information on the limiting behavior of the minimum value of $\mathcal{F}_{s,p}$, by testing the energy $\mathcal{F}_{s,p}$ with ρ_0.

Corollary 6.2. *Let $1 < p < \infty$ and $0 < s\,p < N$. Let $m > p'$ and let $\chi, M > 0$. If $\rho_s \in \mathcal{Y}_M$ is a minimizer of $\mathcal{F}_{s,p}$ over $\mathcal{Y}_M$ for every $s \in (0, N/p)$, then we have*

$$\limsup_{s\to 0} \mathcal{F}_{s,p}(\rho_s) < 0 \quad \text{and} \quad \limsup_{s\to 0} \mathcal{D}_s > 0$$

where $\mathcal{D}_s > 0$ is the constant related to ρ_s appearing in Lemma 5.7*.*

Proof. Let ρ_0 be as in Proposition 6.1. By using Theorem 3.4, we have

$$\lim_{s\to 0} \|K_{s/2} * \rho_0 - \rho_0\|_{p'} = 0.$$

By the minimality of ρ_s and by using the explicit expression of ρ_0, this entails that

$$\limsup_{s\to 0} \mathcal{F}_{s,p}(\rho_s) \le \lim_{s\to 0} \mathcal{F}_{s,p}(\rho_0) = \frac{M}{m-1}\left(\frac{\chi}{p}\right)^{\frac{m-1}{m-p'}} - M\frac{\chi}{p'}\left(\frac{\chi}{p}\right)^{\frac{p'-1}{m-p'}} < 0,$$

where the last inequality follows since $m > p'$. By recalling Remark 5.8, we also get the announced asymptotic behavior of $\mathcal{D}_s$. □

Remark 6.3. Concerning the limit functional $\mathcal{F}_0$ in the case $m = p'$, for any $M > 0$ it is clear that $\inf_{\mathcal{Y}_M} \mathcal{F}_0 = 0$ if $0 < \chi \le p$ and that $\inf_{\mathcal{Y}_M} \mathcal{F}_0 = -\infty$ if $\chi > p$. These properties are obtained by taking dilations ρ^λ for any given $\rho \in \mathcal{Y}_M$ and by sending λ to $+\infty$ and to 0, respectively. The infimum is not realized, except for the trivial case $p = \chi$.

6.2. **The limiting diffusion dominated regime.** Next, we discuss the limiting behavior of the minimizers for $s \searrow 0$, in the case $m > p'$.

Proposition 6.4 (Equiboundedness of ρ_s)**.** *Let $1 < p < \infty$ and $0 < s\,p < N$. Let $m > p'$ and let $\chi, M > 0$. Let $\rho_s \in \mathcal{Y}_M$ be a minimizer of $\mathcal{F}_{s,p}$ over $\mathcal{Y}_M$ for every $s \in (0, N/p)$. Then there exists $s_0 \in (0, N/p)$ such that*

$$\sup_{s\in(0,s_0)} \|\rho_s\|_\infty < \infty \qquad \text{and} \qquad \sup_{s\in(0,s_0)} \|\rho_s\|_m < \infty.$$

Proof. From Corollary 5.6 and Lemma 5.7 we know that ρ_s is a radially symmetric nonincreasing Hölder continuous function. This yields that

$$\|\rho_s\|_\infty = \rho_s(0) \qquad \text{and} \qquad \rho_s(x) \le \frac{M}{\omega_N |x|^N} \qquad \text{for every } x \in \mathbb{R}^N \setminus \{0\}, \tag{6.3}$$

for every s. The first fact is clear, and the second one follows since we have

$$M \ge \int_{B_{|x|}} \rho_s\, dy \ge \int_{B_{|x|}} \rho(x)\, dy = \rho_s(x)\, \omega_N\, |x|^N, \qquad \text{for every } x \in \mathbb{R}^N \setminus \{0\}.$$

