ON THE STRUCTURE OF BUSEMANN SPACES WITH NON-NEGATIVE CURVATURE

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ABSTRACT. We extend the structure theory of Burago–Gromov–Perelman for Alexandrov spaces with curvature bounded below, to the setting of Busemann spaces with non-negative curvature. We prove that any finite-dimensional Busemann space with non-negative curvature satisfying Ohta's S-concavity and local semi-convexity, admits a non-trivial integer-dimensional Hausdorff measure, and satisfies the measure contraction property. We also show that such spaces are rectifiable and that almost every point admits a unique tangent cone isometric to a finite-dimensional Banach space. In addition, under mild control of the uniform smoothness constant, we obtain refined estimates for the Hausdorff dimension of the singular strata. Our results not only enrich the theory of synthetic sectional curvature lower bound for metric spaces, but also provide some useful tools and examples to study Finslerian metric measure spaces.

Keywords: Busemann space, non-negative curvature, non-Riemannian metric space, rectifiability, singular sets, unique Banach tangent cone

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1. Introduction

1.1. **Motivation and object.** The study of how curvature shapes the geometry of spaces is a central theme in both mathematics and physics. While classical differential geometry–particularly Riemannian geometry–has provided deep insights into the interplay between curvature tensor and large-scale geometry of smooth spaces, it is now widely recognized that synthetic notions of curvature bound are powerful tools for understanding the structure of non-smooth spaces. Such non-smooth metric (measure) spaces naturally arise as (measured) Gromov–Hausdorff limits, quotients, gluing, and suspensions of smooth manifolds.

Among the various synthetic notions of curvature bound, those defined via comparison of geodesic triangles with their counterparts in model spaces play a central role in the study of metric spaces with 'generalized' sectional curvature bounds. By demanding that the geodesic triangles in a geodesic space are 'fatter' or 'thinner' than their comparison triangles in space forms, one can define spaces with curvature bounded below (CBB spaces) and curvature bounded above (CBA spaces). This approach, initiated by A.D. Alexandrov [2], has been extensively studied from various perspectives, resulting in a rich and well-developed theory; see for instance [1, 7, 11, 13] and bibliography therein.

After the groundbreaking work of Burago, Gromov, and Perelman [13], a comprehensive structure theory for Alexandrov spaces with curvature bounded below has been extensively developed. This theory, which illuminates how sectional curvature bounds interact with the topological, geometric, and measure-theoretic properties of the underling space, has been further investigated by Otsu and Shioya [39], Perelman [41, 42], Perelman and Petrunin [43, 44], Petrunin [45], Ambrosio and Bertrand [3] and many others. The ideas and tools in the structure theory of Alexandrov with curvature bounded below have recently been extended by Lytchak and Nagano [29, 30] to the setting of Alexandrov spaces with curvature bounded above and the local geodesic extension property (so-called GCBA spaces), leading to a comprehensive structure theory for these spaces. These structure theories have revealed that finite-dimensional Alexandrov spaces possess many properties analogous to those of Riemannian manifolds, implying the *Riemannian nature* of Alexandrov spaces.

Besides Alexandrov geometry, there have been significant efforts to develop synthetic notions of (sectional) curvature for non-Riemannian metric spaces such as general Banach spaces and Finsler manifolds. These efforts can be traced back to the pioneering works of Busemann [14, 15], who introduced a weaker notion of non-positive curvature, characterized by the convexity of distance functions along geodesics, to study Finsler spaces:

Definition 1.1 (Busemann Convex)**.** A complete geodesic space (X, d) is said to be Busemann convex if for any pair of constant-speed geodesics $\gamma, \eta : [0, 1] \to X$, the function

$$t \mapsto d(\gamma(t), \eta(t)) \tag{1.1}$$

is convex on $[0, 1]$.

By triangle inequality, one can equivalently define Busemann convexity as follows.

Definition 1.2 (Busemann Convex)**.** A complete geodesic space (X, d) is said to be Busemann convex if for any pair of constant-speed geodesics $\gamma, \eta : [0, 1] \to X$ starting from a common point $\gamma(0) = \eta(0)$, the function

$$t \mapsto \frac{d(\gamma(t), \eta(t))}{t} \tag{1.2}$$

is non-decreasing on $[0, 1]$.

This synthetic notion of curvature bound, together with another axiomatic notion of geodesic spaces introduced by Busemann, called G-spaces, has aroused great interest in the study of non-Riemannian geometry due to its *Finsler nature*, leading to very rich literature; see for example [40, 49] and bibliography therein.

Similar to Definition 1.2, we have a notion of non-negative (sectional) curvature in the sense of Busemann. However, although we call it 'concave', we do not have an equivalent definition for curvature lower bound, in the same way as Definition 1.1.

Definition 1.3 (Busemann Concave)**.** A complete geodesic space (X, d) is said to be Busemann concave if for any pair of constant-speed geodesics $\gamma, \eta : [0, 1] \to X$ starting from a common point $\gamma(0) = \eta(0)$, the function

$$t \mapsto \frac{d(\gamma(t), \eta(t))}{t} \tag{1.3}$$

is non-increasing on $[0, 1]$.

Notice that any Banach space with strictly convex norm is *both Busemann convex and Busemann concave*. In order to establish fine structure theory as in Alexandrov geometry, we need to impose some natural (intrinsic) assumptions. Taking a normed space $(E, \|\cdot\|)$ as an example, assume that there are two constants $S, C \geq 1$ such that for any $u, v \in E$, it holds

$$\left\|\frac{u+v}{2}\right\|^2 \leq \frac{1}{2}\|u\|^2 + \frac{1}{2}\|v\|^2 - \frac{1}{4C}\|u-v\|^2, \tag{1.4}$$

and

$$\left\|\frac{u+v}{2}\right\|^2 \geq \frac{1}{2}\|u\|^2 + \frac{1}{2}\|v\|^2 - \frac{S}{4}\|u-v\|^2. \tag{1.5}$$

Notice that, when $S = C = 1$, the norm is induced by an inner product. Thus, the inequalities (1.4) and (1.5) are a kind of quantitative continuity conditions, where the constants S, C measure the degree to which the norm $\|\cdot\|$ satisfies the parallel rule. Inspired by this example and Busemann's work, Ohta [34, 37] (see also [26, 38]) introduced two concepts for metric spaces, called C-convexity and S-concavity[1], and found their corresponding geometric quantities in Finsler geometry. For example, as noticed by Shen [47] and Ohta [37, 38], S-concavity is satisfied by Finsler manifolds with non-negative flag curvature whose tangent spaces are uniformly smooth Minkowski spaces. Within this framework, Ohta [34, 37] generalized the Alexandrov–Toponogov comparison theorem to Finsler spaces.

Recently, in an effort to investigate the compatibility of Busemann concavity and a synthetic notion of non-negative Ricci curvature called the Measure Contraction Property (see [35] and [48]), Kell [24] studied the metric measure geometry of Busemann spaces of non-negative curvature (Busemann concave spaces) and uniformly smooth spaces. Under the assumption of existence of a non-trivial Hausdorff measure, Kell proved several geometric and analytic properties for Busemann concave spaces, including the measure contraction property, Poincaré inequality and uniqueness of tangent cones. However, due to the instability of Busemann concavity under Gromov–Hausdorff convergence, it is not clear whether almost all tangent cones are Banach spaces or not.

In light of the seminal work of Burago–Gromov–Perelman [13] and the recent contributions of Kell [24], a natural question arises:

> *do Busemann concave spaces admit a refined structure theory analogous to that of Alexandrov spaces with curvature bounded below, such as rectifiability, uniqueness of Banach tangent cones, and fine measure-theoretic estimates for singular sets?*

[1] They are called 2-uniform convexity and 2-uniform smoothness by Ohta in [37], and renamed as k-convexity and k-concavity in [38].

While some geometric properties of Busemann concave spaces appears straightforward, establishing fine regularity results presents significant challenges. One of the main difficulties is that Busemann concave spaces, such as normed spaces, do not possess a robust notion of angles as in Alexandrov spaces. This difficulty reflects non-Riemannian nature of Busemann concave spaces, which renders their tangent cones considerably more intricate than the metric cone structure in Alexandrov spaces. It becomes more subtle, if we do not have a non-trivial Hausdorff measure. For instance, there is a compact convex subset K in the infinite-dimensional ℓ^p-space with $1 < p < \infty$, such that the Gromov–Hausdorff limit of blow-ups fails to coincide with the tangent cone at any interior point of K [24].

This difficulty is also addressed in a recent work of Fujioka and Gu [18], who make significant progress in the study of topological regularity of Busemann spaces with non-positive curvature and local geodesic extension property (so-called GNPC spaces), building upon the contributions of Lytchak–Nagano [30] and Lytchak–Nagano–Stadler [32]. One of the key ingredients in their study is two distinct notions of angle. These notions of angle, which coincide in GCBA spaces, capture multi-faceted geometric features and non-Riemannian nature of Busemann spaces. Based upon these notions of angle, Fujioka and Gu introduce 'almost orthogonal coordinates' in the context of Busemann spaces of non-positive curvature, namely strainer maps, enabling them to successfully implement the strategy developed in [32]. Concerning Busemann concave spaces, the problem is *more difficult*. Unlike Busemann spaces of non-positive curvature, distance functions are not necessarily differentiable in Busemann concave spaces. Even if we have a non-trivial Hausdorff measure, detecting finer structure theory for Busemann concave spaces beyond the results of Kell [24] and Le Donne [17] appears infeasible, which establish that almost all points in Busemann concave spaces having non-trivial Hausdorff measure admit unique tangent cones isometric to Carnot groups equipped with left-invariant sub-Finsler metrics.

As we can see, in order to obtain structural theorems beyond the results of Kell [24] and Le Donne [17], we need to impose some natural assumptions. A natural assumption is *S-concavity* introduced by Ohta [34, 37], and *local semi-convexity*, which is much weaker than Ohta's *C-convexity* (cf. (1.4) and (1.5)).

The present work is devoted to establishing fine geometric, infinitesimal structural, and measure-theoretic properties for Busemann concave spaces satisfying both S-concavity and local semi-convexity, extending the comprehensive structure theory of Alexandrov spaces with curvature bounded below [13] to the setting of Busemann spaces of non-negative curvature.

1.2. **Main results.** Throughout this part, we always assume that (X, d) is an S-concave, locally semi-convex, Busemann concave space for some $S \geq 1$.

Our first main theorem concerns Hausdorff dimension and Hausdorff measure. A key ingredient is strainer number, which can be interpreted as the maximal dimension of 'local almost orthogonal coordinates' around points in X.

Theorem 1.4 (Proposition 6.6 and Corollary 6.7)**.** *Let (X, d) be an S-concave, local semi-convex, Busemann concave space for some $S \geq 1$. Then X is of finite Hausdorff dimension if and only if it has finite strainer number. In either case, both values coincide with the topological dimension of X. Moreover, X admits a non-trivial Hausdorff measure $\mathcal{H}^n$ of integer dimension n, and the metric measure space $(X, d, \mathcal{H}^n)$ satisfies the measure contraction property $\mathrm{MCP}(0, n)$. In particular, (X, d) is doubling and proper, and contains a topological n-manifold part which is open and dense in X.*

Theorem 1.4 generalizes the well-known results of Burago–Gromov–Perelman [13, Corollary 6.5, 6.7, 6.8] to our broader setting. We remark that, sufficient conditions for the existence of a non-trivial Hausdorff measure on Busemann concave spaces, is an open problem left in [24].

Our second main result shows that if X is of finite Hausdorff dimension n, then the almost regular part $\mathcal{A}(n,\delta)$ forms a subset of X with full measure. We refer to Section 6 for the precise definition of almost regular parts $\mathcal{A}(n,\delta)$, namely the sets of (k,δ)-*strained points*. Roughly speaking, $\mathcal{A}(n,\delta)$ can be thought of as the set of points in X on some of whose neighborhood admits an almost orthogonal coordinate chart of dimension n.

Theorem 1.5 (Theorem 7.6)**.** *Let X be of finite Hausdorff dimension n. Then for any $\delta > 0$, $\mathcal{H}^n(X \setminus \mathcal{A}(n,\delta)) = 0$. In particular, X contains an open dense topological n-manifold part which has full n-Hausdorff measure.*

The next result establishes that X is n-rectifiable, and that $\mathcal{H}^n$-almost every point admits a unique tangent cone, which is isometric to a finite-dimensional Banach space.

Theorem 1.6 (Theorem 7.7 and 7.8)**.** *Moreover, X is n-rectifiable, and for $\mathcal{H}^n$-almost every point in X admits a unique tangent cone (T_xX, d_x, o), which is isometric to a finite-dimensional Banach space.*

Theorem 1.6 generalizes the well-known result for finite-dimensional Alexandrov spaces with curvature bounded below, where $\mathcal{H}^n$-almost every point admits a unique tangent cone isometric to Euclidean space. Under supplementary geometric conditions on distance functions, we provide a refined characterization of these Banach tangent cones, in terms of the geometry of Banach spaces. We refer to Section 7.2 for the definition of local p-uniform convexity in the theorem. For further discussion on the connection with the work of Kell [24], see Remark 7.11.

Theorem 1.7 (Corollary 7.10)**.** *If X further satisfies local p-uniform convexity, then each of these Banach tangent cones possesses a strictly convex norm and is both 2-uniformly smooth and p-uniformly convex.*

The last main result provides a measure-theoretic estimates for singular strata, with respect to the uniform smoothness constant S. This theorem, together with Theorem 1.5, generalizes the well-known result for Hausdorff measure and Hausdorff dimension of singular strata in Alexandrov spaces with curvature bounded below [13].

Theorem 1.8 (Theorem 7.17, Corollary 7.18)**.** *For any $\delta > 0$. There exists a constant $S_1 := S_1(\delta) > 1$ such that the following holds: if X is an n-dimensional S-concave, local semi-convex, Busemann concave space with $S \in [1, S_1]$, then the Hausdorff dimension of the singular set $X \setminus \mathcal{A}(k,\delta)$ is at most $k-1$ for all $k = 1,\ldots,n$. In particular, X admits a stratification $\{X_k\}_{k=0}^n$ such that X is the disjoint union of $\{X_k\}_{k=0}^n$ and $\dim_H(X_k) \leq k$ for all $k = 0,\ldots,n$.*

1.3. **Main ideas and tools.** We now explain new ideas in our proof and compare our methods with those employed in the structure theory of Alexandrov spaces with curvature bounded below [13].

The main geometric tool is an analog of a classical notion from Alexandrov geometry, called *strainer maps*, which can be roughly interpreted as local almost orthogonal coordinate charts around points. The notion of strainer maps is introduced by Burago–Gromov–Perelman [13] in the context of Alexandrov spaces with curvature bounded below, and later generalized by Lytchak and Nagano [29] to GCBA spaces, and recently further extended by Fujioka and Gu [18] to GNPC spaces.

Similar to [18], to establish strainer maps in our setting, we introduce two notions of angle to capture complementary geometric aspects of the underlying space: the first one, called *angles viewed from a fixed point* (Definition 4.1), measures the orthogonality of geodesics, while the second one, called *angles of fixed scale* (Definition 4.13), reflects the infinitesimal structure of the underlying space, which is closely related to the metrics of tangent cones. These two notions of angle, which may not coincide in Busemann setting (see Example 4.15), are implicitly correlated with the *Euclidean comparison angle* by the uniform smoothness constant S as well as the *ratio of side-lengths* of geodesic triangles (see Lemma 4.3 and 4.17). Here we summarize the different notions of angle in Table 1 used in this paper. For more details on different notions of angle, we refer to Section 4.

Notation	Meaning
$\angle px\xi$	angle at x viewed from the point p along the geodesic ξ
$\angle_x(\gamma(t),\eta(s))$	angle of the geodesics γ,η at x of scale t/s
$\tilde{\angle}_x(p,q)$ or $\tilde{\angle}pxq$	comparison angle of Euclidean triangle $\tilde{\Delta}pxq$ at x
$\angle_x((\gamma,t),(\eta,t))$	angle metric on space of directions with common length

TABLE 1. Different notions of angles

Using the first notion of angle, we define strainers and strainer maps. Our definition is adapted from the ones in [13] and [18], to address the non-uniqueness of geodesics between pairs of points and the lack of monotonicity of comparison angles in the Busemann setting. To construct local almost orthogonal coordinate charts with strainer maps, we then prove ε-openness and bi-Lipschitz continuity of strainer maps. The main challenge in our setting arises from the asymmetric nature of strainer maps (see Remark 5.9). This challenge, which also arises in the study of Busemann spaces with non-positive curvature [18], stems from the inherent asymmetry of the first notion of angle, as illustrated in Example 4.6. To overcome this difficulty, we adopt a strategy similar to that of [18] by introducing an anisotropic L^1-norm for the target spaces of strainer maps. Then we are able to establish the properties we need.

In the proof of Theorem 1.4, a key ingredient is the self-improvement property of strainers (Lemma 5.17), which ensures that once a 'less orthogonal' strainer map is found, one can automatically obtain a 'more orthogonal one'. Unfortunately, the 'straightening strategy' to obtain improved strainers used in [13] (see also [12, Proposition 10.8.17]) is not applicable in Busemann setting. In fact, the new k-tuple $(p_1', p_2, \ldots, p_k)$ obtained from the old strainer $(p_1, p_2, \ldots, p_k)$ by straightening procedure is no more a strainer, due to the asymmetry of strainers and angles in Busemann setting. To overcome this difficulty, we adopt a new strategy inspired by computer science, known as 'dequeuing and enqueuing'[2]. This method allows us to improve the orthogonality of strainers step by step, ultimately leading to the desired self-improvement property.

To estimate the Hausdorff measure of almost regular sets (Theorem 1.5), we employ an approach substantially different from that used by [13, Theorem 10.6] for Alexandrov spaces with curvature bounded below, in which the authors make use of the compactness of the space of directions at a point to establish an upper bound for the maximal cardinality of r-separated subsets within singular sets (see [13, Lemma 10.5]). This method relies implicitly on the metric cone structure of tangent cones in Alexandrov spaces, which is not available in our context without any control on the uniform smoothness constant. Instead, we adapt a method from [29] developed for GCBA spaces, leveraging the 'almost extendable' property of geodesics established in Lemma 7.2. Such a property is shared by all metric measure spaces satisfying the measure contraction property [50]. This approach enables us to analyze the infinitesimal behavior of strainer maps when restricted to the singular strata, thereby allowing us to determine the top-dimensional Hausdorff measure of singular sets. It is worth noting that none of our assumptions on spaces imply the local geodesic extension property. This geometric property, which is intimately connected to the

[2] In computer science terminology, to dequeue means to remove the first element from a queue, and to enqueue means to add a new element to the end of the queue.

local geometry and topology of the underlying spaces (see, for example, [31]), is a much stronger condition than the almost extendable property established here.

Based on Theorem 1.5, the n-rectifiability of X follows from the local bi-Lipschitz property of strainer maps and a standard covering argument. The existence and uniqueness of Banach tangent cones at almost every point in X, is a consequence of a well-known structure theorem of Kirchheim, concerning rectifiable sets in metric spaces [25, Theorem 9].

Finally, to estimate the Hausdorff dimension of singular strata (Theorem 1.8), the main difficulty is the absence of a metric cone structure for tangent cones in our setting. We cannot apply the techniques used in Theorem 1.5 either, as the almost extendable property of geodesics only holds up to null measure sets, which prevents us from obtaining finer estimates for the Hausdorff dimension of singular strata. To overcome these difficulties, we introduce a new notion for tangent cones, called the *space of directions with common length* (Definition 4.19), which are naturally equipped with an angle metric (see the last angle in Table 1) based on the second notion of angle. We then establish the uniform compactness of spaces of directions with common lengths (Lemma 4.21), and provide a technical lemma (Lemma 7.13) that quantifies how the uniform smoothness constant S and ratio of side-lengths control the asymmetry of the two notions of angle. This lemma, together with the uniform compactness of spaces of directions with common length, allows us to adapt a strategy similar to that of [13, Lemma 10.5] to derive an upper bound for the maximal cardinality of r-separated subsets of singular sets within small cylindrical regions (Lemma 7.15), thereby enabling us to determine the Hausdorff dimension of singular strata.

1.4. **Organization of the paper.** The paper is organized as follows: in Section 2, we introduce basic terminologies and notations used throughout the paper. In Section 3, we introduce several geometric conditions, including S-concavity, local semi-convexity and Busemann concavity, and gives some examples. In Section 4, we introduce two notions of angle and establish the key properties, including almost comparison inequalities. We further define tangent cones and space of directions with common lengths, using angles in the context of Busemann concave spaces, and prove several properties of tangent cones and spaces of directions with common lengths. We then study strainers and strainer maps for our setting in Section 5, and prove several important properties including ε-openness, bi-Lipschitz continuity, and self-improvement property. In Section 6, we introduce the notion of strainer number and prove Theorem 1.4. In Section 7, we investigate the structure and measure-theoretic properties of Busemann concave spaces, and prove Theorem 1.5–1.8. Finally, in Appendix A, we provide a proof of the well-known criterion for ε-openness for Lipschitz maps from locally complete spaces to geodesic spaces.

2. Preliminaries and notations

In this section, we briefly recall the terminologies and notations used in this manuscript. Our notations are standard and mainly follows from [5, 12, 13, 21].

2.1. **Spaces and maps.** Throughout this paper, we denote by (X, d) a complete, separable metric space with the distance function d. Given two points $x, y \in X$, we often denote their distance $d(x, y)$ by $|xy|$ for simplicity. For $r > 0$, we denote by $B(x, r)$ and $\bar{B}(x, r)$ the open and closed balls centered at x with radius r, respectively.

Given a subset $E \subset X$ and $r > 0$, we say that E is r-*separated* if every pair of distinct points $x, y \in E$ satisfies $|xy| \geq r$. An r-separated subset E is called *maximal* if there does not exist any r-separated subset $E' \subset X$ that properly contains E. For a set E, we denote by $\beta_E(r) \in \mathbb{N} \cup \{\infty\}$ the largest possible cardinality of a maximal r-separated subset of E. A metric space (X, d) is said to be *doubling*, or N-*doubling*, if for any $r > 0$, the number of elements in any maximal $r/2$-separated subset of any ball $B(x, r)$ is at most N, i.e., $\beta_{B(x,r)}(r/2) \le N$ for all $x \in X$ and $r > 0$. Equivalently, any ball $B(x,r) \subset X$ can be covered by at most N balls of radius $r/2$.

For a curve $\gamma \subset X$, we denote its length by $l(\gamma)$. A curve $\gamma : [0,1] \to X$ is called a *constant-speed geodesic* from x to y if it is a length-minimizing curve connecting x and y satisfying that $|\gamma(t)\gamma(s)| = |t-s|\,|xy|$ for all $t, s \in [0,1]$. A length-minimizing curve is called *unit-speed geodesic* or *geodesic* for short, if it is parametrized by arc-length. A metric space (X,d) is said to be geodesic if any pair of distinct points can be connected by a geodesic. It is said to be *uniquely geodesic* if any pair of distinct points can be connected by a unique geodesic. A geodesic space (X,d) is said to be non-branching if for any pair of geodesics $\gamma, \eta : [0,1] \to X$, the condition $\gamma|_{[0,t]} = \eta|_{[0,t]}$ for some $t \in (0,1)$ implies that $\gamma = \eta$. We say a function $f : X \to \mathbb{R}$ is convex if $f \circ \gamma$ is convex on $[0,1]$ for any constant-speed geodesic γ.

For a geodesic space (X,d), we denote by Δxyz a *geodesic triangle* on X with vertices $x, y, z \in X$ which is the union of three geodesics which connect x, y, z pair-wisely. A triangle $\tilde{\Delta} xyz$ is said to be an Euclidean comparison triangle of $\Delta xyz \subset X$ if the triangle $\tilde{\Delta} xyz \subset \mathbb{R}^2$ is on the Euclidean plane with vertices $\tilde{x}, \tilde{y}, \tilde{z}$ and side-lengths $|xy| = |\tilde{x}\tilde{y}|, |yz| = |\tilde{y}\tilde{z}|$ and $|xz| = |\tilde{x}\tilde{z}|$. We denote the angle at vertex $\tilde{x}$ of Euclidean comparison triangle $\tilde{\Delta} xyz$ by $\tilde{\angle}_x(y,z)$, or $\tilde{\angle} yxz$ for simplicity.

