A DUALITY APPROACH FOR VARIATIONAL PROBLEMS IN DOMAINS WITH CRACKS

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Abstract. In this paper we study the asymptotic behaviour of the solutions of some minimization problems for integral functionals with convex integrands, in two-dimensional domains with cracks, under perturbations of the cracks in the Hausdorff metric. In the first part of the paper, we examine conditions for the stability of the minimum problem via duality arguments in convex optimization. In the second part, we study the limit problem in some special cases when there is no stability, using the tool of Γ-convergence.

Key words: capacity, convex optimization, Γ-convergence, Hausdorff distance, integral functional

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Contents

1. Introduction

Let Ω be a bounded connected and simply connected open subset of $\mathbb{R}^2$, let K be a compact subset of $\overline{\Omega}$ and let $g \in W^{1,p}(\Omega)$. We consider the following variational problem:

$$(P) \qquad \min_{w=g \text{ on } \partial_D\Omega\setminus K} \int_{\Omega\setminus K} f(x,\nabla w)\,dx,$$

where $\partial_D\Omega$ is a non-empty part of the boundary of Ω with a finite number of connected components and the function $f : \Omega \times \mathbb{R}^2 \to \mathbb{R}$ is a Borel function which satisfies the following assumptions: there exist positive constants α, β, γ such that, for almost every $x \in \Omega$ and for every $\xi \in \mathbb{R}^2$

$$\alpha|\xi|^p \leq f(x,\xi) \leq \beta|\xi|^p + \gamma; \tag{1.1}$$

$$f(x,\cdot) \text{ is strictly convex.} \tag{1.2}$$

Our purpose in this paper is to study the asymptotic behaviour of the solutions u_K of the problem (P) with respect to the variations of the compact set K in the Hausdorff metric. This problem has been recently studied in [15] for $f(x,\xi) = |\xi|^2/2$ in order to give a precise mathematical formulation for the quasi-static growth of brittle fractures, following Griffith's criterion of crack growth.

The study of the asymptotic behaviour of solutions of variational problems with respect to domain variations is also related to some shape optimization problems, where very often the nonexistence of solutions is due to the non stability of the state equation. By stability of problem (P), more precisely stability of a given compact set K along a sequence (K_h) converging to K in the Hausdorff distance, we mean the convergence in a suitable topology of the sequence of solutions (u_{K_h}) of (P) to the function u_K. It is known that a necessary condition for stability is the convergence of the two-dimensional Lebesgue measure $|K_n|$ of K_n to the two-dimensional Lebesgue measure $|K|$ of K (see [12]).

If $f(x,\cdot)$ is differentiable, then the solution u_K solves a nonlinear mixed type boundary value problem. In the literature there are various results on the asymptotic behaviour of solutions of elliptic PDE with purely Dirichlet boundary conditions, with respect to domains variations. In this case the type of limit problem is known even when there is no stability (see for instance [13], [7], [14]).

Concerning stability results for purely Neumann problems, we can mention for instance the papers [9], [8], [5], [2], [3], [12], where the families of domains satisfy suitable structural assumptions. In the literature there are well known examples showing that without these structural assumptions some additional term (typically depending on jumps on the limit set K) may appear in the limit problem (see [26], [17], [10]). However, unlike Dirichlet problems, there is not a general characterization of the limit problem with Neumann conditions.

In the first part of this paper we prove the following stability result using the duality argument of convex optimization.

Theorem 1.1. *Let (K_h) be a sequence of compact subsets of $\overline{\Omega}$ which converges to a compact set K in the Hausdorff metric. Assume that K_h has a uniformly bounded number of connected components, $|K_h|$ converges to $|K|$, and that the intersection of the limits of two different connected components of $K_h \cup (\partial\Omega \setminus \partial_D\Omega)$ is either empty or has positive $(1,q)$-capacity, where q is the conjugate exponent of p. Then the compact set K is stable for the problem (P) along the sequence (K_h).*

When $p \leq 2$ the stability result follows immediately from [12, Theorem 6.3] even when Ω is not simply connected.

The approach by duality consists in proving the stability of the limit set K for problem (P) from its stability for the dual problem, which is more easy. Indeed, unlike problem (P), the admissible functions in the dual problem for the approximating sequence (K_h) belong all to the same space $W^{1,q}(\Omega)$, with the constraint that these functions are constant on every connected components of $K_h \cup (\partial\Omega \setminus \partial_D\Omega)$. Then the assumptions of Theorem 1.1 give the same constraint for the limit set K.

In the second part of the paper we study several examples of non stability in the case $p > 2$, using the tool of Γ-convergence. For instance, Example 5.2 shows that without the capacitary assumption in Theorem 1.1, we may have non stability even when K_h has just two connected components. In the case of non stability, we do not yet have a general characterization of the limit problem. However, in Example 5.6, we are able to find the limit problem under some geometrical assumptions on the sequence (K_h).

2. Notation and preliminaries

Let Ω be a bounded connected and simply connected open subset of $\mathbb{R}^2$ with Lipschitz continuous boundary $\partial\Omega$. Let $\partial_D\Omega \subset \partial\Omega$ be a (non-empty) relatively open subset of $\partial\Omega$ composed of a finite number of connected components and $\partial_N\Omega := \partial\Omega \setminus \partial_D\Omega$.

Let $\mathcal{K}(\overline{\Omega})$ be the class of compact subsets of $\overline{\Omega}$ and $\mathcal{K}_m(\overline{\Omega})$ be the subset of $\mathcal{K}(\overline{\Omega})$ whose elements have at most m connected components.

For any $x \in \Omega$ and $\rho > 0$, $B_\rho(x)$ denotes the open ball of $\mathbb{R}^2$ centered at x with radius ρ. For any subset E of $\mathbb{R}^2$, 1_E is the characteristic function of E, E^c is the complement of E, and $|E|$ is the Lebesgue measure of E. Given a subset F of some vectorial space X, I_F will denote the indicator function of F, i.e., $I_F(x)$ is equal 0 if $x \in F$ and $+\infty$ otherwise.

Throughout the paper B is an open ball containing $\overline{\Omega}$ and p and q are real numbers, with $1 < p,\, q < +\infty$ and $p^{-1} + q^{-1} = 1$.

2.1. Conjugate function and duality argument in optimization.

In this section we recall the concept of duality for the minimization of convex functionals. For more details, the reader is referred to [20].

Let X be a reflexive Banach space and let X^* be its topological dual. Given a function $F : X \to \overline{\mathbb{R}}$ convex, lower semicontinuous and proper, the conjugate function $F^* : X^* \to \overline{\mathbb{R}}$ of F is defined by:

$$F^*(u^*) := \sup_{u \in X} \{\langle u, u^* \rangle - F(u)\} \quad \forall u^* \in X^*$$

where $\langle \cdot, \cdot \rangle$ denotes the duality brackets between X and X^*.

We recall that for functionals of the type $F(u) = \int_\Omega f(x, u(x))\, dx$, defined in $L^p(\Omega, \mathbb{R}^2)$, where f satisfies for instance the assumptions (2.6)-(2.7) below, the following formula holds (see for instance [20, Proposition 2.1])

$$F^*(u^*) = \int_\Omega f^*(x, u^*(x))\, dx \qquad \forall u^* \in L^q(\Omega, \mathbb{R}^2). \tag{2.1}$$

Now we consider the following minimization problem

$$(P) \qquad \min_{u \in X} F(u).$$

Let Y be a Banach space and let Y^* be its topological dual. The duality argument in the study of Problem (P) is described as follows. We consider a family of perturbations of Problem (P):

$$(P_\xi) \qquad \min_{u \in X} \Phi(u, \xi)$$

where $\Phi : X \times Y \to \overline{\mathbb{R}}$ is a convex, lower semicontinuous and proper function such that

$$\Phi(u, 0) = F(u) \qquad \forall u \in X.$$

The dual problem of (P) with respect to Φ is given by:

$$(P^*) \qquad \sup_{\xi^* \in Y^*} \{-\Phi^*(0, \xi^*)\}.$$

The following proposition is proved in [20, Proposition 2.4].

Proposition 2.1. *Assume that* $\inf_X F$ *is finite, that* F *is coercive and that there exists* $u_0 \in X$ *such that* $\xi \to \Phi(u_0, \xi)$ *takes values in* $\mathbb{R}$ *and is continuous in* 0.

Then the problems (P) *and* (P^*) *each have at least one solution. Moreover*

$$\inf_{u \in X} F(u) = \sup_{\xi^* \in Y^*} \{-\Phi^*(0, \xi^*)\} \tag{2.2}$$

and the following relation is satisfied

$$\Phi(\bar{u}, 0) + \Phi^*(0, \bar{\xi}^*) = 0, \tag{2.3}$$

where $\bar{u}$ *is a solution of* (P) *and* $\bar{\xi}^*$ *is a solution of* (P^*).

Conversely, if $\bar{u} \in X$ *and* $\bar{\xi}^* \in Y^*$ *satisfy (2.3), then* $\bar{u}$ *is a solution of* (P) *and* $\bar{\xi}^*$ *is a solution of* (P^*).

In this paper, we will deal with functionals Φ of this type:

$$\Phi(u, \xi) = F_1(u) + F_2(Au - \xi), \tag{2.4}$$

where $F_1 : X \to \mathbb{R}$ and $F_2 : Y \to \mathbb{R}$ are convex lower semicontinuous functions and $A : X \to Y$ is a linear continuous operator. In this case, we have that

$$\Phi^*(0, \xi^*) = F_1^*(A^*\xi^*) + F_2^*(-\xi^*). \tag{2.5}$$

where $A^* : Y^* \to X^*$ denotes the transpose of the operator A.

2.2. **Deny-Lions spaces.** Given an open subset U of $\mathbb{R}^2$, the Deny-Lions space is defined by

$$L^{1,p}(U) := \{u \in L^p_{\mathrm{loc}}(U) : \nabla u \in L^p(U,\mathbb{R}^2)\}.$$

It is well-known that $L^{1,p}(U)$ coincides with the Sobolev space $W^{1,p}(U)$ whenever U is bounded and has a Lipschitz continuous boundary. It is also known that the set $\{\nabla u : u \in L^{1,p}(U)\}$ is a closed subspace of $L^p(U,\mathbb{R}^2)$. The Deny-Lions spaces $L^{1,p}$ are usually involved in minimization problems of the type (2.8) below in non-smooth domains, where Poincaré inequalities do not hold in general. For further properties of the spaces $L^{1,p}$ we refer the reader to [18] and [25].