By using the Euler-Lagrange equation (4.9) and Hölder's inequality we get

$$\begin{aligned}
\frac{m'}{\chi}\rho_s(0)^{m-1} &\le K_{s/2} * (K_{s/2} * \rho_s)^{p'-1}(0)\\
&= c_{N,s/2} \int_{B_1} |y|^{s-N} \left(K_{s/2} * \rho_s\right)^{p'-1}(y)dy + c_{N,s/2} \int_{B_1^c} |y|^{s-N} \left(K_{s/2} * \rho_s\right)^{p'-1}(y)dy\\
&\le c_{N,s/2} \left(\frac{N\omega_N}{s}\right) \|K_{s/2} * \rho_s\|_\infty^{p'-1} + c_{N,s/2}\, \frac{N\omega_N}{(N-s)\, p' + N}\, \|K_{s/2} * \rho_s\|_{p'}^{\frac{p'}{p}}\\
&\le c_{N,s/2} \left(\frac{N\omega_N}{s}\right) (\alpha_s\, M + \beta_s\, \rho_s(0))^{p'-1} + c_{N,s/2}\, \frac{N\omega_N}{(N-s)\, p' + N}\, H_s^{p'-1}\, \|\rho_s\|_{(p_s^*)'}^{p'-1},
\end{aligned}$$

where in the last line we used Lemma 3.1 with data $q := 1$ and $r := \infty$ and the HLS inequality (3.7). By spending again (6.3), we infer

$$\int_{\mathbb{R}^N} \rho_s^{(p_s^*)'}\, dx = \int_{B_1} \rho_s^{(p_s^*)'}\, dx + \int_{B_1^c} \rho_s^{(p_s^*)'}\, dx \le \omega_N\, \rho_s(0)^{(p_s^*)'} + \left(\frac{M}{\omega_N}\right)^{(p_s^*)'} \frac{1}{N\left((p_s^*)' - 1\right)}$$

By collecting the last two inequalities, we get

$$\begin{aligned}
\frac{m'}{\chi}\rho_s(0)^{m-1} \le & c_{N,s/2} \left(\frac{N\,\omega_N}{s}\right) (\alpha_s\, M + \beta_s\, \rho_s(0))^{p'-1}\\
&+ c_{N,s/2} \frac{N\, \omega_N\, H_s^{p'-1}}{(N-s)\, p' + N} \left(\rho_s(0)^{(p_s^*)'}\omega_N + \left(\frac{M}{\omega_N}\right)^{(p_s^*)'} \frac{1}{N\ \left((p_s^*)' - 1\right)}\right)^{\frac{p'-1}{(p_s^*)'}}.
\end{aligned}$$

By contradiction, we assume now that

$$\limsup_{s\to 0} \rho_s(0) = +\infty,$$

and we divide both sides of the previous inequality by $\rho_s(0)^{p'-1}$. By sending $s \searrow 0$, since $m > p'$ and by recalling the asymptotic behaviors of α_s, β_s, H_s and $c_{N,s/2}$ given respectively in Lemma 3.1, Corollary 3.3 and (3.1), we obtain a contradiction. This proves that there exists $s_0 > 0$ such that $S := \sup_{s\in(0,s_0)} \rho_s(0) < \infty$. We conclude the proof by observing that we can also infer the equiboundendness of $\|\rho_s\|_m$ for $s \in (0, s_0)$, since we have $\|\rho_s\|_m \le S^{\frac{1}{m'}}\, M^{\frac{1}{m}}$, for $s \in (0, s_0)$, in light of the interpolation inequality in L^p spaces. □

Proposition 6.5 (Equiboundedness of $\mathrm{supp}(\rho_s)$)**.** *Let $1 < p < \infty$ and $0 < s\,p < N$. Let $m > p'$ and let $\chi, M > 0$. Let $\rho_s \in \mathcal{Y}_M$ be a minimizer of $\mathcal{F}_{s,p}$ over $\mathcal{Y}_M$ for every $s \in (0, N/p)$. Then there exist $s_0 \in (0, N/p)$ and $R_0 > 0$ such that $\mathrm{supp}(\rho_s) \subseteq B_{R_0}$ for every $0 < s < s_0$.*