A map $F : X \to Y$ between two metric spaces (X, d_X) and (Y, d_Y) is called *Lipschitz* or *L-Lipschitz* if there is $L > 0$ such that $|F(x)F(y)| \le L|xy|$ for any $x, y \in X$ [3]. It is called *L-biLipschitz* for some $L \ge 1$ if $|xy|/L \le |F(x)F(y)| \le L|xy|$ for any $x, y \in X$.

2.2. **Hausdorff measure, dimensions and rectifiability.** Let (X,d) be a complete, separable metric space. For $\delta > 0, \alpha \in [0,\infty)$ and $E \subset X$, $\mathcal{H}^\alpha_\delta(E)$ is defined as

$$\mathcal{H}^\alpha_\delta(E) := \inf \left\{ \omega(\alpha) \sum_{i=1}^\infty \left(\frac{\mathrm{diam}(E_i)}{2} \right)^\alpha : E \subset \bigcup_{i=1}^\infty E_i, \mathrm{diam}(E_i) \le \delta \right\}, \tag{2.1}$$

where $\omega_\alpha := 2^{-\alpha}\pi^{\alpha/2}/\Gamma(\alpha/2+1)$ and Γ is the Gamma function. The *α-Hausdorff measure* $\mathcal{H}^\alpha(E)$ of E is defined as $\mathcal{H}^\alpha(E) := \sup_{\delta>0} \mathcal{H}^\alpha_\delta(E)$. Sometimes we use $\mathcal{H}^\alpha_d$ to emphasize that the α-Hausdorff measure is defined with respect to the metric d.

It is known that the α-Hausdorff measure is an outer measure, and moreover a Borel regular measure on X; see for example [5, Theorem 2.1.4]. Furthermore, it can be shown that the α-Hausdorff measure is a Radon measure on complete metric spaces, see [21, Proposition 3.3.44]. For $n \in \mathbb{N}$, the n-Hausdorff measure on $\mathbb{R}^n$ coincides with the standard n-dimensional Lebesgue measure on $\mathbb{R}^n$.

For a subset $E \subset X$, the *Hausdorff dimension* $\dim_H(E)$ of the subset E is the infimum of numbers $\alpha > 0$ such that $\mathcal{H}^\alpha(E) = 0$ if such numbers exist. Otherwise, we say the Hausdorff dimension of E is infinite. For an L-Lipschitz map $F : X \to Y$, it can be checked from the definition that $\mathcal{H}^\alpha(F(E)) \le L^\alpha \mathcal{H}^\alpha(E)$. In particular, the Hausdorff dimension is invariant under the bi-Lipschitz homeomorphism. The Hausdorff dimension is monotone in the sense that $\dim_H(E) \subset \dim_H(E')$ for each measurable subsets $E \subset E' \subset X$. Furthermore, if $\{E_i\}_i$ is an at most countable covering of X, then $\dim_H(X) = \sup_i \dim_H(E_i)$. For more properties of Hausdorff dimension, we refer to [12, Section 1.7.4].

The *rough dimension* $\dim_r(E)$ of the subset E is defined as the infimum of numbers $\alpha > 0$ such that $\limsup_{r \searrow 0} r^\alpha \beta_E(r) = 0$ if such numbers exist, otherwise we take the rough dimension of E equal infinite. It is well known that the topological dimension (Lebesgue covering dimension), Hausdorff dimension, and rough dimension satisfy the following relationship:

$$\dim_T(E) \le \dim_H(E) \le \dim_r(E), \quad \text{for any } E \subset X. \tag{2.2}$$

[3] For different metric space, we use the same notation $|xy|$ and $|F(x)F(y)|$ to denote the distance $d_X(x,y)$ and $d_Y(F(x),F(y))$ respectively, if there is no ambiguity.

For the proofs of these inequalities, see [20, Theorem 8.14] and [12, Section 10.6.4].

A metric space (X, d) is said to be *n-rectifiable* for $n \in \mathbb{N}$ if it can be covered, up to an $\mathcal{H}^n$-null set, by countably many Lipschitz images of $\mathbb{R}^n$, i.e., there exists a countable family of Lipschitz maps $f_i : E_i \to X$ defined on measurable subsets $E_i \subset \mathbb{R}^n$ such that $\mathcal{H}^n(X \setminus \cup_{i=1}^{\infty} f_i(E_i)) = 0$. For more details on rectifiability of metric spaces, we refer to [4, 9, 25].

2.3. **Gromov–Hausdorff convergence and blow-ups.** In this subsection, we recall the definitions of pointed Gromov–Hausdorff convergence and blow-ups of pointed metric spaces. More details and equivalent definitions of Gromov–Hausdorff convergence can be found in [8, 12, 16, 21].

Given two pointed metric spaces $(X, d_X), x$ and (Y, d_Y, y) and $\varepsilon > 0$, a map $F : (X, d_X, x) \to (Y, d_Y, y)$ is called an *ε-isometry* if it satisfies the following three conditions:

(1) $F(x) = y$;

(2) for any $z, z' \in B(x, 1/\varepsilon) \subset X$, it holds that $||zz'| - |f(z)f(z')|| < \varepsilon$;

(3) for any $r < 1/\varepsilon$, it holds that $B(y, r - \varepsilon) \subset N(f(B(x, r)), \varepsilon)$.

Given a sequence $\{(X_n, d_n, x_n)\}_n$ of pointed metric spaces, we say that (X_n, d_n, x_n) pointed Gromov–Hausdorff converges to a pointed metric space (X, d, x), if for any $\varepsilon > 0$, there exists an $N_0 := N_0(\varepsilon) \in \mathbb{N}$ such that for any $n \geq N_0$, we can find ε-isometries F_n from (X_n, d_n, x_n) to (X, d, x). It is known that if a sequence of pointed metric spaces whose doubling constants are uniformly bounded above, then it contains a subsequence which pointed Gromov–Hausdorff converges to a pointed metric space with the same doubling constant bound, see for example [19, Proposition 2.2].

For a pointed metric space (X, d, x) and a positive number $\lambda \in (0, 1]$, we call the rescaled space $(X, d/\lambda, x)$ a *blow-up* of X at x. We denote by $\mathrm{Tan}(X, d, x)$ the collection of all pointed Gromov–Hausdorff limits of the blow-ups $\{(X, d/\lambda_n, x)\}$ at x for some $\lambda_n \in (0, 1]$ converging to 0. It is known that if (X, d) is doubling, then $\mathrm{Tan}(X, d, x)$ is non-empty.

2.4. **Uniform convexity and smoothness.** In this subsection, we recall the definitions of uniform convexity and smoothness for normed spaces. See [38] for more details.

Let $(E, \|\cdot\|)$ be a normed space and $p \in (1, \infty)$. We say E is *p-uniformly convex* if there exists a constant $C \geq 1$ such that for any $u, v \in E$, it holds

$$\left\| \frac{u+v}{2} \right\|^p \leq \frac{1}{2}\|u\|^p + \frac{1}{2}\|v\|^p - \frac{1}{4C}\|u - v\|^p, \tag{2.3}$$

where the constant C is called the *uniform convexity constant*. This characterizes the convexity of the norm in a quantitative way. Similarly, we say that E is *p-uniformly smooth* if there exists a constant $S \geq 1$ such that for any $u, v \in E$, it holds

$$\left\| \frac{u+v}{2} \right\|^p \geq \frac{1}{2}\|u\|^p + \frac{1}{2}\|v\|^p - \frac{S}{4}\|u - v\|^p. \tag{2.4}$$

The constant S in the p-uniformly smooth inequality (2.4) is called *uniform smoothness constant*, which characterizes the concavity of the norm. Note that any normed space $(E, \|\cdot\|)$ satisfies the inequality $\|(u+v)/2\|^p \leq \frac{1}{2}\|u\|^p + \frac{1}{2}\|v\|^p$ for all $u, v \in E$, duo to the convexity of norm and function $t \mapsto t^p$.

It is well-known that for a finite-dimensional normed space $(\mathbb{R}^n, \|\cdot\|)$, the norm $\|\cdot\|$ is induced by an inner product if and only if $(\mathbb{R}^n, \|\cdot\|)$ is 2-uniformly convex with $C = 1$, which is also equivalent to $(\mathbb{R}^n, \|\cdot\|)$ being 2-uniformly smooth with $S = 1$. Furthermore, it can be shown (see, for example, [6]) by the Clarkson's inequality that any finite dimensional l^p-space $(\mathbb{R}^n, \|\cdot\|_p)$ with $p \in [2, \infty)$ is p-uniformly convex with $C = 1$ as well as being 2-uniformly smooth with $S = p - 1$. By duality, any finite dimensional l^q-space with $q \in (1,2]$ is 2-uniformly convex with $C = (q-1)^{-1}$ as well as being q-uniformly smooth with $S = 1$. Note that among l^p-spaces, only l^2-space is both 2-uniformly convex and smooth. See [38, Section 1.2] for more details. For infinite dimensional L^p-spaces, similar properties also hold due to Beckner's inequality.

3. S-CONCAVITY, LOCAL SEMI-CONVEXITY AND BUSEMANN CONCAVITY

In this section, we introduce S-concavity, local semi-convexity, and Busemann concavity for geodesic spaces. These synthetic sectional curvature bounds for metric spaces, have been well-studied by Ohta [37, 38], Kell [24], and others (see for example [23]), as metric generalizations of curvature bounds for Finsler manifolds. For other synthetic notions of sectional curvature bound such as the Busemann's notion of non-positive curvature, we refer to [14, 22, 40] and more recent [18].

Definition 3.1 (S-concavity)**.** We say that a complete geodesic space (X,d) is *S-concave* for $S \geq 1$, or semi-concave for short, if for any point $p \in X$ and any constant-speed geodesic $\xi : [0,1] \to X$, it holds that

$$|p\xi(t)|^2 \geq (1-t)\,|p\xi(0)|^2 + t\,|p\xi(1)|^2 - S\,t(1-t)\,|\xi(0)\xi(1)|^2\,, \tag{3.1}$$

for any $t \in [0,1]$. If we only require the above inequality to hold for curves satisfying $\sup_{t\in[0,1]} |p\xi(t)| < D$ for some $D > 0$, then we call (X,d) is locally S-concave or locally semi-concave.

Remark 3.2. *We remark that the S-concavity condition (3.1) is equivalent to requiring that the squared distance function from a point is S-concave, this is to say, the function $t \mapsto |p\xi(t)|^2 - St^2|\xi(0)\xi(1)|^2$ is concave on $[0,1]$ for any constant-speed geodesic ξ. Furthermore, no geodesic space can be S-concave for $S < 1$, except for the trivial case when X is a singleton, see [38, Section 8.3].*

The S-concavity condition is a metric generalization of 2-uniform smoothness of Banach spaces, where the uniform smoothness constant S characterizes the infinitesimal concavity of local geometry of the underlying spaces. Such geometric condition plays an important role in the study of the geometry of Banach spaces, metric spaces and Wasserstein spaces (see, for example, [34, 36, 37]). For example, it is known that 1-concave spaces are exactly Alexandrov spaces with non-negative curvature. In the setting of Finsler geometry, it has been proved by Ohta [37, Theorem 4.2] (see also [38, Corollary 8.20]) that any complete Berwald space with non-negative flag curvature and uniform smoothness constant bounded above by S is S-concave in the sense of Definition 3.1. It is clear that any 2-uniformly smooth Banach space with the uniform smoothness constant $S_F \geq S$ is S-concave. In particular, any finite-dimensional l^p-space $(\mathbb{R}^n, \|\cdot\|_p)$ with $p \geq 2$ is S-concave with $S = p-1$.

Definition 3.3 (Local semi-convexity)**.** We say that a complete geodesic space (X,d) is *(C,D)-locally semi-convex*, or locally semi-convex for simplicity, if there are $C \geq 0, D > 0$ such that for any point $p \in X$ and any constant-speed geodesic $\xi : [0,1] \to X$ satisfying $\sup_{t\in[0,1]} |p\xi(t)| < D$, it holds

$$|p\xi(t)|^2 \leq (1-t)\,|p\xi(0)|^2 + t\,|p\xi(1)|^2 + Ct(1-t)\,|\xi(0)\xi(1)|^2\,, \quad \text{for any } t \in [0,1]. \tag{3.2}$$

If $C = 0$, we say that (X,d) is *locally convex*.

By definition, uniquely geodesic Banach spaces are surely locally convex. Locally semi-convex spaces are generalizations of Banach spaces, in a quantitative and non-linear way. There is a stronger convexity condition introduced by Ohta [34, 37], called *k-convexity*, which ask for the following stronger convexity condition:

$$|p\xi(t)|^2 \leq (1-t)\,|p\xi(0)|^2 + t\,|p\xi(1)|^2 - k\,t(1-t)\,|\xi(0)\xi(1)|^2\,, \quad k > 0 \tag{3.3}$$

for any point p and any constant-speed geodesic $\xi : [0, 1] \to X$. Such stronger condition has been studied in the setting of Finsler geometry by Ohta himself [38, Section 8.3], in which it is shown that any forward complete Finsler manifold with flag curvature bounded above by $\kappa \geq 0$ with vanishing T-curvature and uniform convexity constant bounded from above by $C \geq 1$ satisfies the following inequality:

$$\lim_{t\to 0} \frac{|p\xi(-t)|^2 + |p\xi(t)|^2 - 2|p\xi(0)|^2}{2t^2} \geq C^{-1} \frac{\sqrt{\kappa} r \cos(\sqrt{\kappa} r)}{\sin(\sqrt{\kappa} r)}, \tag{3.4}$$

for any point p and unit-speed geodesic $\xi : [-\varepsilon, \varepsilon] \to X$ with $r := |p\xi(0)| < \pi/\sqrt{\kappa}$. In particular, any complete, simply connected Berwald space of non-positive flag curvature and uniform convexity constant bounded from above by $C \geq 1$ is C^{-1}-convex in the sense of (3.3).

Definition 3.4 (Busemann Concave). A complete geodesic space (X, d) is said to be Busemann concave if for any pair of constant-speed geodesics $\gamma, \eta : [0, 1] \to X$ starting from a common point $\gamma(0) = \eta(0)$, the function

$$t \mapsto \frac{d(\gamma(t), \eta(t))}{t} \tag{3.5}$$

is non-increasing on $[0, 1]$.

Remark 3.5. *The Busemann concave space can be defined equivalently by comparison triangles, in the same way as Alexandrov spaces of non-negative curvature. More precisely, for two constant-speed geodesics $\gamma, \eta : [0, 1] \to X$ from x to y, z respectively, the Busemann concavity (3.5) asks for the following comparison inequality:*

$$|\gamma(t)\eta(t)| \geq |\tilde{\gamma}(t)\tilde{\eta}(t)|, \quad \text{for all } t \in [0, 1], \tag{3.6}$$

where $\tilde{\gamma}, \tilde{\eta} : [0, 1] \to \mathbb{R}^2$ are the edges of Euclidean comparison triangle $\tilde{\Delta}xyz$ from $\tilde{x}$ to $\tilde{y}$ and $\tilde{z}$ respectively. We remark that, however, compared to the Busemann concavity, the comparison condition for Alexandrov spaces is much stronger since it requires $|\gamma(t), \eta(s)| \geq |\tilde{\gamma}(t), \tilde{\eta}(s)|$ for any $t, s \in [0, 1]$, rather than only for $t = s$.

The Busemann concavity condition (3.6) is a generalization of Alexandrov's triangle comparison condition. It follows directly from the definition that Busemann concave spaces are non-branching. One can also see that Banach spaces with strictly convex norms and Alexandrov spaces of non-negative curvature are Busemann concave. For further constructions and examples of Busemann concave spaces, we refer the reader to [24, Section 2].

In the following, we provide some basic examples of spaces that satisfy all three conditions introduced above.

Example 3.6. *The following are locally convex, S-concave, Busemann concave spaces:*

(1) Any 2-uniformly smooth Banach space with strictly convex norm and uniform smoothness constant $S_F \leq S$ is a locally convex, S-concave Busemann concave space. In particular, for any $p \in [2, \infty)$, l^p-space and L^p-space are such kind of spaces with uniform smoothness constant $S_F = p - 1$.

(2) A geodesic space satisfying both $\mathrm{CBB}(0)$ condition and $\mathrm{CAT}(\kappa)$ condition for some $\kappa \geq 0$ is a locally convex, 1-concave Busemann concave space.

(3) The product space $Y := X \times E$, where (X, d_X) is a geodesic space satisfying both the $\mathrm{CBB}(0)$ and $\mathrm{CAT}(\kappa)$ conditions for some $\kappa > 0$, and $(E, \|\cdot\|_E)$ is a 2-uniformly smooth Banach space with strictly convex norm and uniform smoothness constant $S > 1$, equipped with the product metric $d_Y = \sqrt{d_X^2 + \|\cdot\|_E^2}$, is also a locally convex, S-concave, Busemann concave space. Note that in this case, Y is not an Alexandrov space with curvature bounded below.

Remark 3.7 (Berwald spaces)**.** *While one can show that Berwald spaces with flag curvature bounded below satisfy Busemann concavity locally, it remains unknown whether all such Berwald spaces satisfy Busemann concavity globally, as pointed out by Kell [24, Section 2.1]. In contrast, it has been shown in [27] that among Berwald spaces, non-positive flag curvature is equivalent to a 'reversed' condition of Busemann concavity, called Busemann convexity. Thus, any Berwald space of non-positive flag curvature satisfies Busemann convexity globally. We refer to [24, Section 2.1] for more details and to [18] for further discussion on Busemann convexity.*

Finally, we emphasize that local semi-convex, S-concave, Busemann concave spaces are very different from the GNPC spaces studied in [18]. While these spaces are non-branching, they do not necessarily possess the local geodesic extension property, which serves as a key assumption in the study of GCBA spaces [29] and GNPC spaces [18]. This geometric condition, which requires that any geodesic segment can be extended as a local geodesic beyond its endpoints, is intimately related to the local homology at a point (see [31, Theorem 1.5]) and provides superior control over both the local geometry and topological regularity of the underlying spaces. Furthermore, any pair of points in our spaces may be connected by multiple geodesics, which is in contrast to the uniquely geodesic property of GNPC spaces.

4. Angles

In this section, we introduce two notions of angle: one defined for S-concave and locally semi-convex spaces, and another on Busemann concave spaces. We then establish several important properties of these angles. We remark that, these notions also play an important role in the study of Busemann convex spaces [18].

The first notion of angle, referred to as *angle viewed from a fixed point*, is related to the S-concavity (3.1) and local semi-convexity (3.2) of distance functions. Such angle measures the orthogonality of geodesics, which serves as a key ingredient for establishing strainers and strainer maps. The second notion of angle, referred to as *angle of fixed scale*, is related to the Busemann concavity (3.5). Such angle reflects the infinitesimal structure of the underlying space and is fundamental to developing the notion of tangent cone.

These two notions of angle are independent and generally do not coincide in S-concave, local semi-convex, Busemann concave spaces, except when $S = 1$, in which case the space reduces to an Alexandrov space of non-negative curvature. This asymmetry between the two notions of angle, as well as the asymmetry inherent to the first notion of angle, is fundamentally rooted in the non-Riemannian character of the spaces under consideration and brings a considerable challenge.

4.1. Angles in S-concave and local semi-convex metric spaces.

In this subsection, we define the notion of angle viewed from a point in S-concave and locally semi-convex geodesic spaces. We show that this notion of angle is well-defined in both contexts, and possesses the properties required for constructing 'almost orthogonal coordinates', namely, strainer maps.

Definition 4.1 (Angles viewed from a fixed point)**.** Let (X, d) be an S-concave geodesic space for some $S \geq 1$ and let $p \in X$ be a point. Let $x \in X$ be a point different from p and let ξ be a unit-speed geodesic from x to $y \in X$. The angle $\angle px\xi$ is defined as

$$\angle px\xi := \lim_{t \searrow 0} \tilde{\angle} px\xi(t), \tag{4.1}$$

where $\tilde{\angle} px\xi(t)$ is the angle at x of Euclidean comparison triangle $\tilde{\Delta} px\xi(t)$. We call $\angle px\xi$ the *angle viewed from the point p at x toward the point y along the geodesic ξ.*

Remark 4.2. *Our notation $\angle px\xi$ is slightly different from the one used in [18] for GNPC spaces, due to the possibly non-uniqueness of geodesics joining x and y in S-concave spaces.*

The lack of monotonicity for comparison angles in S-concave spaces makes it non-trivial to determine whether the limit on the right-hand side of (4.1) exists. Below, we prove a key lemma, following from S-concavity, which shows that $\angle px\xi$ dominates $\tilde{\angle} px\xi(t)$ up to an explicit error term. In particular, this establishes that the angle $\angle px\xi$ is well-defined.

Lemma 4.3. *Let (X, d) be an S-concave geodesic space with $S \geq 1$. Let $p, x \in X$ be two different points and ξ be a unit-speed geodesic starting from x.*

(1) The angle $\angle px\xi$ is well-defined and satisfies the following almost comparison inequality:

$$\tilde{\angle} px\xi(t) \leq \angle px\xi + \delta_S(t; |px|), \quad t \in [0, t_0], \tag{4.2}$$

where $\delta_S(t; |px|) := \arccos(1 - \frac{(S-1)t}{2|px|})$ is a non-negative continuous function defined on $t \in [0, t_0]$ satisfying $\delta_S(t) \to 0$ as $t \to 0$. Here $t_0 > 0$ is a sufficient small constant depending only on the distance $|px|$, the constant S and the length $l(\xi)$.

(2) The function $t \mapsto |p\xi(t)|$ is differentiable at $t = 0$ and satisfies

$$\left.\frac{d}{dt}\right|_{t=0} |p\xi(t)| = -\cos\angle px\xi. \tag{4.3}$$

Remark 4.4. *As shown in Lemma 4.3, $\cos\angle px\xi$ represents the directional derivative of the distance function from p. In the case when X is an Alexandrov space of non-negative curvature, the first variation formula gives that the directional derivative of the distance function from p satisfies*

$$\left.\frac{d}{dt}\right|_{t=0} |p\xi(t)| = D_x d_p\left(\xi'(0)\right) = -\cos \min_{w \in \Uparrow_x^p} \angle(w, \xi'(0)), \tag{4.4}$$

where $\xi'(0) \in \Sigma_x X$ denotes the initial direction of the geodesic ξ, and $\Uparrow_x^p \subset \Sigma_x X$ is the set of all initial directions of geodesics from x to p. In this case, the angle $\angle px\xi$ coincides with the minimal angle between the geodesics from x to p and the direction ξ, i.e., $\angle px\xi = \min_{w \in \Uparrow_x^p} \angle(w, \xi'(0))$. Here, $\angle(\cdot,\cdot)$ denotes the angle metric on the space of directions $\Sigma_x X$.

Proof. (1) By applying the Euclidean law of cosine to the Euclidean comparison triangle $\tilde{\Delta} px\xi(t)$, it follows that

$$\cos\tilde{\angle} px\xi(t) = \frac{|px|^2 + t^2 - |p\xi(t)|^2}{2t|px|} = \frac{|px|^2 - |p\xi(t)|^2 + St^2}{2t|px|} - \frac{(S-1)t}{2|px|}. \tag{4.5}$$

By S-concavity, it follows that the quotient $(|p\xi(t)|^2 - |p\xi(0)|^2 - St^2)/t$ is non-increasing in $t \in [0, l(\xi)]$. Therefore, $\lim_{t \searrow 0}(|px|^2 - |p\xi(t)|^2 + St^2)/t$ exists. By the monotonicity and continuity of the cosine function, it follows from the equality (4.5) that $\lim_{t\to 0}\tilde{\angle} px\xi(t)$ exists and satisfies

$$\begin{aligned}\cos\angle px\xi = \lim_{t\searrow 0}\cos\tilde{\angle} px\xi(t) &= \lim_{t\searrow 0}\frac{|px|^2 - |p\xi(t)|^2 + St^2}{2t|px|} \\ &= \inf_{t>0}\frac{|px|^2 - |p\xi(t)|^2 + St^2}{2t|px|}.\end{aligned} \tag{4.6}$$

Moreover, the monotonicity of $t \mapsto (|px|^2 - |p\xi(t)|^2 + St^2)/t$ implies that

$$\cos\tilde{\angle} px\xi(t) = \frac{|px|^2 - |p\xi(t)|^2 + St^2}{2t|px|} - \frac{(S-1)t}{2|px|} \geq \cos\angle px\xi - \tilde{\delta}_S(t/|px|), \tag{4.7}$$

where $\tilde{\delta}_S(r) := (S-1)r/2$. Take $\delta_S(t;|px|) := \arccos(1-\tilde{\delta}_S(t/|px|))$. By concavity of the cosine function, it can be deduced from the inequality (4.7) that

$$\tilde{\angle} px\xi(t) \le \angle px\xi + \delta_S(t;|px|), \quad \text{for any } t \in [0,t_0], \tag{4.8}$$

where $t_0 > 0$ is a sufficiently small constant such that $t_0 \le l(\xi)$ and $(S-1)t_0/(2|px|) \le 1$.

