HOMOGENISATION OF PHASE-FIELD FUNCTIONALS WITH LINEAR GROWTH

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Abstract. We propose a first rigorous homogenisation procedure in image-segmentation models by analysing the relative impact of (possibly random) fine-scale oscillations and phase-field regularisations for a family of elliptic functionals of Ambrosio and Tortorelli type, when the regularised volume term grows *linearly* in the gradient variable. In contrast to the more classical case of superlinear growth, we show that our functionals homogenise to a free-discontinuity energy whose surface term explicitly depends on the jump amplitude of the limit variable. The convergence result as above is obtained under very mild assumptions which allow us to treat, among other, the case of *stationary random integrands*.

1. Introduction

In this paper we study the combined effect of *homogenisation* and *elliptic regularisation* for phase-field functionals of the form

$$F_\varepsilon(u,v,A)=\int_A v^2 f(\tfrac{x}{\varepsilon},\nabla u)\mathrm{d}x+\int_A\Big(\tfrac{(1-v)^2}{\varepsilon}+\varepsilon|\nabla v|^2\Big)\mathrm{d}x, \tag{1.1}$$

where $\varepsilon>0$ describes both the oscillation and the regularisation scale, and f grows *linearly* in the gradient variable. In (1.1) $A\subset\mathbb{R}^n$ is open, bounded, with Lipschitz boundary, u is a vector-valued function which belongs to $W^{1,1}(A,\mathbb{R}^N)$, while v is a phase-field variable lying in $W^{1,2}(A)$.

As mentioned above, we require that the integrand $f:\mathbb{R}^n\times\mathbb{R}^{N\times n}\to[0,+\infty)$ obeys *linear* growth and coercivity conditions in the second variable; that is

$$C^{-1}|\xi|\le f(x,\xi)\le C(|\xi|+1), \tag{1.2}$$

for every $(x,\xi)\in\mathbb{R}^n\times\mathbb{R}^{N\times n}$ and for some $C\in(0,+\infty)$. Besides (1.2), we work under very mild assumptions on f which *do not include any spatial periodicity* (cf. Definition 2.4). Working in such a general setting allows us to prove a homogenisation result which also covers the case of random *stationary* integrands, as we are going to explain below.

The elliptic functionals in (1.1) are reminiscent of the celebrated phase-field model given by

$$AT_\varepsilon(u,v,A)=\int_A v^2|\nabla u|^2\mathrm{d}x+\int_A\Big(\tfrac{(1-v)^2}{\varepsilon}+\varepsilon|\nabla v|^2\Big)\mathrm{d}x,$$

which was proposed by Ambrosio and Tortorelli in the seminal works [10, 11] to approximate the (relaxed) Mumford-Shah functional [37]. The latter was introduced in the 2d framework of image segmentation to recover shapes in noisy images via curve evolution. In this setting the Ambrosio-Tortorelli functional is employed for implementation by gradient descent, where curves are replaced by a continuous edge-strength function ($1-v$ in our notation) which gives the probability of an object boundary to be present at any point in the image domain. Then, the actual shape boundaries are determined in the form of geodesics defined in a metric determined by v itself (cf. [42, 45]).

After the revisitation of Griffith's brittle-fracture theory due to Francfort and Marigo [33] (see also [19, 18]), a number of variants of the Ambrosio-Tortorelli model have been proposed and extensively used also to approximate brittle fracture models [12, 13, 16, 21, 25, 32], just to mention few examples. The advantage of this kind of approximations is twofold: on the one hand they establish a rigorous connection between variational fracture models and gradient-damage models [39, 38], on the other hand, in most of the cases, they provide efficient algorithms for numerical simulations [18, 19, 33].

If instead in (1.1) we choose $f(x,\xi)=|\xi|$, the corresponding phase-field functionals were originally proposed by Shah [43, 44] as possible regularisations of an image-segmentation model, alternative to the Mumford-Shah functional, which provides a common framework for image segmentation and isotropic curve evolution in Computer Vision. Moreover, Shah's functional overcomes a number of limitations of the earlier models. Loosely speaking, in this framework the domain A is interpreted as a Riemannian manifold endowed with a metric defined by the image properties so that the image-segmentation problem amounts to finding a minimal cut in a Riemannian manifold (cf. [46]).

The main difference between the Ambrosio-Tortorelli functionals (and their more "classical" variants) and (1.1)-(1.2) rests on the growth of the function f: superlinear in the former versus linear in the latter. Such different behaviours lead to some structural differences in the corresponding, attainable limit models. In fact, the weaker gradient penalisation in (1.2) allows for an interaction between the two competing terms in (1.1), as it also typical of free-discontinuity functionals in the linear setting [17, 24]. As a result, the surface energy densities obtained in this case are of *cohesive* type as proven in [6], in the scalar isotropic case, and in [7], in the vector-valued anisotropic case. That is, the resulting limit surface integrands in the linear setting are bounded, increasing, and concave functions of the jump amplitude $[u]$ of the (possibly discontinuous) limit variable u, moreover they exhibit a linear growth at the origin. We observe though, that the linear growth of f for large gradients is not justified in the applications to Fracture Mechanics, so that more recently other variants of the Ambrosio-Tortorelli functional related to the gradient-damage models in [39, 38] were designed to provide a variational approximation of cohesive energies (cf. [27, 47, 48, 31, 28, 29, 34, 3, 26]). It is also worth mentioning that in these models the parameters can be tuned to approximate prescribed cohesive laws (satisfying suitable assumptions) as shown in [4] (see also [5] for applications to an engineering problem).

Furthermore, we observe that the coercivity assumption in (1.2) yields the "weaker" lower bound

$$C^{-1}\int_A v^2|\nabla u|\mathrm{d}x+\int_A\Big(\tfrac{(1-v)^2}{\varepsilon}+\varepsilon|\nabla v|^2\Big)\mathrm{d}x\leq F_\varepsilon(u,v,A), \tag{1.3}$$

where the functionals on the left-hand side are those proposed by Shah and studied in [6]. Hence, from (1.3) and the analysis in [6] (see also [7]) we readily deduce that if $(u_\varepsilon)\subset W^{1,1}(A,\mathbb{R}^N)$ is a sequence with equi-bounded energy which additionally satisfy $\sup_\varepsilon\|u_\varepsilon\|_{L^q}<+\infty$, for some $q>1$, then (up to subsequences) $u_\varepsilon\to u$ with respect to the strong $L^1(A,\mathbb{R}^N)$-convergence, for some $u\in(G)BV(A,\mathbb{R}^N)$. Therefore, in the linear setting, the limit functional shall contain a term depending on the Cantor part of the measure derivative Du. These features are in sharp contrast with the case where f grows superlinearly in the gradient variable. Indeed in this case the limit functional is defined on the smaller space $(G)SBV(A,\mathbb{R}^N)$. Additionally, the superlinear growth of f in $|\nabla u|$ makes it energetically unfavourable to approximate a pure jump function with elastic deformations, so that the only surface energy densities which can be obtained in the limit are necessarily independent of the jump amplitude of u, as recently proven in [14, 15, 16].

Motivated by the applications to anisotropic curve evolution [43, 44, 46], in this paper we study the homogenisation of the phase-field functionals in (1.1) which encompass the case of highly oscillating, possibly random metrics. Moreover, since image-segmentation models are highly sensitive to the presence of heterogeneities in regions or objects due to noise, it is in general of great importance to incorporate a homogenisation procedure in these models and in their phase-field counterparts. ln fact, the presence of noise can cause random variations in the image intensity values, which in turn produce false detections in the image so that homogenisation may help to reduce the impact of noise, shadows, and changes in the illumination intensity, which usually make it difficult to accurately segment the image into its relevant parts. Therefore, in practice, by removing such, it can be easier to detect boundaries between different objects in the image, and to distinguish between foreground and background regions. More specifically, in the present work we rigorously analyse the interplay between fine-scale oscillations and phase-field approximations in linear models as in (1.1). Due to the presence of microscopic heterogeneities, as ε tends to zero we expect to obtain an effective model where the (cohesive) energy density depends both on the homogenised integrand $f_{\rm hom}$ (through its recession function) and on the regularised surface-term in (1.1). On the other hand, on the account of the analysis in [6, 7] we also expect a limit volume energy which only depends on the first term in (1.1) and therefore in this case on $f_{\rm hom}$. A central feature of our analysis is that we study the homogenisation of F_ε without imposing any periodicity of f in the spatial variable. In fact, in the same spirit as in [23, 24], we work under more general assumptions which, notably, are satisfied in the random stationary case.

Finally, it is also worth noticing that the homogenisation problem analysed in this paper can be seen as a case study of a homogenisation problem for the gradient-damage models proposed in [27, 47, 48, 31, 28, 29, 34, 3, 26] to approximate cohesive energies in Fracture Mechanics. Indeed, on account of the analysis performed in these papers, also in this case we expect effective surface integrands defined by asymptotic minimisation problems in which all the terms in the approximating functionals interact with one another. Moreover, when working with such approximations, a more technically demanding analysis shall be expected due to the superlinear growth of the bulk energy density and to the more complex, parameter-dependent, choice of the degenerate function multiplying f.

Below we briefly outline the proof strategy employed to get our homogenisation result. Loosely speaking, this strategy consists of two main steps: a purely deterministic one, where we devise sufficient conditions (on f) leading to homogenisation and a probabilistic step, where we show that if f is a stationary random variable, then the sufficient conditions mentioned above are indeed fulfilled. Therefore a stochastic homogenisation result readily follows as a corollary of the deterministic analysis.

1.1. **Deterministic homogenisation.** Here we assume that f satisfies the assumptions listed in Definition 2.4. Besides (1.2) these require that the recession function f^∞ is defined at every point. We stress here that we do not require any continuity of f in the spatial variable, since this would be unnatural for the applications.

Under these general assumptions, using the localisation method of Γ-convergence [30], we can prove the existence of a subsequence (ε_j) such that, for every $A \subset \mathbb{R}^n$ open and bounded, the functionals $F_{\varepsilon_j}(\cdot,\cdot,A)$ Γ-converge to an abstract functional $\widehat{F}(\cdot,\cdot,A)$. Furthermore, the latter has the property that for every $u \in BV_{\rm loc}(\mathbb{R}^n,\mathbb{R}^N)$ the set function $A \mapsto \widehat{F}(u,1,A)$ is the restriction to the open subsets of $\mathbb{R}^n$ of a Borel measure (cf. Theorem 5.2). We observe that since we do not assume any spatial periodicity of f, the continuity of $z \mapsto \widehat{F}(u(\cdot - z),1,A+z)$ may fail and therefore we cannot directly use the integral representation result in BV [17] to deduce the form of $\widehat{F}$. Our integral representation result is then obtained under some additional assumptions, which are however more general than periodicity. We require that the limits of some scaled minimisation problems, defined in terms of f and f^∞, exist and are independent of the spatial variable. These limits will then define the volume and surface integrands of $\widehat{F}$. Eventually, the Cantor integrand will be automatically identified due to the lower semicontinuity of $\widehat{F}$.

Specifically, we make the two following assumptions. If $Q_r(rx)$ denotes the open cube with side-length r centred at rx and $\ell_\xi(x) = \xi x$, the first assumption amounts to asking that for every

$\xi \in \mathbb{R}^{N\times n}$ the limit

$$\lim_{r\to\infty} \frac{1}{r^n} \inf\left\{\int_{Q_r(rx)} f(y,\nabla u)\mathrm{d}y \colon u\in W^{1,1}(Q_r(rx),\mathbb{R}^N),\ u=\ell_\xi \text{ on } \partial Q_r(rx)\right\} \tag{1.4}$$

exists and it is independent of $x\in\mathbb{R}^n$. The value of (1.4) is denoted by $f_{\rm hom}(\xi)$.

Moreover, if $Q_r^\nu(rx)$ denotes the open cube with side-length r centred at rx, one side orthogonal to $\nu\in\mathbb{S}^{n-1}$, and

$$u_{rx,\zeta,\nu}(y)=\begin{cases}\zeta & \text{if } (y-rx)\cdot\nu\geq 0\\ 0 & \text{if } (y-rx)\cdot\nu<0,\end{cases}$$

we also require that for every $\zeta\in\mathbb{R}^N$ and $\nu\in\mathbb{S}^{n-1}$ the limit

$$\lim_{r\to+\infty}\frac{1}{r^{n-1}}\inf\bigg\{\int_{Q_r^\nu(rx)}\big(v^2 f^\infty(y,\nabla u)+(1-v)^2+|\nabla v|^2\big)\mathrm{d}y\colon u\in W^{1,1}(Q_r^\nu(rx),\mathbb{R}^N),$$
$$v\in W^{1,2}(Q_r^\nu(rx)) \text{ and } (u,v)=(u_{rx,\zeta,\nu},1) \text{ on } \partial Q_r^\nu(rx)\bigg\} \tag{1.5}$$

exists and is independent of $x\in\mathbb{R}^N$. The value of (1.5) is denoted by $g_{\rm hom}(\zeta,\nu)$.

It is worth mentioning here that $f_{\rm hom}$ and $g_{\rm hom}$ satisfy a number of properties (cf. Section 4) which ensure, in particular, that they are Borel measurable.

Then, assuming (1.4) and (1.5) we resort to the blow-up technique in BV [17] to show that for every $u\in BV(A,\mathbb{R}^N)$ the following identities hold true

$$\frac{\mathrm{d}\widehat{F}(u,1,\cdot)}{\mathrm{d}\mathcal{L}^n}(x)=f_{\rm hom}(\nabla u(x)) \quad \text{for } \mathcal{L}^n\text{-a.e. } x\in A,$$
$$\frac{\mathrm{d}\widehat{F}(u,1,\cdot)}{\mathrm{d}|D^c u|}(x)=f_{\rm hom}^\infty(\nabla u(x)) \quad \text{for } |D^c u|\text{-a.e. } x\in A,$$
$$\frac{\mathrm{d}\widehat{F}(u,1,\cdot)}{\mathrm{d}\mathcal{H}^{n-1}}(x)=g_{\rm hom}([u](x),\nu_u(x)) \quad \text{for } \mathcal{H}^{n-1}\text{-a.e. } x\in J_u\cap A.$$

In their turn, these allow us to represent $\widehat{F}$ in an integral form first on BV, and then by standard truncation arguments on the domain of the Γ-limit, that is, on GBV. Furthermore, since in the equalities above the right-hand side does not depend on the subsequence (ε_j), under assumptions (1.4) and (1.5) we obtain a Γ-convergence result for the whole sequence (F_ε) (see Theorem 5.1).

1.2. **Stochastic homogenisation.** Here we consider an underlying complete probability space $(\Omega,\mathcal{T},P)$ endowed with a group of P-preserving transformations, and allow the integrand f to additionally depend on $\omega\in\Omega$, in a suitable measurable way. Then, if f is a *stationary random integrand* in the sense of Definition 2.7, we show that assumptions (1.4) and (1.5) are automatically satisfied for P-a.e. $\omega\in\Omega$, that is, *almost surely.* As it is by-now costumery (see [23, 24]) this is done by appealing to the Ackoglu and Krengel Subadditive Ergodic Theorem [1]. More specifically, the proof that (1.4) holds is standard and follows as in [41]. On the other hand, the verification of (1.5) is highly non trivial, as it is always the case when working with “surface terms” where there is a dimensional mismatch between the domain of integration and the scaling, and, moreover, a boundary datum which is inherently inhomogeneous (cf. 1.5).

Once (1.4) and (1.5) are shown to hold (cf. Proposition 6.1 and Proposition 6.3) we can immediately resort to the deterministic analysis to deduce that the random functionals

$$F_\varepsilon(\omega)(u,v,A)=\int_A v^2 f(\omega,\tfrac{x}{\varepsilon},\nabla u)\mathrm{d}x+\int_A\Big(\tfrac{(1-v)^2}{\varepsilon}+\varepsilon|\nabla v|^2\Big)\mathrm{d}x,$$

homogenise, almost surely, to the random, autonomous, free-discontinuity functional

$$F_{\text{hom}}(\omega)(u,v,A) = \int_A f_{\text{hom}}(\omega,\nabla u)\mathrm{d}x + \int_A f^\infty_{\text{hom}}(\omega, \tfrac{\mathrm{d}D^c u}{\mathrm{d}|D^c u|})\mathrm{d}|D^c u| + \int_{J_u\cap A} g_{\text{hom}}(\omega,[u],\nu_u)\mathrm{d}\mathcal{H}^{n-1},$$

if $u \in GBV(A,\mathbb{R}^N)$ and $v = 1$ $\mathcal{L}^n$-a.e in A, where f_{hom} and g_{hom} are defined, respectively, by (1.4) and (1.5) while f^∞_{hom} is the recession function of f_{hom} (cf. Theorem 3.3 and Theroem 3.4). Eventually, if f is stationary with respect to an ergodic group of P-preserving transformations on $(\Omega,\mathcal{T},P)$, then the homogenisation procedure becomes effective and thus F_{hom} is deterministic.

2. Preliminaries and set up

2.1. **Notation.** We introduce some notation which will be used throughout the paper.

(a) Let $n,N\in\mathbb{N}$ be fixed with $n\geq 2$. For $x,y\in\mathbb{R}^n$ and $\zeta\in\mathbb{R}^N$, $x\cdot y := x_1y_1+\cdots+x_ny_n$ is the euclidean scalar product of x and y, while $\zeta\otimes x := (\zeta_i x_j)_{ij}\in\mathbb{R}^{N\times n}$ is the tensor product of ζ and x.

(b) For $x\in\mathbb{R}^n$ and $\nu\in\mathbb{S}^{n-1}$, we set $\Pi^\nu := \{y\in\mathbb{R}^n : y\cdot\nu = 0\}$ and $\Pi^\nu_x := x+\Pi^\nu$.

(c) For $\xi\in\mathbb{R}^{N\times n}$, ℓ_ξ denotes the linear function from $\mathbb{R}^n$ to $\mathbb{R}^N$ with gradient ξ.

(d) For $k\in\mathbb{N}$ and $x=(x_1,\dots,x_k)\in\mathbb{R}^k$, $|x| := \sqrt{x_1^2+\cdots+x_k^2}$ is the euclidean norm of the vector x. $\mathbb{S}^{k-1} := \{x\in\mathbb{R}^k \mid |x|=1\}$ is the $k-1$-dimensional sphere centered in the origin and $\hat{\mathbb{S}}^{k-1}_\pm := \{x\in\mathbb{S}^{k-1} \mid \pm x_{i(x)}>0\}$, where $i(x)$ is the largest $i\in\{1,\dots,k\}$ such that $x_i\neq 0$. Note that $\mathbb{S}^{k-1} = \hat{\mathbb{S}}^{k-1}_+\cup\hat{\mathbb{S}}^{k-1}_-$ and $\hat{\mathbb{S}}^{k-1}_\pm$ is a Borel set.

(e) For $\nu\in\mathbb{S}^{n-1}$, let R_ν be an orthogonal $n\times n$ matrix such that $R_\nu e_n=\nu$; we assume that the restriction of the function $\nu\mapsto R_\nu$ to the sets $\hat{\mathbb{S}}^{n-1}_\pm$, defined in (d) of the notation list, are continuous and that $R_{-\nu}Q_1 = R_\nu Q_1$; moreover we assume that $R_\nu\in\mathbb{Q}^{n\times n}$ if $\nu\in\mathbb{Q}^n$. A map $\nu\mapsto R_\nu$ satisfying these properties is provided in [22, Example A.1 and Remark A.2].

(f) For $x\in\mathbb{R}^n$ and $\rho>0$ we set $B_\rho(x) := \{y\in\mathbb{R}^n :\ |y-x|<\rho\}$ and $Q_\rho(x) := \{y\in\mathbb{R}^n :\ |(y-x)\cdot e_i|<\rho/2 \text{ for } i=1,\dots,n\}$, where $\{e_1,\dots,e_n\}$ is the standard basis of $\mathbb{R}^n$. Moreover B_ρ and Q_ρ stand, respectively, for $B_\rho(0)$ and $Q_\rho(0)$.

For $x\in\mathbb{R}^n$, $\rho>0$, and $\nu\in\mathbb{S}^{n-1}$ we set

$$Q^\nu_\rho(x) := x+R_\nu Q_\rho.$$

For $k\in\mathbb{N}$ we define the rectangle

$$Q^{\nu,k}_\rho(x) := x+Q^{\nu,k}_\rho$$

where $Q^{\nu,k}_\rho := R_\nu((-\frac{k\rho}{2},\frac{k\rho}{2})^{n-1}\times(-\frac{\rho}{2},\frac{\rho}{2}))$. Moreover we set

$$\partial^\perp Q^{\nu,k}_\rho(x) := \partial Q^{\nu,k}_\rho(x)\cap R_\nu((-\tfrac{k\rho}{2},\tfrac{k\rho}{2})^{n-1}\times\mathbb{R})$$

$$\partial^\parallel Q^{\nu,k}_\rho(x) := \partial Q^{\nu,k}_\rho(x)\cap R_\nu(\mathbb{R}^{n-1}\times(-\tfrac{\rho}{2},\tfrac{\rho}{2})).$$

(g) $\mathscr{A}$ and $\mathscr{A}_\infty$ denotes the collection of all bounded open sets and of all bounded open Lipschitz sets of $\mathbb{R}^n$ respectively; if $A,B\in\mathscr{A}$, by $A\subset\subset B$ we mean that exists a compact set K such that $A\subset K\subset B$. For every $C\in\mathscr{A}$, we define $\mathscr{A}(C) := \{A\in\mathscr{A} \mid A\subseteq C\}$ and $\mathscr{A}_\infty(C) := \{A\in\mathscr{A}_\infty \mid A\subseteq C\}$.

(h) For every topological space X, $\mathscr{B}(X)$ denotes its Borel σ-algebra. For every integer $k\geq 1$, $\mathscr{B}^k$ is the Borel σ-algebra of $\mathbb{R}^k$, while $\mathscr{B}^n_S$ denotes the Borel σ-algebra of $\mathbb{S}^{n-1}$.

(i) $\mathcal{L}^k$ and $\mathcal{H}^{k-1}$ denote respectively the Lebesgue and the $(k-1)$-dimensional Hausdorff measure on $\mathbb{R}^k$.

(j) Let μ and λ two Radon measures on $A \in \mathscr{A}$, with values in a finite dimensional real vector space X and in $[0,+\infty]$, respectively; then $\frac{\mathrm{d}\mu}{\mathrm{d}\lambda} := \frac{\mathrm{d}\mu^a}{\mathrm{d}\lambda} \in L^1_{\mathrm{loc}}(A,X)$, where $\mu^a \ll \lambda$, $\mu^a + \mu^s$ is the Radon-Nykodym decomposition of μ respect to λ and $\mu^a(B) = \int_B \frac{\mathrm{d}\mu^a}{\mathrm{d}\lambda}\mathrm{d}\lambda$ for every Borel set $B \subseteq A$.

(k) For $u \in BV(A,\mathbb{R}^N)$, with $A \in \mathscr{A}$, the jump of u on the jump set J_u is denoted by $[u] := u^+ - u^-$, while ν_u denotes the normal to J_u. The distributional gradient Du, is a $\mathbb{R}^{N\times n}$-valued Radon measure on A, whose absolutely continuous part with respect to $\mathcal{L}^n$, denoted by $D^a u$, has density $\nabla u \in L^1(A,\mathbb{R}^{N\times n})$ (which coincides with that approximate gradient of u), while the singular part $D^s u$ can be decomposed as $D^s u = D^j u + D^c u$ where the jump part $D^j u$ is given by $D^j u = [u]\otimes \nu_u \mathcal{H}^{n-1} \llcorner J_u$, and the Cantor part $D^c u$ is a $\mathbb{R}^{N\times n}$-valued Radon measure on A which vanishes on all Borel sets $B \subseteq A$ with $\mathcal{H}^{n-1}(B) < +\infty$.

We refer to the book [9] for all the properties of $(G)BV$ and $(G)SBV$ functions, giving precise references.

(l) For $x \in \mathbb{R}^n$, $\zeta \in \mathbb{R}^m$, $\nu \in \mathbb{S}^{n-1}$ and $\varepsilon > 0$ we define the function $u_{x,\zeta,\nu}, \overline{u}^\varepsilon_{x,\zeta,\nu} : \mathbb{R}^n \to \mathbb{R}^N$ as

$$u_{x,\zeta,\nu}(y) := \begin{cases} \zeta & \text{if } (y-x)\cdot\nu \geq 0 \\ 0 & \text{if } (y-x)\cdot\nu < 0, \end{cases} \quad \text{and} \quad \overline{u}^\varepsilon_{x,\zeta,\nu}(y) := \zeta\overline{\mathrm{u}}(\tfrac{1}{\varepsilon}(y-x)\cdot\nu)$$

where $\overline{\mathrm{u}} : \mathbb{R} \to [0,1]$ is a fixed smooth cut-off function such that $\overline{\mathrm{u}} \equiv 1$ on $[1/2,+\infty)$ and $\overline{\mathrm{u}} \equiv 0$ on $(-\infty,-1/2]$.

We also use the shorthand notation $u_{\zeta,\nu} := u_{0,\zeta,\nu}$, $\overline{u}_{x,\zeta,\nu} := \overline{u}^1_{x,\zeta,\nu}$ and $\overline{u}_{\zeta,\nu} := \overline{u}_{0,\zeta,\nu}$.

(m) We define the truncation functions $\mathcal{T}_k \in C^1_c(\mathbb{R}^N,\mathbb{R}^N)$ satisfying

$$\mathcal{T}_k(\zeta) := \begin{cases} \zeta & \text{if } |\zeta| \leq a_k, \\ 0 & \text{if } |\zeta| \geq a_{k+1}, \end{cases} \tag{2.1}$$

and

$$\mathrm{Lip}(\mathcal{T}_k) \leq 1 \quad \text{and} \quad |\mathcal{T}_k(\zeta)| \leq a_{k+1} \ \text{ for every } \zeta \in \mathbb{R}^n, \tag{2.2}$$

for some diverging and strictly increasing sequence of positive numbers (a_k).

(n) Given $h : \mathbb{R}^{N\times n} \to [0,+\infty]$ its recession function $h^\infty : \mathbb{R}^{N\times n} \to [0,+\infty]$ is defined as

$$h^\infty(\xi) := \limsup_{t\to+\infty} \frac{h(t\xi)}{t}.$$

2.2. **The subadditive ergodic Theorem.** In this subsection we recall a variant of the pointwise subaddtive ergodic Theorem of Ackoglu and Krengel [1, Theorem 2.7] which is useful for our purposes (cf. [36, Theorem 4.1]).

Let $d \in \mathbb{N}$. Let $(\Omega,\mathcal{F},P)$ be a probability space and let $\tau := (\tau_z)_{z\in\mathbb{Z}^d}$ denote a group of P-preserving transformations on $(\Omega,\mathcal{F},P)$, that is, τ is a family of measurable mappings $\tau_z : \Omega \to \Omega$ satisfying the following properties:

- $\tau_z\tau_{z'} = \tau_{z+z'}$, $\tau_z^{-1} = \tau_{-z}$, for every $z,z' \in \mathbb{Z}^d$;
- τ preserves the probability measure P; *i.e.*, $P(\tau_z E) = P(E)$, for every $z \in \mathbb{Z}^d$ and every $E \in \mathcal{F}$;

If in addition every τ-invariant set $E \in \mathcal{F}$ has either probability 0 or 1, then τ is called *ergodic*.

For every $a,b \in \mathbb{R}^d$ with $a_i < b_i$ for $i = 1,\dots,d$, we define

$$[a,b) = \{x \in \mathbb{R}^d \ : \ a_i \leq x_i < b_i \ \text{ for } i = 1,\dots,d\},$$

and we set

$$\mathcal{I}_d := \{[a,b) \ : \ a,b \in \mathbb{R}^d, \ a_i < b_i \ \text{ for } i = 1,\dots,d\}.$$

Definition 2.1 (Subadditive process). *Let $\tau := (\tau_z)_{z\in\mathbb{Z}^d}$ be a group of P-preserving transformations on $(\Omega, \mathcal{F}, P)$. A d-dimensional subadditive process is a function $\mu : \Omega \times \mathcal{I}_d \to \mathbb{R}$ satisfying the following properties:*

(a) for every $A \in \mathcal{I}_d$ the map $\omega \mapsto \mu(\omega, A)$ is $\mathcal{T}$-measurable;

(b) for every $\omega \in \Omega$, $A \in \mathcal{I}_d$, and $z \in \mathbb{Z}^d$ we have $\mu(\omega, A+z) = \mu(\tau_z\omega, A)$;

(c) for every $A \in \mathcal{I}_d$ and for every finite family $(A_i)_{i\in I}$ in $\mathcal{I}_d$ of pairwise disjoint sets such that $\cup_{i\in I} A_i = A$, we have

$$\mu(\omega, A) \leq \sum_{i\in I} \mu(\omega, A_i),$$

for every $\omega \in \Omega$;

(d) there exists $c > 0$ such that

$$0 \leq \mu(\omega, A) \leq c\mathcal{L}^d(A),$$

for every $\omega \in \Omega$ and every $A \in \mathcal{I}_d$.

Definition 2.2 (Regular family of sets). *A family of sets $(A_t)_{t>0}$ in $\mathcal{I}_d$ is called regular with constant $M \in (0, +\infty)$ if there exists another family of sets $(A'_t)_{t>0}$ in $\mathcal{I}_d$ such that:*

- *$A_t \subset A'_t$ for every $t > 0$;*
- *$A'_s \subset A'_t$ whenever $0 < s < t$;*
- *$0 < \mathcal{L}^d(A'_t) \leq M\mathcal{L}^d(A_t)$ for every $t > 0$;*
- *$\bigcup_{t>0} A'_t = \mathbb{R}^d$.*

Theorem 2.3 (Subadditive Ergodic Theorem). *Let $\tau = (\tau_z)_{z\in\mathbb{Z}^d}$ be a group of P-preserving transformations on $(\Omega, \mathcal{T}, P)$. Let $\mu : \Omega \times \mathcal{I}_d \to [0, +\infty)$ be a d-dimensional subadditive process. Then there exist a $\mathcal{T}$-measurable function $\varphi : \Omega \to [0, +\infty)$ and a set $\Omega' \in \mathcal{T}$ with $P(\Omega') = 1$ such that*

$$\lim_{t\to+\infty} \frac{\mu(\omega, A_t)}{\mathcal{L}^d(A_t)} = \varphi(\omega)$$

for every regular family of sets $(A_t)_{t>0}$ in $\mathcal{I}_d$ and for every $\omega \in \Omega'$. If in addition τ is ergodic, then φ is constant P-a.e.

2.3. **Assumptions.** In this subsection we introduce the class of the admissible random integrands.

Definition 2.4 (Admissible integrand). *Let $C \geq 1$ and $\alpha \in (0, 1)$ be given, then $\mathcal{F}(C, \alpha)$ denotes the collection of all functions $f : \mathbb{R}^n \times \mathbb{R}^{N\times n} \to [0, +\infty)$ with the following properties:*

(f1) (measurability) *f is $\mathscr{B}^n \otimes \mathscr{B}^{N\times n}$-measurable;*

(f2) (linear growth) *for every $x \in \mathbb{R}^n$ and every $\xi \in \mathbb{R}^{N\times n}$*

$$C^{-1}|\xi| \leq f(x, \xi) \leq C(|\xi| + 1);$$

(f3) (continuity) *for every $x \in \mathbb{R}^n$ the maps $\xi \mapsto f(x, \xi)$ and $\xi \mapsto f^\infty(x, \xi)$ are continuous;*

(f4) (recession function) *for every $x \in \mathbb{R}^n$ every $\xi \in \mathbb{R}^{N\times n}$ and every $t > 0$*

$$\Big| f^\infty(x, \xi) - \frac{f(x, t\xi)}{t} \Big| < \frac{C}{t}(1 + f(x, t\xi)^{1-\alpha}).$$

Remark 2.5. *Let $f \in \mathcal{F}(C, \alpha)$, then thanks to (f2) and (f4), for every $x \in \mathbb{R}^n$ and every $\xi \in \mathbb{R}^{N\times n}$ we have that there exists the limit*

$$\lim_{t\to+\infty} \frac{f(x, t\xi)}{t} = f^\infty(x, \xi),$$

and

$$C^{-1}|\xi| \leq f^\infty(x, \xi) \leq C|\xi|. \tag{2.3}$$

Moreover, for every $L > 0$ there exists $M > 0$ such that for every $x \in \mathbb{R}^n$, $\xi \in \mathbb{R}^{N\times n}$ with $|\xi| = 1$ and $t > L$ we have that

$$\Big|f^\infty(x,\xi) - \frac{f(x,t\xi)}{t}\Big| \le \frac{M}{t^\alpha}. \tag{2.4}$$

Definition 2.6 (Random integrand)**.** *A function $f : \Omega\times\mathbb{R}^n\times\mathbb{R}^{N\times n} \to [0,+\infty)$ is called a random integrand if*

- *(s-f1) f is $\mathcal{T}\otimes\mathscr{B}^n\otimes\mathscr{B}^{N\times n}$-measurable;*
- *(s-f2) $f(\omega,\cdot,\cdot) \in \mathcal{F}(C,\alpha)$ for every $\omega \in \Omega$, where $\mathcal{F}(C,\alpha)$ is as in Definition 2.4.*

If f is a random integrand then $f^\infty : \Omega \times \mathbb{R}^n \times \mathbb{R}^{N\times n} \to [0,+\infty)$ is given by

$$f^\infty(\omega,x,\xi) = \lim_{t\to+\infty} \frac{f(\omega,x,t\xi)}{t},$$

where the existence of the limit above is ensured by the very definition of random integrand together with Remark 2.5.

Definition 2.7 (Stationary random integrand)**.** *A random integrand f is stationary if there exists $\tau = (\tau_z)_{z\in\mathbb{Z}^n}$ n-dimensional group of P-preserving transformation on $(\Omega,\mathcal{T},P)$ such that*

$$f(\omega,x+z,\xi) = f(\tau_z\omega,x,\xi)$$

for every $\omega\in\Omega$, $x\in\mathbb{R}^n$, $z\in\mathbb{R}^n$, and $\xi\in\mathbb{R}^{N\times n}$.

If in addition τ is ergodic we call f an ergodic random integrand.