Proof. Let s_0 given by Proposition 6.4. We set $B_{R_s} = \mathrm{supp}(\rho_s)$ and by contradiction, we assume that

$$\limsup_{s\to 0} R_s = +\infty.$$

This entails that $R_s > 1$, for small value of s. We preliminary observe that, from Corollary 3.3 and Proposition 6.4, we have

$$\|K_{s/2} * \rho_s\|_{p'} \le H_s \|\rho_s\|_{(p_s^*)'} \le H_s\, M^{\vartheta}\, \|\rho_s\|_\infty^{1-\vartheta} \le L, \qquad \text{for } s \in (0, s_0), \tag{6.4}$$

where $\vartheta = 1/(p_s^*)'$ and $L > 0$ is a constant dending only on s_0. We take $x \in \partial B_{R_s}$, and by using Lemma 5.7 we get

$$\begin{aligned} \frac{1}{\chi}\,\mathcal{D}_s = K_{s/2} * \left(K_{s/2} * \rho_s\right)^{p'-1}(x) &= c_{N,s/2} \int_{B_1} |y|^{s-N} \left(\left(K_{s/2} * \rho_s\right)(x-y)\right)^{p'-1} dy \\ &+ c_{N,s/2} \int_{B_1^c} |y|^{s-N} \left(\left(K_{s/2} * \rho_s\right)(x-y)\right)^{p'-1} dy \\ &=: \mathcal{A}_1 + \mathcal{A}_2. \end{aligned} \tag{6.5}$$

We estimate the last two integrals separately, starting from the second one. By using Hölder's inequality (we observe that, by assumption $p' > N/(N-s)$, so we have $(s-N)\,p' + N < 0$) we have

$$\begin{aligned} \mathcal{A}_2 &\le c_{N,s/2} \left(\int_{B_1^c} |y|^{(s-N)p'}\, dy \right)^{\frac{1}{p'}} \left(\int_{B_1^c} \left((K_{s/2} * \rho_s)(x-y)\right)^{p'} dy \right)^{\frac{1}{p}} \\ &\le c_{N,s/2}\, \frac{N\omega_N}{(N-s)\,p' + N}\, \|K_{s/2} * \rho_s\|_{p'}^{\frac{p'}{p}}. \end{aligned}$$

So by using (3.1) and (6.4) we get

$$\lim_{s\to 0} \mathcal{A}_2 = 0. \tag{6.6}$$

We now consider the first integral

$$\mathcal{A}_1 = c_{N,s/2} \int_{B_1} |y|^{s-N} \left(\left(K_{s/2} * \rho_s\right)(x-y)\right)^{p'-1} dy$$

Since ρ_s is nonincreasing, so is $K_{s/2} * \rho_s$ (see [13]). Then, by using also (6.4), we get

$$L \ge \left(\int_{\mathbb{R}^N} |K_{s/2} * \rho_s|^{p'}\, dy \right)^{\frac{1}{p'}} \ge \left(\int_{B_{|x|}} |K_{s/2} * \rho_s|^{p'}\, dy \right)^{\frac{1}{p'}} \ge \left(\omega_N |x|^N\right)^{\frac{1}{p'}} \left(K_{s/2} * \rho_s\right)(x),$$

for every $x \in \mathbb{R}^N \setminus \{0\}$. Since $|x| = R_s > 1$, by the triangle inequality we have

$$|x-y| \ge |x| - |y| \ge R_s - 1, \qquad \text{for } y \in B_1,$$

thus

$$\begin{aligned} \mathcal{A}_1 \le L\,\omega_N^{-\frac{1}{p}}\, c_{N,s/2} \int_{B_1} |y|^{s-N} |x-y|^{-\frac{N}{p}}\, dy &\le \frac{L\,\omega_N^{-\frac{1}{p}}}{(R_s-1)^{\frac{N}{p}}}\, c_{N,s/2} \int_{B_1} |y|^{s-N}\, dy \\ &= \frac{L\,\omega_N^{-\frac{1}{p}}}{(R_s-1)^{\frac{N}{p}}}\, c_{N,s/2} \left(\frac{N\omega_N}{s}\right). \end{aligned}$$