(2) By (4.5), it follows that

$$\begin{aligned}\cos\tilde{\angle} px\xi(t) &= \frac{|px|^2+t^2-|p\xi(t)|^2}{2t|px|} = \frac{(|px|-|p\xi(t)|)2|px|}{2t|px|} + \frac{t^2}{2t|px|}\\ &\quad + \frac{(|px|-|p\xi(t)|)(|px|+|p\xi(t)|-2|px|)}{2t|px|}\\ &= \frac{|px|-|p\xi(t)|}{t} + \delta(t;\xi),\end{aligned} \tag{4.9}$$

where $\delta(t;\xi) := \frac{t}{2|px|} - \frac{(|p\xi(t)|-|px|)^2}{2t|px|}$. Note that from $0 \le \delta(t;\xi) \le \frac{t}{2|px|} \to 0$ as $t \searrow 0$ and $\lim_{t\to 0}\cos\tilde{\angle} px\xi(t)$ exits, we know that $\lim_{t\to 0}\frac{|px|-|p\xi(t)|}{t}$ exists and

$$\lim_{t\searrow 0}\frac{|px|-|p\xi(t)|}{t} = \lim_{t\searrow 0}\cos\tilde{\angle} px\xi(t) = \cos\angle px\xi. \tag{4.10}$$

□

Remark 4.5. *We emphasize that the error function δ_S in the almost comparison inequality (4.2) depends only on the ratio $t/|px|$ and the uniform smoothness constant S and does not depend on x and the geodesic ξ. In particular, for any $\varepsilon > 0$, we can find a sufficiently small neighborhood U of x and sufficiently small constant $l > 0$ such that any unit-geodesic ξ starting from $z \in U$, it holds*

$$\tilde{\angle} pz\xi(t) \le \angle pz\xi + \delta_S(t;|pz|) < \angle px\xi + \varepsilon, \quad \textit{for any } t \in [0,l]. \tag{4.11}$$

It is worth to mention that the angle viewed from a fixed point exhibits an asymmetry, as demonstrated by the following example:

Example 4.6 (Asymmetry of angles viewed from a fixed point)**.** *Consider $\mathbb{S}^2$ equipped with the spherical distance $d_{\mathbb{S}^2}$, and let p and x be the north and south poles, respectively. Let ξ be a geodesic from x to p, and let $p' \in \xi$ and $q \notin \xi$ be points near p and x, respectively. Let η be a geodesic from x to q. One can readily verify that the angle viewed from p at x along η towards q equals 0, i.e., $\angle px\eta = 0$, while the angle viewed from the different point p' satisfies $\angle p'x\eta > 0$. Moreover, the angle viewed from q at x towards p is strictly positive, i.e., $\angle qx\xi > 0$, further illustrating the asymmetric nature of angles viewed from fixed points.*

Next we prove several fundamental properties of angles viewed from a fixed point. We first show the lower semi-continuity of angle $\xi \mapsto \angle px\xi$ with respect to the pointwise convergence.

Lemma 4.7 (Lower semi-continuity of $\angle px\xi$)**.** *Let (X,d) be an S-concave space with $S \ge 1$. Let $p \in X$ be an arbitrary point and ξ_i be sequence of constant-speed geodesics on X such that ξ_i converges pointwisely to a non-trivial geodesic ξ. Let η_i and η be the geodesics of ξ_i and ξ parametrized by arc-length. Then it holds*

$$\angle px\eta \le \liminf_{i\to\infty} \angle px_i\eta_i, \tag{4.12}$$

where $x_i = \eta_i(0)$ and $x = \eta(0)$.

Proof. Let $0 < t < l(\eta)$ be sufficiently small. From the assumption that ξ_i converges pointwisely to the geodesic ξ, it can be seen that $\eta_i(t) \to \eta(t)$ as i goes to infinity. In particular, it holds that

$$|p\eta_i(t)| \to |p\eta(t)|, \quad |p\eta_i(0)| \to |p\eta(0)|, \quad |\eta_i(0)\eta_i(t)| \to |\eta(0)\eta(t)|, \tag{4.13}$$

as i goes to infinity. From the Euclidean law of cosine, it follows that $\tilde{\angle} px_i\eta_i(t)$ converges to $\tilde{\angle} px\eta(t)$. On the other hand, by the choice of the error function δ_S in the almost comparison (4.2), we have $\lim_{i\to\infty} \delta_S(t; |px_i|) = \delta_S(t; |px|)$. Therefore, by the almost comparison inequality (4.2), it follows that

$$\begin{aligned} \tilde{\angle} px\eta(t) = \lim_{i\to\infty} \tilde{\angle} px_i\eta_i(t) &\leq \liminf_{i\to\infty} \angle px_i\eta_i + \lim_{i\to\infty} \delta_S(t; |px_i|) \\ &= \liminf_{i\to\infty} \angle px_i\eta_i + \delta_S(t; |px|). \end{aligned} \tag{4.14}$$

By taking $t \to 0$ on both sides of the inequality above, our claim follows. □

Next we show that the sum of angles viewed from a common point p at x toward two reversed directions along the geodesic passing through x is bounded above by π.

Lemma 4.8. *Let (X, d) be an S-concave space with $S \geq 1$. Let $p, x \in X$ and $\gamma : [-\varepsilon, \varepsilon] \to X$ be a unit-speed geodesic passing through $x := \xi(0)$, and let ξ, η be the reparametrization by arc-length of the geodesic segments of γ restricted to $[0, \varepsilon]$ and $[-\varepsilon, 0]$ respectively with the common starting point $\xi(0) = \eta(0) = x$. Then it holds*

$$\angle px\xi + \angle px\eta \leq \pi. \tag{4.15}$$

Proof. Let $t \in (0, \varepsilon)$. Applying the S-concave inequality (3.1) to the point p and the geodesic $\xi : [-\varepsilon, \varepsilon] \to X$, it follows that

$$|px|^2 \geq \frac{1}{2} |p\gamma(-t)|^2 + \frac{1}{2} |p\gamma(t)|^2 - \frac{S}{4} (2t)^2 . \tag{4.16}$$

By plugging the inequality above into the Euclidean law of cosine, we obtain that

$$\begin{aligned} \cos \tilde{\angle} px\xi(t) + \cos \tilde{\angle} px\eta(t) &= \frac{t^2 + |px|^2 - |p\gamma(t)|^2}{2|px|t} \\ &\quad + \frac{t^2 + |px|^2 - |p\gamma(-t)|^2}{2|px|t} \geq \frac{(1-S)t}{|px|}. \end{aligned} \tag{4.17}$$

Taking $t \to 0$ on both sides of the inequality above, we obtain that

$$\cos \angle px\xi + \cos \angle px\eta \geq 0. \tag{4.18}$$

This implies that $\angle px\xi + \angle px\eta \leq \pi$. □

Remark 4.9. *We remark that it is not clear whether, in S-concave spaces with $S > 1$, the sum of angles viewed from two different points p, q at x along a common geodesic γ can be bounded above by π, even when p, q lie on a common geodesic passing through x.*

We now discuss angles in locally semi-convex spaces (cf. Definition 3.3). It turns out by the following lemma, similar to Lemma 4.3, that angles in Definition 4.1 is also well-defined in locally semi-convex spaces if the distance $|px|$ is small enough.

Lemma 4.10. *Let (X, d) be a (C, D)-locally semi-convex space. Then the angle $\angle px\xi$ defined in Definition 4.1 is well-defined for any unit-speed geodesic ξ starting from x if the distance $|px| < D$. Furthermore, it satisfies the following almost comparison inequality:*

$$\tilde{\angle} px\xi(t) \geq \angle px\xi - \delta_C(t; |px|), \quad \text{for any } t \in [0, t_0], \tag{4.19}$$

where $\delta_C(t; |px|) := \arccos(1 - (C+1)t/(2|px|))$ is a non-negative function defined on $[0, t_0]$, and $t_0 > 0$ is a constant depending only on $|px|$, the constants C, D and the length $l(\xi)$.

Proof. The proof is similar to Lemma 4.3. Let $p, x \in X$ be two points such that $|px| < D$, and let $\xi : [0, l] \to X$ be an arbitrary unit-speed geodesic starting from x. By restricting ξ to a smaller domain, we may assume that $\sup_{t\in[0,l]} |p\xi(t)| < D$. Let $t \in (0, l)$. By applying the (C,D)-local semi-convexity (3.2) to the point p and the geodesic ξ, it follows that

$$t \mapsto \frac{|p\xi(t)|^2 + Ct^2 - |p\xi(0)|^2}{t} \tag{4.20}$$

is non-decreasing on $[0,l]$. Therefore, the limit of $(|px|^2 - |p\xi(t)|^2 - Ct^2)/(2t|px|)$ as $t \searrow 0$ exists and satisfies

$$\lim_{t\searrow 0} \frac{|px|^2 - |p\xi(t)|^2 - Ct^2}{2t|px|} = \sup_{t\in(0,l)} \frac{|px|^2 - |p\xi(t)|^2 - Ct^2}{2t|px|}. \tag{4.21}$$

By the Euclidean law of cosine, it follows that the limit of $\cos \tilde{\angle} px\xi(t)$ as $t \searrow 0$ exists and satisfies

$$\begin{aligned}\lim_{t\searrow 0} \cos \tilde{\angle} px\xi(t) = \lim_{t\searrow 0} \left(\frac{|px|^2 - |p\xi(t)|^2 - Ct^2}{2t|px|} + \frac{(C+1)t}{2|px|} \right) \\ = \sup_{t\in(0,l)} \frac{|px|^2 - |p\xi(t)|^2 - Ct^2}{2t|px|}.\end{aligned} \tag{4.22}$$

By the monotonicity of cosine function and the definition of $\angle px\xi$, we obtain that $\angle px\xi = \lim \tilde{\angle} px\xi(t)$ exists. Furthermore, from the monotonicity of $t \mapsto (|px|^2 - |p\xi(t)|^2 - Ct^2)/(2t|px|)$, it follows that

$$\begin{aligned}\cos \tilde{\angle} px\xi(t) &= \frac{|px|^2 - |p\xi(t)|^2 - Ct^2}{2t|px|} + \frac{(C+1)t}{2|px|} \\ &\le \sup_{s\in(0,l)} \left(\frac{|px|^2 - |p\xi(s)|^2 - Cs^2}{2s|px|} \right) + \frac{(C+1)t}{2|px|} = \cos \angle px\xi + \frac{(C+1)t}{2|px|}.\end{aligned} \tag{4.23}$$

Take $\delta_C(t;|px|) = \arccos(1-(C+1)t/(2|px|))$. From the monotonicity of derivative of cosine function, we obtain that

$$\tilde{\angle} px\xi \ge \angle px\xi - \delta_C(t;|px|), \quad \text{for any } t \in [0,t_0], \tag{4.24}$$

where $t_0 > 0$ is a sufficiently small number such that $t_0 \le l$ such that $t_0 + |px| < D$ and $(C+1)t_0/(2|px|) \le 1$. □

We conclude this subsection by establishing several properties of angles in locally semi-convex spaces, which are counterparts to Lemma 4.7 and Lemma 4.8. Since the proofs follow identical arguments to those of Lemma 4.7 and Lemma 4.8, we omit them here.

Lemma 4.11 (Upper semi-continuity of $\angle px\xi$)**.** *Let (X,d) be a (C,D)-locally semi-convex space. Let $p \in X$ be an arbitrary point and ξ_i be sequence of constant-speed geodesics on X such that ξ_i converges pointwisely to a non-trivial geodesic ξ, and that $\sup_i |p\xi_i(0)| < D$. Let η_i and η be the geodesics of ξ_i and ξ parametrized by arc-length. Then it holds*

$$\angle px\eta \ge \limsup_{i\to\infty} \angle px_i\eta_i, \tag{4.25}$$

where $x_i = \eta_i(0)$ and $x = \eta(0)$.

Lemma 4.12. *Let (X,d) be a (C,D)-locally semi-convex space. Let $p \in X$ and $\gamma : [-\varepsilon,\varepsilon] \to X$ be a unit-speed geodesic passing through $x := \gamma(0)$ such that $|px| < D$. Let ξ and η be the reparametrizations by arc-length of the geodesic segments of γ restricted to $[0,\varepsilon]$ and $[-\varepsilon,0]$, respectively, both starting at x. Then it holds*

$$\angle px\xi + \angle px\eta \ge \pi. \tag{4.26}$$

4.2. **Angles and tangent cones in Busemann concave spaces.** In this subsection, we introduce the notion of angle of fixed scale and the concept of tangent cones in Busemann concave spaces. These notions, which are closely related to the Busemann concavity, have been previously introduced and studied by Kell [24]. After recalling several important results concerning tangent cones in Busemann concave spaces from [24], we define the *space of directions at a point with common length* and establish that, when the underlying Busemann concave space is doubling, these spaces of directions with common length are uniformly compact.

Definition 4.13 (Angles of fixed scale)**.** Let (X, d) be a Busemann concave space and let $x \in X$ be a point. Let γ, η be two non-trivial unit-speed geodesics starting from x. For any $t, s > 0$, the angle $\angle_x(\gamma(t), \eta(s))$ is defined as

$$\angle_x(\gamma(t), \eta(s)) := \sup_{\substack{\theta\in(0,1],\\ \max\{\theta t,\theta s\}\leq a}} \tilde{\angle}_x(\gamma(\theta t), \eta(\theta s)), \tag{4.27}$$

where $a > 0$ is an arbitrary positive number such that γ, η are both defined on $I_a := [0, a]$. We call $\angle_x(\gamma(t), \eta(s))$ *the angle of fixed scale*. In the case $t = s$, we call $\angle_x(\gamma(t), \eta(t))$ *the angle of common scale*.

Remark 4.14. *The definition of angle of fixed scale in [18] is slightly different from our Definition 4.13, where we do not assume that $\gamma(t)$ and $\eta(s)$ are well-defined. We adopt the current definition to maintain consistency with the definition of tangent cones in Busemann concave spaces.*

It follows from the Busemann concavity (3.5) that the function $\theta \mapsto \tilde{\angle}_x(\gamma(\theta t), \eta(\theta s))$ is non-increasing on the interval $(0, \min\{1, a/t, a/s\}]$. In particular, this monotonicity implies that the value on the right-hand side of (4.27) does not depend on the choice of a, and the supremum in (4.27) is actually a limit. Therefore, the angle $\angle_x(\gamma(t), \eta(s))$ is well-defined.

We emphasize, however, that in contrast to the angles in Alexandrov spaces, the angle $\angle_x(\gamma(t), \eta(s))$ in Busemann concave spaces generally depends on the choice of the length parameters $t, s > 0$. Furthermore, these angles do not necessarily coincide with the angles viewed from a fixed point in Definition 4.1 for S-concave Busemann concave spaces with $S > 1$, as illustrated by the following example.

Example 4.15 (Asymmetry of two notions of angle)**.** *Let $\gamma, \eta \subset (\mathbb{R}^2, \|\cdot\|_p)$ be two line segments in the two-dimensional l^p-space with $p \in [2, \infty)$, both starting from the origin o and ending at the points $u := (0, 1)$ and $v := (1, 0)$, respectively. A direct computation shows that $\angle_o(\gamma(1), \eta(1)) = \arccos(1 - 2^{2/p-1})$, while $\angle_o(\gamma(1), \eta(1/2)) = \arccos\left(5/4 + (1 + 1/2^p)^{2/p}\right)$, and $\angle uo\eta = \lim_{t\searrow 0} \arccos\left(\frac{1+t^2-(1+t^p)^{2/p}}{2t}\right)$. It is straightforward to verify that these three angles are generally distinct unless $p = 2$.*

We will subsequently demonstrate that the angle of fixed scale $\angle_x(\gamma(t), \eta(s))$ depends solely on the ratio t/s, rather than on the individual values of t and s.

We now introduce the notion of tangent cones in Busemann concave spaces, following the terminology and notations of [24].

Definition 4.16 (Pre-tangent cone and tangent cone)**.** Let (X, d) be a Busemann concave space and let Γ_x denote the set of all non-trivial maximal unit-speed geodesics starting from $x \in X$. The pre-tangent cone at x, denoted by $\hat{T}_x X$, is defined as the set $\Gamma_x \times [0, \infty)/\sim$, where all points of the form $(\gamma, 0)$, for $\gamma \in \Gamma_x$, are identified as a single point. The metric d_x on $\hat{T}_x X$ is defined as follows: given any $(\gamma, t), (\eta, s) \in \hat{T}_x X$, let $I_a := [0, a]$ be an interval such that both γ and η are defined on I_a. The distance d_x is defined as

$$d_x((\gamma, t), (\eta, s)) := \sup_{\substack{\theta\in(0,1],\\ \max\{\theta t,\theta s\}\leq a}} \frac{|\gamma(\theta t), \eta(\theta s)|}{\theta}. \tag{4.28}$$

The tangent cone (T_xX, d_x) at x is defined as the completion of the pre-tangent cone $\hat{T}_xX$ with respect to the metric d_x.

It has been shown in [24, Lemma 2.17] that the metric d_x on the pre-tangent cone $\hat{T}_xX$ is well-defined and the supremum in (4.28) is in fact a limit, due to the Busemann concavity. Furthermore, the metric d_x satisfies the following positive homogeneity property:

$$d_x((\gamma, \lambda t), (\eta, \lambda s)) = \lambda d_x((\gamma, t), (\eta, s)), \quad \text{for any } \lambda > 0, (\gamma, t), (\eta, s) \in \hat{T}_xX. \tag{4.29}$$

The metric d_x on the tangent cone T_xX is related to the angle of fixed scale in Definition 4.13 through the Euclidean law of cosine, as shown in the following lemma.

Lemma 4.17. *Let (X, d) be a Busemann concave space and let $x \in X$ be an arbitrary point. Let γ, η be two non-trivial unit-speed geodesics, and $\bar{\gamma}, \bar{\eta}$ be their maximal extensions. Then for any $t, s > 0$, we have*

$$d_x((\bar{\gamma}, t), (\bar{\eta}, s))^2 = t^2 + s^2 - 2ts \cos \angle_x(\gamma(t), \eta(s)). \tag{4.30}$$

Moreover, the angle of fixed scale is positive scaling-invariant; that is, for any $\lambda > 0$,

$$\angle_x(\gamma(\lambda t), \eta(\lambda s)) = \angle_x(\gamma(t), \eta(s)). \tag{4.31}$$

In particular, the angle $\angle_x(\gamma(t), \eta(s))$ depends only on the ratio t/s, and not on the individual values of t and s.

Proof. Let $a > 0$ be a positive number such that γ and η are both defined on the interval $I_a = [0, a]$. Note that from the definition, it holds that the angle $\angle_x(\bar{\gamma}(t), \bar{\eta}(s))$ actually coincides with the angle $\angle_x(\gamma(t), \eta(s))$, since $\tilde{\angle}_x(\bar{\gamma}(\theta t), \bar{\eta}(\theta s)) = \tilde{\angle}_x(\gamma(\theta t), \eta(\theta s))$ when $\theta > 0$ is sufficiently small. Now for any sufficiently small $\theta > 0$, it follows by the Euclidean law of cosine that

$$|\bar{\gamma}(\theta t)\bar{\eta}(\theta s)|^2 = (\theta t)^2 + (\theta s)^2 - 2\theta^2 ts \cos \tilde{\angle}_x(\gamma(\theta t), \eta(\theta s)). \tag{4.32}$$

By first dividing θ^2 and then letting $\theta \searrow 0$ on both sides of the equality above, we obtain the equality (4.30). The second claim just follows from the positive homogeneity of d_x together with the equality (4.30). □

We now discuss the relationship between the tangent cone (T_xX, d_x) and the pointed Gromov–Hausdorff limit of the blow-ups $\{(X, d/\lambda, x)\}_{\lambda>0}$. It is important to note that, in general, the tangent cone (T_xX, d_x) of a Busemann concave space X does not necessarily coincide with the pointed Gromov–Hausdorff limit of the blow-ups $\{(X, d/\lambda, x)\}_{\lambda>0}$ at x, as discussed in [24, Section 2.3]. However, the following proposition, due to [24, Lemma 2.20, Corollary 2.21], shows that if the Busemann concave space X is doubling, then the tangent cone and the pointed Gromov–Hausdorff limit of the blow-ups at x do indeed coincide.

Proposition 4.18 (Kell, [24])**.** *Let (X, d) be a Busemann concave space. If X is (locally) doubling, then the tangent cone (T_xX, d_x) at x is locally compact and coincides with the unique pointed Gromov–Hausdorff limit of the blow-ups $\{(X, d/\lambda, x)\}_\lambda$ as $\lambda \to 0$.*

While the angle of fixed scale $\angle_x(\gamma(t), \eta(s))$ is well-defined and possesses the desirable relationship with the metric d_x on the tangent cone T_xX, it fundamentally depends on the ratio t/s of the side-lengths of the geodesics. This dependence prevents a direct identification of the space of directions at x with the set of equivalence classes of geodesics emanating from x equipped with the angle metric, as in Alexandrov spaces. Consequently, the structure of tangent cones in Busemann concave spaces is inherently more intricate than the metric cone over the space of directions, reflecting the subtleties of non-Riemannian geometry of Busemann concave spaces. In the following, we introduce a distinguished subset of the tangent cone, referred to as the *space of directions with common length*. This subset can be naturally equipped with the angle of common scale as a metric.

Definition 4.19 (Space of directions with common length)**.** Let (X, d) be a Busemann concave space and $x \in X$ be an arbitrary point. Given $l > 0$, we denoted by $\hat{\Sigma}_x^l X$ the subset of the pre-tangent cone $\hat{T}_x X$ consisting of elements of the form (γ, l). We define the angle metric $\angle_x(\cdot, \cdot)$ on $\hat{\Sigma}_x^l X$ as the angle of fixed scale restricted to the subset $\hat{\Sigma}_x^l X$, that is,

$$\angle_x((\gamma, l), (\eta, l)) := \angle_x(\gamma(l), \eta(l)), \quad \text{for any } (\gamma, l), (\eta, l) \in \hat{\Sigma}_x^l X. \tag{4.33}$$

The space of directions at x with common length l, denoted by $\Sigma_x^l X$, is defined as the completion of $\hat{\Sigma}_x^l X$ with respect to the angle metric $\angle_x$.

Lemma 4.20. *The angle metric $\angle_x$ on $\hat{\Sigma}_x^l X$ is well-defined. Furthermore, a sequence in $(\Sigma_x^l X, \angle_x)$ is convergent with respect to the angle metric $\angle_x$ if and only if it is convergent with respect to the metric d_x. In particular, we can identify the elements in $(\Sigma_x^l, \angle_x)$ as the ones in $(T_x X, d_x)$. Finally, $\angle_x$ is positive scaling-invariant; that is,*

$$\angle_x((\gamma, \lambda l), (\eta, \lambda l)) = \angle_x((\gamma, l), (\eta, l)), \tag{4.34}$$

where we identify $\angle_x$ on the left-hand side of (4.34) as the metric on $\Sigma_x^{\lambda l} X$.

