AHLFORS REGULARITY IN CARNOT-CARATHÉODORY SPACES

LUCA CAPOGNA AND NICOLA GAROFALO

ABSTRACT. We study the relationship between the geometry of C^2 hypersurfaces in a Carnot-Carathéodory space and the Ahlfors regularity of the corresponding perimeter measure. In addition, we find a sharp geometric condition on the nature of the characteristic set which implies the 1-Ahlfors regularity. As a corollary, we have that all C^2 hypersurfaces in a Carnot-Carathéodory space of rank two (or less) are 1−Ahlfors regular. In particular, in the notation used by David and Semmes [DS2], in a Carnot group of step 2 with homogeneous dimension Q, all $C^{1,1}$ hypersurfaces are (Ahlfors) regular of dimension $Q-1$.

1. **Introduction.**

A Borel measure μ, on a complete metric space (S,d) is said to be $s-$Ahlfors regular, if there exist two positive constants s and C, such that

$$C^{-1}\, r^s \ \le\ \mu(B(x,r))\ \le\ C\, r^s\ , \tag{1.1}$$

for all $x \in X$ and $0 < r < diam(X)$. Here $B(x,r)$ denotes the metric ball centered at x and with radius $r > 0$. If a complete metric space admits a $s-$Ahlfors regular measure, then the $s-$dimensional Hausdorff measure is $s-$Ahlfors regular as well, see [DS2], Lemma 1.6. David and Semmes call such spaces: *(Ahlfors) regular of dimension s*. As an important example consider $S = \partial\Omega$ a Lipschitz hypersurface (boundary of an open set $\Omega \subset \mathbb{R}^n$), and let d be the Euclidean distance. If we denote by $P(\Omega;\cdot)$ the perimeter measure in the sense of De Giorgi (see [DG1], [DG2]) relative to Ω, then $P(\Omega;\cdot)$ is $(n-1)-$Ahlfors regular (see for instance [AFP]).

The main goal of this paper is to establish the Ahlfors regularity of the intrinsic perimeter for hypersurfaces in a class of metric spaces known as *Carnot-Carathéodory (CC) spaces.* We recall that a CC space is a Riemannian manifold (M^n, g) endowed with a distance d different from the Riemannian one $d_{\mathcal{R}}$ generated by the metric tensor g. Such distance d is the control metric associated with a given subbundle HM^n of the tangent bundle TM^n. If $X = \{X_1, ..., X_m\}$ is a smooth distribution of vector fields (locally) describing HM^n, then the basic assumption is that X satisfy Hörmander's finite rank condition [H]

$$rank\ Lie[X_1, ..., X_m]\ \equiv\ n\ . \tag{1.2}$$

Condition (1.2) is equivalent to saying that at every point $x \in M^n$, the vector fields $X_1, ..., X_m$ and their iterated brackets $[X_i, X_j]$, $[[X_i, X_j], X_k]$,..., up to a certain order $r \in \mathbb{N}$, span the tangent space T_xM^n (in the rest of the paper we will refer to such spaces also as CC spaces of rank r). Under such hypothesis the theorem of Chow-Rashevsky [Ch], [Ra] guarantees, if

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M^n is connected, that for any $x, y \in M^n$ there exists at least one absolutely continuous path connecting x to y whose tangent vector $\gamma'(t)$ lies in $H_{\gamma(t)}M^n$, for every t at which $\gamma'(t)$ exists. Minimizing on the lengths of all such paths, one obtains the CC distance $d(x,y)$. The class of CC spaces encompasses of course all Riemannian manifolds. Less trivial examples include Euclidean $\mathbb{R}^n$ with a system of C^∞ vector fields satisfying (1.2), but also the Gromov-Hausdorff limit of some sequences of Riemannian manifolds, see [Gro]. Moreover, tangent spaces of CC spaces are themselves CC spaces endowed with a special nonabelian structure. They are graded nilpotent Lie groups, also known as *Carnot groups*, or quotient spaces of Carnot groups.