By recalling (3.1) and (6.2), we eventually get that $\lim_{s\to 0}\mathcal{A}_1 = 0$, and so also $\limsup_{s\to 0}\mathcal{D}_s = 0$, by (6.5) and (6.6). This contradicts Corollary 6.2 and gives the desired result. □

Proposition 6.6. *Let $1 < p < \infty$ and $0 < s\,p < N$. Let $m > p'$, let $\chi, M > 0$ and let $\rho_s \in \mathcal{Y}_M$ be a minimizer of $\mathcal{F}_{s,p}$ over $\mathcal{Y}_M$ for every $s \in (0, N/p)$. There exists $s_0 \in (0, N/p)$ such that the family of minimizers $(\rho_s)_{s\in(0,s_0)}$ is equibounded in $W^{1,1}(\mathbb{R}^N)$. Moreover, if $(s_k)_{k\in\mathbb{N}} \subseteq (0, s_0)$ converges to 0, then the family $(\rho_{s_k})_{k\in\mathbb{N}}$ admits limit points in the strong $L^1(\mathbb{R}^N)$ topology, and if ρ is a limit point along a not relabeled subsequence we have $\rho \in \mathcal{Y}_M \cap L^\infty(\mathbb{R}^N)$ and*

$$\lim_{k\to\infty} \|\rho_{s_k} - \rho\|_q = 0, \qquad \text{for every } q \in [1, \infty).$$

Proof. The proof is the same of [28, Lemma 3.7] by using Proposition 6.4 and Proposition 6.5. □

The main result of this section regarding the case $m > p'$ reads as follows

Theorem 6.7. *Let $1 < p < \infty$ and $0 < s\,p < N$. Let $m > p'$, let $\chi, M > 0$ and let $\rho_s \in \mathcal{Y}_M$ be a minimizer of $\mathcal{F}_{s,p}$ over $\mathcal{Y}_M$ for every $s \in (0, N/p)$. Then*

$$\lim_{s\to 0} \|\rho_s - \rho_0\|_q = 0, \qquad \text{for every } q \in [1, \infty),$$

where ρ_0 is given by (6.1).

Proof. By using Proposition 6.6, there exists a sequence $(s_k)_{k\in\mathbb{N}}$ converging to 0 and a function $\rho \in \mathcal{Y}_M \cap L^\infty(\mathbb{R}^N)$ such that

$$\lim_{k\to\infty} \|\rho_{s_k} - \rho\|_m = 0. \tag{6.7}$$

By the triangle inequality and the HLS-type inequality (3.9), we get

$$\begin{aligned}\|K_{s_k/2} * \rho_{s_k} - \rho\|_{p'} &\le \|K_{s_k/2} * (\rho_{s_k} - \rho)\|_{p'} + \|K_{s_k/2} * \rho - \rho\|_{p'} \\ &\le H_s\,(2\,M)^{\vartheta_0}\, \|\rho_{s_k} - \rho\|_m^{1-\vartheta_0} + \|K_{s_k/2} * \rho - \rho\|_{p'}.\end{aligned}$$

where ϑ_0 is given by (3.10), it depends on s_k and converges to $1 - m'/p > 0$ as $k \to +\infty$. By Theorem 3.4, we have

$$\lim_{k\to\infty} \|K_{s_k/2} * \rho - \rho\|_{p'} = 0,$$

thus from the previous inequality, from (6.7) and from the bound on H_s by Corollay 3.3, we get

$$\lim_{k\to\infty} \|K_{s_k/2} * \rho_{s_k} - \rho\|_{p'} = 0.$$

This entails that

$$\lim_{k\to\infty} \mathcal{F}_{s_k,p}(\rho_{s_k}) = \frac{\|\rho\|_m^m}{m-1} - \frac{\chi}{p'}\|\rho\|_{p'} = \mathcal{F}_0(\rho).$$