Proof. Let $l > 0$ be arbitrary and let $(\gamma, l), (\eta, l) \in \hat{\Sigma}_x^l X$. We claim that $\angle_x$ is a metric on $\hat{\Sigma}_x^l X$. Indeed, the symmetry and triangle inequality just follow from the definition of $\angle_x(\gamma(l), \eta(l))$ and triangle inequality of Euclidean comparison angles. Thus, it suffices to show that $\angle_x((\gamma, l), (\eta, l)) = 0$ implies that γ coincides with η in Γ_x. Indeed, if $\angle_x((\gamma, l), (\eta, l)) = 0$, then it follows that for some sufficiently small $\theta \in (0, 1)$, it holds that $\gamma(t) = \eta(t)$ for all $t \in [0, \theta l]$. Then the non-branching property of Busemann concave spaces implies that $\gamma(s) = \eta(s)$ for all $s > 0$ if $\gamma(s)$ and $\eta(s)$ are defined. Since γ and η are both maximal geodesics, they must coincide. This shows that $\angle_x$ is a well-defined metric on $\hat{\Sigma}_x^l X$. For the second claim, it follows from Lemma 4.17 that

$$d_x((\gamma, l), (\eta, l))^2 = 2l^2 - 2l^2 \cos \angle_x((\gamma, l), (\eta, l)). \tag{4.35}$$

This implies that $\{(\gamma_n, l)\}_n \subset \hat{\Sigma}_x^l X$ is a convergent sequence with respect to the angle metric $\angle_x$ if and only if it is also a convergent sequence with respect to d_x. Therefore, the second claim follows. Finally, the positive scaling-invariance directly follows from the positive scaling-invariance of angles of fixed scale. □

We conclude this subsection by showing that the family of spaces of directions at x with common lengths, $\{(\Sigma_x^l X, \angle_x)\}_{l>0}$, are uniformly compact whenever the Busemann concave space (X, d) is doubling. Our proof follows a similar strategy to that of [12, Proposition 10.9.1] for the spaces of directions in Alexandrov spaces.

Lemma 4.21. *Let (X, d) be a Busemann concave space and let $x \in X$ be an arbitrary point. If X is doubling, then the family $\{(\Sigma_x^l X, \angle_x)\}_{l>0}$ is uniformly compact; that is, for any $\varepsilon > 0$, there exists $N_0(\varepsilon) > 0$, depending only on the doubling constant of X, such that every ε-separated subset of $(\Sigma_x^l X, \angle_x)$ contains at most $N_0(\varepsilon)$ elements.*

Proof. For any $r > 0$ and $\varepsilon > 0$, let $N(\varepsilon) > 0$ denote the maximal cardinality of any εr-separated subset of a ball of radius r in X. We claim that the cardinality of any ε-separated set in $\Sigma_x^l X$ is at most $N(\varepsilon/4)$ for any $l > 0$. Indeed, let $\{v_i\}_{i=1}^m \subset \Sigma_x^l X$ be an ε-separated set in $(\Sigma_x^l X, \angle_x)$. We may assume that each v_i can be represented as $(\gamma_i, l) \in \hat{\Sigma}_x^l X$. By the definition of $\angle_x$, it follows that we can find a sufficiently small number $\theta \in (0, 1)$ such that

$$\tilde{\angle}_x(\gamma_i(\theta l), \gamma_j(\theta l)) > \frac{1}{2} \angle_x((\gamma_i, l), (\gamma_j, l)) \geq \frac{\varepsilon}{2}, \quad \text{for all } 1 \leq i < j \leq m. \tag{4.36}$$

Let $t_l := \theta l$. Then from the fundamental geometry of triangle, it follows that

$$|\gamma_i(t_l)\gamma_j(t_l)| \geq 2t_l \sin \frac{\varepsilon}{4} \geq \frac{\varepsilon}{4} t_l, \quad \text{for any } 1 \leq i < j \leq m. \tag{4.37}$$

This implies that the family $\{\gamma_i(t_l)\}_{i=1}^m$ is an $\varepsilon t_l/4$-separated subset of the t_l-ball centered at x in X. By the doubling property of X, it follows that $m \leq N(\varepsilon/4)$. The claim follows by choosing the constant $N_0(\varepsilon) := N(\varepsilon/4)$. □

5. Strainers and strainer maps

In this section, we develop strainers and strainer maps for S-concave spaces that also satisfies local semi-convexity. The strainer maps, which can be regarded as 'almost orthogonal coordinates', play a key role in studying the structure theory.

After introducing the definitions of strainers and strainer maps, we proceed to establish the ε-openness of strainer maps defined on suitable open domains. Furthermore, we demonstrate that such strainer maps are bi-Lipschitz provided that no point in the possibly smaller domain admits an additional strainer which, together with the original strainer, would yield a strainer map mapping into a higher-dimensional target space. Finally, we establish a key property known as the self-improvement property of strainers. This property plays an important role in the next section, where we analyze the dimension of S-concave Busemann concave spaces that satisfy local semi-convexity.

We emphasize that, we do not need the Busemann concavity in the construction of strainers and strainer maps.

5.1. **Definitions of strainers and strainer maps.**

Definition 5.1 ($(1,\delta)$-strainer)**.** Let X be an S-concave space and let $x \in X$. Given $0 < \delta < 1/2$, a point $p \in X \setminus \{x\}$ is called a $(1,\delta)$-*strainer at* x if there exists a point $q \in X \setminus \{x\}$ such that the Euclidean comparison angle $\tilde{\angle} pxq > \pi - \delta$ and that

$$\bar{\delta}_S(|qx|;|px|) := \arccos\left(1 - S\frac{|qx|}{2|px|}\right) < \delta. \tag{5.1}$$

In this case, we refer to the point q as an *opposite strainer* of p at x, and to the pair (p,q) a $(1,\delta)$-*strainer pair*. We say that x is a $(1,\delta)$-*strained point* if it admits a $(1,\delta)$-strainer at itself.

Remark 5.2. *Our definition of strainer differs from the original one in [13] for Alexandrov spaces with curvature bounded below, by imposing an additional control on the error function with respect to the ratio of distances $|qx|/|px|$. We adopt such a modified version to overcome the absence of monotonicity of comparison angles.*

The next useful lemma demonstrates that a $(1,\delta)$-strainers p at x is in fact implicitly connected to the notion of angle viewed from a fixed point.

Lemma 5.3. *Let X be an S-concave space with $S \geq 1$. If the angle viewed from $p \in X$ at $x \in X$ along the unit-speed geodesic ξ satisfies that $\angle px\xi > \pi - \delta$ for some small $\delta > 0$, then p is a $(1,\delta)$-strainer at x with an opposite strainer q which can be selected arbitrarily close to x. Conversely, if p is a $(1,\delta)$-strainer at x with q as an opposite strainer, then for any unit-speed geodesic ξ from x to q, it holds*

$$\angle px\xi > \pi - 2\delta. \tag{5.2}$$

Proof. Let p, x, ξ be defined as the assumptions in lemma. From the definition of $\angle px\xi$, it follows that we can find $t_0 > 0$ such that

$$\tilde{\angle} px\xi(t) > \pi - \delta, \quad \text{for any } t \in (0, t_0). \tag{5.3}$$

By taking t_0 even smaller if necessary, we can assume that $\bar{\delta}_S(t;|px|) < \delta$ for any $t \in (0, t_0)$. This implies that p is a $(1,\delta)$-strainer at x with $\xi(t)$ as an opposite strainer for any $t \in (0, t_0)$. For the second statement, let η be an arbitrary unit-speed geodesic from x to q, and let $l := l(\xi) = |qx|$. Then it follows from the almost comparison inequality (4.2) that

$$\angle px\xi \geq \tilde{\angle} px\xi(l) - \delta_S(l;|px|) > \tilde{\angle} pxq - \bar{\delta}_S(|qx|;|px|) > \pi - 2\delta. \tag{5.4}$$

□

Next we show that the set of $(1, \delta)$-strained points is open in an S-concave space.

Lemma 5.4. *Let X be an S-concave space with $S \geq 1$. Let $x \in X$ be $(1, \delta)$-strained point at which p is a $(1, \delta)$-strainer with q as an opposite strainer. Then there exists an open neighborhood U of x at each point of which p is a $(1, \delta)$-strainer with q as an opposite strainer.*

Proof. The statement directly follows from the continuity of Euclidean comparison angles and the function $\bar{\delta}_S$. Indeed, since $\lim_{y\to x} \tilde{\angle} pyq = \tilde{\angle} pxq$, it follows that we can find a small open neighborhood U of x such that $\tilde{\angle} pzq > \pi - \delta$ for any $z \in U$. On the other hand, since the function $z \mapsto \bar{\delta}_S(|qz|; |pz|)$ is continuous if only $z \neq p$, we can shrink U to a smaller neighborhood of x if necessary so that $\bar{\delta}_S(|qz|; |pz|) < \delta$ for all $z \in U$. Thus, the open subset U is the desired open neighborhood. □

Next, using a similar inductive procedure as [18, Definition 5.2], we define (k, δ)-strainers on S-concave spaces that further satisfy local semi-convexity.

Definition 5.5 ((k, δ)-strainer)**.** Let X be an S-concave space that satisfies the (C, D)-local semi-convexity with $S \geq 1$, and let $0 < \delta < 1/2$. Given $k \geq 1$, we call a k-tuple $(p_1, \cdots, p_k)$ of points in X a (k, δ)-*strainer* at x if $|p_1 x| < D$ and the following inductive conditions hold:

(1) $(p_1, \cdots, p_{k-1})$ is a $(k-1, \delta)$-strainer at x.

(2) p_k is a $(1, \delta)$-strainer at x, with the distances satisfying $\bar{\delta}_{S,C}(|p_k x|; |p_i x|) < \delta$ for each $i = 1, \dots, k-1$, where $\bar{\delta}_{S,C}(r; t) := \arccos(1 - (S + C)r/(2t))$.

(3) There exists an opposite $(1, \delta)$-strainer q_k of p_k in the sense of Definition 5.1 such that

$$\left|\tilde{\angle} p_i x p_k - \pi/2\right| < \delta, \quad \left|\tilde{\angle} p_i x q_k - \pi/2\right| < \delta, \tag{5.5}$$

for any $i = 1, \cdots, k-1$.

In this case, we call the k-tuple $(q_1, \cdots, q_k)$ an *opposite* (k, δ)-*strainer* of $(p_1, \cdots, p_k)$ at x. A point x is said to be a (k, δ)-*strained point* if it admits a (k, δ)-strainer. If $(p_1, \cdots, p_k)$ is a (k, δ)-strainer at each point of subset U, we call the map

$$f(\cdot) := (d_{p_1}(\cdot), \cdots, d_{p_k}(\cdot)) : U \subset X \to \mathbb{R}^k \tag{5.6}$$

a (k, δ)-strainer map on U associated with $(p_1, \cdots, p_k)$, where $d_p(\cdot) := d(p, \cdot)$.

We emphasize that for the case $k = 1$, we always assume that the $(1, \delta)$-strainer p_1 at x on a (C, D)-locally semi-convex S-concave space satisfies $|p_1 x| < D$.

Remark 5.6. *Our definition of (k, δ)-strainers differs from the definition [18, Definition 5.2] for GNPC spaces, in which the almost orthogonality condition (5.5) is imposed for angles viewed from fixed points, rather than comparison angles. The advantage of our definition is that we do not need the local compactness of the underling space.*

Remark 5.7. *Our definition of (k, δ)-strainers differs from the definition in [13, Definition 5.2] for Alexandrov spaces with curvature bounded below. We intentionally adopt this modified definition based on an inductive procedure, incorporating additional control over distances from strainers, to address the obstacles posed by the absence of monotonicity of comparison angles. Additionally, we impose upper bound controls on comparison angles in the almost orthogonality condition (5.5) to address issues arising from the absence of the quadruple condition (see [12, Proposition 10.1.1]). This quadruple condition, which provides an equivalent characterization for Alexandrov spaces with curvature bounded below, automatically derives the upper bounds on comparison angles from the lower bounds.*

The next lemma demonstrates the 'almost orthogonality' property of (k, δ)-strainers in terms of angles viewed from the strainer points.

Lemma 5.8. *Let X be an S-concave space that satisfies the (C, D)-local semi-convexity with $S \geq 1$. Given $0 < \delta < 1/2$, let $(p_1, \cdots, p_k)$ be a (k, δ)-strainer at $x \in X$ in the sense of Definition 5.5 with the opposite strainer $(q_1, \cdots, q_k)$. Then for any $1 \leq i < j \leq k$ and any unit-speed geodesic ξ_j, η_j from x to p_j and q_j respectively, the following almost orthogonality holds:*

$$|\angle p_i x \xi_j - \pi/2| < 2\delta, \quad |\angle p_i x \eta_j - \pi/2| < 2\delta. \tag{5.7}$$

Proof. Note that $\sup_{i=1,\cdots,k} |p_i x| < D$. Indeed, the definition of (k, δ)-strainer implies that $|p_1 x| < D$. This fact, together with the inequality

$$\frac{|p_j x|}{2|p_1 x|} \leq \arccos\left(1 - (S + C)\frac{|p_j x|}{2|p_1 x|}\right) < \delta < \frac{1}{2}, \quad \text{for any } j = 2, \ldots, k, \tag{5.8}$$

implies that $|p_j x| < |p_1 x| < D$. Thus, the claim follows directly from the almost comparison inequality Lemma 4.3, Lemma 4.10 and the assumption that $\bar{\delta}_{S,C}(|p_j x|; |p_i x|) < \delta$ for all $1 \leq i < j \leq k$. □

Remark 5.9 (Asymmetry of strainers and strainer maps)**.** *We remark that (k, δ)-strainers exhibits asymmetry in two aspects. On the one hand, a k-tuple $(p_1, \ldots, p_j, \ldots, p_i, \ldots, p_k)$ reordered from a (k, δ)-strainer $(p_1, \ldots, p_k)$ is not a (k, δ)-strainer anymore due to the inductive nature of Definition 5.5. On the other hand, the almost orthogonality condition (5.5) in the definition provides only partial control over the orthogonality. More specifically, the associated strainer map $f = (d_{p_1}, \ldots, d_{p_k})$, which serves as an 'almost orthogonal coordinates chart' in a small neighborhood of x, only controls the orthogonality of each coordinate with respect to those of lower indices: while the i-th coordinate $|p_i x|$ remains nearly unchanged as x moves along the k-th coordinate toward either p_k or q_k (due to the almost orthogonality (5.7)), there is no corresponding control over the k-th coordinate $|p_k x|$ when x approaches the point p_i. This asymmetry fundamentally stems from the asymmetry nature of angels viewed from a fixed point. For a similar asymmetry phenomenon in GNPC spaces, we refer to [18, page 22].*

We conclude this subsection with a lemma concerning the openness of sets of (k, δ)-strained points. The proof, being analogous to Lemma 5.4, is omitted here.

Lemma 5.10. *Let X be an S-concave space that satisfies local semi-convexity with $S \geq 1$. Let k-tuple $(p_1, \ldots, p_k)$ be a (k, δ)-strainer at x with $(q_1, \ldots, q_k)$ as an opposite strainer. Then there exists an open neighborhood U of x at each point of which $(p_1, \ldots, p_k)$ is a (k, δ)-strainer with $(q_1, \ldots, q_k)$ as an opposite strainer.*

5.2. **Openness and bi-Lipschitz of strainer maps.** We begin by recalling the following definition of ε-open maps. For further discussion and alternative formulations of ε-openness and local ε-openness, see [16, 28, 29].

Definition 5.11 (ε-open maps)**.** A Lipschitz map $F : U \to Y$ from an open subset $U \subset X$ to a metric space Y is said to be *ε-open* if for any point $x \in U$, there exists $r > 0$ such that the closed ball $\bar{B}(x, \varepsilon^{-1} r) \subset U$ is complete, and for any $v \in B(F(x), r) \subset Y$, there exists a point $y \in U$ such that $F(y) = v$ and $\varepsilon |yx| \leq |F(x)v|$. In particular, for any $s \in (0, r]$, we have the inclusion that $B(F(x), s) \subset f(B(x, \varepsilon^{-1} s))$.

We need the following useful criterion for ε-open maps. This criterion is classical and is nearly identical to [18, Lemma 5.15], with the exception that here X is assumed to be locally complete rather than locally compact. For the sake of completeness, we provide the proof in Appendix A.

Lemma 5.12 (criterion for ε-open maps)**.** *Let $f : X \to Y$ be a locally Lipschitz map from a locally complete*[4] *metric space X to a geodesic space Y. Suppose there exists $\varepsilon > 0$ such that the following holds: for every $x \in X$ and every $v \in Y \setminus \{f(x)\}$ sufficiently close to $f(x)$, there exists $y \in X$ such that*

$$|f(y)v| - |f(x)v| \leq -\varepsilon|xy|. \tag{5.9}$$

Then f is an ε'-open map for any $0 < \varepsilon' < \varepsilon$.

Remark 5.13. *In [18, Lemma 5.15], the endpoint case $\varepsilon' = \varepsilon$ can be achieved. We cannot reach the endpoint case due to the lack of local compactness of the space.*

Next, we show that the (k, δ)-strainer map is ε-open when $\delta > 0$ is small enough. The main difficulty lies in the asymmetry of (k, δ)-strainer maps (see Remark 5.9). This drawback prevents us from applying the standard procedure to show the ε-openness of strainer map. To overcome the difficulty, we follow a similar idea as [18, Proposition 5.17] and introduce an anisotropic L^1-norm for the target space $\mathbb{R}^k$ so that the asymmetry concerning almost orthogonality can be made up for.

Proposition 5.14. *Let X be an S-concave space that satisfies (C, D)-local semi-convexity with $S \geq 1$. There exists a constant $\delta_k > 0$ depending only on $k \in \mathbb{N}$ such that the following holds: let $f : U \to \mathbb{R}^k$ be a (k, δ)-strainer map with $\delta < 1/2$ on an open subset $U \subset X$ associated with the (k, δ)-strainer $(p_1, \dots, p_k)$ with $(q_1, \dots, q_k)$ as an opposite strainer, where $\mathbb{R}^k$ is equipped with the L^1-norm. If $\delta < \delta_k$, then f is $\varepsilon_k(\delta)$-open. Here we can take $\varepsilon_k(\delta) := \frac{(1-2\delta)}{4^{k-1}}$ and $\delta_k = k^{-1}2^{-2k-1}$.*

Proof. Let $f_i := d_{p_i}$ and $f_{[i]} := (f_1, \dots, f_i)$ for $i = 1, \dots, k$. We show the statement by induction.

STEP 1: We first show that f_1 is $\varepsilon_1(\delta)$-open. Let $x \in U$ be an arbitrary point and $v \in \mathbb{R} \setminus f(x)$ be a point close to $f_1(x)$ such that $|v - f_1(x)| < r_x$, where $r_x \in (0, R_x)$ is a small constant to be determined later, and R_x is the supremum of all radius $r > 0$ such that $B(x, 2r) \subset U$. Let η, ξ be arbitrary unit-speed geodesics from x to p_1 and q_1, respectively. In the case that $v > f_1(x)$, since $\angle px\xi > \pi - 2\delta$ from Lemma 5.3, it follows that

$$\lim_{t \searrow 0} \frac{|p_1\xi(t)| - |p_1x|}{t} = -\cos\angle px\xi > \cos 2\delta > 1 - 2\delta. \tag{5.10}$$

It follows that we can find a sufficiently small $t_0 > 0$ such that $f_1(\xi(t)) = |p_1\xi(t)| \geq f_1(x) + (1 - 2\delta)t$ for all $t \in [0, t_0]$. Therefore, if $f_1(x) < v < f_1(x) + (1 - 2\delta)t_0$, from the continuity of f_1 we can find $t \in (0, t_0)$ such that $v = f_1(\xi(t))$. Taking $y = \xi(t)$, we obtain that $v = f_1(y)$ and

$$(1 - 2\delta)|xy| \leq |p_1y| - |px| = |f_1(y) - f_1(x)|\,. \tag{5.11}$$

In the case that $v < f_1(x)$ with $0 < f_1(x) - v < |p_1x|/2$, we can take $y = \eta(t)$ for some $t \in (0, |p_1x|/2)$ such that $v = f_1(y)$ and that $|yx| = f_1(x) - f_1(y)$. In both cases, we can find the point y in either ξ or η such that $v = f_1(y)$ and

$$(1 - 2\delta)|xy| \leq |f_1(x) - v|, \tag{5.12}$$

if $|v - f_1(x)| < r_x := \min\{t_0, |px|/2, R_x\}$. This shows that f_1 is a $(1 - 2\delta)$-open map from U to $\mathbb{R}$. We can take $\varepsilon_1(\delta) := 1 - 2\delta$ for $\delta < \delta_1 := 1/8$.

STEP 2: Suppose that the $(k - 1, \delta)$-strainer map $f_{[k-1]}$ is an $\varepsilon_{k-1}(\delta)$-open map for $\delta < \delta_{k-1}$ from U to $\mathbb{R}^{k-1}$ equipped with the L^1-norm, where $\varepsilon_{k-1}(\delta) := (1 - 2\delta)/4^{k-2}$. For simplicity, we denote $\varepsilon_{k-1}(\delta)$ by ε_{k-1}. Note that from Step 1, it holds that f_k is a $(1 - 2\delta)$-open map.

[4] A metric space (X, d) is said to be locally complete if every point $x \in X$ admits a complete neighborhood, see [28].

We aim to apply Lemma 5.12. To do that, we first introduce the following anisotropic norm $\|\cdot\|$ on $\mathbb{R}^k$:

$$\|v\| := \left\|v_{[k-1]}\right\|_1 + \frac{\varepsilon_{k-1}}{2}|v_k|, \quad v = (v_1 \dots, v_k) \in \mathbb{R}^k, \tag{5.13}$$

where $v_{[k-1]} := (v_1, \dots, v_{k-1}) \in \mathbb{R}^k$ is the first $(k-1)$-coordinates of v, and $\|\cdot\|_1$ denotes the L^1-norm on the Euclidean space. Let $x \in U$ be fixed. It suffices to show that we can find $y \in U$ satisfying the inequality (5.9) whenever $v \in \mathbb{R}^k \setminus \{f(x)\}$ is sufficiently close to $f(x)$. To find such a point, we consider the following two different cases:

Case 1: $f_{[k-1]}(x) \neq v_{[k-1]}$. In this case, by the assumption that $f_{[k-1]}$ is an ε_{k-1}-open map, it follows that we can find $y \in U$ such that

$$f_{[k-1]}(y) = v_{[k-1]}, \quad \varepsilon_{k-1}|xy| \leq \|f_{[k-1]}(x) - v_{[k-1]}\|_1. \tag{5.14}$$

This implies that

$$\begin{aligned}\|f(y) - v\| - \|f(x) - v\| &= \left\|f_{[k-1]}(y) - v_{[k-1]}\right\|_1 + \frac{\varepsilon_{k-1}}{2}|f_k(y) - v_k| \\ &\quad - \left(\left\|f_{[k-1]}(x) - v_{[k-1]}\right\|_1 + \frac{\varepsilon_{k-1}}{2}|f_k(x) - v_k|\right) \\ &\leq \frac{\varepsilon_{k-1}}{2}|f_k(x) - f_k(y)| - \varepsilon_{k-1}|xy| \leq -\frac{\varepsilon_{k-1}}{2}|xy|, \end{aligned} \tag{5.15}$$

where we use the triangle inequality and the fact $f_{[k-1]}(y) = v_{[k-1]}$ in the first inequality, and the 1-Lipschitz of distance functions in the last inequality.