Given a CC space we will denote by $B(x,r) = \{y \in M^n \mid d(x,y) < r\}$ the open ball centered at x with radius r in the control metric d. If v_g indicates the volume form on M^n, attached to the metric tensor g, we let $|E| = \int_E dv_g$ denote the ordinary Lebesgue measure of the measurable set $E \subset M^n$. We recall that the Lebesgue measure $|B(x,r)|$ of the CC balls was studied in a fundamental paper by Nagel, Stein and Wainger [NSW]. Their main result states that for every bounded set $K \subset M^n$ there exist $C, R_o > 0$, depending on K such that for every $x \in K$ and $r < R_o$ one has

$$C\ \Lambda(x,r)\ \le\ |B(x,r)|\ \le\ C^{-1}\ \Lambda(x,r)\ . \tag{1.3}$$

Here, the Nagel-Stein-Wainger polynomial

$$\Lambda(x,r)\ =\ \sum_I\ |a_I(x)|\ r^{d(I)}\ , \tag{1.4}$$

is defined as follows: For every $x \in M^n$ denote by $Y_1, ..., Y_l$ the collection of the X_j's and of those commutators which are needed to generate T_xM^n. A "degree" is assigned to each Y_i, namely the corresponding order of the commutator. If $I = (i_1, ..., i_n)$, $1 \le i_j \le l$, is a n-tuple of integers, one defines $d(I) = \sum_{j=1}^n deg(Y_{i_j})$, and $a_I(x) = \det\ (Y_{i_1}, ..., Y_{i_n})$.

As one can easily infer from (1.4), apart from special CC structures (for instance Carnot groups), the Lebesgue measure is not $s-$Ahlfors regular, for any choice of s, in the sense of (1.1). This observation leads us to introduce a slightly different notion of Ahlfors regularity: given a CC space (M^n, g, d) denote by $\mathcal{B}$ the class of Borel measures on it.

Definition 1.1. *Given $s \ge 0$, a measure $\mu \in \mathcal{B}$ will be called an* upper s-Ahlfors measure with respect to the CC distance *if there exist $M, R_o > 0$, such that for $x \in M^n$, $0 < r \le R_o$, one has*

$$\mu(B(x,r)) \le\ M\ \frac{|B(x,r)|}{r^s}. \tag{1.5}$$

We will say that μ is a lower s-Ahlfors measure, *if for some $M, R_o > 0$ one has instead for x and r as above*

$$\mu(B(x,r))\ \ge\ M^{-1}\ \frac{|B(x,r)|}{r^s}. \tag{1.6}$$

When μ is both an upper and lower s-Ahlfors measure, then we say that it is a s-Ahlfors measure on M^n with respect to the CC distance.

Next, we recall the definition of perimeter measure in a CC space, see [CDG1] and [GN1]). Let M^n be a CC space with respect to a given subbundle $HM^n \subset TM^n$ which we assume generated by a system of smooth vector fields $X = \{X_1, ..., X_m\}$. Given an open set $\Omega \subset M^n$, we denote by $\mathcal{F}(\Omega)$ the set of all vector fields $\zeta \in C_o^1(\Omega, HM^n)$ such that $|\zeta| \leq 1$. If $f \in L^1(\Omega)$, then the X-variation of f is defined by

$$Var_X(f;\Omega) \ = \ \sup_{\zeta\in\mathcal{F}(\Omega)} \int_\Omega f \ \sum_{j=1}^m X_j^* \zeta_j \ dv_g \ .$$