For every $\widetilde{\rho} \in \mathcal{Y}_M$, by using the previous equality, the minimality of ρ_{s_k} and Theorem 3.4, we infer

$$\mathcal{F}_0(\rho) = \lim_{k\to\infty} \mathcal{F}_{s_k,p}(\rho_{s_k}) \le \lim_{k\to\infty} \mathcal{F}_{s_k,p}(\widetilde{\rho}) = \mathcal{F}_0(\widetilde{\rho}).$$

This proves that ρ must be a minimizer of $\mathcal{F}_0$ in $\mathcal{Y}_M$ and so by Proposition 6.1 it must coincide with ρ_0. By the arbitrariness of $(s_k)_{k\in\mathbb{N}}$, we eventually get that the whole family (ρ_s) strongly converges to ρ_0 in $L^q(\mathbb{R}^N)$, for every $q \in [1, \infty)$. □

We further have the following $\Gamma-$convergence result.

Proposition 6.8. *Let $1 < p < \infty$ and $\chi, M > 0$. Let $m > p'$. For $s \to 0$, the functional $\mathcal{F}_{s,p}$ $\Gamma-$converges to $\mathcal{F}_0$ on $\mathcal{Y}_M$ with respect to the strong convergence in $L^{p'}(\mathbb{R}^N)$.*

Proof. Let $(\rho_s)_{s\in(0,N/p)} \subset \mathcal{Y}_M$ and $\rho \in \mathcal{Y}_M$ be such that

$$\lim_{s\to 0} \|\rho_s - \rho\|_{p'} = 0. \tag{6.8}$$

By the triangle inequality and the Hardy-Littlewood-Sobolev inequality (3.7) we have

$$\begin{aligned}\|K_{s/2} * \rho_s - \rho\|_{p'} &\le \|K_{s/2} * (\rho_s - \rho)\|_{p'} + \|K_{s/2} * \rho - \rho\|_{p'}\\ &\le H_s \|\rho_s - \rho\|_{(p_s^*)'} + \|K_{s/2} * \rho - \rho\|_{p'}\\ &\le H_s \|\rho_s - \rho\|_1^{\vartheta} \|\rho_s - \rho\|_{p'}^{1-\vartheta} + \|K_{s/2} * \rho - \rho\|_{p'},\end{aligned}$$

where $\vartheta = 1 - p/p_s^*$. By Theorem 3.4, we have

$$\lim_{s\to 0} \|K_{s/2} * \rho - \rho\|_{p'} = 0,$$

wich entails that

$$\lim_{s\to 0} \|K_{s/2} * \rho_s - \rho\|_{p'} = 0,$$

where we also used our assumption (6.8). By Fatou's lemma and by spending the last information, we infer

$$\mathcal{F}_0(\rho) \le \liminf_{s\to 0} \mathcal{F}_{s,p}(\rho_s).$$

On the other hand, let $\rho \in \mathcal{Y}_M$ we set $\rho_s := \rho$, for every s. By using again Theorem 3.4 we have

$$\limsup_{s\to 0} \mathcal{F}_{s,p}(\rho_s) = \mathcal{F}_0(\rho),$$

From the last two facts we obtain the claimed result. □

6.3. **The limiting fair competition regime.** We now analyze the limiting behavior of the minimizers in the remaining case $m = p'$ thus concluding the proof of Theorem 2.2. We treat separately the cases $\chi \ne p$ and $\chi = p$, staring from the first one.

Theorem 6.9. *Let $1 < p < \infty$ and $0 < s\,p < N$. Let $M > 0$ and $m = p'$. If $\rho_s \in \mathcal{Y}_M$ is a minimizer of $\mathcal{F}_{s,p}$ over $\mathcal{Y}_M$ for every $s \in (0, N/p)$, then we have*

$$\lim_{s\to 0} \|\rho_s\|_\infty = -\lim_{s\to 0} \mathcal{F}_{s,p}(\rho_s) = \begin{cases} 0, & \text{if } 0 < \chi < p,\\ \\ +\infty, & \text{if } \chi > p.\end{cases}$$

Moreover, if $\chi > p$ we have $\rho_s \to M\delta_0$ in the sense of measures as $s \to 0$.

