A new characterization of Sobolev spaces on Lipschitz differentiability spaces

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Abstract

We prove a new characterization of metric Sobolev spaces, in the spirit of Brezis–Van Schaftingen–Yung's asymptotic formula. A new feature of our work is that we do not need Poincaré inequality which is a common tool in the literature. Another new feature is that we find a direct link between Brezis–Van Schaftingen–Yung's asymptotic formula and Cheeger's Lipschitz differentiability.

Keywords: Sobolev space, metric measure space, Lipschitz differentiability space, asymptotic formula

Contents

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1 Introduction

Recently, Brezis, Van Schaftingen and Yung [BVSY21] discover an asymptotic formula for Sobolev norms:

$$\lim_{\lambda\to+\infty} \lambda^p \mathcal{L}^{2N}(E^{\mathbb{R}^N}_{\lambda,u}) = \frac{k_{p,N}}{N}\|\nabla u\|^p_{L^p}, \quad \forall u \in W^{1,p}(\mathbb{R}^N), \tag{BVY}$$

where

$$E^{\mathbb{R}^N}_{\lambda,u} = \left\{(x,y) \in \mathbb{R}^N \times \mathbb{R}^N : x \neq y,\ |u(x)-u(y)| \geq \lambda |x-y|^{\frac{N}{p}+1}\right\},$$

and $k_{p,N} = \int_{\mathbb{S}^{N-1}} |e\cdot w|^p \, dw$, $e \in \mathbb{S}^{N-1}$. As a variant of the well-known asymptotic formula of Bourgain–Brezis–Mironescu (cf. [BBM01]), this new formula has received a lot of attention since its discovery and has been generalized to different settings such as ball Banach function spaces [DLY+23], Orlicz spaces [KMS23, IS24], convex bodies [GH23], Triebel Lizorkin spaces [BSY23], metric measure spaces [DLY+22], and one parameter families of operators [DM22].

Naturally, one may ask whether this formula is valid on more general spaces or not. A more interesting question is, *once we have such a formula, what can we say about the structure of the underling space?*

Inspired by recent papers [DMS19] and [DLY+22], in which the authors proved a characterization of Sobolev spaces in PI spaces in the spirit of Bourgain–Brezis–Mironescu's formula and Brezis–Van Schaftingen–Yung's formula respectively, we can partially answer the above questions and reads as follows. Denote by $\mathrm{lip}(u)$ and $\mathrm{Lip}(u)$ the lower and upper pointwise Lipschitz constants. We refer to Section 2 for other missing definitions.

Theorem 1.1 (Theorem 3.1, Theorem 3.2 and Theorem 3.6)**.** *Let $p \geq 1$ and $(X, \mathrm{d}, \mathfrak{m})$ be a metric measure space equipped with a β-doubling measure $\mathfrak{m}$. Denote by N the doubling dimension $\frac{\log\beta}{\log 2}$. Assume there are constants $b > a > 0$, such that the lower and upper density functions satisfy $a \leq \theta^-_N \leq \theta^+_N \leq b$. Then there exist universal constants C_1 and C_2 such that*

$$\varliminf_{\lambda\to+\infty} \lambda^p (\mathfrak{m}\times\mathfrak{m})(E_{\lambda,u}) \geq C_1 \int_X \frac{\big(\mathrm{lip}(u)(x)\big)^{N+p}}{\big(\mathrm{Lip}(u)(x)\big)^N}\, \mathrm{d}\mathfrak{m}(x),$$
$$\varlimsup_{\lambda\to+\infty} \lambda^p (\mathfrak{m}\times\mathfrak{m})(E_{\lambda,u}) \leq C_2 \int_X \big(\mathrm{Lip}(u)(x)\big)^p\, \mathrm{d}\mathfrak{m}(x)$$

for any $u \in \mathrm{Lip}_b(X, \mathrm{d})$, where

$$E_{\lambda,u} = \left\{(x,y) \in X\times X : x \neq y,\ |u(x)-u(y)| \geq \lambda \big(\mathrm{d}(x,y)\big)^{\frac{N}{p}+1}\right\}. \tag{1.1}$$

Furthermore, the parameter N appearing in (1.1) cannot be replaced by any other constant.

In the seminal paper [Che99], Cheeger studied the differentiability of Lipschitz function in the setting of metric measure space. In PI spaces (i.e. it is doubling and it satisfies a p-Poincaré inequality), he proved that any Lipschitz function u is differentiable and $\mathrm{lip}(u) = \mathrm{Lip}(u)$ almost everywhere. In the next decade, this "generalized Rademacher's theorem" was further studied by Keith [Kei04] and several authors, and was formed into a concept, called *Lipschitz differentiability space* named by Bate [Bat15]. A characterization of Lipschitz differentiability spaces, obtained by Keith and Bate, tells us that $(X, \mathrm{d}, \mathfrak{m})$ is a Lipschitz differentiability space if and only if $\mathrm{Lip}(u)$ and $\mathrm{lip}(u)$ are comparable for any Lipschitz function u.

Surprisingly, recently Bate, Eriksson-Bique and Soultanis [BEBS24] prove that a metric measure space $(X, \mathrm{d}, \mathfrak{m})$ is a Lipschitz differentiability space if and only if for any $u \in \mathrm{Lip}(X, \mathrm{d})$,

$$\mathrm{Lip}(u) = \mathrm{lip}(u) = |\mathrm{D}u|_* \quad \mathfrak{m}\text{-a.e.}$$

where $|\mathrm{D}u|_*$ denotes the minimal $*$-upper gradient of u introduced in [BEBS24]. Applying their theorem to Lipschitz differentiability spaces, we get the following characterization of Sobolev norms. To the best of our knowledge, this is the first asymptotic formula of this type without Poincaré inequality (we refer to [DMS19] for a characterization of Sobolev spaces in PI spaces in the spirit of the Bourgain–Brezis–Mironescu and Nguyen).

Corollary 1.2 (Corollary 3.3)**.** *Let $(X, \mathrm{d}, \mathfrak{m})$ be a Lipschitz differentiability space. Using the same notations and assumptions as Theorem 1.1, we have*

$$\begin{aligned}
\varliminf_{\lambda \to +\infty} \lambda^p (\mathfrak{m} \times \mathfrak{m})(E_{\lambda,u}) &\geq C_1 \| |\mathrm{D}u|_* \|^p_{L^p(X,\mathfrak{m})}, \\
\varlimsup_{\lambda \to +\infty} \lambda^p (\mathfrak{m} \times \mathfrak{m})(E_{\lambda,u}) &\leq C_2 \| |\mathrm{D}u|_* \|^p_{L^p(X,\mathfrak{m})}
\end{aligned}$$

for any $u \in \mathrm{Lip}_b(X, \mathrm{d})$.

In [Han24], via a metric geometry approach (after Górny [Gór22]), the first author finds a unified proof to Bourgain–Brezis–Mironescu's asymptotic formulas on some important spaces, including finite dimensional Banach spaces and equiregular sub-Riemannian manifolds. Concerning the asymptotic formula (BVY), such geometric approach still works. We remark that one can also prove more asymptotic formulas, by replacing $E_{\lambda,u}$ by $\{(x, y) \in X \times X : x \neq y, |u(x) - u(y)| \geq \rho_\lambda(x, y)\}$ for suitable mollifiers $\{\rho_\lambda\}_{\lambda>0}$.

Theorem 1.3 (Theorem 3.5)**.** *Let $p > 1, N \in \mathbb{N}$ and $(X, \mathrm{d}, \mathfrak{m})$ be a metric measure space satisfying the hypothesis in Proposition 3.4. Then for any $u \in \mathrm{Lip}_b(X, \mathrm{d})$, we have*

$$\lim_{\lambda \to +\infty} \lambda^p (\mathfrak{m} \times \mathfrak{m})(E_{\lambda,u}) = \|\nabla u\|^p_{K_{p,\mathfrak{C}}},$$

where

$$\|\nabla u\|^p_{K_{p,\mathfrak{C}}} = \int_X \int_{S_1^{\mathfrak{C}}} \frac{|u_{0,x}(w)|^p}{N} \mathrm{d}\mathfrak{m}^+_{\mathfrak{C}}(w) \mathrm{d}\mathfrak{m}(x), \tag{1.2}$$

where $S_1^{\mathfrak{C}}$ is the unit ball centered at the origin in the tangent space $\mathfrak{C}$, and $u_{0,x}$ denotes the uniform limit of the rescaling functions $u_{\delta,x}(y) := \frac{u(y)-u(x)}{\delta}, \delta > 0$.