3. Statements of the main results

Let f be a given stationary random integrand. For $\varepsilon > 0$ we consider the phase-field functionals $F_\varepsilon(\omega) : L^1_{\rm loc}(\mathbb{R}^n,\mathbb{R}^{N+1}) \times \mathscr{A} \longrightarrow [0,+\infty]$ defined as

$$F_\varepsilon(\omega)(u,v,A) := \begin{cases} \displaystyle\int_A \big(v^2 f(\omega,\tfrac{x}{\varepsilon},\nabla u) + \tfrac{(1-v)^2}{\varepsilon} + \varepsilon|\nabla v|^2\big)\mathrm{d}x, & (u,v) \in W^{1,1}(A,\mathbb{R}^N)\times W^{1,2}(A) \\ +\infty & \text{otherwise.} \end{cases} \tag{3.1}$$

Remark 3.1. *For $v \in W^{1,2}(A)$ set $\tilde v := \min\{\max\{0,v\},1\}$. We notice that for every $\varepsilon > 0$, $\omega\in\Omega$ there holds*

$$F_\varepsilon(\omega)(u,\tilde v,A) \le F_\varepsilon(\omega)(u,v,A),$$

for every $(u,v) \in W^{1,1}(A,\mathbb{R}^N)\times W^{1,2}(A)$ and $A \in \mathscr{A}$. Therefore it is not restrictive to assume that the phase-filed variable v satisfies the pointwise bounds $0 \le v \le 1$ for $\mathcal{L}^n$-a.e. $x \in A$.

Remark 3.2 (Equi-coercivity)**.** *The coercivity assumption in (f2) immediately gives that*

$$C^{-1}\int_A v^2|\nabla u|\mathrm{d}x + \int_A \Big(\tfrac{(1-v)^2}{\varepsilon} + \varepsilon|\nabla v|^2\Big)\mathrm{d}x \le F_\varepsilon(\omega)(u,v,A)$$

where the functionals on the left-hand side are those studied in [6] *(see also* [7]*). Hence, up to considering the perturbed functionals*

$$F_\varepsilon(\omega)(u,v,A) + \|u\|_{L^q(A,\mathbb{R}^N)},$$

for some $q > 1$, we can appeal to [7, Lemma 7.1] *to deduce that if $(u_\varepsilon,v_\varepsilon) \subset W^{1,1}(A,\mathbb{R}^N)\times W^{1,2}(A,[0,1])$ satisfies*

$$\sup_{\varepsilon>0}\Big(F_\varepsilon(\omega)(u_\varepsilon,v_\varepsilon,A) + \|u_\varepsilon\|_{L^q(A,\mathbb{R}^N)}\Big) < +\infty,$$

then, up to subsequences, $(u_\varepsilon, v_\varepsilon) \to (u, 1)$ strongly in $L^1(A, \mathbb{R}^{N+1})$ for some $u \in GBV(A, \mathbb{R}^N)$. For this reason, in what follows we are going to study the Γ-convergence of F_ε with respect to the strong L^1-convergence.

Before stating our main results we need some additional notation. Let $h : \mathbb{R}^n \times \mathbb{R}^{n\times N} \to [0, \infty)$ satisfy (f2) and (f1). For $A \in \mathscr{A}_\infty$ and $(u, v) \in W^{1,1}(A, \mathbb{R}^N) \times W^{1,2}(A, [0, 1])$ consider the following auxiliary integral functionals

$$E^h(u, A) := \int_A h(x, \nabla u)\mathrm{d}x,$$

and

$$S^h(u, v, A) := \int_A (v^2 h(x, \nabla u) + (1 - v)^2 + |\nabla v|^2)\mathrm{d}x.$$

Moreover, let $w \in BV_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{R}^N)$ and define the minimisation problems

$$m^h_{\mathrm{b}}(w, A) := \inf\{E^h(u, A) \ : \ u \in W^{1,1}(A, \mathbb{R}^N),\ u = w \text{ on } \partial A\}$$

and

$$m^h_{\mathrm{s}}(w, A) := \inf\{S^h(u, v, A) \colon u \in W^{1,1}(A, \mathbb{R}^N),\ v \in W^{1,2}(A, [0, 1]),\ (u, v) = (w, 1) \text{ on } \partial A\},$$

where $u = w$ on ∂A has to be intended in the sense of traces and inner traces for u and w, respectively. If $A \subseteq \mathbb{R}^n$ is a set such that $\mathrm{int}A \in \mathscr{A}_\infty$ then we use the following convention $m^h_{\mathrm{b}}(w, A) := m^h_{\mathrm{b}}(w, \mathrm{int}A)$ and $m^h_{\mathrm{s}}(w, A) := m^h_{\mathrm{s}}(w, \mathrm{int}A)$.

The main result of this paper is contained in Theorem 3.4, below, and provides an *almost sure* Γ-convergence result for the functionals F_ε defined in (3.1). In order to state this result we preliminarily need to state a theorem which guarantees the almost sure existence of the integrands of the Γ-limit. Namely, the next theorem establishes the existence and spatial homogeneity of the limits defining the asymptotic cell formulas appearing in Theorem 3.4 below.

Throughout the paper we adopt the following shorthand notation.

$$m^{f_\omega}_{\mathrm{b}} := m^{f(\omega,\cdot,\cdot)}_{\mathrm{b}} \quad \text{and} \quad m^{f^\infty_\omega}_{\mathrm{s}} := m^{f^\infty(\omega,\cdot,\cdot)}_{\mathrm{s}}.$$

Theorem 3.3 (Homogenisation formulas)**.** *Let f be a stationary random integrand. Then there exists $\Omega' \in \mathcal{T}$ with $P(\Omega') = 1$, such that for every $\omega \in \Omega'$*

(i) every $x \in \mathbb{R}^n$, $\nu \in \mathbb{S}^{n-1}$, $k \in \mathbb{N}$, and $\xi \in \mathbb{R}^{N\times n}$, the limit

$$\lim_{r\to+\infty} \frac{m^{f_\omega}_{\mathrm{b}}(\ell_\xi, Q^{\nu,k}_r(rx))}{k^{n-1}r^n}$$

exists and it is independent of x, ν and k;

(ii) every $x \in \mathbb{R}^n$, $\zeta \in \mathbb{R}^N$, and $\nu \in \mathbb{S}^{n-1}$, the limit

$$\lim_{r\to+\infty} \frac{m^{f^\infty_\omega}_{\mathrm{s}}(u_{rx,\zeta,\nu}, Q^\nu_r(rx))}{r^{n-1}}$$

exists and it is independent of x.

More precisely there exist a $\mathcal{T} \otimes \mathscr{B}^{N\times n}$-measurable function $f_{\mathrm{hom}} : \Omega \times \mathbb{R}^{N\times n} \to [0, \infty)$ and a $\mathcal{T} \otimes \mathscr{B}^N \otimes \mathscr{B}^n_S$-measurable function $g_{\mathrm{hom}} : \Omega \times \mathbb{R}^N \times \mathbb{S}^{n-1} \to [0, +\infty)$ such that for every $\omega \in \Omega'$, $x \in \mathbb{R}^n$, $\xi \in \mathbb{R}^{N\times n}$, $\zeta \in \mathbb{R}^N$, and $\nu \in \mathbb{S}^{n-1}$

$$\begin{aligned} f_{\mathrm{hom}}(\omega, \xi) &= \lim_{r\to+\infty} \frac{m^{f_\omega}_{\mathrm{b}}(\ell_\xi, Q^{\nu,k}_r(rx))}{k^{n-1}r^n} \\ &= \lim_{r\to+\infty} \frac{m^{f_\omega}_{\mathrm{b}}(\ell_\xi, Q_r(rx))}{r^n} = \lim_{r\to+\infty} \frac{m^{f_\omega}_{\mathrm{b}}(\ell_\xi, Q_r)}{r^n}, \end{aligned}$$

$$f^\infty_{\rm hom}(\omega,\xi) = \lim_{r\to+\infty}\frac{m_{\rm b}^{f^\infty_\omega}(\ell_\xi, Q_r^{\nu,k}(rx))}{k^{n-1}r^n}$$
$$= \lim_{r\to+\infty}\frac{m_{\rm b}^{f^\infty_\omega}(\ell_\xi, Q_r(rx))}{r^n} = \lim_{r\to+\infty}\frac{m_{\rm b}^{f^\infty_\omega}(\ell_\xi, Q_r)}{r^n},$$
$$g_{\rm hom}(\omega,\zeta,\nu) = \lim_{r\to+\infty}\frac{m_{\rm s}^{f^\infty_\omega}(u_{rx,\zeta,\nu}, Q_r^\nu(rx))}{r^{n-1}} = \lim_{r\to+\infty}\frac{m_{\rm s}^{f^\infty_\omega}(u_{\zeta,\nu}, Q_r^\nu)}{r^{n-1}},$$

where $f^\infty_{\rm hom}$ denotes the recession function of $f_{\rm hom}$.

If we additionally assume that f is ergodic, then $f_{\rm hom}$ and $g_{\rm hom}$ are independent of ω and

$$f_{\rm hom}(\xi) = \lim_{r\to+\infty}\frac{1}{r^n}\int_\Omega m_{\rm b}^{f_\omega}(\ell_\xi, Q_r)\mathrm{d}P(\omega),$$
$$f^\infty_{\rm hom}(\xi) = \lim_{r\to+\infty}\frac{1}{r^n}\int_\Omega m_{\rm b}^{f^\infty_\omega}(\ell_\xi, Q_r)\mathrm{d}P(\omega),$$
$$g_{\rm hom}(\zeta,\nu) = \lim_{r\to+\infty}\frac{1}{r^n}\int_\Omega m_{\rm s}^{f^\infty_\omega}(u_{\zeta,\nu}, Q_r)\mathrm{d}P(\omega).$$

We are now in a position to state the main result of this paper.

Theorem 3.4 (Almost sure Γ-convergence)**.** *Let f be a stationary random integrand. For $\varepsilon>0$ and $\omega\in\Omega$ let $F_\varepsilon(\omega)$ be the functionals defined in (3.1). Then, there exists $\Omega'\in\mathcal{T}$ with $P(\Omega')=1$ such that for every $\omega\in\Omega'$, $A\in\mathscr{A}$, and $(u,v)\in L^1_{\rm loc}(\mathbb{R}^n,\mathbb{R}^N)$ we have*

$$\Gamma(L^1_{\rm loc}(\mathbb{R}^n,\mathbb{R}^{N+1}))\text{-}\lim_{\varepsilon\to0}F_\varepsilon(\omega)(u,v,A) = F_{\rm hom}(\omega)(u,v,A),$$

where $F_{\rm hom}(\omega): L^1_{\rm loc}(\mathbb{R}^n,\mathbb{R}^{N+1})\times\mathscr{A}\longrightarrow[0,\infty]$ is defined as

$$F_{\rm hom}(\omega)(u,v,A) := \begin{cases}\displaystyle\int_A f_{\rm hom}(\omega,\nabla u)\mathrm{d}x + \int_A f^\infty_{\rm hom}(\omega,\tfrac{\mathrm{d}D^cu}{\mathrm{d}|D^cu|})\mathrm{d}|D^cu| + \int_{J_u\cap A} g_{\rm hom}(\omega,[u],\nu_u)\mathrm{d}\mathcal{H}^{n-1} \\ \qquad \text{if } u\in GBV(A,\mathbb{R}^N) \text{ and } v=1 \text{ for } \mathcal{L}^n\text{-a.e. } x\in A\\ +\infty \qquad \text{otherwise}\end{cases}$$

with $f_{\rm hom}$ and $g_{\rm hom}$ as in Theorem 3.3.

If in addition f is ergodic, then the functional $F_{\rm hom}$ is deterministic.

The proof of Theorem 3.4 will be carried out in a number of steps in the next sections.

4. Properties of the homogenized integrands

In this section we prove a number of structural properties of the homogenized integrands $f_{\rm hom}$ and $g_{\rm hom}$.

For later use, it is convenient to work in a deterministic framework where the dependence of $f_{\rm hom}$ and $g_{\rm hom}$ on ω is not taken into account. Then, as a consequence, we need to assume that the limits defining $f_{\rm hom}$ and $g_{\rm hom}$ exist and are spatially homogeneous.

We start with $f_{\rm hom}$.

Proposition 4.1. *Let $f\in\mathcal{F}(C,\alpha)$ and assume that for every $x\in\mathbb{R}^n$ and $\xi\in\mathbb{R}^{N\times n}$ the limit*

$$\lim_{r\to+\infty}\frac{m_{\rm b}^f(\ell_\xi, Q_r(rx))}{r^n} =: f_{\rm hom}(\xi) \tag{4.1}$$

exists (and is independent of x). Then, $f_{\rm hom}$ satisfies the following properties:

(i) $f_{\rm hom}$ is quasi-convex;

(ii) *for every* $\xi_1, \xi_2 \in \mathbb{R}^{N\times n}$

$$|f_{\rm hom}(\xi_1) - f_{\rm hom}(\xi_2)| \le K|\xi_1 - \xi_2|,$$

where K *is a constant that depends only on* n, N *and* C*;*

(iii) *for every* $\xi \in \mathbb{R}^{N\times n}$

$$C^{-1}|\xi| \le f_{\rm hom}(\xi) \le C(|\xi|+1). \tag{4.2}$$

Proof. (i) The quasi-convexity of $f_{\rm hom}$ defined as in (4.1) is shown in [41, Proposition 5.5 Step 2].

(ii) Let $\xi_1, \xi_2 \in \mathbb{R}^{N\times n}$ and $r > 0$ be fixed. For every $u \in BV(Q_r, \mathbb{R}^N)$ and $A \in \mathscr{A}(Q_r)$ consider the auxiliary functional defined as

$$J(u,A) := \begin{cases} \displaystyle\int_A f(y, \nabla u) {\rm d}y & \text{if } u \in W^{1,1}(Q_r, \mathbb{R}^N) \\ +\infty & \text{in } BV(Q_r, \mathbb{R}^N) \setminus W^{1,1}(Q_r, \mathbb{R}^N), \end{cases}$$

as well as $\overline{J}(\cdot, A) := sc^-(L^1)J(\cdot, A)$.

By (f2) we have that $\overline{J}(u,A) \le C(|Du|(A) + \mathcal{L}^n(A))$, therefore thanks to [17, Lemma 3.1 and Lemma 4.1.2] we get

$$\begin{aligned} |m_{\overline{J}}(\ell_{\xi_1}, Q_r) - m_{\overline{J}}(\ell_{\xi_2}, Q_r)| &\le C\|\ell_{\xi_1} - \ell_{\xi_2}\|_{L^1(\partial Q_r^\nu(rx))} \\ &\le C\widehat{K}|\xi_1 - \xi_2| \int_{\partial Q_r} |y| {\rm d}\mathcal{H}^{n-1}(y) \le \frac{C\widehat{K}\sqrt{n}}{2}\mathcal{H}^{n-1}(\partial Q_1)|\xi_1 - \xi_2| r^n \end{aligned} \tag{4.3}$$

where $\widehat{K}$ depends only on n and N, while

$$m_{\overline{J}}(\ell_\xi, Q_r) := \inf\{\overline{J}(u, Q_r) \ : \ u \in BV(Q_r, \mathbb{R}^N) \ \text{ with } u = \ell_\xi \text{ on } \partial Q_r\}.$$

Appealing to [17, Lemma 4.1.3] we deduce that

$$m_{\overline{J}}(\ell_{\xi_1}, Q_r) = m^f_{\rm b}(\xi_1, Q_r) \quad \text{and} \quad m_{\overline{J}}(\ell_{\xi_2}, Q_r) = m^f_{\rm b}(\xi_2, Q_r)$$

therefore, combining (4.1) and (4.3) readily gives

$$|f_{\rm hom}(\xi_1) - f_{\rm hom}(\xi_2)| \le K|\xi_1 - \xi_2|,$$

with $K := \frac{C\widehat{K}\sqrt{n}}{2}\mathcal{H}^{n-1}(\partial Q_1)$.

(iii) Let $\xi \in \mathbb{R}^{N\times n}$, $r > 0$, and $u \in W^{1,1}(Q_r, \mathbb{R}^N)$ with $u = \ell_\xi$ on ∂Q_r be arbitrary and fixed. By (f2) we have

$$C^{-1}|\xi| r^n = C^{-1}\Big| \int_{Q_r} \nabla u \ {\rm d}x \Big| \le C^{-1} \int_{Q_r} |\nabla u| {\rm d}x \le \int_{Q_r} f(x, \nabla u) {\rm d}x,$$

therefore passing to the inf on u we immediately get

$$C^{-1}|\xi| r^n \le m^f_{\rm b}(\ell_\xi, Q_r)$$

for every $\xi \in \mathbb{R}^{N\times n}$ and $r > 0$. Hence the second inequality in (4.2) follows by (4.1).

The second inequality in (4.2) is a consequence of the trivial inequality

$$m^f_{\rm b}(\ell_\xi, Q_r) \le \int_{Q_r} f(x, \nabla \ell_\xi) {\rm d}x \le C(|\xi|+1) r^n$$

which, in turn, is implied by (f2). □

Below we prove that $f^\infty_{\rm hom}$ can be equivalently expressed as the limit of suitable (scaled) minimisation problems. To prove it we make use of the following lemma.

Lemma 4.2. *Let $g \in \mathcal{F}(C,\alpha)$, $A \in \mathscr{A}$, $(u,v) \in W^{1,1}(A,\mathbb{R}^N) \times W^{1,2}(A,[0,1])$, then for every $t>0$ we have that*

$$\int_A \Big| v^2 g^\infty(y,\nabla u) - v^2 \frac{g(y,t\nabla u)}{t}\Big| \mathrm{d}y \le \frac{K}{t}\mathcal{L}^n(A) + \frac{K}{t^\alpha}\mathcal{L}^n(A)^\alpha \Big(\int_A v^2|\nabla u| \mathrm{d}y\Big)^{1-\alpha},$$

where K is a positive constant depending only on C and α.

Proof. Thanks to (f4), $0 \le v \le 1$ and $\alpha \in (0,1)$ we have that

$$\int_A \Big| v^2 g^\infty(y,\nabla u) - v^2 \frac{g(y,t\nabla u)}{t}\Big| \mathrm{d}y \le \frac{C}{t}\mathcal{L}^n(A) + \frac{C}{t}\int_A v^{2(1-\alpha)} g(y,t\nabla u)^{1-\alpha} \mathrm{d}y\,,$$

thus by Jensen's Inequality we deduce that

$$\frac{C}{t}\int_A v^{2(1-\alpha)} g(y,t\nabla u)^{1-\alpha}\mathrm{d}y \le \frac{C}{t}\mathcal{L}^n(A)^\alpha \Big(\int_A v^2 g(y,t\nabla u)\mathrm{d}y\Big)^{1-\alpha}.$$

Eventually, we conclude by (f2). □

Proposition 4.3. *Let $f \in \mathcal{F}(C,\alpha)$ and assume that for every $x \in \mathbb{R}^n$, $\nu \in \mathbb{S}^{n-1}$, $k \in \mathbb{N}$, and $\xi \in \mathbb{R}^{N\times n}$*

$$\lim_{r\to+\infty} \frac{m_{\mathrm{b}}^f(\ell_\xi, Q_r^{\nu,k}(rx))}{k^{n-1}r^n} = f_{\mathrm{hom}}(\xi) \tag{4.4}$$

where f_{hom} is as in (4.1). Let f_{hom}^∞ be the recession function of f_{hom}, then for every $x \in \mathbb{R}^n$, $\xi \in \mathbb{R}^{n\times N}$, $\nu \in \mathbb{S}^{n-1}$ and $k \in \mathbb{N}$ we have

$$f_{\mathrm{hom}}^\infty(\xi) = \lim_{r\to+\infty} \frac{m_{\mathrm{b}}^{f^\infty}(\ell_\xi, Q_r^{\nu,k}(rx))}{k^{n-1}r^n},$$

hence, in particular, $f_{\mathrm{hom}}^\infty = (f^\infty)_{\mathrm{hom}}$.

Proof. Let $x \in \mathbb{R}^n$, $\xi \in \mathbb{R}^{n\times N}$, $\nu \in \mathbb{S}^{n-1}$, $k \in \mathbb{N}$ and $\eta \in (0,1)$ be fixed. By (**??**), for every $r>0$ there exists $u_r \in W^{1,1}(Q_r^{\nu,k}(rx))$ with $u_r = \ell_\xi$ on $\partial Q_r^{\nu,k}(rx)$, such that

$$E^{f^\infty}(u_r, Q_r^{\nu,k}(rx)) \le m_{\mathrm{b}}^{f^\infty}(\ell_\xi, Q_r^{\nu,k}(rx)) + \eta k^{n-1}r^n, \tag{4.5}$$

and

$$C^{-1}\int_{Q_r^{\nu,k}(rx)} |\nabla u_r| \mathrm{d}y \le m_{\mathrm{b}}^{f^\infty}(\ell_\xi, Q_r^{\nu,k}(rx)) + \eta k^{n-1}r^n \le C(|\xi|+1)k^{n-1}r^n.$$

In particular, by Lemma 4.2, for every $t \ge 1$ we obtain

$$\begin{aligned}\int_{Q_r^{\nu,k}(rx)} &|f^\infty(y,\nabla u_r) - \frac{1}{t}f(y,t\nabla u_r)|\mathrm{d}y \\ &\le \frac{K}{t}k^{n-1}r^n + \frac{K}{t^\alpha}(k^{n-1}r^n)^\alpha\Big(\int_{Q_r^{\nu,k}(rx)}|\nabla u_r|\mathrm{d}y\Big)^{1-\alpha} \le \frac{1}{t^\alpha}\hat{K}k^{n-1}r^n\,,\end{aligned}$$

where $\hat{K}$ depends only on C, α and ξ. Hence, for $t \ge 1$,

$$E^{f_t}(u_r, Q_r^{\nu,k}(rx)) \le E^{f^\infty}(u_r, Q_r^{\nu,k}(rx)) + \frac{1}{t^\alpha}\hat{K}k^{n-1}r^n,$$

where $f_t(y,\xi) := \frac{f(y,t\xi)}{t}$ and consequently, by (4.5),

$$\frac{m_{\mathrm{b}}^{f_t}(\ell_\xi, Q_r^{\nu,k}(rx))}{k^{n-1}r^n} \le \frac{m_{\mathrm{b}}^{f^\infty}(\ell_\xi, Q_r^{\nu,k}(rx))}{k^{n-1}r^n} + \eta + \frac{\hat{K}}{t^\alpha}. \tag{4.6}$$

Observing that $m_{\mathrm{b}}^{f_t}(\ell_\xi, Q_r^{\nu,k}(rx)) = \frac{1}{t}m_{\mathrm{b}}^f(\ell_{t\xi}, Q_r^{\nu,k}(rx))$, thanks to the linearity of $\xi \mapsto \ell_\xi$, we get

$$\lim_{r\to\infty}\frac{m_{\mathrm{b}}^{f_t}(\ell_\xi, Q_r^{\nu,k}(rx))}{k^{n-1}r^n} = \lim_{r\to\infty}\frac{m_{\mathrm{b}}^{f}(\ell_{t\xi}, Q_r^{\nu,k}(rx))}{tk^{n-1}r^n} = \frac{f_{\mathrm{hom}}(t\xi)}{t}, \tag{4.7}$$

by (4.4). Hence, from (4.6) and (4.7), letting $\eta \to 0$, we have

$$\limsup_{t\to+\infty} \frac{f_{\hom}(t\xi)}{t} \leq \liminf_{r\to+\infty} \frac{m_{\mathrm{b}}^{f^\infty}(\ell_\xi, Q_r^{\nu,k}(rx))}{k^{n-1}r^n}.$$

Exchanging the role of f_t and f^∞ and arguing analogously, we obtain

$$\limsup_{r\to+\infty} \frac{m_{\mathrm{b}}^{f^\infty}(\ell_\xi, Q_r^{\nu,k}(rx))}{k^{n-1}r^n} \leq \liminf_{t\to+\infty} \frac{f_{\hom}(t\xi)}{t}. \qquad \square$$

To prove the properties satisfied by $g_{\hom}$, we first establish some technical results in the spirit of [17, Section 3].

Lemma 4.4. *Let $x \in \mathbb{R}^n$, $r > 1$, $\nu \in \mathbb{S}^{n-1}$, and $w_1, w_2 \in BV_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{R}^N)$, then we have that*

$$|m_{\mathrm{s}}^{f^\infty}(w_1, Q_r^\nu(x)) - m_{\mathrm{s}}^{f^\infty}(w_2, Q_r^\nu(x))| \leq C \int_{\partial Q_r^\nu(x)} |w_1 - w_2| \mathcal{H}^{n-1}.$$

Proof. First we observe that for every $w \in BV_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{R}^N)$, $x \in \mathbb{R}^n$, $\nu \in \mathbb{S}^{n-1}$, and $r > 1$ there holds $m_{\mathrm{s}}^{f^\infty}(w, Q_r^\nu(rx)) = m_{\mathrm{s}}^{f^\infty,*}(w, Q_r^\nu(rx))$, where

$$\begin{aligned} m_{\mathrm{s}}^{f^\infty,*}(w, Q_r^\nu(rx)) := \inf\{ S^{f^\infty}(u, v, Q_r^\nu(rx)) \ : \ u \in W^{1,1}(Q_r^\nu(rx), \mathbb{R}^N),\ v \in W^{1,2}(Q_r^\nu(rx), [0,1]), \\ (u,v) = (w,1) \text{ on } \partial Q_r^\nu(rx),\ \ v \geq \eta \text{ for some } \eta \in (0,1)\}, \end{aligned} \tag{4.8}$$

with S^{f^∞} defined in (**??**). In fact, given $\eta \in (0,1)$, $u \in W^{1,1}(Q_r^\nu(x), \mathbb{R}^N)$ and $v \in W^{1,2}(Q_r^\nu(x), [0,1])$, $v_\eta := v \vee \eta \in W^{1,2}(Q_r^\nu(x), [0,1])$ with $v_\eta = 1$ on $\partial Q_r^\nu(x)$, and

$$\int_{Q_r^\nu(x)} (1 - v_\eta)^2 + |\nabla v_\eta|^2 \mathrm{d}y \leq \int_{Q_r^\nu(x)} (1 - v)^2 + |\nabla v|^2 \mathrm{d}y,$$

and

$$\lim_{\eta\to 0^+} \int_{Q_r^\nu(x)} v_\eta^2 f^\infty(y, \nabla u) \mathrm{d}y = \int_{Q_r^\nu(x)} v^2 f^\infty(y, \nabla u) \mathrm{d}y.$$

Let $v \in W^{1,2}(Q_r^\nu(x), [0,1])$ with $v = 1$ on $\partial Q_r^\nu(x)$ and $v \geq \eta$ for some $\eta \in (0,1)$. Define the functional $\mathcal{F}_v : BV(Q_r^\nu(x), \mathbb{R}^N) \times \mathscr{A}(Q_r^\nu(x)) \longrightarrow [0, +\infty]$ as

$$\mathcal{F}_v(u, B) := \begin{cases} \displaystyle\int_B v^2 f^\infty(y, \nabla u) \mathrm{d}y & \text{if } u \in W^{1,1}(Q_r^\nu(x), \mathbb{R}^N) \\ +\infty & \text{otherwise.} \end{cases}$$

Consider its relaxation $\overline{\mathcal{F}}_v := sc^-(L^1)\mathcal{F}_v : BV(Q_r^\nu(x), \mathbb{R}^N) \times \mathscr{A}(Q_r^\nu(x)) \to [0, +\infty]$. [17, Lemma 3.1 and Lemma 4.1.2] and $\overline{\mathcal{F}}_v(u, B) \leq C|Du|(B)$ imply that

$$|m_{\overline{\mathcal{F}}_v}(w_1, Q_r^\nu(x)) - m_{\overline{\mathcal{F}}_v}(w_2, Q_r^\nu(x))| \leq C \int_{\partial Q_r^\nu(x)} |w_1 - w_2| \mathcal{H}^{n-1}, \tag{4.9}$$

where we recall that $m_{\overline{\mathcal{F}}_v}(w, Q_r^\nu(x))$ is defined in (5.9) using the functional $\overline{\mathcal{F}}_v$.

In addition, $m_{\overline{\mathcal{F}}_v}(w_i, Q_r^\nu(x)) = m_{\mathcal{F}_v}(w_i, Q_r^\nu(x))$, $i \in \{1,2\}$, by [17, Lemma 4.1.3]. Therefore, using (4.8) we can rewrite $m_{\mathrm{s}}^{f^\infty}(w, Q_r^\nu(rx))$ as

$$\begin{aligned} m_{\mathrm{s}}^{f^\infty}(w, Q_r^\nu(rx)) = \inf\{ m_{\overline{\mathcal{F}}_v}(w, Q_r^\nu(rx)) + \textstyle\int_{Q_r^\nu(rx)} ((1-v)^2 + |\nabla v|^2) \mathrm{d}y : \\ v \in W^{1,2}(Q_r^\nu(rx), [0,1])\ v \geq \eta \ \text{ for some } \eta \in (0,1)\}, \end{aligned}$$

and thus we can conclude thanks to (4.9). $\square$

The following result readily follows from Lemma 4.4.

Corollary 4.5. *Let $x \in \mathbb{R}^n$, $r > 1$, $\nu \in \mathbb{S}^{n-1}$, and $\zeta \in \mathbb{R}^N$ then we have that*

$$|m_{\rm s}^{f^\infty}(u_{x,\zeta,\nu}, Q_r^\nu(x)) - m_{\rm s}^{f^\infty}(\overline{u}_{x,\zeta,\nu}, Q_r^\nu(x))| \leq 2C|\zeta| r^{n-2},$$

where $u_{x,\zeta,\nu}$ and $\overline{u}_{x,\zeta,\nu}$ are defined in (l).

The next lemma will be widely used in Section 6.

Lemma 4.6. *Let $x, z \in \mathbb{R}^n$, $\nu_1, \nu_2 \in \mathbb{S}^{n-1}$, and $r_2 > r_1 > 2R \geq 1$ be such that $Q_{r_1}^{\nu_1}(x) \subset\subset Q_{r_2}^{\nu_2}(z)$ and $|(y-x)\cdot \nu_1| \leq \frac{1}{2}$ imply*

$$|(y-z)\cdot\nu_2| \leq R \qquad \text{for every } y \in Q_{r_2}^{\nu_2}(z). \tag{4.10}$$

Then, for every $\zeta \in \mathbb{R}^N$ and $\eta > 0$ the following statements hold true: for every $r_3 \geq r_2$ such that $Q_{r_2}^{\nu_2}(z) \subset\subset Q_{r_3}^{\nu_1}(x)$, then

$$m_{\rm s}^{f^\infty}(\overline{u}_{z,\zeta,\nu_2}, Q_{r_2}^{\nu_2}(z)) - m_{\rm s}^{f^\infty}(\overline{u}_{x,\zeta,\nu_1}, Q_{r_1}^{\nu_1}(x)) \leq \eta r_1^{n-1} + \tilde{K}((r_3 - r_1) r_3^{n-2} + R|\zeta| r_2^{n-2});$$

with $\tilde{K}$ depending only on n and C.

Proof. Fix $\zeta \in \mathbb{R}^N$, $\eta > 0$ and let $(u,v) \in W^{1,1}(Q_{r_1}^{\nu_1}(x), \mathbb{R}^N) \times W^{1,2}(Q_{r_1}^{\nu_1}(x), [0,1])$, with $(u,v) = (\overline{u}_{x,\zeta,\nu}, 1)$ on $\partial Q_{r_1}^{\nu_1}(x)$, such that

$$S^{f^\infty}(u,v,Q_{r_1}^{\nu_1}(x)) = \int_{Q_{r_1}^{\nu_1}(x)} v^2 (f^\infty(y,\nabla u) + (1-v)^2 + |\nabla v|^2) \mathrm{d}y \leq m_{\rm s}^{f^\infty}(\overline{u}_{x,\zeta,\nu}, Q_{r_1}^{\nu_1}(x)) + \eta r_1^{n-1}. \tag{4.11}$$

Define $(\hat{u}, \hat{v})$ by

$$(\hat{u}(y), \hat{v}(y)) := \begin{cases} (u(y), v(y)) & \text{if } y \in Q_{r_1}^{\nu_1}(x) \\ (\overline{u}_{x,\zeta,\nu}, 1) & \text{if } y \in Q_{r_2}^{\nu_2}(z) \setminus Q_{r_1}^{\nu_1}(x)\,, \end{cases}$$

and note that $(\hat{u}, \hat{v}) \in W^{1,1}(Q_{r_2}^{\nu_2}(z), \mathbb{R}^N) \times W^{1,2}(Q_{r_2}^{\nu_2}(z), [0,1])$ with $(\hat{u}, \hat{v}) = (\overline{u}_{x,\zeta,\nu}, 1)$ on $\partial Q_{r_2}^{\nu_2}(z)$.