2.3. **The minimization problem.** Let $f : \Omega \times \mathbb{R}^2 \to \mathbb{R}$ be a Borel function which satisfies the following assumptions: there exist positive constants α, β, γ such that, for almost every $x \in \Omega$ and for every $\xi \in \mathbb{R}^2$

$$\alpha|\xi|^p \leq f(x,\xi) \leq \beta|\xi|^p + \gamma; \tag{2.6}$$

$$f(x,\cdot) \text{ is strictly convex.} \tag{2.7}$$

Given $K \in \mathcal{K}(\overline{\Omega})$ and a function $g \in W^{1,p}(\Omega)$, we consider the following minimization problem

$$\min_w \left\{ \int_{\Omega\setminus K} f(x,\nabla w)\,dx : \ w \in L^{1,p}(\Omega \setminus K)\,, \quad w = g \ \text{ on } \ \partial_D\Omega \setminus K \right\} \tag{2.8}$$

whose solution exits from direct methods of the calculus of variations and is unique in the sense of gradients.

2.4. **Γ-convergence.** Let us recall the definition of De Giorgi's Γ-*convergence* in metric spaces. Let (X,d) be a metric space. We say that a sequence $F_h : X \to [-\infty,+\infty]$ Γ-converges to $F : X \to [-\infty,+\infty]$ (as $h \to \infty$) if for all $u \in X$ we have

(i) (*lower limit inequality*) for every sequence (u_h) converging to u in X,
$$\liminf_{h\to\infty} F_h(u_h) \geq F(u);$$
(ii) (*existence of a recovery sequence*) there exists a sequence (u_h) converging to u in X, such that
$$\limsup_{h\to\infty} F_h(u_h) \leq F(u).$$

The function F is called Γ-limit of (F_h) (with respect to d), and we write $F = \Gamma\text{-}\lim_h F_h$. The peculiarity of this type of convergence is its variational character explained in the following proposition.

Proposition 2.2. *Assume that $\{F_h\}$ Γ-converges to F and that there exists a compact set $K \subseteq X$ such that*

$$\inf_{u\in K} F_h(u) = \inf_{u\in X} F_h(u) \qquad \forall h \in \mathbb{N}.$$

Then

(i) $\inf_X F_h$ *converges as* $h \to \infty$ *to* $\min_X F$ *and any limit point of any sequence* (u_h) *such that*

$$\lim_{h\to\infty}\Big(F_h(u_h) - \inf_{u\in X} F_h(u)\Big) = 0$$

is a minimizer of F.

(ii) $(F_h + G)$ Γ*-converges to* $F + G$ *for any* $G : X \to]-\infty, +\infty[$ *continuous.*

We refer the reader to [11] for an exhaustive treatment of this topic.

2.5. **Hausdorff convergence.** The *Hausdorff distance* between two closed subsets K_1 and K_2 of $\overline{\Omega}$ is defined by

$$d_H(K_1, K_2) := \max\left\{\sup_{x\in K_1} \operatorname{dist}(x, K_2)\,,\ \sup_{x\in K_2} \operatorname{dist}(x, K_1)\right\},$$

with the conventions $\operatorname{dist}(x, \emptyset) = \operatorname{diam}(\Omega)$ and $\sup \emptyset = 0$, so that

$$d_H(\emptyset\,, K) = \begin{cases} 0 & \text{if } K = \emptyset, \\ \operatorname{diam}(\Omega) & \text{if } K \neq \emptyset. \end{cases}$$

Let (K_h) be a sequence of compact subsets of $\overline{\Omega}$. We say that (K_h) converges to K in the *Hausdorff metric* if $d_H(K_h\,, K)$ converges to 0. It is well-known (see e.g., [22, Blaschke's Selection Theorem]) that $\mathcal{K}(\overline{\Omega})$ and $\mathcal{K}_m(\overline{\Omega})$ are compact with respect to the Hausdorff convergence.

In order to study the continuity of the solution u of (2.8) with respect to the variations of the compact set K, we should be able to compare two solutions defined in two different domains. This is why, throughout this paper, given a function $u \in L^{1,p}(\Omega \setminus K)$, we extend ∇u in Ω by setting $\nabla u = 0$ in $\Omega \cap K$.

2.6. **Capacity.** Let $1 < r < \infty$. We recall that B is a fixed open ball containing $\overline{\Omega}$. For every subset E of B, the $(1, r)$-capacity of E in B, denoted by $C_r(E, B)$ or simply by $C_r(E)$ (when there is no ambiguity), is defined as the infimum of $\int_B |\nabla u|^r\, dx$ over the set of all functions $u \in W^{1,r}_0(B)$ such that $u \geq 1$ a.e. in a neighborhood of E. If $r > 2$, then $C_r(E) > 0$ for every nonempty set E. On the contrary, if $r = 2$ there are nonempty sets E with $C_r(E) = 0$ (for instance, $C_r(\{x\}) = 0$ for every $x \in B$).

We say that a property $\mathcal{P}(x)$ holds C_r*-quasi everywhere* (abbreviated C_r*-q.e.*) in a set E if it holds for all $x \in E$ except a subset N of E with $C_r(N) = 0$. We recall that the expression *almost everywhere* (abbreviated *a.e.*) refers, as usual, to the Lebesgue measure.

A function $u : E \to \overline{\mathbb{R}}$ is said to be *quasi-continuous* if for every ε there exists $A_\varepsilon \subset E$, with $C_r(A_\varepsilon) < \varepsilon$, such that the restriction of u to $E \setminus A_\varepsilon$ is continuous. If $r > 2$ every quasi-continuous function is continuous, while for $r = 2$ there are quasi-continuous functions that are not continuous. It is well known that, for every open subset U with $\overline{U} \subset B$, any function $u \in L^{1,r}(U)$ has a *quasi-continuous representative* $\overline{u} : U \cup \partial_L U \to \mathbb{R}$ which satisfies

$$\lim_{\rho\to 0^+} \fint_{B_\rho(x)\cap U} |u(y) - \overline{u}(x)|\, dy = 0 \quad \text{for } C_r\text{-q.e. } x \in U \cup \partial_L U,$$

where $\partial_L U$ denotes the Lipschitz part of the boundary ∂U of U. We recall that if u_h converges to u strongly in $W^{1,r}(U)$, then a subsequence of $\overline{u}_h$ converges to $\overline{u}$ pointwise C_r-q.e. on $U \cup \partial_L U$. To simplify the notation we shall always identify throughout the paper each function $u \in L^{1,r}(U)$ with its quasi-continuous representative $\overline{u}$.

For these and other properties on quasi-continuous representatives the reader is referred to [21], [24], [25], [27].

The following lemma is proved in [15, Lemma 4.1] for $p = 2$. The case $p \neq 2$ can be proved in the same way.

Lemma 2.3. *Let (K_h) be a sequence in $\mathcal{K}(\overline{\Omega})$ which converges to a compact set K in the Hausdorff metric. Let $u_h \in L^{1,p}(\Omega \setminus K_h)$ be a sequence such that $u_h = 0$ C_p-q.e. on $\partial_D\Omega \setminus K_h$ and (∇u_h) is bounded in $L^p(\Omega, \mathbb{R}^2)$. Then, there exists a function $u \in L^{1,p}(\Omega \setminus K)$ with $u = 0$ C_p-q.e. on $\partial_D\Omega \setminus K$ such that, up to a subsequence, ∇u_h converges weakly to ∇u in $L^p(A, \mathbb{R}^2)$ for every $A \subset\subset \Omega \setminus K$. If in addition $|K_h|$ converges to $|K|$, then ∇u_h converges weakly to ∇u in $L^p(\Omega, \mathbb{R}^2)$.*

The following lemma will be crucial in the proof of our main results.

Lemma 2.4. *Let $(K_h) \subset \mathcal{K}_m(\overline{\Omega})$ be a sequence which converges to a compact set K in the Hausdorff metric, and let (v_h) be a sequence in $W^{1,q}(\Omega)$ which converges to a function v weakly in $W^{1,q}(\Omega)$. Assume that the intersection of the limits of two different connected components of K_h is either empty or has positive C_q-capacity and that every function v_h is constant C_q-q.e. in each connected component of K_h. Then v is constant C_q-q.e. in each connected component of K.*

Proof. By extending both functions v_h and v in the open ball B containing $\overline{\Omega}$ such that $v_h \rightharpoonup v$ weakly in $W^{1,q}(B)$ and arguing as in [12, Lemma 3.5] we obtain that v is constant C_q-q.e. in the limit of each connected component of K_h. Now using the assumption that the intersection of the limits of two different connected components of K_h is either empty or has positive C_q-capacity, we get that v is constant C_q-q.e. in each connected component of K. □

The following lemma will be used in order to get the strong convergence of solutions in our main results.

Lemma 2.5. *Let $f : \Omega \times \mathbb{R}^2 \to R$ be a Borel function which satisfies the assumptions (2.6) and (2.7), and let (ξ_h) be a sequence in $L^p(\Omega, \mathbb{R}^2)$ weakly converging to some $\xi \in L^p(\Omega, \mathbb{R}^2)$. If $\int_\Omega f\big(x, \xi_h(x)\big)\, dx$ converges to $\int_\Omega f\big(x, \xi(x)\big)\, dx$, then $\big(\xi_h\big)$ converges to ξ strongly in $L^p(\Omega, \mathbb{R}^2)$.*

Proof. By the convexity of f, we have the following lower semicontinuity inequality

$$\int_\Omega f\big(x, \xi(x)\big)\, dx \leq \liminf_h \int_\Omega f\big(x, (\xi + \xi_h(x))/2\big)\, dx.$$

Hence,

$$\limsup_h \int_\Omega \left[\frac{1}{2}f\big(x,\xi(x)\big)+\frac{1}{2}f\big(x,\xi_h(x)\big)-f\big(x,(\xi(x)+\xi_h(x))/2\big)\right]\,dx \le$$

$$\le \limsup_h \left[\frac{1}{2}\int_\Omega f\big(x,\xi_h(x)\big)\,dx-\frac{1}{2}\int_\Omega f\big(x,\xi(x)\big)\,dx\right]=0 \tag{2.9}$$

On the other hand, by the convexity of $f(x,\cdot)$ we have that

$$\frac{1}{2}f\big(x,\xi(x)\big)+\frac{1}{2}f\big(x,\xi_h(x)\big)-f\big(x,(\xi(x)+\xi_h(x))/2\big)$$

is non negative, and thus

$$\frac{1}{2}f\big(x,\xi(x)\big)+\frac{1}{2}f\big(x,\xi_h(x)\big)-f\big(x,(\xi(x)+\xi_h(x))/2\big)\to 0 \quad \text{strongly in } L^1(\Omega). \tag{2.10}$$

Up to a subsequence, we have

$$\frac{1}{2}f\big(x,\xi(x)\big)+\frac{1}{2}f\big(x,\xi_h(x)\big)-f\big(x,(\xi(x)+\xi_h(x))/2\big)\to 0 \quad \text{a.e. in } \Omega.$$