The rest of this paper is organized as follows. In Section 2 we recall some preliminaries on calculus on metric measure spaces. In Section 3, we prove our characterization of metric Sobolev spaces.

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2 Preliminaries

In this paper, (X, d) is a Polish space, and $\mathfrak{m}$ is a σ-finite Radon measure on X with full support. The triple $(X, \mathrm{d}, \mathfrak{m})$ is called a metric measure space.

Given a function $u : X \to \mathbb{R}$, the upper pointwise Lipschitz constant $\mathrm{Lip}(u)(x) : X \to [0, +\infty]$ is defined as

$$\mathrm{Lip}(u)(x) := \varlimsup_{r\to 0^+} \sup_{y\in B_r(x)} \frac{|u(y)-u(x)|}{r} = \varlimsup_{y\to x} \frac{|u(y)-u(x)|}{\mathrm{d}(x,y)},$$

and the lower pointwise Lipschitz constant $\mathrm{lip}(u)(x) : X \to [0, +\infty]$ is defined as

$$\mathrm{lip}(u)(x) := \varliminf_{r\to 0^+} \sup_{y\in B_r(x)} \frac{|u(y)-u(x)|}{r},$$

where $B_r(x) := \{y \in X : \mathrm{d}(x, y) < r\}$. The (global) Lipschitz constant is defined as

$$\mathbf{Lip}(u) := \sup_{x\neq y} \frac{|u(y)-u(x)|}{\mathrm{d}(x,y)}.$$

If $\mathbf{Lip}(u) < +\infty$, we call u a Lipschitz function and write $u \in \mathrm{Lip}(X, \mathrm{d})$. We denote by $\mathrm{Lip}_b(X, \mathrm{d})$ the collection of Lipschitz functions with bounded support. The following property provides an alternative definition of $\mathrm{Lip}(u)(x)$ and $\mathrm{lip}(u)(x)$.

Proposition 2.1 ([Kei04]). *For any $u \in \mathrm{Lip}(X, \mathrm{d})$ and $r > 0$, define*

$$l_r u(x) = \inf_{0<s\le r} \sup_{y\in B_s(x)} \frac{|u(y)-u(x)|}{s},$$

and

$$L_r u(x) = \sup_{0<s\le r} \sup_{y\in B_s(x)} \frac{|u(y)-u(x)|}{s}.$$

Then for any $x \in X$,

$$\mathrm{lip}(u)(x) = \lim_{r\to 0^+} l_r u(x) \quad \text{and} \quad \mathrm{Lip}(u)(x) = \lim_{r\to 0^+} L_r u(x).$$

In particular, $\mathrm{lip}(u)$ and $\mathrm{Lip}(u)$ are Borel measurable.

Definition 2.2 (Doubling measure)**.** We say $\mathfrak{m}$ is β-doubling if there exists a constant $\beta > 1$, called doubling constant, such that for any $x \in \text{supp}(\mathfrak{m})$ and $r > 0$, it holds

$$\mathfrak{m}\big(B_{2r}(x)\big) \le \beta\mathfrak{m}\big(B_r(x)\big).$$

Lemma 2.3. *Let $\mathfrak{m}$ be a β-doubling measure. Then for any $x_0 \in X$, $r_0 > 0$, $x \in B_{r_0}(x_0)$, we have the following sharp estimate:*

$$\frac{\mathfrak{m}\big(B_r(x)\big)}{\mathfrak{m}\big(B_{r_0}(x_0)\big)} \ge \frac{1}{\beta^2}\left(\frac{r}{r_0}\right)^{\frac{\log\beta}{\log 2}}, \qquad \forall r \in (0, r_0).$$

In particular, the exponent $\frac{\log\beta}{\log 2}$ is optimal and we call it doubling dimension.

Definition 2.4 (Density functions)**.** For any $N > 0$, the upper and lower density functions $\theta_N^{\pm}(x) : X \to [0, +\infty]$ are defined as

$$\theta_N^+(x) := \varlimsup_{r\to 0^+} \frac{\mathfrak{m}\big(B_r(x)\big)}{r^N} \quad \text{and} \quad \theta_N^-(x) := \varliminf_{r\to 0^+} \frac{\mathfrak{m}\big(B_r(x)\big)}{r^N}.$$

On a metric measure space, we can also talk about the differentiability of Lipschitz functions. Recall the following definition proposed by Cheeger [Che99]:

Definition 2.5. Let $n \in \mathbb{N}$ and (X, d) be a metric space. We say a Borel set $U \subset X$ and a Lipschitz function $\phi : X \to \mathbb{R}^n$ form a chart (U, ϕ) of dimension n and that a function $u : X \to \mathbb{R}$ is differentiable at $x_0 \in U$ with respect to (U, ϕ) if there exists a unique $\mathrm{d}_{x_0}u \in \mathbb{R}^n$ such that

$$\varlimsup_{x\to x_0} \frac{\big|u(x) - u(x_0) - \mathrm{d}_{x_0}u \cdot \big(\phi(x) - \phi(x_0)\big)\big|}{\mathrm{d}(x, x_0)} = 0.$$

Definition 2.6 (Lipschitz differentiability space)**.** We say a metric measure space $(X, \mathrm{d}, \mathfrak{m})$ is a Lipschitz differentiability space if there exists a countable decomposition of X into charts such that any $u \in \text{Lip}(X, \mathrm{d})$ is differentiable at almost every point of every chart.

It is proved by Cheeger [Che99] that PI spaces are Lipschitz differentiability spaces. Furthermore, Lipschitz differentiability can be characterized by the following "Lip-lip condition", introduced by Keith in [Kei04].

Definition 2.7. We say a metric measure space $(X, \mathrm{d}, \mathfrak{m})$ satisfies Lip-lip condition if there exists a countable Borel decomposition $X = \bigcup_i X_i$ and for each $i \in \mathbb{N}$, there exists $\delta_i > 0$ such that for any $u \in \text{Lip}(X, \mathrm{d})$, we have

$$\text{Lip}(u)(x) \le \delta_i \text{lip}(u)(x), \quad \text{for } \mathfrak{m}\text{-a.e. } x \in X_i.$$

Theorem 2.8 ([Bat15, Gon12, Kei04])**.** *If $(X, \mathrm{d}, \mathfrak{m})$ is a metric measure space equipped with a doubling measure $\mathfrak{m}$, then $(X, \mathrm{d}, \mathfrak{m})$ is a Lipschitz differentiability space if and only if it satisfies the* Lip-lip *condition.*

Surprisingly, the inequality in Lip-lip condition can be improved to an equality. For any $u \in \mathrm{Lip}(X, \mathrm{d})$, there exists a minimal $*$-upper gradient, denoted by $|\mathrm{D}u|_*$. We refer to [BEBS24, Section 2] (and [Sch16, CKS16]) for details.

Theorem 2.9 ([BEBS24]). *$(X, \mathrm{d}, \mathfrak{m})$ is a Lipschitz differentiability space if and only if for any $u \in \mathrm{Lip}(X, \mathrm{d})$,*

$$\mathrm{Lip}(u) = \mathrm{lip}(u) = |\mathrm{D}u|_*, \quad \mathfrak{m}\text{-}a.e..$$

Remark 2.10. There are some different notions about weak gradient in metric measure setting. In general, we can not replace $|\mathrm{D}u|_*$ by Cheeger's minimal p-weak upper gradient $|\mathrm{D}u|_p$ in the above theorem. Consider a Lipschitz differentiability space $(K, |\cdot|, \mathcal{L}^n|_K)$, where $K \subseteq \mathbb{R}^n$ is a Cantor set with $\mathcal{L}^n(K) > 0$. In such a space, $|\mathrm{D}u|_p = 0$ for any $u \in \mathrm{Lip}_b(K)$ and $p \geq 1$. We refer to [AGS13, HKST15, IPS22] for more discussions about this topic.

3 Main results

3.1 Lower bound estimate

Theorem 3.1. *Let $p \geq 1$ and $(X, \mathrm{d}, \mathfrak{m})$ be a metric measure space equipped with a β-doubling measure $\mathfrak{m}$. Assume $\theta_N^- \geq a$ for some $a > 0$. Then for any $u \in \mathrm{Lip}_b(X, \mathrm{d})$, we have*

$$\varliminf_{\lambda \to +\infty} \lambda^p (\mathfrak{m} \times \mathfrak{m})(E_{\lambda,u}) \geq C \int_X \frac{\big(\mathrm{lip}(u)(x)\big)^{N+p}}{\big(\mathrm{Lip}(u)(x)\big)^N} \mathrm{d}\mathfrak{m}(x),$$

where $E_{\lambda,u}$ is defined in (1.1), $N = \frac{\log \beta}{\log 2}$, $C = C(\beta, a, p) > 0$.