From (4.10) we have that $\overline{u}_{z,\zeta,\nu_2}(y) = \overline{u}_{x,\zeta,\nu_1}(y)$ for every $y \in Q_{r_2}^{\nu_2}(z)$ such that $|(y-z)\cdot\nu_2| > R$; in particular Lemma 4.4 yields that

$$\begin{aligned} |m_{\rm s}^{f^\infty}(\overline{u}_{z,\zeta,\nu_2}, Q_{r_2}^{\nu_2}(z)) - m_{\rm s}^{f^\infty}(\overline{u}_{x,\zeta,\nu_1}, Q_{r_2}^{\nu_2}(z))| &\leq \int_{\partial Q_{r_2}^{\nu_2}(z)} |\overline{u}_{z,\zeta,\nu_2} - \overline{u}_{x,\zeta,\nu_1}| \mathrm{d}\mathcal{H}^{n-1} \\ = \int_{\partial Q_{r_2}^{\nu_2}(z) \cap \Sigma_{\hat{K}}} |\overline{u}_{z,\zeta,\nu_2} - \overline{u}_{x,\zeta,\nu_1}| \mathrm{d}\mathcal{H}^{n-1} &\leq 8R(n-1)|\zeta| r_2^{n-2}, \end{aligned} \tag{4.12}$$

where $\Sigma_{\nu_2,R} := \{|(y-z)\cdot\nu_2| \leq R\}$. Furthermore, setting $\Sigma_{\nu_1,1/2} := \{|(y-x)\cdot\nu_1| \leq \frac{1}{2}\}$, from (l), (2.3) and (4.11) we get

$$\begin{aligned} m_{\rm s}^{f^\infty}&(\overline{u}_{x,\zeta,\nu_1}, Q_{r_2}^{\nu_2}(z)) \leq S^{f^\infty}(\hat{u}, \hat{v}, Q_{r_2}^{\nu_2}(z)) \\ &\leq S^{f^\infty}(u,v,Q_{r_1}^{\nu_1}(x)) + \int_{Q_{r_2}^{\nu_2}(z)\setminus Q_{r_1}^{\nu_1}(x)} f^\infty(\nabla\hat{u}) \mathrm{d}y \\ &\leq m_{\rm s}^{f^\infty}(\overline{u}_{x,\zeta,\nu_1}, Q_{r_1}^{\nu_1}(x)) + \eta r_1^{n-1} + C\|\overline{\mathrm{u}}'\|_{L^\infty(\mathbb{R})} \mathcal{L}^n((Q_{r_2}^{\nu_2}(z) \setminus Q_{r_1}^{\nu_1}(x)) \cap \Sigma_{\nu_1}) \\ &\leq m_{\rm s}^{f^\infty}(\overline{u}_{x,\zeta,\nu_1}, Q_{r_1}^{\nu_1}(x)) + \eta r_1^{n-1} + C\|\overline{\mathrm{u}}'\|_{L^\infty(\mathbb{R})} \mathcal{L}^n((Q_{r_3}^{\nu_1}(z) \setminus Q_{r_1}^{\nu_1}(x)) \cap \Sigma_{\nu_1}) \\ &\leq m_{\rm s}^{f^\infty}(\overline{u}_{x,\zeta,\nu_1}, Q_{r_1}^{\nu_1}(x)) + \eta r_1^{n-1} + C\|\overline{\mathrm{u}}'\|_{L^\infty(\mathbb{R})} (r_3^{n-1} - r_1^{n-1}) \\ &\leq m_{\rm s}^{f^\infty}(\overline{u}_{x,\zeta,\nu_1}, Q_{r_1}^{\nu_1}(x)) + \eta r_1^{n-1} + C(n-1)\|\overline{\mathrm{u}}'\|_{L^\infty(\mathbb{R})} (r_3 - r_1) r_3^{n-2}. \end{aligned} \tag{4.13}$$

Therefore, recollecting (4.12) and (4.13), we deduce

$$\begin{aligned} m_{\rm s}^{f^\infty}&(\overline{u}_{z,\zeta,\nu_2}, Q_{r_2}^{\nu_2}(z)) \leq m_{\rm s}^{f^\infty}(\overline{u}_{x,\zeta,\nu_1}, Q_{r_2}^{\nu_2}(z)) + 8R(n-1)|\zeta| r_2^{n-2} \\ &\leq m_{\rm s}^{f^\infty}(\overline{u}_{x,\zeta,\nu_1}, Q_{r_1}^{\nu_1}(x)) + \eta r_1^{n-1} + C(n-1)\|\overline{\mathrm{u}}'\|_{L^\infty(\mathbb{R})} (r_3 - r_1) r_3^{n-2} + 8R(n-1)|\zeta| r_2^{n-2} \end{aligned}$$

and therefore the claim. □

We are now in a position to establish some properties satisfied by $g_{\rm hom}$.

Proposition 4.7. *Let $f \in \mathcal{F}(C,\alpha)$ and assume that for every $x \in \mathbb{R}^n$, $\zeta \in \mathbb{R}^N$, and $\nu \in \mathbb{S}^{n-1}$ the limit*

$$\lim_{r\to+\infty} \frac{m_{\rm s}^{f^\infty}(u_{rx,\zeta,\nu}, Q_r^\nu(rx))}{r^{n-1}} =: g_{\rm hom}(\zeta,\nu) \tag{4.14}$$

exists (and is independent of x). Then, $g_{\rm hom}$ satisfies the following properties:

(i) for every $\zeta_1, \zeta_2 \in \mathbb{R}^N$ and every $\nu \in \mathbb{S}^{n-1}$

$$|g_{\rm hom}(\zeta_1,\nu) - g_{\rm hom}(\zeta_2,\nu)| \le C\mathcal{H}^{n-1}(\partial Q_1)|\zeta_1-\zeta_2|; \tag{4.15}$$

(ii) $g_{\rm hom} : \mathbb{R}^N \times \hat{\mathbb{S}}^{n-1}_{\pm} \to [0,+\infty)$ is continuous;

(iii) for every $\zeta \in \mathbb{R}^N$ and every $\nu \in \mathbb{S}^{n-1}$

$$\frac{2|\zeta|}{C(|\zeta|+2)} \le g_{\rm hom}(\zeta,\nu) \le \frac{2C|\zeta|}{|\zeta|+2}; \tag{4.16}$$

(iv) for every $\zeta \in \mathbb{R}^N$ and every $\nu \in \mathbb{S}^{n-1}$

$$g_{\rm hom}(\zeta,\nu) = g_{\rm hom}(-\zeta,-\nu).$$

Proof. To prove (i) fix $\nu \in \mathbb{S}^{n-1}$ and $\zeta_1,\zeta_2 \in \mathbb{R}^{N\times n}$. Thanks to Lemma 4.4 we get

$$|m_{\rm s}^{f^\infty}(\overline{u}_{\zeta_1,\nu}, Q_r^\nu) - m_{\rm s}^{f^\infty}(\overline{u}_{\zeta_2,\nu}, Q_r^\nu)| \le C\int_{\partial Q_r^\nu} |\overline{u}_{\zeta_1,\nu} - \overline{u}_{\zeta_2,\nu}| \mathrm{d}\mathcal{H}^{n-1}.$$

By the definition of $\overline{u}_{\zeta,\nu}$, using Lemma 4.4 we have

$$\int_{\partial Q_r^\nu} |\overline{u}_{\zeta_1,\nu} - \overline{u}_{\zeta_2,\nu}| \mathrm{d}\mathcal{H}^{n-1} = \int_{\partial Q_r^\nu} |\zeta_1-\zeta_2|\overline{\mathrm{u}}\,(y\cdot\nu)\,\mathrm{d}\mathcal{H}^{n-1}(y) \le \mathcal{H}^{n-1}(\partial Q_1)|\zeta_1-\zeta_2|r^{n-1}.$$

Then we conclude by (4.14) also noticing that by Corollary 4.5 we have

$$g_{\rm hom}(\zeta,\nu) = \lim_{r\to+\infty} \frac{m_{\rm s}^{f^\infty}(\overline{u}_{rx,\zeta,\nu}, Q_r^\nu(rx))}{r^{n-1}},$$

for every $x \in \mathbb{R}^n$, $\zeta \in \mathbb{R}^N$, and $\nu \in \mathbb{S}^{n-1}$.

To prove (ii) we preliminarily show that $g_{\rm hom}(\zeta,\cdot) : \hat{\mathbb{S}}^{n-1}_{\pm} \to [0,+\infty)$ is continuous for every $\zeta \in \mathbb{R}^N$. Fix $\zeta \in \mathbb{R}^N$, $\nu \in \hat{\mathbb{S}}^{n-1}_{\pm}$ and a sequence $(\nu_j)_{j\in\mathbb{N}}$ in $\hat{\mathbb{S}}^{n-1}_{\pm}$ such that $\nu_j \to \nu$ as $j \to +\infty$. For every $\delta \in (0,1/2)$, by the continuity of the map $\nu \mapsto R_\nu$ on $\hat{\mathbb{S}}^{n-1}_{\pm}$ (cf. (e) of the notation list), there exists j_δ such that for every $r>0$ and every $j \ge j_\delta$

$$Q_r^\nu \subset\subset Q^{\nu_j}_{(1+\delta)r} \subset\subset Q^\nu_{(1+2\delta)r}\,. \tag{4.17}$$

Setting $\kappa_j := \max\{|R_{\nu_j}(e_i)\cdot\nu| \;:\; i=1,\dots,n-1\}$, we have that $\kappa_j \to 0$ as $j \to +\infty$, by the continuity of the map $\nu \mapsto R_\nu$ on $\hat{\mathbb{S}}^{n-1}_{\pm}$. Letting $y \in \overline{Q^{\nu_j}_{r(1+\delta)}}$, then $y = y' + (y\cdot\nu_j)\nu_j$ where

$$y' \in R_{\nu_j}\Big(\big[-\frac{r}{2}(1+\delta), \frac{r}{2}(1+\delta)\big]^{n-1} \times \{0\}\Big).$$

In particular $(y\cdot\nu_j)(\nu\cdot\nu_j) = y\cdot\nu - y'\cdot\nu$ and thus, if $|y\cdot\nu| \le \frac{1}{2}$ and j is large enough, we get

$$|y\cdot\nu_j| \le \tfrac{|y'\cdot\nu|}{|\nu_j\cdot\nu|} + \tfrac{1}{2\nu_j\cdot\nu} \le \tfrac{(n-1)\kappa_j r(1+\delta)+1}{2(1-\delta)} = K(\delta)r\kappa_j + 1, \tag{4.18}$$

where $K(\delta) := \frac{(n-1)(1+\delta)}{2(1-\delta)}$. Applying Lemma 4.6 with $R = K(\delta) r\kappa_j + 1$, we deduce

$$m_{\rm s}^{f^\infty}(\overline{u}_{\zeta,\nu_j}, Q^{\nu_j}_{r(1+\delta)}) \le m_{\rm s}^{f^\infty}(\overline{u}_{\zeta,\nu}, Q^{\nu}_{r}) + \eta r^{n-1}$$
$$+ \tilde K(2\delta(1+2\delta)^{n-2} r^{n-1} + (K(\delta) r\kappa_j + 1)|\zeta|(1+\delta)^{n-2} r^{n-2}),$$

where $\tilde K$ depends only on n and C. Consequently, letting the $r \to +\infty$, appealing to (4.14) and to Lemma 4.4, we get

$$(1+\delta)^{n-1} g_{\rm hom}(\zeta,\nu_j) \le g_{\rm hom}(\zeta,\nu) + \eta + \tilde K(1+2\delta)^{n-2}(2\delta + K(\delta)\kappa_j|\zeta|).$$

Taking the lim sup for $j \to +\infty$ we have

$$(1+\delta)^{n-1} \limsup_{j\to+\infty} g_{\rm hom}(\zeta,\nu_j) \le g_{\rm hom}(\zeta,\nu) + \eta + 2\tilde K\delta(1+2\delta)^{n-2}$$

thus letting $\eta, \delta \to 0$ we obtain

$$\limsup_{j\to+\infty} g_{\rm hom}(\zeta,\nu_j) \le g_{\rm hom}(\zeta,\nu).$$

An analogous argument, using the cube $Q^{\nu_j}_{(1-\delta)r}$, shows that

$$g_{\rm hom}(\zeta,\nu) \le \liminf_{j\to+\infty} g_{\rm hom}(\zeta,\nu_j)$$

and hence the claim.

To establish the continuity with respect to both variables, consider a sequence $(\zeta_j)_{j\in\mathbb{N}}$ in $\mathbb{R}^{N\times n}$ such that $\zeta_j \to \zeta$. Thanks to (4.15), we have that

$$|g_{\rm hom}(\zeta,\nu) - g_{\rm hom}(\zeta_j,\nu_j)| \le |g_{\rm hom}(\zeta,\nu) - g_{\rm hom}(\zeta,\nu_j)| + |g_{\rm hom}(\zeta,\nu_j) - g_{\rm hom}(\zeta_j,\nu_j)|$$
$$\le |g_{\rm hom}(\zeta,\nu) - g_{\rm hom}(\zeta,\nu_j)| + C\mathcal{H}^{n-1}(\partial Q_1)|\zeta - \zeta_j|$$

and therefore we get (ii).

To prove (iii) fix $\zeta \in \mathbb{R}^N$ and $\nu \in \mathbb{S}^{n-1}$, and recall that by (4.14) and the spatial homogeneity of $g_{\rm hom}$ we have that

$$g_{\rm hom}(\zeta,\nu) = \lim_{r\to+\infty} \frac{m_{\rm s}^{f^\infty}(u_{\zeta,\nu}, Q^\nu_r)}{r^{n-1}}\,. \tag{4.19}$$

We notice that for every $r > 0$ and $M \in \mathbb{N}$ we have

$$\frac{m_{\rm s}^{f^\infty}(u_{\zeta,\nu}, Q^\nu_{Mr})}{Mr^{n-1}} \le \frac{m_{\rm s}^{f^\infty}(u_{\zeta,\nu}, Q^\nu_r)}{r^{n-1}}\,.$$

Indeed, assume for simplicity $\nu = e_n$, then if (u,v) is a competitor for $m_{\rm s}^{f^\infty}(u_{\zeta,e_n}, Q^{e_n}_r)$ then (u_M, v_M) defined by $(u,v)(x - r\mathtt{i})$ for $x \in r\mathtt{i} + Q^{e_n}_r$, for $\mathtt{i} \in \mathbb{Z}^{n-1}\times\{0\}$ with components in $[-M+1, M-1]$, and equal to u_{ζ,e_n} otherwise on $Q^{e_n}_{Mr}$ is a competitor for $m_{\rm s}^{f^\infty}(u_{\zeta,e_n}, Q^{e_n}_{Mr})$ with $S^{f^\infty}(u_M, v_M, Q^{e_n}_{Mr}) = M^{n-1} S^{f^\infty}(u, v, Q^{e_n}_r)$. Thus, we infer that

$$\lim_{r\to+\infty} \frac{m_{\rm s}^{f^\infty}(u_{\zeta,\nu}, Q^\nu_r)}{r^{n-1}} = \inf_{r>0} \frac{m_{\rm s}^{f^\infty}(u_{\zeta,\nu}, Q^\nu_r)}{r^{n-1}}\,.$$

Moreover, by (f2) and $C \ge 1$, we have that

$$C^{-1} G_\varepsilon(u,v,Q^\nu_r) \le F_\varepsilon(u,v,Q^\nu_r) \le C G_\varepsilon(u,v,Q^\nu_r)\,,$$

where $G_\varepsilon : L^1_{\rm loc}(\mathbb{R}^n, \mathbb{R}^{N+1}) \times \mathscr{A} \to [0,+\infty]$ is given by

$$G_\varepsilon(u,v,B) := \begin{cases} \displaystyle\int_B (v^2|\nabla u| + \tfrac{(1-v)^2}{\varepsilon} + \varepsilon|\nabla v|^2)\mathrm{d}y & \text{if } (u,v) \in W^{1,1}(B,\mathbb{R}^N)\times W^{1,2}(B,[0,1]) \\ +\infty & \text{otherwise.} \end{cases}$$

Therefore, we conclude that

$$C^{-1}\inf_{r>0}\frac{m_{\rm s}^{|\cdot|}(u_{\zeta,\nu},Q_r^\nu)}{r^{n-1}}\leq g_{\rm hom}(\zeta,\nu)\leq C\inf_{r>0}\frac{m_{\rm s}^{|\cdot|}(u_{\zeta,\nu},Q_r^\nu)}{r^{n-1}}\,.$$

Finally, [7, Lemma 3.8 and Remark 3.9] yield that

$$\inf_{r>0}\frac{m_{\rm s}^{|\cdot|}(u_{\zeta,\nu},Q_r^\nu)}{r^{n-1}}=g(\zeta):=\frac{2|\zeta|}{|\zeta|+2}.$$

where using the same notation as in [7]

$$g(s):=\min_{[0,1]}\{t^2s+4\textstyle\int_t^1(1-\lambda)\mathrm{d}\lambda\}=\dfrac{2s}{s+2}\,.$$

Eventually, (iv) is a direct consequence of the identity $R_\nu(Q_1)=R_{-\nu}(Q_1)$ and of the fact that $u=u_{-\zeta,-\nu}$ on ∂Q_r^ν if and only if $u+\zeta=u_{\zeta,\nu}$ on ∂Q_r^ν for every $\zeta\in\mathbb{R}^N$, $\nu\in\mathbb{S}^{n-1}$, $r>0$, and $u\in W^{1,1}(Q_r^\nu,\mathbb{R}^N)$. □

5. Deterministic homogenisation

To prove the stochastic homogenisation result in Theorem 3.4 we follow the same proof strategy as in [23, 24]. To this end, we preliminarily work in a deterministic framework (where $\omega\in\Omega$ is regarded as fixed) and prove a homogenisation result without assuming any periodicity of the integrand. Then, in Section 6, the deterministic homogenisation result at fixed ω will be used in combination with the Subadditive Ergodic Theorem, Theorem 2.3, to derive an almost sure Γ-convergence result for the random functionals $F_\varepsilon(\omega)$.

The main result of this section is stated in the following theorem.

Theorem 5.1 (Deterministic homogenisation)**.** *Let $f\in\mathcal{F}(C,\alpha)$ and consider the phase-field functionals $F_\varepsilon: L^1_{\rm loc}(\mathbb{R}^n,\mathbb{R}^{N+1})\times\mathscr{A}\longrightarrow[0,+\infty]$ given by*

$$F_\varepsilon(u,v,A):=\begin{cases}\displaystyle\int_A(v^2f(\tfrac{x}{\varepsilon},\nabla u)+\tfrac{(1-v)^2}{\varepsilon}+\varepsilon|\nabla v|^2)\mathrm{d}x, & (u,v)\in W^{1,1}(A,\mathbb{R}^N)\times W^{1,2}(A,[0,1])\\ +\infty & \text{otherwise.}\end{cases}\tag{5.1}$$

Assume that

(i) for every $x\in\mathbb{R}^n$, $\xi\in\mathbb{R}^{N\times n}$, $\nu\in\mathbb{S}^{n-1}$ and $k\in\mathbb{N}$ the limit

$$\lim_{r\to+\infty}\frac{m_{\rm b}^f(\ell_\xi,Q_r^{\nu,k}(rx))}{k^{n-1}r^n}=:f_{\rm hom}(\xi)\tag{5.2}$$

exists and is independent of x,ν and k;

(ii) for every $x\in\mathbb{R}^n$, $\zeta\in\mathbb{R}^N$ and $\nu\in\mathbb{S}^{n-1}$ the limit

$$\lim_{r\to+\infty}\frac{m_{\rm s}^{f^\infty}(u_{rx,\zeta,\nu},Q_r^\nu(rx))}{r^{n-1}}=:g_{\rm hom}(\zeta,\nu)\tag{5.3}$$

exists and is independent of x.

Let, moreover, $f^\infty_{\rm hom}$ be the recession function of $f_{\rm hom}$. Then, for every $A\in\mathscr{A}$ and every $(u,v)\in L^1_{\rm loc}(\mathbb{R}^n,\mathbb{R}^{N+1})$ we have

$$\Gamma(L^1_{\rm loc}(\mathbb{R}^n,\mathbb{R}^{N+1}))\text{-}\lim_{\varepsilon\to0}F_\varepsilon(u,v,A)=F_{\rm hom}(u,v,A),$$

where $F_{\rm hom} : L^1_{\rm loc}(\mathbb{R}^n, \mathbb{R}^{N+1}) \times \mathscr{A} \longrightarrow [0, +\infty]$ *is the functional defined by*

$$F_{\rm hom}(u, v, A) := \int_A f_{\rm hom}(\nabla u) \mathrm{d}x + \int_A f^\infty_{\rm hom}\big(\tfrac{\mathrm{d}D^c u}{\mathrm{d}|D^c u|}\big) \mathrm{d}|D^c u| + \int_{J_u \cap A} g_{\rm hom}([u], \nu_u) \mathrm{d}\mathcal{H}^{n-1}, \tag{5.4}$$

if $u \in GBV(A, \mathbb{R}^N)$ *and* $v = 1$ $\mathcal{L}^n$*-a.e in* A, $F_{\rm hom}(u, v, A) = +\infty$, *otherwise.*

To prove Theorem 5.1 we use a standard approach in homogenisation theory based on the compactness of Γ-convergence and on the so-called localization method (cf. [20, 30]). Namely, we first show that for every infinitesimal sequence $(\varepsilon_j)_{j\in\mathbb{N}}$, up to a subsequence, the functionals F_{ε_j}, defined in (5.1), Γ-converge to some abstract functional $\widehat{F}$. Then, we prove that $\widehat{F}$ admits an integral representation as in (5.4) on $BV(A, \mathbb{R}^N)$, for every $A \in \mathscr{A}$. Eventually, thanks to (5.2) and (5.3) we deduce that $\widehat{F}$ does not depend on the extracted subsequence, and hence the homogenisation result for (F_ε) follows by the Uryshon property of Γ-convergence.

We start by proving the following abstract Γ-convergence result.

Theorem 5.2 (Γ-convergence and properties of the Γ-limit)**.** *Let* $f \in \mathcal{F}(C, \alpha)$ *and* F_ε *be as in* (5.1)*, then there exists a subsequence* $(\varepsilon_j)_{j\in\mathbb{N}}$ *and a functional* $\widehat{F} : L^1_{\rm loc}(\mathbb{R}^n, \mathbb{R}^{N+1}) \times \mathscr{A} \longrightarrow [0, +\infty]$ *such that, for every* $A \in \mathscr{A}$ *and every* $u \in L^1_{\rm loc}(\mathbb{R}^n, \mathbb{R}^N)$ *with* $u \in BV(A, \mathbb{R}^N)$

$$\Gamma(L^1_{\rm loc}(\mathbb{R}^n, \mathbb{R}^{N+1}))\text{-}\lim_{j\to+\infty} F_{\varepsilon_j}(u, 1, A) = \widehat{F}(u, 1, A). \tag{5.5}$$

Moreover $\widehat{F}$ *satisfies the following properties:*

*(i) (*locality*)* $\widehat{F}(u_1, v_1, A) = \widehat{F}(u_2, v_2, A)$ *for every* $A \in \mathscr{A}$ *and every* $(u_1, v_1), (u_2, v_2) \in L^1_{\rm loc}(\mathbb{R}^n, \mathbb{R}^{N+1})$ *such that* $(u_1, v_1) = (u_2, v_2)$ $\mathcal{L}^n$*-a.e in* A*;*

*(ii) (*semicontinuity*) for every* $A \in \mathscr{A}$ *the functional* $\widehat{F}(\cdot, 1, A) : L^1_{\rm loc}(\mathbb{R}^n, \mathbb{R}^N) \longrightarrow [0, +\infty]$ *is lower semicontinuous;*

*(iii) (*upper bound*) for every* $A \in \mathscr{A}$ *and every* $u \in L^1_{\rm loc}(\mathbb{R}^n, \mathbb{R}^N)$ *with* $u \in BV(A, \mathbb{R}^N)$ *there holds*

$$\widehat{F}(u, 1, A) \le C(\mathcal{L}^n(A) + |Du|(A)); \tag{5.6}$$

*(iv) (*lower bound*) for every* $M > 0$ *there exists* $C_M > 0$ *such that for every* $A \in \mathscr{A}$ *and every* $u \in L^1_{\rm loc}(\mathbb{R}^n, \mathbb{R}^N)$ *with* $u \in BV(A, \mathbb{R}^N)$ *and* $\|u\|_{L^\infty(A,\mathbb{R}^N)} \le M$ *we have*

$$C_M|Du|(A) \le \widehat{F}(u, 1, A); \tag{5.7}$$

*(v) (*measure property*) for every* $A \in \mathscr{A}$*, every* $u \in L^1_{\rm loc}(\mathbb{R}^n, \mathbb{R}^N)$ *such that* $u \in BV(A, \mathbb{R}^N)$*, the set function* $\widehat{F}(u, 1, \cdot) : \mathscr{A}(A) \to [0, +\infty]$ *is the restriction of a finite Radon measure on* A*;*

*(vi) (*translation invariance in u*) for every* $A \in \mathscr{A}$ *and every* $(u, v) \in L^1_{\rm loc}(\mathbb{R}^n, \mathbb{R}^{N+1})$ *we have*

$$\widehat{F}(u + s, v, 1, A) = \widehat{F}(u, v, 1, A),$$

for every $s \in \mathbb{R}^N$*.*

Proof. Given any sequence of positive real numbers decreasing to zero [30, Theorem 16.9] provides us with a subsequence (ε_j) such that

$$\overline{\Gamma}\text{-}\lim_{j\to+\infty} F_{\varepsilon_j} = \widehat{F},$$

where $\widehat{F} : L^1_{\rm loc}(\mathbb{R}^n, \mathbb{R}^{N+1}) \times \mathscr{A} \longrightarrow [0, +\infty]$ is increasing, inner regular, and superadditive as a set function and lower semicontinuous in $L^1_{\rm loc}(\mathbb{R}^n, \mathbb{R}^{N+1})$ as a functional. By definition of $\overline{\Gamma}$-convergence, we have

$$\widehat{F}'_- = \widehat{F} = \widehat{F}''_-, \tag{5.8}$$

where

$$\widehat{F}'(\cdot, A) = \Gamma\text{-}\liminf_{j\to\infty} F_{\varepsilon_j}(\cdot, A), \quad \widehat{F}''(\cdot, A) = \Gamma\text{-}\limsup_{j\to\infty} F_{\varepsilon_j}(\cdot, A),$$

and

$$\widehat{F}'_{-}(\cdot, A) := \sup_{A'\subset\subset A} \widehat{F}'(\cdot, A'), \quad \widehat{F}''_{-}(\cdot, A) := \sup_{A'\subset\subset A} \widehat{F}''(\cdot, A').$$

The locality property and the translation invariance of $\widehat{F}$ are direct consequences of (5.8), and of the locality and translation invariance of $\widehat{F}'$ and $\widehat{F}''$.

Arguing exactly as in [7, Lemma 5.1], for every $A \in \mathscr{A}$, every $u \in L^1_{\rm loc}(\mathbb{R}^n, \mathbb{R}^N)$ such that $u \in BV(A, \mathbb{R}^N)$, every $A', A'' \in \mathscr{A}(A)$ and every $B' \subset\subset A'$, with $B' \in \mathscr{A}(A)$ we can obtain that

$$\widehat{F}''(u, 1, B' \cup A'') \leq \widehat{F}''(u, 1, A') + \widehat{F}''(u, 1, A''),$$

from which we can easily deduce that the inner regular envelope $\widehat{F}''_{-}(u, 1, \cdot)$ is subadditive on $\mathscr{A}(A)$. Therefore, thanks to the De Giorgi-Letta Criterion, we infer that the set function $\widehat{F}(u, 1, \cdot) : \mathscr{A}(A) \to [0, +\infty]$ is the restriction to the open sets of a Borel measure on A.

For every $A \in \mathscr{A}$, and every $u \in L^1_{\rm loc}(\mathbb{R}^n, \mathbb{R}^N)$ with $u \in BV(A, \mathbb{R}^N)$, in view of (f2), we obtain

$$\widehat{F}''(u, 1, A) \leq C(|Du|(A) + \mathcal{L}^n(A)).$$

Hence, in particular

$$\widehat{F}(u, 1, A) \leq C(|Du|(A) + \mathcal{L}^n(A)),$$

so that using [30, Proposition 18.6] we get

$$\widehat{F}'(u, 1, A) = \widehat{F}(u, 1, A) = \widehat{F}''(u, 1, A).$$

The latter eventually provides the Γ-convergence statement in (5.5). Eventually, the lower bound is a consequence of (f2), $C^{-1} \leq 1$ and [6, Theorem 4.1 and Remark 4.2] and [7, Section 3.1 and Proposition 4.1]. □

The next three subsections are devoted to the proof of Theorem 5.1. Namely, in subsections 5.1 - 5.3 we identify, respectively, the three measure derivatives

$$\frac{\mathrm{d}\widehat{F}(u,1,\cdot)}{\mathrm{d}\mathcal{L}^n}, \quad \frac{\mathrm{d}\widehat{F}(u,1,\cdot)}{\mathrm{d}\mathcal{H}^{n-1}\llcorner J_u}, \quad \text{and} \quad \frac{\mathrm{d}\widehat{F}(u,1,\cdot)}{\mathrm{d}|D^c u|}.$$

In fact, we will prove that under the assumptions of Theorem 5.1, for every $A \in \mathscr{A}$ and $u \in BV(A, \mathbb{R}^N)$ the following three equalities hold:

$$\frac{\mathrm{d}\widehat{F}(u,1,\cdot)}{\mathrm{d}\mathcal{L}^n}(x) = f_{\rm hom}(\nabla u(x)) \quad \text{for } \mathcal{L}^n\text{-a.e. } x \in A,$$

$$\frac{\mathrm{d}\widehat{F}(u,1,\cdot)}{\mathrm{d}\mathcal{H}^{n-1}}(x) = g_{\rm hom}([u](x), \nu_u(x)) \quad \text{for } \mathcal{H}^{n-1}\text{-a.e. } x \in J_u \cap A,$$

$$\frac{\mathrm{d}\widehat{F}(u,1,\cdot)}{\mathrm{d}|D^c u|}(x) = f^\infty_{\rm hom}(\nabla u(x)) \quad \text{for } |D^c u|\text{-a.e. } x \in A.$$

Since in the equalities above the right-hand sides do not depend on the subsequence $(\varepsilon_j)_{j\in\mathbb{N}}$, we will be able to conclude that $\widehat{F}$ is subsequence independent and therefore the Γ-convergence result holds for the whole sequence (F_ε) (cf. Theorem 5.1).

The strategy to prove the identities above uses, on one hand, the global method for relaxation in BV [17] and, on the other hand, a direct (although involved) comparison argument.

For later use it is useful to recall the following notation: let $U \in \mathscr{A}_\infty$ and let $G : BV(U,\mathbb{R}^N) \times \mathscr{A}(U) \longrightarrow [0,\infty)$; for every $(w,A) \in BV(U,\mathbb{R}^N) \times \mathscr{A}_\infty(U)$ set

$$m_G(w,A) := \inf\{G(u,A) \; : \; u \in L^1_{\rm loc}(\mathbb{R}^n,\mathbb{R}^N),\; u \in BV(A,\mathbb{R}^N),\; u = w \text{ on } \partial A\}. \tag{5.9}$$

In addition, we use the notation $sc^-(L^1)G$ for the relaxation of G with respect to the L^1 convergence, namely $sc^-(L^1)G(u,A) := \Gamma(L^1)\text{-}\lim_j G(u;A)$ (cf. [30]).

In what follows we will use in several instances a truncation lemma that follows from De Giorgi's slicing and averaging argument on the codomain (see for instance [7, Proposition 6.2] and [28, Proposition 3.2]). We give here a detailed proof of it since the statement is slightly different from the standard one. In particular, in Propositions 5.4 and 5.9 we choose $v \equiv 1$, while in Proposition 5.7 it is important that the constant γ in the growth condition below equals 0. We recall the notation $\mathcal{T}_k$ for the smooth truncation operators and a_k for the related sequence introduced in (m).

Lemma 5.3. *Let $A \in \mathscr{A}$ and $\mathcal{G} : GBV(A,\mathbb{R}^N) \times L^1(A,[0,1]) \to [0,\infty]$ be the functional defined by*

$$\mathcal{G}(u,v) := \int_A v(x) g(x,\nabla u(x)) \mathrm{d}x$$

where $g : \mathbb{R}^n \times \mathbb{R}^{N\times n} \to [0,\infty)$ is a Borel function for which there exists $\gamma \in [0,\infty)$ such that

$$c^{-1}|\xi| \le g(x,\xi) \le c(|\xi| + \gamma) \tag{5.10}$$

for every $(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^{N\times n}$, and for some $c > 0$.

Then for every $M \in \mathbb{N}$ and $(u,v) \in GBV(A,\mathbb{R}^N)\times L^1(A,[0,1])$ there exists $k \in \{M+1,\dots,2M\}$ such that $\mathcal{T}_k(u) \in BV \cap L^\infty(A,\mathbb{R}^N)$ with $\|\mathcal{T}_k(u)\|_{L^\infty} \le a_{k+1}$, $\mathcal{H}^{n-1}(J_{\mathcal{T}_k(u)} \cap A) \le \mathcal{H}^{n-1}(J_u \cap A)$ and

$$\mathcal{G}(\mathcal{T}_k(u),v) \le \Big(1 + \frac{c^2}{M}\Big)\mathcal{G}(u,v) + \gamma C \mathcal{L}^n(\{|u| > a_M\})\,.$$

Proof. Let us fix $M \in \mathbb{N}$ and $(u,v) \in GBV(A,\mathbb{R}^N) \times L^1(A,[0,1])$ with $\mathcal{G}(u,v) < \infty$, otherwise the claim follows trivially. By averaging, there exists $k \in \{M+1,\dots,2M\}$ such that

$$\int_{A\cap\{a_k \le |u| < a_{k+1}\}} v(x) g(x,\nabla u(x)) \mathrm{d}x \le \frac{1}{M}\mathcal{G}(u,v). \tag{5.11}$$

By the properties of GBV functions and the very definition of $\mathcal{T}_k$, we have that $\mathcal{T}_k(u)$ belongs to $BV(A,\mathbb{R}^N) \cap L^\infty(A,\mathbb{R}^N)$ with $\|\mathcal{T}_k(u)\|_{L^\infty} \le a_{k+1}$, $J_{\mathcal{T}_k(u)} \cap A \subseteq J_u \cap A$ and $\nabla(\mathcal{T}_k(u))(x) = \nabla\mathcal{T}_k(u(x))\nabla u(x)$ for $\mathcal{L}^n$-a.e. $x \in A$. Furthermore, being $\mathrm{Lip}(\mathcal{T}_k) \le 1$, we can check for every $y, v \in \mathbb{R}^N$ with $|v| = 1$ that $|(\nabla\mathcal{T}_k(y))v| = |\partial_v \mathcal{T}_k(y)| \le 1$ that provides $\|\nabla\mathcal{T}_k(y)\|_2 \le 1$ for every $y \in \mathbb{R}^N$ (here $\|\cdot\|_2$ stands for the matrix norm on $\mathbb{R}^{N\times N}$ induced by $|\cdot|$ on $\mathbb{R}^N$) and consequently

$$|\nabla(\mathcal{T}_k(u))(x)| \le |\nabla u(x)| \qquad \text{for } \mathcal{L}^n\text{-a.e. } x \in A. \tag{5.12}$$

In particular, in virtue of $\mathcal{T}_k(y) = u$ on $\{|y| < a_k\}$ and $\mathcal{T}_k(y) = 0$ on $\{|y| \ge a_{k+1}\}$, we obtain

$$\begin{aligned}
\mathcal{G}(\mathcal{T}_k(u),v) &= \int_{A\cap\{|u|<a_k\}} v(x) g(x,\nabla u)\mathrm{d}x + \int_{A\cap\{a_k\le|u|<a_{k+1}\}} v(x) g(x,\nabla(\mathcal{T}_k(u)))\mathrm{d}x + \\
&+ \int_{A\cap\{|u|\ge a_{k+1}\}} v(x) g(x,0)\mathrm{d}x \le \mathcal{G}(u) + c\int_{A\cap\{a_k\le|u|<a_{k+1}\}} v(x)|\nabla(\mathcal{T}_k(u))|\mathrm{d}x \\
&+ c\gamma\mathcal{L}^n(|u| \ge a_k) \le \mathcal{G}(u) + c\int_{A\cap\{a_k\le|u|<a_{k+1}\}} v(x)|\nabla(u)|\mathrm{d}x + c\gamma\mathcal{L}^n(|u| > a_M) \\
&\le \Big(1 + \frac{c^2}{M}\Big)\mathcal{G}(u,v) + c\gamma\mathcal{L}^n(|u| > a_M),
\end{aligned}$$

where in the first inequality we used (5.10), in the second one (5.12), and finally in the last one (5.10) and (5.11). □

5.1. **Identification of the volume term.** This section is devoted to identify the measure derivative $\frac{\mathrm{d}\widehat{F}(u,1,\cdot)}{\mathrm{d}\mathcal{L}^n}$ with f_{hom}.