Proof. By Corollary 5.6, we have that ρ_s is radially symmetric and nonincreasing. We discuss separately two cases.

If $\boxed{0 < \chi < p}$, we argue by contradiction and assume that

$$\limsup_{s\to 0} \rho_s(0) > 0.$$

By arguing as in the proof of Proposition 6.4, we infer that

$$\frac{p}{\chi}\,\rho_s(0)^{p'-1} \le c_{N,s/2}\left(\frac{N\omega_N}{s}\right)(\alpha_s\,M+\beta_s\,\rho_s(0))^{p'-1}$$
$$+\,c_{N,s/2}\,\frac{N\,\omega_N\,H_s^{p'-1}}{(N-s)\,p'+N}\left(\rho_s(0)^{(p_s^*)'}\,\omega_N+\left(\frac{M}{\omega_N}\right)^{(p_s^*)'}\frac{1}{N\,((p_s^*)'-1)}\right)^{\frac{p'-1}{(p_s^*)'}}.$$

By dividing both sides of the previous inequality by $\rho_s(0)^{p'-1}$, we can rewrite it as

$$\frac{p}{\chi}-c_{N,s/2}\left(\frac{N\omega_N}{s}\right)\left(\frac{\alpha_s\,M}{\rho_s(0)}+\beta_s\right)^{p'-1}$$
$$\le c_{N,s/2}\frac{N\omega_N\,H_s^{p'-1}}{(N-s)\,p'+N}\left(\omega_N+\left(\frac{M}{\omega_N\,\rho_s(0)}\right)^{(p_s^*)'}\frac{1}{N\,((p_s^*)'-1)}\right)^{\frac{p'-1}{(p_s^*)'}}=o(s).$$

By sending $s\to 0$ and by using (3.1), (3.2) and (3.3), we get a contradiction. Therefore ρ_s converges uniformly to zero as $s\to 0$, which also implies $\mathcal{F}_{s,p}(\rho_s)\to 0$, by Corollary 3.3.

If $\boxed{\chi>p}$, we consider the limit functional $\mathcal{F}_0$, given by (2.7) with $m=p'$. For $\rho\in\mathcal{Y}_M$, we have

$$\lim_{\lambda\to\infty}\mathcal{F}_0(\rho^\lambda)=\lim_{\lambda\to\infty}\lambda^{N\,(p'-1)}\left(\frac{1}{p'-1}-\frac{\chi}{p'}\right)\|\rho\|_{p'}^{p'}=-\infty,$$

where $\rho^\lambda\in\mathcal{Y}_M$ is the mass invariant dilation of ρ by factor λ, given by (1.5). This entails that

$$\inf_{\rho\in\mathcal{Y}_M}\mathcal{F}_0=-\infty.$$

Then if $\beta<0$, we can take $\overline{\rho}\in\mathcal{Y}_M$ such that $\mathcal{F}_0(\overline{\rho})<\beta$. By using that

$$\lim_{s\to 0}\|K_{s/2}*\overline{\rho}-\overline{\rho}\|_{p'}=0,$$

thanks to Theorem 3.4, and by the minimality of ρ_s we then obtain

$$\limsup_{s\to 0}\mathcal{F}_{s,p}(\rho_s)\le\limsup_{s\to 0}\mathcal{F}_0(\overline{\rho})=\mathcal{F}_0(\overline{\rho})<\beta.$$

By the arbitrariness of β and by using Remark 5.8, this yields

$$\lim_{s\to 0}\mathcal{F}_{s,p}(\rho_s)=-\infty\quad\text{and}\quad\lim_{s\to 0}\mathcal{D}_s=+\infty. \tag{6.9}$$