By the strict convexity of $f(x,\cdot)$, it easily follows that

$$\xi_h(x)\to\xi(x) \quad \text{a.e. in } \Omega$$

and hence $f(x,\xi_h(x))\to f(x,\xi(x))$ a.e. in Ω. Then by Fatou's Lemma we get

$$\liminf_h \int_\Omega \left[f\big(x,\xi_h(x)\big)+f\big(x,\xi(x)\big)-\left|f\big(x,\xi_h(x)\big)-f\big(x,\xi(x)\big)\right|\right]dx \ge 2\int_\Omega f\big(x,\xi(x)\big)dx \tag{2.11}$$

from which it follows that $\limsup_h \int_\Omega \left|f\big(x,\xi_h(x)\big)-f\big(x,\xi(x)\big)\right|dx\le 0$, that is

$$f\big(x,\xi_h(x)\big)\to f\big(x,\xi(x)\big) \text{ strongly in } L^1(\Omega). \tag{2.12}$$

Now from (2.12) and by assumption (2.6), we have (up to a subsequence) that (ξ_h) is dominated in $L^p(\Omega,\mathbb{R}^2)$, which together with the pointwise convergence above imply that $\xi_h\to\xi$ strongly in $L^p(\Omega,\mathbb{R}^2)$. □

3. The dual problem

According to the notation of Section 2.1, we set

$$X:=L^{1,p}(\Omega\setminus K),\quad Y:=L^p(\Omega\setminus K,\mathbb{R}^2);$$

$$F_1(u):=I_{\{w:\ w=g \text{ on } \partial_D\Omega\setminus K\}}(u)\quad \forall u\in L^{1,p}(\Omega\setminus K);$$

$$F_2(\xi):=\int_{\Omega\setminus K} f(x,\xi)\,dx \quad \forall\xi\in L^p(\Omega\setminus K,\mathbb{R}^2);$$

$$Au:=\nabla u\quad \forall u\in L^{1,p}(\Omega\setminus K).$$

So, the functional to minimize in (2.8) is of the type (2.4), that is

$$F(u)=F_1(u)+F_2(Au).$$

According to formula (2.5), we need to compute F_1^* and F_2^*. First of all, for every $u^* \in (L^{1,p}(\Omega \setminus K))^*$ there exists some $\xi^* \in L^q(\Omega \setminus K, \mathbb{R}^2)$ such that

$$\langle u^*, u\rangle = \int_{\Omega\setminus K} \xi^* \nabla u \, dx \quad \forall u \in L^{1,p}(\Omega \setminus K). \tag{3.1}$$

Note that

$$A^* \xi^* = u^* \tag{3.2}$$

Using this representation, we have that

$$F_1^*(u^*) = \sup_{\substack{u \in L^{1,p}(\Omega\setminus K)\\ u=g \text{ on } \partial_D \Omega\setminus K}} \Big[\int_{\Omega\setminus K} \xi^* \nabla u \, dx\Big] = \sup_{\substack{u \in L^{1,p}(\Omega\setminus K)\\ u=0 \text{ on } \partial_D \Omega\setminus K}} \Big[\int_{\Omega\setminus K} \xi^* \nabla u \, dx + \int_{\Omega\setminus K} \xi^* \nabla g \, dx\Big].$$

So, by the fact that the supremum of an affine function on a vector space is equal to 0 or to ∞, we obtain

$$F_1^*(u^*) = F_1^*(A^*\xi^*) = \begin{cases} \displaystyle\int_{\Omega\setminus K} \xi^* \nabla g \, dx & \text{if } \begin{cases} \int_{\Omega\setminus K} \xi^* \nabla \varphi \, dx = 0 & \forall \varphi \in L^{1,p}(\Omega \setminus K), \\ \varphi = 0 \text{ on } \partial_D \Omega \setminus K, \end{cases} \\ +\infty & \text{otherwise .} \end{cases} \tag{3.3}$$

Note that the condition $\int_{\Omega\setminus K} \xi^* \nabla \varphi \, dx = 0 \ \forall \varphi \in L^{1,p}(\Omega \setminus K)$ with $\varphi = 0$ on $\partial_D \Omega \setminus K$, is the weak formulation of

$$\begin{cases} \operatorname{div} \xi^* = 0 & \text{in } \Omega \setminus K, \\ \xi^* \cdot \nu = 0 & \text{on } \partial_N \Omega \cup \partial K. \end{cases}$$

On the other hand, from (2.1) we have also

$$F_2^*(\xi^*) = \int_{\Omega\setminus K} f^*(x, \xi^*) \, dx \quad \forall \xi^* \in L^q(\Omega \setminus K, \mathbb{R}^2). \tag{3.4}$$

Finally, formula (2.5) in this case gives

$$\Phi^*(0, \xi^*) = \begin{cases} \displaystyle\int_{\Omega\setminus K} \big[f^*(x, -\xi^*) + \xi^* \nabla g\big] \, dx & \text{if } \begin{cases} \int_{\Omega\setminus K} \xi^* \nabla \varphi \, dx = 0 & \forall \varphi \in L^{1,p}(\Omega \setminus K), \\ \varphi = 0 \text{ on } \partial_D \Omega \setminus K, \end{cases} \\ +\infty & \text{otherwise.} \end{cases} \tag{3.5}$$

The duality formula (2.2) in this case is given by

$$\min_{u \in L^{1,p}(\Omega\setminus K)} F(u) = \sup_{\substack{\xi^* \in L^q(\Omega\setminus K, \mathbb{R}^2)\\ \int_{\Omega\setminus K} \xi^* \nabla \varphi \, dx = 0 \\ \forall \varphi \in L^{1,p}(\Omega\setminus K),\ \varphi=0 \text{ on } \partial_D \Omega\setminus K}} \int_{\Omega\setminus K} \big[-f^*(x, -\xi^*) - \xi^* \nabla g\big] \, dx \, . \tag{3.6}$$

Note that all the results above are actually valid in any dimension, while in two dimensional domains, the dual problem in the right hand-side of (3.6) can be rewritten as a maximum problem in some suitable subspace of $W^{1,q}(\Omega)$.

To this aim, let R be the rotation on $\mathbb{R}^2$ defined by

$$R(y_1, y_2) := (-y_2, y_1)$$

and let $i : W^{1,q}(\Omega) \to L^q(\Omega, \mathbb{R}^2)$ be the mapping defined by

$$i(v) := R\nabla v \quad \forall v \in W^{1,q}(\Omega). \tag{3.7}$$

For every compact set $K \subset \overline{\Omega}$ we set

$$W^{1,q}_K(\Omega) := \left\{ v \in W^{1,q}(\Omega),\ \int_\Omega v\, dx = 0 \text{ and } v \text{ is constant } C_q\text{−q.e. in every } \mathcal{C}.\mathcal{C}. \text{ of } K \right\}$$

where the notation $\mathcal{C}.\mathcal{C}.$ means connected component.
The following proposition establishes a bijection between the subspace $W^{1,q}_{K\cup\partial_N\Omega}(\Omega)$ and the set of admissible functions for the dual problem in the right hand-side of (3.6).

Proposition 3.1. *Assume that the compact set K has a finite number of connected components. Then the mapping i defined in (3.7) establishes a bijection between the subspace $W^{1,q}_{K\cup\partial_N\Omega}(\Omega)$ and the set of functions*

$$\left\{ \xi^* \in L^q(\Omega \setminus K, \mathbb{R}^2) : \int_{\Omega\setminus K} \xi^* \nabla\varphi\, dx = 0 \quad \forall \varphi \in L^{1,p}(\Omega \setminus K),\ \varphi = 0 \text{ on } \partial_D\Omega \setminus K \right\}.$$

Proof. Let $v \in W^{1,q}_{K\cup\partial_N\Omega}(\Omega)$. Let $C^1, \dots, C^l$ be the connected components of $K \cup \partial_N\Omega$. Since $v = c^i$ C_q-q.e on C^i, by [24, Theorem 4.5] we can approximate v strongly in $W^{1,q}(\Omega)$ by a sequence of functions $v_n \in C^\infty_c(\mathbb{R}^2)$ that are constant in a suitable neighborhood V^i_n of C^i. Let $\varphi \in L^{1,p}(\Omega \setminus K)$ with $\varphi = 0$ on $\partial_D\Omega \setminus K$ and let $\varphi_n \in W^{1,p}_0(\Omega \setminus K)$ such that $\varphi_n = \varphi$ in $\Omega \setminus \bigcup_i V^i_n$. Then we have that

$$\int_{\Omega\setminus K} R\nabla v_n \nabla\varphi\, dx = \int_{\Omega\setminus K} R\nabla v_n \nabla\varphi_n dx = 0, \tag{3.8}$$

where the last equality follows from the fact that the vector field $R\nabla v_n$ is divergence free. Then passing to the limit in (3.8) for $n \to \infty$, we get

$$\int_{\Omega\setminus K} R\nabla v \nabla\varphi\, dx = 0 \quad \forall \varphi \in L^{1,p}(\Omega \setminus K) \text{ with } \varphi = 0 \text{ on } \partial_D\Omega \setminus K.$$

So, i maps the space $W^{1,q}_{K\cup\partial_N\Omega}(\Omega)$ in the set of admissible function in the dual problem.
Now, let $\xi^* \in L^q(\Omega \setminus K, \mathbb{R}^2)$ be such that $\int_{\Omega\setminus K} \xi^* \nabla\varphi\, dx = 0 \quad \forall \varphi \in L^{1,p}(\Omega \setminus K),\ \varphi = 0$ on $\partial_D\Omega \setminus K$. By extending ξ^* by zero on K and still denoting this extension by ξ^*, we obtain

$$\int_\Omega \xi^* \nabla\varphi\, dx = 0 \quad \forall \varphi \in \mathcal{D}(\Omega),$$

i.e., $\operatorname{div} \xi^* = \operatorname{curl}(R\xi^*) = 0$ in $\mathcal{D}'(\Omega)$. Since Ω is simply connected, there exists $v \in W^{1,q}(\Omega)$ such that $R\xi^* = \nabla v$ a.e. in Ω. It is not restrictive to assume that $\int_\Omega v\, dx = 0$. So, we have to prove that v is constant on every connected component of $K \cup \partial_N\Omega$. Let C a connected component of $K \cup \partial_N\Omega$ and, for every $\varepsilon > 0$, let

$$C_\varepsilon := \{x \in \overline{\Omega}\colon \operatorname{dist}(x, C) < \varepsilon\} \quad K_\varepsilon := K \cup \partial_N\Omega \cup \overline{C_\varepsilon}.$$

Let ξ_ε^* be the solution of the dual problem in the right hand of (3.6) with K replaced by K_ϵ. By the monotonicity of $\Omega \setminus K_\varepsilon$, it is easy to see that $\xi_\varepsilon^* \to \xi^*$ strongly in $L^q(\Omega, \mathbb{R}^2)$ when $\varepsilon \to 0$. As above, let $v_\varepsilon \in W^{1,q}(\Omega)$ be such that $\int_\Omega v_\epsilon \, dx = 0$ and $R\xi_\varepsilon^* = \nabla v_\varepsilon$ a.e. in Ω. By the fact that $\nabla v_\varepsilon = 0$ in C_ε, $v_\varepsilon \to v$ strongly in $W^{1,q}(\Omega)$, and that $C \subset\subset C_\varepsilon$, we get that v is constant C_q-q.e. on C, so $v \in W^{1,q}_{K\cup\partial_N\Omega}(\Omega)$. $\square$

Using Proposition 3.1, the dual problem can be rewritten as

$$\sup_{v\in W^{1,q}_{K\cup\partial_N\Omega}(\Omega)} \int_{\Omega\setminus K} \left[-f^*(x, R\nabla v) + R\nabla v \nabla g\right] dx\,. \tag{3.9}$$

So, the duality formula (3.6) in two dimensional domains becomes

$$\min_{u\in L^{1,p}(\Omega\setminus K)} F(u) = \sup_{v\in W^{1,q}_{K\cup\partial_N\Omega}(\Omega)} \int_{\Omega\setminus K} \left[-f^*(x, R\nabla v) + R\nabla v \nabla g\right] dx\,. \tag{3.10}$$

Let $u \in L^{1,p}(\Omega \setminus K)$ be a solution of the left hand-side of (3.10). A solution $v \in W^{1,q}_{K\cup\partial_N\Omega}(\Omega)$ of the right hand-side of (3.10) is called *conjugate* of u.