Case 2: $f_k(x) \neq v_k$. In this case, note that from Step 1, we can find $y \in U$ lying in either a geodesic ξ_k from x to p_k or a geodesic η_k from x to q_k such that

$$f_k(y) = v_k, \quad (1 - 2\delta)|xy| \leq |f_k(x) - v_k|. \tag{5.16}$$

This implies that

$$\begin{aligned}\|f(y) - v\| - \|f(x) - v\| &= \left\|f_{[k-1]}(y) - v_{[k-1]}\right\|_1 + \frac{\varepsilon_{k-1}}{2}|f_k(y) - v_k| \\ &\quad - \left(\left\|f_{[k-1]}(x) - v_{[k-1]}\right\|_1 + \frac{\varepsilon_{k-1}}{2}|f_k(x) - v_k|\right) \\ &\leq \left\|f_{[k-1]}(y) - f_{[k-1]}(x)\right\|_1 - \frac{\varepsilon_{k-1}}{2}(1 - 2\delta)|xy|, \end{aligned} \tag{5.17}$$

where we use the triangle inequality and the fact $f_k(y) = v_k$ in the first inequality. To estimate the first term on the right-hand side of the inequality (5.17), we note from Lemma 5.8 that

$$|\cos\angle p_i x \xi_k| < 2\delta, \quad |\cos\angle p_i x \eta_k| < 2\delta, \quad \text{for } i = 1, \dots, k-1. \tag{5.18}$$

This implies that $|f_i(y) - f_i(x)| \leq 2\delta|xy|$ for $i = 1, \dots, k-1$ if v_k is sufficient close to $f_k(x)$. Summing over indices i from 1 to $k-1$, it follows that

$$\left\|f_{[k-1]}(y) - f_{[k-1]}(x)\right\|_1 = \sum_{i=1}^{k-1} |f_i(x) - f_i(y)| \leq 2\delta(k-1)|xy|. \tag{5.19}$$

Plugging it into the inequality (5.17), we obtain that

$$\|f(y) - v\| - \|f(x) - v\| \leq -\left(\frac{\varepsilon_{k-1}}{2}(1 - 2\delta) - 2\delta(k-1)\right)|xy|. \tag{5.20}$$

Let $\delta_k := k^{-1}2^{-2k-1}$. One can check that for any $0 < \delta < \delta_k$, it holds that

$$\|f(y) - v\| - \|f(x) - v\| \leq -\frac{5}{16}\varepsilon_{k-1}|xy| < -\frac{1}{4}\varepsilon_{k-1}|xy|. \tag{5.21}$$

Combining two cases above, we find a $y \in U$ such that the inequality (5.9) holds with $\varepsilon = 5/16\varepsilon_{k-1}$. By Lemma 5.12, it follows that f is an $\varepsilon_{k-1}/4$-open map from U to $\mathbb{R}^k$ equipped with the anisotropic norm $\|\cdot\|$. One can readily check that f is an $\varepsilon_{k-1}/4$-open map as well from U to $\mathbb{R}^k$ equipped with the L^1-norm since L^1-norm dominates the anisotropic norm $\|\cdot\|$. By induction, it follows that f is an $\varepsilon_k(\delta)$-open map for $\delta < \delta_k$, where $\varepsilon_k(\delta) := \frac{(1-2\delta)}{4^{k-1}}$ and $\delta_k = k^{-1}2^{-2k-1}$.

□

The following result is a direct implication from Lemma 5.10 and Proposition 5.14.

Corollary 5.15. *Let X be an S-concave space that satisfies local semi-convexity with $S \geq 1$. Let $(p_1, \cdots, p_k)$ be a (k, δ)-strainer at x for some $\delta < \delta_k$. Then there exists an open neighborhood U of x such that the associated (k, δ)-strainer map $f : U \to \mathbb{R}^k$ is $\sqrt{k}$-Lipschitz and $\bar{\varepsilon}(\delta)$-open. In particular, the Hausdorff dimension of U_x is at least k. Here $\bar{\varepsilon}_k(\delta) := \varepsilon_k(\delta)/\sqrt{k}$, and ε_k, δ_k are the constants from Proposition 5.14.*

We now present the second important property of strainer maps: a (k, δ)-strainer map defined on some neighborhood of x is bi-Lipschitz if no point near x admits a $(k + 1, \delta)$-strainer.

Proposition 5.16. *Let X be an S-concave concave space that satisfies (C, D)-local semi-convexity for $S \geq 1$. Let $(p_1, \cdots, p_k)$ be a (k, δ)-strainer at x with $\delta < \delta_k$. If there is no point near x admitting a $(k + 1, 2\delta)$-strainer, then the (k, δ)-strainer map $f := (d_{p_1}, \ldots, d_{p_k})$ is a bi-Lipschitz homeomorphism from some neighborhood of x to a domain of $\mathbb{R}^k$.*

Proof. We note that it suffices to show the injectivity of the (k, δ)-strainer map f on some neighborhood of x. The proof that a Lipschitz map being both injective and ε-open is locally bi-Lipschitz is standard, and we refer to, for example, [18, Theorem 5.30].

We now show that f is injective on some neighborhood of x. Let $r_0 > 0$ be a sufficient small constant such that f is a (k, δ)-strainer map on $B(x, r)$ and $r_0 + |p_1 x| < D$ and $\bar{\delta}_{S,C}(2r_0; |p_k x|) < 0.01\delta$ and no point in $U := B(x, r_0)$ admits $(k + 1, 2\delta)$-strainers.

We show by contraction that f is injective on U. Suppose that the claim does not hold. Then we can find $y \in U$ such that $f(x) = f(y)$, which implies that $|p_i x| = |p_i y|$ for $i = 1, \ldots, k$. Let $\xi : [0, l] \to X$ be a unit-speed geodesic from x to y, where $l = l(\xi)$.

In the following, we show that there is a point $z = \xi(t)$ sufficiently close to x such that $(p_1, \ldots, p_k, y)$ is a $(k + 1, 2\delta)$-strainer at z. But this contradicts our assumption on U.

STEP 1: We first check that $(p_1, \ldots, p_k, y)$ satisfies the first two conditions in Definition 5.5. Firstly, it follows from our choice of r_0 that the distance from p_1 of any point in U is less than D and that the k-tuple $(p_1, \cdots, p_k)$ is a (k, δ)-strainer at each point of U with an opposite strainer $(q_1, \cdots, q_k)$. Secondly, we choose $t > 0$ sufficiently small such that $\bar{\delta}_{S,C}(t; l - t) < \delta/8$. Let $z := \xi(t)$. It is easy to check that the Euclidean comparison triangle $\tilde{\Delta}xzy$ is degenerated so that $\tilde{\angle}yzx = \pi > \pi - \delta$, and $\bar{\delta}_S(|zx|; |zy|) \leq \bar{\delta}_{S,C}(|zx|; |zy|) < \delta/8$. This shows that y is a $(1, \delta)$-strainer at z with x as an opposite strainer. Moreover, from the $\bar{\delta}_{S,C}(2r_0; |p_k x|) < 0.01\delta$, it follows that the point y satisfies that $\bar{\delta}_{S,C}(|yz|; |p_i z|) < 0.01\delta$ for all $i = 1, ., k$. Therefore, the conditions (1)-(2) in Definition 5.5 are satisfied for the $(k + 1)$-tuple $(p_1, \ldots, p_k, y)$ at z.

STEP 2: We are left to show that $(p_1, \cdots, p_k, y)$ satisfies the almost orthogonality condition (3) in Definition 5.5.

We first show that $|\tilde{\angle}p_i zy - \pi/2| < \delta$. Indeed, the conditions $\bar{\delta}_{S,C}(|p_j z|; |p_i z|) < \delta$ for $1 \leq i < j \leq k$ in the definition of (k, δ)-strainer maps implies that $|p_j z| \leq |p_i z|$ for $1 \leq i < j \leq k$. Combing with the assumption that $\bar{\delta}_{S,C}(2r_0; |p_k x|) < 0.01\delta$ and the inequality $r \leq \arccos(1 - r)$, we obtain that $r_0 \leq 0.01\delta \min_{i=1\ldots,k} |p_i x|$. Applying the

Euclidean law of cosine to the comparison triangle $\tilde{\Delta} p_i z y$, it follows that

$$\left|\cos \tilde{\angle} p_i z y\right| = \left|\frac{|p_i z|^2 + |zy|^2 - |p_i y|^2}{2|p_i z||zy|}\right| \le \frac{|zy|}{2|p_i z|} + \frac{(|p_i z| + |p_i x|)\,||p_i z| - |p_i x||}{2|p_i z||zy|}$$
$$\le \frac{r_0}{2(1-0.01\delta)|p_i x|} + \frac{(2+0.01\delta)|p_i x||xz|}{2(1-0.01\delta)|p_i x||zy|} \le 0.1\delta + 1.1\,\frac{t}{l-t}, \tag{5.22}$$

where we use the assumption that $|p_i x| = |p_i y|$ in the first inequality. From our choice of t that $\bar{\delta}_{S,C}(t; l-t) < \delta/8$, we can derive that $|\cos \tilde{\angle} p_i z y| < \delta/2$. This implies that $|\tilde{\angle} p_i z y - \pi/2| < \delta$.

Finally, we show that $|\tilde{\angle} p_i z x - \pi/2| < 2\delta$. Let γ, η be the reparametrization by arc-length of geodesic segments of the geodesic ξ from z to y, x respectively. Combing the inequality $|\tilde{\angle} p_i z y - \pi/2| < \delta$ with the almost comparison inequalities (Lemma 4.3 and Lemma 4.10) and the fact $\bar{\delta}_{S,C}(|yz|; |p_i z|) < 0.01\delta$, it follows that $|\angle p_i x \gamma - \pi/2| < 1.5\delta$. Note that Lemma 4.8 and Lemma 4.12 implies that $\angle p_i z \gamma + \angle p_i z \eta = \pi$. This together with the fact $|\angle p_i x \gamma - \pi/2| < 1.5\delta$ implies that $|\angle p_i x \eta - \pi/2| < 1.5\delta$. By again applying the almost comparison inequality (4.2) and (4.19) and noting that $\bar{\delta}_{S,C}(|xz|; |p_i z|) \le \bar{\delta}_{S,C}(|yz|; |p_i z|) \le \bar{\delta}_{S,C}(2r_0; |p_i x|) < 0.01\delta$, we obtain that $|\tilde{\angle} p_i z x - \pi/2| < 2\delta$.

We have verified that the conditions (1) – (3) in Definition 5.5 are all satisfied for the $(k+1)$-tuple $(p_1, \cdots, p_k, y)$ at the point $z = \xi(t)$. Therefore, it is a $(k+1, 2\delta)$-strainer at $z \in U$. But this contradicts our assumption that no point in U admits $(k+1, 2\delta)$-strainer. Hence, f is injective on U, which ends the proof. □

5.3. **Self-improvement of strainers.** In this subsection, we establish a self-improvement property for (k,δ)-strainers on S-concave spaces that satisfy local semi-convexity: any (k,δ)-strainer can be improved to a (k,δ')-strainer for any $0 < \delta' < \delta$. This property is analogous to the one [13, Lemma 5.9] for Alexandrov spaces (see also [12, Lemma 10.8.17]), whose proof is based on a straightening strategy. However, we cannot apply this strategy directly in our setting due to the asymmetry of strainer maps: the new k-tuple $(p_1, \ldots, p'_i, \ldots, p_k)$ obtained by straightening the strainer pair (p_i, q_i) from the original (k,δ)-strainer $(p_1, \ldots, p_k)$ at x is not a (k,δ)-strainer anymore.

To overcome this difficulty, we adopt a new 'dequeuing and enqueuing' strategy: we first straighten the first strainer pair (p_1, q_1) to obtain a new strainer pair (p'_1, q'_1). We then remove the old strainer point p_1 and add the new strainer point p'_1 to the end of the strainer list, forming a new k-tuple $(p_2, \ldots, p_k, p'_1)$. This new k-tuple turns out to be a 'mixed' strainer in the following sense: while $(p_2, \ldots, p_k, p'_1)$ is still a (k,δ)-strainer, the point p'_1 is a $(1,\delta')$-strainer with a better parameter $\delta' < \delta$.

We continue this process of dequeuing and enqueuing in the same manner. At the end of each step, we obtain a new (k,δ)-strainer $(p_j, \ldots, p_k, p'_1, \ldots, p'_{j-1})$ at some point z near x, whose last $j-1$ elements form a $(j-1,\delta')$-strainer at z. By repeating this procedure for k steps, we obtain the desired (k,δ')-strainer.

Lemma 5.17 (Self-improvement of a strainer)**.** *Let X be an S-concave space that satisfies (C,D)-local semi-convexity for $S \ge 1$. Let x be a (k,δ)-strained point for $\delta < \delta_k$. Then for any $\delta' < \delta$, any open neighborhood of x contains a (k,δ')-strained point.*

Proof. Let $(p_1, \cdots, p_k)$ be a (k,δ)-strainer at x with $(q_1, \ldots, q_k)$ as the opposite strainer. Let $f := (d_{p_1}, \cdots, d_{p_k})$ be the associated strainer map, and $0 < \delta' < \delta$ be arbitrary. The proof for the case $k = 1$ is trivial, since we can pick a point y in some geodesic joining x and p_1 sufficiently close to x such that y is a $(1,\delta')$-strained point with (p_1, x) as its $(1,\delta')$-strainer pair. Therefore, we can assume $k \ge 2$. We will dequeue and enqueue the strainer points p_i from $i = 1$ to k inductively to construct the desired (k,δ')-strainer and strained point.

STEP 1: We first dequeue and enqueue the point p_1. Let $r_1 > 0$ be an arbitrary small constant such that the strainer map f is ε_k-open on $B(x, r_1)$ and $\bar{\delta}_{S,C}(2r_1; |p_i x|) < 0.01\delta'/3$ for $i = 2, \ldots, k$. Since f is open on $U_1 := B(x, r_1)$, it follows that we can find a point $v \in f(U_1)$ such that $v_i = |p_i x|$ for $i = 2, \cdots, k$. Let $y \in U_1$ such that $f(y) = v$, and ξ be an arbitrary geodesic from x to y. We take $z := \xi(t)$ where $t \in (0, l(\xi))$ is to be determined later. By applying the Euclidean law of cosine to $\tilde{\Delta} p_i z y$ and following the same argument as the inequality (5.22), it follows that

$$|\cos \tilde{\angle} p_i z y| \leq 0.1\frac{\delta'}{3} + 1.1\frac{t}{l-t}, \tag{5.23}$$

for $i = 2, \cdots, k$. By taking $t > 0$ sufficient small, it follows from the same argument as the one in Step 2 of Proposition 5.16 that $\tilde{\angle} yzx = \pi > \pi - \delta'$ and

$$\left|\tilde{\angle} p_i z y - \pi/2\right| < \frac{2}{3}\delta', \quad \left|\tilde{\angle} p_i z x - \pi/2\right| < \delta', \quad i = 2, \ldots, k. \tag{5.24}$$

We claim that the new k-tuple $(p_2, \ldots, p_k, y)$ is a (k, δ)-strainer at z with the opposite strainer $(q_2, \ldots, q_k, x)$. Moreover, y is a $(1, \delta')$-strainer at z such that $\bar{\delta}_{S,C}(|zy|; |p_k z|) < \delta'$ and

$$|\tilde{\angle} p_i z x - \pi/2| < \delta', \quad |\tilde{\angle} p_i z y - \pi/2| < \delta'. \tag{5.25}$$

Indeed, the almost orthogonality condition (5.25) follows automatically from the inequality (5.24). Moreover, since $(p_1, \ldots, p_k)$ is (k, δ)-strainer on U_1, it is easy to see from definition that $(p_2, \ldots, p_k)$ is a $(k-1, \delta)$-strainer on U_1, since removing the top strainer point p_1 does not change the rest angles for the strainer $(p_2, \ldots, p_k)$ in Definition 5.5. Finally, it follows from $\bar{\delta}_{S,C}(2r_1; |p_i x|) < 0.01\delta'/3$ that

$$\bar{\delta}_{S,C}(|zy|; |p_k z|) \leq \bar{\delta}_{S,C}(r_1; |p_k x| - r_1) \leq \bar{\delta}_{S,C}(2r_1; |p_k x|) < 0.01\delta'/3 < \delta'. \tag{5.26}$$

Together with what we have shown above, our claim follows. We take $p_1' := y$ and $q_1' := x$ and obtain a new (k, δ)-strainer $(p_2, \ldots, p_k, p_1')$ at z. Note that $|zx| < r_1$.

STEP 2: Suppose that for $2 \leq j < k$, we have obtained a (k, δ)-strainer $(p_j, \cdots, p_k, p_1', \cdots, p_{j-1}')$ at x' such that $(p_1', \cdots, p_{j-1}')$ is a $(j-1, \delta')$-strainer at x' with that $|xx'| < \sum_{i=1}^{j-1} r_1/2^{i-1}$. Let f be the associated (k, δ)-strainer map. Let $r_j > 0$ be a sufficient small constant such that f is an ε_k-open (k, δ)-strainer map on $U_j := B(x', r_j)$ and

$$\bar{\delta}_{S,C}(2r_j; |p_{j-1}' x'|) < \frac{0.01}{3}\delta'. \tag{5.27}$$

We further take r_j even smaller such that $r_j \leq r_1/2^{j-1}$ if necessary. By the openness of f, we can find $y' \in U_j$ such that $|p_i x'| = |p_i y'|$ and $|p_l' x'| = |p_l' y'|$ for all $i = j + 1, \cdots, k$ and $l = 1, \cdots, j-1$. Let ξ be an arbitrary geodesic from x' to y', and let $z' := \xi(t)$ for $t \in (0, l(\xi))$ to be determined in the following. Now by taking $t > 0$ sufficient small and following the same argument as Step 1, one can readily check that the k-tuple $(p_{j+1}, \cdots, p_k, p_1', \cdots, p_{j-1}', y')$ is a (k, δ)-strainer at z' such that $(p_1', \cdots, p_{j-1}', y')$ is a (j, δ')-strainer at z'. Moreover, by our choice of r_j, it follows that $|x'z'| < r_j \leq r_1/2^{j-1}$ and therefore $|xz'| < \sum_{i=1}^{j} r_1/2^{i-1}$. We take $p_j' := y'$ and $q_j' := z'$ and obtain a new (k, δ)-strainer $(p_{j+1}, \cdots, p_k, p_1', \cdots, p_j')$ such that $(p_1', \cdots, p_j')$ is a (j, δ')-strainer at z'.

STEP 3: Repeating the steps for k times, we obtain a new k-tuple $(p_1', \cdots, p_k')$. By induction, it is a (k, δ')-strainer at some point z such that $|zx| < 2r_1$. Since $r_1 > 0$ can be arbitrary small, the statement follows.

□

6. DIMENSION

In this section, we investigate the dimension theory of an S-concave Busemann concave space (X, d) that satisfies local semi-convexity. We prove that the Hausdorff dimension of the space coincides with a constant, known as the *strainer number*. Furthermore, if X has either finite Hausdorff dimension or finite strainer number, then X is proper, its topological dimension agrees with its Hausdorff dimension, and X carries a non-trivial n-dimensional Hausdorff measure for some $n \in \mathbb{N}$. In particular, X has integer Hausdorff dimension, and the metric measure space $(X, d, \mathcal{H}^n)$ satisfies a synthetic curvature-dimension condition, called the measure contraction property $\mathrm{MCP}(0, n)$.

We first recall the following definition of strainer number from [13, Section 6], see also [12, Definition 10.8.11].

Definition 6.1 (Strainer number)**.** Let (X, d) be an S-concave Busemann concave space that satisfies local semi-convexity for some $S \geq 1$. A natural number $m \in \mathbb{N}$ is called the *strainer number* at a point $x \in X$ if, for every sufficiently small $\delta < \delta_m$ and every sufficiently small neighborhood U_x of x, there exists a (m, δ)-strained point $y \in U_x$, but the analogous property fails for $m + 1$. If no such number exists, the strainer number at x is defined to be ∞. The strainer number of X is the supremum of all m such that there exists a point $x \in X$ with strainer number m.

The following lemma shows that the notion of strainer number at a point is well-defined.

Lemma 6.2. *The strainer number at a point of an S-concave Busemann concave space (X, d) that satisfies local semi-convexity is well-defined.*

Proof. Let $x \in X$ be an arbitrary point. We claim that the strainer number at x is either a unique natural number or ∞, and cannot take two distinct values. Indeed, suppose for contradiction that $m < m'$ are both strainer numbers at x. By the definition of strainer number, there exists a neighborhood U of x and some $\delta < \delta_{m+1}$ such that U contains no $(m + 1, \delta)$-strained point. However, since m' is also a strainer number at x possibly taking value ∞, it follows from the definition of strainer number that we can find a natural number $n \in \mathbb{N}$ with $m < n \leq m'$ such that for any sufficiently small $\delta' < \min\{\delta_n, \delta\}$, every neighborhood of x must contain a (n, δ')-strained point, which in particular yields a $(m + 1, \delta)$-strained point. This contradiction shows that the strainer number at x can only take one values, which implies that the strainer number at x is well-defined. □

Remark 6.3. *We note that if a point x has a neighborhood U containing no $(m + 1, \delta)$-strained point for some $\delta < \delta_{m+1}/2$, then U does not contain any $(m + 1, 2\delta)$-strained point either, since otherwise, we can construct a sequence of $(m + 1, \delta)$-strained points converging to some $(m + 1, 2\delta)$-strained point in U by the self-improvement property of strainers (Lemma 5.17), which is a contradiction.*

The following lemma shows that the strainer number at any point of an S-concave Busemann concave space that satisfies local semi-convexity is at least 1.

Lemma 6.4. *Let X be an S-concave Busemann concave space that satisfies (C, D)-local semi-convexity for some $S \geq 1$. Then any neighborhood of any point admits a $(1, \delta)$-strained point. In particular, the strainer number at any point is at least 1.*

Proof. Let $\delta > 0$ be any small number and let $x, p \in X$ be an arbitrary point such that $|px| < D$. Let ξ be a unit-speed geodesic from x to p. We take $t > 0$ arbitrary small such that $\bar{\delta}_S(t; |px| - t) < \delta$. Let $z := \xi(t)$. Then it is easy to check that $\tilde{\angle} pzx = \pi > \pi - \delta$ and that $\bar{\delta}_S(|zx|; |pz|) < \delta$. This implies that p is a $(1, \delta)$-strainer at $z = \xi(t)$ with x as an opposite strainer, and therefore $z = \xi(t)$ is a $(1, \delta)$-strained point. Since t can be taken arbitrarily small, the assertion follows. □

We now show that the strainer number of an S-concave Busemann concave space X satisfying local semi-convexity is precisely equal to the Hausdorff dimension of the space. In particular, this implies that the Hausdorff dimension of X is necessarily either an integer or infinite. Our proof is based on the constancy of the Hausdorff dimensions of Busemann concave spaces established by Kell [24, Lemma 2.22], which shows that all open subsets of a Busemann concave space have the same Hausdorff dimension.

Lemma 6.5. *Let X be an S-concave Busemann concave space that satisfies local semi-convexity for some $S \geq 1$. Then the strainer number at any point of X is equal to the Hausdorff dimension of any open neighborhood of that point. In particular, the strainer numbers at all points are equal and coincides with the Hausdorff dimension of space. If the strainer numbers are finite, then they coincide with the topological dimension of space.*

Proof. Let U be an arbitrary open neighborhood of an arbitrary point $x \in X$. We show that the Hausdorff dimension of U is equal to the strainer number m at x. We first show the case that m equals infinity. By the definition of strainer number, for any integer $k \in \mathbb{N}$, any $\delta < \delta_k$, and any sufficiently small neighborhood U of x, there exists a (k, δ)-strained point in U. Let $y \in U$ be a (k, δ)-strained point. By Corollary 5.15, it follows that we can find a small neighborhood $V \subset U$ of y such that the (k, δ)-strainer map $f : V \to \mathbb{R}^k$, which is associated with the (k, δ)-strainer at y, is $\sqrt{k}$-Lipschitz and $\bar{\varepsilon}_k(\delta)$-open. Since f is Lipschitz and open, it follows that

$$k = \dim_H(f(V)) \leq \dim_H(V) \leq \dim_H(U). \tag{6.1}$$

Since $k \in \mathbb{N}$ is arbitrary, it follows that $\dim_H(U) = \infty$.

We now show the case that the strainer number m at x is finite. By the definition of strainer number, we can find a small neighborhood $U_x \subset U$ of x which does not contain any $(m+1, \delta')$-strained points for some $\delta' < \delta_{m+1}$. Let $\delta > 0$ be a constant such that $\delta < \delta_{m+1}/2$. Since U_x admits no $(m+1, \delta')$-strained point, by the self-improvement of strainers (Lemma 5.17), it follows that U_x does not contain any $(m+1, 2\delta)$-strained point either (see Remark 6.3). Now let $y \in U_x$ be a (m, δ)-strained point and f be the strainer map associated with a (m, δ)-strainer at y. Since there is no $(m+1, 2\delta)$-strained point near y, it follows by Proposition 5.16 that we can find an open neighborhood $V \subset U_x$ of y such that $f : V \to f(V) \subset \mathbb{R}^m$ is a bi-Lipschitz homeomorphism. Since all open subsets have the same Hausdorff dimension on Busemann concave spaces (see [24, Lemma 2.22]), it follows that

$$m = \dim_H(f(V)) = \dim_H(V) = \dim_H(U_x) = \dim_H(U). \tag{6.2}$$

This shows that the strainer number m at x is equal to the Hausdorff dimension of U. Since U and x are arbitrary, our first assertion follows. The second assertion just follows from the first assertion that the strainer number at any point is equal to the Hausdorff dimension of the whole space X.