By contradiction, we assume that

$$R_0:=\limsup_{s\to 0}R_s>0.$$

We take a sequence $(s_k)_{k\in\mathbb{N}}\subseteq(0,1)$ converging to zero and such that

$$\lim_{k\to\infty}R_{s_k}=\limsup_{s\to 0}R_s=R_0. \tag{6.10}$$

We set $\overline{R} = R_0/2$ and $B_{R_{s_k}} := \mathrm{supp}(\rho_{s_k})$. For $x \in \partial B_{R_{s_k}}$ by using Lemma 5.7 we have

$$\begin{aligned}\frac{1}{\chi}\,\mathcal{D}_{s_k} &= K_{s_k/2} * \left(K_{s_k/2} * \rho_{s_k}\right)^{p'-1}(x)\\ &= c_{N,s_k/2}\int_{B_{\overline{R}}} |y|^{s_k-N}\left(\left(K_{s_k/2} * \rho_{s_k}\right)(x-y)\right)^{p'-1} dy\\ &\quad + c_{N,s_k/2}\int_{B^c_{\overline{R}}} |y|^{s_k-N}\left(\left(K_{s_k/2} * \rho_{s_k}\right)(x-y)\right)^{p'-1} dy =: \mathcal{A}_1 + \mathcal{A}_2.\end{aligned} \tag{6.11}$$

For $\mathcal{A}_1$, by arguing as in Proposition 6.5, we have that

$$\left(\omega_N |x|^N\right)^{\frac{1}{p'}}\left(K_{s_k/2} * \rho_{s_k}\right)(x) \le L, \qquad \text{for every } x \in \mathbb{R}^N \setminus \{0\},$$

and, since $|x| = R_{s_k} > \overline{R}$ for k large enough, by the triangle inequality we have

$$|x-y| \ge |x| - |y| \ge R_{s_k} - \overline{R}, \qquad \text{for } y \in B_{\overline{R}},$$

This entails that

$$\mathcal{A}_1 \le c_{N,s_k/2}\left(\frac{N\omega_N}{s_k}\right)\frac{L\,\omega_N^{-\frac{1}{p}}}{(R_{s_k} - \overline{R})^{\frac{N}{p}}}. \tag{6.12}$$

For $\mathcal{A}_2$, by using Hölder's inequality we have

$$\begin{aligned}\mathcal{A}_2 &\le c_{N,s_k/2}\left(\int_{B_1^c} |y|^{(s_k-N)\,p'}\,dy\right)^{\frac{1}{p'}}\left(\int_{B_1^c}\left((K_{s_k/2} * \rho_{s_k})(x-y)\right)^{p'}\,dy\right)^{\frac{1}{p}}\\ &\le c_{N,s_k/2}\,\frac{N\,\omega_N}{(N-s_k)\,p' + N}\,\|K_{s_k/2} * \rho_{s_k}\|_{p'}^{\frac{p'}{p}}\\ &\le E\,c_{N,s_k/2}\left(\frac{\mathcal{D}_{s_k}}{s_k}\right)^{\frac{1}{p}},\end{aligned} \tag{6.13}$$

where in the last equality we used Remark 5.8 and we set

$$E = E\,(N, p, \chi, M) := \frac{N\omega_N}{(N-1)\,p' + N}\left(\frac{M}{\chi}\,\frac{N}{p}\right)^{\frac{1}{p}}.$$

By spending (6.12) and (6.13) in (6.11) and by dividing for $\mathcal{D}_{s_k}^{\frac{1}{p}}$, we get

$$\frac{1}{\chi}\,\mathcal{D}_{s_k}^{1-\frac{1}{p}} \le c_{N,s_k/2}\left(\frac{N\omega_N}{s_k}\right)\frac{L\,\omega_N^{-\frac{1}{p}}}{(R_{s_k} - \overline{R})^{\frac{N}{p}}} + E\,c_{N,s_k/2}\left(\frac{1}{s_k}\right)^{\frac{1}{p}}.$$

In light of (3.1), (6.9) and (6.10), by sending $k \to \infty$, the last inequality yields a contradiction. This implies that

$$\lim_{s\to 0} R_s = 0,$$

thus, since $\rho_s \in \mathcal{Y}_M$, we must have that ρ_s converges to a point mass at the origin as $s \to 0$, as desired. Along with (6.9), this concludes the proof. ☐