Given a measurable set $E \subset \mathbb{R}^n$ we define the *X-perimeter* of E with respect to Ω as

$$P_X(E,\Omega) \ = \ Var_X(\chi_E;\Omega) \ ,$$

where χ_E denotes the characteristic function of E. We also refer the reader to the papers [BM] and [FSS1] where related definitions of variation and perimeter were independently set forth. In the Euclidean geometry, if Ω is a smooth set then the perimeter is equivalent to the surface measure. The situation is quite different in the CC case. Let $\Omega = \{x \in M^n | \ \phi(x) < 0\}$, where $\phi : M^n \to \mathbb{R}$ is a C^1 function. The horizontal gradient of ϕ is given by $X\phi = X_1\phi X_1 + ... + X_m\phi X_m$. Define a new measure supported on $\partial\Omega$ by letting for every Borel set $E \subset M^n$

$$\mu(E) \ \stackrel{def}{=} \ \int_{E\cap\partial\Omega} |X\phi| \ d\sigma \ , \tag{1.7}$$

where $\sigma = H_{n-1}\lfloor\partial\Omega$, and as before H_{n-1} indicates the $(n-1)$-dimensional Hausdorff measure on M^n constructed with the Riemannian distance $d_{\mathcal{R}}$. A key fact, see Theorem 5.8 in [DGN1], is the existence of $C = C(\Omega) > 0$ such that for every $g \in \partial\Omega$ and $r > 0$ one has

$$C \ \mu(B(x,r)) \ \leq \ P_X(\Omega; B(x,r)) \ \leq \ C^{-1} \ \mu(B(x,r)) \ . \tag{1.8}$$

It is clear that when $M^n = \mathbb{R}^n$, if $X = \{\frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_n}\}$ is the standard basis of $\mathbb{R}^n$, then $d(x,y) = |x-y|$, and $d\mu = |D\phi| d\sigma$ is equivalent to $d\sigma$. In the sub-Riemannian case, however, the *angle function* $|X\phi|$ vanishes on a subset of $\partial\Omega$, the so-called *characteristic set* of $\partial\Omega$. The existence of characteristic points makes controlling the measure $d\mu$, and thereby $P_X(\Omega;\cdot)$, a very delicate task.

In this paper we will study the interplay between the geometry of a minimally smooth hypersurface $S = \partial\Omega \subset M^n$ in a CC space and the $s-$Ahlfors regularity of its perimeter measure $P_X(\Omega;\cdot)$. We will, in fact, mainly focus on the following $1-$Ahlfors regularity

$$M^{-1} \ \frac{|B(x,R)|}{R} \ \leq \ P_X(\Omega; B(x,R)) \ \leq \ M \ \frac{|B(x,R)|}{R} \ , \tag{1.9}$$

due to the relevance of this property in the study of boundary value problems and in geometric measure theory. In a general Carnot group one should not expect such property. In fact, the restriction of the vector fields $X_1, ..., X_m$ to the boundary of a smooth domain gives rise to a general CC metric on such space, and the Nagel-Stein-Wainger estimates (1.3), (1.4) show that the best one can hope for is some kind of $s-$Ahlfors regularity, where the exponent $s = s(x)$ is an integer-valued function defined on the boundary of the domain. In order to describe the function $s(x)$ we introduce the notion of *type*. Henceforth, to distinguish the model setting of a Carnot group $\boldsymbol{G}$ from that of a general CC manifold M^n, we will indicate points in $\boldsymbol{G}$ with the letters g, g_o, etc., whereas points in M^n will be denoted by x, x_o, etc. With this in mind, if $\boldsymbol{G}$ is a Carnot group, and

$$\Omega \ = \ \{g \in \boldsymbol{G} \ \mid \phi(g) < 0\} \tag{1.10}$$

is a C^1 bounded domain in $\boldsymbol{G}$, then we define the *type* of a point $g_o \in \partial\Omega$ as the smallest order of commutators which are transversal to the $\partial\Omega$ at g_o, see Definition 3.1. We stress that this definition depends only on the first order Taylor polynomial at $g_o \in \partial\Omega$ of the defining function ϕ. It will be helpful to the reader to keep in mind the following example. If $\boldsymbol{G}$ has step r, with Lie algebra $\mathfrak{g} = V_1 \oplus ... \oplus V_r$, let $m_j = dim(V_j)$ and denote by $\xi_j = (x_{j,m_1}, ..., x_{j,m_j})$, $j = 1, ..., r$, the projection of the exponential coordinates onto the j-th layer of the Lie algebra of $\mathfrak{g}$, see Section 2.1. For a fixed $j \in \{1, ..., r\}$ consider the "hyperplane" passing through the group identity e