Proposition 5.4 (Homogenised volume integrand)**.** *Let $f \in \mathcal{F}(C,\alpha)$ satisfy (4.1). Let $\widehat{F}$ be as in (5.5). Then, for every $A \in \mathscr{A}$ and every $u \in L^1_{\mathrm{loc}}(\mathbb{R}^n,\mathbb{R}^N)$, with $u \in BV(A,\mathbb{R}^N) \cap L^\infty(A,\mathbb{R}^N)$ there holds*

$$\frac{\mathrm{d}\widehat{F}(u,1,\cdot)}{\mathrm{d}\mathcal{L}^n}(x) = f_{\mathrm{hom}}(\nabla u(x)) \quad \text{for } \mathcal{L}^n\text{-a.e. } x \in A,$$

where f_{hom} is as in (4.1).

To prove Proposition 5.4, we need the two following technical lemmas.

Lemma 5.5. *Let $g \in \mathcal{F}(C,\alpha)$ be given and define $\hat{g}: \mathbb{R}^n \times \mathbb{R}^{N\times n} \to [0,\infty)$ as*

$$\hat{g}(x,\xi) := \limsup_{\eta\to 0} \inf\left\{\int_Q g(x+\eta z, \nabla w)\mathrm{d}z \ : \ w - \ell_\xi \in W^{1,1}_0(Q,\mathbb{R}^N)\right\}, \tag{5.13}$$

*Let $A \in \mathscr{A}_\infty$ and let $E^g(\cdot,A)$ and $E^{\hat{g}}(\cdot,A)$ be defined as in (**??**) with h replaced by g and $\hat{g}$, respectively. Moreover, consider the functionals $F^g, F^{\hat{g}} : L^1(A,\mathbb{R}^N) \longrightarrow [0,\infty]$ given by*

$$F^g(u) := \begin{cases} E^g(u,A) & \text{if } u \in W^{1,1}(A,\mathbb{R}^N) \\ +\infty & \text{otherwise} \end{cases}, \quad F^{\hat{g}}(u) := \begin{cases} E^{\hat{g}}(u,A) & \text{if } u \in W^{1,1}(A,\mathbb{R}^N) \\ +\infty & \text{otherwise.} \end{cases}$$

Then the following statements hold:

(i) if g is 1-homogeneous in ξ, then the same holds for $\hat{g}$;

(ii) there exists $H \subseteq \mathbb{R}^n$ with $\mathcal{L}^n(H) = 0$ such that for every $x \in \mathbb{R}^n \setminus H$ and every $\xi \in \mathbb{R}^{N\times n}$

$$\hat{g}(x,\xi) \le g(x,\xi);$$

(iii) for every $u \in W^{1,1}(A,\mathbb{R}^N)$

$$sc^-(L^1)F^g(u) = F^{\hat{g}}(u);$$

(iv) for every $u \in BV(A,\mathbb{R}^N)$

$$\left| sc^-(L^1)F^g(u) - \int_A \hat{g}(x,\nabla u)\mathrm{d}x \right| \le C|D^s u|(A);$$

(v) for every $u \in L^1(A,\mathbb{R}^N)$

$$sc^-(L^1)F^g(u) = sc^-(L^1)F^{\hat{g}}(u);$$

(vi) for every $\xi \in \mathbb{R}^{N\times n}$

$$m^g_{\mathrm{b}}(\ell_\xi, A) = m^{\hat{g}}_{\mathrm{b}}(\ell_\xi, A).$$

Proof. Property (i) readily follows from the definition of $\hat{g}$. Instead, (iii) and (iv) are a direct consequence of [17, Theorem 4.1.4].

To prove (ii) let $\xi \in \mathbb{Q}^{N\times n}$ be fixed, by definition we get

$$\hat{g}(x,\xi) \le \limsup_{\eta\to 0} \int_Q g(x+\eta z, \xi_j)\mathrm{d}z = \limsup_{\eta\to 0} \frac{1}{\eta^n} \int_{Q_\eta(x)} g(z,\xi)\mathrm{d}z$$

for every $x \in \mathbb{R}^n$. Then, the Lebesgue Differentiation Theorem provides us with a set $H_\xi \subset \mathbb{R}^n$ such that $\mathcal{L}^n(H_\xi) = 0$ and $\hat{g}(x,\xi) \le g(x,\xi)$ for every $x \in \mathbb{R}^n \setminus H_\xi$. Therefore, we conclude by setting

$$H := \bigcup_{\xi \in \mathbb{Q}^{N\times n}} H_\xi ,$$

and invoking the continuity of g, (f3), and the lower semicontinuity of $\hat{g}$ as the bulk energy density of the functional $sc^-(L^1)F^{\hat{g}}$.

The proof of (v) follows straightforwardly from (ii) and (iii).

To conclude the proof, we are left to show (vi). We start noticing that in view of (ii) we only need to prove that

$$m^g_{\mathrm{b}}(\ell_\xi, A) \le m^{\hat{g}}_{\mathrm{b}}(\ell_\xi, A).$$

for every $\xi \in \mathbb{R}^{N\times n}$. To prove the inequality above, fix $\xi \in \mathbb{R}^{N\times n}$ and let $u \in W^{1,1}(A,\mathbb{R}^N)$ satisfy $u = \ell_\xi$ on ∂A. By (iii) we can infer the existence of a sequence $(u_j)_{j\in\mathbb{N}} \subset W^{1,1}(A,\mathbb{R}^N)$ such that $u_j \to u$ in $L^1(A,\mathbb{R}^N)$ as $j \to \infty$ and

$$\lim_{j\to\infty} E^g(u_j, A) = E^{\hat{g}}(u, A).$$

By [17, Lemma 2.6 and Remark 2.7] we can find a sequence $(w_j)_{j\in\mathbb{N}} \subset W^{1,1}(A,\mathbb{R}^N)$ satisfying $w_j = \ell_\xi$ on ∂A such that $w_j \to u$ in $L^1(A,\mathbb{R}^N)$ as $j \to \infty$ and

$$\limsup_{j\to\infty} E^g(w_j, A) \le \liminf_{j\to\infty} E^g(u_j, A) = E^{\hat{g}}(u, A),$$

therefore the claim follows by the arbitrariness of u. □

Using a classical argument of Ambrosio, in the following lemma we prove a truncation result in the same spirit as in [28, Lemma 4.4].

Lemma 5.6. *Let F_ε be the functionals defined in* (5.1)*. Then, for every $\delta \in (0,1)$, $A \in \mathscr{A}$, and $(u,v) \in L^1_{\mathrm{loc}}(\mathbb{R}^n,\mathbb{R}^{N+1})$ with $u \in W^{1,1}(A,\mathbb{R}^N) \cap L^\infty(A,\mathbb{R}^N)$ and $W^{1,2}(A,[0,1])$, there exists $u^\delta \in L^1_{\mathrm{loc}}(\mathbb{R}^n,\mathbb{R}^N) \cap SBV(A,\mathbb{R}^N)$ (also depending on A) such that for every $\varepsilon > 0$*

$$H^\delta_\varepsilon(u^\delta, A) \le F_\varepsilon(u, v, A) + C\mathcal{L}^n(\{v \le \delta\} \cap A), \tag{5.14}$$

where $H^\delta_\varepsilon : L^1_{\mathrm{loc}}(\mathbb{R}^n,\mathbb{R}^N) \times \mathscr{A} \longrightarrow [0,+\infty]$ is the functional given by

$$H^\delta_\varepsilon(w, A) := \begin{cases} \alpha_\delta \displaystyle\int_A f(\tfrac{x}{\varepsilon}, \nabla w)\mathrm{d}x + \beta_\delta \mathcal{H}^{n-1}(J_w \cap A) & \text{if } w \in SBV(A,\mathbb{R}^N) \\ +\infty & \text{otherwise,} \end{cases}$$

with $\alpha_\delta, \beta_\delta > 0$ such that

$$\lim_{\delta\to 1} \alpha_\delta = 1 \quad \text{and} \quad \lim_{\delta\to 1} \beta_\delta = 0.$$

Moreover, if $(u_\varepsilon, v_\varepsilon) \to (u, 1)$ in $L^1(A,\mathbb{R}^{N+1})$ as $\varepsilon \to 0$, then the corresponding (u^δ_ε) satisfies

$$u^\delta_\varepsilon \to u \quad \text{in } L^1(A,\mathbb{R}^N) \quad \text{as } \varepsilon \to 0. \tag{5.15}$$

Proof. Let $\delta \in (0,1)$, $\varepsilon > 0$, $A \in \mathscr{A}$, and $(u,v) \in W^{1,1}(A,\mathbb{R}^N) \times W^{1,2}(A,[0,1])$ be given. We have

$$F_\varepsilon(u, v, A) \ge \int_{\{v \ge \delta^2\}} \alpha_\delta f(\tfrac{x}{\varepsilon}, \nabla u)\mathrm{d}x + \int_A (1-v)|\nabla v|\mathrm{d}x \tag{5.16}$$

where $\alpha_\delta := \min_{t\in[\delta^2,1]} t^2 = \delta^4$. Set

$$\Phi(t) := \int_0^t (1-s)\,ds = t - \frac{t^2}{2} \quad \text{and} \quad \Phi_v := \Phi \circ v \in W^{1,2}(A).$$

By the Coarea Formula we can infer that

$$\int_A (1-v)|\nabla v| \mathrm{d}x = \int_A |\nabla \Phi_v| \mathrm{d}x \geq \int_{\Phi(\delta^2)}^{\Phi(\delta)} \mathcal{H}^{n-1}(A \cap \partial^*(\{\Phi_v > t\}))\mathrm{d}t,$$

therefore, there exists $t^\delta \in (\Phi(\delta^2), \Phi(\delta))$ such that

$$\int_A (1-v)|\nabla v| \mathrm{d}x \geq (\Phi(\delta) - \Phi(\delta^2))\mathcal{H}^{n-1}(A \cap \partial^*(\{\Phi_v > t^\delta\})). \tag{5.17}$$

Set $u^\delta := u\chi_{\{v > \Phi^{-1}(t^\delta)\}}$; we notice that $u^\delta \in L^1_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{R}^N) \cap SBV(A, \mathbb{R}^N)$ since $\{v > \Phi^{-1}(t^\delta)\}$ is a set of finite perimeter in A and $u \in L^\infty(A, \mathbb{R}^N)$. Since by definition $J_{u^\delta} \cap A \subseteq \partial^*(\{\Phi_v > t^\delta\}) \cap A$, (5.17) becomes

$$\int_A (1-v)|\nabla v| \mathrm{d}x \geq \beta_\delta \mathcal{H}^{n-1}(J_{u^\delta} \cap A), \tag{5.18}$$

where $\beta_\delta := \Phi(\delta) - \Phi(\delta^2)$.

Moreover, by the strict monotonicity of Φ on $[0,1]$, we get

$$\int_{\{v \geq \delta^2\}} f(\tfrac{x}{\varepsilon}, \nabla u)\mathrm{d}x \geq \int_{\{v > \Phi^{-1}(t^\delta)\}} f(\tfrac{x}{\varepsilon}, \nabla u^\delta)\mathrm{d}x,$$

so that thanks to (f2), we obtain

$$\alpha_\delta \int_{\{v \geq \delta^2\}} f(\tfrac{x}{\varepsilon}, \nabla u)\mathrm{d}x + C\mathcal{L}^n(\{v \leq \delta\} \cap A) \geq \alpha_\delta \int_A f(\tfrac{x}{\varepsilon}, \nabla u^\delta)\mathrm{d}x. \tag{5.19}$$

Eventually, (5.14) follows by gathering (5.16), (5.18), and (5.19).

Now let $(u_\varepsilon, v_\varepsilon) \to (u, 1)$ in $L^1(A, \mathbb{R}^{N+1})$ as $\varepsilon \to 0$ and consider

$$u_\varepsilon^\delta := u_\varepsilon \chi_{\{v_\varepsilon > \Phi^{-1}(t^\delta)\}}.$$

We observe that

$$\|u_\varepsilon - u_\varepsilon^\delta\|_{L^1(A)} \leq \|u_\varepsilon\|_{L^1(\{v_\varepsilon \leq \delta\} \cap A)}. \tag{5.20}$$

Therefore (5.15) follows by (5.20) in view of the equi-integrability of (u_ε) and the convergence in measure of (v_ε) to 1. □

We are now ready to identify the Radon-Nikodym derivative of $\widehat{F}$ with respect to the Lebesgue measure with f_{hom}.

Proof of Proposition 5.4. Fix $A \in \mathscr{A}$ and $u \in L^1_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{R}^N)$ with $u \in BV(A, \mathbb{R}^N) \cap L^\infty(A, \mathbb{R}^N)$. We divide the proof into two steps.

Step 1: We claim that

$$\frac{\mathrm{d}\widehat{F}(u, 1, \cdot)}{\mathrm{d}\mathcal{L}^n}(x) \leq f_{\mathrm{hom}}(\nabla u(x)) \quad \text{for } \mathcal{L}^n\text{-a.e. } x \in A.$$

For every $A \in \mathscr{A}$ and $q \in \mathbb{Q} \cap (0,1)$ set $\widehat{F}_q(u, 1, A) := \widehat{F}(u, 1, A) + q|Du|(A)$. Thanks to [17, Lemma 3.5] we obtain that for $\mathcal{L}^n$-a.e. $x \in A$

$$\frac{\mathrm{d}\widehat{F}(u, 1, \cdot)}{\mathrm{d}\mathcal{L}^n}(x) + q|\nabla u(x)| = \lim_{\rho \to 0} \frac{m_{\widehat{F}_q}(\ell_{\nabla u(x)}, Q_\rho(x))}{\rho^n} \tag{5.21}$$

where where $m_{\widehat{F}_q}$ is as in (5.9). Let $x \in A$ be that (5.21) holds, and set $\xi := \nabla u(x)$. In view of (4.1), for every $\rho > 0$ we have

$$f_{\mathrm{hom}}(\xi) = \lim_{r \to +\infty} \frac{m_{\mathrm{b}}^f(\ell_\xi, Q_r(\frac{r}{\rho}x))}{r^n}. \tag{5.22}$$

Fix $\eta \in (0,1)$. By (5.22), for every $\rho, r > 0$ there exists $w_r^\rho \in W^{1,1}(Q_r(\frac{r}{\rho}x), \mathbb{R}^N)$ with $w_r^\rho = \ell_\xi$ on $\partial Q_r(\frac{r}{\rho}x)$, such that

$$\int_{Q_r(\frac{r}{\rho}x)} f(y, \nabla w_r^\rho)\mathrm{d}y \le m_{\mathrm{b}}^f(\ell_\xi, Q_r(\tfrac{r}{\rho}x)) + \eta r^n\,. \tag{5.23}$$

Thus, for every $\rho > 0$, (5.22) and (5.23) yield

$$\limsup_{r\to+\infty} \frac{1}{r^n} \int_{Q_r(\frac{r}{\rho}x)} f(y, \nabla w_r^\rho)\mathrm{d}y \le f_{\mathrm{hom}}(\xi) + \eta.$$

Now, let $(\varepsilon_j)_{j\in\mathbb{N}}$ be as in (5.5) and set $r = \frac{\rho}{\varepsilon_j}$. Define $u_{\varepsilon_j}^\rho : \mathbb{R}^n \to \mathbb{R}^N$ as

$$u_{\varepsilon_j}^\rho(y) := \begin{cases} \varepsilon_j w_r^\rho\left(\frac{y}{\varepsilon_j}\right) & \text{if } y \in Q_\rho(x) \\ \ell_\xi(y) & \text{if } y \in \mathbb{R}^n \setminus Q_\rho(x), \end{cases}$$

therefore $u_{\varepsilon_j}^\rho \in W_{\mathrm{loc}}^{1,1}(\mathbb{R}^n, \mathbb{R}^N)$ with $u_{\varepsilon_j}^\rho = \ell_\xi$ on $\mathbb{R}^n \setminus Q_\rho(x)$. Changing variables and again invoking (5.23), for every $\rho > 0$ we get

$$\limsup_{j\to+\infty} \frac{1}{\rho^n} \int_{Q_{\rho(1+\eta)}(x)} f(\tfrac{y}{\varepsilon_j}, \nabla u_{\varepsilon_j}^\rho)\mathrm{d}y \le f_{\mathrm{hom}}(\xi) + \eta + C(|\xi|+1)((1+\eta)^n - 1) \tag{5.24}$$

where we also used (f2), the fact that $\nabla u_{\varepsilon_j}^\rho = \xi$ on $Q_{\rho(1+\eta)}(x) \setminus \overline{Q_\rho(x)}$, and (5.5).

Appealing to (5.24), (f2), and the Poincaré Inequality, for every ρ we can find a subsequence of $(\varepsilon_j)_{j\in\mathbb{N}}$ (not relabeled) such that $u_{\varepsilon_j}^\rho$ converges in $L_{\mathrm{loc}}^1(\mathbb{R}^n, \mathbb{R}^N)$ to some $u^\rho \in L_{\mathrm{loc}}^1(\mathbb{R}^n, \mathbb{R}^N) \cap BV(Q_{\rho(1+\eta)}(x), \mathbb{R}^N)$ with $u^\rho = \ell_\xi$ on $\partial Q_{\rho(1+\eta)}(x)$. Moreover, by (5.5), (5.24), and (f2), for every $\rho > 0$, we have that

$$\begin{aligned} \frac{m_{\widehat{F}}(\ell_\xi, Q_{\rho(1+\eta)}(x))}{\rho^n} &\le \frac{\widehat{F}(u^\rho, 1, Q_{\rho(1+\eta)}(x)) + q|Du^\rho|(Q_{\rho(1+\eta)}(x))}{\rho^n} \\ &\le \liminf_{j\to+\infty} \Big(\tfrac{F_{\varepsilon_j}(u_{\varepsilon_j}^\rho, 1, Q_{\rho(1+\eta)}(x))}{\rho^n} + \tfrac{q}{\rho^n} \textstyle\int_{Q_{\rho(1+\eta)}(x)} |\nabla u_{\varepsilon_j}^\rho|\mathrm{d}y \Big) \\ &\le \big(f_{\mathrm{hom}}(\xi) + \eta + C(|\xi|+1)((1+\eta)^n - 1)\big)(1+qC). \end{aligned}$$

Eventually, by (5.21) and taking the limit as $\rho \to 0$ we get

$$\begin{aligned} (1+\eta)^n \Big(\frac{\mathrm{d}\widehat{F}(u,1,\cdot)}{\mathrm{d}\mathcal{L}^n}(x) + q\xi \Big) &= \lim_{\rho\to 0} \frac{m_{\widehat{F}}(\ell_\xi, Q_{\rho(1+\eta)}(x))}{\rho^n} \\ &\le \big(f_{\mathrm{hom}}(\xi) + \eta + C(|\xi|+1)((1+\eta)^n - 1)\big)(1+qC), \end{aligned}$$

hence the claim follows by letting $\eta, q \to 0$.

Step 2: We claim that

$$\frac{\mathrm{d}\widehat{F}(u,1,\cdot)}{\mathrm{d}\mathcal{L}^n}(x) \ge f_{\mathrm{hom}}(\nabla u(x)) \quad \text{for } \mathcal{L}^n\text{-a.e. } x \in A.$$

Let $A' \in \mathscr{A}(A)$, by Theorem 5.2, we can find a sequence $(u_j, v_j)_{j\in\mathbb{N}} \in L_{\mathrm{loc}}^1(\mathbb{R}^n, \mathbb{R}^N)$ such that $(u_j, v_j) \in W^{1,1}(A', \mathbb{R}^N) \times W^{1,2}(A', [0,1])$, $(u_j, v_j) \to (u, 1)$ in $L_{\mathrm{loc}}^1(\mathbb{R}^n, \mathbb{R}^{N+1})$, $v_j(x) \to 1$ for $\mathcal{L}^n$-a.e. $x \in A'$ as $j \to +\infty$ and

$$\lim_{j\to+\infty} F_{\varepsilon_j}(u_j, v_j, A') = \widehat{F}(u, 1, A'). \tag{5.25}$$

Let $\delta \in (0,1)$ be fixed; by Lemma 5.6 we have

$$H^\delta_{\varepsilon_j}(u^\delta_j, A') \le F_{\varepsilon_j}(u_j, v_j, A') + C\mathcal{L}^n(\{v_j \le \delta\} \cap A'),$$

where $(u^\delta_j) \subset SBV(A', \mathbb{R}^N)$ with $u^\delta_j \to u$ in $L^1(A', \mathbb{R}^N)$. Therefore by (5.25) we get

$$\liminf_{j\to+\infty} H^\delta_{\varepsilon_j}(u^\delta_j, A') \le \widehat{F}(u, 1, A'), \tag{5.26}$$

since $(v_j)_{j\in\mathbb{N}}$ converges in measure to 1 on A'.

We now consider the measures μ^δ_j defined on A' as follows

$$\mu^\delta_j := \alpha_\delta f(\tfrac{x}{\varepsilon_j}, \nabla u^\delta_j)\mathcal{L}^n \llcorner A' + \beta_\delta \mathcal{H}^{n-1} \llcorner (J_{u^\delta_j} \cap A').$$

Note that by (5.26), there is a subsequence (not relabeled) and a finite Radon measure μ^δ on A' such that $\mu^\delta_j \overset{*}{\rightharpoonup} \mu^\delta$ as $j \to +\infty$.

Now let $x_0 \in A'$ be a point of approximate differentiability of u, and additionally assume that

$$\lim_{\rho\to 0} \frac{\mu^\delta(Q_\rho(x_0))}{\rho^n} = \frac{\mathrm{d}\mu^\delta}{\mathrm{d}\mathcal{L}^n}(x_0). \tag{5.27}$$

Such conditions determine a subset of full measure in A'. Then, consider the rescaled function $u^\rho : Q_1 \to \mathbb{R}^N$ given by

$$u^\rho(y) := \frac{u(x_0 + \rho y) - u(x_0)}{\rho},$$

thanks to [9, Remark 3.72] we have $u^\rho \to \ell_\xi$ in $L^1(Q_1, \mathbb{R}^N)$, where $\xi := \nabla u(x_0)$.

By the weak*-convergence of μ^δ_j towards μ^δ we have

$$\begin{aligned}\frac{\mathrm{d}\mu^\delta}{\mathrm{d}\mathcal{L}^n}(x_0) &= \lim_{\rho\to 0} \frac{\mu^\delta(Q_\rho(x_0))}{\rho^n} = \lim_{\rho\to 0,\, \rho\in I(x_0)} \lim_{j\to+\infty} \frac{\mu^\delta_j(Q_\rho(x_0))}{\rho^n} \\ &= \lim_{\rho\to 0,\, \rho\in I(x_0)} \lim_{j\to+\infty} \rho^{-n}\Big(\alpha_\delta \int_{Q_\rho(x_0)} f(\tfrac{x}{\varepsilon_j}, \nabla u^\delta_j)\mathrm{d}x + \beta_\delta \mathcal{H}^{n-1}(J_{u^\delta_j} \cap Q_\rho(x_0))\Big)\end{aligned} \tag{5.28}$$

where $I(x_0) := \{\rho \in (0, \frac{2}{\sqrt{n}}\mathrm{dist}(x_0, \partial A'))\colon \mu^\delta(\partial Q_\rho(x_0)) = 0\}$.

For every ρ and j, define the rescalings $u^\rho_j \in SBV(A', \mathbb{R}^N)$ by

$$u^\rho_j(y) := \frac{u^\delta_j(x_0 + \rho y) - u(x_0)}{\rho},$$

then $u^\rho_j \to u^\rho$ in $L^1(Q_1, \mathbb{R}^N)$ as $j \to +\infty$. Furthermore, thanks to (5.28) we get

$$\frac{\mathrm{d}\mu^\delta}{\mathrm{d}\mathcal{L}^n}(x_0) = \lim_{\rho\to 0,\, \rho\in I(x_0)} \lim_{j\to+\infty} \Big(\alpha_\delta \int_{Q_1} f(\tfrac{x_0+\rho y}{\varepsilon_j}, \nabla u^\rho_j)\mathrm{d}y + \frac{\beta_\delta}{\rho}\mathcal{H}^{n-1}(J_{u^\rho_j} \cap Q_1)\Big). \tag{5.29}$$

Fix $M \in \mathbb{N}$, for every ρ and j, we apply Lemma 5.3 with $v \equiv 1$ so that there is $k_{\rho,j} \in \{M+1, \dots, 2M\}$ such that $\hat{u}^\rho_j := \mathcal{T}_{k_{\rho,j}}(u^\rho_j) \in SBV(Q_1, \mathbb{R}^N)$,

$$\int_{Q_1} f(\tfrac{x_0+\rho_i y}{\varepsilon_i}, \nabla \hat{u}^\rho_j)\mathrm{d}y \le \Big(1 + \frac{C^2}{M}\Big) \int_{Q_1} f(\tfrac{x_0+\rho_i y}{\varepsilon_i}, \nabla u^\rho_j)\mathrm{d}y + C\mathcal{L}^n(\{|u^\rho_j| \ge a_M\}). \tag{5.30}$$

Up to subsequences (not relabeled) we can assume that $k_{\rho,j} \in \{M+1, \dots, 2M\}$ actually depends only on ρ. If we choose $a_M > \sup_{y\in Q_1} \ell_\xi(y)$ we get that

$$\lim_{\rho\to 0} \lim_{j\to+\infty} \hat{u}^\rho_j = \ell_\xi \quad \text{in } L^1(Q_1, \mathbb{R}^N)$$

and

$$\lim_{\rho\to 0,\ \rho\in I(x_0)} \limsup_{j\to+\infty} \mathcal{L}^n(\{|u_j^\rho| \geq a_M\}) = 0, \tag{5.31}$$

since we also have that $\lim\limits_{\rho\to 0} \lim\limits_{j\to+\infty} u_j^\rho = \ell_\xi$ in $L^1(Q_1,\mathbb{R}^N)$. In particular, for M is large enough, by combining (5.29), (5.30), and (5.31) we can infer

$$\Big(1+\frac{C^2}{M}\Big)\frac{\mathrm{d}\mu^\delta}{\mathrm{d}\mathcal{L}^n}(x_0) \geq \limsup_{\rho\to 0,\ \rho\in I(x_0)} \limsup_{j\to+\infty} \alpha_\delta \int_{Q_1} f(\tfrac{x_0+\rho y}{\varepsilon_j}, \nabla \hat{u}_j^\rho)\mathrm{d}y, \tag{5.32}$$

and

$$\lim_{\rho\to 0,\rho\in I(x_0)} \limsup_{j\to+\infty} \mathcal{H}^{n-1}(J_{\hat{u}_j^\rho} \cap Q_1) = 0, \tag{5.33}$$

since $\mathcal{T}_{k_{\rho,j}} \in C^1(\mathbb{R}^N,\mathbb{R}^N)$, $\frac{\mathrm{d}\mu^\delta}{\mathrm{d}\mathcal{L}^n}(x_0)$ is finite, and $\beta_\delta > 0$. Set

$$\tau_{\rho,j} := \|\hat{u}_j^\rho - \ell_\xi\|_{L^1(Q_1,\mathbb{R}^N)} + \frac{\rho}{j};$$

then, $\lim\limits_{\rho\to 0} \lim\limits_{j\to+\infty} \tau_{\rho,j} = 0$. Thus, for every $\rho > 0$ small and every j large (depending on ρ) we have $\tau_{\rho,j} \in (0,1)$. Therefore, thanks to the Coarea formula and to the properties of the traces of BV functions on rectifiable sets (see [9, Theorem 3.77]), there exists $\hat{r}_{\rho,j} \in (1-\tau_{\rho,j}^{1/2}, 1)$ such that

$$\int_{\partial Q_{\hat{r}_{\rho,j}}} |(\hat{u}_j^\rho)^- - \ell_\xi|\mathrm{d}\mathcal{H}^{n-1} \leq \tau_{\rho,j}^{-1/2}\|\hat{u}_j^\rho - \ell_\xi\|_{L^1(Q_1,\mathbb{R}^N)} \leq \tau_{\rho,j}^{1/2}, \tag{5.34}$$

where $(\hat{u}_j^\rho)^-$ is the inner trace of $\hat{u}_j^\rho$ on $\partial Q_{\hat{r}_{\rho,j}}$. Therefore, defining the functions $w_j^\rho \in SBV(Q_1,\mathbb{R}^N)$ as

$$w_j^\rho(y) := \begin{cases} \hat{u}_j^\rho(y) & \text{if } y \in Q_{\hat{r}_{\rho,j}} \\ \ell_\xi(y) & \text{if } y \in Q_1 \setminus Q_{\hat{r}_{\rho,j}}, \end{cases}$$

thanks to (f2) we have that

$$\alpha_\delta \int_{Q_1} f(\tfrac{x_0+\rho y}{\varepsilon_j}, \nabla \hat{u}_j^\rho)\mathrm{d}y + \alpha_\delta(C|\xi|+1)\mathcal{L}^n(Q_1\setminus Q_{\hat{r}_{\rho,j}}) \geq \alpha_\delta \int_{Q_1} f(\tfrac{x_0+\rho y}{\varepsilon_j}, \nabla w_j^\rho)\mathrm{d}y$$

and, since $\lim\limits_{\rho\to 0} \lim\limits_{j\to+\infty} \hat{r}_{\rho,j} = 1$, from (5.32) we obtain

$$\Big(1+\frac{C^2}{M}\Big)\frac{\mathrm{d}\mu^\delta}{\mathrm{d}\mathcal{L}^n}(x_0) \geq \limsup_{\rho\to 0,\ \rho\in I(x_0)} \limsup_{j\to+\infty} \alpha_\delta \int_{Q_1} f(\tfrac{x_0+\rho y}{\varepsilon_j}, \nabla w_j^\rho)\mathrm{d}y. \tag{5.35}$$

Furthermore, thanks to (5.34), (2.2), and to the definition of $\hat{u}_j^\rho$ we can estimate the singular part of Dw_j^ρ as follows

$$|D^s w_j^\rho|(Q_1) \leq \int_{J_{w_j^\rho}\cap Q_1} |[w_j^\rho]|\mathrm{d}\mathcal{H}^{n-1} \leq \tau_{\rho,j}^{1/2} + 2a_{2M+1}\mathcal{H}^{n-1}(J_{\hat{u}_j^\rho}\cap Q_1). \tag{5.36}$$

Now, for every $\rho >$ and $j \in \mathbb{N}$, consider functional $F_{\rho,j} : L^1(Q_1,\mathbb{R}^N) \longrightarrow [0,\infty]$ given by

$$F_{\rho,j}(w) := \begin{cases} \displaystyle\int_{Q_1} f(\tfrac{x_0+\rho y}{\varepsilon_j}, \nabla w)\mathrm{d}y & \text{if } w \in W^{1,1}(Q_1,\mathbb{R}^N) \\ \infty & \text{otherwise.} \end{cases}$$

In view of Lemma 5.5 (iv), for every $w \in SBV(Q_1,\mathbb{R}^N)$ we have

$$\Big|sc^-(L^1)F_{\rho,j}(w) - \int_{Q_1} f_{\rho,j}(y,\nabla w)\mathrm{d}y\Big| \leq C\int_{J_w\cap Q_1} |[w]|\mathrm{d}\mathcal{H}^{n-1}, \tag{5.37}$$

where $f_{\rho,j} := \hat{g}$, with $g(x,\xi) := f\left(\frac{x_0+\rho x}{\varepsilon_j},\xi\right)$ for every $(x,\xi)\in\mathbb{R}^n\times\mathbb{R}^{N\times n}$ (cf. (5.13)). By [17, Lemma 2.6], for every ρ and j we can find $\hat{w}_j^\rho\in W^{1,1}(Q_1,\mathbb{R}^N)$ with $\hat{w}_j^\rho=\ell_\xi$ on ∂Q_1 and such that

$$\left|sc^-(L^1)F_{\rho,j}(w_j^\rho)-\int_{Q_1}f_{\rho,j}(y,\nabla\hat{w}_j^\rho)\mathrm{d}y\right|\le\frac{\rho}{j}. \tag{5.38}$$

In particular, from (5.36), (5.37) and (5.38) and the equality $f_{\rho,j}(y,\xi)=\hat{f}(\frac{x_0+\rho y}{\varepsilon_j},\xi)$ which follows from formula (5.13), we infer

$$\begin{aligned}\int_{Q_1}f(\tfrac{x_0+\rho y}{\varepsilon_j},\nabla w_j^\rho)\mathrm{d}y&\ge\int_{Q_1}f_{\rho,j}(y,\nabla w_j^\rho)\mathrm{d}y\ge sc^-(L^1)F_{\rho,j}(w_j^\rho)-C\int_{J_{w_j^\rho}\cap Q_1}|[w_j^\rho]|\mathrm{d}\mathcal{H}^{n-1}\\&\ge\int_{Q_1}f_{\rho,j}(y,\nabla\hat{w}_j^\rho)\mathrm{d}y-\frac{\rho}{j}-C(\tau_{\rho,j}^{1/2}+2a_{2M+1}\mathcal{H}^{n-1}(J_{\hat{u}_j^\rho}\cap Q_1))\\&=\int_{Q_1}\hat{f}(\tfrac{x_0+\rho y}{\varepsilon_j},\nabla\hat{w}_j^\rho)\mathrm{d}y-\tfrac{\rho}{j}-C(\tau_{\rho,j}^{1/2}+2a_{2M+1}\mathcal{H}^{n-1}(J_{\hat{u}_j^\rho}\cap Q_1)).\end{aligned} \tag{5.39}$$

Setting

$$r_{\rho,j}=\frac{\rho}{\varepsilon_j}\quad\text{and}\quad\overline{w}_j^\rho(x):=r_{\rho,j}\hat{w}_j^\rho(\tfrac{x}{r_{\rho,j}}-\tfrac{x_0}{\rho})$$

we have $\overline{w}_j^\rho\in W^{1,1}(Q_{r_{\rho,j}}(\frac{r_{\rho,j}}{\rho}x_0),\mathbb{R}^N)$ with $\overline{w}_j^\rho=\ell_\xi-\frac{1}{\varepsilon_j}x_0$ on $\partial Q_{r_{\rho,j}}(\frac{r_{\rho,j}}{\rho}x_0)$ and

$$\int_{Q_1}\hat{f}(\tfrac{x_0+\rho y}{\varepsilon_j},\nabla\hat{w}_j^\rho)\mathrm{d}y=\frac{1}{r_{\rho,j}^n}\int_{Q_{r_{\rho,j}}(\frac{r_{\rho,j}}{\rho}x_0)}\hat{f}(x,\nabla\overline{w}_j^\rho)\mathrm{d}x.$$

In particular, Lemma 5.5 (vi) gives

$$\int_{Q_{r_{\rho,j}}(\frac{r_{\rho,j}}{\rho}x_0)}\hat{f}(x,\nabla\overline{w}_j^\rho)\mathrm{d}x\ge m_{\mathrm{b}}^{\hat{f}}(\ell_\xi,Q_{r_{\rho,j}}(\tfrac{r_{\rho,j}}{\rho}x_0))=m_{\mathrm{b}}^{f}(\ell_\xi,Q_{r_{\rho,j}}(\tfrac{r_{\rho,j}}{\rho}x_0)).$$

Therefore, (5.35), (5.39) and (4.1) yield

$$\begin{aligned}\Big(1+\frac{C^2}{M}\Big)\frac{\mathrm{d}\mu^\delta}{\mathrm{d}\mathcal{L}^n}(x_0)&\ge\limsup_{\rho\to0,\ \rho\in I(x_0)}\ \limsup_{j\to+\infty}\alpha_\delta\frac{m_{\mathrm{b}}^f(\ell_\xi,Q_{r_{\rho,j}}(\frac{r_{\rho,j}}{\rho}x_0))}{r_{\rho,j}^n}\\&=\alpha_\delta f_{\mathrm{hom}}(\xi)=\alpha_\delta f_{\mathrm{hom}}(\nabla u(x_0))\end{aligned}$$

and thus

$$\frac{\mathrm{d}\mu^\delta}{\mathrm{d}\mathcal{L}^n}(x_0)\ge\alpha_\delta f_{\mathrm{hom}}(\nabla u(x_0))$$

by letting $M\to\infty$. Hence, recalling (5.26), we deduce that

$$\widehat{F}(u,1,A')\ge\liminf_{j\to\infty}H_{\varepsilon_j}^\delta(u_j^\delta,A')=\liminf_{j\to\infty}\mu_j^\delta(A')\ge\mu^\delta(A')\ge\alpha_\delta\int_{A'}f_{\mathrm{hom}}(\nabla u)\mathrm{d}x.$$

Eventually, the claim follows by letting $\delta\to0$ and by the arbitrariness of $A'\in\mathscr{A}(A)$. □

5.2. **Identification of the surface term.** In this subsection we show that the Radon Nikodym derivative of $\widehat{F}$ with respect to $\mathcal{H}^{n-1}\llcorner J_u$ equals to g_{hom} for every $u\in BV$.