The duality relation between u and v is

$$\int_{\Omega\setminus K} f(x, \nabla u)\, dx = \int_{\Omega\setminus K} \left[-f^*(x, R\nabla v) + R\nabla v \nabla g\right] dx. \tag{3.11}$$

Remark 3.2. Since $u = g$ on $\partial_D\Omega \setminus K$ and $v \in W^{1,q}_{K\cup\partial_N\Omega}(\Omega)$ it follows that

$$\int_{\Omega\setminus K} R\nabla v \nabla g \, dx = \int_{\Omega\setminus K} R\nabla v \nabla u \, dx.$$

Hence (3.11) becomes

$$\int_{\Omega\setminus K} \left[f(x, \nabla u) + f^*(x, R\nabla v) - R\nabla v \nabla u\right] dx = 0. \tag{3.12}$$

Since the integrand in (3.12) is positive, we get

$$f(x, \nabla u) + f^*(x, R\nabla v) - R\nabla v \nabla u = 0 \quad \text{a.e. in } \Omega \setminus K.$$

That is

$$R\nabla v \in \partial_\xi f(x, \nabla u)$$

where $\partial_\xi f(x, \nabla u)$ denotes the subdifferential of $f(x, \cdot)$ at the point ∇u. Whenever $f(x, \cdot)$ is also of class C^1, then $f^*(x, \cdot)$ is strictly convex and hence the dual problem as a unique solution v such that $R\nabla v = \nabla_\xi f(x, \nabla u)$.

For $f(\xi) := \frac{1}{p}|\xi|^p$ we obtain

$$R\nabla v = |\nabla u|^{p-2}\nabla u \quad \text{a.e. in } \Omega \setminus K.$$

In particular for $p = 2$ we obtain the classical notion of harmonic conjugate.

4. Stability for the minimum problem

Let $(K_h) \subset \mathcal{K}_m(\overline{\Omega})$ be a sequence which converges to a compact set K in the Hausdorff metric. We say that K is stable for the problem (P) along the sequence (K_h) if for every function f that satisfies conditions (2.6)-(2.7) and for every $g \in W^{1,p}(\Omega)$, we have

$$\nabla u_h \to \nabla u \quad \text{strongly in } L^p(\Omega, \mathbb{R}^2),$$

where u_h and u are solutions of (2.8) in $\Omega \setminus K_h$ and in $\Omega \setminus K$ respectively.

In the following theorem, we prove the equivalence between the stability of K for the minimum problem (2.8) and for its dual under the condition that $f(x, \cdot)$ is of class C^1.

Theorem 4.1. *Let $(K_h) \subset \mathcal{K}_m(\overline{\Omega})$ be a sequence which converges to a compact set K in the Hausdorff metric and such that $|K_h|$ converges to $|K|$. Assume that $f(x, \cdot)$ is of class C^1. Let $g \in W^{1,p}(\Omega)$. Let u_h and u be solutions of (2.8) in $\Omega \setminus K_h$ and in $\Omega \setminus K$ respectively. Let v_h and v be the solutions of the problem (3.9) in $\Omega \setminus K_h$ and in $\Omega \setminus K$ respectively. Then*

$$\nabla u_h \to \nabla u \text{ strongly in } L^p(\Omega, \mathbb{R}^2) \quad \text{if and only if} \quad \nabla v_h \to \nabla v \text{ strongly in } L^q(\Omega, \mathbb{R}^2).$$

Proof. Assume that $\nabla v_h \to \nabla v$ strongly in $L^q(\Omega, \mathbb{R}^2)$. By (3.11), we have

$$\int_{\Omega\setminus K_h} f(x, \nabla u_h)\, dx = \int_{\Omega\setminus K_h} \left[-f^*(x, R\nabla v_h) + R\nabla v_h \nabla g\right] dx. \tag{4.1}$$

By the growth assumptions (2.6) on the function f, we have that ∇u_h is bounded in $L^p(\Omega, \mathbb{R}^2)$. So applying Lemma 2.3 to $u_h - g$, we obtain that ∇u_h converges (up to a subsequence) to $\nabla \tilde{u}$ weakly in $L^p(\Omega, \mathbb{R}^2)$ for some function $\tilde{u} \in L^{1,p}(\Omega \setminus K)$ with $\tilde{u} = g$ on $\partial_D \Omega \setminus K$. So passing to the limit in (4.1) we get

$$\begin{aligned}
\int_{\Omega\setminus K} f(x, \nabla \tilde{u})\, dx &\le \liminf_{h\to\infty} \int_{\Omega\setminus K_h} f(x, \nabla u_h)\, dx \le \limsup_{h\to\infty} \int_{\Omega\setminus K_h} f(x, \nabla u_h)\, dx \\
&= \lim_{h\to\infty} \int_{\Omega\setminus K_h} \left[-f^*(x, R\nabla v_h) + R\nabla v_h \nabla g\right] dx = \int_{\Omega\setminus K} \left[-f^*(x, R\nabla v) + R\nabla v \nabla g\right] dx \\
&= \int_{\Omega\setminus K} f(x, \nabla u)\, dx
\end{aligned} \tag{4.2}$$

where the last equality follows from the duality relation between u and v. From (4.2) and the fact that $f(x, \cdot)$ is strictly convex, we get $\nabla u = \nabla \tilde{u}$ a.e. in Ω, and then all the inequalities in (4.2) are equalities. Therefore, all the sequence (∇u_h) converges weakly in $L^p(\Omega, \mathbb{R}^2)$ and

$$\lim_{h\to\infty} \int_{\Omega\setminus K_h} f(x, \nabla u_h)\, dx = \int_{\Omega\setminus K} f(x, \nabla u)\, dx.$$

Using the convention $\nabla u_h = 0$ on K_h, $\nabla u = 0$ on K, and the fact that $|K_h| \to |K|$, we get also

$$\lim_{h\to\infty} \int_{\Omega} f(x, \nabla u_h)\, dx = \int_{\Omega} f(x, \nabla u)\, dx. \tag{4.3}$$

Now using the strict convexity of $f(x, \cdot)$, we get from (4.3) and from Lemma 2.5 that (∇u_h) converges to ∇u strongly in $L^p(\Omega, \mathbb{R}^2)$.

Viceversa, suppose that $\nabla u_h \to \nabla u$ strongly in $L^p(\Omega, \mathbb{R}^2)$. Since $f(x, \cdot)$ is of class C^1, from Remark 3.2, we have $R\nabla v_h = f_\xi(x, \nabla u_h)$ a.e. in $\Omega \setminus K_h$ and $R\nabla v = f_\xi(x, \nabla u)$ a.e. in $\Omega \setminus K$. Then from the growth assumptions on f we obtain that $\nabla v_h \to \nabla v$ strongly in $L^q(\Omega, \mathbb{R}^2)$. □

In the following theorem, we give sufficient conditions on the sequence (K_h) which guarantee the stability for Problem (2.8).

Theorem 4.2. *Let $(K_h) \subset \mathcal{K}_m(\overline{\Omega})$ be a sequence which converges to a compact set K in the Hausdorff metric and such that $|K_h|$ converges to $|K|$. Let $g \in W^{1,p}(\Omega)$. Let u_h and u be solutions of (2.8) in $\Omega \setminus K_h$ and in $\Omega \setminus K$ respectively. Assume that the intersection of the limits of two different connected components of $K_h \cup \partial_N \Omega$ is either empty or has positive $(1, q)$-capacity. Then ∇u_h converges strongly to ∇u in $L^p(\Omega, \mathbb{R}^2)$.*

Proof. Let $v_h \in W^{1,q}_{K_h \cup \partial_N \Omega}(\Omega)$ and $v \in W^{1,q}_{K \cup \partial_N \Omega}(\Omega)$ be conjugates of u_h and u respectively. Up to a subsequence, $\nabla v_h \rightharpoonup \nabla \tilde{v}$ weakly in $W^{1,q}(\Omega)$ for some $\tilde{v} \in W^{1,q}(\Omega)$. By the fact that the intersection of the limits of two different connected components of $K_h \cup \partial_N \Omega$ is either empty or has positive $(1, q)$-capacity, it follows from Lemma 2.4 that $\tilde{v} \in W^{1,q}_{K \cup \partial_N \Omega}(\Omega)$.