$$\Pi_j \;=\; \{x_{j,m_s} \;=\; 0\} \;, \tag{1.11}$$

where $s \in \{1, ..., m_j\}$ is fixed. An elementary calculation shows that the point e is of type j. Thus for instance for any of the m_1 hyperplanes Π_1 the identity is of type 1, and therefore it is non-characteristic (one can easily recognize that, in fact, such hyperplanes do not possess any characteristic point). This example shows that the type of a point can be any integer ranging from 1 to the step r of the group.

Having introduced the notion of type in a Carnot group, we now state in this setting our main results regarding the Ahlfors regularity of the perimeter measure. The reason for starting with this situation is twofold. First, the Ahlfors estimates in Carnot groups are more precise than those in a general CC manifold. Secondly, the analysis of this special situation constitutes the backbone of the general case.

Theorem 1.2. *Let $\boldsymbol{G}$ be a Carnot group. Consider a bounded, open set Ω as in* (1.10)*, with $\phi \in C^{1,1}(\boldsymbol{G})$. For every $g_o \in \partial\Omega$, there exist $M = M(\boldsymbol{G}, X, \Omega, g_o) > 0$ and $R_o = R_o(\boldsymbol{G}, X, \Omega, g_o) > 0$ depending continuously on g_o, such that for for any $0 < R < R_o$ one has*

$$P_X(\Omega; B(g_o, R)) \;\le\; M\; \frac{|B(g_o, R)|}{R^{s(g_o)}} \;, \tag{1.12}$$

with

$$s(g_o) = \begin{cases} type(g_o) - 1, & \text{if } g_o \text{ is characteristic} \;, \\ 1, & \text{if } g_o \text{ is non-characteristic} \;. \end{cases} \tag{1.13}$$

Furthermore, if $\phi \in C^2(\boldsymbol{G})$, there exist $M = M(\boldsymbol{G}, X, \Omega, g_o) > 0$ and $R_o = R_o(\boldsymbol{G}, X, \Omega, g_o) > 0$ as above such that if $g_o \in \partial\Omega$ is of type ≤ 2, then we also have

$$P_X(\Omega; B(g_o, R)) \;\ge\; M^{-1}\; \frac{|B(g_o, R)|}{R} \;, \tag{1.14}$$

for $0 < R < R_o$.

More in general, we prove that Theorem 1.2 continues to be valid in a CC space M^n, when the subbundle HM^n is generated by a system of free vector fields, see Theorems 6.10 and 6.8. If Ω is a domain whose boundary points are all of type ≤ 2, then (1.13) gives $s(g_o) \equiv 1$ on $\partial\Omega$. We thus obtain the following important corollary of Theorem 1.2.

Corollary 1.3. *Let Ω be a C^2 domain in a Carnot group $\boldsymbol{G}$. If every point $g_o \in \partial\Omega$ is of type ≤ 2, then the perimeter $P_X(\Omega; \cdot)$ is a $1-$Ahlfors measure.*

In particular, the measure $P(\Omega;\cdot)$ is doubling, i.e., we have for any $g_o \in \partial\Omega$ and $0 < R < R_o/2$, $P_X(\Omega; B(g_o, 2R)) \leq C_1\, P_X(\Omega; B(g_o, R))$, with $C_1 = 2^Q M^2$.

Hereafter, the number Q will denote the so-called homogeneous dimension of the group $\boldsymbol{G}$, see (2.10). Regarding the sharpness of our assumptions, we recall that in the Heisenberg group $\mathbb{H}^n$, for any $0 < \alpha < 1$ there exist $C^{1,\alpha}$ domains for which the lower 1-Ahlfors regularity of $P_X(\Omega;\cdot)$ in Theorem 1.2 is not true, see Section 7.4 in [DGN2]. In Theorem 4.10 we show an alternative approach to the lower Ahlfors regularity which yields the $1-$Ahlfors regularity for $C^{1,1}$ hypersurfaces in Carnot groups of step two. Our results are also sharp in terms of the "*type condition*". In fact, in section 5 we construct an example of a C^∞ domain of type 3 in a Carnot group of step 3 for which the upper $1-$Ahlfors regularity of the X-perimeter fails.