Proof. For any $u \in \mathrm{Lip}_b(X, \mathrm{d})$, denote

$$L_i = \left\{ x \in X : \frac{1}{2^{i+1}} \mathrm{Lip}(u)(x) < \mathrm{lip}(u)(x) \leq \frac{1}{2^i} \mathrm{Lip}(u)(x) \right\}, \quad i \in \mathbb{N}, \tag{3.1}$$

then $X = \bigcup_{i=0}^\infty L_i \bigcup \{\mathrm{lip}(u)(x) = 0\}$. By Proposition 2.1, for any $i \in \mathbb{N}$, L_i is Borel. We divide the proof into six steps.

Step 1: Let $i_0 \in \mathbb{N}$ be such that $\mathfrak{m}(L_{i_0}) > 0$. Given $0 < \epsilon < \min \left\{ \frac{1}{\beta^2} \left(\frac{1}{8T_{i_0}} \right)^{\frac{\log \beta}{\log 2}}, \frac{1}{2}\mathfrak{m}(L_{i_0}) \right\}$ satisfying

$$\mathfrak{m}\{x \in L_{i_0} : \mathrm{lip}(u)(x) > 2^{i_0+3}\epsilon\} > \frac{2}{3}\mathfrak{m}(L_{i_0}), \tag{3.2}$$

where T_{i_0} depends only on i_0 to be determined. We claim:

- there is $K_\epsilon \subseteq L_{i_0}$ such that

$$\mathfrak{m}(L_{i_0} \setminus K_\epsilon) < \epsilon, \tag{3.3}$$

- there is $r_\epsilon > 0$ such that for any $0 < r \le r_\epsilon$ and $x \in K_\epsilon$,

$$\mathrm{lip}(u)(x) - \epsilon \le \sup_{y \in B_r(x)} \frac{|u(y) - u(x)|}{r} < 2^{i_0+1}\mathrm{lip}(u)(x) + \epsilon, \tag{3.4}$$

- for any $x, y \in K_\epsilon$ with $\mathrm{d}(x, y) \le r_\epsilon$,

$$|\mathrm{lip}(u)(x) - \mathrm{lip}(u)(y)| < \epsilon, \tag{3.5}$$

- for any $x \in K_\epsilon$ and $0 < r \le r_\epsilon$,

$$\frac{\mathfrak{m}\big(B_r(x) \cap K_\epsilon\big)}{\mathfrak{m}\big(B_r(x)\big)} > 1 - \epsilon. \tag{3.6}$$

Firstly, by Proposition 2.1, $\mathrm{lip}(u) = \lim_{r \to 0^+} l_r u$ pointwisely. By Egoroff's theorem, there is $K_\epsilon^1 \subseteq L_{i_0}$ with $\mathfrak{m}\left(L_{i_0} \setminus K_\epsilon^1\right) < \frac{\epsilon}{4}$ such that $l_r u$ converges to $\mathrm{lip}(u)$ uniformly on K_ϵ^1. So there is r_ϵ^1 such that for any $0 < r \le r_\epsilon^1$ and $x \in K_\epsilon^1$, it holds

$$l_r u(x) = \inf_{0 < s \le r} \sup_{y \in B_s(x)} \frac{|u(y) - u(x)|}{s} \ge \mathrm{lip}(u)(x) - \epsilon,$$

so that

$$\sup_{y \in B_r(x)} \frac{|u(y) - u(x)|}{r} \ge \mathrm{lip}(u)(x) - \epsilon.$$

Similarly, there is $K_\epsilon^2 \subseteq L_{i_0}$, such that $\mathfrak{m}\left(L_{i_0} \setminus K_\epsilon^2\right) < \frac{\epsilon}{4}$ and $L_r u$ converges to $\mathrm{Lip}(u)$ uniformly on K_ϵ^2. So there is r_ϵ^2 such that for any $0 < r \le r_\epsilon^2$ and $x \in K_\epsilon^2$, it holds

$$L_r u(x) = \sup_{0 < s \le r} \sup_{y \in B_s(x)} \frac{|u(y) - u(x)|}{s} \le \mathrm{Lip}(u)(x) + \epsilon.$$

Note that $x \in L_{i_0}$, we also have

$$\sup_{y \in B_r(x)} \frac{|u(y) - u(x)|}{r} \le \mathrm{Lip}(u)(x) + \epsilon < 2^{i_0+1}\mathrm{lip}(u)(x) + \epsilon.$$

Next, by Lusin's theorem, there exists a compact set $K_\epsilon^3 \subseteq L_{i_0}$ such that $\mathfrak{m}\left(L_{i_0} \setminus K_\epsilon^3\right) < \frac{\epsilon}{4}$ and $\mathrm{lip}(u)$ is continuous on K_ϵ^3. By Heine–Cantor theorem, there is r_ϵ^3 such that for any $x, y \in K_\epsilon^3$ with $\mathrm{d}(x, y) \le r_\epsilon^3$, we have $|\mathrm{lip}(u)(x) - \mathrm{lip}(u)(y)| \le \epsilon$.

By Lebesgue differentiation theorem, for $\mathfrak{m}$-a.e. $x \in K_\epsilon^3$, we have

$$\lim_{r \to 0^+} \frac{\mathfrak{m}\big(B_r(x) \cap K_\epsilon^3\big)}{\mathfrak{m}\big(B_r(x)\big)} = 1.$$

By Egoroff's theorem again, there is $K_\epsilon^4 \subseteq K_\epsilon^3 \subseteq L_{i_0}$ such that $\mathfrak{m}\left(K_\epsilon^3 \setminus K_\epsilon^4\right) < \frac{\epsilon}{4}$ and $\frac{\mathfrak{m}\left(B_r(x) \cap K_\epsilon^3\right)}{\mathfrak{m}\left(B_r(x)\right)}$ converges to 1 uniformly on K_ϵ^4. Thus there exists r_ϵ^4 such that for any $0 < r \le r_\epsilon^4$ and $x \in K_\epsilon^4$, it holds

$$\frac{\mathfrak{m}\big(B_r(x) \cap K_\epsilon^3\big)}{\mathfrak{m}\big(B_r(x)\big)} \ge 1 - \epsilon.$$

Choosing $r_\epsilon = \min\{r_\epsilon^1, r_\epsilon^2, r_\epsilon^3, r_\epsilon^4\}$, $K_\epsilon = K_\epsilon^1 \cap K_\epsilon^2 \cap K_\epsilon^3 \cap K_\epsilon^4$, we get the claim.

Step 2: Denote

$$A_{\epsilon,i_0} = \left\{x \in K_\epsilon \subset L_{i_0} : \mathrm{lip}(u)(x) > 2^{i_0+3}\epsilon\right\}. \tag{3.7}$$

We know that A_{ϵ,i_0} is non-empty. Indeed, if A_{ϵ,i_0} is empty, by the choice of ϵ and (3.3), we have

$$\mathfrak{m}(\left\{x \in K_\epsilon \subset L_{i_0} : \mathrm{lip}(u)(x) \le 2^{i_0+3}\epsilon\right\}) = \mathfrak{m}(K_\epsilon) > \frac{1}{2}\mathfrak{m}(L_{i_0}),$$

which contradicts (3.2).

Fix $x \in A_{\epsilon,i_0}$ and $0 < r \le r_\epsilon$. By (3.4), there is $\bar{y}_{x,r} \in \overline{B_{\frac{r}{2}}(x)}$ such that

$$|u(x) - u(\bar{y}_{x,r})| \ge \frac{r}{2}\big(\mathrm{lip}(u)(x) - \epsilon\big) \ge \left(\frac{1}{2} - \frac{1}{2^{i_0+4}}\right)\mathrm{lip}(u)(x)r. \tag{3.8}$$

Let $T_{i_0} > 2$ and consider geodesic balls $B_{\frac{r}{T_{i_0}}}(\bar{y}_{x,r})$ and $B_{\frac{r}{2T_{i_0}}}(\bar{y}_{x,r})$. Assume by contradiction that $B_{\frac{r}{T_{i_0}}}(\bar{y}_{x,r}) \setminus B_{\frac{r}{2T_{i_0}}}(\bar{y}_{x,r}) \subseteq K_\epsilon^c$. Note $B_{\frac{r}{T_{i_0}}}(\bar{y}_{x,r}) \subset B_r(x)$, by (3.6), we have

$$\frac{\mathfrak{m}\big(B_{\frac{r}{T_{i_0}}}(\bar{y}_{x,r}) \setminus B_{\frac{r}{2T_{i_0}}}(\bar{y}_{x,r})\big)}{\mathfrak{m}\big(B_r(x)\big)} \le \frac{\mathfrak{m}\big(B_r(x) \cap K_\epsilon^c\big)}{\mathfrak{m}\big(B_r(x)\big)} \le \epsilon.$$