Proposition 5.7 (Homogenised surface integrand)**.** *Let $f\in\mathcal{F}(C,\alpha)$ satisfy* (4.14)*. Let $\widehat{F}$ be as in* (5.5)*. Then, for every $A\in\mathscr{A}$ and every $u\in L^1_{\mathrm{loc}}(\mathbb{R}^n,\mathbb{R}^N)$, with $u\in BV(A,\mathbb{R}^N)\cap L^\infty(A,\mathbb{R}^N)$ there holds*

$$\frac{\mathrm{d}\widehat{F}(u,1,\cdot)}{\mathrm{d}\mathcal{H}^{n-1}\llcorner J_u}(x)=g_{\mathrm{hom}}([u](x),\nu_u(x))\quad\text{for }\mathcal{H}^{n-1}\text{-a.e. }x\in J_u\cap A,$$

where g_{hom} is as in (4.14)*.*

To prove Proposition 5.7 we need a preliminary lemma which is an extension to our setting of some results contained in [17].

Lemma 5.8. *Let $U \in \mathscr{A}$ be fixed and let $G : BV(U,\mathbb{R}^N) \times \mathscr{A}(U) \longrightarrow [0,\infty)$ be such that*

(1) for every $u \in BV(U,\mathbb{R}^N)$ the set function $G(u,\cdot)$ is the restriction to $\mathscr{A}(U)$ of a finite Radon measure on U;
(2) for every $A \in \mathscr{A}(U)$ the functional $G(\cdot, A)$ is $L^1(A,\mathbb{R}^n)$-lower semicontinuous;
(3) there exists $K \in (0,\infty)$ such that

$$G(u,A) \le K(\mathcal{L}^n(A) + |Du|(A))$$

for every $u \in BV(U,\mathbb{R}^N)$ and every $A \in \mathscr{A}(U)$.
(4) For every $M \in (0,\infty)$ there exists $K_M \in (0,\infty)$ such that

$$K_M|Du|(A) \le G(u,A)$$

for every $u \in BV(U,\mathbb{R}^N)$ with $\|u\|_{L^\infty(U)} \le M$ and every $A \in \mathscr{A}(U)$.

Then, if $w \in BV(U,\mathbb{R}^N)$ is such that $2\|w\|_{L^\infty(U)} \le M$ we have that for $\mathcal{H}^{n-1}$-a.e. $x \in J_w$

$$\frac{\mathrm{d}G(w,\cdot)}{\mathrm{d}\mathcal{H}^{n-1} \llcorner J_w}(x) = \lim_{\rho\to 0} \frac{m^M_G(w, Q^{\nu_w(x)}_\rho(x))}{\rho^{n-1}} = \lim_{\rho\to 0} \frac{m^M_G(u_{x,[w](x),\nu_w(x)}, Q^{\nu_w(x)}_\rho(x))}{\rho^{n-1}} \tag{5.40}$$

where

$$m^M_G(w,A) := \inf\{G(v,A) :\ v \in BV(A,\mathbb{R}^N),\, v = w \text{ on } \partial A,\, \|v\|_{L^\infty(A,\mathbb{R}^N)} \le M\}. \tag{5.41}$$

Proof. The proof follows by combining a number of arguments from [17, Section 3] which we briefly summarize. Appealing to [17, Lemma 3.5 and formula (3.17) in Theorem 3.7] the equality in (5.40) can be established for functionals G satisfying assumptions (1)-(3) above, and the stronger growth condition

$$C|Du|(A) \le G(u,A), \tag{5.42}$$

for every $u \in BV(A,\mathbb{R}^N)$.

In their turn, [17, Lemma 3.5 and formula (3.17) in Theorem 3.7] are a consequence of [17, Lemmata 3.1 and 3.3]. Namely, [17, Lemmata 3.1] establishes the Lipschitz continuity of m_G as in (5.9), with respect to the traces and is stated under the sole positivity of G. It is easy to check that an analogous result holds true for m^M_G as in (5.41).

Moreover, (5.42) is used in [17, Lemma 3.3] to prove the equality $G(u,A) = \sup_{\delta>0} m_{G,\delta}(u,A)$, where

$$m_{G,\delta}(u,A) := \inf\Big\{\sum_{i\in\mathbb{N}} m_G(u, Q^{\nu_i}_{r_i}(x_i)) : Q^{\nu_i}_{r_i}(x_i) \subset A, Q^{\nu_i}_{r_i}(x_i) \cap Q^{\nu_j}_{r_j}(x_j) = \emptyset, i \ne j,$$
$$\mathrm{diam} Q^{\nu_i}_{r_i}(x_i) < \delta,\, \mu(A \setminus \cup_{i\in\mathbb{N}} Q^{\nu_i}_{r_i}(x_i)) = 0\Big\},$$

with $\mu := \mathcal{L}^n + |D^s u|$.

Then to conclude we notice that the same identity holds true for m^M_G under the assumptions (1)-(4). In fact, one inequality is trivial, while the other can be obtained by exhibiting a competitor with the same L^∞ bound. □

We are now ready to show Proposition 5.7.

Proof of Proposition 5.7. Let us fix $A \in \mathscr{A}$ and $u \in L^1_{\mathrm{loc}}(\mathbb{R}^n,\mathbb{R}^N)$ with $u \in BV(A,\mathbb{R}^N) \cap L^\infty(A,\mathbb{R}^N)$. We divide the proof into two steps.

Step 1: We claim that

$$\frac{\mathrm{d}\widehat{F}(u,1,\cdot)}{\mathrm{d}\mathcal{H}^{n-1}\llcorner J_u}(x)\le g_{\mathrm{hom}}([u](x),\nu_u(x))\quad\text{for }\mathcal{H}^{n-1}\text{-a.e. }x\in J_u\cap A.$$

Thanks to Theorem 5.2 and Lemma 5.8, we obtain for $\mathcal{H}^{n-1}$-a.e. $x\in J_u\cap A$ that

$$\frac{\mathrm{d}\widehat{F}(u,1,\cdot)}{\mathrm{d}\mathcal{H}^{n-1}\llcorner J_u}(x)=\lim_{\rho\to0}\frac{m^L_{\widehat{F}}(u,Q^{\nu_u(x)}_\rho(x))}{\rho^{n-1}}=\lim_{\rho\to0}\frac{m^L_{\widehat{F}}(u_{x,[u](x),\nu_u(x)},Q^{\nu_u(x)}_\rho(x))}{\rho^{n-1}},\tag{5.43}$$

for every $L>0$ with $2\|u\|_{L^\infty(A)}\le L$.

Fix $x\in A\cap J_u$ such that (5.43) holds and set $\zeta=[u](x)$ and $\nu_u(x)=\nu$. By combining (4.14) and Corollary 4.5, for every $\rho>0$ we get

$$g_{\mathrm{hom}}(\zeta,\nu)=\lim_{r\to+\infty}\frac{m^{f^\infty}_{\mathrm{s}}(\overline{u}_{\frac{r}{\rho}x,\zeta,\nu},Q^\nu_r(\frac{r}{\rho}x))}{r^{n-1}}.\tag{5.44}$$

Let $\eta\in(0,1)$ be fixed. By (5.44), for every ρ and every r large enough, we have that

$$m^{f^\infty}_{\mathrm{s}}(\overline{u}_{\frac{r}{\rho}x,\zeta,\nu},Q^\nu_r(\tfrac{r}{\rho}x))\le g_{\mathrm{hom}}(\zeta,\nu)r^{n-1}+\eta r^{n-1}.$$

Therefore there exist $w^\rho_r\in W^{1,1}(Q^\nu_r(\frac{r}{\rho}x),\mathbb{R}^N)$ and $v^\rho_r\in W^{1,2}(Q^\nu_r(\frac{r}{\rho}x),[0,1])$ with $(w^\rho_r,v^\rho_r)=(\overline{u}_{\frac{r}{\rho}x,\zeta,\nu},1)$ on $\partial Q^\nu_r(\frac{r}{\rho}x)$, such that

$$\begin{aligned}S^{f^\infty}(w^\rho_r,v^\rho_r,Q^\nu_r(\tfrac{r}{\rho}x))&=\int_{Q^\nu_r(\frac{r}{\rho}x)}\big((v^\rho_r)^2f^\infty(y,\nabla w^\rho_r)+(1-v^\rho_r)^2+|\nabla v^\rho_r|^2\big)\,\mathrm{d}y\le\\&\le m^{f^\infty}_{\mathrm{s}}(\overline{u}_{\frac{r}{\rho}x,\zeta,\nu},Q^\nu_r(\tfrac{r}{\rho}x))+\eta r^{n-1}\le g_{\mathrm{hom}}(\zeta,\nu)r^{n-1}+2\eta r^{n-1}.\end{aligned}\tag{5.45}$$

Next apply Lemma 5.3 with $\gamma=0$ to infer that $\mathcal{T}_{k^\rho_r}(w^\rho_r)=:\hat{w}^\rho_r\in W^{1,1}(Q^\nu_r(\frac{r}{\rho}x),\mathbb{R}^N)$ satisfies

$$\begin{aligned}S^{f^\infty}(\hat{w}^\rho_r,v^\rho_r,Q^\nu_r(\tfrac{r}{\rho}x))&\le\Big(1+\frac{C^2}{M}\Big)S^{f^\infty}(w^\rho_r,v^\rho_r,Q^\nu_r(\tfrac{r}{\rho}x))\\&\le\Big(1+\frac{C^2}{M}\Big)(g_{\mathrm{hom}}(\zeta,\nu)r^{n-1}+2\eta r^{n-1}).\end{aligned}\tag{5.46}$$

Moreover, if $M\in\mathbb{N}$ is such that $(2|\zeta|+1)+2\|u\|_{L^\infty(A)}\le a_{2M}$ then $\hat{w}^\rho_r=\overline{u}_{\frac{r}{\rho}x,\zeta,\nu}$ on $\partial Q^\nu_r(\frac{r}{\rho}x)$ and $\|\hat{w}^\rho_r\|_{L^\infty(Q^\nu_r(\frac{r}{\rho}x))}\le a_{2M+1}$.

In particular, by (5.46) we have that

$$\frac{1}{r^{n-1}}\int_{Q^\nu_r(\frac{r}{\rho}x)}(v^\rho_r)^2|\nabla\hat{w}^\rho_r|\mathrm{d}y\le C\Big(1+\frac{C^2}{M}\Big)(g_{\mathrm{hom}}(\zeta,\nu)+2\eta).\tag{5.47}$$

By Lemma 4.2 and (5.47), given $\rho>0$, for every r large enough we have

$$\begin{aligned}&\frac{1}{r^{n-1}}\int_{Q^\nu_r(\frac{r}{\rho}x)}|(v^\rho_r)^2f^\infty(y,\nabla\hat{w}^\rho_r)-(v^\rho_r)^2\tfrac{\rho}{r}f(y,\tfrac{r}{\rho}\nabla\hat{w}^\rho_r)|\mathrm{d}y\\&\le K\rho+\frac{K\rho^\alpha}{r^{(n-1)(1-\alpha)}}\Big(\int_{Q^\nu_r(\frac{r}{\rho}x)}(v^\rho_r)^2|\nabla\hat{w}^\rho_r|\mathrm{d}y\Big)^{1-\alpha}\\&\le K\rho+K\rho^\alpha C^{1-\alpha}\Big(1+\frac{C^2}{M}\Big)^{1-\alpha}(g_{\mathrm{hom}}(\zeta,\nu)+2\eta)^{1-\alpha}.\end{aligned}\tag{5.48}$$

Therefore, recollecting (5.44), (5.45) and (5.48), for every $\rho > 0$ we get the following

$$\limsup_{r\to+\infty}\frac{1}{r^{n-1}}\int_{Q^\nu_r(\frac{r}{\rho}x)}((v^\rho_r)^2\tfrac{\rho}{r}f(y,\tfrac{r}{\rho}\nabla\hat{w}^\rho_r)+(1-v^\rho_r)^2+|\nabla v^\rho_r|^2)\mathrm{d}y$$
$$\leq\Big(1+\frac{C^2}{M}\Big)(g_{\mathrm{hom}}(\zeta,\nu)+2\eta)+\tilde{K}(M,\rho,\eta),$$

where $\tilde{K}(M,\rho,\eta):=K\rho+K\rho^\alpha C^{1-\alpha}\big(1+\frac{C^2}{M}\big)^{1-\alpha}(g_{\mathrm{hom}}(\zeta,\nu)+2\eta)^{1-\alpha}$.

Given $\varepsilon>0$ and $\rho>0$, we define $(\hat{u}^\rho_\varepsilon,\hat{v}^\rho_\varepsilon):\mathbb{R}^n\to\mathbb{R}^{N+1}$ as follows

$$\hat{u}^\rho_\varepsilon(y):=\begin{cases}\hat{w}^\rho_r(\frac{ry}{\rho}) & \text{if } y\in Q^\nu_\rho(x)\\ \overline{u}^\varepsilon_{x,\zeta,\nu} & \text{if } y\in\mathbb{R}^n\setminus Q^\nu_\rho(x)\end{cases}\qquad \hat{v}^\rho_\varepsilon(y):=\begin{cases}\hat{v}^\rho_r(\frac{ry}{\rho}) & \text{if } y\in Q^\nu_\rho(x)\\ 1 & \text{if } y\in\mathbb{R}^n\setminus Q^\nu_\rho(x)\end{cases}$$

with $r=\frac{\rho}{\varepsilon}$. Thereby $\hat{u}^\rho_\varepsilon\in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^n,\mathbb{R}^N)$ with $\|\hat{u}^\rho_\varepsilon\|_{L^\infty(\mathbb{R}^n)}\leq a_{2M+1}$ and $\hat{v}^\rho_\varepsilon\in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^n,[0,1])$. Changing variables it is immediate to get

$$\limsup_{j\to+\infty}\Big(\frac{1}{\rho^{n-1}}\int_{Q^\nu_\rho(x)}((\hat{v}^\rho_{\varepsilon_j})^2f(\tfrac{y}{\varepsilon_j},\nabla\hat{u}^\rho_{\varepsilon_j})+\tfrac{(1-\hat{v}^\rho_{\varepsilon_j})^2}{\varepsilon_j}+\varepsilon_j|\nabla\hat{v}^\rho_{\varepsilon_j}|^2)\mathrm{d}y\Big)$$
$$\leq\Big(1+\frac{C^2}{M}\Big)(g_{\mathrm{hom}}(\zeta,\nu)+2\eta)+\tilde{K}(M,\rho,\eta),$$

where $(\varepsilon_j)_{j\in\mathbb{N}}$ is the sequence in Theorem 5.2 along which the Γ-convergence of $(F_\varepsilon)_{\varepsilon>0}$ holds. Moreover, we observe that

$$\frac{1}{\rho^{n-1}}\int_{Q^\nu_{\rho(1+\eta)}(x)\setminus Q^\nu_\rho(x)}((\hat{v}^\rho_{\varepsilon_j})^2f(\tfrac{y}{\varepsilon_j},\nabla\hat{u}^\rho_{\varepsilon_j})+\tfrac{(1-\hat{v}^\rho_{\varepsilon_j})^2}{\varepsilon_j}+\varepsilon_j|\nabla\hat{v}^\rho_{\varepsilon_j}|^2)\mathrm{d}y$$
$$\leq\frac{C}{\rho^{n-1}}\int_{Q^\nu_{\rho(1+\eta)}(x)\setminus Q^\nu_\rho(x)}(1+|\nabla\overline{u}^{\varepsilon_j}_{x,\zeta,\nu}|)\mathrm{d}y$$
$$\leq C\rho\,((1+\eta)^n-1)+\frac{C}{\rho^{n-1}}\int_{(Q^\nu_{\rho(1+\eta)}(x)\setminus Q^\nu_\rho(x))\cap\{|(y-x)\cdot\nu|\leq\varepsilon_j/2\}}|\nabla\overline{u}^{\varepsilon_j}_{x,\zeta,\nu}|\mathrm{d}y$$
$$\leq C\rho\,((1+\eta)^n-1)+C|\zeta|\|\overline{\mathrm{u}}'\|_{L^\infty(\mathbb{R})}\big((1+\eta)^{n-1}-1\big)\leq C((1+\eta)^n-1)(\rho+|\zeta|\|\overline{\mathrm{u}}'\|_{L^\infty(\mathbb{R})}).$$

Therefore, for every ρ, we have that

$$\sup_{j\in\mathbb{N}}\int_{Q^\nu_{\rho(1+\eta)}(x)}((\hat{v}^\rho_{\varepsilon_j})^2|\nabla\hat{u}^\rho_{\varepsilon_j}|+\tfrac{(1-\hat{v}^\rho_{\varepsilon_j})^2}{\varepsilon_j}+\varepsilon_j|\nabla\hat{v}^\rho_{\varepsilon_j}|^2\mathrm{d}y)<+\infty.$$

From $\|\hat{u}^\rho_{\varepsilon_j}\|_{L^\infty(Q^\nu_\rho(x))}\leq a_{2M+1}$ and [7, Lemma 7.1] there exists a subsequence (not relabeled) of $(\varepsilon_j)_{j\in\mathbb{N}}$ and $u^\rho\in L^1_{\mathrm{loc}}(\mathbb{R}^n,\mathbb{R}^N)$ such that $(\hat{u}^\rho_{\varepsilon_j},\hat{v}^\rho_{\varepsilon_j})\to(u^\rho,1)$ in $L^1_{\mathrm{loc}}(\mathbb{R}^n,\mathbb{R}^{N+1})$, $u^\rho\in BV(Q^\nu_{\rho(1+\eta)}(x),\mathbb{R}^N)$, $u^\rho(y)=u_{x,\zeta,\nu}(y)$ for $\mathcal{L}^n$-a.e. $y\in\mathbb{R}^n\setminus Q^\nu_\rho(x)$. By assumption (ii) In Theorem 5.1 (cf. formula (5.2)), it follows that for every ρ

$$\frac{m^{a_{2M+1}}_{\widehat{F}}(u_{x,\zeta,\nu},Q^\nu_{(1+\eta)\rho}(x))}{\rho^{n-1}}\leq\frac{\widehat{F}(u^\rho,1,Q^\nu_{(1+\eta)\rho}(x))}{\rho^{n-1}}\leq\liminf_{j\to+\infty}\frac{F_{\varepsilon_j}(\hat{u}^\rho_{\varepsilon_j},\hat{v}^\rho_{\varepsilon_j},Q^\nu_{(1+\eta)\rho}(x))}{\rho^{n-1}}$$
$$\leq\Big(1+\frac{C^2}{M}\Big)(g_{\mathrm{hom}}(\zeta,\nu)+2\eta)+\tilde{K}(M,\rho,\eta)+C((1+\eta)^n-1)(\rho+|\zeta|\|\overline{\mathrm{u}}'\|_{L^\infty(\mathbb{R})}).$$

In particular, thanks to (5.43) we have that

$$(1+\eta)^{n-1}\frac{\mathrm{d}\widehat{F}(u,1,\cdot)}{\mathrm{d}\mathcal{H}^{n-1}\llcorner J_u}(x)=\lim_{\rho\to 0}\frac{m_{\widehat{F}}^{a_{2M+1}}(u_{x,\zeta,\nu},Q^{\nu}_{(1+\eta)\rho}(x))}{\rho^{n-1}}$$
$$\leq\Big(1+\frac{C^2}{M}\Big)(g_{\mathrm{hom}}(\zeta,\nu)+2\eta)+C|\zeta|\|\overline{\mathrm{u}}'\|_{L^\infty(\mathbb{R})}\left((1+\eta)^n-1\right)$$

and thereby, letting $M\to+\infty$ and $\eta\to 0$, we can conclude.

Step 2: We claim that

$$\frac{\mathrm{d}\widehat{F}(u,1,\cdot)}{\mathrm{d}\mathcal{H}^{n-1}\llcorner J_u}(x)\geq g_{\mathrm{hom}}([u](x),\nu_u(x))\quad\text{for }\mathcal{H}^{n-1}\text{-a.e. } x\in J_u\cap A.$$

By Theorem 5.2 there exists $(u_j,v_j)\in L^1_{\mathrm{loc}}(\mathbb{R}^n,\mathbb{R}^N)$ with $u_j\in W^{1,1}(A,\mathbb{R}^N)$ and $v_j\in W^{1,2}(A,[0,1])$, such that $v_j(x)\to 1$ for $\mathcal{L}^n$-a.e. $x\in A$ as $j\to+\infty$,

$$(u_j,v_j)\to(u,1)\quad\text{in } L^1_{\mathrm{loc}}(\mathbb{R}^n,\mathbb{R}^{N+1})\quad\text{and}\quad\lim_{j\to+\infty}F_{\varepsilon_j}(u_j,v_j,A)=\widehat{F}(u,1,A).$$

For $\mathcal{H}^n$-a.e. $x\in J_u\cap A$ (cf. [9, Theorem 3.77 and Proposition 3.92]) we have

$$\lim_{\rho\to 0}\frac{1}{\rho^{n-1}}|Du|(Q^{\nu_u(x)}_\rho(x))=|[u](x)|\neq 0, \tag{5.49}$$

$$\lim_{\rho\to 0}\frac{1}{\rho^n}\int_{Q^{\nu_u(x)}_\rho(x)}|u(y)-u_{x,[u](x),\nu_u(x)}(y)|\mathrm{d}y=0, \tag{5.50}$$

$$\lim_{\rho\to 0}\frac{\widehat{F}(u,1,Q^{\nu_u(x)}_\rho(x))}{\rho^{n-1}}=\frac{\mathrm{d}\widehat{F}(u,1,\cdot)}{\mathrm{d}\mathcal{H}^{n-1}\llcorner J_u}(x)<+\infty. \tag{5.51}$$

Let us fix $x\in J_u\cap A$ such that (5.49)-(5.51) are satisfied, and set $\zeta:=[u](x)$ and $\nu:=\nu_u(x)$. Using the lower bound inequality in the Γ-convergence of $(F_\varepsilon)_{\varepsilon>0}$ on $Q^{\nu_u(x)}_\rho(x)$ and $A\setminus Q^{\nu_u(x)}_\rho(x)$, the super-additivity of the inferior limit operator implies that $\lim_{j\to+\infty}F_{\varepsilon_j}(u_j,v_j,Q^\nu_\rho(x))=\widehat{F}(u,1,Q^\nu_\rho(x))$ for every $\rho\in I(x):=\{\rho\in(0,\frac{2}{\sqrt{n}}\mathrm{dist}(x,\partial A))\ :\ \widehat{F}(u,1,\partial Q^\nu_\rho(x))=0\}$. Hence, we deduce that

$$\frac{\mathrm{d}\widehat{F}(u,1\cdot)}{\mathrm{d}\mathcal{H}^{n-1}\llcorner J_u}(x)=\lim_{I(x)\ni\rho\to 0}\lim_{j\to+\infty}\frac{1}{\rho^{n-1}}\int_{Q^\nu_\rho(x)}(v_j^2 f(\tfrac{y}{\varepsilon_j},\nabla u_j)+\tfrac{(1-v_j)^2}{\varepsilon_j}+\varepsilon_j|\nabla v_j|^2)\mathrm{d}y. \tag{5.52}$$

Now, we consider the rescalings $(u^\rho_j,v^\rho_j),(u^\rho,v^\rho):Q^\nu_1\to\mathbb{R}^{N+1}$ given by

$$(u^\rho_j(y),v^\rho_j(y)):=(u_j(x+\rho y),v_j(x+\rho y))\quad\text{and}\quad(u^\rho(y),v^\rho(y)):=(u(x+\rho y),1)$$

Then $u^\rho_j\in W^{1,1}(Q^\nu_1,\mathbb{R}^N)$, $v^\rho_j\in W^{1,2}(Q^\nu_1,[0,1])$, $u^\rho\in BV(Q^\nu_1,\mathbb{R}^N)$, and $(u^\rho_j,v^\rho_j)\to(u^\rho,1)$ in $L^1(Q^\nu_1,\mathbb{R}^{N+1})$, $u^\rho\to u_{\zeta,\nu}$ in $L^1(Q^\nu,\mathbb{R}^N)$ by (5.50), and $v^\rho_j\to 1$ in $L^2(Q^\nu)$ for every ρ by(5.52). Changing variables, formula (5.52) rewrites as

$$\frac{\mathrm{d}\widehat{F}(u,1\cdot)}{\mathrm{d}\mathcal{H}^{n-1}\llcorner J_u}(x)=\lim_{I(x)\ni\rho\to 0}\lim_{j\to+\infty}\int_{Q^\nu_1}(\rho(v^\rho_j)^2 f(\tfrac{x+\rho y}{\varepsilon_j},\tfrac{1}{\rho}\nabla u^\rho_j)+\tfrac{\rho}{\varepsilon_j}(1-v^\rho_j)^2+\tfrac{\varepsilon_j}{\rho}|\nabla v^\rho_j|^2)\mathrm{d}y\,,$$

thus by (f2) we infer that

$$\frac{\mathrm{d}\widehat{F}(u,1\cdot)}{\mathrm{d}\mathcal{H}^{n-1}\llcorner J_u}(x)\geq C\limsup_{I(x)\ni\rho\to 0}\limsup_{j\to+\infty}\int_{Q^\nu_1}(v^\rho_j)^2|\nabla u^\rho_j|\mathrm{d}y. \tag{5.53}$$

Lemma 4.2 implies that

$$\rho\int_{Q_1^\nu}(v_j^\rho)^2 f(\tfrac{x+\rho y}{\varepsilon_j},\tfrac{1}{\rho}\nabla u_j^\rho)\mathrm{d}y \geq \int_{Q_1^\nu}(v_j^\rho)^2 f^\infty(\tfrac{x+\rho y}{\varepsilon_j},\nabla u_j^\rho)\mathrm{d}y - K\rho - K\rho^\alpha\Big(\int_{Q_1^\nu}(v_j^\rho)^2|\nabla u_j^\rho|\mathrm{d}y\Big)^{1-\alpha} \tag{5.54}$$

where K is a constant that depends only on C and α. Thanks to (5.51), (5.53) and (5.54) we get

$$\frac{\mathrm{d}\widehat{F}(u,1\cdot)}{\mathrm{d}\mathcal{H}^{n-1}\llcorner J_u}(x) \geq \limsup_{I(x)\ni\rho\to 0}\limsup_{j\to+\infty} F^{\infty,x}_{\rho,\varepsilon_j}(u_j^\rho, v_j^\rho, Q_1^\nu), \tag{5.55}$$

where

$$F^{\infty,x}_{\rho,\varepsilon}(u,v,A) := \int_A (v^2 f^\infty(\tfrac{x+\rho y}{\varepsilon},\nabla u) + \tfrac{\rho}{\varepsilon}(1-v)^2 + \tfrac{\varepsilon}{\rho}|\nabla v|^2)\mathrm{d}y.$$

Now, for every ρ and j we consider the sequences $(a_j^\rho)_{j\in\mathbb{N}}$, $(b_j^\rho)_{j\in\mathbb{N}}$ and $(s_j^\rho)_{j\in\mathbb{N}}$ given by

$$a_j^\rho := \rho + \|u_j^\rho - w^\rho\|^{\frac12}_{L^1(Q_1^\nu)} + \|v_j^\rho - 1\|^{\frac12}_{L^2(Q_1^\nu)}, \quad b_j^\rho := \left\lfloor\frac{\rho}{\varepsilon_j}\right\rfloor \quad \text{and} \quad s_j^\rho := \frac{a_j^\rho}{b_j^\rho}, \tag{5.56}$$

where (w^ρ) is a sequence in $W^{1,1}(Q_1^\nu,\mathbb{R}^N)$ such that $w^\rho = u_{\zeta,\nu}$ on ∂Q_1^ν for every ρ,

$$|Dw^\rho|(Q_1^\nu) \to |Du_{\zeta,\nu}|(Q_1^\nu) \quad \text{and} \quad w^\rho \to u_{\zeta,\nu} \ \text{ in } L^1(Q_1^\nu,\mathbb{R}^N) \text{ as } \rho\to 0, \tag{5.57}$$

(see [17, Lemma 2.5]), where $\lfloor s\rfloor$ denotes the integer part of $s\in\mathbb{R}$. Fix ρ small enough, such that $\rho + \|u^\rho - w^\rho\|^{\frac12}_{L^1(Q_1^\nu)} < \frac14$ and then fix j large enough such that $0 < a_j^\rho < \frac12$ and $2 < b_j^\rho$. For every $i = 0,\dots,b_j^\rho$ we define $Q^\nu_{\rho,j,i}$ as

$$Q^\nu_{\rho,j,i} := (1 - a_j^\rho + i s_j^\rho)Q_1^\nu,$$

while for every $i = 1,..,b_j^\rho$ we consider the cut-off function $\phi^\rho_{j,i}\in C_c^\infty(Q^\nu_{\rho,j,i})$ such that $0\leq\phi^\rho_{j,i}\leq 1$, $\phi^\rho_{j,i}\equiv 1$ on $Q^\nu_{\rho,j,i-1}$ and $\|\nabla\phi^\rho_{j,i}\|_{L^\infty(\mathbb{R}^n)}\leq 2(s_j^\rho)^{-1}$. Set for $i=1,..,b_j^\rho$

$$u^\rho_{j,i} := \phi^\rho_{j,i-1}u_j^\rho + (1-\phi^\rho_{j,i-1})w^\rho \quad \text{and} \quad v^\rho_{j,i} := \phi^\rho_{j,i}v_j^\rho + (1-\phi^\rho_{j,i}).$$

Then $u^\rho_{j,i}\in W^{1,1}(Q_1^\nu,\mathbb{R}^N)$, $v^\rho_{j,i}\in W^{1,2}(Q_1^\nu,[0,1])$ with $(u^\rho_{j,i},v^\rho_{j,i}) = (u_{\zeta,\nu},1)$ on ∂Q_1^ν. Moreover, for every $i = 2,..,b_{\rho,j}$ we have the following

$$\begin{aligned}
&F^{\infty,x}_{\rho,\varepsilon_j}(u^\rho_{j,i},v^\rho_{j,i},Q_1^\nu) \leq F^{\infty,x}_{\rho,\varepsilon_j}(u_j^\rho,v_j^\rho,Q^\nu_{\rho,j,i-2})\\
&+\int_{Q^\nu_{\rho,j,i-1}\setminus Q^\nu_{\rho,j,i-2}}(v_j^\rho)^2 f^\infty(\tfrac{x+\rho y}{\varepsilon_j},\nabla u^\rho_{j,i})\mathrm{d}y + \int_{Q^\nu_{\rho,j,i-1}\setminus Q^\nu_{\rho,j,i-2}}(\tfrac{\rho}{\varepsilon_j}(1-v_j^\rho)^2 + \tfrac{\varepsilon_j}{\rho}|\nabla v_j^\rho|^2)\mathrm{d}y\\
&+\int_{Q_1^\nu\setminus Q^\nu_{\rho,j,i-1}}(v^\rho_{j,i})^2 f^\infty(\tfrac{x+\rho y}{\varepsilon_j},\nabla w^\rho)\mathrm{d}y + \int_{Q^\nu_{\rho,j,i}\setminus Q^\nu_{\rho,j,i-1}}(\tfrac{\rho}{\varepsilon_j}(1-v^\rho_{j,i})^2 + \tfrac{\varepsilon_j}{\rho}|\nabla v^\rho_{j,i}|^2)\mathrm{d}y.
\end{aligned}$$

We estimate separately the terms appearing above. We start with

$$F^{\infty,x}_{\rho,\varepsilon_j}(u_j^\rho,v_j^\rho,Q^\nu_{\rho,j,i-2}) + \int_{Q^\nu_{\rho,j,i-1}\setminus Q^\nu_{\rho,j,i-2}}(\tfrac{\rho}{\varepsilon_j}(1-v_j^\rho)^2 + \tfrac{\varepsilon_j}{\rho}|\nabla v_j^\rho|^2)\mathrm{d}y \leq F^{\infty,x}_{\rho,\varepsilon_j}(u_j^\rho,v_j^\rho,Q^\nu).$$