By the growth assumptions (2.6) on the function f, we have that ∇u_h is bounded in $L^p(\Omega, \mathbb{R}^2)$. So applying Lemma 2.3 to $u_h - g$, we obtain that ∇u_h converges (up to a subsequence) to $\nabla \tilde{u}$ weakly in $L^p(\Omega, \mathbb{R}^2)$ for some function $\tilde{u} \in L^{1,p}(\Omega \setminus K)$ with $\tilde{u} = g$ on $\partial_D \Omega \setminus K$.
Since $v \in W^{1,q}_{K \cup \partial_N \Omega}(\Omega)$, using [24, Theorem 4.5] we can approximate strongly in $W^{1,q}(\Omega)$ the function v with smooth functions w_n which are constant in a suitable neighborhood of any connected component of $K \cup \partial_N \Omega$, and hence constant in any connected component of $K_h \cup \partial_N \Omega$ for h big enough. So there exists a subsequence of integers (h_n) such that $w_n \in W^{1,q}_{K_{h_n} \cup \partial_N \Omega}(\Omega)$ and w_n converges strongly in $W^{1,q}(\Omega)$ to the function v as $n \to \infty$. Therefore,

$$\int_{\Omega \setminus K} \left[-f^*(x, R\nabla v) + R\nabla v \nabla g\right] dx = \lim_{n \to \infty} \int_{\Omega \setminus K_{h_n}} \left[-f^*(x, R\nabla w_n) + R\nabla w_n \nabla g\right] dx$$

$$\leq \limsup_{n \to \infty} \int_{\Omega \setminus K_{h_n}} \left[-f^*(x, R\nabla v_{h_n}) + R\nabla v_{h_n} \nabla g\right] dx \leq \int_{\Omega \setminus K} \left[-f^*(x, R\nabla \tilde{v}) + R\nabla \tilde{v} \nabla g\right] dx.$$

Therefore, since v is a maximizer of the dual problem in $\Omega \setminus K$, all inequalities in the previous formula are equalities, so we obtain

$$\limsup_{n \to \infty} \int_{\Omega \setminus K_{h_n}} \left[-f^*(x, R\nabla v_{h_n}) + R\nabla v_{h_n} \nabla g\right] dx = \int_{\Omega \setminus K} \left[-f^*(x, R\nabla v) + R\nabla v \nabla g\right] dx.$$

Now, using the duality relations between the functions v, u on one hand, and v_{h_n}, u_{h_n} on the other hand, and then passing to the limit, we obtain

$$\begin{aligned} \int_{\Omega\setminus K} f(x,\nabla u)\,dx \;&=\; \int_{\Omega\setminus K} \big[-f^*(x,R\nabla v)+R\nabla v\nabla g\big]\,dx \\ = \; \limsup_{n\to\infty}\int_{\Omega\setminus K_{h_n}} \big[-f^*(x,R\nabla v_{h_n})+R\nabla v_{h_n}\nabla g\big]\,dx \;&=\; \limsup_{n\to\infty}\int_{\Omega\setminus K_{h_n}} f(x,\nabla u_{h_n})\,dx \\ \geq \; \liminf_{n\to\infty}\int_{\Omega\setminus K_{h_n}} f(x,\nabla u_{h_n})\,dx \;&\geq\; \int_{\Omega\setminus K} f(x,\nabla\tilde u)\,dx. \end{aligned}$$

Since $f(x,\cdot)$ is strictly convex, we get that $\nabla\tilde u=\nabla u$ a.e. in Ω. Therefore, all the sequence (∇u_h) converges weakly in $L^p(\Omega,\mathbb{R}^2)$ to ∇u. Now, using that $\int_\Omega f(x,\nabla u_h)\,dx \to \int_\Omega f(x,\nabla u)\,dx$, by Lemma 2.5 we get that (∇u_h) converges to ∇u strongly in $L^p(\Omega,\mathbb{R}^2)$. □

Remark 4.3. When $f(x,\cdot)$ is of class C^1, Theorem 4.2 is a consequence of Theorem 4.1. Indeed, the assumption that the intersection of the limits of two different connected components of $K_h\cup\partial_N\Omega$ is either empty or has positive $(1,q)$-capacity easily guarantees the stability for the dual problem, and hence using Theorem 4.1 the stability for Problem (2.8) also follows.

5. Some examples of non stability

In this section we study some examples for $f(x,\xi)=\frac{1}{p}|\xi|^p$. Throughout the section we assume that $p>2$.

5.1. **Limit problem via Γ-convergence.** In the following example, the assumptions of Theorem 4.2 hold. We show in this case that the stability result follows also by Γ-convergence arguments.

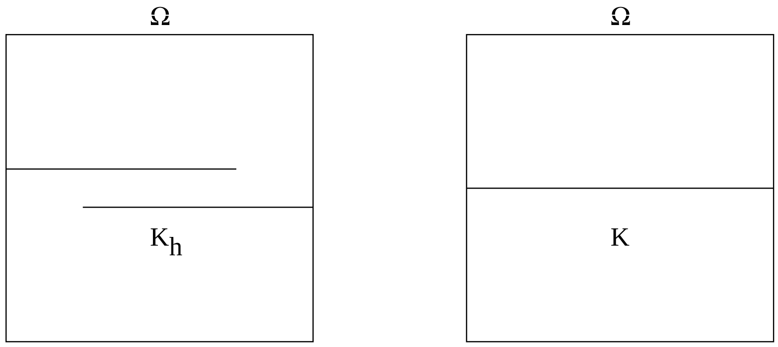

Fig. 1

Example 5.1. Let $\Omega := (-1,1)\times(-1,1)$, $\partial_D\Omega := (-1,1)\times\{-1,1\}$, $K = [-1,1]\times\{0\}$ and let

$$K_h := \left[-1,\frac{1}{2}\right]\times\left\{\frac{1}{h}\right\}\cup\left[-\frac{1}{2},1\right]\times\left\{-\frac{1}{h}\right\},$$

(see Fig. 1). We consider the sequence of functionals F_h defined in $L^p(\Omega)$ by:

$$F_h(u) := \begin{cases} \dfrac{1}{p}\displaystyle\int_{\Omega\setminus K_h} |\nabla u|^p dxdy & \text{if } u\in W^{1,p}(\Omega\setminus K_h) \text{ and } u=g \text{ on } \partial_D\Omega,\\ +\infty & \text{otherwise.}\end{cases} \tag{5.1}$$

Then, F_h Γ-converges to F_∞ in the strong topology of $L^p(\Omega)$, where

$$F_\infty(u) := \begin{cases} \dfrac{1}{p}\displaystyle\int_{\Omega\setminus K} |\nabla u|^p\, dxdy & \text{if } u\in W^{1,p}(\Omega\setminus K) \text{ and } u=g \text{ on } \partial_D\Omega,\\ +\infty & \text{otherwise.}\end{cases}$$

Hence in this case the conclusion of Theorem 4.2 follows from a general result on convergence of minima (see Proposition 2.2).

Proof. **(i) Γ-liminf**: Let $u_h\to u$ strongly in $L^p(\Omega)$, we want to prove that $\liminf_{h\to\infty} F_h(u_h)\geq F(u)$. We can assume that $\liminf_{h\to\infty} F_h(u_h) = \lim_{h\to\infty} F_h(u_h) < \infty$. So, for any $\Omega'\subset\subset\Omega\setminus K$ with $\partial_D\Omega\subset\partial\Omega'$, we have that $u_h\in W^{1,p}(\Omega')$ for h big enough, $u_h\rightharpoonup u$ in $W^{1,p}(\Omega')$ and $u=g$ on $\partial_D\Omega$. Now from the lower semicontinuity of the L^p−norm of the gradients and from the arbitrariness of Ω' we get that $u\in W^{1,p}(\Omega\setminus K)$ and the Γ-liminf inequality holds.
(ii) Γ-limsup: Let $u\in L^p(\Omega)$. We want to construct a sequence $(u_h)\subset L^p(\Omega)$ converging strongly to u in $L^p(\Omega)$ such that $\lim_h F_h(u_h)\leq F(u)$. We can assume that $u\in W^{1,p}(\Omega\setminus K)$ and $u=g$ on $\partial_D\Omega$. We set $u_h := u$ in $\Omega\setminus R_h$ where $R_h := (-1,1)\times\left[-\frac{1}{h},\frac{1}{h}\right]$. Now let us define the function u_h in R_h. To this aim, we consider the function v_h defined in R_h by

$$v_h(x,y) := u\Big(x, \frac{2}{h}sgn(x) - y\Big)$$

where $sgn(x)$ denotes the sign of x. In other words, the function v_h is obtained from u by symmetry with respect to the segment $[0,1]\times\{\frac{1}{h}\}$ for x positive and by symmetry with respect to the segment $[-1,0]\times\{-\frac{1}{h}\}$ for x negative.

Such a function v_h may jump on the segment $\{0\}\times\left[-\frac{1}{h},\frac{1}{h}\right]$. So, we consider the function $\varphi\in C^0(R_h)$ defined by

$$\varphi(x,y) := \begin{cases} 1 & \text{if } |x|>\frac{1}{2},\\ -2x & \text{if } -\frac{1}{2}\leq x\leq 0,\\ 2x & \text{if } 0\leq x\leq\frac{1}{2}.\end{cases}$$

Now we set $u_h := \varphi v_h$ on R_h. For this choice of u_h, it is easy to see that $u_h\in W^{1,p}(\Omega\setminus K_h)$ with $u_h = g$ on $\partial_D\Omega$ and that the Γ-limsup inequality holds. □

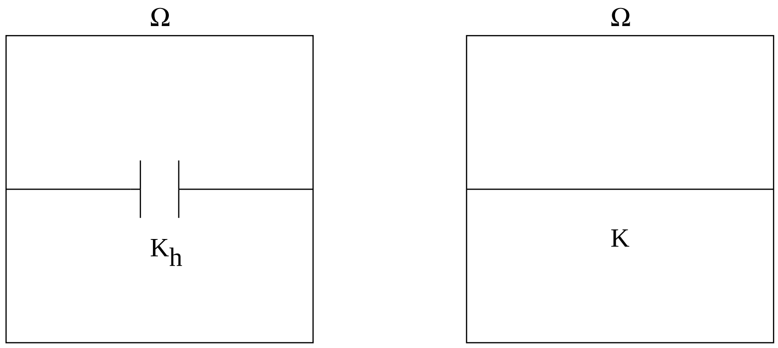

Fig.2

In the following example, we consider a sequence of compact sets K_h along which the problem (P) is not stable. More precisely in the limit problem, that is the problem solved by the limit function u, there is an additional term involving the jump of u on a point of K.

Example 5.2. Let Ω, $\partial_D\Omega$ and K be as in the previous example and let

$$K_h := \Big[-1,-\frac{a_h}{2}\Big]\times\Big\{0\Big\}\cup\Big[\frac{a_h}{2},1\Big]\times\Big\{0\Big\}\cup\Big\{-\frac{a_h}{2}\Big\}\times\Big[-\frac{b_h}{2},\frac{b_h}{2}\Big]\cup\Big\{\frac{a_h}{2}\Big\}\times\Big[-\frac{b_h}{2},\frac{b_h}{2}\Big]$$

be as in Fig. 2 with (a_h) and (b_h) being two sequences of positive numbers converging to 0. In this way (K_h) converges to K in the Hausdorff metric. Let F_h be defined as in (5.1).

Assume that the sequence $(\frac{1}{p}\,a_h\,b_h^{1-p})$ converges to some $c\in[0,+\infty]$. Then F_h Γ-converges in the strong topology of $L^p(\Omega)$ to F_∞ defined in $L^p(\Omega)$ in the following way (with the convention that $0\cdot\infty = 0$).

$$F_\infty(u) := \begin{cases} \dfrac{1}{p}\displaystyle\int_{\Omega\setminus K}|\nabla u|^p dxdy + c\,\big|u^+(0,0)-u^-(0,0)\big|^p & \text{if } \begin{cases} u\in W^{1,p}(\Omega\setminus K) \text{ and} \\ u=g \text{ on } \partial_D\Omega, \end{cases} \\ +\infty & \text{otherwise,} \end{cases} \tag{5.2}$$

where $u^+(0,0)$ and $u^-(0,0)$ are respectively the values in $(0,0)$ of the traces of $u|_{\Omega^+}$ and $u|_{\Omega^-}$ on K, Ω^+ and Ω^- being respectively the upper and the lower connected components of $\Omega\setminus K$.