However, there is $z \in B_r(x)$ such that $\mathrm{d}(z, \bar{y}_{x,r}) = \frac{3r}{4T_{i_0}}$ and $B_{\frac{r}{8T_{i_0}}}(z) \subset B_{\frac{r}{T_{i_0}}}(\bar{y}_{x,r}) \setminus B_{\frac{r}{2T_{i_0}}}(\bar{y}_{x,r})$. By Lemma 2.3, we get

$$\frac{\mathfrak{m}\big(B_{\frac{r}{T_{i_0}}}(\bar{y}_{x,r}) \setminus B_{\frac{r}{2T_{i_0}}}(\bar{y}_{x,r})\big)}{\mathfrak{m}\big(B_r(x)\big)} \ge \frac{\mathfrak{m}\big(B_{\frac{r}{8T_{i_0}}}(z)\big)}{\mathfrak{m}\big(B_r(x)\big)} \ge \frac{1}{\beta^2}\left(\frac{1}{8T_{i_0}}\right)^{\frac{\log\beta}{\log 2}} > \epsilon,$$

which is a contradiction. So there is $y_{x,r} \in B_{\frac{r}{T_{i_0}}}(\bar{y}_{x,r}) \setminus B_{\frac{r}{2T_{i_0}}}(\bar{y}_{x,r}) \cap K_\epsilon$. Then

$$\begin{aligned} \sup_{z \in B_{\frac{r}{T_{i_0}}}(y_{x,r})} |u(z) - u(y_{x,r})| &\overset{(3.4)}{<} \frac{r}{T_{i_0}}\big(2^{i_0+1}\mathrm{lip}(u)(y_{x,r}) + \epsilon\big) \\ &\overset{(3.5)}{\le} \frac{r}{T_{i_0}}\big(2^{i_0+1}\mathrm{lip}(u)(x) + 2^{i_0+1}\epsilon + \epsilon\big) \\ &\overset{(3.7)}{\le} \frac{2^{i_0+2}r}{T_{i_0}}\mathrm{lip}(u)(x). \end{aligned} \tag{3.9}$$

Step 3: In this step we will estimate $|u(y_{x,r}) - u(\bar{y}_{x,r})|$. Denote $y = y_{x,r}$, $\bar{y} = \bar{y}_{x,r}$, and $s = \mathrm{d}(y, \bar{y}) \in \left[\frac{r}{2T_{i_0}}, \frac{r}{T_{i_0}}\right]$. By (3.4), (3.5), (3.7) and the fact $\mathrm{d}(y, x) < r$, we have

$$\begin{aligned} |u(y) - u(\bar{y})| &\le \sup_{z \in B_s(y)} |u(z) - u(y)| \\ &\le s\big(2^{i_0+1}\mathrm{lip}(u)(y) + \epsilon\big) \\ &\le s\big(2^{i_0+1}\mathrm{lip}(u)(x) + 2^{i_0+1}\epsilon + \epsilon\big) \\ &< 2^{i_0+2}\mathrm{lip}(u)(x)\mathrm{d}(y, \bar{y}). \end{aligned}$$

Step 4: Denote $S_{i_0}(x,r) = \left\{z \in B_r(x) : x \in A_{\epsilon,i_0}, |u(z)-u(x)| \geq \frac{1}{8}\mathrm{lip}(u)(x)\mathrm{d}(z,x)\right\}$. We claim, for T_{i_0} large enough,

$$B_{\frac{r}{T_{i_0}}}(y_{x,r}) \subseteq S_{i_0}(x,r), \qquad \forall x \in A_{\epsilon,i_0}, r < r_\epsilon. \tag{3.10}$$

Recall that $\mathrm{d}(y_{x,r}, \bar{y}_{x,r}) \leq \frac{r}{T_{i_0}}$. By Step 3, we have

$$|u(y_{x,r}) - u(\bar{y}_{x,r})| \leq \frac{2^{i_0+2}}{T_{i_0}}\mathrm{lip}(u)(x)r. \tag{3.11}$$

For $T_{i_0} = 2^{i_0+5}$, we have

$$|u(y_{x,r}) - u(\bar{y}_{x,r})| \leq \frac{1}{8}\mathrm{lip}(u)(x)r. \tag{3.12}$$

Thus, by (3.8) and (3.12),

$$|u(x) - u(y_{x,r})| \geq |u(x) - u(\bar{y}_{x,r})| - |u(y_{x,r}) - u(\bar{y}_{x,r})| > \frac{1}{4}\mathrm{lip}(u)(x)r. \tag{3.13}$$

By (3.9) and (3.13), for any $z \in B_{\frac{r}{T_{i_0}}}(y_{x,r}) \subset B_r(x)$,

$$|u(z) - u(x)| \geq |u(x) - u(y_{x,r})| - |u(z) - u(y_{x,r})| > \frac{1}{8}\mathrm{lip}(u)(x)\mathrm{d}(z,x). \tag{3.14}$$

Therefore, $B_{\frac{r}{T_{i_0}}}(y_{x,r}) \subseteq S_{i_0}(x,r)$.

Step 5: Fix $x \in A_{\epsilon,i_0}$. Let $\lambda, r > 0$ be such that $8\lambda r^{N/p} = \mathrm{lip}(u)(x)$. Denote

$$\hat{E}_{\lambda,u}(x) = \left\{z \in X : |u(x) - u(z)| \geq \lambda\big(\mathrm{d}(x,z)\big)^{\frac{N}{p}+1}\right\}.$$

We can see that

$$B_{\frac{r}{T_{i_0}}}(y_{x,r}) \subseteq S_{i_0}(x,r) \subseteq \hat{E}_{\lambda,u}(x). \tag{3.15}$$

Then by Fatou's lemma, we have

$$\begin{aligned}
& \varliminf_{\lambda\to+\infty} \lambda^p \int_{L_{i_0}}\int_X \chi_{E_{\lambda,u}}(x,y)\mathrm{d}\mathfrak{m}(y)\mathrm{d}\mathfrak{m}(x) \\
\overset{(3.15)}{\geq} & \varliminf_{\lambda\to+\infty} \lambda^p \int_{A_{\epsilon,i_0}} \left(\frac{r}{T_{i_0}}\right)^N \left(\frac{T_{i_0}}{r}\right)^N \mathfrak{m}\left(B_{\frac{r}{T_{i_0}}}(y_{x,r})\right)\mathrm{d}\mathfrak{m}(x) \\
\geq & \frac{a}{2^{(i_0+5)N}8^p}\int_{A_{\epsilon,i_0}} |\mathrm{lip}(u)(x)|^p\mathrm{d}\mathfrak{m}(x),
\end{aligned}$$

where we use $\theta_N^- \geq a$ in the last inequality.

Letting $\epsilon \to 0$, we obtain

$$\varliminf_{\lambda\to+\infty} \lambda^p \int_{L_{i_0}}\int_X \chi_{E_{\lambda,u}}(x,y)\mathrm{d}\mathfrak{m}(y)\mathrm{d}\mathfrak{m}(x) \geq \frac{a}{2^{(i_0+5)N}8^p}\int_{L_{i_0}} |\mathrm{lip}(u)(x)|^p\mathrm{d}\mathfrak{m}(x).$$

Step 6: Finally,

$$\begin{aligned}
& \lim_{\lambda\to+\infty} \lambda^p(\mathfrak{m}\times\mathfrak{m})(E_{\lambda,u}) \\
= & \lim_{\lambda\to+\infty} \lambda^p \int_X\int_X \chi_{E_{\lambda,u}}(x,y)\mathrm{d}\mathfrak{m}(y)\mathrm{d}\mathfrak{m}(x) \\
\geq & \sum_{i=0}^{+\infty} \lim_{\lambda\to+\infty} \lambda^p \int_{L_i}\int_X \chi_{E_{\lambda,u}}(x,y)\mathrm{d}\mathfrak{m}(y)\mathrm{d}\mathfrak{m}(x) \\
\geq & \sum_{i=0}^{+\infty} \int_{L_i} \frac{a}{2^{(i+5)N}8^p} |\mathrm{lip}(u)(x)|^p \,\mathrm{d}\mathfrak{m}(x) \\
\overset{(3.1)}{\geq} & \frac{a}{2^{5N}8^p}\int_X \frac{\big(\mathrm{lip}(u)(x)\big)^{N+p}}{\big(\mathrm{Lip}(u)(x)\big)^N}\mathrm{d}\mathfrak{m}(x),
\end{aligned}$$

which is the thesis.