Moreover, since $\nabla u^\rho_{j,i} = \phi^\rho_{j,i-1}\nabla u^\rho_j + (1-\phi^\rho_{j,i-1})\nabla w^\rho + \nabla\phi^\rho_{j,i-1}\otimes(u^\rho_j - w^\rho)$, we have that

$$\begin{aligned}
&\int_{Q^\nu_{\rho,j,i-1}\setminus Q^\nu_{\rho,j,i-2}} (v^\rho_j)^2 f^\infty(\tfrac{x+\rho y}{\varepsilon_j},\nabla u^\rho_{j,i})\mathrm{d}y\\
&\leq C\int_{Q^\nu_{\rho,j,i-1}\setminus Q^\nu_{\rho,j,i-2}} (v^\rho_j)^2(|\nabla u^\rho_j| + |\nabla w^\rho| + |\nabla\phi^\rho_{j,i-1}||u^\rho_j - w^\rho|)\mathrm{d}y\\
&\leq C^2\int_{Q^\nu_{\rho,j,i-1}\setminus Q^\nu_{\rho,j,i-2}} (v^\rho_j)^2 f^\infty(\tfrac{x+\rho y}{\varepsilon_j},\nabla u^\rho_j)\mathrm{d}y + C\int_{Q^\nu_{\rho,j,i-1}\setminus Q^\nu_{\rho,j,i-2}}|\nabla w^\rho|\mathrm{d}y\\
&+\frac{2C}{s^\rho_j}\int_{Q^\nu_{\rho,j,i-1}\setminus Q^\nu_{\rho,j,i-2}}|u^\rho_j - w^\rho|\mathrm{d}y.
\end{aligned}$$

Analogously, we obtain

$$\int_{Q^\nu_1\setminus Q^\nu_{\rho,j,i-1}} (v^\rho_{j,i})^2 f^\infty(\tfrac{x+\rho y}{\varepsilon_j},\nabla w^\rho)\mathrm{d}y \leq C\int_{Q^\nu_1\setminus Q^\nu_{\rho,j,i-1}}|\nabla w^\rho|\mathrm{d}y.$$

Since $\nabla v^\rho_{j,i} = \phi^\rho_{j,i}\nabla v^\rho_j + (v^\rho_j - 1)\nabla\phi^\rho_{j,i}$, we have that

$$\begin{aligned}
&\int_{Q^\nu_{\rho,j,i}\setminus Q^\nu_{\rho,j,i-1}} (\tfrac{\rho}{\varepsilon_j}(1-v^\rho_{j,i})^2 + \tfrac{\varepsilon_j}{\rho}|\nabla v^\rho_{j,i}|^2)\mathrm{d}y\\
&\leq \frac{\rho}{\varepsilon_j}\mathcal{L}^n(Q^\nu_{\rho,j,i}\setminus Q^\nu_{\rho,j,i-1}) + 2\int_{Q^\nu_{\rho,j,i}\setminus Q^\nu_{\rho,j,i-1}}\frac{\varepsilon_j}{\rho}|\nabla v^\rho_j|^2\mathrm{d}y + \frac{8\varepsilon_j}{\rho(s^\rho_j)^2}\int_{Q^\nu_{\rho,j,i}\setminus Q^\nu_{\rho,j,i-1}}|v^\rho_j - 1|^2\mathrm{d}y\\
&\leq \frac{\rho}{\varepsilon_j}\mathcal{L}^n(Q^\nu_{\rho,j,i}\setminus Q^\nu_{\rho,j,i-1}) + 2F^{\infty,x}_{\rho,\varepsilon_j}(u^\rho_j, v^\rho_j, Q^\nu_{\rho,j,i}\setminus Q^\nu_{\rho,j,i-1}) + \frac{8\varepsilon_j}{\rho(s^\rho_j)^2}\int_{Q^\nu_{\rho,j,i}\setminus Q^\nu_{\rho,j,i-1}}|v^\rho_j - 1|^2\mathrm{d}y\,.
\end{aligned}$$

In particular, thanks to the previous calculations and and recalling the definition of s^ρ_j, there exists $i^\rho_j \in \{2,\dots,b^\rho_j\}$ such that

$$\begin{aligned}
&F^{\infty,x}_{\rho,\varepsilon_j}(u^\rho_{j,i^\rho_j}, v^\rho_{j,i^\rho_j}, Q^\nu_1) \leq \frac{1}{b^\rho_j - 1}\sum_{i=2}^{b^\rho_j} F^{\infty,x}_{\rho,\varepsilon_j}(u^\rho_{j,i}, v^\rho_{j,i}, Q^\nu_1)\\
&\leq F^{\infty,x}_{\rho,\varepsilon_j}(u^\rho_j, v^\rho_j, Q^\nu_1) + \frac{2}{b^\rho_j - 1}F^{\infty,x}_{\rho,\varepsilon_j}(u^\rho_j, v^\rho_j, Q^\nu_1) + C\int_{Q^\nu_1\setminus Q^\nu_{0,\rho,j}}|\nabla w^\rho|\mathrm{d}y\\
&+\frac{2Cb^\rho_j}{(b^\rho_j - 1)a^\rho_j}\int_{Q^\nu_1\setminus Q^\nu_{\rho,j,0}}|u^\rho_j - w^\rho|\mathrm{d}y + \frac{\rho}{\varepsilon_j(b^\rho_j - 1)}\mathcal{L}^n(Q^\nu_1\setminus Q^\nu_{\rho,j,0})\\
&+\frac{8\varepsilon_j(b^\rho_j)^2}{\rho(a^\rho_j)^2(b^\rho_j - 1)}\int_{Q^\nu_1\setminus Q^\nu_{\rho,j,0}}|v^\rho_j - 1|^2\mathrm{d}y.
\end{aligned}$$

Hence, by the definition of a^ρ_j and b^ρ_j in (5.56) we deduce that

$$\begin{aligned}
&F^{\infty,x}_{\rho,\varepsilon_j}(u^\rho_{j,i^\rho_j}, v^\rho_{j,i^\rho_j}, Q^\nu) \leq \Big(1 + \frac{2}{b^\rho_j - 1}\Big)F^{\infty,x}_{\rho,\varepsilon_j}(u^\rho_j, v^\rho_j, Q^\nu)\\
&+ C\int_{Q^\nu\setminus Q^\nu_{\rho,j,0}}|\nabla w^\rho|\mathrm{d}y + 4Ca^\rho_j + 3\mathcal{L}^n(Q^\nu\setminus Q^\nu_{\rho,j,0}) + 16(a^\rho_j)^2.
\end{aligned}$$

Thus, setting where $\kappa_\rho = 1 - \|u^\rho - w^\rho\|_{L^1(Q^\nu)}$, from (5.55) and (5.56) we obtain

$$\limsup_{I(x)\ni\rho\to0}\ \limsup_{j\to+\infty} F^{\infty,x}_{\rho,\varepsilon_j}(u^\rho_{j,i^\rho_j}, v^\rho_{j,i^\rho_j}, Q^\nu) \leq \frac{\mathrm{d}\widehat{F}(u,1\cdot)}{\mathrm{d}\mathcal{H}^{n-1}\llcorner J_u}(x) + C\limsup_{I(x)\ni\rho\to0}|Dw^\rho|(Q^\nu\setminus Q^\nu_{\kappa_\rho}),$$

As $w^\rho \to u_{\zeta,\nu}$ strictly in BV (cf. (5.57)), we have that $|Dw^\rho|(Q^\nu \setminus Q^\nu_{\kappa_\rho}) \to 0$ as $\rho \to 0$, and thus

$$\frac{\mathrm{d}\widehat{F}(u,1\cdot)}{\mathrm{d}\mathcal{H}^{n-1}\llcorner J_u}(x) \geq \limsup_{I(x)\ni\rho\to 0}\ \limsup_{j\to+\infty} F^{\infty,x}_{\rho,\varepsilon_j}(u^\rho_{j,i^\rho_j}, v^\rho_{j,i^\rho_j}, Q^\nu). \tag{5.58}$$

By the change of variable and the 1-homogeneity of f^∞ , we have

$$F^{\infty,x}_{\rho,\varepsilon_j}(u^\rho_{j,i^\rho_j}, v^\rho_{j,i^\rho_j}, Q^\nu) = \frac{1}{r^{n-1}_{\rho,j}} \int_{Q^\nu_{r_{\rho,j}}(\frac{r_{\rho,j}}{\rho}x)} \big(\overline{v}^2_{\rho,j} f^\infty(y, \nabla \overline{u}_{\rho,j}) + (1-\overline{v}_{\rho,j})^2 + |\nabla \overline{v}_{\rho,j}|^2\big)\,\mathrm{d}y,$$

where $r_{\rho,j} := \frac{\rho}{\varepsilon_j}$, $\overline{u}^\rho_j(y) := u^\rho_{j,i^\rho_j}(\frac{y}{r_{\rho,j}} - \frac{x}{\rho})$ and $\overline{v}^\rho_j(y) := v^\rho_{j,i^\rho_j}(\frac{y}{r_{\rho,j}} - \frac{x}{\rho})$. In this way $\overline{u}^\rho_j \in W^{1,1}(Q^\nu_{r_{\rho,j}}(\frac{r_{\rho,j}}{\rho}x), \mathbb{R}^N)$, $\overline{v}^\rho_j \in W^{1,2}(Q^\nu_{r_{\rho,j}}(\frac{r_{\rho,j}}{\rho}x), [0,1])$ with $(\overline{u}^\rho_j, \overline{v}^\rho_j) = (u_{\frac{r_{\rho,j}}{\rho}x,\zeta,\nu}, 1)$ on $\partial Q^\nu_{r_{\rho,j}}(\frac{r_{\rho,j}}{\rho}x)$. In particular, by (5.58), the definition of $m^{f^\infty}_{\mathrm{s}}$ in (**??**) and the assumption (b) of Theorem 5.1 we obtain

$$\frac{\mathrm{d}\widehat{F}(u,1\cdot)}{\mathrm{d}\mathcal{H}^{n-1}\llcorner J_u}(x) \geq \limsup_{\tilde{I}(x)\ni\rho\to 0}\ \limsup_{j\to+\infty} \frac{m^{f^\infty}_{\mathrm{s}}(u_{\frac{r_{\rho,j}}{\rho}x,\zeta,\nu}, Q^\nu_{r_{\rho,j}}(\frac{r_{\rho,j}}{\rho}x))}{r^{n-1}_{\rho,j}} = g_{\mathrm{hom}}(\zeta,\nu),$$

deducing the claim. ☐

5.3. **Identification of the Cantor term.** Eventually, in this subsection we identify the density of the Cantor part of the Γ-limit $\widehat{F}$.

Proposition 5.9 (Homogenised Cantor integrand)**.** *Let $f \in \mathcal{F}(C,\alpha)$ satisfy* (4.4)*. Let $\widehat{F}$ be as in* (5.5)*. Then for every $A \in \mathscr{A}$ and every $u \in L^1_{\mathrm{loc}}(\mathbb{R}^n,\mathbb{R}^N)$, with $u \in BV(A,\mathbb{R}^N)\cap L^\infty(A,\mathbb{R}^N)$, we have that*

$$\frac{\mathrm{d}\widehat{F}(u,1,\cdot)}{\mathrm{d}|D^c u|}(x) = f^\infty_{\mathrm{hom}}\Big(\frac{\mathrm{d}D^c u}{\mathrm{d}|D^c u|}\Big) \quad \text{for } |D^c u|\text{-a.e. } x \in A,$$

where f^∞_{hom} is the recession function of f_{hom} as in (4.1)*.*

Proof. Let us fix $A \in \mathscr{A}$ and $u \in L^1_{\mathrm{loc}}(\mathbb{R}^n,\mathbb{R}^N)$ with $u \in BV \cap L^\infty(A,\mathbb{R}^N)$. We divide the proof into two steps.

Step 1: We claim that

$$\frac{\mathrm{d}\widehat{F}(u,1,\cdot)}{\mathrm{d}|D^c u|}(x) \leq f^\infty_{\mathrm{hom}}\Big(\frac{\mathrm{d}D^c u}{\mathrm{d}|D^c u|}\Big) \quad \text{for } |D^c u|\text{-a.e. } x \in A.$$

By Alberti's Rank-one Theorem [2] we know that for $|D^c u|$-a.e. $x \in A$ we have

$$\frac{\mathrm{d}D^c u}{\mathrm{d}|D^c u|}(x) = a(x)\otimes\nu(x) \tag{5.59}$$

where $(a(x),\nu(x)) \in \mathbb{R}^N \times \mathbb{S}^{n-1}$. By Theorem 5.2 and by [17, Lemma 3.9] we have that for $|D^c u|$-a.e. $x \in A$ there exists a doubly indexed positive sequence $(t_{\rho,k})$, with $\rho \in (0,\infty)$ and $k \in \mathbb{N}$, such that for every $k \in \mathbb{N}$

$$t_{\rho,k} \to +\infty \quad \text{ and } \quad \rho t_{k,\rho} \to 0 \quad \text{ as } \rho \to 0, \tag{5.60}$$

and for every $q \in \mathbb{Q}\cap(0,1)$

$$\frac{\mathrm{d}\widehat{F}(u,1,\cdot)}{\mathrm{d}|D^c u|}(x) + q a(x)\otimes\nu(x) = \lim_{k\to+\infty}\ \limsup_{\rho\to 0} \frac{m_{\widehat{F}_q}(\ell_{t_{\rho,k}a(x)\otimes\nu(x)}, Q^{\nu(x),k}_r(\frac{r}{\rho}x)}{k^{n-1}\rho^n t_{\rho,k}}\,, \tag{5.61}$$

where for every $A \in \mathscr{A}$ and $q \in \mathbb{Q}\cap(0,1)$ let $\widehat{F}_q(u,1,A) := \widehat{F}(u,1,A) + q|Du|(A)$, and $Q^{\nu,k}_r(z)$ is the parallelepiped defined in (f) of the notation list. Let $x \in A$ be such that (5.59)-(5.61) hold true, and set $a := a(x)$ and $\nu := \nu(x)$. Thanks to Proposition 4.3, for every $\rho > 0$ and every $k \in \mathbb{N}$ we have

$$f^{\infty}_{\rm hom}(a\otimes\nu) = \lim_{r\to\infty} \frac{m_{\rm b}^{f^\infty}(\ell_{a\otimes\nu}, Q^{\nu,k}_r(\frac{r}{\rho}x))}{k^{n-1}r^n}. \tag{5.62}$$

Let us fix $\eta \in (0,1)$. By the very definition of $m_{\rm b}^{f^\infty}$, for every $k \in \mathbb{N}$, $\rho \in (0,1)$ and $r \in (0,\infty)$ there exists a function $\hat u^{\rho,k}_r \in W^{1,1}(Q^{\nu,k}_r(\frac{r}{\rho}x),\mathbb{R}^N)$ with $\hat u^{\rho,k}_r = \ell_{a\otimes\nu}$ on $\partial Q^{\nu,k}_r(\frac{r}{\rho}x)$ such that

$$E^{f^\infty}\Big(\hat u^{\rho,k}_r, Q^{\nu,k}_r(\tfrac{r}{\rho}x)\Big) \le m_{\rm b}^{f^\infty}\Big(\ell_{a\otimes\nu}, Q^{\nu,k}_r(\tfrac{r}{\rho}x)\Big) + \eta k^{n-1}r^n \le C|a|k^{n-1}r^n + \eta k^{n-1}r^n \tag{5.63}$$

and in particular

$$\int_{Q^{\nu,k}_r(\frac{r}{\rho}x)} |\nabla \hat u^{\rho,k}_r|\mathrm{d}y \le C^2(|a|+1)k^{n-1}r^n.$$

Therefore, by Lemma 4.2, we have that

$$\begin{aligned}
&\frac{1}{k^{n-1}r^n}\int_{Q^{\nu,k}_r(\frac{r}{\rho}x)} \Big| f^\infty(y,\nabla\hat u^{\rho,k}_r) - \frac{1}{t_{\rho,k}} f(y, t_{\rho,k}\nabla \hat u^{\rho,k}_r)\Big|\mathrm{d}y \qquad (5.64)\\
&\le \frac{K}{t_{\rho,k}}k^{n-1}r^n + \frac{K}{t^\alpha_{\rho,k}}(k^{n-1}r^n)^\alpha\Big(\int_{Q^{\nu,k}_r(\frac{r}{\rho}x)}|\nabla u_r|\mathrm{d}y\Big)^{1-\alpha} \le \frac{\hat K}{t^\alpha_{\rho,k}}k^{n-1}r^n
\end{aligned}$$

where $\hat K$ depends only on C, α and a. Collecting (5.62)-(5.64), we infer that

$$\limsup_{r\to+\infty} \frac{1}{k^{n-1}r^n t_{\rho,k}}\int_{Q^{\nu,k}_r(\frac{r}{\rho}x)} f(y,t_{\rho,k}\nabla\hat u^{\rho,k}_r)\mathrm{d}y \le f^\infty_{\rm hom}(a\otimes\nu) + \eta + \frac{\hat K}{t^\alpha_{\rho,k}}.$$

For $k \in \mathbb{N}$, $\rho \in (0,1)$ and $\varepsilon \in (0,\infty)$ we define the function $u^{\rho,k}_\varepsilon : \mathbb{R}^n \to \mathbb{R}^N$ given by

$$u^{\rho,k}_\varepsilon(y) := \begin{cases} \varepsilon t_{\rho,k}\hat u^{\rho,k}_r(\frac{y}{\varepsilon}) & \text{if } y \in Q^{\nu,k}_\rho(x)\\ t_{\rho,k}\ell_{a\otimes\nu} & \text{if } y \in \mathbb{R}^n\setminus Q^{\nu,k}_\rho(x),\end{cases}$$

where $r := \frac{\rho}{\varepsilon}$. Thus $u^{\rho,k}_\varepsilon \in W^{1,1}_{\rm loc}(\mathbb{R}^n,\mathbb{R}^N)$ with $u^{\rho,k}_\varepsilon = t_{\rho,k}\ell_{a\otimes\nu}$ on $\partial Q^{\nu,k}_\rho(x)$ and changing variable we obtain

$$\begin{aligned}
&\limsup_{\varepsilon\to0} \frac{1}{k^{n-1}\rho^n t_{\rho,k}}\int_{Q^{\nu,k}_{\rho(1+\eta)}(x)} f\left(\tfrac{x}{\varepsilon},\nabla u^{\rho,k}_\varepsilon\right)\mathrm{d}y\\
&\qquad\le f^\infty_{\rm hom}(a\otimes\nu) + \eta + \frac{\hat K}{t^\alpha_{\rho,k}} + C\left(\frac{1}{t_{\rho,k}} + |a|\right)((1+\eta)^n - 1),
\end{aligned} \tag{5.65}$$

since $u^{\rho,k}_\varepsilon$ coincides with $t_{\rho,k}\ell_{a\otimes\nu}$ on $Q^{\nu,k}_{\rho(1+\eta)}(x)\setminus Q^{\nu,k}_\rho(x)$. By (5.65) and Poincaré inequality, we can extract a subsequence (not relabelled) of $(\varepsilon_j)_{j\in\mathbb{N}}$, for every $\rho \in (0,1)$ and $k \in \mathbb{N}$, such that $u^{\rho,k}_{\varepsilon_j} \to u^{\rho,k}$ in $L^1_{\rm loc}(\mathbb{R}^n,\mathbb{R}^N)$, where $u^{\rho,k} \in BV(Q^{\nu,k}_{\rho(1+\eta)}(x),\mathbb{R}^N)$ with $u^{\rho,k} = t_{\rho,k}\ell_{a\otimes\nu}$ on $Q^{\nu,k}_{\rho(1+\eta)}(x)\setminus Q^{\nu,k}_\rho(x)$. As a consequence of the Γ-convergence stated in Theorem 5.2, of the superadditivity of the inferior limit operator, of ((f2)) and of estimate (5.65) we obtain

$$\begin{aligned}\frac{m_{\widehat{F}_q}(\ell_{t_{\rho,k}a\otimes\nu}, Q^{\nu,k}_{\rho(1+\eta)}(x))}{k^{n-1}\rho^n t_{\rho,k}} &\leq \frac{\widehat{F}_q(u^{\rho,k}, 1, Q^{\nu,k}_{\rho(1+\eta)}(x))}{k^{n-1}\rho^n t_{\rho,k}} \\ &\leq \liminf_{j\to+\infty}\Big(\frac{F_{\varepsilon_j}(u^{\rho,k}_{\varepsilon_j}, 1, Q^{\nu,k}_{\rho(1+\eta)}(x))}{k^{n-1}\rho^n t_{\rho,k}} + \frac{q}{k^{n-1}\rho^n t_{\rho,k}}\int_{Q^{\nu,k}_{\rho(1+\eta)}(x)}|\nabla u^{\rho,k}_{\varepsilon_j}|\mathrm{d}y\Big) \\ &\leq (1+qC)\Big(f^\infty_{\hom}(a\otimes\nu) + \eta + \frac{\hat{K}}{t^\alpha_{\rho,k}} + C\left(\frac{1}{t_{\rho,k}} + |a|\right)((1+\eta)^n - 1)\Big).\end{aligned}$$

We can pass to the limit in the last inequality for $\rho\to 0$ and then for $k\to+\infty$, using (5.60) and (5.61) we arrive to

$$(1+\eta)^n\Big(\frac{\mathrm{d}\widehat{F}(u,1,\cdot)}{\mathrm{d}|D^c u|}(x) + qa\otimes\nu\Big) \leq (1+qC)(f^\infty_{\hom}(a\otimes\nu) + \eta + C|a|((1+\eta)^n - 1))\,.$$

The claim follows by letting η and $q\to 0$.

Step 2: We claim that

$$\frac{\mathrm{d}\widehat{F}(u,1,\cdot)}{\mathrm{d}|D^c u|}(x) \geq f^\infty_{\hom}\Big(\frac{\mathrm{d}D^c u}{\mathrm{d}|D^c u|}\Big) \quad \text{for } |D^c u|\text{-a.e. } x\in A\,. \tag{5.66}$$

Let $A'\in\mathscr{A}(A)$, then by Theorem 5.2 we can find a sequence $(u_j,v_j)_{j\in\mathbb{N}}\in L^1_{\rm loc}(\mathbb{R}^n,\mathbb{R}^{N+1})$ such that $(u_j,v_j)\in W^{1,1}(A',\mathbb{R}^N)\times W^{1,2}(A',[0,1])$, $(u_j,v_j)\to(u,1)$ in $L^1_{\rm loc}(\mathbb{R}^n,\mathbb{R}^{N+1})$, $v_j\to 1$ for $\mathcal{L}^n$-a.e. $x\in A'$ as $j\to\infty$ and

$$\lim_{j\to\infty} F_{\varepsilon_j}(u_j,v_j,A') = \widehat{F}(u,1,A').$$

Fix $\delta\in(0,1)$; by Lemma 5.6 we have

$$H^\delta_{\varepsilon_j}(u^\delta_j, A') \leq F_{\varepsilon_j}(u_j,v_j,A') + C\mathcal{L}^n(\{v_j\leq\delta\}\cap A'),$$

where $u^\delta_j\in SBV(A',\mathbb{R}^N)$ with $u^\delta_j\to u$ in $L^1(A',\mathbb{R}^N)$ as $j\to\infty$, and therefore

$$\liminf_{j\to\infty} H^\delta_{\varepsilon_j}(u^\delta_j, A') \leq \widehat{F}(u,1,A'). \tag{5.67}$$

Define on A' the measures μ^δ_j given by

$$\mu^\delta_j := \alpha_\delta f\left(\tfrac{x}{\varepsilon_j}, \nabla u^\delta_j\right)\mathcal{L}^n\llcorner A' + \beta_\delta\mathcal{H}^{n-1}\llcorner(J_{u^\delta_j}\cap A')\,.$$

By definition of H^δ_ε and the compactness of Radon measures, there exists subsequence (not relabeled) of $(\varepsilon_j)_{j\in\mathbb{N}}$ and a finite Radon measure μ^δ on A' such that $\mu^\delta_j\to\mu^\delta$ weakly* in the sense of measures on A' as $j\to\infty$. For $|D^c u|$-a.e. $x\in A'$ (cf. [9, Proposition 3.92 and Theorem 3.94]) there exists $a(x)\in\mathbb{R}^N$ and $\nu(x)\in\mathbb{S}^{n-1}$ such that for every $k\in\mathbb{N}$ we have

$$\lim_{\rho\to 0}\frac{Du(Q^{\nu(x),k}_\rho(x))}{|Du|(Q^{\nu(x),k}_\rho(x))} = \frac{\mathrm{d}D^c u}{\mathrm{d}|D^c u|}(x) = a(x)\otimes\nu(x) \tag{5.68}$$

$$\lim_{\rho\to 0}\frac{|Du|(Q^{\nu(x),k}_\rho(x))}{\rho^n} = \infty \tag{5.69}$$

$$\lim_{\rho\to 0}\frac{|Du|(Q^{\nu(x),k}_\rho(x))}{\rho^{n-1}} = 0 \tag{5.70}$$

$$\frac{\mathrm{d}\mu^\delta}{\mathrm{d}|Du|}(x) = \frac{\mathrm{d}\mu^\delta}{\mathrm{d}|D^c u|}(x). \tag{5.71}$$

Fix now $x_0 \in A'$ such that (5.68)-(5.71) hold true and set $a := a(x_0)$ and $\nu := \nu(x_0)$. For $k \in \mathbb{N}$ and ρ set

$$t_{\rho,k} := \frac{|Du|(Q_\rho^{\nu,k}(x_0))}{k^{n-1}\rho^n},$$

therefore

$$t_{\rho,k} \to \infty \quad \text{and} \quad \rho t_{\rho,k} \to 0 \quad \text{as } \rho \to 0. \tag{5.72}$$

and

$$\frac{\mathrm{d}\mu^\delta}{\mathrm{d}|D^c u|}(x_0) = \lim_{\rho\to 0} \frac{\mu^\delta(Q_\rho^{\nu,k}(x_0))}{t_{\rho,k}k^{n-1}\rho^n}$$

By the weak*-convergence of $(\mu^\delta_{\varepsilon_j})_{j\in\mathbb{N}}$ to μ^δ we infer that

$$\frac{\mathrm{d}\mu^\delta}{\mathrm{d}|D^c u|}(x_0) = \lim_{\rho\to 0,\rho\in I(x_0)} \lim_{j\to\infty} \frac{\mu_j^\delta(Q_\rho^{\nu,k}(x_0))}{t_{\rho,k}k^{n-1}\rho^n}, \tag{5.73}$$

where $I(x_0) := \{\rho \in (0,1) : \mu^\delta(\partial Q_\rho^{\nu,k}(x_0)) = 0 \text{ for every } k \in \mathbb{N} \text{ s.t. } Q_\rho^{\nu,k}(x_0) \subset\subset A'\}$. Note that $I(x_0)$ has full measure in $(0,1)$. Fix $k \in \mathbb{N}$ and consider the rescaled functions $u_j^{\rho,k}$, $u^{\rho,k} : Q^{\nu,k} \to \mathbb{R}^N$

$$u_j^{\rho,k}(y) = \frac{1}{k^{n-1}\rho t_{\rho_i,k}}\Big(u_j^\delta(x_0+\rho_i y) - \frac{1}{k^{n-1}\rho_i^n}\int_{Q_\rho^{\nu,k}(x_0)} u_j^\delta(z)\mathrm{d}z\Big)$$

$$u^{\rho,k}(y) = \frac{1}{k^{n-1}\rho t_{\rho,k}}\Big(u(x_0+\rho y) - \frac{1}{k^{n-1}\rho^n}\int_{Q_\rho^{\nu,k}(x_0)} u(z)\mathrm{d}z\Big).$$

From now on we work at $k \in \mathbb{N}$ fixed and this will tend to ∞ only at the very end of the proof. Therefore, for those parameters infinitesimal as $j \to \infty$ and $\rho \to 0$ the possible dependence on k will not be highlighted. For every ρ small enough, depending on k, $u_j^{\rho,k} \in SBV(Q_1^{\nu,k},\mathbb{R}^N)$, $u^{\rho,k} \in BV(Q_1^{\nu,k},\mathbb{R}^N)$, $u_j^{\rho,k} \to u^{\rho,k}$ in $L^1(Q_1^{\nu,k},\mathbb{R}^N)$ as $j \to \infty$, and the function $u^{\rho,k}$ satisfies the following

$$\int_{Q^{\nu,k}} u^{\rho,k}(y)\mathrm{d}y = 0 \quad \text{with} \quad Du^{\rho,k}(Q_1^{\nu,k}) = \frac{Du(Q_\rho^{\nu,k}(x_0))}{|Du|(Q_\rho^{\nu,k}(x_0))} \to a\otimes\nu \quad \text{as } \rho \to 0.$$

Moreover, from (5.73) we obtain

$$\begin{aligned}
\frac{\mathrm{d}\mu^\delta}{\mathrm{d}|D^c u|}(x_0) &= \lim_{\rho\to 0,\rho\in I(x_0)} \lim_{j\to\infty} \frac{\mu_j^\delta(Q_\rho^{\nu,k}(x_0))}{k^{n-1}\rho^n t_{\rho,k}} \\
&= \lim_{\rho\to 0,\rho\in I(x_0)} \lim_{j\to\infty} \Big(\frac{\alpha_\delta}{k^{n-1}\rho^n t_{\rho,k}}\int_{Q_\rho^{\nu,k}(x_0)} f(\tfrac{x}{\varepsilon_j},\nabla u_j^\delta)\mathrm{d}x + \frac{\beta_\delta}{k^{n-1}\rho^n t_{\rho,k}}\mathcal{H}^{n-1}(J_{u_j^\delta}\cap Q_\rho^{\nu,k}(x_0))\Big) = \\
&= \lim_{\rho\to 0,\rho\in I(x_0)} \lim_{j\to\infty} \Big(\frac{\alpha_\delta}{k^{n-1}t_{\rho,k}}\int_{Q_1^{\nu,k}} f(\tfrac{x_0+\rho y}{\varepsilon_j}, k^{n-1}t_{\rho,k}\nabla u_j^{\rho,k})\mathrm{d}x + \frac{\beta_\delta}{k^{n-1}\rho t_{\rho,k}}\mathcal{H}^{n-1}(J_{u_j^{\rho,k}}\cap Q_1^{\nu,k})\Big).
\end{aligned} \tag{5.74}$$

By [8, Theorem 2.3] and [35, Lemma 5.1] there exists a subsequence (not relabeled), depending on k, such that $u^{\rho,k} \to u^k$ in $L^1(Q_1^{\nu,k},\mathbb{R}^N)$ as $\rho \to 0$, where $u^k \in BV(Q_1^{\nu,k},\mathbb{R}^N)$, $u^k(y) = \chi_k(y\cdot\nu)a$ for every $y \in Q_1^{\nu,k}$, $\chi_k : [-1/2,1/2] \to \mathbb{R}$ is a nondecreasing function such that $D\chi_k((-1/2,1/2)) = \chi_k(1/2) - \chi_k(-1/2) = \frac{1}{k^{n-1}}$, $-\frac{1}{k^{n-1}} \le \chi_k(-1/2) \le 0 \le \chi_k(1/2) \le \frac{1}{k^{n-1}}$, and

$$\int_{-1/2}^{1/2} \chi_k(t)\mathrm{d}t = 0.$$

Furthermore, being χ_k continuous in $-1/2$ and $1/2$, thanks to the trace's properties of BV functions, we have that the inner trace of u^k satisfies $u^k = \ell^k$ on $\partial^\perp Q_1^{\nu,k}$ (cf. (f) of the notation list) where

$$\ell^k(y) := \frac{1}{k^{n-1}}\ell_{a\otimes\nu}(y) + \Big(\chi_k(1/2) - \frac{1}{2k^{n-1}}\Big)a.$$

To obtain a uniform L^∞-bound on the scaled sequence, we let $M \in \mathbb{N}$ and use Lemma 5.3 with $v \equiv 1$ to get for every $k \in \mathbb{N}$, every ρ small enough and every j (up to a subsequence), $m_\rho \in \{M+1, \dots, 2M\}$ such that

$$\Big(1 + \frac{C^2}{M}\Big)\frac{\mathrm{d}\mu^\delta}{\mathrm{d}|D^c u|}(x_0) \geq \limsup_{\rho\to 0,\ \rho\in I(x_0)} \limsup_{j\to\infty} \frac{\alpha_\delta}{k^{n-1}t_{\rho,k}} \int_{Q_1^{\nu,k}} f\left(\tfrac{x_0+\rho y}{\varepsilon_j}, k^{n-1}t_{\rho,k}\nabla \hat{u}_j^{\rho,k}\right)\mathrm{d}y, \quad (5.75)$$

and

$$\mathcal{H}^{n-1}(J_{\hat{u}_j^{\rho,k}} \cap Q_1^{\nu,k}) + \|\hat{u}_j^{\rho,k} - u^k\|_{L^1(Q_1^{\nu,k})} \leq \mathcal{H}^{n-1}(J_{u_j^\rho} \cap Q_1^{\nu,k}) + \|u_j^{\rho,k} - u^k\|_{L^1(Q_1^{\nu,k})}$$

where $\hat{u}_j^{\rho,k} := \mathcal{T}_{m_\rho}(u_j^{\rho,k}) \in SBV(Q_1^{\nu,k}, \mathbb{R}^N)$. Therefore, choosing M such that $a_M > |a|$, it follows from (5.72) and (5.74) that

$$\lim_{\rho\to 0, \rho\in I(x_0)} \lim_{j\to\infty} (\mathcal{H}^{n-1}(J_{\hat{u}_j^{\rho,k}} \cap Q_1^{\nu,k}) + \|\hat{u}_j^{\rho,k} - u^k\|_{L^1(Q_1^{\nu,k})}) = 0\,. \quad (5.76)$$