Returning to Theorem 1.2, it should be obvious from its statement that there is a marked discrepancy between upper and lower Ahlfors regularity of the X-perimeter. The latter is subtler than the former, as one has to control the zeroes of the *angle function* $|X\phi|$, where ϕ is as in (1.10). What complicates matters even further is the fact that while the upper Ahlfors regularity is influenced only by the type of the base point $g_o \in \partial\Omega$, and the latter notion depends in turn only on the first-order term in the Taylor expansion of ϕ at g_o, the lower Ahlfors regularity not only depends on the type of g_o, but possibly also on terms of order two or higher in the Taylor expansion. To make this precise we mention that in Section 4 we show that if $\Pi \subset \boldsymbol{G}$ is *any* "hyperplane" passing through the group identity $e \in \boldsymbol{G}$, and Ω denotes one of the two half-spaces with boundary Π, then there exists a constant $M > 0$ such that the 1-Ahlfors estimate (1.9) holds at $g = e$ for any $R > 0$. Since as we saw above for the hyperplanes Π_j in (1.11) the type of the identity can be any integer ranging from 1 to the step of the group, it is clear that the lower Ahlfors estimates cannot depend only on the first-order terms in the Taylor expansion of ϕ. As a consequence of these remarks, the lower estimates cannot hold with the same exponent $s(\cdot)$ as in the upper ones in (1.13).

Having discussed the model setting of Carnot groups, we now turn to that of a general CC manifold M^n. As we mentioned, our focus is on the 1-Ahlfors regularity of the X-perimeter measure on a hypersurface, a detailed study of the lower $s-$Ahlfors regularity being deferred to a forthcoming study. Given a CC manifold with generating distribution $\{X_1, ..., X_m\}$, consider a C^1 domain $\Omega = \{x \in M^n \mid \phi(x) < 0\}$. We say that a point $x_o \in \partial\Omega$ is of type ≤ 2 if either there exists $j_o \in \{1, ..., m\}$ such that $X_{j_o}\phi(x_o) \neq 0$, i.e., x_o is non-characteristic, or there exist indices $i_o, j_o \in \{1, ..., m\}$ such that $[X_{i_o}, X_{j_o}]\phi(x_o) \neq 0$. We say that Ω is of type ≤ 2 if every point $x_o \in \partial\Omega$ is of type ≤ 2. It is important to stress that when M^n is a CC space of rank $r \leq 2$, then every C^1 domain is automatically of type ≤ 2. Let in fact $X = \{X_1, ..., X_m\}$ be a collection of vector fields as in (1.2), and suppose that for some $x_o \in \partial\Omega$ one has $X_i\phi(x_o) = 0$, for every $i = 1, ..., m$. If we also had $[X_i, X_j]\phi(x_o) = 0$ for $i, j = 1, ..., m$, then we would conclude for the Riemannian gradient $\nabla\phi(x_o)$ that it does not belong to the $span\{X_i\phi(x_o), [X_i, X_j]\phi(x_o) \mid i, j = 1, ..., m\}$. But this contradicts the fact that the vector fields X_i and their first commutators generate the tangent space $\mathbb{R}^n$.

Having made this point we can now state our main results. We begin with the upper Ahlfors regularity of the perimeter measure.