□

3.2 Upper bound estimate

Theorem 3.2. *Let $p\geq 1$ and $(X,\mathrm{d},\mathfrak{m})$ be a metric measure space equipped with a β-doubling measure $\mathfrak{m}$. Assume $\theta_N^+\leq b$ for some $b>0$. Then for any $u\in \mathrm{Lip}_b(X,\mathrm{d})$, we have*

$$\varlimsup_{\lambda\to+\infty} \lambda^p(\mathfrak{m}\times\mathfrak{m})(E_{\lambda,u}) \leq 2b\int_X \big(\mathrm{Lip}(u)(x)\big)^p \,\mathrm{d}\mathfrak{m}(x),$$

where $E_{\lambda,u}$ is defined in (1.1) and $N=\frac{\log\beta}{\log 2}$.

Proof. Note that if $x,y\notin \operatorname{supp} u$, then $(x,y)\notin E_{\lambda,u}$. We just need to consider $x\in\operatorname{supp} u$. Denote $\hat{E}_{\lambda,u}(x)=\{y\in X: y\neq x, |u(x)-u(y)|\geq \lambda\big(\mathrm{d}(x,y)\big)^{\frac{N}{p}+1}\}$. For any $\epsilon>0$, there exists $\delta>0$ such that for any $y\in B_\delta(x)$,

$$\frac{|u(x)-u(y)|}{\mathrm{d}(x,y)}\leq \mathrm{Lip}(u)(x)+\epsilon.$$

Note also that there exists λ_1, such that for any $\lambda>\lambda_1$,

$$\left(\frac{\mathrm{Lip}(u)(x)+\epsilon}{\lambda}\right)^{\frac{p}{N}} \leq \left(\frac{\mathbf{Lip}(u)+\epsilon}{\lambda}\right)^{\frac{p}{N}} <\delta.$$

So for any $y\in \hat{E}_{\lambda,u}(x)$, we have

$$\lambda\delta^{\frac{N}{p}} > \mathbf{Lip}(u)+\epsilon \geq \frac{|u(x)-u(y)|}{\mathrm{d}(x,y)} \geq \lambda\big(\mathrm{d}(x,y)\big)^{\frac{N}{p}}.$$

Thus $\hat{E}_{\lambda,u}(x)\subset B_\delta(x)$ and

$$\begin{aligned}
\mathfrak{m}\left(\hat{E}_{\lambda,u}(x)\right) &= \mathfrak{m}\left(\hat{E}_{\lambda,u}(x)\cap B_\delta(x)\right) \\
&\leq \mathfrak{m}\left(\left\{y\in B_\delta(x):\ \mathrm{Lip}(u)(x)+\epsilon\geq \lambda\big(\mathrm{d}(x,y)\big)^{\frac{N}{p}}\right\}\right) \\
&= \mathfrak{m}\left(B_{\left(\frac{\mathrm{Lip}(u)(x)+\epsilon}{\lambda}\right)^{\frac{p}{N}}}(x)\right).
\end{aligned} \tag{3.16}$$

Since $\mathfrak{m}$ is β-doubling, by Lemma 2.3, for any $0 < r < \left(\frac{\mathbf{Lip}(u)+\epsilon}{\lambda}\right)^{\frac{p}{N}}$, we have

$$\frac{\mathfrak{m}\left(B_{\left(\frac{\mathbf{Lip}(u)+\epsilon}{\lambda}\right)^{\frac{p}{N}}}(x)\right)}{\lambda^{-p}} \leq \beta^2 \left(\frac{\left(\frac{\mathbf{Lip}(u)+\epsilon}{\lambda}\right)^{\frac{p}{N}}}{r}\right)^N \frac{\mathfrak{m}(B_r(x))}{\lambda^{-p}} = \beta^2 \big(\mathbf{Lip}(u)+\epsilon\big)^p \frac{\mathfrak{m}(B_r(x))}{r^N}.$$

Letting $r \to 0^+$, we obtain

$$\lambda^p \mathfrak{m}\left(B_{\left(\frac{\mathrm{Lip}(u)(x)+\epsilon}{\lambda}\right)^{\frac{p}{N}}}(x)\right) \leq \beta^2 \big(\mathbf{Lip}(u)+\epsilon\big)^p \varlimsup_{r\to 0^+} \frac{\mathfrak{m}(B_r(x_0))}{r^N} \leq \beta^2 \big(\mathbf{Lip}(u)+\epsilon\big)^p b.$$

Therefore, by Fubini's theorem and Fatou's lemma, we have

$$\begin{aligned}
&\varlimsup_{\lambda\to+\infty} \lambda^p (\mathfrak{m}\times\mathfrak{m})(E_{\lambda,u}) \\
=&\varlimsup_{\lambda\to+\infty} \lambda^p \int_X \int_X \chi_{E_{\lambda,u}}(x,y)\mathrm{d}\mathfrak{m}(y)\mathrm{d}\mathfrak{m}(x) \\
\leq& 2\int_{\mathrm{supp}\,u} \varlimsup_{\lambda\to+\infty} \lambda^p \mathfrak{m}\left(B_{\left(\frac{\mathrm{Lip}(u)(x)+\epsilon}{\lambda}\right)^{\frac{p}{N}}}(x)\right)\mathrm{d}\mathfrak{m}(x) \\
\leq& 2b\int_X \big(\mathrm{Lip}(u)(x)+\epsilon\big)^p \mathrm{d}\mathfrak{m}(x),
\end{aligned} \tag{3.17}$$

where we use $\theta_N^+ \leq b$ in the last inequality.

Letting $\epsilon \to 0$, then we prove the theorem. □

Concerning Lipschitz differentiability spaces, by Theorem 2.9, we have:

Corollary 3.3. *Let $p \geq 1$ and $(X, \mathrm{d}, \mathfrak{m})$ be a Lipschitz differentiability space equipped with a β-doubling measure $\mathfrak{m}$. If $a \leq \theta_N^- \leq \theta_N^+ \leq b$ for some $b > a > 0$, then for any $u \in \mathrm{Lip}_b(X, \mathrm{d})$, there exist universal constants C_1 and C_2 depending on β, a, b, p, such that*

$$\varliminf_{\lambda\to+\infty} \lambda^p (\mathfrak{m}\times\mathfrak{m})(E_{\lambda,u}) \geq C_1 \||\mathrm{D}u|_*\|^p_{L^p(X,\mathfrak{m})},$$
$$\varlimsup_{\lambda\to+\infty} \lambda^p (\mathfrak{m}\times\mathfrak{m})(E_{\lambda,u}) \leq C_2 \||\mathrm{D}u|_*\|^p_{L^p(X,\mathfrak{m})}.$$

3.3 Asymptotic formula

The following geometric characterizations of finite-dimensional Banach spaces and equi-regular sub-Riemannian manifolds can be find in [Han24, Section 4].

Proposition 3.4. *Let $(X, \mathrm{d}, \mathfrak{m})$ and $(\mathfrak{C}, \mathrm{d}_{\mathfrak{C}}, \mathfrak{m}_{\mathfrak{C}})$ be one of the following pairs of metric measure spaces:*

- *$(X, \mathrm{d}, \mathfrak{m})$ is an m-dimensional equi-regular sub-Riemannian manifold with homogeneous dimension $N \geq m$, equipped with the Carnot–Carathéodory metric d and the associated Hausdorff measure $\mathfrak{m} = \mathcal{H}^N_{\mathrm{d}}$, and $(\mathfrak{C}, \mathrm{d}_{\mathfrak{C}}, \mathfrak{m}_{\mathfrak{C}})$ is the Carnot group $(\mathbb{R}^m, \mathrm{d}_C, \mathcal{H}^N_{\mathrm{d}_C})$ with homogeneous dimension N.*

- *both* $(X, \mathrm{d}, \mathfrak{m})$ *and* $(\mathfrak{C}, \mathrm{d}_\mathfrak{C}, \mathfrak{m}_\mathfrak{C})$ *are isometric to an* N*-dimensional Banach space* $(\mathbb{R}^N, ||\cdot||, \mathcal{L}^N)$.