Proof. **(i) Γ-liminf**: Let $u_h\to u$ in $L^p(\Omega)$, we want to prove that $\liminf_{h\to\infty}F_h(u_h)\geq F(u)$. We can assume that $\liminf_{h\to\infty}F_h(u_h)=\lim_{h\to\infty}F_h(u_h)<\infty$. So, for any $\Omega'\subset\subset\Omega\setminus K$ with $\partial_D\Omega\subset\partial\Omega'$, we have that $u_h\in W^{1,p}(\Omega')$ for h big enough, $u_h\rightharpoonup u$ in $W^{1,p}(\Omega')$ and $u=g$ on $\partial_D\Omega$. Now from the lower semicontinuity of the L^p-norm of the gradients and from the arbitrariness of Ω' we get that $u\in W^{1,p}(\Omega\setminus K)$.

We set $R_h := \left(-\frac{a_h}{2}, \frac{a_h}{2}\right) \times \left(-\frac{b_h}{2}, \frac{b_h}{2}\right)$. We have

$$\begin{aligned} F_h(u_h) &= \frac{1}{p}\int_{\Omega\setminus K_h} |\nabla u_h|^p\,dxdy = \frac{1}{p}\int_{\Omega\setminus R_h} |\nabla u_h|^p\,dxdy + \frac{1}{p}\int_{R_h} |\nabla u_h|^p\,dxdy \\ &\geq \frac{1}{p}\int_{\Omega\setminus R_h} |\nabla u_h|^p\,dxdy + \frac{1}{p}a_h b_h^{1-p} \fint_{-\frac{a_h}{2}}^{\frac{a_h}{2}} \left|u_h\Big(x,-\frac{b_h}{2}\Big) - u_h\Big(x,\frac{b_h}{2}\Big)\right|^p dx. \quad (5.3)\end{aligned}$$

Now let us fix $\Omega' \subset\subset \Omega\setminus K$. We have that $\Omega' \subset\subset \Omega\setminus R_h$ definitively, and

$$\liminf_{h\to\infty}\int_{\Omega\setminus R_h} |\nabla u_h|^p\,dxdy \geq \liminf_{h\to\infty}\int_{\Omega'} |\nabla u_h|^p\,dxdy \geq \int_{\Omega'} |\nabla u|^p\,dxdy.$$

By the arbitrariness of Ω', we get

$$\liminf_{h\to\infty}\int_{\Omega\setminus R_h} |\nabla u_h|^p\,dxdy \geq \int_{\Omega\setminus K} |\nabla u|^p\,dxdy. \quad (5.4)$$

Let us consider the functions $\tilde{u}_h^1$ defined in $(-1,1)\times(0,1-\frac{b_h}{2})$ by

$$\tilde{u}_h^1(x,y) := u_h|_{(-1,1)\times(\frac{b_h}{2},1)}\big(x, y + \frac{b_h}{2}\big)$$

and $\tilde{u}_h^2$ defined in $(-1,1)\times(-1+\frac{b_h}{2},0)$ by

$$\tilde{u}_h^2(x,y) := u_h|_{(-1,1)\times(-1,-\frac{b_h}{2})}\big(x, y - \frac{b_h}{2}\big).$$

We extend $\tilde{u}_h^1$ and $\tilde{u}_h^2$ respectively in Ω^+ and Ω^- in such a way those extensions converge weakly to u respectively in $W^{1,p}(\Omega^+)$ and in $W^{1,p}(\Omega^-)$. Recalling that $p > 2$, We have the uniform convergence of their traces on K. So,

$$\begin{aligned} \frac{1}{p}a_h b_h^{1-p} \fint_{-\frac{a_h}{2}}^{\frac{a_h}{2}} \left|u_h\Big(x,-\frac{b_h}{2}\Big) - u_h\Big(x,\frac{b_h}{2}\Big)\right|^p dx &= \frac{1}{p}a_h b_h^{1-p} \fint_{-\frac{a_h}{2}}^{\frac{a_h}{2}} \left|\tilde{u}_h^1(x,0) - \tilde{u}_h^2(x,0)\right|^p dx \\ &= \frac{1}{p}a_h b_h^{1-p} \fint_{-\frac{a_h}{2}}^{\frac{a_h}{2}} \left|u^+(x,0) - u^-(x,0) + w_h(x)\right|^p dx,\end{aligned}$$

with (w_h) converging uniformly to 0 on K. From this, it follows that

$$\lim_{h\to\infty}\frac{1}{p}a_h b_h^{1-p} \fint_{-\frac{a_h}{2}}^{\frac{a_h}{2}} \left|u_h\Big(x,-\frac{b_h}{2}\Big) - u_h\Big(x,\frac{b_h}{2}\Big)\right|^p dx = c\left|u^+(0,0) - u^-(0,0)\right|^p. \quad (5.5)$$

Therefore, the Γ-liminf inequality follows from (5.3), (5.4) and (5.5).

(ii) Γ-limsup: Let $u \in L^p(\Omega)$. We want to construct a sequence $(u_h) \subset L^p(\Omega)$ which converges to u such that $\lim_h F_h(u_h) \leq F(u)$. We can assume that $u \in W^{1,p}(\Omega\setminus K)$ and $u = g$ on $\partial_D\Omega$.

We set $u_h = u$ in $(\Omega\setminus K)\setminus R_h$ and we modify suitably u in $R_h\setminus K$ in order to get a new function which does not jump on $K\cap R_h$. To this aim let $R_h^1 := R_h \cap \{y > 0\}$ and $R_h^2 := R_h \cap \{y < 0\}$. Let us define u_h in R_h^1. We set

$$v_h := u|_{\left(-\frac{a_h}{2},\frac{a_h}{2}\right)\times\left(\frac{b_h}{2},b_h\right)}$$

and

$$\tilde{u}_h(x,y) := v_h(x, b_h - y) \quad \text{for any } (x,y) \in R_h^1.$$

In other words, $\tilde{u}_h$ is defined by taking the reflection of the restriction of u on the rectangle symmetric to R_h^1 with respect to the horizontal line $y = \frac{b_h}{2}$. Now we consider the linear function $\varphi_1(x,y) := \frac{2}{b_h}\, y$. For any $(x,y) \in R_h^1$ we set

$$u_h(x,y) := \varphi_1(x,y)\Big(\tilde{u}_h(x,y) \;-\; \frac{u^+(0,0) + u^-(0,0)}{2}\Big) \;+\; \frac{u^+(0,0) + u^-(0,0)}{2}.$$

In the similar way, we define u_h in R_h^2 using

$$u|_{\left(-\frac{a_h}{2},\frac{a_h}{2}\right)\times\left(-b_h,-\frac{b_h}{2}\right)} \quad \text{and} \quad \varphi_2(x,y) := -\frac{2}{b_h}\, y.$$

It is easy to check that $u_h \in W^{1,p}(\Omega \setminus K_h)$, $u_h = g$ on $\partial_D\Omega$ and by construction

$$\lim_{h\to\infty} \frac{1}{p}\int_{R_h^1} |\nabla u_h|^p \, dxdy \;=\; \lim_{h\to\infty} \frac{1}{p}\int_{R_h^1} |\nabla u_h|^p \, dxdy \;=\; \frac{c}{2}\left|u^+(0,0) - u^-(0,0)\right|^p.$$

Therefore,

$$\begin{aligned}
\lim_h F_h(u_h) \;&=\; \lim_h \frac{1}{p}\int_{\Omega\setminus K_h} |\nabla u_h|^p \, dxdy \\
&=\; \lim_h \frac{1}{p}\int_{(\Omega\setminus K)\setminus R_h} |\nabla u_h|^p \, dxdy \;+ \lim_h \frac{1}{p}\int_{R_h^1} |\nabla u_h|^p \, dxdy \;+\; \lim_h \frac{1}{p}\int_{R_h^2} |\nabla u_h|^p \, dxdy \\
&=\; \frac{1}{p}\int_{\Omega\setminus K} |\nabla u|^p \, dxdy \;+\; c\left|u^+(0,0) - u^-(0,0)\right|^p \;=\; F_\infty(u).
\end{aligned}$$

□

Remark 5.3. Note that if the constant c in the previous example is equal to zero, then we have the stability of K for the minimization problem (2.8) along the sequence K_h even if the intersection of the limit of the two connected components of K_h is the point $(0,0)$ whose $(1,q)-$capacity is equal to zero (recall that $q < 2$). So, in Theorem 4.2 the assumption that the limit of two connected components of K_h is either empty or has positive $(1,q)-$capacity is not a necessary condition. Although, it can not be removed as shown by the case $c > 0$.

Remark 5.4. Starting from Example 5.2, one can construct examples in which the Γ-limit involves traces at the origin from more than two subdomains, as shown in fig. 3.

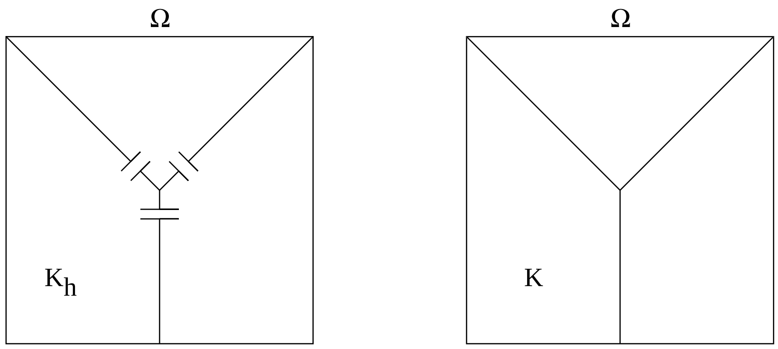

Fig. 3

In this case, we can obtain a Γ-limit of the form

$$F_\infty(u) := \begin{cases} \dfrac{1}{p}\displaystyle\int_{\Omega\setminus K} |\nabla u|^p dxdy + \sum_{1\le i<j\le 3} c_{i,j}\, \left|u^i(0,0) - u^j(0,0)\right|^p & \text{if } \begin{cases} u \in W^{1,p}(\Omega\setminus K) \\ u = g \text{ on } \partial\Omega \end{cases} \\ +\infty & \text{otherwise ,} \end{cases}$$

where $u^i(0,0)$ is the value at $(0,0)$ of the trace of $u|_{\Omega_i}$, Ω_i being the connected components of $\Omega\setminus K$.