Theorem 1.4. *Let (M^n, d) be a CC space and consider a bounded $C^{1,1}$ domain $\Omega \subset M^n$. For every point $x_o \in \partial\Omega$ of type ≤ 2 there exist $M = M(\Omega, x_o) > 0$ and $R_o = R_o(\Omega, x_o) > 0$, depending continuously on x_o, such that for any $0 < r < R_o$ one has*

$$P_X(\Omega; B(x_o, r)) \;\leq\; M \;\frac{|B(x_o, r)|}{r} \;. \tag{1.15}$$

We actually establish a slightly stronger estimate, namely: If $\Omega = \{x \in M^n |\ \phi(x) < 0\}$, and σ denotes the surface measure on $\partial\Omega$, then

$$\left(\sup_{B(x_o,r)\cap\partial\Omega} |X\phi| \right) \sigma(B(x_o, r) \cap \partial\Omega) \;\leq\; M \;\frac{|B(x_o, r)|}{r} \;, \tag{1.16}$$

for any $0 < r < R_o$.

A compactness argument and the analysis of special examples yields the following global version of the theorem.

Theorem 1.5. *Let (M^n, d) be a CC space and consider a bounded $C^{1,1}$ domain $\Omega \subset M^n$ of type ≤ 2. There exist $M = M(\Omega) > 0$ and $R_o = R_o(\Omega) > 0$ such that for any $x_o \in \partial\Omega$ and $0 < r < R_o$, one has that* (1.15) *holds with the uniform constants M and R_o. Moreover, there exist CC spaces (M^n, d) and smooth bounded domains $\Omega \subset M^n$ with points of type* 3 *or higher, for which* (1.15) *does not hold.*

In the next theorem we generalize the second part of Theorem 1.2 to arbitrary sub-Riemannian manifolds.

Theorem 1.6. *Let (M^n, d) be a CC space and consider a bounded C^2 domain $\Omega \subset M^n$. For every point $x_o \in \partial\Omega$ of type ≤ 2 there exist $M = M(\Omega, x_o) > 0$ and $R_o = R_o(\Omega, x_o) > 0$ depending continuously on x_o, such that for any $0 < r < R_o$, one has*

$$P_X(\Omega; B(x_o, r)) \;\geq\; M^{-1} \;\frac{|B(x_o, r)|}{r} \;. \tag{1.17}$$

A compactness argument then yields the global version.

Theorem 1.7. *Let (M^n, d) be a CC space and consider a bounded C^2 domain $\Omega \subset M^n$ of type ≤ 2. There exist $M = M(\Omega) > 0$ and $R_o = R_o(\Omega) > 0$ such that for any $x_o \in \partial\Omega$ and $0 < r < R_o$, one has that* (1.17) *holds with the uniform constants M and R_o.*

Corollary 1.8. *The X-perimeter measure $P_X(\Omega; \cdot)$ of a C^2 domain of type ≤ 2 in a CC space is a* 1*-Ahlfors measure. In particular, such property holds generically for C^2 domains in a CC manifold of rank* 2.

Define $\Delta_2 \;=\; \{x_o \in \partial\Omega \mid\ type(x_o) \;\leq\; 2\}$. Notice that the closed set $\partial\Omega \setminus \Delta_2$, is essentially the set where the Ahlfors estimates fail. Such set is small in the following sense:

$$H_{n-1}(\partial\Omega \setminus \Delta_2) \;=\; 0 \;. \tag{1.18}$$

When M^n is a Carnot group the following stronger information is available

$$\mathcal{H}^{Q-1}(\partial\Omega \setminus \Delta_2) \ = \ 0 \ . \tag{1.19}$$

Here we denote by H_s the s-dimensional Hausdorff measure constructed with the Riemannian distance $d_{\mathcal{R}}$ of M^n, and with the notation $\mathcal{H}^s$ we indicate instead the s-dimensional Hausdorff measure constructed with the CC distance d. Equation (1.18) is essentially due to Derridj [De]. Although he actually proved that the complement of the characteristic set has zero H_{n-1}-dimensional measure for C^∞ domains, his ideas can be adapted to cover the case of C^2 domains. Equation (1.19) instead, follows from the recent work of Magnani [Ma].

Theorems 1.2, Corollary 1.3, 1.5 and 1.7 find numerous applications to the development of function spaces and potential theory on lower dimensional manifolds in CC spaces [DGN1], [DGN2], Dirichlet and Neumann problems for sub-Laplacians [CGN1], [CGN2], [CGN3], [DGN2], [DGN3], geometric measure theory in CC spaces [DGN4], [DGN5].