Next we change the boundary datum $\hat{u}_j^{\rho,k}$ on a neighborhood of $\partial^\perp Q^{\nu,k}$ with u^k. For every ρ small enough and every j large enough (depending on ρ), we have that

$$\|\hat{u}_j^{\rho,k} - u^k\|_{L^1(Q_1^{\nu,k})} + \frac{\rho}{j} =: \tau_{\rho,j} \in (0,1)$$

and thanks to Fubini's Theorem and the trace properties of BV functions on rectifiable sets, there exists $q_{\rho,j} \in (1/2 - \tau_{\rho,j}^{1/2}, 1/2)$

$$\int_{\partial^\perp R_\nu(B^k_{\rho,j})} |(\hat{u}_j^{\rho,k})^- - u^k|\mathrm{d}\mathcal{H}^{n-1} \leq \frac{\|\hat{u}_j^\rho - \ell_\xi\|_{L^1(Q_1^{\nu,k})}}{\tau_{\rho,j}^{1/2}} \leq \tau_{\rho,j}^{1/2}\,, \quad (5.77)$$

where $(\hat{u}_j^{\rho,k})^-$ is the inner trace of $\hat{u}_j^{\rho,k}$ on $R_\nu(B^k_{\rho,j})$, where $B^k_{\rho,j} := (-\frac{k}{2}, \frac{k}{2})^{n-1} \times (-q_{\rho,j}, q_{\rho,j})$ and R_ν is the rotation in (e) of the notation list. Defining the functions $w_j^{\rho,k} \in BV(Q_1^{\nu,k}, \mathbb{R}^N)$ as

$$w_j^{\rho,k}(y) = \begin{cases} \hat{u}_j^{\rho,k}(y) & \text{if } y \in R_\nu(B^k_{\rho,j}) \\ u^k & \text{if } y \in Q_1^{\nu,k} \setminus R_\nu(B^k_{\rho,j}), \end{cases}$$

we have that

$$\begin{aligned}&\frac{\alpha_\delta}{k^{n-1}t_{\rho,k}} \int_{Q_1^{\nu,k}} f(\tfrac{x_0+\rho y}{\varepsilon_j}, k^{n-1}t_{\rho,k}\nabla \hat{u}_j^{\rho,k})\mathrm{d}y + \alpha_\delta C\|\nabla u^k\|_{L^1(Q_1^{\nu,k}\setminus R_\nu(B^k_{\rho,j}))} + \\ &+ \frac{\alpha_\delta}{k^{n-1}t_{\rho,k}}\mathcal{L}^n(Q_1^{\nu,k} \setminus R_\nu(B^k_{\rho,j})) \geq \frac{\alpha_\delta}{k^{n-1}t_{\rho,k}} \int_{Q_1^{\nu,k}} f(\tfrac{x_0+\rho y}{\varepsilon_j}, k^{n-1}t_{\rho,k}\nabla w_j^{\rho,k})\mathrm{d}y.\end{aligned}$$

Since $\lim_{\rho\to 0}\lim_{j\to\infty} \mathcal{L}^n(Q_1^{\nu,k} \setminus R_\nu(B^k_{\rho,j})) = 0$, we get from (5.75)

$$\Big(1 + \frac{C^2}{M}\Big)\frac{\mathrm{d}\mu^\delta}{\mathrm{d}|D^c u|}(x_0) \geq \limsup_{\rho\to 0,\ \rho\in I(x_0)} \limsup_{j\to\infty} \frac{\alpha_\delta}{k^{n-1}t_{\rho,k}} \int_{Q_1^{\nu,k}} f(\tfrac{x_0+\rho y}{\varepsilon_j}, k^{n-1}t_{\rho,k}\nabla w_j^{\rho,k})\mathrm{d}y\,. \quad (5.78)$$

In particular, we infer that

$$\Big(1 + \frac{C^2}{M}\Big)\frac{\mathrm{d}\mu^\delta}{\mathrm{d}|D^c u|}(x_0) \geq C\alpha_\delta \limsup_{\rho\to 0,\ \rho\in I(x_0)} \limsup_{j\to\infty} \int_{Q_1^{\nu,k}} |\nabla w_j^{\rho,k}|\mathrm{d}y\,, \quad (5.79)$$

and thus by (5.77) we conclude

$$|D^s w_j^{\rho,k}|(Q_1^{\nu,k}) \le |D^s \hat{u}_j^{\rho,k}|(R_\nu(B^k_{\rho,j})) + |D^s u^k|(Q_1^{\nu,k} \setminus R_\nu(B^k_{\rho,j})) + \tau_{\rho,j}^{\frac{1}{2}}$$
$$\le 2a_{2M+1}\mathcal{H}^{n-1}(J_{\hat{u}_j^{\rho,k}} \cap Q_1^{\nu,k}) + |D^s u^k|(Q^{\nu,k} \setminus R_\nu(B^k_{\rho,j})) + \tau_{\rho,j}^{\frac{1}{2}}$$

and therefore

$$\lim_{\rho\to 0, \rho\in I(x_0)} \lim_{j\to\infty} |D^s w_j^{\rho,k}|(Q_1^{\nu,k}) = 0. \tag{5.80}$$

In addition, thanks to Lemma 4.2 and (5.78) we deduce that

$$\Big(1+\frac{C^2}{M}\Big)\frac{\mathrm{d}\mu^\delta}{\mathrm{d}|D^c u|}(x_0) \ge \limsup_{\rho\to 0,\ \rho\in I(x_0)} \limsup_{j\to\infty} \alpha_\delta \int_{Q_1^{\nu,k}} f^\infty(\tfrac{x_0+\rho y}{\varepsilon_j}, \nabla w_j^{\rho,k})\mathrm{d}y. \tag{5.81}$$

We now change the boundary datum $w_j^{\rho,k}$ on a neighborhood of $\partial^{\|} Q_1^{\nu,k}$ with ℓ_k. Let $h_k \in (0,k)$ be such that $\mathcal{L}^n(Q_1^{\nu,k} \setminus Q_1^{\nu,h_k}) \to 0$ as $k\to\infty$, necessarily $h_k \to \infty$. Then, by Fubini's Theorem there exists $\lambda^k_{\rho,j} \in (k-h_k, k)$ such that

$$\int_{\partial^{\|} Q_1^{\nu,\lambda^k_{\rho,j}}} |(w_j^{\rho,k})^- - \ell_k|\mathcal{H}^{n-1} \le \frac{2}{h_k}\int_{Q_1^{\nu,k}} |w_j^{\rho,k} - \ell_k|\mathrm{d}y, \tag{5.82}$$

where $(w_j^{\rho,k})^-$ is the inner trace of $w_j^{\rho,k}$ on $\partial^{\|} Q_1^{\nu,\lambda^k_{\rho,j}}$. Furthermore, since $w_j^{\rho,k} = u^k = \ell_k$ on $\partial^\perp Q_1^{\nu,k}$, using Poincarè inequality on the one-dimensional restrictions of $w_j^{\rho,k}$ in the ν direction, we obtain that

$$\int_{Q_1^{\nu,k}} |w_j^{\rho,k} - \ell_k|\mathrm{d}y \le 2\Big|Dw_j^{\rho,k} - \frac{a\otimes\nu}{k^{n-1}}\Big|(Q_1^{\nu,k}) \le 2|Dw_j^{\rho,k}|(Q_1^{\nu,k}) + 2|a|\,.$$

Therefore, by (5.82) we infer that

$$\int_{\partial^{\|} Q_1^{\nu,\lambda^k_{\rho,j}}} |(w_j^{\rho,k})^- - \ell_k|\mathcal{H}^{n-1} \le \frac{4}{h_k}|Dw_j^{\rho,k}|(Q_1^{\nu,k}) + \frac{4}{h_k}|a|. \tag{5.83}$$

Defining $\hat{w}_j^{\rho,k}$ as

$$\hat{w}_j^{\rho,k}(y) = \begin{cases} w_j^{\rho,k} & \text{if } y \in Q_1^{\nu,\lambda^k_{\rho,j}} \\ \ell_k(y) & \text{if } y \in Q_1^{\nu,k} \setminus Q_1^{\nu,\lambda^k_{\rho,j}} \end{cases}$$

we have that $\hat{w}_j^{\rho,k} \in BV(Q_1^{\nu,k},\mathbb{R}^N)$, $\hat{w}_j^{\rho,k} = \ell_k$ on $\partial Q_1^{\nu,k}$, and by (5.83)

$$|D^s \hat{w}_j^{\rho,k}|(Q_1^{\nu,k}) \le |D^s w_j^{\rho,k}|(Q_1^{\nu,k}) + \frac{4}{h_k}|Dw_j^{\rho,k}|(Q_1^{\nu,k}) + \frac{4}{h_k}|a|$$
$$\le 2|D^s w_j^{\rho,k}|(Q_1^{\nu,k}) + \frac{4}{h_k}\|\nabla w_j^{\rho,k}\|_{L^1(Q_1^{\nu,k})} + \frac{4}{h_k}|a|.$$

In particular, by (5.80), we obtain

$$\limsup_{\rho\to 0, \rho\in I(x_0)} \limsup_{j\to\infty} |D^s \hat{w}_j^{\rho,k}|(Q_1^{\nu,k}) = \frac{4}{h_k}\limsup_{\rho\to 0, \rho\in I(x_0)} \limsup_{j\to\infty} \|\nabla w_j^{\rho,k}\|_{L^1(Q_1^{\nu,k})} + \frac{4|a|}{h_k} \tag{5.84}$$

and, from (5.81),

$$\Big(1+\frac{C^2}{M}\Big)\frac{\mathrm{d}\mu^\delta}{\mathrm{d}\mathcal{L}^n}(x_0) + \frac{C|a|\mathcal{L}^n(Q_1^{\nu,k} \setminus Q_1^{\nu,h_k})}{k^{n-1}}$$
$$\ge \limsup_{\rho\to 0,\ \rho\in I(x_0)} \limsup_{j\to\infty} \alpha_\delta \int_{Q_1^{\nu,k}} f^\infty(\tfrac{x_0+\rho y}{\varepsilon_j}, \nabla \hat{w}_j^{\rho,k})\mathrm{d}y. \tag{5.85}$$

We now argue as in Proposition 5.4 (cf. (5.37), (5.38)), and for every ρ and j fixed we use Lemma 5.5 to infer the existence of $\overline{w}_j^{\rho,k} \in W^{1,1}(Q_1^{\nu,k}, \mathbb{R}^N)$ such that $\overline{w}_j^{\rho,k} = \ell_k$ on $\partial Q_1^{\nu,k}$ and

$$\begin{aligned}\int_{Q_1^{\nu,k}} f^\infty(\tfrac{x_0+\rho y}{\varepsilon_j}, \nabla \hat{w}_j^{\rho,k})\mathrm{d}y &\geq \int_{Q_1^{\nu,k}} \widehat{f^\infty}(\tfrac{x_0+\rho y}{\varepsilon_j}, \nabla \hat{w}_j^{\rho,k})\mathrm{d}y \\ &\geq sc^-(L^1)F^\infty_{\rho,j}(\hat{w}_j^{\rho,k}) - C|D^s\hat{w}_j^{\rho,k}|(Q_1^{\nu,k}) \\ &\geq \int_{Q_1^{\nu,k}} \widehat{f^\infty}(\tfrac{x_0+\rho y}{\varepsilon_j}, \nabla \overline{w}_j^{\rho,k})\mathrm{d}y - C|D^s\hat{w}_j^{\rho,k}|(Q_1^{\nu,k}) - \frac{\rho}{j}\,.\end{aligned}$$

Therefore, from (5.79), (5.84), (5.85) we conclude form the last inequality above that

$$\begin{aligned}\Big(1+\frac{C^2}{M}-\frac{4}{h_k}\Big)\frac{\mathrm{d}\mu^\delta}{\mathrm{d}\mathcal{L}^n}(x_0) + \frac{C|a|\mathcal{L}^n(Q_1^{\nu,k}\setminus Q_1^{\nu,h_k})}{k^{n-1}} + C\alpha_\delta\frac{4|a|}{h_k} \\ \geq \limsup_{\rho\to 0,\ \rho\in I(x_0)}\ \limsup_{j\to\infty} \alpha_\delta \int_{Q^{\nu,k}} \widehat{f^\infty}(\tfrac{x_0+\rho y}{\varepsilon_j}, \nabla\overline{w}_j^{\rho,k})\mathrm{d}y\,. \qquad (5.86)\end{aligned}$$

Set $r_{\rho,j} := \frac{\rho}{\varepsilon_j}$ and $\tilde{w}_j^{\rho,k}(x) := r_{\rho,j}\overline{w}_j^{\rho,k}\left(\frac{x}{r_{\rho,j}} - \frac{x_0}{\rho}\right)$, we have that $\tilde{w}_j^{\rho,k} \in W^{1,1}(Q^{\nu,k}_{r_{\rho,j}}\left(\frac{r_{\rho,j}}{\rho}x_0\right), \mathbb{R}^N)$ with $\tilde{w}_j^{\rho,k} = \ell_k - \frac{x_0}{\varepsilon_j}$ on $\partial Q_{r_{\rho,j}}\left(\frac{r_{\rho,j}}{\rho}x_0\right)$ and, thanks to the 1-homogeneity of $\widehat{f^\infty}$ (cf. item (i) in Lemma 5.5), we infer that

$$\int_{Q^{\nu,k}} \widehat{f^\infty}\left(\frac{x_0+\rho y}{\varepsilon_j}, \nabla\overline{w}_j^{\rho,k}\right)\mathrm{d}y = \frac{1}{k^{n-1}r^n_{\rho,j}}\int_{Q^{\nu,k}_{r_{\rho,j}}\left(\frac{r_{\rho,j}}{\rho}x_0\right)} \widehat{f^\infty}(y, k^{n-1}\nabla\tilde{w}_j^{\rho,k})\mathrm{d}y.$$

In particular, since $k^{n-1}\tilde{w}_j^{\rho,k} = \ell_{a\otimes\nu} - \frac{k^{n-1}}{\varepsilon_j}x_0$ on $\partial Q^{\nu,k}_{r_{\rho,j}}\left(\frac{r_{\rho,j}}{\rho}x_0\right)$, thanks to (5.86) we obtain

$$\begin{aligned}\Big(1+\frac{C^2}{M}-\frac{4}{h_k}\Big)&\frac{\mathrm{d}\mu^\delta}{\mathrm{d}\mathcal{L}^n}(x_0) + \frac{C|a|\mathcal{L}^n(Q_1^{\nu,k}\setminus Q_1^{\nu,h_k})}{k^{n-1}} + C\alpha_\delta\frac{4|a|}{h_k} \\ &\geq \limsup_{\rho\to 0,\ \rho\in I(x_0)}\ \limsup_{j\to\infty} \frac{\alpha_\delta}{k^{n-1}r^n_{\rho,j}}\int_{Q^{\nu,k}_{r_{\rho,j}}\left(\frac{r_{\rho,j}}{\rho}x_0\right)} \widehat{f^\infty}(y, k^{n-1}\nabla\tilde{w}_j^{\rho,k})\mathrm{d}y \\ &\geq \limsup_{\rho\to 0,\ \rho\in I(x_0)}\ \limsup_{j\to\infty} \alpha_\delta \frac{m_{\mathrm{b}}^{\widehat{f^\infty}}\left(\ell_{a\otimes\nu}, Q^{\nu,k}_{r_{\rho,j}}\left(\frac{r_{\rho,j}}{\rho}x_0\right)\right)}{k^{n-1}r^n_{\rho,j}} \\ &= \limsup_{\rho\to 0,\ \rho\in I(x_0)}\ \limsup_{j\to\infty} \alpha_\delta \frac{m_{\mathrm{b}}^{f^\infty}(\ell_{a\otimes\nu}, Q^{\nu,k}_{r_{\rho,j}}\left(\frac{r_{\rho,j}}{\rho}x_0\right))}{k^{n-1}r^n_{\rho,j}} = \alpha_\delta f^\infty_{\mathrm{hom}}(a\otimes\nu)\,,\end{aligned}$$

where the last-but-one equality follows from Lemma 5.5 (vi), and the last equality follow from Proposition 4.3. Then, taking $k\to\infty$ and $M\to\infty$ in this order, we infer that

$$\frac{\mathrm{d}\mu^\delta}{\mathrm{d}|D^c u|}(x_0) \geq \alpha_\delta f^\infty_{\mathrm{hom}}(a\otimes\nu) = \alpha_\delta f^\infty_{\mathrm{hom}}\Big(\frac{\mathrm{d}D^c u}{\mathrm{d}|D^c u|}(x_0)\Big).$$

Therefore, using (5.67) we conclude that

$$\widehat{F}(u,1,A') \geq \liminf_{j\to\infty} H^\delta_{\varepsilon_j}(u_j^\delta, A') = \liminf_{j\to\infty}\mu_j^\delta(A') \geq \mu^\delta(A') \geq \alpha_\delta\int_{A'} f^\infty_{\mathrm{hom}}(\tfrac{\mathrm{d}D^c u}{\mathrm{d}|D^c u|}(x))\mathrm{d}|D^c u|,$$

thus (5.66) follows by letting $\delta\to 0$ recalling that $\alpha_\delta\to 1$ and by the arbitrariness of $A' \in \mathscr{A}(A)$. □

Finally, we are in a position to prove the deterministic homogenisation result Theorem 5.1.

Proof of Theorem 5.1. Theorem 5.2 implies that from any strictly positive infinitesimal sequence we can extract a subsequence (ε_j) such that

$$\Gamma(L^1_{\rm loc}(\mathbb{R}^n,\mathbb{R}^{N+1}))\text{-}\lim_{j\to\infty} F_{\varepsilon_j}(u,1,A) = \widehat{F}(u,1,A)$$

with $\widehat{F} : L^1_{\rm loc}(\mathbb{R}^n,\mathbb{R}^{N+1}) \times \mathscr{A} \longrightarrow [0,+\infty]$. Moreover, $\widehat{F}(u,1,\cdot)$ is the restriction to open sets of a finite Radon measure on A and $\widehat{F}(u,1,A) \le C(|Du|(A) + \mathcal{L}^n(A))$ for every $A \in \mathscr{A}$ and every $u \in L^1_{\rm loc}(\mathbb{R}^n,\mathbb{R}^N)$ such that $u \in BV(A,\mathbb{R}^N)$. Therefore, $\widehat{F}(u,1,\cdot)$ is absolutely continuous respect to the measure $\mathcal{L}^n \llcorner A + |D^c u| \llcorner A + \mathcal{H}^{n-1} \llcorner J_u \cap A$. Since $\mathcal{L}^n \llcorner A, |D^c u| \llcorner A, \mathcal{H}^{n-1} \llcorner J_u \cap A$ are mutually singular, by the properties of Radon-Nikodym derivatives, for every $B \in \mathscr{A}(A)$ we have that

$$\widehat{F}(u,1,B) = \int_B \frac{\mathrm{d}\widehat{F}(u,1,\cdot)}{\mathrm{d}\mathcal{L}^n}\mathrm{d}x + \int_B \frac{\mathrm{d}\widehat{F}(u,1,\cdot)}{\mathrm{d}|D^c u|}\mathrm{d}|D^c u| + \int_{J_u\cap B} \frac{\mathrm{d}\widehat{F}}{\mathrm{d}\mathcal{H}^{n-1}\llcorner J_u}\mathrm{d}\mathcal{H}^{n-1}\,.$$

In particular, if $A \in \mathscr{A}$ and $u \in L^1_{\rm loc}(\mathbb{R}^n,\mathbb{R}^N)$ with $u \in BV(A,\mathbb{R}^N) \cap L^\infty(A,\mathbb{R}^N)$, Propositions 5.4, 5.7 and 5.9 give that

$$\Gamma(L^1_{\rm loc}(\mathbb{R}^n,\mathbb{R}^{N+1}))\text{-}\lim_{j\to\infty} F_{\varepsilon_j}(u,1,A) = \widehat{F}(u,1,A) = F_{\rm hom}(u,1,A),$$

where $F_{\rm hom}$ is as in (5.4). From (f2), for every $A \in \mathscr{A}$ and every $(u,v) \in L^1_{\rm loc}(\mathbb{R}^n,\mathbb{R}^{N+1})$ with $(u,v) \in W^{1,1}(A,\mathbb{R}^N) \times W^{1,2}(A,[0,1])$ we have that

$$\int_A \Big(C^{-1}v^2|\nabla u| + \frac{(1-v)^2}{\varepsilon_j} + \varepsilon_j|\nabla v|^2\Big)\mathrm{d}x \le F_{\varepsilon_j}(u,v,A),$$

hence, by [6, Theorem 4.1] and [7, Remark 3.5], for every $(u,v) \in L^1_{\rm loc}(\mathbb{R}^n,\mathbb{R}^{N+1})$ such that $u \notin GBV(A,\mathbb{R}^N)$ or $v \neq 1$ on A we get

$$\Gamma(L^1_{\rm loc}(\mathbb{R}^n,\mathbb{R}^{N+1}))\text{-}\liminf_{j\to\infty} F_{\varepsilon_j}(u,v,A) = F_{\rm hom}(u,v,A) = +\infty\,.$$

Eventually, arguing exactly as in [7, Section 6] we obtain

$$\Gamma(L^1_{\rm loc}(\mathbb{R}^n,\mathbb{R}^{N+1}))\text{-}\lim_{j\to\infty} F_{\varepsilon_j}(u,1,A) = F_{\rm hom}(u,1,A)$$

for every $A \in \mathscr{A}$ and every $u \in L^1_{\rm loc}(\mathbb{R}^n,\mathbb{R}^N)$ such that $u \in GBV(A,\mathbb{R}^N)$. Indeed, the lower bound inequality for general GBV maps follows easily from Lemma 5.3 and the result in the $BV \cap L^\infty$-setting. Instead, the upper bound inequality is a consequence of the latter together with both the $L^1_{\rm loc}(\mathbb{R}^n,\mathbb{R}^{N+1})$ lower semicontinuity of $\Gamma\text{-}\limsup_{j\to\infty} F_{\varepsilon_j}(\cdot,1,A)$ and the continuity of $F_{\rm hom}$ along sequences of maps obtained via the smooth truncations $(\mathcal{T}_k)_{k\in\mathbb{N}}$, namely $F_{\rm hom}(\mathcal{T}_k(u),1,A) \to F_{\rm hom}(u,1,A)$ as $k \to \infty$ for every $u \in GBV(A,\mathbb{R}^N)$ (cf. [7, Lemma 6.1]).

Since the Γ-limit does not depend on the extracted subsequence Uryshon's property of Γ-convergence yields the claim. □

6. Stochastic homogenisation

This section is devoted to the proof of the stochastic homogenisation result stated in Theorem 3.4. The proof will be achieved by showing that if f is a stationary random integrand in the sense of Definition 2.7, then the assumptions of Theorem 5.1 are satisfied for P-a.e. $\omega \in \Omega$. Here a pivotal role is played by the Subadditive Ergodic Theorem, Theorem 2.3.

The following proposition establishes the existence and spatial homogeneity of $f_{\rm hom}$. The proof can be found in [24, Proposition 9.1] and in [41, Lemma 4.1].

Proposition 6.1 (Homogenized random volume integrand)**.** *Let f be a stationary random integrand. Then there exist $\Omega' \in \mathcal{T}$, with $P(\Omega') = 1$ and a $\mathcal{T} \otimes \mathscr{B}^{N\times n}$-measurable function $f_{\rm hom} : \Omega \times \mathbb{R}^{N\times n} \to [0, +\infty)$ such that for every $\omega \in \Omega'$, $x \in \mathbb{R}^n$, $\xi \in \mathbb{R}^{N\times n}$, $\nu \in \mathbb{S}^{n-1}$ and $k \in \mathbb{N}$*

$$f_{\rm hom}(\omega, \xi) = \lim_{r\to+\infty} \frac{m_{\rm b}^{f_\omega}(\ell_\xi, Q_r^{\nu,k}(rx))}{k^{n-1}r^n} = \lim_{r\to+\infty} \frac{m_{\rm b}^{f_\omega}(\ell_\xi, Q_r)}{r^n}.$$

If in addition f is ergodic, then $f_{\rm hom}$ is independent of ω and

$$f_{\rm hom}(\xi) = \lim_{r\to+\infty} \frac{1}{r^n} \int_\Omega m_{\rm b}^{f_\omega}(\ell_\xi, Q_r) \mathrm{d}P(\omega).$$

Propositions 4.3 and 6.1 readily imply the following result.

Proposition 6.2 (Homogenized random Cantor integrand)**.** *Let f be a stationary random integrand. Then there exist $\Omega' \in \mathcal{T}$, with $P(\Omega') = 1$ and a $\mathcal{T} \otimes \mathscr{B}^{N\times n}$-measurable function $f^\infty_{\rm hom} : \Omega \times \mathbb{R}^{N\times n} \to [0, +\infty)$ such that for every $\omega \in \Omega'$, every $\xi \in \mathbb{R}^{N\times n}$ every $k \in \mathbb{N}$, every $x \in \mathbb{R}^n$ and every $\nu \in \mathbb{S}^{n-1}$*

$$f^\infty_{\rm hom}(\omega, \xi) = \lim_{t\to+\infty} \frac{f_{\rm hom}(\omega, t\xi)}{t}$$

and

$$f^\infty_{\rm hom}(\omega, \xi) = \lim_{r\to+\infty} \frac{m_{\rm b}^{f^\infty_\omega}(\ell_\xi, Q_r^{\nu,k}(rx))}{k^{n-1}r^n} = \lim_{r\to+\infty} \frac{m_{\rm b}^{f^\infty_\omega}(\ell_\xi, Q_r)}{r^n}.$$

If in addition f is ergodic, then $f^\infty_{\rm hom}$ is independent of ω and

$$f^\infty_{\rm hom}(\xi) = \lim_{r\to+\infty} \frac{1}{r^n} \int_\Omega m_{\rm b}^{f^\infty_\omega}(\ell_\xi, Q_r) \mathrm{d}P(\omega). \tag{6.1}$$

The analogous result for the surface integrand is more involved and requires a new proof.

Proposition 6.3 (Homogenized random surface integrand)**.** *Let f be a stationary random integrand. Then there exist $\Omega' \in \mathcal{T}$, with $P(\Omega') = 1$ and a $\mathcal{T} \otimes \mathscr{B}^N \otimes \mathscr{B}^n_S$-measurable function $g_{\rm hom} : \Omega \times \mathbb{R}^N \times \mathbb{S}^{n-1} \to [0, +\infty)$ such that for every $\omega \in \Omega'$, $x \in \mathbb{R}^n$, $\zeta \in \mathbb{R}^N$ and $\nu \in \mathbb{S}^{n-1}$*

$$g_{\rm hom}(\omega, \zeta, \nu) = \lim_{r\to+\infty} \frac{m_{\rm s}^{f^\infty_\omega}(u_{rx,\zeta,\nu}, Q^\nu_r(rx))}{r^{n-1}} = \lim_{r\to+\infty} \frac{m_{\rm s}^{f^\infty_\omega}(u_{\zeta,\nu}, Q^\nu_r)}{r^{n-1}}.$$

If in addition f is ergodic, then $g_{\rm hom}$ is independent of ω and

$$g_{\rm hom}(\zeta, \nu) = \lim_{r\to+\infty} \frac{1}{r^{n-1}} \int_\Omega m_{\rm s}^{f^\infty_\omega}(u_{\zeta,\nu}, Q^\nu_r) \mathrm{d}P(\omega). \tag{6.2}$$

Proof. We divide the proof into a number of steps.

Step 1: Let $\overline{u}_{\zeta,\nu}$ be as in (l) of the notation list. In this step we prove that for every $\zeta \in \mathbb{Q}^N$ and $\nu \in \mathbb{S}^{n-1} \cap \mathbb{Q}^n$ and for P-a.e. $\omega \in \Omega$ there exists the limit

$$\lim_{r\to+\infty} \frac{m_{\rm s}^{f^\infty_\omega}(\overline{u}_{\zeta,\nu}, Q^\nu_r)}{r^{n-1}}$$

and defines an x-independent random variable.

To prove the claim let $\nu \in \mathbb{S}^{n-1} \cap \mathbb{Q}^n$ and $\zeta \in \mathbb{Q}^N$ be fixed, $R_\nu \in O(n) \cap \mathbb{Q}^{n\times n}$ be the orthogonal matrix as in (e) of the notation list, and M_ν be a positive integer such that $M_\nu R_\nu \in \mathbb{Z}^{n\times n}$, so that $M_\nu R_\nu(z', 0) \in \Pi^\nu_0 \cap \mathbb{Z}^n$. Given $A' = [a_1, b_1) \times \cdots \times [a_{n-1}, b_{n-1}) \in \mathcal{I}_{n-1}$ we define the n-dimensional interval $T_\nu(A')$ as

$$T_\nu(A') := M_\nu R_\nu(A' \times [-c, c)), \quad \text{with } c := \frac{1}{2} \max_{1\le j\le n-1} (b_j - a_j). \tag{6.3}$$

For every $\omega \in \Omega$ and every $A' \in \mathcal{I}_{n-1}$ we set

$$\mu_{\zeta,\nu}(\omega, A') := \frac{1}{M_\nu^{n-1}} m_{\rm s}^{f_\omega^\infty}(\overline{u}_{\zeta,\nu}, T_\nu(A')). \tag{6.4}$$

We now show that $\mu_{\zeta,\nu} : \Omega \times \mathcal{I}_{n-1} \to [0, +\infty)$ defines an $(n-1)$-dimensional subadditive process on $(\Omega, \mathcal{T}, P)$. The separability and completeness of $W^{1,1}(A, \mathbb{R}^N) \times W^{1,2}(A, [0,1])$ for every $A \in \mathscr{A}$ combined with [40, Lemma C.2] and $(f2)$ in Definition 2.4 give the $\mathcal{T}$-measurability of the map $\omega \mapsto m_{\rm s}^{f_\omega^\infty}(\overline{u}_{\zeta,\nu}, T_\nu(A'))$ for every $A' \in \mathcal{I}_{n-1}$.

Next, we prove that $\mu_{\zeta,\nu}$ is stationary with respect to an $(n-1)$-dimensional group of P-preserving transformations $(\tau^\nu_{z'})_{z' \in \mathbb{Z}^{n-1}}$. To this end, fix $z' \in \mathbb{Z}^{n-1}$ and $A' \in \mathcal{I}_{n-1}$. By (6.3) we have that

$$T_\nu(A' + z') = M_\nu R_\nu(A' \times [-c, c)) + M_\nu R_\nu(z', 0) = T_\nu(A') + z'_\nu,$$

where $z'_\nu := M_\nu R_\nu(z', 0) \in \Pi^\nu \cap \mathbb{Z}^n$. Thus by (6.4) we get

$$\mu_{\zeta,\nu}(\omega, A' + z') = \frac{1}{M_\nu^{n-1}} m_{\rm s}^{f_\omega^\infty}(\overline{u}_{\zeta,\nu}, T_\nu(A' + z')) = \frac{1}{M_\nu^{n-1}} m_{\rm s}^{f_\omega^\infty}(\overline{u}_{\zeta,\nu}, T_\nu(A') + z'_\nu). \tag{6.5}$$

Now let u, v be test functions in the definition of $m_{\rm s}^{f_\omega^\infty}(\overline{u}_{\zeta,\nu}, T_\nu(A') + z'_\nu)$ and for $x \in T_\nu(A')$ set

$$\tilde{u}(x) := u(x + z'_\nu) \quad \text{and} \quad \tilde{v}(x) := v(x + z'_\nu).$$

Then, a change of variables together with the stationarity of f yield

$$S^{f^\infty(\omega)}(u, v, T_\nu(A') + z'_\nu) = S^{f^\infty(\tau_{z'_\nu}\omega)}(\tilde{u}, \tilde{v}, T_\nu(A')).$$

Set $(\tau^\nu_{z'})_{z' \in \mathbb{Z}^{n-1}} := (\tau_{z'_\nu})_{z' \in \mathbb{Z}^{n-1}}$; we notice that $(\tau^\nu_{z'})_{z' \in \mathbb{Z}^{n-1}}$ is well defined since $z'_\nu \in \mathbb{Z}^n$ and it defines a group of P-preserving transformations on $(\Omega, P, \mathcal{T})$. Then, the equality above can be rewritten as

$$S^{f^\infty(\omega)}(u, v, T_\nu(A') + z'_\nu) = S^{f^\infty(\tau^\nu_{z'}\omega)}(\tilde{u}, \tilde{v}, T_\nu(A')). \tag{6.6}$$

Moreover, since $z'_\nu \in \Pi^\nu \cap \mathbb{Z}^n$ we also have that $\tilde{u} = \bar{u}_{\zeta,\nu}$ on $\partial T_\nu(A')$. Thus gathering (6.5) and (6.6), by the arbitrariness of $\tilde{u}, \tilde{v}$ we infer

$$\mu_{\zeta,\nu}(\omega, A' + z') = \mu_{\zeta,\nu}(\tau^\nu_{z'}\omega, A'),$$

and hence the stationarity of $\mu_{\zeta,\nu}$ with respect to $(\tau^\nu_{z'})_{z' \in \mathbb{Z}^{n-1}}$.

To show that $\mu_{\zeta,\nu}$ is subadditive in $\mathcal{I}_{n-1}$, fix $\omega \in \Omega$ and $A' \in \mathcal{I}_{n-1}$ and let $(A'_i)_{1 \le i \le M} \subset \mathcal{I}_{n-1}$ be a finite family of pairwise disjoint sets such that $A' = \cup_{i=1}^M A'_i$. For every $\eta > 0$ and $i \in \{1, \dots, M\}$, let $(u_i, v_i) \in W^{1,1}(T_\nu(A'_i), \mathbb{R}^N) \times W^{1,2}(T_\nu(A'_i), [0,1])$ with $(u_i, v_i) = (\overline{u}_{\zeta,\nu}, 1)$ on $\partial T_\nu(A'_i)$ such that

$$\int_{T_\nu(A'_i)} \big(v_i^2 f^\infty(\omega, y, \nabla u_i) + (1 - v_i)^2 + |\nabla v_i|^2\big)\, dy \le m_{\rm s}^{f_\omega^\infty}(\overline{u}_{\zeta,\nu}, T_\nu(A'_i)) + \eta.$$

Note that by construction we always have $\cup_{i=1}^M T_\nu(A'_i) \subseteq T_\nu(A')$, thus we define

$$(u(y), v(y)) := \begin{cases} (u_i(y), v_i(y)) & \text{if } y \in T_\nu(A'_i) \\ (\overline{u}_{\zeta,\nu}, 1) & \text{if } y \in T_\nu(A') \setminus \cup_i T_\nu(A'_i). \end{cases}$$

In particular, $(u, v) \in W^{1,1}(T_\nu(A'), \mathbb{R}^N) \times W^{1,2}(T_\nu(A'), [0,1])$ with $(u, v) = (\overline{u}_{\zeta,\nu}, 1)$ on $\partial T_\nu(A')$. Hence, we get

$$\begin{aligned} m_{\rm s}^{f_\omega^\infty}(\overline{u}_{\zeta,\nu}, T_\nu(A')) &\le \int_{T_\nu(A')} \big(v^2 f^\infty(\omega, y, \nabla u) + (1 - v)^2 + |\nabla v|^2\big)\, dy \\ &= \sum_{i=1}^M \int_{T_\nu(A'_i)} \big(v_i^2 f^\infty(\omega, y, \nabla u_i) + (1 - v_i)^2 + |\nabla v_i|^2\big)\, dy \le \sum_{i=1}^M m_{\rm s}^{f_\omega^\infty}(\overline{u}_{\zeta,\nu}, T_\nu(A'_i)) + M\eta \end{aligned}$$

and the subadditivity follows by the arbitrariness of $\eta > 0$.