Proposition 6.10 (Case $\chi = p$)**.** *Let $1 < p < \infty$ and $0 < s\,p < N$. Let $M > 0$, $m = p'$ and $\chi = p$. If $\rho_s \in \mathcal{Y}_M$ is a minimizer of $\mathcal{F}_{s,p}$ over $\mathcal{Y}_M$ for every $s \in (0, N/p)$, then we have*

$$\lim_{s\to 0} \mathcal{F}_{s,p}(\rho_s) = 0 \qquad \text{and} \qquad \lim_{s\to 0} \mathcal{D}_s = 0.$$

Proof. We recall that by Proposition 5.3, we have

$$\limsup_{s\to 0} \mathcal{F}_{s,p}(\rho_s) \le \limsup_{s\to 0} \mu_s \le 0.$$

We claim that

$$\limsup_{s\to 0} \|\rho_s\|_{p'} < \infty. \tag{6.14}$$

Since we have

$$\mathcal{F}_{s,p}(\rho_s) = -\frac{s\,p}{N - s\,p}\,\|\rho_s\|_{p'}^{p'},$$

by Remark 5.8, this would entail that

$$\liminf_{s\to 0} \mathcal{F}_{s,p}(\rho_s) = 0,$$

and so the desired result. By recalling (4.1) and Corollary 5.6, we have that

$$H_s^{p'} \ge H^*_{p',s} = \frac{\|K_{s/2} * \rho_s\|_{p'}^{p'}}{M^{p'\frac{s}{N}}\,\|\rho_s\|_{p'}^{p'\,(1-\frac{s}{N})}} = \frac{p_s^*}{p}\left(\frac{\|\rho_s\|_{p'}^{p'}}{M^{p'}}\right)^{\frac{s}{N}},$$

where H_s is given as in Corollary 3.3 and the last equality follows from (5.6). Our claim (6.14), will follow by proving that

$$\limsup_{s\to 0} H_s^{\frac{1}{s}} < \infty. \tag{6.15}$$

By (3.8), we have

$$\begin{aligned} H_s^{\frac{1}{s}} &\le \mathcal{B}_s^{\frac{1}{s}}\left(\frac{N-s}{N}\right)^{\frac{N-s}{N\,s}}\left(\frac{(p')^{\frac{N-s}{N}} + (p_s^*)^{\frac{N-s}{N}}}{(p_s^*)'\,p}\right)^{\frac{1}{s}} \\ &\le \mathcal{B}_s^{\frac{1}{s}}\left(\frac{N}{N-s\,p}\right)^{\frac{1}{s}} \le \mathcal{B}_s^{\frac{1}{s}}\left(1 + \frac{s\,p}{N-p}\right)^{\frac{1}{s}}, \end{aligned} \tag{6.16}$$

where we set

$$\mathcal{B}_s := \frac{c_{N,s/2}}{s}\,N\,\omega_N^{\frac{N-s}{N}}.$$

By (3.1), we have $\lim_{s\to 0} \mathcal{B}_s = 1$, and we can write

$$\mathcal{B}_s = \frac{1}{2}\,\frac{\pi^{-\frac{N}{2}}\,2^{-s}\,\Gamma\left(\frac{N-s}{2}\right)}{\Gamma\left(\frac{s}{2}+1\right)}\,N\left(\frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}+1\right)}\right)^{\frac{N-s}{N}},$$

for every $s \in (0, N/p)$. Since $\mathcal{B}_s$ is smooth in a neighborhood of $s = 0$, we have

$$\lim_{s\to 0} \frac{\mathcal{B}_s - 1}{s} < \infty.$$

By spending this information in (6.16), we get (6.15) which in turn implies (6.14). Eventually, by recalling Remark 5.8 we infer the desired asymptotic behavior also for $\mathcal{D}_s$. □

Proof of Theorem 2.2. If $m > p'$, the claimed result is contained in Theorem 6.7. If $m = p'$, the conclusion follows by Theorem 6.9, Proposition 6.10 and Remark 6.3. □

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