In Theorem 4.2, the assumption that Ω is simply connected cannot be removed. In fact we will consider in the next example a sequence of connected compact sets K_h converging to K and along which the stability of K for the problem (2.8) does not hold.

Example 5.5. Let $\Omega := Q_2 \setminus \overline{Q}_1$ with Q_1 and Q_2 as in fig. 4. and let K_h and K be as in fig. 4.

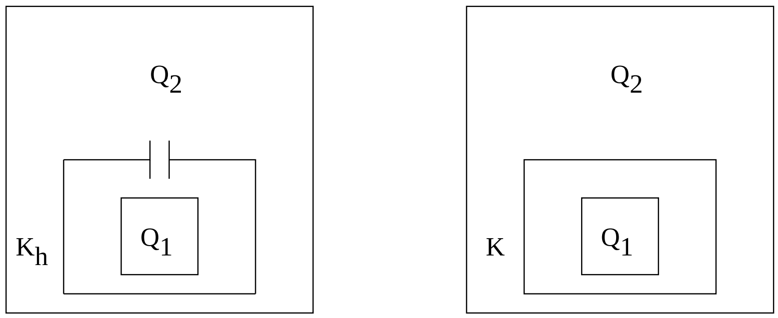

Fig. 4

In this case, arguing as in Example 5.2 we have that (F_h) Γ-converge in the strong topology of $L^p(\Omega)$ to the functional

$$F_\infty(u) := \begin{cases} \frac{1}{p}\int_{\Omega\setminus K} |\nabla u|^p dxdy + c\,\left|u^+(0,0) - u^-(0,0)\right|^p & \text{if } \begin{cases} u \in W^{1,p}(\Omega\setminus K) \text{ and} \\ u = g \text{ on } \partial_D\Omega, \end{cases} \\ +\infty & \text{otherwise.} \end{cases} \tag{5.6}$$

Let $\partial_D\Omega := \partial\Omega = \partial Q_1 \cup \partial Q_2$ and let

$$g = \begin{cases} 0 & \text{on } \partial Q_1, \\ 1 & \text{on } \partial Q_2. \end{cases}$$

Let $u_h \in W^{1,p}(\Omega \setminus K_h)$ be a solution of problem (2.8) in $\Omega \setminus K_h$. Then $\nabla u_h \rightharpoonup \nabla\tilde{u}$ weakly in $L^p(\Omega, \mathbb{R}^2)$, where $\tilde{u}$ minimizes the functional F_∞ among all functions $w \in W^{1,p}(\Omega \setminus K)$ with $w = g$ on $\partial_D\Omega$. Now, let $\varphi \in C^1(\Omega)$ with $\varphi = g$ on $\partial_D\Omega$. We can always assume that the sequence (K_h) is such that $c > \int_\Omega |\nabla\varphi|^p dx$. It is easy to see that the solution u of Problem (2.8) with data g has gradient equal to 0. Then

$$F_\infty(u) = c > \int_\Omega |\nabla\varphi|^p dx = F_\infty(\varphi) \geq F_\infty(\tilde{u}).$$

So, $u \neq \tilde{u}$ and hence K is not stable along the sequence K_h for Problem (2.8).

5.2. **Limit problem obtained by duality.** In this section we examine, by a duality approach, the problem solved by the limit function u, even when there is not stability. Let $(K_h) \subset \mathcal{K}(\overline{\Omega})$ be such that K_h converges to K in the Hausdorff metric and $|K_h|$ converges to $|K|$. Let u_h be solution of (2.8) in $(\Omega \setminus K_h)$ and v_h its conjugate which in this case satisfies

$$R\nabla v_h = |\nabla u_h|^{p-2}\nabla u_h \text{ a.e in } \Omega. \tag{5.7}$$

From Lemma 2.3 it follows that, up to a subsequence, $\nabla u_h \rightharpoonup \nabla u$ weakly in $L^p(\Omega, \mathbb{R}^2)$ for some function $u \in L^{1,p}(\Omega \setminus K)$ and $\nabla v_h \rightharpoonup \nabla v$ weakly in $L^q(\Omega, \mathbb{R}^2)$ for some function $v \in W^{1,q}(\Omega)$. Using the fact that for every $\Omega' \subset\subset \Omega \setminus K$, we have $\operatorname{div}(|\nabla u_h|^{p-2}\nabla u_h) = 0$ in $\mathcal{D}'(\Omega')$ for h big enough, it follows from the result in [1] that $\nabla u_h \to \nabla u$ a.e. in Ω'. So by the arbitrariness of Ω', we get $\nabla u_h \to \nabla u$ a.e. in $\Omega \setminus K$. Hence, using the fact $|K_h| \to |K|$ we can pass to the limit in (5.7) and obtain

$$R\nabla v = |\nabla u|^{p-2}\nabla u \text{ a.e in } \Omega, \tag{5.8}$$

through which we will find the limit problem solved by the function u in the next example.

To this aim, we call contact point, any point of $K \cup \partial_N\Omega$ which is limit of at least two sequences belonging to two different connected components of $K_h \cup \partial_N\Omega$.

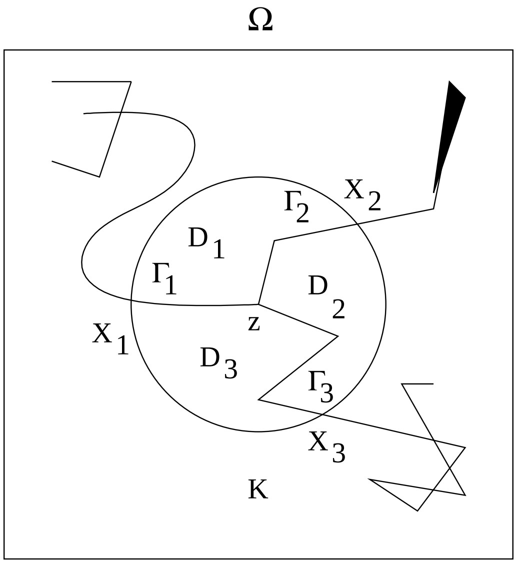

Fig. 5

Example 5.6. Let $(K_h) \subset \mathcal{K}_m(\overline{\Omega})$ be a sequence which converges to a compact set K in the Hausdorff metric and such that $|K_h|$ converges to $|K|$ with K having only one contact point $z \in \Omega$. Assume that there exists $r > 0$ such that $B_r(z) \cap K = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$, with Γ_i Lipschitz simple curves such that $\Gamma_i \cap \Gamma_j = z$ for $i \neq j$ and $\Gamma_i \cap \partial B_r(z) = x_i$ for every i (see fig.5). Suppose that $B_r(z) \setminus K = \bigcup_{i=1}^3 D_i$ with D_i Lipschitz domains.

If u_h is a solution of (2.8), then the sequence (∇u_h) converges weakly to ∇u in $L^p(\Omega, \mathbb{R}^2)$, where the function u solves a minimization problem of the type:

$$\min_w \left\{ \begin{array}{r} \dfrac{1}{p} \displaystyle\int_{\Omega\setminus K} |\nabla w|^p dx - \big[a_1(w_3 - w_1) + a_2(w_1 - w_2) + a_3(w_2 - w_3)\big] \\ w \in L^{1,p}(\Omega \setminus K),\ w = g \text{ on } \partial_D \Omega \setminus K \end{array} \right\} \tag{5.9}$$

where w_j is the trace of $w|_{D_j}$ evaluated at z and a_j coincides with the value taken on Γ_j by the continuous representative of the limit v of the conjugates v_h.

Proof. First of all from Lemma 2.3 it follows that $u \in L^{1,p}(\Omega \setminus K)$ and $u = g$ on $\partial_D \Omega \setminus K$. Now let $\varphi \in C^1(\Omega \setminus K) \cap L^{1,p}(\Omega \setminus K)$ with $\varphi|_{D_j} \in C^1(\overline{D_j})$ $\forall j$ and such that $\varphi = 0$ on $\partial_D \Omega \setminus K$. Using (5.8), we have that

$$\begin{aligned} \int_{\Omega\setminus K} |\nabla u|^{p-2} \nabla u \nabla \varphi \, dx &= \int_{\Omega\setminus K} R\nabla v \, \nabla \varphi \, dx \\ &= \int_{(\Omega\setminus K)\setminus B_r(z)} R\nabla v \, \nabla \varphi \, dx + \sum_{j=1}^{3} \int_{D_j} R\nabla v \nabla \varphi \, dx \end{aligned} \tag{5.10}$$

Since z is the only contact point of $K \cup \partial_N \Omega$, we have that any connected component of $K \cup \partial_N \Omega \setminus B_r(z)$ is contained in a limit of some connected component of $K_h \cup \partial_N \Omega$. Therefore v is constant on every connected components of $K \cup \partial_N \Omega \setminus B_r(z)$. In order to integrate by parts outside $B_r(z)$, where K in general is not regular, it is useful (from [24, Theorem 4.5]) to approximate strongly in $W^{1,q}(\Omega \setminus \overline{B_r(z)})$ the function v with smooth functions w_n that are constant in a suitable neighborhood of any connected component of $(K \cup \partial_N \Omega) \setminus B_r(z)$. As $\operatorname{div}(R\nabla w_n) = 0$, integrating by parts we get

$$
\begin{aligned}
\int_{(\Omega\setminus K)\setminus\overline{B_r(z)}} R\nabla v\,\nabla\varphi\,dx \;&=\; \lim_{n\to\infty}\int_{(\Omega\setminus K)\setminus B_r(z)} R\nabla w_n\,\nabla\varphi\,dx \\
=\; \lim_{n\to\infty}\int_{\partial B_r(z)} (R\nabla w_n)\nu\,\varphi\,d\mathcal{H}^1 \;&=\; -\lim_{n\to\infty}\int_{\partial B_r(z)} \frac{\partial w_n}{\partial\tau}\varphi\,d\mathcal{H}^1 \\
=\; +\int_{\partial B_r(z)} v\frac{\partial\varphi}{\partial\tau}\,d\mathcal{H}^1 &- (a_1\varphi|_{D_1}(x_1) - a_2\varphi|_{D_1}(x_2)) - (a_2\varphi|_{D_2}(x_2) - a_3\varphi|_{D_2}(x_3)) \\
&-(a_3\varphi|_{D_3}(x_3) - a_1\varphi|_{D_3}(x_1))
\end{aligned}
\tag{5.11}
$$

where ν is the unit vector outer normal to $B_r(z)$ and $\tau = -R\nu$ is the corresponding tangential unit vector, so that $\frac{\partial\varphi}{\partial\tau}$ denotes the tangential derivative of the function φ.