Then we have:

- ***(Tangent space: metric)*** *For* $\mathfrak{m}$*-a.e.* $x \in X$*, there is a family of maps* $\{\phi_\delta\}_{\delta>0}$ *from* X *to* $\mathfrak{C}$ *satisfying* $\phi_\delta(x) = 0 \in \mathfrak{C}$ *and*

$$\left|\frac{\frac{1}{\delta}\mathrm{d}(y,z)}{\mathrm{d}_\mathfrak{C}\big(\phi_\delta(y),\phi_\delta(z)\big)} - 1\right| < \eta(\delta), \qquad \forall y, z \in B_\delta(x),\ \delta \in (0,1), \tag{3.18}$$

where $\eta : (0,1) \to (0,1)$ *is an increasing function with* $\lim_{\delta\searrow 0}\eta(\delta) = 0$.

- ***(Tangent space: measure)*** *For any* $\epsilon > 0$*, there exists* $\delta_0 > 0$ *such that for any* $\delta < \delta_0$*, we have*

$$(1-\epsilon)\delta^N \mathfrak{m}_\mathfrak{C}|_{\phi_\delta(B_\delta(x))} < (\phi_\delta)_\sharp\left(\mathfrak{m}|_{B_\delta(x)}\right) < (1+\epsilon)\delta^N \mathfrak{m}_\mathfrak{C}|_{\phi_\delta(B_\delta(x))}. \tag{3.19}$$

- ***(Homogeneity)*** *For any* $\delta > 0$*, there is a dilation map* D_δ*, which is an isometry between* $(\mathfrak{C}, \mathrm{d}_\mathfrak{C}, \mathfrak{m}_\mathfrak{C})$ *and* $(\mathfrak{C}, \delta^{-1}\mathrm{d}_\mathfrak{C}, \delta^{-N}\mathfrak{m}_\mathfrak{C})$ *such that* $D_\delta(0) = 0$ *and* $D_\delta \circ D_{\delta^{-1}} = \mathrm{Id}$*. In particular,*

$$(D_\delta)_\sharp \mathfrak{m}_\mathfrak{C} = \delta^{-N}\mathfrak{m}_\mathfrak{C} \tag{3.20}$$

and

$$\mathfrak{m}^+_\mathfrak{C}|_{S^\mathfrak{C}_\delta} = \delta^{N-1}(D_\delta)_\sharp\left(\mathfrak{m}^+_\mathfrak{C}|_{S^\mathfrak{C}_1}\right), \tag{3.21}$$

where $S^\mathfrak{C}_\delta$ *is the boundary of the ball centered at the origin in the tangent space* $\mathfrak{C}$ *with radius* δ*, and* $\mathfrak{m}^+_\mathfrak{C}$ *is the boundary measure.*

- ***(Rademacher's theorem)*** *For any* $u \in \mathrm{Lip}_b(X, \mathrm{d})$ *and* $\mathfrak{m}$*-a.e.* $x \in X$*, there is a unique function* $u_{0,x}$ *on* $(\mathfrak{C}, \mathrm{d}_\mathfrak{C})$*, such that the rescaling functions*

$$u_{\delta,x}(y) := \frac{u(y) - u(x)}{\delta}, \quad y \in X, \quad \delta > 0$$

converge to $u_{0,x}$ *in the following sense: there is a function* $\alpha(\delta)$ *satisfying* $\alpha(\delta) \searrow 0$ *as* $\delta \searrow 0$ *such that*

$$\|u_{0,x} \circ \phi_\delta - u_{\delta,x}\|_{L^\infty(B_\delta(x),\mathfrak{m})} \le \alpha(\delta), \quad \forall \delta > 0. \tag{3.22}$$

Then we can prove the following asymptotic formula. We remark that our proof also works for general mollifiers. Interested readers could find more asymptotic formulas by combining our theorems and some carefully selected mollifiers (cf. [Han24, §3.2]).

Theorem 3.5. *Let* $p > 1, N \in \mathbb{N}$ *and* $(X, \mathrm{d}, \mathfrak{m})$ *be a metric measure space satisfying the hypothesis in Proposition 3.4. Then for any* $u \in \mathrm{Lip}_b(X, \mathrm{d})$*, we have*

$$\lim_{\lambda\to+\infty} \lambda^p (\mathfrak{m}\times\mathfrak{m})(E_{\lambda,u}) = \|\nabla u\|^p_{K_{p,\mathfrak{C}}},$$

where

$$\|\nabla u\|^p_{K_{p,\mathfrak{C}}} = \int_X \int_{S^\mathfrak{C}_1} \frac{|u_{0,x}(w)|^p}{N} \mathrm{d}\mathfrak{m}^+_\mathfrak{C}(w)\mathrm{d}\mathfrak{m}(x). \tag{3.23}$$

Proof. For $x \in X$, $\delta < \delta_0$ and $\lambda > 0$, we write

$$E_{\lambda,u,\delta}(x) = \underbrace{\big\{(x,y) \in E_{\lambda,u}\big\} \cap B_\delta(x)}_{E^1_{\lambda,u,\delta}(x)} \cup \underbrace{\big\{(x,y) \in E_{\lambda,u}\big\} \cap B^c_\delta(x)}_{E^2_{\lambda,u,\delta}(x)}.$$

If $y \in E^2_{\lambda,u,\delta}(x)$, we have

$$\lambda \leq \frac{|u(x) - u(y)|}{\big(\mathrm{d}(x,y)\big)^{\frac{N}{p}+1}} \leq \frac{\mathbf{Lip}(u)}{\delta^{\frac{N}{p}}}.$$

So $\mathfrak{m}\big(E^2_{\lambda,u,\delta}(x)\big) = 0$ for large λ and

$$\varlimsup_{\lambda\to+\infty} \lambda^p \int_X \chi_{E_{\lambda,u,\delta}(x)}(y)\mathrm{d}\mathfrak{m}(y) = \varlimsup_{\lambda\to+\infty} \lambda^p \underbrace{\int_X \chi_{E^1_{\lambda,u,\delta}(x)}(y)\mathrm{d}\mathfrak{m}(y)}_{I_1(\lambda,u,\delta,x)}. \tag{3.24}$$

Concerning $I_1(\lambda, u, \delta, x)$ we have

$$\begin{aligned}
& I_1(\lambda, u, \delta, x) \\
= \;& \mathfrak{m}\left(\left\{y \in B_\delta(x) : |u(x) - u(y)| \geq \lambda\big(\mathrm{d}(x,y)\big)^{\frac{N}{p}+1}\right\}\right) \\
= \;& \mathfrak{m}\left(\left\{y \in B_\delta(x) : |u_{\delta,x}(y)| \geq \frac{\lambda}{\delta}\big(\mathrm{d}(x,y)\big)^{\frac{N}{p}+1}\right\}\right) \\
\overset{(3.22)}{\leq} \;& \mathfrak{m}\left(\left\{y \in B_\delta(x) : |u_{0,x}(\phi_\delta(y))| + \alpha(\delta) \geq \frac{\lambda}{\delta}\big(\mathrm{d}(x,y)\big)^{\frac{N}{p}+1}\right\}\right) \\
\overset{(3.18)}{\leq} \;& \mathfrak{m}\left(\left\{y \in B_\delta(x) : |u_{0,x}(\phi_\delta(y))| + \alpha(\delta) \geq \frac{\lambda}{\delta}\big(\mathrm{d}_\mathfrak{C}(\phi_\delta(y),0)\delta(1-\eta(\delta))\big)^{\frac{N}{p}+1}\right\}\right).
\end{aligned}$$

Letting $v = \phi_\delta(y)$, we get

$$\begin{aligned}
& I_1(\lambda, u, \delta, x) \\
\leq \;& (\phi_\delta)_\sharp\mathfrak{m}\left(\left\{v \in \phi_\delta\big(B_\delta(x)\big) : |u_{0,x}(v)| + \alpha(\delta) \geq \frac{\lambda}{\delta}\big(\mathrm{d}_\mathfrak{C}(v,0)\delta(1-\eta(\delta))\big)^{\frac{N}{p}+1}\right\}\right) \\
\overset{(3.19)}{\leq} \;& (1+\epsilon)\delta^N\mathfrak{m}_\mathfrak{C}\left(\left\{v \in \phi_\delta\big(B_\delta(x)\big) : |u_{0,x}(v)| + \alpha(\delta) \geq \frac{\lambda}{\delta}\big(\mathrm{d}_\mathfrak{C}(v,0)\delta(1-\eta(\delta))\big)^{\frac{N}{p}+1}\right\}\right) \\
\leq \;& \underbrace{(1+\epsilon)\delta^N\mathfrak{m}_\mathfrak{C}\left(\left\{v \in B^\mathfrak{C}_1(0) : |u_{0,x}(v)| + \alpha(\delta) \geq \frac{\lambda}{\delta}\big(\mathrm{d}_\mathfrak{C}(v,0)\delta(1-\eta(\delta))\big)^{\frac{N}{p}+1}\right\}\right)}_{I^a_1(\lambda,u,\delta,x)} \\
+ \;& \underbrace{(1+\epsilon)\delta^N\mathfrak{m}_\mathfrak{C}\left(\left\{v \in \phi_\delta\big(B_\delta(x)\big)\backslash B^\mathfrak{C}_1(0) : |u_{0,x}(v)| + \alpha(\delta) \geq \frac{\lambda}{\delta}\big(\mathrm{d}_\mathfrak{C}(v,0)\delta(1-\eta(\delta))\big)^{\frac{N}{p}+1}\right\}\right)}_{I^b_1(\lambda,u,\delta,x)}.
\end{aligned}$$