The paper is structured as follows. After recalling a few preliminary results and definitions in Section 2, we prove Theorem 1.2 in Sections 3 and 4. The sharpness of the type condition is established with examples in Section 5. In Section 6 we implement a technique of Rothschild and Stein [RS] in which a system of free Hörmander vector fields can be approximated locally by a nilpotent Lie algebra. The perimeter estimates for free Hörmander vector fields will then follow from the results in sections 3 and 4, and from the study of the higher order error term in the Rothschild-Stein approximation. In section 7, we complete the proof of Theorems 1.4 and 1.6, using another fundamental technique introduced in [RS], the *lifting* of a general system $X_1, ..., X_m$ of Hörmander vector fields to a system $\tilde{X}_1, ..., \tilde{X}_m$ of free vector fields. Finally, in section 8 we analyze the connection between the 1-Ahlfors regularity of the X-perimeter and boundary value problems for sub-Laplacians. We show that the former property implies the regularity of the relevant domain with respect to the Dirichlet problem. This fact, combined with some examples of Hansen and Hueber [HH], gives another (indirect) proof of the impossibility of the 1-Ahlfors estimates when the domain is of type ≥ 3.

Regarding previous results on this subject, we mention that when M^n is the Heisenberg group, the upper 1-Ahlfors regularity of $P_X(\Omega;\cdot)$ was established in [DGN1]. The same result was subsequently generalized to Carnot groups of step 2 in [CGN2]. The lower 1-Ahlfors regularity for C^2 domains in Carnot groups of step 2 has been recently established in [DGN2].

We also recall that for C^∞ domains in a CC space, Monti and Morbidelli [MM1] have recently proved the $1-$Ahlfors regularity of the ordinary surface measure $d\sigma$ away from characteristic points. The approach in [MM1], however, substantially differs from ours and fails to work in a neighborhood of characteristic points. For other regularity results in this vein, see [MM2], and [MM3].

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2. Carnot-Carathéodory spaces

In this section we recall the definition of CC manifolds and of their tangent spaces, the class of Carnot groups. The relation between CC spaces and Carnot groups is described in a series of papers by Rothschild and Stein [RS], Folland [F], Nagel, Stein and Wainger [NSW], and Sanchez-Calle [SC]. One should also see [Mi], [F], [FS], [FSC], [Str], [P], [Be], [Gro], and [Mon].

Let (M^n, g) be a smooth Riemannian manifold, with $n \geq 3$, with volume form dv_g. Denote by $d_{\mathcal{R}}$ the Riemannian distance on M^n, and by $|E| = \int_E dv_g$ the standard Lebesgue measure of a measurable set $E \subset M^n$. We consider a given subbundle $HM^n \subset TM^n$ of the tangent bundle. Let $X = \{X_1, ..., X_m\}$ be a system of C^∞ vector fields which locally generate HM^n, and consider the system of differential equations

$$\gamma' \ = \ \sum_{j=1}^{m} u_j(t)\ X_j(\gamma)\ , \tag{2.1}$$

where the *control* $u = (u_1, ..., u_m)$ is assumed to belong to $L^1([a,b], \mathbb{R}^m)$. If the path $\gamma : [a,b] \to M^n$ solves the above system and if $\gamma(a) = x$, $\gamma(b) = y$, then one says that the control u steers the system from the state x to the state y. The length of γ is defined by

$$l(\gamma) \ = \ \int_a^b \sqrt{u_1(t)^2 \ + \ ... \ + \ u_m(t)^2}\ dt\ .$$

Next, for $x \in M^n$ and $v \in T_xM^n$ we let

$$h_x(v) \ = \ \inf\ \{||u||^2 = u_1^2 + ... + u_m^2 \mid u_1X_1(x) + ... + u_mX_m(x) = v\}\ .$$