On the other hand, as $\operatorname{div}(R\nabla\varphi) = 0$, we have

$$
\sum_{j=1}^{3}\int_{D_j} R\nabla v\nabla\varphi\,dx \;=\; -\sum_{j=1}^{3}\int_{D_j} \nabla v R\nabla\varphi\,dx \;=\; -\sum_{j=1}^{3}\int_{\partial D_j} v\frac{\partial\varphi}{\partial\tau}\,d\mathcal{H}^1. \tag{5.12}
$$

Let us compute

$$
\begin{aligned}
\int_{\partial D_1} v\frac{\partial\varphi}{\partial\tau}\,d\mathcal{H}^1 \;&=\; \int_{\partial B_r(z)\cap\partial D_1} v\frac{\partial\varphi}{\partial\tau}\,d\mathcal{H}^1 + \int_{\Gamma_1} v\frac{\partial\varphi}{\partial\tau}\,d\mathcal{H}^1 + \int_{\Gamma_2} v\frac{\partial\varphi}{\partial\tau}\,d\mathcal{H}^1 \\
=\; &\int_{\partial B_r(z)\cap\partial D_1} v\frac{\partial\varphi}{\partial\tau}\,d\mathcal{H}^1 - a_1(\varphi|_{D_1}(x_1) - \varphi|_{D_1}(z)) - a_2(\varphi|_{D_1}(z) - \varphi|_{D_1}(x_2)).
\end{aligned}
$$

In a similar way

$$
\int_{\partial D_2} v\frac{\partial\varphi}{\partial\tau}\,d\mathcal{H}^1 = \int_{\partial B_r(z)\cap\partial D_2} v\frac{\partial\varphi}{\partial\tau}\,d\mathcal{H}^1 - a_2(\varphi|_{D_2}(x_2) - \varphi|_{D_2}(z)) - a_3(\varphi|_{D_2}(z) - \varphi|_{D_2}(x_3))
$$

$$
\int_{\partial D_3} v\frac{\partial\varphi}{\partial\tau}\,d\mathcal{H}^1 = \int_{\partial B_r(z)\cap\partial D_3} v\frac{\partial\varphi}{\partial\tau}\,d\mathcal{H}^1 - a_3(\varphi|_{D_3}(x_3) - \varphi|_{D_3}(z)) - a_1(\varphi|_{D_3}(z) - \varphi|_{D_3}(x_1)).
$$

Then, we have by (5.12) that

$$
\begin{aligned}
\sum_{j=1}^{3}\int_{D_j} R\nabla v\nabla\varphi\,dx \;=\; &-\int_{\partial B_r(z)} v\frac{\partial\varphi}{\partial\tau}\,d\mathcal{H}^1 + a_1(\varphi|_{D_1}(x_1)-\varphi|_{D_1}(z))\\
&+a_2(\varphi|_{D_1}(z)-\varphi|_{D_1}(x_2)) + a_2(\varphi|_{D_2}(x_2)-\varphi|_{D_2}(z)) + a_3(\varphi|_{D_2}(z)-\varphi|_{D_2}(x_3))\\
&+a_3(\varphi|_{D_3}(x_3)-\varphi|_{D_3}(z)) + a_1(\varphi|_{D_3}(z)-\varphi|_{D_3}(x_1))\\
&= -\int_{\partial B_r(z)} v\frac{\partial\varphi}{\partial\tau}\,d\mathcal{H}^1 + a_1(\varphi|_{D_3}(z)-\varphi|_{D_1}(z)) + a_2(\varphi|_{D_1}(z)-\varphi|_{D_2}(z))\\
&+a_3(\varphi|_{D_2}(z)-\varphi|_{D_3}(z)) + [(a_1\varphi|_{D_1}(x_1)-a_2\varphi|_{D_1}(x_2)) + (a_2\varphi|_{D_2}(x_2)-a_3\varphi|_{D_2}(x_3))\\
&\quad+(a_3\varphi|_{D_3}(x_3)-a_1\varphi|_{D_3}(x_1))]\,.
\end{aligned}
\tag{5.13}
$$

Therefore from (5.10), (5.11) and (5.13) we get the identity

$$\int_{\Omega\setminus K}|\nabla u|^{p-2}\nabla u\nabla\varphi\,dx = a_1(\varphi|_{D_3}(z)-\varphi|_{D_1}(z))+a_2(\varphi|_{D_1}(z)-\varphi|_{D_2}(z))+a_3(\varphi|_{D_2}(z)-\varphi|_{D_3}(z))$$

which is the weak formulation of the Euler-Lagrange equation for the minimization problem (5.9) and the conclusion follows from the convexity of the functional in (5.9). □

Remark 5.7. Note that the function u solves a problem of the type (5.9) whenever K is as in Fig.5 and z is the only contact point of K, independently of the sequence (K_h). However the constants a_i are related to the particular sequence (K_h) and to the particular boundary data g. Indeed, the constants a_i are the limits (as $h\to\infty$) of the values taken by the conjugates v_h on the connected components of K_h.

Note also that the functional to minimize in (5.9) can be rewritten as

$$\frac{1}{p}\int_{\Omega\setminus K}|\nabla w|^p dx - \big[w_1(a_2-a_1)+w_2(a_3-a_2)+w_3(a_1-a_3)\big].$$

Remark 5.8. The example above can easily be extended to cases in which there are finitely many contact points where a finite number of curves intersect each other as above. In particular, this method applied to Example 5.2 gives the following minimization problem in the limit

$$\min_w\left\{\begin{array}{l}\displaystyle\frac{1}{p}\int_{\Omega\setminus K}|\nabla w|^p dx-(a_1-a_2)(u^+(0,0)-u^-(0,0))\\ \hfill w\in L^{1,p}(\Omega\setminus K),\ w=g \text{ on } \partial_D\Omega\setminus K\end{array}\right\}. \tag{5.14}$$

Now from the weak Euler-Lagrange equations of (5.14) and of the minimization problem involving the functional F_∞ in (5.2) we get for every $\varphi\in L^{1,p}(\Omega\setminus K)$ with $\varphi=0$ on $\partial_D\Omega\setminus K$

$$
\begin{aligned}
pc|u^+(0,0)-u^-(0,0)|^{p-2}(u^+(0,0)-u^-(0,0))(\varphi^+(0,0)-\varphi^-(0,0)) \;=\\
=\;(v^+(0,0)-v^-(0,0))(\varphi^+(0,0)-\varphi^-(0,0)).
\end{aligned}
$$

By the arbitrariness of φ we get

$$pc|u^+(0,0)-u^-(0,0)|^{p-2}(u^+(0,0)-u^-(0,0)) = (v^+(0,0)-v^-(0,0))$$

which can be interpreted as a discrete version of the duality relation (5.8).

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References

[1] Boccardo L., Murat F.: Almost everywhere convergence of the gradients of solutions to elliptic and parabolic equations, *Nonlinear Anal. TMA* **19** (1992), 581-597.

[2] Bucur D., Varchon N.: Boundary variation for a Neumann problem, *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)* **29** (2000), 807-821.

[3] Bucur D., Varchon N.: A duality approach for the boundary variation of Neumann problems, Preprint Univ. Franche-Comté, 2000.

[4] Bucur D., Trebeschi P.: Shape optimization problem governed by a nonlinear state equation, *Proc. Roy. Soc. Edinburgh Sect. A* **128** (1998), 949-963.

[5] Bucur, D., Zolesio, J.-P.: Continuité par rapport au domaine dans le problème de Neumann, *C. R. Acad. Sci. Paris Sér. I* **319** (1994), 57-60.

[6] Bucur, D., Zolesio, J.-P.: Shape continuity for Dirichlet-Neumann problems, Chipot, M. (ed.) et al., *Progress in Partial Differential Equations: the Metz Surveys 4. (Metz, 1994-95).* Pitman Res. Notes in Math. 345, Longman, Harlow (1996), 53-65.

[7] Buttazzo G., Dal Maso G.: Shape optimization for Dirichlet problems: relaxed formulation and optimality conditions, *Appl. Math. Optim.* **23** (1991), 17-49.

[8] Chambolle A., Doveri F.: Continuity of Neumann linear elliptic problems on varying two-dimensional bounded open sets, *Comm. Partial Differential Equations* **22** (1997), 811-840.

[9] Chenais D.: On the existence of a solution in domain identification problem, *J. Math Anal. Appl.* **52** (1975), 189-289.

[10] Cortesani G.: Asymptotic behaviour of a sequence of Neumann problems, *Comm. Partial Differential Equations* **22** (1997), 1691-1729.

[11] Dal Maso G.: *An Introduction to* Γ*-convergence,* Birkhäuser, Boston, 1993.

[12] Dal Maso G., Ebobisse F., Ponsiglione M.: A stability result for nonlinear Neumann problems under boundary variations, Preprint SISSA (June 2002), to appear in *J. Math. Pures Appl. (9)*

[13] Dal Maso G., Mosco U.: Wiener criteria and energy decay for relaxed Dirichlet problems, *Arch. Rational Mech. Anal.* **95** (1986), 345-387.

[14] Dal Maso G., Murat F.: Asymptotic behaviour and correctors for Dirichlet problems in perforated domains with homogeneous monotone operators, *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)* **XXIV** 1997, 239-290.

[15] Dal Maso G., Toader R.: A model for the quasi-static growth of brittle fractures: existence and approximation results, *Arch. Ration. Mech. Anal.* **162** (2002), 101-135.

[16] Dal Maso G., Toader R.: A model for the quasi-static growth of brittle fractures based on local minimization, Preprint SISSA (June 2002).

[17] Damlamian A.: Le problème de la passoire de Neumann, *Rend. Sem. Mat. Univ. Politec. Torino* **43** (1985), 427-450.

[18] Deny J., Lions J.L.: Les espaces du type Beppo Levi, *Ann. Inst. Fourier (Grenoble)* **5** (1953), 305-370.

[19] Ebobisse F., Ponsiglione M.: Stability of some unilateral Free-discontinuity problems in two-dimensional domains, Preprint SISSA (July 2002).

[20] Ekeland I., Temam R.: Convex analysis and variational problems, Amsterdam: North-Holland, 1976.

[21] Evans L.C., Gariepy R.F.: *Measure Theory and Fine Properties of Functions*, CRC Press, Boca Raton, 1992.

[22] Falconer K.J.: *The Geometry of Fractal Sets*, Cambridge University Press, Cambridge, 1985.

[23] Francfort G.A., Marigo J.-J.: Revisiting brittle fracture as energy minimization problem, *J. Mech. Phys. Solids* **46** (1998), 1319-1342.
[24] Heinonen J., Kipelanen T., Martio O.: *Nonlinear Potential Theory of Degenerate Elliptic Equations*, Clarendon Press, Oxford, 1993.
[25] Maz'ya V.G.: *Sobolev Spaces*, Springer-Verlag, Berlin, 1985.
[26] Murat F.: The Neumann sieve, *Nonlinear Variational Problems (Isola d'Elba, 1983)*, Marino A. *et al.*, editor, Res. Notes in Math. 127, Pitman, London, (1985), 24-32.
[27] Ziemer W. *Weakly Differentiable Functions*, Springer-Verlag, Berlin, 1989.

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