For the last assertion, let the strainer number m at x be finite. On the one hand, the topological dimension of U is not greater than the Hausdorff dimension of U. On the other hand, from the proof of the first assertion, we can find a neighborhood $V \subset U$ which is bi-Lipschitz homeomorphic to some open subset of $\mathbb{R}^m$. These imply that

$$m = \dim_T(V) \leq \dim_T(U) \leq \dim_H(U) = m. \tag{6.3}$$

This shows that the topological dimension of U is equal to m. Together with the constancy of the Hausdorff dimensions of Busemann concave spaces, our last assertion follows. □

We now present our main result of this section. The following proposition establishes that for an S-concave Busemann concave space satisfying local semi-convexity, if it has either finite Hausdorff dimension or finite strainer number, then it is proper and satisfies the measure contraction property $\mathrm{MCP}(0, n)$ when equipped with the Hausdorff measure.

Proposition 6.6 (Measure contraction property)**.** *Let X be an S-concave Busemann concave space that satisfies local semi-convexity for some $S \geq 1$. Then X has finite Hausdorff dimension if and only if it has finite strainer number. In either case, both values coincide* *with the topological dimension of X, and X admits a non-trivial n-Hausdorff measure. Moreover, the metric measure space $(X, d, \mathcal{H}^n)$ satisfies the measure contraction property* $\mathrm{MCP}(0, n)$ *and the Bishop–Gromov volume comparison inequality* $\mathrm{BG}(0, n)$. *In particular, (X, d) is doubling and proper*[5].

Proof. For the first assertion, Lemma 6.5 implies that the strainer numbers at all points must coincide with the Hausdorff dimension of X. This shows that the Hausdorff dimension n of X has to be an integer, which by Lemma 6.5 again implies that the topological dimension of X is equal to n. To show the existence of the non-trivial n-Hausdorff measure, by following the same argument as in the proof of Lemma 6.5, we can find an open subset $U \subset X$ that is bi-Lipschitz homeomorphic to an open subset of $\mathbb{R}^n$. This implies that $\mathcal{H}^n(U) > 0$, and therefore X admits a non-trivial n-dimensional Hausdorff measure $\mathcal{H}^n$.

For the second assertion, both the measure contraction property and the Bishop–Gromov volume comparison inequality follow directly from the first assertion and [24, Proposition 2.23]. Finally, the Bishop–Gromov volume comparison inequality $\mathrm{BG}(0, n)$ implies that $\mathcal{H}^n$ is a doubling measure on X, which in turn implies that (X, d) is proper (see [10, Proposition 3.1]) and doubling (see for example [21, page 102]). □

In light of the equality between the strainer number and the Hausdorff dimension, we say that an S-concave Busemann concave space that satisfies local semi-convexity is *finite dimensional* if it has either finite Hausdorff dimension or finite strainer number.

We conclude this section by showing that an S-concave Busemann concave space that satisfies local semi-convexity contains an open dense topological manifold part.

Corollary 6.7 (Topological manifold part)**.** *Let X be an n-dimensional S-concave Busemann concave space that satisfies local semi-convexity with $S \geq 1$. Then for any $\delta > 0$, the set $\mathcal{A}(n, \delta)$ of (n, δ)-strained points in X is open and dense in X. Moreover, $\mathcal{A}(n, \delta)$ is a topological n-manifold if* $0 < \delta < \delta_{n+1}/2$.

Proof. The openness of $\mathcal{A}(n, \delta)$ follows directly from the openness of (n, δ)-strainer maps (see Corollary 5.15). For the denseness of $\mathcal{A}(n, \delta)$, the fact that the strainer number of space equals n, together with Lemma 6.5, implies that any neighborhood of any point contains a point in $\mathcal{A}(n, \delta)$.

For the second assertion, if $0 < \delta < \delta_{n+1}/2$, it follows from the same argument as Lemma 6.5 that for any point x in $\mathcal{A}(n, \delta)$, we can find an open neighborhood U of x which does not contain any $(n+1, 2\delta)$-strained point. By applying Proposition 5.16, we can find a possibly smaller neighborhood $V \subset \mathcal{A}(n, \delta)$ of x which is bi-Lipschitz homeomorphic to some open subset of $\mathbb{R}^n$. This shows that $\mathcal{A}(n, \delta)$ is a topological n-manifold. □

Remark 6.8. *An n-dimensional S-concave Busemann concave space X that satisfies local semi-convexity admits a natural stratification $\{X_k\}_{k=0}^n$, where $X_n := \mathcal{A}(n, \delta)$ and $X_k := \mathcal{A}(k, \delta) \setminus \mathcal{A}(k+1, \delta)$ for $k = 0, \ldots, n-1$, and $\delta < \delta_n$. In the next section, we will study the Hausdorff measure and Hausdorff dimension of singular stratum X_k. For more detail of the stratification of Alexandrov spaces, see [12, Section 10.10] and [44, Section 2].*

7. HAUSDORFF MEASURE AND HAUSDORFF DIMENSION OF SINGULAR SETS

In this section, we investigate the structure of n-dimensional S-concave Busemann concave spaces that satisfy local semi-convexity.

Section 7.1 is devoted to the study of the n-Hausdorff measure of the singular strata induced by the sets of (k, δ)-strained points. In particular, we prove that the set of (n, δ)-strained points has full n-Hausdorff measure.

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[5]A metric space is called *proper* if all closed and bounded subsets are compact.

Building on these results, Section 7.2 establishes that every n-dimensional S-concave Busemann concave space satisfying local semi-convexity is n-rectifiable, and that $\mathcal{H}^n$-almost every point admits a unique tangent cone, which is isometric to a finite-dimensional Banach space. Under the additional assumption of local p-uniform convexity of distance functions, we further characterize the Banach tangent cones.

Finally, in Section 7.3, we estimate the Hausdorff dimension of the singular strata induced by the sets of (k, δ)-strained points, under suitable restriction on the uniform smoothness constant S of the space.

7.1. **Hausdorff measure of singular sets.** We recall the following notion of cut points in geodesic spaces. See [46] for more discussion on cut points and conjugate points in length spaces.

Definition 7.1. Let X be a geodesic space, and $x \in X$ be a point. Let I_x be the set of all points y that are connected to x by at least one extendable geodesic, i.e.,

$$I_x := \{\gamma(t) : t \in [0, 1), \quad \gamma \text{ any constant-speed geodesic starting at } x\}. \tag{7.1}$$

Let $T_x := X \setminus I_x$ denote the set of cut points of x, i.e., those points that do not lie in the interior of any geodesic starting at x.

The next lemma shows that the cut locus of any point in a finite-dimensional S-concave Busemann concave space that satisfies local semi-convexity has null top-dimensional Hausdorff measure.

Lemma 7.2. *Let X be an n-dimensional S-concave Busemann concave space that satisfies (C, D)-local semi-convexity with $S \geq 1$. Then $\mathcal{H}^n(T_x) = 0$ for all $x \in X$. In particular, $\mathcal{H}^n(X \setminus \mathcal{A}(1, \delta)) = 0$ for any $\delta > 0$*

Proof. The first assertion follows directly from the measure contraction property $\mathrm{MCP}(0, n)$ of the metric measure space $(X, d, \mathcal{H}^n)$ and the well-known fact (see [50, Section 3.3]) that in any metric measure space satisfying the measure contraction property, the cut locus of any point has zero measure. Therefore, $\mathcal{H}^n(T_x) = 0$ for all $x \in X$.

For the second assertion, let $p \in X$ be arbitrary. By the first assertion, for $\mathcal{H}^n$-almost every point $x \in B(p, D)$, there exists an extendable geodesic γ starting from p that can be extended beyond x to some point $y \in B(p, D)$, with the distance $|xy|$ satisfying $\bar{\delta}_S(|xy|; |px|) < \delta$. It is straightforward to verify that x is a $(1, \delta)$-strained point with p as its strainer. Covering X by at most countably many balls $B(p_i, D)$, the second assertion follows. □

Remark 7.3. *Similar result for Alexandrov spaces with curvature bounded below has already been shown by Otsu and Shioya [39, Proposition 3.1], based on the Alexandrov convexity.*

Next we prove a key technical lemma, which states that for any $1 \leq k \leq n - 1$, the set of points admitting a (k, δ)-strainer but not a $(k + 1, 4\delta)$-strainer has zero n-Hausdorff measure in any n-dimensional S-concave Busemann concave space satisfying local semi-convexity. A similar, but stronger, result concerning the Hausdorff dimension of singular strata was proved in [13, Lemma 10.5 and Theorem 10.6] for Alexandrov spaces. Their argument relies implicitly on the metric cone structure of tangent cones in Alexandrov spaces, which is not available in our setting. To overcome this difficulty, we utilize the 'almost extendable' property of geodesics, established in Lemma 7.2, to analyze the infinitesimal behavior $\liminf_{y \to x} |f(y)f(x)|/|xy|$ of strainer maps restricted to the singular sets. This infinitesimal behavior together with the classical result from Lytchak [28, Lemma 3.1] enables us to determine the top-dimensional Hausdorff measure of the singular sets.

Lemma 7.4. *Let X be an n-dimensional S-concave Busemann concave space that satisfies (C,D)-local semi-convexity with $S \geq 1$. Given $1 \leq k \leq n-1$, let $(p_1,\dots,p_k)$ be a (k,δ)-strainer on an open set U for some $\delta < \delta_k$. Let $E \subset U$ be the set of points that do not admit any $(k+1, 4\delta)$-strainer. Then $\mathcal{H}^n(E) = 0$.*

Proof. Let $f := (d_{p_1}, \cdots, d_{p_k}) : U \to \mathbb{R}^k$ be the associated (k,δ)-strainer map. We show the claim by contradiction. Suppose that $\mathcal{H}^n(E) > 0$.

STEP 1: Let $F \subset E$ be the set of density points of E with respect to the n-Hausdorff measure $\mathcal{H}^n$. From Proposition 6.6, we know that $\mathcal{H}^n$ is a doubling measure on X. Then by the Lebesgue differentiation theorem, it follows that $\mathcal{H}^n(E \setminus F) = 0$. In particular, it holds that $\mathcal{H}^n(B(x,r) \cap E) > 0$ for any $x \in F$ and any $r > 0$. By the inner regularity of the n Hausdorff measure[6], we can find a sequence of increasing compact subsets $K_n \subset F$ such that $\mathcal{H}^n(F \setminus \cup_n K_n) = 0$. Let $K := \cup_n K_n$. It is clear that $\mathcal{H}^n(E \setminus K) = 0$. In particular, it holds that $\mathcal{H}^n(B(x,r) \cap K) > 0$ for any $x \in K$ and $r > 0$.

From Lemma 7.2, it holds that I_x has full $\mathcal{H}^n$-measure of X for any $x \in K$. Then since $\mathcal{H}^n(B(y,r) \cap K) > 0$ for any $y \in K$, it follows that $I_x \cap B(y,r) \cap K \neq \emptyset$ for any $y \in K$ and $r > 0$. In particular, this implies that $I_x \cap K$ is dense in K for any $x \in K$.

STEP 2: In this step, we show that $\liminf_{K \ni y \to x} \frac{|f(y)f(x)|}{|yx|} \geq \delta$ for any $x \in K$.

We first show that for any $x \in K$, it holds that

$$\lim_{r \to \infty} \inf_{\substack{y \in B(x,r) \cap K, \\ y \in I_x}} \frac{|f(y)f(x)|}{|xy|} \geq \delta. \tag{7.2}$$

Note first that the set $B(x,r) \cap K \cap I_x$ is not empty for any $x \in K$ and $r > 0$. Therefore, the infimum on the left-hand side of inequality (7.2) is not equal to infinity for any $r > 0$, and is bounded above by Lipschitz constant of f. Now suppose that the claim does not hold. Then we can find a point $x \in K$ and a sequence $\{x_j\}_j \subset K \cap I_x$ such that $x_j \to x$ and

$$\lim_{j \to \infty} \frac{|f(x)f(x_j)|}{|xx_j|} < \delta. \tag{7.3}$$

It follows from the above inequality and the definition of f that

$$\lim_{j \to \infty} \frac{||p_i x_j| - |p_i x||}{t_j} = \lim_{j \to \infty} \frac{|f_i(x_j) - f_i(x)|}{|xx_j|} < \delta, \tag{7.4}$$

for all $i = 1, \dots, k$. Choose j sufficiently large such that the distance $t_j = |xx_j|$ is small enough such that $t_j + |p_1 x_j| < D$ and $\bar{\delta}_{S,C}(2t_j; |p_i x|) < 0.01\delta$ and $|f_i(x_j) - f_i(x)|/t_j < \delta$ for all $i = 1, \dots, k$.

In the following, we show that the point x_j is a $(k+1, 4\delta)$-strained point, which then contradicts the assumption that $x_j \in K \subset E$ and E does not contain any $(k+1, 4\delta)$-strained point. Indeed, since $x_j \in I_x$, it follows that we can find a unit-speed geodesic $\eta_j : [0, t_j + \varepsilon] \to X$ starting from x such that x_j is lies in its interior. We choose an $s_j \in (0, \varepsilon)$ sufficient small such that $\bar{\delta}_{S,C}(s_j; t_j) < \delta$. Let $y := \eta(t_j + s_j)$.

We claim that the $(k+1)$-tuple $(p_1, \dots, p_k, x)$ is a $(k+1, 4\delta)$-strainer at x_j. Indeed, note first that $(p_1, \dots, p_k)$ is a (k,δ)-strainer at $x_j \in U$ by assumption. By our choice of y, it follows that the comparison angle $\tilde{\angle} x x_j y = \pi$ and that $\bar{\delta}_S(|yx_j|; |xx_j|) \leq \bar{\delta}_{S,C}(s_j; t_j) < \delta$, which shows by Definition 5.1 that x is a $(1,\delta)$-strainer at x_j with the opposite strainer y. Furthermore, from the choice of t_j, it follows that

$$\bar{\delta}_{S,C}(|xx_j|; |p_i x_j|) \leq \bar{\delta}_{S,C}(t_j; |p_i x| - t_j) \leq \bar{\delta}_{S,C}(2t_j; |p_i x|) < \delta. \tag{7.5}$$

These shows that the conditions (1)–(2) in Definition 5.5 of $(k+1,\delta)$-strainer are satisfied for $(p_1, \dots, p_k, x)$ at x_j.

[6]Recall that Hausdorff measures on complete metric spaces are Radon measures.

It is left to check the condition (3) in Definition 5.5 for $(p_1, \dots, p_k, x)$ at x_j. By applying the Euclidean law of cosine to the comparison triangle $\tilde{\Delta} p_i x x_j$ and re-organizing the equality as (4.9) in Lemma 4.3, we obtain that

$$\cos \tilde{\angle} p_i x_j x = \frac{|p_i x_j| - |p_i x|}{t_j} + \delta(t_j; \xi_j), \tag{7.6}$$

where ξ_j is an arbitrary geodesic from x_j to x, and δ is an error function in the equality (4.9) satisfying $0 \le \delta(t_j; \xi_j) \le t_j/(2|p_i x|)$. Then from our choice of t_j, it follows that

$$\left|\cos \tilde{\angle} p_i x_j x\right| \le \left|\frac{|p_i x_j| - |p_i x|}{t_j}\right| + \frac{t_j}{2|p_i x|} < \delta + \frac{\delta}{2} = \frac{3}{2}\delta. \tag{7.7}$$

This implies that $|\tilde{\angle} p_i x_j x - \pi/2| < 3\delta$ for all $i = 1, \cdots, k$. Thus, the first inequality in (5.5) of Definition 5.5 is satisfied. Finally, following the same argument as the final argument in the proof of Step 2 of Proposition 5.16, together with the fact η_j is a unit-speed geodesic passing through x_j and our choice of t_j and y, we obtain that

$$|\tilde{\angle} p_i x_j y - \frac{\pi}{2}| < 4\delta, \tag{7.8}$$

which shows that the second inequality in (5.5) of Definition 5.5 is satisfied. To sum up, we have shown that the $(k+1)$-tuple $(p_1, \dots, p_k, x)$ is a $(k+1, 4\delta)$-strainer at x_j. But this contradicts our assumptions. Therefore, the inequality (7.2) holds for any $x \in K$.

Finally, since $I_x \cap K$ is dense in K for any $x \in K$, one can readily check that $I_x \cap K \cap B(x, r)$ is also dense in $B(x, r) \cap K$ for any $r > 0$. This implies that

$$\inf_{y \in B(x,r) \cap K} \frac{|f(y) f(x)|}{|xy|} = \inf_{\substack{y \in B(x,r) \cap K \\ y \in I_x}} \frac{|f(y) f(x)|}{|xy|}. \tag{7.9}$$

By taking $r \to \infty$ on both sides, we obtain from the (7.2) and (7.9) that

$$\lim_{r \to \infty} \inf_{y \in B(x,r) \cap K} \frac{|f(y), f(x)|}{|x, y|} \ge \delta, \tag{7.10}$$

for any $x \in K$.

STEP 3: We now prepare to derive a contradiction to our assumption that $\mathcal{H}^n(E) > 0$. The claim is in fact the consequence of the infinitesimal behavior of the map f established by inequality (7.10), as shown in [28, Lemma 3.1]. Indeed, we have shown in Step 2 that the restriction $f : (K, d|_K) \to \mathbb{R}^k$ is a Lipschitz map such that $\liminf_{y \to x} f \ge \delta$ for any $x \in K$. Let $Z_m \subset K$ be the set of points $x \in K$ satisfying that if $z \in K$ with $|zx| < 1/m$, then $|f(x) f(z)| \ge \delta |xz|/2$. One can check that Z_m is closed in K. Furthermore, the property $\liminf_{y \to x} f \ge \delta$ on K implies that $K = \cup_m Z_m$. Therefore, we obtain a decomposition $\{Z_m\}_m$ of $(K, d|_K)$ such that f is locally bi-Lipschitz on each Z_m. This implies that the Hausdorff dimension of Z_m is at most k and so is the Hausdorff dimension of K. The isometric embedding $i : (K, d|_K) \to (X, d)$ implies that the subset K, as a subset of X, has at most the Hausdorff dimension k, and therefore $\mathcal{H}^n(K) = 0$ since $k \le n - 1$. However, this contradicts our assumption that $\mathcal{H}^n(K) = \mathcal{H}^n(E) > 0$. Hence, our assumption that $\mathcal{H}^n(E) > 0$ does not hold, and so $\mathcal{H}^n(E) = 0$.

□

Remark 7.5. *We note that Lemma 7.4 holds without any restriction on the uniform smoothness constant S of the space.*

Now we can prove our main theorem. Recall that $\mathcal{A}(n, \delta)$ denotes the set of (k, δ)-strained points.

Theorem 7.6. *Let X be an n-dimensional S-concave Busemann concave space that satisfies local semi-convexity with $S \geq 1$. Then for any $\delta > 0$, $\mathcal{H}^n(X \setminus \mathcal{A}(n, \delta)) = 0$. In particular, X contains an open dense topological n-manifold part which has full n-Hausdorff measure.*

Proof. It suffices to show the first claim for $0 < \delta < \delta_n$. We show the claim by induction. Let $\delta_1' < \delta/4^{n-1}$ be a small number such that $\delta_k' := 4^{k-1}\delta_1' < \delta_k$ for all $k = 1, \cdots, n$. For $k = 1$, the claim that $\mathcal{H}^n(X \setminus \mathcal{A}(1, \delta_1')) = 0$ directly follows from Lemma 7.2.

Suppose that $\mathcal{H}^n(X \setminus \mathcal{A}(k-1, \delta_{k-1}')) = 0$. We claim that $\mathcal{H}^n(\mathcal{A}(k-1, \delta_{k-1}') \setminus \mathcal{A}(k, \delta_k')) = 0$. Indeed, for each $x \in \mathcal{A}(k-1, \delta_{k-1})$, by the openness of strainer maps, we can choose $r_x > 0$ sufficiently small such that $B(x, r_x) \subset \mathcal{A}(k-1, \delta_{k-1})$ and that the $(k-1)$-tuple $(p_1, \ldots, p_{k-1})$ is a $(k-1, \delta_{k-1}')$-strainer on $B(x, r_x)$. Now Since X is proper, by approximating $\mathcal{A}(k-1, \delta_{k-1}')$ by a sequence of increasing compact subsets $\{K_j\}_j$ and then covering K_j by finitely many balls $B(x, r_x)$ with $x \in K_j$, we can cover $\mathcal{A}(k-1, \delta_{k-1}')$ by countably many balls $\{B(x_l, r_l)\}_l$. By Lemma 7.4, it follows that $\mathcal{H}^n(B(x_l, r_l) \setminus \mathcal{A}_x(k, \delta_k')) = 0$ for all $l \in \mathbb{N}$. This together with covering $\mathcal{A}(k-1, \delta_{k-1}')$ by countably many balls implies that $\mathcal{H}^n(\mathcal{A}(k-1, \delta_{k-1}') \setminus \mathcal{A}(k, \delta_k')) = 0$. By assumption, it follows that $\mathcal{H}^n(X \setminus \mathcal{A}(k, \delta_k')) = 0$. By induction, we obtain that $\mathcal{H}^n(X \setminus \mathcal{A}(n, \delta_n')) = 0$. Our claim just follows from this fact and the inclusion that $\mathcal{A}(n, \delta_n') \subset \mathcal{A}(n, \delta)$.

For the second assertion, let $\delta < \delta_{n+1}/2$. Then by Corollary 6.7 and the first claim, the set $\mathcal{A}(n, \delta)$ is a topological n-manifold, which is open and dense in X and has full n-Hausdorff measure. □

7.2. **Rectifiability and Banach tangent cones.** In this part, building upon the measure-theoretic results established in the previous subsection, we derive several structural results of finite-dimensional S-concave Busemann concave spaces that satisfy local semi-convexity.

We first show the n-rectifiability of such spaces. For general results about rectifiability for spaces satisfying different notions of curvature-dimension condition, we refer to [33], and refer to [8, 9, 16] for further characterizations of rectifiability of general metric measure spaces.

Theorem 7.7 (n-rectifiable)**.** *Let (X, d) be an n-dimensional S-concave Busemann concave space that satisfies local semi-convexity with $S \geq 1$. Then X is n-rectifiable.*

Proof. Let $\delta < \delta_{n+1}/2$ be a small number. For any $x \in \mathcal{A}(n, \delta)$, let $r_x > 0$ be a sufficiently small radius such that $B(x, r_x) \subset \mathcal{A}(n, \delta)$ and that there exists a (n, δ)-strainer map f_x on $B(x, r_x)$. Since the strainer number equals the Hausdorff dimension n, it follows that no point sufficiently close to x is a $(n+1, 2\delta)$-strained point. Thus, by Proposition 5.16, we can find possibly smaller neighborhood $U_x \subset B(x, r_x)$ such that the associated (n, δ)-strainer map f_x is bi-Lipschitz homeomorphism from U_x onto $f_x(U_x)$. Now since X is proper, by first approximating $\mathcal{A}(n, \delta)$ from interior by a sequence of increasing compact subsets and then cover each compact subset by finitely many open subsets $U_i := U_{x_i}$, we can cover $\mathcal{A}(n, \delta)$ by at most countably many open subsets U_i. Let $A_i = f_i(U_i)$, where $f_i := f_{x_i} : U_i \to A_i$ is the associated (n, δ)-strainer map, and let $g_i := f_i^{-1} : A_i \to U_i$ be the inverse map of f_i. Then by Theorem 7.6, we obtain that $\mathcal{H}^n(X \setminus \cup_i g_i(A_i)) = 0$, which shows that X is n-rectifiable. □

The next theorem, regarding uniqueness of Banach tangent cone, is a direct consequence of n-rectifiability and Kirchheim's local structure theorem of rectifiable sets in metric spaces [25, Theorem 9], which is well-known to experts in the field. For the sake of completeness, we provide a detailed proof here.