By considering the dilation map D_δ and a change of variable, we get

$$
\begin{aligned}
& I_1^a(\lambda,u,\delta,x) \\
= \ & (1+\epsilon)\delta^N (D_\delta)_\sharp \mathfrak{m}_{\mathfrak{C}}\left(\left\{w\in B_\delta^{\mathfrak{C}}(0): \left|u_{0,x}\left(D_{\delta^{-1}}(w)\right)\right| + \alpha(\delta) \geq \frac{\lambda}{\delta}\left(\mathrm{d}_{\mathfrak{C}}(w,0)(1-\eta(\delta))\right)^{\frac{N}{p}+1}\right\}\right) \\
\overset{(3.20)}{=} \ & (1+\epsilon)\mathfrak{m}_{\mathfrak{C}}\left(\left\{w\in B_\delta^{\mathfrak{C}}(0): \left|u_{0,x}\left(D_{\delta^{-1}}(w)\right)\right| + \alpha(\delta) \geq \frac{\lambda}{\delta}\left(\mathrm{d}_{\mathfrak{C}}(w,0)(1-\eta(\delta))\right)^{\frac{N}{p}+1}\right\}\right) \\
= \ & (1+\epsilon)\int_0^\delta \int_{S_r^{\mathfrak{C}}} \chi_{\left\{|u_{0,x}(D_{\delta^{-1}}(w))|+\alpha(\delta)\geq\frac{\lambda}{\delta}(r(1-\eta(\delta)))^{\frac{N}{p}+1}\right\}} \mathrm{d}\mathfrak{m}_{\mathfrak{C}}^+(w)\mathrm{d}r,
\end{aligned}
$$

where $\mathfrak{m}_{\mathfrak{C}}^+$ is the boundary measure. By the definition of $u_{0,x}$, the homogeneity of $\mathfrak{C}$ and Fubini's theorem, we know that

$$
\begin{aligned}
& I_1^a(\lambda,u,\delta,x) \\
= \ & (1+\epsilon)\int_0^\delta \int_{S_r^{\mathfrak{C}}} \chi_{\left\{\delta^{-1}r|u_{0,x}(D_{r^{-1}}(w))|+\alpha(\delta)\geq\frac{\lambda}{\delta}(r(1-\eta(\delta)))^{\frac{N}{p}+1}\right\}} \mathrm{d}\mathfrak{m}_{\mathfrak{C}}^+(w)\mathrm{d}r \\
\overset{(3.21)}{=} \ & (1+\epsilon)\int_0^\delta r^{N-1}\int_{S_1^{\mathfrak{C}}} \chi_{\left\{r|u_{0,x}(w)|+\alpha(\delta)\geq\lambda(r(1-\eta(\delta)))^{\frac{N}{p}+1}\right\}} \mathrm{d}\mathfrak{m}_{\mathfrak{C}}^+(w)\mathrm{d}r \\
= \ & (1+\epsilon)\int_0^\delta r^{N-1}\int_{S_1^{\mathfrak{C}}} \chi_{\left\{r^{\frac{N}{p}}\leq\frac{|u_{0,x}(w)|+\alpha(\delta)}{\lambda(1-\eta(\delta))^{\frac{N}{p}+1}}\right\}} \mathrm{d}\mathfrak{m}_{\mathfrak{C}}^+(w)\mathrm{d}r \\
= \ & (1+\epsilon)\int_{S_1^{\mathfrak{C}}}\int_0^\delta r^{N-1} \chi_{\left\{r^{\frac{N}{p}}\leq\frac{|u_{0,x}(w)|+\alpha(\delta)}{\lambda(1-\eta(\delta))^{\frac{N}{p}+1}}\right\}} \mathrm{d}r\mathrm{d}\mathfrak{m}_{\mathfrak{C}}^+(w) \\
= \ & \frac{1+\epsilon}{N}\int_{S_1^{\mathfrak{C}}} A(\lambda,u,\delta,x,w)^N \mathrm{d}\mathfrak{m}_{\mathfrak{C}}^+(w),
\end{aligned}
$$

where

$$
A(\lambda,u,\delta,x,w) = \min\left\{\delta, \left(\frac{|u_{0,x}(w)|+\alpha(\delta)}{\lambda\left(1-\eta(\delta)\right)^{\frac{N}{p}+1}}\right)^{\frac{p}{N}}\right\}.
$$

For λ large enough, we have

$$
I_1^a(\lambda,u,\delta,x) = \frac{1+\epsilon}{N}\int_{S_1^{\mathfrak{C}}}\left(\frac{|u_{0,x}(w)|+\alpha(\delta)}{\lambda\left(1-\eta(\delta)\right)^{\frac{N}{p}+1}}\right)^p \mathrm{d}\mathfrak{m}_{\mathfrak{C}}^+(w). \tag{3.25}
$$

Consider $I_1^b(\lambda,u,\delta,x)$. Note $v\in\phi_\delta\left(B_\delta(x)\right)\backslash B_1^{\mathfrak{C}}(0)$ and $u\in\mathrm{Lip}_b(X,\mathrm{d})$. We have $\mathrm{d}_{\mathfrak{C}}(v,0)\geq 1$ and $|u_{0,x}(v)|\leq\mathbf{Lip}(u)$. So for λ large enough, we have

$$
I_1^b(\lambda,u,\delta,x) = 0. \tag{3.26}
$$

Combining (3.24)-(3.26), we have

$$\begin{aligned}
&\overline{\lim_{\lambda\to+\infty}}\lambda^p\int_X\int_X\chi_{E_{\lambda,u}}(x,y)\mathrm{d}\mathfrak{m}(y)\mathrm{d}\mathfrak{m}(x)\\
\le&\overline{\lim_{\lambda\to+\infty}}\int_X\Big(\lambda^pI_1(\lambda,u,\delta,x)\Big)\mathrm{d}\mathfrak{m}(x)\\
\le&\overline{\lim_{\lambda\to+\infty}}\int_X\Big(\lambda^pI_1^a(\lambda,u,\delta,x)\Big)\mathrm{d}\mathfrak{m}(x)\\
=&\int_X\frac{1+\epsilon}{N}\int_{S_1^{\mathfrak{C}}}\left(\frac{|u_{0,x}(w)|+\alpha(\delta)}{\big(1-\eta(\delta)\big)^{\frac{N}{p}+1}}\right)^p\mathrm{d}\mathfrak{m}_{\mathfrak{C}}^+(w)\mathrm{d}\mathfrak{m}(x).
\end{aligned}$$

Letting $\delta\to0$ and $\epsilon\to0$, we get

$$\overline{\lim_{\lambda\to+\infty}}\lambda^p(\mathfrak{m}\times\mathfrak{m})(E_{\lambda,u})\le\|\nabla u\|^p_{K_{p,\mathfrak{C}}}. \tag{3.27}$$