Finally, we show that $\mu_{\zeta,\nu}$ is bounded. To this end we observe that for every $A' \in \mathcal{I}_{n-1}$ and every $\omega \in \Omega$ we have

$$\mu_{\zeta,\nu}(\omega, A') = \frac{1}{M_\nu^{n-1}} m_{\rm s}^{f_\omega^\infty}(\overline{u}_{\zeta,\nu}, T_\nu(A')) \leq \frac{C}{M_\nu^{n-1}} \int_{T_\nu(A')} |\nabla \overline{u}_{\zeta,\nu}| \mathrm{d}y \leq C|\zeta| \|\overline{\mathrm{u}}'\|_{L^\infty(\mathbb{R})} \mathcal{L}^{n-1}(A'),$$

where we used $T_\nu(A') \cap \Pi^\nu = M_\nu R_\nu(A' \times \{0\})$ and $\{|\nabla(\overline{u}_{\zeta,\nu}(y)| > 0\} \subseteq \{|y \cdot \nu| \leq 1/2\}$.

Therefore, for every fixed for every $\zeta \in \mathbb{Q}^N$ and $\nu \in \mathbb{S}^{n-1} \cap \mathbb{Q}^n$ defines a subadditive process. Then, we can apply Theorem 2.3 to deduce the existence of a $\mathcal{T}$-measurable function $\psi_{\nu,\zeta} : \Omega \to [0, +\infty)$, and a set $\Omega_{\zeta,\nu} \in \mathcal{T}$ with $P(\Omega_{\zeta,\nu}) = 1$, such that for every $\omega \in \Omega_{\zeta,\nu}$

$$M_\nu^{n-1} \psi_{\zeta,\nu}(\omega) = M_\nu^{n-1} \lim_{r \to +\infty} \frac{\mu_{\zeta,\nu}(\omega, Q'_r)}{r^{n-1}} = \lim_{r \to +\infty} \frac{m_{\rm s}^{f_\omega^\infty}(\overline{u}_{\zeta,\nu}, Q_r^\nu)}{r^{n-1}}.$$

where $Q'_r := Q_r \cap \{x_n = 0\}$, $r > 0$.

Step 2: In this step we prove the existence of $\tilde{\Omega} \in \mathcal{T}$ with $P(\tilde{\Omega}) = 1$ such that for every $\omega \in \tilde{\Omega}$ and for every $\zeta \in \mathbb{R}^N$ and $\nu \in \mathbb{S}^{n-1}$ the following limit exists

$$\lim_{r \to +\infty} \frac{m_{\rm s}^{f_\omega^\infty}(\overline{u}_{\zeta,\nu}, Q_r^\nu)}{r^{n-1}}$$

and defines an x-independent $\mathcal{T} \otimes \mathscr{B}^N \otimes \mathscr{B}_S^n$-measurable function.

To prove the claim let $\tilde{\Omega}$ denote the intersection of the sets $\Omega_{\zeta,\nu}$, as in Step 1, for $\zeta \in \mathbb{Q}^N$ and $\nu \in \mathbb{S}^{n-1} \cap \mathbb{Q}^n$. Clearly, $\tilde{\Omega} \in \mathcal{T}$ and $P(\tilde{\Omega}) = 1$. Let $\underline{g}, \overline{g} : \tilde{\Omega} \times \mathbb{R}^N \times \mathbb{S}^{n-1} \to [0, +\infty]$ be the functions given by

$$\underline{g}(\omega, \zeta, \nu) := \liminf_{r \to +\infty} \frac{m_{\rm s}^{f_\omega^\infty}(\overline{u}_{\zeta,\nu}, Q_r^\nu)}{r^{n-1}}, \qquad \overline{g}(\omega, \zeta, \nu) := \limsup_{r \to +\infty} \frac{m_{\rm s}^{f_\omega^\infty}(\overline{u}_{\zeta,\nu}, Q_r^\nu)}{r^{n-1}}.$$

By Step 1, for every $\omega \in \tilde{\Omega}$, every $\zeta \in \mathbb{Q}^N$ and every $\nu \in \mathbb{S}^{n-1} \cap \mathbb{Q}^n$ we have that

$$\underline{g}(\omega, \zeta, \nu) = \overline{g}(\omega, \zeta, \nu). \tag{6.7}$$

Furthermore, fixed $\omega \in \tilde{\Omega}$ and $\nu \in \mathbb{S}^{n-1}$, arguing as in Proposition 4.7 (i) we have

$$|\underline{g}(\omega, \zeta_1, \nu) - \underline{g}(\omega, \zeta_2, \nu)| + |\overline{g}(\omega, \zeta_1, \nu) - \overline{g}(\omega, \zeta_2, \nu)| \leq 2C\mathcal{H}^{n-1}(\partial Q_1)|\zeta_1 - \zeta_2| \tag{6.8}$$

for every $\zeta_1, \zeta_2 \in \mathbb{R}^N$. From (6.7) and (6.8) we deduce that for every $\omega \in \tilde{\Omega}$, every $\zeta \in \mathbb{R}^N$ and every $\nu \in \mathbb{S}^{n-1} \cap \mathbb{Q}^n$

$$\underline{g}(\omega, \zeta, \nu) = \overline{g}(\omega, \zeta, \nu), \tag{6.9}$$

and that $\underline{g}(\cdot, \zeta, \nu) : \tilde{\Omega} \to [0, +\infty)$ is $\mathcal{T}$-measurable for every $\zeta \in \mathbb{R}^N$ and every $\nu \in \mathbb{S}^{n-1} \cap \mathbb{Q}^n$.

We now claim that for every $\omega \in \tilde{\Omega}$ and every $\zeta \in \mathbb{R}^N$, the restrictions of the functions $\nu \mapsto \underline{g}(\omega, \zeta, \nu)$ and $\nu \mapsto \overline{g}(\omega, \zeta, \nu)$ to the sets $\hat{\mathbb{S}}_\pm^{n-1}$ are continuous. We show only the continuity of $\underline{g}$ on $\hat{\mathbb{S}}_+^{n-1}$, the proof for $\overline{g}$ is analogous. To this end, let $\omega \in \tilde{\Omega}$, $\zeta \in \mathbb{R}^N$, $\nu \in \hat{\mathbb{S}}_+^{n-1}$, then by density let $(\nu_j)_{j \in \mathbb{N}} \subset \hat{\mathbb{S}}_+^{n-1} \cap \mathbb{Q}^n$ be such that $\nu_j \to \nu$ as $j \to +\infty$. By the continuity of $\nu \mapsto R_\nu$ on $\hat{\mathbb{S}}_+^{n-1}$, for every $\delta \in (0, 1/2)$ there exists a $j_\delta \in \mathbb{N}$ such that

$$Q_r^\nu \subset\subset Q_{(1+\delta)r}^{\nu_j} \subset\subset Q_{(1+2\delta)r}^\nu \tag{6.10}$$

for every $j \geq j_\delta$ and every $r > 0$.

Setting $\kappa_j := \max\{|R_{\nu_j}(e_i)\cdot\nu| \,:\, i=1,\dots,n-1\}$ we have that $\kappa_j \to 0$ as $j\to+\infty$, thanks to the continuity of $\nu\mapsto R_\nu$ on $\hat{\mathbb{S}}^{n-1}_\pm$. We observe that for every $y\in\overline{Q^{\nu}_{r(1+\delta)}}$, we have $y=y'+(y\cdot\nu_j)\nu_j$ where

$$y'\in R_{\nu_j}\Big(\big(-\frac{r}{2}(1+\delta),\frac{r}{2}(1+\delta)\big)^{n-1}\times\{0\}\Big)$$

and in particular, if in addition $|y\cdot\nu|\le\frac12$, for j large enough depending only on δ, we get

$$|y\cdot\nu_j|\le\tfrac{|y'\cdot\nu|}{|\nu_j\cdot\nu|}+\tfrac{1}{2(1-\delta)}<\tfrac{(n-1)\kappa_j r(1+\delta)}{2(1-\delta)}+1=K(\delta)r\kappa_j+1,$$

where $K(\delta):=\frac{(n-1)(1+\delta)}{2(1-\delta)}$. Then, by applying Lemma 4.6, with $R=K(\delta)r\kappa_j+1$, we obtain

$$\begin{aligned}m_{\rm s}^{f^\infty_\omega}(\overline{u}_{\zeta,\nu_j},Q^{\nu_j}_{r(1+\delta)})&\le m_{\rm s}^{f^\infty_\omega}(\overline{u}_{\zeta,\nu},Q^{\nu}_{r})+\eta r^{n-1}\\&\quad+\tilde{K}(2\delta(1+2\delta)^{n-2}r^{n-1}+(K(\delta)r\kappa_j+1)|\zeta|(1+\delta)^{n-2}r^{n-2}).\end{aligned}$$

Therefore, dividing by r^{n-1}, passing to the liminf as $r\to+\infty$, and to the limsup as $j\to+\infty$, and finally letting $\eta,\delta\to0$ we obtain

$$\limsup_{j\to+\infty}\underline{g}(\omega,\zeta,\nu_j)\le\underline{g}(\omega,\zeta,\nu).$$

An analogous argument using the cubes $Q^{\nu_j}_{(1-\delta)r}$ shows that

$$\underline{g}(\omega,\zeta,\nu)\le\liminf_{j\to+\infty}\underline{g}(\omega,\zeta,\nu_j)$$

implying the claim.

In particular, thanks to (6.9) we deduce that for every $\omega\in\tilde\Omega$, $\zeta\in\mathbb{R}^N$ and $\nu\in\mathbb{S}^{n-1}$

$$\underline{g}(\omega,\zeta,\nu)=\overline{g}(\omega,\zeta,\nu)\,. \tag{6.11}$$

The $\mathcal{T}$-measurability of $\underline{g}(\cdot,\zeta,\nu):\tilde\Omega\to[0,+\infty)$ for every $\zeta\in\mathbb{R}^N$ and $\nu\in\mathbb{S}^{n-1}$ follows from the analogous property for $\nu\in\mathbb{S}^{n-1}\cap\mathbb{Q}^n$. Furthermore, the map $\underline{g}(\omega,\cdot,\cdot):\mathbb{R}^N\times\hat{\mathbb{S}}^{n-1}_\pm\to[0,+\infty)$ is continuous for every $\omega\in\tilde\Omega$ thanks to (6.8).

Thus, defining $g_{\rm hom}:\Omega\times\mathbb{R}^N\times\mathbb{S}^{n-1}\to[0,+\infty)$ by

$$g_{\rm hom}(\omega,\zeta,\nu):=\begin{cases}\underline{g}(\omega,\zeta,\nu) & \text{if } \omega\in\tilde\Omega\\ \frac{2|\zeta|}{C(|\zeta|+2)} & \text{if } \omega\notin\tilde\Omega,\end{cases}$$

we have that $g_{\rm hom}$ is $\mathcal{T}\otimes\mathscr{B}^N\otimes\mathscr{B}^n_S$-measurable and, thanks to Corollary 4.5,

$$g_{\rm hom}(\omega,\zeta,\nu)=\lim_{r\to+\infty}\frac{m_{\rm s}^{f^\infty_\omega}(\overline{u}_{\zeta,\nu},Q^\nu_r)}{r^{n-1}}=\lim_{r\to+\infty}\frac{m_{\rm s}^{f^\infty_\omega}(u_{\zeta,\nu},Q^\nu_r)}{r^{n-1}} \tag{6.12}$$

for every $\omega\in\tilde\Omega$, every $\zeta\in\mathbb{R}^N$ and every $\nu\in\mathbb{S}^{n-1}$.

Step 3: In this step we show the existence of $\Omega'\in\mathcal{T}$ with $\Omega'\subseteq\tilde\Omega$ and $P(\Omega')=1$, such that for every $\omega\in\Omega'$, $z\in\mathbb{Z}^n$, $\zeta\in\mathbb{Q}^N$, $\nu\in\mathbb{S}^{n-1}\cap\mathbb{Q}^n$, and for every integer sequence (r_k) with $r_k\ge k$ for every k

$$\lim_{k\to+\infty}\frac{m_{\rm s}^{f^\infty_\omega}(\overline{u}_{-kz,\zeta,\nu},Q^\nu_{r_k}(-kz))}{r_k^{n-1}}=g_{\rm hom}(\omega,\nu,\zeta)\,. \tag{6.13}$$

Let $z\in\mathbb{Z}^n$, $\zeta\in\mathbb{Q}^N$, $\nu\in\mathbb{S}^{n-1}\cap\mathbb{Q}^n$, $\eta>0$ and $\delta\in(0,1/4)$. Arguing exactly as in [23, Theorem 6.1] we can prove the existence of a set $\Omega^{\zeta,\nu,\eta}_z\in\mathcal{T}$, with $\Omega^{\zeta,\nu,\eta}_z\subseteq\tilde\Omega$, $P(\Omega^{\zeta,\nu,\eta}_z)=1$, and an integer $m_0=m_0(\zeta,\nu,\eta,z,\omega,\delta)>\frac1\delta$ satisfying the following property: for every $\omega\in\Omega^{\zeta,\nu,\eta}_z$ and for every integer $m \geq m_0$ there exists $i = i(\zeta, \nu, \eta, z, \omega, \delta, m) \in \{m+1, \ldots, m+\ell\}$, with $\ell := \lfloor 5m\delta \rfloor$, such that

$$\Big| \frac{m_{\rm s}^{f_\omega^\infty}(\overline{u}_{-iz,\zeta,\nu}, Q_h^\nu(-iz))}{h^{n-1}} - g_{\rm hom}(\omega,\zeta,\nu) \Big| \leq \eta \quad \text{for every } h \in \mathbb{N} \text{ with } h \geq j_0, \tag{6.14}$$

where $j_0 = j_0(\zeta, \nu, \eta, z, \omega, \delta)$, and $\lfloor s \rfloor$ denotes the integer part of $s \in \mathbb{R}$.

Define Ω' as the intersection of the sets $\Omega_z^{\zeta,\nu,\eta}$ for $\zeta \in \mathbb{Q}^N$, $\nu \in \mathbb{S}^{n-1} \cap \mathbb{Q}^n$, $\eta \in \mathbb{Q}$, with $\eta > 0$ and $z \in \mathbb{Z}^n$. Thus $\Omega' \subseteq \tilde{\Omega}$ and $P(\Omega') = 1$. Let $\omega \in \Omega'$ and r_k be as required, $\delta > 0$ with $20\delta(|z|+1) < 1$ and $\eta \in \mathbb{Q}$ with $\eta > 0$. For every $k \geq 2m_0(\zeta, \nu, \eta, z, \omega, \delta)$, let $\underline{r}_k, \overline{r}_k \in \mathbb{N}$ be defined as

$$\underline{r}_k := r_k - 2(i_k - k)\lfloor |z|+1 \rfloor \quad \text{and} \quad \overline{r}_k := r_k + 2(i_k - k)\lfloor |z|+1 \rfloor,$$

where

$$i_k = i(\zeta, \nu, \eta, z, \omega, \delta, k) \in \{k+1, \ldots, k + \lfloor 5k\delta \rfloor\}, \tag{6.15}$$

therefore, by construction, we have that $Q_{\underline{r}_k}^\nu(-i_k z) \subset\subset Q_{r_k}^\nu(-kz) \subset\subset Q_{\overline{r}_k}^\nu(-i_k z)$.

Since $20\delta(|z|+1) < 1$, $k \leq r_k$ and $i_k - k \leq 5k\delta$ by (6.15), for every $y \in Q_{\overline{r}_k}^\nu(-i_k z)$ such that $|(y + i_k z) \cdot \nu| \leq \frac{1}{2}$, we obtain that

$$\begin{aligned} |(y + kz) \cdot \nu| &= |(y + i_k z) \cdot \nu + (kz - i_k z) \cdot \nu| \leq \frac{1}{2} + |(kz - i_k z) \cdot \nu| \\ &\leq (i_k - k)|z| + \frac{1}{2} \leq 5k\delta|z| + \frac{1}{2} \leq 5r_k\delta|z| + \frac{1}{2}, \end{aligned}$$

and $r_k - \underline{r}_k = 2(i_k - k)\lfloor |z|+1 \rfloor \leq 10k\delta\lfloor |z|+1 \rfloor \leq 10r_k\delta\lfloor |z|+1 \rfloor < \frac{r_k}{2}$. Applying Lemma 4.6, with $R = 5r_k\delta|z| + \frac{1}{2}$, we obtain

$$\begin{aligned} m_{\rm s}^{f_\omega^\infty}(\overline{u}_{-kz,\zeta,\nu}, Q_{r_k}^\nu(-kz)) &\leq m_{\rm s}^{f_\omega^\infty}(\overline{u}_{-i_k z,\zeta,\nu}, Q_{\underline{r}_k}^\nu(-i_k z)) + \eta \underline{r}_k^{n-1} \\ &+ \frac{\tilde{K}}{2}(10\delta(|z|+1)r_k^{n-1} + (10r_k\delta|z|+1)|\zeta|r_k^{n-2}). \end{aligned}$$

In particular, from the latter estimate, (6.14) and $\underline{r}_k \leq r_k$, for every k large enough such that $\underline{r}_k \geq j_0(\zeta, \nu, \eta, z, \omega, \delta)$, we obtain

$$\begin{aligned} g_{\rm hom}(\omega,\zeta,\nu) + \eta &\geq \frac{m_{\rm s}^{f_\omega^\infty}(\overline{u}_{-i_k z,\zeta,\nu}, Q_{\underline{r}_k}^\nu(-i_k z))}{\underline{r}_k^{n-1}} \geq \frac{m_{\rm s}^{f_\omega^\infty}(\overline{u}_{-i_k z,\zeta,\nu}, Q_{\underline{r}_k}^\nu(-i_k z))}{r_k^{n-1}} \\ &\geq \frac{m_{\rm s}^{f_\omega^\infty}(\overline{u}_{-kz,\zeta,\nu}, Q_{r_k}^\nu(-kz))}{r_k^{n-1}} - \eta - \frac{\tilde{K}}{2}(10\delta(|z|+1) + (10\delta|z| + \frac{1}{r_k})|\zeta|) \end{aligned}$$

and thus, taking the limsup for $k \to +\infty$ and letting $\eta, \delta \to 0$, we get

$$g_{\rm hom}(\omega,\zeta,\nu) \geq \limsup_{k\to+\infty} \frac{m_{\rm s}^{f_\omega^\infty}(\overline{u}_{-kz,\zeta,\nu}, Q_{r_k}^\nu(-kz))}{r_k^{n-1}}.$$

Arguing analogously with the external cubes $Q_{\overline{r}_k}^\nu(-i_k z)$ we get

$$g_{\rm hom}(\omega,\zeta,\nu) \leq \liminf_{k\to+\infty} \frac{m_{\rm s}^{f_\omega^\infty}(\overline{u}_{-kz,\zeta,\nu}, Q_{r_k}^\nu(-kz))}{r_k^{n-1}},$$

obtaining the claim.

Step 4: Let Ω' be the set introduced in Step 3, then for every $\omega \in \Omega'$, $x \in \mathbb{R}^n$, $\zeta \in \mathbb{Q}^N$ and $\nu \in \mathbb{S}^{n-1} \cap \mathbb{Q}^n$ there holds

$$\lim_{r\to+\infty} \frac{m_{\rm s}^{f_\omega^\infty}(\overline{u}_{rx,\zeta,\nu}, Q_r^\nu(rx))}{r^{n-1}} = g_{\rm hom}(\omega,\zeta,\nu)\,. \tag{6.16}$$

Fix ω, x, ζ, ν as required, $\eta \in (0, \frac{1}{2})$, $q \in \mathbb{Q}^n$ with $|x-q| < \eta$, and $h \in \mathbb{Z}$ such that $z := hq \in \mathbb{Z}^n$. Consider a sequence of real numbers $t_k \to +\infty$ as $k \to +\infty$ and let $s_k := \frac{t_k}{h}$. Fixing an integer $j > 2|z|+1$ and setting $r_k := \lfloor t_k + 2\eta t_k \rfloor + j$ we have that $Q^\nu_{t_k}(t_k x) \subset\subset Q^\nu_{r_k}(\lfloor s_k \rfloor z)$. Since $|(t_k x - \lfloor s_k \rfloor z)\cdot \nu| \le |t_k x - t_k q| + |s_k z - \lfloor s_k \rfloor z| \le t_k \eta + |z|$, for every $y \in Q^\nu_{r_k}(\lfloor s_k \rfloor z)$ such that $|(y - t_k x)\cdot\nu| \le \frac{1}{2}$ we have that $|(y - \lfloor s_k \rfloor z)\cdot \nu| \le t_k\eta + |z| + \frac{1}{2}$. In particular, for k large enough depending only on z, we can apply Lemma 4.6, with $R = t_k \eta + |z| + \frac{1}{2}$, to obtain

$$m_{\rm s}^{f_\omega^\infty}(\overline{u}_{\lfloor s_k \rfloor z, \zeta, \nu}, Q^\nu_{r_k}(\lfloor s_k \rfloor z)) \le m_{\rm s}^{f_\omega^\infty}(\overline{u}_{t_k x, \zeta, \nu}, Q^\nu_{t_k}(t_k x)) + \eta r_k^{n-1}$$
$$+ \tilde{K}((2\eta r_k + j) r_k^{n-2} + (r_k \eta + |z| + \frac{1}{2})|\zeta| r_k^{n-2}), \tag{6.17}$$

where we used $r_k \ge t_k$. From (6.17), dividing by t_k^{n-1} and recalling that $r_k \ge t_k \ge s_k \ge \lfloor s_k \rfloor$, we obtain that

$$\frac{m_{\rm s}^{f_\omega^\infty}(\overline{u}_{\lfloor s_k \rfloor z, \zeta, \nu}, Q^\nu_{r_k}(\lfloor s_k \rfloor z))}{r_k^{n-1}} - \eta - \tilde{K}\Big((2\eta + \frac{j}{r_k}) + (\eta + \frac{|z|}{r_k} + \frac{1}{2r_k})|\zeta|\Big) \le \frac{m_{\rm s}^{f_\omega^\infty}(\overline{u}_{t_k x, \zeta, \nu}, Q^\nu_{t_k}(t_k x))}{t_k^{n-1}}.$$

Since $\omega \in \Omega'$ and $r_k \ge \lfloor s_k \rfloor$, we can apply (6.13), taking the lim inf as $k \to \infty$ and letting $\eta \to 0$ we obtain

$$g_{\rm hom}(\omega, \zeta, \nu) \le \liminf_{k\to+\infty} \frac{m_{\rm s}^{f_\omega^\infty}(\overline{u}_{t_k x, \zeta, \nu}, Q^\nu_{t_k}(t_k x))}{t_k^{n-1}}.$$

Arguing analogously we obtain

$$g_{\rm hom}(\omega, \zeta, \nu) \ge \limsup_{k\to+\infty} \frac{m_{\rm s}^{f_\omega^\infty}(\overline{u}_{t_k x, \zeta, \nu}, Q^\nu_{t_k}(t_k x))}{t_k^{n-1}}.$$

deducing the claim, thenks to the generality of the sequence $(t_k)_{k\in\mathbb{N}}$.

Step 5: Let Ω' be the set introduced in Step 3, then for every $\omega \in \Omega'$, $x \in \mathbb{R}^n$, $\zeta \in \mathbb{R}^N$, and $\nu \in \mathbb{S}^{n-1}$

$$g_{\rm hom}(\omega, \zeta, \nu) = \lim_{r\to+\infty} \frac{m_{\rm s}^{f_\omega^\infty}(\overline{u}_{rx, \zeta, \nu}, Q^\nu_r(rx))}{r^{n-1}}.$$

For ω, x, ζ, ν as above define

$$\underline{g}(\omega, x, \zeta, \nu) := \liminf_{r\to+\infty} \frac{m_{\rm s}^{f_\omega^\infty}(\overline{u}_{rx, \zeta, \nu}, Q^\nu_r(rx))}{r^{n-1}}, \qquad \overline{g}(\omega, x, \zeta, \nu) := \limsup_{r\to+\infty} \frac{m_{\rm s}^{f_\omega^\infty}(\overline{u}_{rx, \zeta, \nu}, Q^\nu_r(rx))}{r^{n-1}}.$$

Arguing exactly as in Proposition 4.7 (i) and in Step 2, we obtain from Step 4 that

$$\underline{g}(\omega, x, \zeta, \nu) = g_{\rm hom}(\omega, \zeta, \nu) = \overline{g}(\omega, x, \zeta, \nu) \tag{6.18}$$

for every $\omega \in \Omega'$, $x \in \mathbb{R}^n$, $\zeta \in \mathbb{R}^N$, and $\nu \in \mathbb{S}^{n-1} \cap \mathbb{Q}^n$.

Now let $\omega \in \Omega'$, $x \in \mathbb{R}^n$, $\zeta \in \mathbb{R}^N$ and $\nu \in \hat{\mathbb{S}}^{n-1}_+$, by density there is $(\nu_j)_{j\in\mathbb{N}}$ in $\hat{\mathbb{S}}^{n-1}_+ \cap \mathbb{Q}^n$ such that $\nu_j \to \nu$ as $j \to +\infty$. Thanks to the continuity on $\hat{\mathbb{S}}^{n-1}_+$ of the map $\nu \mapsto R_\nu$, for every $\delta \in (0, \frac{1}{2})$ there exists j_δ, such that

$$Q^\nu_r(rx) \subset\subset Q^{\nu_j}_{(1+\delta)r}(rx) \subset\subset Q^\nu_{(1+2\delta)r}(rx) \tag{6.19}$$

for every $j \ge j_\delta$ and every $r > 0$. Let us fix $j \ge j_\delta$, $r > 0$ and $\eta > 0$. Setting $c_j := \max\{|R_{\nu_j}(e_i)\cdot\nu| : i = 1, \dots, n-1\}$ we have that $c_j \to 0$ as $j \to +\infty$, by continuity of $\nu \mapsto R_\nu$ on $\hat{\mathbb{S}}^{n-1}_\pm$, and recalling that $R_\nu \in O(n)$ and $R_\nu e_n = \nu$ (cf. (e) of the notation list). For every $y \in \overline{Q^\nu_{r(1+\delta)}(rx)}$ we have that $y - rx = y' + ((y - rx)\cdot \nu_j)\nu_j$ where

$$y' \in R_{\nu_j}\Big(\big(-\frac{r}{2}(1+\delta), \frac{r}{2}(1+\delta)\big)^{n-1} \times \{0\}\Big),$$

with, if j is large enough depending only on δ,

$$|(y-rx)\cdot\nu_j| \le \tfrac{|y'\cdot\nu|}{|\nu_j\cdot\nu|} + \tfrac{1}{2(1-\delta)} < \tfrac{(n-1)c_j r(1+\delta)}{2(1-\delta)} + 1 = K(\delta)rc_j + 1,$$

where $K(\delta) := \frac{(n-1)(1+\delta)}{2(1-\delta)}$, if in addition $|(y-rx)\cdot\nu| \le \frac{1}{2}$. Therefore, we can apply Lemma 4.6, with $R = K(\delta)rc_j + 1$, and we get

$$\begin{aligned} m_{\rm s}^{f_\omega^\infty}(\overline{u}_{rx,\zeta,\nu_j}, Q^{\nu_j}_{r(1+\delta)}(rx)) &\le m_{\rm s}^{f_\omega^\infty}(\overline{u}_{rx,\zeta,\nu}, Q^{\nu}_{r}(rx)) + \eta r^{n-1} \\ &\quad + \tilde{K}(2\delta(1+2\delta)^{n-2}r^{n-1} + (K(\delta)rc_j+1)|\zeta|(1+\delta)^{n-2}r^{n-2}). \end{aligned}$$

Dividing by r^{n-1} and letting $r\to+\infty$, we obtain

$$\begin{aligned} (1+\delta)^{n-1}\underline{g}\big(\omega, \frac{x}{1+\delta}, \zeta, \nu_j\big) &\le \underline{g}(\omega, x, \zeta, \nu) \\ &\quad + \eta + \tilde{K}(2\delta(1+2\delta)^{n-2} + K(\delta)c_j|\zeta|(1+\delta)^{n-2}). \end{aligned}$$

Hence, we may use (6.18) as $\nu_j \in \mathbb{S}^{n-1}\cap\mathbb{Q}^n$ and deduce by taking the superior limit as $j\to+\infty$ and letting $\eta\to 0$ in the latter estimate

$$\begin{aligned} (1+\delta)^{n-1}\limsup_{j\to+\infty} g_{\rm hom}(\omega,\zeta,\nu_j) &= (1+\delta)^{n-1}\limsup_{j\to+\infty}\underline{g}\big(\omega, \frac{x}{1+\delta}, \zeta, \nu_j\big) \\ &\le \underline{g}(\omega, x, \zeta, \nu) + 2\tilde{K}\delta(1+2\delta)^{n-2}. \end{aligned}$$

Therefore, by the continuity of $g_{\rm hom}$ established in Step 2, letting $\delta\to 0$ we obtain

$$g_{\rm hom}(\omega,\zeta,\nu) \le \underline{g}(\omega,x,\zeta,\nu).$$

Arguing analogously we have $\overline{g}(\omega,x,\zeta,\nu) \le g_{\rm hom}(\omega,\zeta,\nu)$, and recalling Corollary 4.5 we conclude.

Step 5: In this step we show that if f is ergodic then $g_{\rm hom}$ is deterministic.

Set $\hat{\Omega} = \bigcap_{z\in\mathbb{Z}^n}\tau_z(\tilde{\Omega})$; we clearly have that $\hat{\Omega}\in\mathcal{T}$, $\hat{\Omega}\subseteq\tilde{\Omega}$ and $\tau_z(\hat{\Omega}) = \hat{\Omega}$ for every $z\in\mathbb{Z}^n$. Moreover, since τ_z is a P-preserving transformation and $P(\tilde{\Omega}) = 1$, we have that $P(\hat{\Omega}) = 1$. We claim that

$$g_{\rm hom}(\tau_z\omega,\zeta,\nu) \le g_{\rm hom}(\omega,\zeta,\nu) \tag{6.20}$$

for every $\omega\in\hat{\Omega}$, every $\zeta\in\mathbb{R}^N$ and every $\nu\in\mathbb{S}^{n-1}$. Fix $z\in\mathbb{Z}^n$, $\omega\in\Omega$ and $\nu\in\mathbb{S}^{n-1}$. For every $r > 3|z|$, let $(u_r, v_r)\in W^{1,1}(Q_r^\nu,\mathbb{R}^N)\times W^{1,2}(Q_r^\nu,[0,1])$, with $(u_r,v_r) = (\overline{u}_{\zeta,\nu},1)$ on ∂Q_r^ν such that

$$\int_{Q_r^\nu}\big(v_r^2 f^\infty(\omega,y,\nabla u_r) + (1-v_r)^2 + |\nabla v_r|^2\big)\,{\rm d}y \le m_{\rm s}^{f_\omega^\infty}(\overline{u}_{\zeta,\nu}, Q_r^\nu) + 1. \tag{6.21}$$

By the stationarity of f (and hence of f^∞) we infer that

$$m_{\rm s}^{f^\infty_{\tau_z\omega}}(\overline{u}_{\zeta,\nu}, Q_r^\nu) = m_{\rm s}^{f_\omega^\infty}(\overline{u}_{z,\zeta,\nu}, Q_r^\nu(z)). \tag{6.22}$$

Observe that $Q_r^\nu \subset\subset Q^\nu_{r+3|z|}(z)$ for every $r>3|z|$, and for every $y\in Q^\nu_{r+3|z|}(z)$ such that $|y\cdot\nu|\le\frac{1}{2}$ we have that

$$1 \ge \frac{1}{2} + |\nu\cdot z| \ge |y\cdot\nu| = |(y-z)\cdot\nu + z\cdot\nu| + |\nu\cdot z| \ge |(y-z)\cdot\nu|.$$

Then we can apply Lemma 4.6, with $R = 1$, and for every $\eta > 0$ we obtain

$$\begin{aligned} m_{\rm s}^{f_\omega^\infty}(\overline{u}_{z,\zeta,\nu}, Q^\nu_{r+3|z|}(z)) &\le m_{\rm s}^{f_\omega^\infty}(\overline{u}_{\zeta,\nu}, Q_r^\nu) + \eta r^{n-1} \\ &\quad + \tilde{K}(r+3|z|)^{n-2}(3|z| + |\zeta|). \end{aligned}$$

Therefore, by definition of g_{hom}, $\hat\Omega \subseteq \tilde\Omega$, and (6.22) we obtain

$$g_{\mathrm{hom}}(\tau_z\omega,\zeta,\nu) = \lim_{r\to+\infty}\frac{m_{\mathrm{s}}^{f^\infty_{\tau_z(\omega)}}(\overline{u}_{\zeta,\nu},Q_r^\nu)}{r^{n-1}} = \lim_{r\to+\infty}\frac{m_{\mathrm{s}}^{f^\infty_{\omega}}(\overline{u}_{z,\zeta,\nu},Q_r^\nu(z))}{r^{n-1}}$$

$$= \lim_{r\to+\infty}\frac{m_{\mathrm{s}}^{f^\infty_{\omega}}(\overline{u}_{z,\zeta,\nu},Q_{r+3|z|}^\nu(z))}{r^{n-1}} \le \lim_{r\to+\infty}\frac{m_{\mathrm{s}}^{f^\infty_{\omega}}(\overline{u}_{\zeta,\nu},Q_r^\nu)}{r^{n-1}} \le g_{\mathrm{hom}}(\omega,\zeta,\nu)$$

thus deducing the claim.

By (6.20) and the properties of $(\tau_z)_{z\in\mathbb{Z}^n}$, we clearly infer that

$$g_{\mathrm{hom}}(\tau_z\omega,\zeta,\nu) = g_{\mathrm{hom}}(\omega,\zeta,\nu)$$

and hence, using the same argument as in [23, Corollary 6.3], if $(\tau_z)_{z\in\mathbb{Z}^n}$ is ergodic we deduce that g_{hom} does not depend on ω and thus is deterministic. To conclude, we just observe that the representation of $g_{\mathrm{hom}}(\zeta,\nu)$ as in (6.2) is a direct consequence of (6.12), and the Dominated Convergence theorem (cf. (4.16)). □

Finally, we are in aposition to prove the main result of this paper, Theorem 3.4.

Proof of Theorem 3.4. The proof readily follwos by combining Theorem 5.1, Proposition 6.1, 6.2, and 6.3. □

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