Theorem 7.8 (Unique Banach tangent cones)**.** *Let (X, d) be an n-dimensional S-concave Busemann concave space that satisfies local semi-convexity with $S \geq 1$. Then $\mathcal{H}^n$-almost* *every point admits a unique tangent cone (T_xX, d_x, o), which is isometric to a finite-dimensional Banach space.*

Proof. Our proof closely follows the first part of [8, Theorem 6.6]. Since X is n-rectifiable, it follows from Kirchheim's local structure theorem [25, Theorem 9] that for $\mathcal{H}^n$-a.e. point x, we can find a norm $\|\cdot\|_x$ on $\mathbb{R}^n$, a map $f_x : X \to \mathbb{R}^n$ and a closed subset $C_x \subset X$ such that $f_x(x) = 0$ and x is a density point of C_x and

$$\limsup_{r\to 0}\left\{\left|1-\frac{\|f_x(y)-f_x(z)\|_x}{|y,z|}\right| : y\neq z, y,z\in B(x,r)\cap C_x\right\}=0. \tag{7.11}$$

Now we fix such a point $x \in X$. Let ε_r be defined as

$$\varepsilon_r := \sup\left\{\left|1-\frac{\|f_x(y)-f_x(z)\|_x}{|y,z|}\right| : y\neq z, y,z\in B(x,\sqrt{r})\cap C_x\right\}. \tag{7.12}$$

Let $C_{x,r} := B(x,\sqrt{r}) \cap C_x$ and $f_{x,r} := f_x/r$. Then for any $y, z \in C_{x,r}$ with $|yz| \le \min\{r^{-1/2}, \varepsilon_r^{-1/2}\}$, it follows that

$$\left|\frac{|y,z|}{r} - \|f_r(y)-f_r(z)\|_x\right| \le \varepsilon_r\frac{|y,z|}{r} \le \varepsilon_r^{1/2}. \tag{7.13}$$

This implies that the map $f_{x,r} : (C_{x,r}, d/r, x) \to (\mathbb{R}^n, \|\cdot\|_x, 0^n)$ is a bi-Lipschitz map with the bi-Lipschitz constant L_r satisfying

$$L_r := \frac{1+\varepsilon_r}{1-\varepsilon_r^2} \to 1, \quad \text{as } r\to 0. \tag{7.14}$$

Thus, by the bi-Lipschitz continuity of $f_{x,r}$, it follows that for any $R > 0$,

$$\begin{aligned}
\mathcal{H}^n_{\|\cdot\|_x}(B(0^n,R)\setminus f_{x,r}(C_{x,r})) &\le \mathcal{H}^n_{\|\cdot\|_x}(B(0^n,R)\setminus f_{x,r}(C_{x,r}\cap B_X(x,Rr/L_r)))\\
&\le \mathcal{H}^n_{\|\cdot\|_x}(B(0^n,R)) - \mathcal{H}^n_{\|\cdot\|_x}(f_{x,r}(C_{x,r}\cap B_X(x,Rr/L_r)))\\
&\le \mathcal{H}^n_{\|\cdot\|_x}(B(0^n,R)) - L_r^n\mathcal{H}^n_{d/r}(C_{x,r}\cap B_X(x,rR/L_r))\\
&\le \mathcal{H}^n_{\|\cdot\|_x}(B(0^n,R)) - \frac{L_r^n}{r^n}\mathcal{H}^n_{d}(C_{x,r}\cap B_X(x,rR/L_r))\\
&= \mathcal{H}^n_{\|\cdot\|_x}(B(0^n,R)) - (\omega_nR^n)\frac{\mathcal{H}^n_d(C_{x,r}\cap B_X(x,rR/L_r))}{\omega_n(rR/L_r)^n}\\
&\to \omega_nR^n-\omega_nR^n=0, \quad \text{as } r\to 0,
\end{aligned} \tag{7.15}$$

where we use the identity that $\mathcal{H}^n_{\|\cdot\|}(B_{\mathbb{R}^n}(0,R)) = \omega_nR^n$ for any norm on $\mathbb{R}^n$ (see [25, Lemma 6]) and the fact that x is a density point of C_x in the last step. This implies that for any $R > 0$ and $\delta > 0$, we can find $r > 0$ small enough such that the ball $B_{\|\cdot\|_x}(0^n, R)$ is contained into the δ-neighborhood of $f_{x,r}(C_{x,r})$. Together with the inequality (7.13), it follows that $f_{x,r} : (C_x, d/r, x) \to (\mathbb{R}^n, \|\cdot\|_x, 0^n)$ is an ε'_r-isometry for some ε'_r with $\varepsilon'_r \to 0$ as $r \to 0$. Thus, it follows that $(\mathbb{R}^n, \|\cdot\|_x, 0^n)$ is the unique pointed Gromov–Hausdorff limit of the blow-ups $\{C_x, d/r, x\}_r$.

On the other hand, since the point x is a density point of C_x and the Hausdorff measure $\mathcal{H}^n$ is an outer measure, it follows from [17, Proposition 3.1] that

$$\mathrm{Tan}(C_x, d, x) = \mathrm{Tan}(X, d, x) = \{(T_xX, d_x, o)\}. \tag{7.16}$$

Thus, we obtain that (T_xX, d_x, o) is equal to the Banach space $(\mathbb{R}^n, \|\cdot\|_x, 0^n)$, up to an isometry. □

Remark 7.9. *It remains an open question for us whether, under the assumptions of Theorem 7.8, all Banach tangent cones are isometric to the same finite-dimensional Banach space. In contrast, it has been shown in [22, Theorem 1.2] that, within the class of Finsler manifolds, Busemann spaces of non-positive curvature are precisely those with Berwald metrics of non-positive flag curvature. In particular, all tangent cones of connected Finsler* *manifolds of Busemann non-positive curvature are isometric to the same normed vector space.*

We conclude this subsection with a further characterization of the geometry of Banach tangent cones under an additional assumption on the distance functions from a point.

Corollary 7.10. *Let X be an n-dimensional S-concave Busemann concave space that satisfies (C, D)-local semi-convexity with $S \geq 1$. Suppose that for every point $x \in X$, there exists $r_x > 0$ such that for every constant-speed geodesic $\xi : [0,1] \to B(x, r_x)$ and every point $y \in B(x, r_x)$, the distance function from y satisfies the p-uniform convexity inequality*

$$|y\xi(1/2)|^p \leq \frac{1}{2}|y\xi(0)|^p + \frac{1}{2}|y\xi(1)|^p - \frac{C'}{4}\,|\xi(0)\xi(1)|^p\,, \tag{7.17}$$

for some $p > S+1$ and $C' \geq 2^{2-p}$. Then for $\mathcal{H}^n$-almost every point x, the unique tangent cone (T_x, d_x, o) is isometric to a Banach space $(E_x, \|\cdot\|_x, o_x)$ with strictly convex norm $\|\cdot\|_x$, which is both 2-uniformly smooth and p-uniformly convex.

Proof. Let $x \in X$ be an arbitrary point whose tangent cone is isometric to a finite-dimensional Banach space $(E_x, \|\cdot\|_x, o_x)$. By Theorem 7.8, it suffices to show that $(E_x, \|\cdot\|_x, o_x)$ is a 2-uniformly smooth and p-uniformly convex Banach space with the strictly convex norm $\|\cdot\|_x$.

We first show that E_x is uniquely geodesic. Let $u, v \in E_x$ be two arbitrary points different from the origin o_x, and $\xi : [0,1] \to E_x$ be an arbitrary constant-speed geodesic from u to v and let $w := \xi(1/2)$ be the midpoint of ξ. Let $a := d_x(u,w) = \|u-v\|_x/2$. For simplicity, we naturally identify the points of T_xX with E_x by the same notation. From the construction of tangent cones, we can find sequences of points $u_n := (\gamma_n, t_n), v_n := (\eta_n, s_n), w_n := (\xi_n, l_n)$ such that $u_n \to u, v_n \to v, w_n \to w$ in E_x. For $\theta \in [0,1]$, let $x_{n,\theta} := \xi_n(\theta l_n)$ and $c_{n,\theta} : [0,1] \to X$ be an arbitrary constant-speed geodesic from $\gamma_n(\theta t_n)$ to $\eta_n(\theta s_n)$. Note that we can take $\theta_0 \in (0,1)$ sufficiently small and N sufficiently large such that $x_{n,\theta}$ and $c_{n,\theta}$ are contained in $B(x, r_x)$ for all $\theta \in (0, \theta_0]$ and $n \geq N$. By applying the p-uniformly convex inequality (7.17) to the point $x_{n,\theta}$ and the geodesic $c_{n,\theta}$, and then dividing θ^p on both sides, it follows that

$$|x_{n,\theta}c_{n,\theta}(1/2)|_\theta^p \leq \frac{1}{2}\,|x_{n,\theta}\gamma_n(\theta t_n)|_\theta^p + \frac{1}{2}\,|x_{n,\theta}\eta_n(\theta s_n)|_\theta^p - \frac{C'}{4}\,|\gamma_n(\theta t_n)\eta_n(\theta s_n)|_\theta^p\,, \tag{7.18}$$

where $|\cdot|_\theta := d/\theta$ denotes the scaled distance function. Since $X_\theta := (X, d/\theta, x)$ pointed Gromov-Hausdorff converges to (T_xX, d_x, o), one can readily check that $\gamma_n(\theta t_n) \to u_n, \eta_n(\theta s_n) \to v_n, x_{n,\theta} \to w_n$ as $\theta \to 0$. Furthermore, it follows from Ascoli-Arzela lemma that $c_{n,\theta} \subset X_\theta$ uniformly converges to some constant-speed geodesic $c_n \subset E_x$ from u_n to v_n, up to a subsequence of θ. By taking $\theta \to 0$ along the subsequence on both sides of the inequality above, it follows that

$$\|w_n - c_n(1/2)\|_x^p \leq \frac{1}{2}\,\|w_n - u_n\|_x^p + \frac{1}{2}\,\|w_n - v_n\|_X^p - \frac{C'}{4}\,\|u_n - v_n\|_x^p\,. \tag{7.19}$$

By letting $n \to \infty$ and applying Ascoli–Arzela lemma again to the geodesic c_n, it follows that

$$\|w - c(1/2)\|_x^p \leq \frac{1}{2}\|w-u\|_x^p + \frac{1}{2}\|w-v\|_x^p - \frac{C'}{4}\|u-v\|_x^p \leq a^p - \frac{C'}{4}\,(2a)^p \leq 0, \tag{7.20}$$

where c_n converges to some constant-speed geodesic c from u to v, up to some subsequence of n. This implies that $w = c(1/2)$. Since the choice of subsequence of n and θ is independent of w, and w is the midpoint of arbitrary geodesic from u to v, it follows that all midpoints of geodesics from u to v coincide. This further implies that all geodesics from u to v coincide with the linear geodesic from u to v. Thus, $(E_x, \|\cdot\|_x, o_x)$ is uniquely geodesic. By [40, Proposition 7.2.1], it follows that $(E_x, \|\cdot\|_x, o_x)$ is a strictly convex Banach space.

For the 2-uniform smoothness and p-uniform convexity of E_x, by applying the S-concave inequality to the point x and the geodesic $c_{n,\theta}$, it follows that

$$|xc_{n,\theta}(1/2)|^2 \geq \frac{1}{2}|xc_{n,\theta}(0)|^2 + \frac{1}{2}|xc_{n,\theta}(1)|^2 - \frac{S}{4}|c_{n,\theta}(0)c_{n,\theta}(1)|^2. \tag{7.21}$$

By taking $\theta \to 0$ first and then $n \to \infty$, we obtain that

$$\left\|\frac{u+v}{2}\right\|_x^2 = \|w\|_x^2 \geq \frac{1}{2}\|u\|_x^2 + \frac{1}{2}\|v\|_x^2 - \frac{S}{4}\|u-v\|_x^2. \tag{7.22}$$

By the same argument, it follows from the inequality (7.17) that

$$\left\|\frac{u+v}{2}\right\|_x^p \leq \frac{1}{2}\|u\|_x^p + \frac{1}{2}\|v\|_x^p - \frac{C'}{4}\|u-v\|_x^p. \tag{7.23}$$

Since u and v are arbitrary points in E_x, our claim follows. □

Remark 7.11. *The same proof in fact allows us to show that for a Busemann concave space, if it admits a non-trivial n-Hausdorff measure and satisfies the p-uniform convexity inequality (7.17) above locally, then $\mathcal{H}^n$-almost every point admits a unique tangent cone which is isometric to a finite-dimensional Busemann concave Carnot group equipped with Carnot–Caratheodory metric (see [17, Lemma 4.1]). In particular, in this case, all such tangent cones are Banach spaces with strictly convex norm (see [24, Proposition 2.5]).*

7.3. **Hausdorff dimension of singular strata.** In this subsection, we investigate the Hausdorff dimension of the singular strata in n-dimensional S-concave Busemann concave spaces that satisfy local semi-convexity. As noted above, the approach developed in [13] for Alexandrov spaces cannot be directly applied here due to the absence of a metric cone structure for tangent cones in our setting. Furthermore, the technique employed in Lemma 7.4 is not sufficient to determine the Hausdorff dimension of singular sets, since the 'almost extendable' property of geodesics holds only up to an $\mathcal{H}^n$-null set and does not provide more refined measure-theoretic information.

To deal with the difficulty, we make use of the relationship between two notions of angles we introduce in Section 4, namely angles of fixed scale and angles viewed from a fixed point. These two notions of angles are generally not equal, as illustrated in Example 4.15. They may even differ greatly if the underlying space is far from the Alexandrov spaces. However, under some control on the uniform smoothness constant S of the underlying space, angles of common scale[7] are almost equal to angles viewed from a fixed point, up to some error. This observation enables us to utilize the space of directions with common length introduced in Definition 4.19 in place of the space of directions in Alexandrov spaces, thereby obtaining the desired dimension estimates for the singular strata.

We firs recall the following lemma, which is nearly the same as [29, Lemma 10.2] and [13, Lemma 10.3].

Lemma 7.12. *For any $N, L \geq 1$ and any natural number $M \geq 1$, there exists a constant $\bar{K} = \bar{K}(N, L, M)$ with the following property: if X is an N-doubling metric space and $E \subset X$ is any subset containing at least K elements, then there exist M points $\{x_i\}_{i=0}^{M-1} \subset E$ such that $|x_{i+1}x_0| \geq L|x_ix_0|$ for each $i = 1, \ldots, M-2$.*

Proof. Our proof is adapted from [29, Lemma 10.2]. Let $N, L \geq 1$ be fixed. Let $C := C(N, L)$ be a constant such that any set $E \subset X$ of diameter D can be covered by at most C subsets of diameter at most $D/(2L)$.

We show the claim by induction. We show that the claim holds for the number $\bar{K} = C^{M-1}$. The case $M = 1$ is trivial. Suppose that the claim holds for the case $k = M - 1$.

[7] That is, the angles of the form $\angle_x(\gamma(l), \eta(l))$.

Let $E \subset X$ be a subset containing at least C^{M-1} points. By removing the redundant elements, we may assume that E is bounded with diameter $D > 0$. We cover E by at most C subsets of diameter at most $D/(2L)$. Then it follows that there exists at least one subset $\tilde{E}$ in this covering that contains at least C^{M-2} elements. By the assumption of induction, we can find $M-1$ points $\{x_i\}_{i=0}^{M-2} \subset \tilde{E}$ satisfying that $|x_0x_{i+1}| \geq L|x_0x_i|$ for $i = 1, \cdots, M-3$. Now, select an arbitrary point $x \in E$ such that $|xx_0| \geq D/2$. This is always possible; otherwise, the diameter of E would be less than D. Let $x_{M-1} := x$. The collection $\{x_i\}_{i=0}^{M-1}$ satisfies the required property because

$$|x_0x_{M-1}| \geq \frac{D}{2} \geq L\frac{D}{2L} \geq L|x_0x_i|, \tag{7.24}$$

for all $i = 1, \cdots, M-2$. □

We now present an important lemma, which roughly quantifies how the uniform smoothness constant S and the ratio of side-lengths in angles of fixed scale control the discrepancy between angles of common scale and angles viewed from a fixed point.

Lemma 7.13. *Let $\delta > 0$ be sufficiently small number. There exists a constant $L_0 = L_0(\delta) \geq 1$ such that for any $\bar{L} \geq L_0$, there is a constant $S_0 = S_0(\delta, \bar{L}) \in (1, 4]$ with the following property: let X be an S-concave Busemann concave space with $S \in [1, S_0]$. Given any point $x \in X$, let γ, η be unit-speed geodesics from x to y and z, respectively, such that $L_0 \leq |xz|/|xy| \leq \bar{L}$. If the angle $\angle_x(\gamma(r), \eta(r)) < \delta$ for some $r > 0$, then the comparison angle $\tilde{\angle}_x(y, z) < 2\delta$. In particular, z is a $(1, 3\delta)$-strainer at y with x as the opposite strainer.*

Proof. We first derive an upper bound of the angle $\tilde{\angle}_x(y, z)$ in terms of an upper bound of the angle $\angle_x(\gamma(r), \eta(r))$, the uniform smoothness constant S and the ratio of side-lengths $L = |xz|/|xy|$. Let $x \in X$ and γ, η be any geodesics from x to y, z respectively such that the angle of common scale $\angle_x(\gamma(r), \eta(r)) < \delta$ for some $r > 0$ and $\delta > 0$. Without loss of generality, we may assume that the ratio $L := |xz|/|xy| \geq 1$. Furthermore, by the positive scaling-invariance of angles of fixed scale, Lemma 4.17, we may also assume that $r = |xy|$. Let $z' := \eta(r)$. By applying the Euclidean law of cosine to the triangle $\Delta yxz'$, it follows that

$$|yz'|^2 = |xy|^2 + |xz'|^2 - 2|xy|\,|xz'| \cos\tilde{\angle}_x(y, z') = 2r^2 - 2r^2\cos\tilde{\angle}_x(\gamma(r), \eta(r)). \tag{7.25}$$

On the other hand, by applying the S-concave inequality (3.1) to the point y and the geodesic η, it follows that

$$|yz'|^2 = |y\eta(r)|^2 \geq \left(1 - \frac{1}{L}\right)|yx|^2 + \frac{1}{L}|yz|^2 - \frac{S}{L}\left(1 - \frac{1}{L}\right)|xz|^2. \tag{7.26}$$

Plugging the equality (7.25) into the inequality above, we obtain that

$$\begin{aligned} |yz|^2 &\leq L|yz'|^2 - (L-1)\,|yx|^2 + \frac{S(L-1)}{L}|xz|^2 \\ &\leq 2Lr^2\left(1 - \cos\tilde{\angle}_x(\gamma(r), \eta(r))\right) - (L-1)r^2 + SL(L-1)r^2. \end{aligned} \tag{7.27}$$

By applying again the Euclidean law of cosine to the angle Δyxz, it follows that

$$\cos\tilde{\angle}_x(y, z) = \frac{|xy|^2 + |xz|^2 - |yz|^2}{2|xy||xz|} \geq -\frac{(S-1)(L-1)}{2} + \cos\tilde{\angle}_x(\gamma(r), \eta(r)). \tag{7.28}$$

Now if $\angle_x(\gamma(r), \eta(r)) < \delta$, then from the definition of angles of fixed scale, it follows that

$$\cos\tilde{\angle}_x(y, z) \geq -\frac{(S-1)(L-1)}{2} + \cos\delta := F(S, L; \delta). \tag{7.29}$$

We now choose L_0 and S_0 such that the following holds: let $L_0 := L_0(\delta) > 1$ be sufficiently large such that $\arccos(1 - 2/(L_0 - 1)) < \delta$ and $\arcsin(1/L_0) < \delta$. For any $\bar{L} \geq L_0$, we take $S_0 := S_0(\delta, \bar{L}) \in (1, 4]$ such that $F(S_0, \bar{L}; \delta) \geq \cos(2\delta)$.

We now show that L_0 and S_0 satisfy the claim of lemma. Indeed, if $L = |xz|/|xy| \in [L_0, \bar{L}]$ and $S \in [1, S_0]$, it follows from the inequality (7.29) that $\cos \tilde{\angle}_x(z, y) \geq \cos 2\delta$, which implies that $\tilde{\angle}_x(y, z) \leq 2\delta$. As for the angle $\tilde{\angle}_z(x, y)$, it follows from the elementary triangle geometry that $\tilde{\angle}_z(x, y) \leq \arcsin(|xy|/|xz|) \leq \delta$ This, together with $\tilde{\angle}_x(y, z) \leq 2\delta$, implies that $\tilde{\angle}_y(x, z) \geq \pi - 3\delta$. From our choice of L_0, it follows that

$$\bar{\delta}_S(|xy|; |yz|) = \arccos\left(1 - S\frac{|xy|}{2|yz|}\right) \leq \arccos\left(1 - \frac{2}{L_0 - 1}\right) < \delta. \tag{7.30}$$

From the Definition 5.1 of $(1, \delta)$-strainer, our claim follows. □

We now present a key technical lemma in this subsection, which provides an estimate for the maximal possible cardinality of a $\delta^{-1}r$-separated subset within a cylindrical region of a small ball. Before we state the lemma, we recall that $\beta_E(r)$ denotes the largest possible cardinality of any maximal r-separated subset of E. Furthermore, we need the following definition of *R-long* strainers introduced in [13].

Definition 7.14 (R-long strainers)**.** A (k, δ)-strainer $(p_1 \ldots, p_k)$ at a point x with the opposite strainer $(q_1, \ldots, q_k)$ is said to be *R-long* if $\min_{i=1,\ldots,k}\{|p_i x|, |q_i x|\} > \delta^{-1}R$. We denote by $\mathcal{A}(k, \delta, R) \subset \mathcal{A}(k, \delta)$ the set of (k, δ)-strained points which admits an R-long (k, δ)-strainer.

We are now in a position to state our main technical lemma. The proof of lemma follows the similar strategy as [13, Lemma 10.5], based on the uniform compactness of space of directions with common length (Lemma 4.21) and our quantification result of asymmetry of angles (Lemma 7.13).

Lemma 7.15. *Let $\delta < \delta_k$ be an arbitrary small number. Then there exists a constant $S_1 = S_1(\delta) \in (1, 4]$ with the following property: let X be an n-dimensional S-concave Busemann concave space that satisfies (C, D)-local semi-convexity for $S \in [1, S_1]$. Suppose that $(p_1, \cdots, p_k)$ be an R-long strainer on some open neighborhood U of $x \in X$. Then there exists a small ball $B(x, r_x) \subset U$ such that for any $0 < r \leq \min\{\delta R, \delta r_x\}$ and any $m = (m_1, \ldots, m_k) \in \mathbb{Z}^k$, we have*

$$\beta_{D_x(r,m)\setminus\mathcal{A}(k+1,10\delta,r)}(\delta^{-1}r) \leq K(N, \delta), \tag{7.31}$$

where $D_x(r, m) := \{z \in B(x, r_x) : 0.1(m_i - 1)r \leq |p_i x| - |p_i z| \leq 0.1 m_i r, i = 1, \cdots, k\}$ is a cylindrical region in $B(x, r_x)$, and $K(N, \delta)$ is a constant depending only on the doubling constant N of X and δ, and not on r.

Remark 7.16. *Compared with [13, Lemma 10.5], our Lemma 7.15 gives cardinality estimates for all cylindrical regions $D_x(r, m)$ in a small ball rather than only the cylindrical region near the center of the ball.*

Proof. Let $0 < \delta < \delta_k$ and $R > 0$ be fixed. We first choose the desired r_x and S_1. Let $r_0 > 0$ be a sufficient small constant such that $B(x, r_0) \subset U$, and $r_0 \leq R/2$, and

$$\arccos\left(1 - \frac{(2 + C/2)\, r_0}{\delta^{-1}R - r_0}\right) < \delta. \tag{7.32}$$

Let $r_x := r_0/2$. Note that from our choice of r_0 and r_x, it follows that $\bar{\delta}_{S,C}(|yz|; |p_i y|) < \delta$ for any $y, z \in B(x, r_x)$ and all $i = 1, \ldots, k$. We now select the constants L_1 and S_1 such that $L_1 := L_1(\delta) = L_0(\delta) + 2$ and $S_1 := S_1(\delta) = S_0(\delta, L_1)$, where L_0, S_0 are the constants from Lemma 7.13.