Similarly, we can see

$$\begin{aligned}
&I_1(\lambda,u,\delta,x)\\
=\;&\mathfrak{m}\left(\left\{y\in B_\delta(x):|u_{\delta,x}(y)|\ge\frac{\lambda}{\delta}\big(\mathrm{d}(x,y)\big)^{\frac{N}{p}+1}\right\}\right)\\
\overset{(3.22)}{\ge}\;&\mathfrak{m}\left(\left\{y\in B_\delta(x):|u_{0,x}(\phi_\delta(y))|-\alpha(\delta)\ge\frac{\lambda}{\delta}\big(\mathrm{d}(x,y)\big)^{\frac{N}{p}+1}\right\}\right)\\
\overset{(3.18)}{\ge}\;&\mathfrak{m}\left(\left\{y\in B_\delta(x):|u_{0,x}(\phi_\delta(y))|-\alpha(\delta)\ge\frac{\lambda}{\delta}\big(\mathrm{d}_{\mathfrak{C}}(\phi_\delta(y),0)\delta(1+\eta(\delta))\big)^{\frac{N}{p}+1}\right\}\right)\\
\overset{(3.19)}{\ge}\;&(1-\epsilon)\delta^N\mathfrak{m}_{\mathfrak{C}}\left(\left\{v\in\phi_\delta\big(B_\delta(x)\big):|u_{0,x}(v)|-\alpha(\delta)\ge\frac{\lambda}{\delta}\big(\mathrm{d}_{\mathfrak{C}}(v,0)\delta(1+\eta(\delta))\big)^{\frac{N}{p}+1}\right\}\right)\\
\ge\;&\underbrace{(1-\epsilon)\delta^N\mathfrak{m}_{\mathfrak{C}}\left(\left\{v\in B_1^{\mathfrak{C}}(0):|u_{0,x}(v)|-\alpha(\delta)\ge\frac{\lambda}{\delta}\big(\mathrm{d}_{\mathfrak{C}}(v,0)\delta(1+\eta(\delta))\big)^{\frac{N}{p}+1}\right\}\right)}_{II_1^a(\lambda,u,\delta,x)}\\
-\;&\underbrace{(1-\epsilon)\delta^N\mathfrak{m}_{\mathfrak{C}}\left(\left\{v\in B_1^{\mathfrak{C}}(0)\backslash\phi_\delta\big(B_\delta(x)\big):|u_{0,x}(v)|-\alpha(\delta)\ge\frac{\lambda}{\delta}\big(\mathrm{d}_{\mathfrak{C}}(v,0)\delta(1+\eta(\delta))\big)^{\frac{N}{p}+1}\right\}\right)}_{II_1^b(\lambda,u,\delta,x)},
\end{aligned}$$

and

$$II_1^a(\lambda,u,\delta,x)=\frac{1-\epsilon}{N}\int_{S_1^{\mathfrak{C}}}\left(\frac{|u_{0,x}(w)|-\alpha(\delta)}{\lambda\big(1+\eta(\delta)\big)^{\frac{N}{p}+1}}\right)^p\mathrm{d}\mathfrak{m}_C^+(w).$$

By (3.18) we have

$$B^{\mathfrak{C}}_{\frac{1}{1+\eta(\delta)}}(x)\subset\phi_\delta\big(B_\delta(x)\big)\subset B^{\mathfrak{C}}_{\frac{1}{1-\eta(\delta)}}(x).$$

Since $v\in B_1^{\mathfrak{C}}(0)\backslash\phi_\delta\big(B_\delta(x)\big)$ and $u\in\mathrm{Lip}_b(X,\mathrm{d})$, we have $\mathrm{d}_{\mathfrak{C}}(v,0)\ge\frac{1}{1+\eta(\delta)}$ and $|u_{0,x}(v)|\le\mathbf{Lip}(u)$. So for λ large enough, we have

$$II_1^b(\lambda,u,\delta,x)=0.$$

Letting $\delta \to 0$ and $\epsilon \to 0$, we get

$$\varliminf_{\lambda \to +\infty} \lambda^p (\mathfrak{m} \times \mathfrak{m})(E_{\lambda,u}) \geq \|\nabla u\|^p_{K_{p,\mathfrak{C}}}. \tag{3.28}$$

Combining (3.27) and (3.28), we prove the theorem. □

3.4 Optimality of N

We can see that the parameter N is sharp:

Theorem 3.6. *Let $p \geq 1, \bar{N} > 0$ and $(X, \mathrm{d}, \mathfrak{m})$ be a metric measure space equipped with a β-doubling measure $\mathfrak{m}$. If $a \leq \theta^-_N \leq \theta^+_N \leq b$ for $N = \frac{\log \beta}{\log 2}$ and some $b > a > 0$, then the following statements are equivalent:*

a) *There exists $u \in \mathrm{Lip}_b(X, \mathrm{d})$ with $\mathfrak{m}(\{x \in X : \mathrm{lip}(u)(x) > 0\}) > 0$ such that*

$$\begin{aligned} &\varliminf_{\lambda \to +\infty} \lambda^p (\mathfrak{m} \times \mathfrak{m})(\bar{E}_{\lambda,u}) > 0, \\ &\varlimsup_{\lambda \to +\infty} \lambda^p (\mathfrak{m} \times \mathfrak{m})(\bar{E}_{\lambda,u}) < +\infty, \end{aligned} \tag{3.29}$$

where

$$\bar{E}_{\lambda,u} = \left\{ (x, y) \in X \times X : x \neq y, |u(x) - u(y)| \geq \lambda \big(\mathrm{d}(x, y)\big)^{\frac{\bar{N}}{p}+1} \right\}.$$

b) *For any $u \in \mathrm{Lip}_b(X, \mathrm{d})$ with $\mathfrak{m}(\{x \in X : \mathrm{lip}(u)(x) > 0\}) > 0$, (3.29) holds.*

c) $\bar{N} = N = \frac{\log \beta}{\log 2}$.

Proof. **c)** ⇒ **b)** is a consequence of Theorem 3.1 and Theorem 3.2, **b)** ⇒ **a)** is trivial. We will prove **a)** ⇒ **c)** by contradiction.

Case 1: $\bar{N} < N$. Similar to the proof of Theorem 3.2, we have

$$\begin{aligned} &\varlimsup_{\lambda \to +\infty} \lambda^p (\mathfrak{m} \times \mathfrak{m})(\bar{E}_{\lambda,u}) \\ =&\varlimsup_{\lambda \to +\infty} \lambda^p \int_X \int_X \chi_{\bar{E}_{\lambda,u}}(x, y) \mathrm{d}\mathfrak{m}(y) \mathrm{d}\mathfrak{m}(x) \\ \leq& 2 \varlimsup_{\lambda \to +\infty} \lambda^p \int_{\mathrm{supp}\, u} \mathfrak{m} \left(B_{\left(\frac{\mathrm{Lip}(u)(x)+\epsilon}{\lambda}\right)^{\frac{p}{N}}}(x) \right) \mathrm{d}\mathfrak{m}(x) \\ \leq& 2 \varlimsup_{\lambda \to +\infty} \lambda^{p\left(1-\frac{N}{\bar{N}}\right)} \int_{\mathrm{supp}\, u} \lambda^{\frac{pN}{\bar{N}}} \mathfrak{m} \left(B_{\left(\frac{\mathrm{Lip}(u)(x)+\epsilon}{\lambda}\right)^{\frac{p}{\bar{N}}}}(x) \right) \mathrm{d}\mathfrak{m}(x) \\ =&0, \end{aligned}$$

which is a contradiction.

Case 2: $\bar{N} > N$. We adopt the same notations and the same argument as in the proof of Theorem 3.1. For any i_0 with $\mathfrak{m}(L_{i_0}) > 0$, set $T_{i_0} = 2^{i_0+5}$ and $0 < \epsilon <$

$\min\left\{\frac{1}{\beta^2}\left(\frac{1}{8T_{i_0}}\right)^{\frac{\log\beta}{\log 2}}, \frac{1}{2}\mathfrak{m}(L_{i_0})\right\}$ satisfying $\mathfrak{m}\{x \in L_{i_0} : \mathrm{lip}(u)(x) > 2^{i_0+3}\epsilon\} > \frac{2}{3}\mathfrak{m}(L_{i_0})$, $x \in A_{\epsilon,i_0}$ and $8\lambda r^{\bar{N}/p} = \mathrm{lip}(u)(x)$. We have

$$\begin{aligned}
&\varliminf_{\lambda\to+\infty} \lambda^p(\mathfrak{m}\times\mathfrak{m})(\bar{E}_{\lambda,u})\\
\geq &\varliminf_{\lambda\to+\infty} \lambda^p \int_{L_{i_0}}\int_X \chi_{\bar{E}_{\lambda,u}}(x,y)\mathrm{d}\mathfrak{m}(y)\mathrm{d}\mathfrak{m}(x)\\
\geq &\varliminf_{\lambda\to+\infty} \lambda^p \int_{A_{\epsilon,i_0}} \left(\frac{r}{T_{i_0}}\right)^N \left(\frac{T_{i_0}}{r}\right)^N \mathfrak{m}\left(B_{\frac{r}{T_{i_0}}}(y_{x,r})\right)\mathrm{d}\mathfrak{m}(x)\\
\geq &\varliminf_{\lambda\to+\infty} \frac{a\lambda^{p\left(1-\frac{N}{\bar{N}}\right)}}{2^{(i_0+5)N}8^{\frac{pN}{\bar{N}}}} \int_{A_{\epsilon,i_0}} |\mathrm{lip}(u)(x)|^{\frac{pN}{\bar{N}}}\mathrm{d}\mathfrak{m}(x)\\
= &+\infty,
\end{aligned}$$

which is a contradiction.

In conclusion, $\bar{N} = N$.

□

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