We fix an arbitrary $r \in (0, \min\{\delta R, \delta r_x\}]$ and $m \in \mathbb{Z}^k$. In the following, we show by contradiction that for any $S \in [1, S_1(\delta)]$, the largest possible cardinality of any $\delta^{-1}r$-separated subset of $D_x(r, m)\setminus\mathcal{A}(k+1, 10\delta, r)$ is bounded above by the constant $K(N, \delta) :=$

$\bar{K}(N, L_1, M)$, where $\bar{K}(N, L, M)$ is the constant from Lemma 7.12, N is the doubling constant of X, and $M := M(\delta) = N_0(\delta) + 2$, where $N_0(\delta)$ is the constant from Lemma 4.21, depending only on the doubling constant of X.

Suppose that our claim does not hold. Then it follows that we can find a $\delta^{-1}r$-separated subset of $D_x(r, m) \setminus \mathcal{A}(k+1, 10\delta, r)$ containing at least $\bar{K}(N, L_1, M)$ points. By Lemma 7.12, it follows that we can find M points $\{x_i\}_{i=0}^{M-1} \subset D_x(r, m) \setminus \mathcal{A}(k+1, 10\delta, r)$ such that $|x_0 x_{i+1}| \geq L_1 |x_0 x_i|$ for all $i = 1, \dots, M-2$. Let $\bar{l} := \min_{i=1,\dots,M-1}\{|x_0 x_i|\}$. Let ξ_i be arbitrary unit-speed geodesics from x_0 to x_i and $\bar{\xi}_i$ be the maximal extension of the geodesic ξ_i for all $i = 1, \cdots, M-1$. From the uniform compactness of $\Sigma_{x_0}^{\bar{l}} X$ (Lemma 4.21) and our choice of $M = N_0(\delta) + 2$, it follows that we can find two indices $1 \leq i < j \leq M-1$ such that $|x_0 x_j| \geq L_1 |x_0 x_i|$ and

$$\angle_{x_0}\left((\bar{\xi}_i, \bar{l}), (\bar{\xi}_j, \bar{l})\right) = \angle_{x_0}\left(\xi_i(\bar{l}), \xi_j(\bar{l})\right) < \delta. \tag{7.33}$$

We take $z \in \xi_j$ the point in the geodesic ξ_j such that $|x_0 z| = (L_0 + 1)|x_0 x_i|$.

In the following, we show that the $(k+1)$-tuple $(p_1, \cdots, p_k, z)$ is an r-long $(k+1, 10\delta)$-strainer at x_i. This contradicts our choice of x_i that $x_i \notin \mathcal{A}(k+1, 10\delta, r)$.

Step 1: In this step, we check the first two conditions in Definition 5.5 for the $(k+1)$-tuple $(p_1, \cdots, p_k, z)$ at x_i. Note that $(p_1, \cdots, p_k)$ is obviously an r-long (k, δ)-strainer at x_i, since $|p_l x_i| > \delta^{-1}R > \delta^{-1}r$ for all $l = 1, \dots, k$. Furthermore, since $\{x_i\}_{i=1}^{M-1}$ is $\delta^{-1}r$-separated, then it follows that $|x_0 x_i| \geq \delta^{-1}r$, which further by the triangle inequality implies that

$$|z x_i| \geq |x_0 z| - |x_0 x_i| = L_0 |x_0 x_i| \geq |x_0 x_i| \geq \delta^{-1} r. \tag{7.34}$$

By Lemma 7.13 together with the inequality (7.33), it follows that z is an r-long $(1, 3\delta)$-strainer at x_i with the opposite strainer x_0. We also observe that, since z lies in the geodesic from x_0 to x_j, it holds that $|z x_i| \leq L_1 |x_0 x_i| \leq 2 r_x = r_0$. Therefore, by our choice of r_0 and the inequality (7.32), it follows that $\bar{\delta}_{S,C}(|z x_i|; |p_l x_i|) < \delta$ for all $l = 1, \dots, k$. Thus, we have verified the first two conditions in Definition 5.5 for the $(k+1)$-tuple $(p_1, \dots, p_k, z)$ at x_i.

Step 2: We are left to verify the almost orthogonality condition in Definition 5.5 for $\tilde{\angle} p_l x_i z$ and $\tilde{\angle} p_l x_i x_0$ for $l = 1, \dots, k$. For simplicity of notation, in the following steps, we denote $p := p_l$ for an arbitrary $l \in \{1, \dots, k\}$.

In this step, we establish an auxiliary result concerning the almost orthogonality of $\tilde{\angle} p x_0 z$. This auxiliary estimate will be instrumental in deriving the almost orthogonality of $\tilde{\angle} p x_i z$ and $\tilde{\angle} p x_i x_0$ in the subsequent step.

We first establish the almost orthogonality for the comparison angles $\tilde{\angle} p x_0 x_j$. The almost orthogonality for $\tilde{\angle} p x_0 z$ then follows by the almost comparison inequalities. Indeed, by applying the Euclidean law of cosine to the comparison triangle $\tilde{\Delta} p x_0 x_j$, it follows that

$$\begin{aligned}\cos \tilde{\angle} p x_0 x_j &= \frac{|p x_0|^2 + |x_0 x_j|^2 - |p x_j|^2}{2|p x_0||x_0 x_j|} \\ &= \frac{(|p x_0| + |p x_j|)(|p x_0| - |p x_j|)}{2|p x_0||x_0 x_j|} + \frac{|x_0 x_j|}{2|p x_0|}.\end{aligned} \tag{7.35}$$

For the first term of right-hand side of the inequality (7.35), note that from $x_0, x_j \in D_x(r, m)$, it follows that $||p x_0| - |p x_j|| < 0.1r \leq 0.1\delta R \leq 0.1\delta^2 |p x_0|$, where we use the fact that $(p_1, \cdots, p_k)$ is R-long on U. Since $|x_0 x_j| \geq \delta^{-1} r$, it follows that

$$\left| \frac{(|p x_0| + |p x_j|)(|p x_0| - |p x_j|)}{2|p x_0||x_0 x_j|} \right| \leq \frac{(2 + 0.1\delta^2)}{2} \frac{0.1r}{\delta^{-1} r} < \frac{1}{4}\delta. \tag{7.36}$$

For the second term of right-hand side of the inequality (7.35), by our choice of r_0 that $|x_0x_j| < 2r_x = r_0 < R/2$, it follows that

$$\frac{|x_0x_j|}{2|px_0|} \le \frac{r_0}{2\delta^{-1}R} < \frac{1}{4}\delta. \tag{7.37}$$

Together, we obtain that $|\cos\tilde{\angle}px_0x_j| \le \delta/2$. This implies that $|\tilde{\angle}px_0x_j - \pi/2| < \delta$.

For the angle $\tilde{\angle}px_0z$, note that the almost comparison inequalities and our choice of r_0 imply that $|\angle px_0\xi_j - \tilde{\angle}px_0x_j| < \bar{\delta}_{S,C}(|x_jx_0|;|px_0|) < \delta$ and $|\angle px_0\xi_j - \tilde{\angle}px_0z| < \bar{\delta}_{S,C}(|x_0z|;|px_0|) < \delta$. This implies that $|\tilde{\angle}px_0x_j - \tilde{\angle}px_0z| < 2\delta$. Thus, we obtain that $|\tilde{\angle}px_0z - \pi/2| < 3\delta$.

Step 3: In this final step, we show the almost orthogonality for $\tilde{\angle}px_iz$ and $\tilde{\angle}px_ix_0$.

We first establish the almost orthogonality for $\tilde{\angle}px_iz$. Indeed, by applying the Euclidean law of cosine to the comparison triangle $\tilde{\Delta}px_iz$ and $\tilde{\Delta}px_0z$, it follows that

$$\begin{aligned}\cos\tilde{\angle}px_iz &= \frac{|px_i|^2 + |x_iz|^2 - |pz|^2}{2|px_i||x_iz|} \\ &= \frac{|px_0|^2 + |x_0z|^2 - |pz|^2}{2|px_i||x_iz|} + \frac{|px_i|^2 - |px_0|^2}{2|px_i||x_iz|} + \frac{|x_iz|^2 - |x_0z|^2}{2|px_i||x_iz|} \\ &= \cos\tilde{\angle}px_0z + \frac{|px_i|^2 - |px_0|^2}{2|px_i||x_iz|} + \frac{|x_iz|^2 - |x_0z|^2}{2|px_i||x_iz|},\end{aligned} \tag{7.38}$$

which implies that

$$\left|\cos\tilde{\angle}px_iz - \cos\tilde{\angle}px_0z\right| \le \left|\frac{|px_i|^2 - |px_0|^2}{2|px_i||x_iz|}\right| + \left|\frac{|x_iz|^2 - |x_0z|^2}{2|px_i||x_iz|}\right|. \tag{7.39}$$

For the first term in the right-hand side of the inequality (7.39), since $x_0, x_i \in D_x(r,m)$, it holds that $||px_0| - |px_i|| < 0.1r \le 0.1\delta R \le 0.1\delta^2|px_i|$. Then by the triangle inequality, it follows that

$$\frac{(|px_0| + |px_i|)\,||px_0| - |px_i||}{2|px_i||x_iz|} \le \left(1 + 0.1\delta^2\right)\frac{|x_0x_i|}{L_0|x_0x_i|} \le \frac{1.1}{L_0}. \tag{7.40}$$

By the choice of $L_0 = L_0(\delta)$ in Lemma 7.13, it holds that $\arccos(1 - 2/(L_0 - 1)) < \delta$. Using the inequality that $2/L_0 \le \arccos(1 - 2/L_0) \le \arccos(1 - 2/(L_0 - 1))$, we obtain that

$$\left|\frac{|px_i|^2 - |px_0|^2}{2|px_i||x_iz|}\right| < \delta. \tag{7.41}$$

For the second term in the right-hand side of the inequality (7.39), since $|x_0z| = (L_0 + 1)|x_0x_i|$, it follows that

$$\frac{|x_iz| + |x_0z|}{2|x_iz|} \le \frac{(L_0+2)|x_0x_i| + (L_0+1)|x_0x_i|}{2L_0|x_ix_0|} = \frac{2L_0+3}{2L_0}, \tag{7.42}$$

and that

$$\frac{||x_iz| - |x_0z||}{|px_i|} \le \frac{|x_ix_0|}{|px_i|} \le \frac{r_0}{\delta^{-1}R} \le \frac{1}{2}\delta. \tag{7.43}$$

Thus, together with the two inequalities above, we obtain that

$$\left|\frac{|x_iz|^2 - |x_0z|^2}{2|px_i||x_iz|}\right| \le \left(1 + \frac{1.5}{L_0}\right)\frac{\delta}{2} \le \delta, \tag{7.44}$$

where we use that fact $L_0 > 2$ in the last inequality, which is derived from $2/L_0 \le \arccos(1 - 2/(L_0 - 1)) < \delta < 1$. Together with the inequalities (7.41) and (7.44), we obtain that

$$\left|\cos\tilde{\angle}px_iz - \cos\tilde{\angle}px_0z\right| < 2\delta. \tag{7.45}$$

From our previous result that $|\tilde{\angle}px_0z - \pi/2| < 3\delta$, we obtain that $|\tilde{\angle}px_iz - \pi/2| < 10\delta$.

Finally, for the angle $\tilde{\angle} px_i x_0$, by applying the Euclidean law of cosine to $\tilde{\Delta} px_0 x_i$ and following the same argument as showing the almost orthogonality of $\tilde{\angle} px_0 x_j$, we obtain that $|\tilde{\angle} px_i x_0 - \pi/2| < \delta$.

Combining Step 1-Step 3 together, we have shown that $(p_1, \cdots, p_k, z)$ is an r-long $(k+1, 10\delta)$-strainer at x_i. However, this contradicts the fact that $x_i \in D_x(r,m) \setminus \mathcal{A}(k+1, 10\delta, r)$. Hence, our assumption does not hold and our claim follows. Thus, we have found the desired $S_1(\delta)$ and the small ball $B(x, r_x)$ such that

$$\beta_{D_x(r,m)\setminus \mathcal{A}(k+1,10\delta,r)}(\delta^{-1} r) \leq K(N, \delta) = \bar{K}(N, L_1, M). \tag{7.46}$$

□

We are now prepared to present our final main theorem. This result provides a Hausdorff dimension estimate for the singular strata, which is sharper than Theorem 7.6, under the additional assumption that the uniform smoothness constant S of the underlying space is sufficiently close to 1. A similar result for Alexandrov spaces was established in [13, Theorem 10.6].

Theorem 7.17. *Let $\delta < \delta_n$ be a small number. Let X be an n-dimensional S-concave Busemann concave space that satisfies local semi-convexity with $S \in [1, S_1(\delta)]$, where $S_1(\delta) > 1$ is the constant from Lemma 7.15. Then $\dim_H(X \setminus \mathcal{A}(k, \delta)) \leq k - 1$ for all $k = 1, \ldots, n$.*

Proof. Let $S_1 := S_1(\delta)$ be the constant from Lemma 7.15, and let S be any arbitrary number in $[1, S_1]$.

We show the claim by induction. Let $\delta'_k := \delta/10^{n-k}, k = 1, \ldots, n-1$. The case that $k = 0$ is trivial, since $X = \mathcal{A}(0, \delta')$ for any $\delta' > 0$. Suppose that $\dim_H(X \setminus \mathcal{A}(k, \delta'_k)) \leq k - 1$. We claim that the Hausdorff dimension of the set $\mathcal{A}(k, \delta'_k) \setminus \mathcal{A}(k+1, \delta'_{k+1})$ is not greater than k. Indeed, for any $x \in \mathcal{A}(k, \delta'_k)$, we can choose $R_x > 0$ small enough such that x admits a $4R_x$-long (k, δ'_k)-strainer $(p_1, \ldots, p_k)$. By Lemma 7.15, it follows that we can choose a radius $r_x > 0$ sufficiently small such that $(p_1, \ldots, p_k)$ is an R_x-long (k, δ'_k)-strainer on $B(x, r_x)$. Now following the same argument as in the proof of Theorem 7.6, we can cover the open set $\mathcal{A}(k, \delta'_k)$ by countable many balls $B(x, r_x)$.

It suffices to show that the Hausdorff dimension of $B(x, r_x) \setminus \mathcal{A}(k+1, \delta'_{k+1})$ is at most k. Let $D_x(r, m)$, $m \in \mathbb{Z}^k$ denote the same cylindrical regions in $B(x, r_x)$ as defined in Lemma 7.15. It is straightforward to verify that the ball $B(x, r_x)$ can be covered by the covering $\{D_x(r,m)\}_{m \in \mathbb{Z}^k}$, with the number of non-empty sets in this covering bounded above by $(2r_x/(0.1r))^k$. For each non-empty $D_x(r, m)$, it follows by Lemma 7.15 that

$$\beta_{D_x(r,m)\setminus \mathcal{A}(k+1,\delta'_{k+1},r)}\left((\delta'_k)^{-1} r\right) \leq K(N, \delta'_k). \tag{7.47}$$

Note that the maximal cardinality of any r-separated subset of a set is bounded above by the sum of the maximal cardinalities of r-separated subsets over all non-empty sets in any covering of the set. Therefore, it follows that

$$\begin{aligned}\beta_{B(x,r_x)\setminus \mathcal{A}(k+1,\delta'_{k+1})}\left((\delta'_k)^{-1} r\right) &\leq \sum_{\substack{m \in \mathbb{Z}^k \\ D_x(r,m) \neq \emptyset}} \beta_{D_x(r,m)\setminus \mathcal{A}(k+1,\delta'_{k+1})}\left((\delta'_k)^{-1} r\right) \\ &\leq \sum_{\substack{m \in \mathbb{Z}^k \\ D_x(r,m) \neq \emptyset}} \beta_{D_x(r,m)\setminus \mathcal{A}(k+1,\delta'_k,r)}\left((\delta'_k)^{-1} r\right) \leq \left(\frac{2r_x}{0.1r}\right)^k K(N, \delta'_k). \end{aligned} \tag{7.48}$$

Thus, for any $\varepsilon > 0$, it follows from the inequality (7.48) that

$$\limsup_{r \to 0} \left((\delta'_k)^{-1} r\right)^{k+\varepsilon} \beta_{B(x,r_x)\setminus \mathcal{A}(k+1,\delta'_{k+1})}\left((\delta'_k)^{-1} r\right) = 0. \tag{7.49}$$

This implies that the rough dimension (see Section 2.2) of $B(x,r_x)\setminus\mathcal{A}(k+1,\delta'_{k+1})$ is at most k. Since the Hausdorff dimension is not greater than the rough dimension, we obtain that $\dim_H(B(x,r_x)\setminus\mathcal{A}(k+1,\delta'_{k+1}))\leq k$.

Thus, by covering $\mathcal{A}(k,\delta'_k)$ by at most countable many balls $B(x,r_x)$, it follows that

$$\dim_H(\mathcal{A}(k,\delta'_k)\setminus\mathcal{A}(k+1,\delta'_{k+1}))\leq k. \tag{7.50}$$

From the assumption of induction, it follows that

$$\begin{aligned}&\dim_H\left(X\setminus\mathcal{A}(k+1,\delta'_{k+1})\right)\\&\quad=\max\left\{\dim_H\left(\mathcal{A}(k,\delta'_k)\setminus\mathcal{A}(k+1,\delta'_{k+1})\right),\dim_H\left(X\setminus\mathcal{A}(k,\delta'_k)\right)\right\}\leq k.\end{aligned}\tag{7.51}$$

By induction, our claim follows. □

Based on Theorem 7.17, we can now deduce the following result of stratification of X.

Corollary 7.18. *Let $\delta<\delta_{n+1}$ be a small number. Let X be an n-dimensional S-concave Busemann concave space that satisfies local semi-convexity with $S\geq 1$ sufficiently close to 1. Then X admits a stratification $\{X_k\}_{k=0}^n$ such that X is the disjoint union of $\{X_k\}_{k=0}^n$ and $\dim_H(X_k)\leq k$.*

Proof. First, observe that $\mathcal{A}(n+1,\delta)$ is empty. Indeed, if it were not, then by the self-improvement property of strainers (Lemma 5.17), X would contain $(n+1,\delta')$-strained points for any $\delta'>0$. This would imply that the strainer number of X is greater than n, contradicting the assumption that X is n-dimensional. Therefore, it is clear that X can be decomposed into the disjoint union of sets $X_k:=\mathcal{A}(k,\delta)\setminus\mathcal{A}(k+1,\delta)$, $k=0\ldots,n-1$ and $X_n:=\mathcal{A}(n,\delta)$. Now the claim follows from Theorem 7.17 that $\dim_H(X_k)\leq\dim_H(X\setminus\mathcal{A}(k+1,\delta))\leq k$ for $k=0,\ldots,n-1$ and $\dim_H(X_n)=n$. □

Appendix A. Criterion for open maps

In this appendix, we provide a proof of the criterion for ε-open maps. Our proof follows a similar argument as [29, Lemma 8.1], based on the classical result of Lytchak [28, Lemma 4.1].

Lemma A.1. *Let $f:X\to Y$ be a locally Lipschitz map from a locally complete metric space X to a geodesic space Y. Suppose there exists an $\varepsilon>0$ such that the following holds: for every $x\in X$ and $v\in Y\setminus\{f(x)\}$ sufficient close to $f(x)$, there exists $y\in X$ such that*

$$|f(y)v|-|f(x)v|\leq-\varepsilon|xy|. \tag{A.1}$$

Then f is an ε'-open map for any $0<\varepsilon'<\varepsilon$.

Proof. Let $x\in X$ and $r>0$ be sufficiently small such that $\bar{B}(x,\varepsilon^{-1}r)$ is complete.

We first show that $B(f(x),s)\subset f(\bar{B}(x,\varepsilon^{-1}s))$ for any $0<s\leq r$. Let $v\in B(f(x),s)\setminus\{f(x)\}$ and $l_v:=|f(x)v|<s$. Let $h:X\to Y$ be the function defined as $h(z):=l_v-|f(z)v|$. Note that $h(x)=0$ and that in order to find $y\in\bar{B}(x,\varepsilon^{-1}s)$ such that $f(y)=v$, it suffices to find y such that $h(y)=l_v$. In order to apply [28, Lemma 4.1], we need to show that $\limsup_{z'\to z}\frac{h(z')-h(z)}{|z'z|}\geq\varepsilon$ for any $z\in\bar{B}(x,\varepsilon^{-1}s)$. Indeed, for any $z\in\bar{B}(x,\varepsilon^{-1}s)$, let γ be a unit-speed geodesic connecting $f(z)$ and v. For a sequence of points $w_n\in\gamma$ converging to $f(z)$, it follows from the assumption (A.1) that there exists a sequence $z'_n\in X$ such that

$$|f(z'_n)w_n|-|f(z)w_n|\leq-\varepsilon|z'_nz|. \tag{A.2}$$

It follows that $|z'_nz|\leq\varepsilon^{-1}|f(z)w_n|\to 0$. Furthermore, since w_n is on the geodesic connecting $f(z)$ and v, it follows that

$$\begin{aligned}|f(z'_n)v|-|f(z)v|&=|f(z'_n)v|-|f(z)w_n|-|w_nv|\\&\leq|f(z'_n)w_n|-|f(z)w_n|\leq-\varepsilon|z'_nz|.\end{aligned}\tag{A.3}$$

This implies that we can find a sequence of point z'_n converging to z such that $h(z'_n) - h(z) \geq \varepsilon|z'_n z|$. Thus, it follows that

$$|\nabla_z h| := \limsup_{z' \to z} \frac{h(z') - h(z)}{|z'z|} \geq \limsup_{n \to \infty} \frac{h(z'n) - h(z)}{|z'_n z|} \geq \varepsilon. \tag{A.4}$$

Thus, we can apply [28, Lemma 4.1] to the function h. Since $\bar{B}(x, \varepsilon^{-1}s) \subset \bar{B}(x, \varepsilon^{-1}r)$ is complete, it follows from [28, Lemma 4.1] that for any $0 < \varepsilon' < \varepsilon$, we can find a $z \in \bar{B}(x, \varepsilon^{-1}s)$ such that $h(z) = \varepsilon'\varepsilon^{-1}s$. Therefore, by choosing $\varepsilon' = \varepsilon l_v/s < \varepsilon$, we obtain that there exists $z \in \bar{B}(x, \varepsilon^{-1}s)$ such that $h(z) = l_v$, which implies that $v = f(z)$. This implies that $B(f(x), s) \subset f(\bar{B}(x, \varepsilon^{-1}s))$.

We now show that f is ε'-open for any $0 < \varepsilon' < \varepsilon$ in the sense of Definition 5.11. Let $0 < \varepsilon' < \varepsilon$ be fixed. Let $0 < r' < r$ be sufficiently small such that $\bar{B}(x, (\varepsilon')^{-1}r') \subset \bar{B}(x, \varepsilon^{-1}r)$ is complete. For any $v \in B(f(x), r')$, let $s := |f(x)v| < r'$. Then for any $n \in \mathbb{N}$ large enough, since $v \in B(f(x), s + 1/n) \subset f(\bar{B}(x, \varepsilon^{-1}(s + 1/n)))$, it follows that we can find a sequence of points z_n such that $v = f(z_n)$ and that

$$|xz_n| \leq \varepsilon^{-1}\left(|f(x)v| + \frac{1}{n}\right). \tag{A.5}$$

By Choosing $n \in \mathbb{N}$ sufficiently large, it holds that

$$|xz_n| \leq \varepsilon^{-1}\left(|f(x)v| + \frac{1}{n}\right) \leq (\varepsilon')^{-1}|f(x)v|. \tag{A.6}$$

Thus, we have shown that for any $x \in X$, there exists $r' > 0$ such that $\bar{B}(x, (\varepsilon')^{-1}r')$ is complete, and that for any $v \in B(f(x), r')$, we can find a $z \in X$ such that $f(z) = v$ and

$$\varepsilon'|xz| \leq |f(x)v|. \tag{A.7}$$

This implies that f is an ε'-open map.

□

DECLARATION

The authors declare that there is no conflict of interest, and this paper has no associated data.

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