If v lies outside H_xM^n, then one lets $h_x(v) = +\infty$. In this way, on each section H_xM^n of the subbundle $HM^n \subset TM^n$ we have defined a quadratic form h_x. The *sub-Riemannian metric* associated with the subbundle HM^n is given by the assignment $x \to h_x$. We set

$$||v||_{H,x} \ = \ \sqrt{h_x(v)}\ ,$$

and define the *horizontal length* of an absolutely continuous path $\gamma : [a,b] \to M^n$ as

$$l_H(\gamma) \ = \ \int_a^b ||\gamma'(t)||_{H,\gamma(t)}\ dt\ .$$

An absolutely continuous path γ is called *horizontal* (or *controlled*), if it satisfies (2.1) for a measurable control $u(t) = (u_1(t), ..., u_m(t))$. Given an open set $\Omega \subset M^n$, and two points $x, y \in \Omega$, we denote by $\mathcal{H}_\Omega(x,y)$ the collection (possibly empty) of all horizontal paths $\gamma : [a,b] \to \Omega$ joining x to y.

The accessibility Theorem of Chow-Rashevsky [Ch], [Ra] states that if at every $x \in M^n$ the system $X = \{X_1, ..., X_m\}$ which locally describes HM^n satisfies the finite rank condition (1.2), then if $\Omega \subset M^n$ is connected one has $\mathcal{H}_\Omega(x,y) \neq \varnothing$ for every $x, y \in \Omega$. This basic result allows to define the *Carnot-Carathéodory* (or *control*) *distance* between x and y as

$$d_\Omega(x,y) \ = \ \inf\ \{l_H(\gamma) \mid \gamma \in \mathcal{H}_\Omega(x,y)\}\ .$$

When $\Omega = M^n$, we write $d(x,y)$ instead of $d_{M^n}(x,y)$. It is clear that $d(x,y) \leq d_\Omega(x,y)$, $x, y \in \Omega$, for every connected open set $\Omega \subset M^n$. In [NSW] it was proved that for every connected $\Omega \subset\subset M^n$ there exist $C, \epsilon > 0$ such that

$$C\ d_{\mathcal{R}}(x,y)\ \leq d_{\Omega}(x,y)\ \leq C^{-1}\ d_{\mathcal{R}}(x,y)^{\epsilon}, \qquad x,y \in \Omega. \tag{2.2}$$

This gives in particular

$$d(x,y)\ \leq\ C^{-1}\ d_{\mathcal{R}}(x,y)^{\epsilon}\ , \qquad x,y \in \Omega\ ,$$

and therefore

$$i : (M^n, d_{\mathcal{R}}) \to (M^n, d) \qquad \textit{is continuous}.$$

It is easy to see that also the continuity of the opposite inclusion holds [GN1], hence the Riemannian and the metric topologies are compatible.

The study of the Lebesgue measure of CC balls $B(x,r)$ was undertaken by Nagel, Stein and Wainger in their seminal paper [NSW]. We have already recalled in the introduction their fundamental contribution (1.3).

2.1. **Carnot groups.** Next, we describe in detail a special subclass of CC spaces which plays a basic role in the development of the general theory. A *Carnot group* of step r is a connected, simply connected Lie group $\boldsymbol{G}$ whose Lie algebra $\mathfrak{g}$ admits a stratification

$$\mathfrak{g} = V_1 \oplus ... \oplus V_r$$

which is r-nilpotent, i.e.,

$$[V_1, V_j]\ =\ V_{j+1}\ , \quad j = 1, ..., r-1\ , \qquad [V_j, V_r]\ =\ \{0\}\ , \quad j = 1, ..., r\ .$$

By these assumptions one immediately sees that any basis of the *horizontal layer* V_1 satisfies the finite rank condition (1.2). A trivial example of (an abelian) Carnot group is $\boldsymbol{G} = \mathbb{R}^n$, whose Lie algebra admits the trivial stratification $\mathfrak{g} = V_1 = \mathbb{R}^n$. The simplest non-abelian example is the Heisenberg group $\mathbb{H}^n$, already described in the introduction, whose Lie algebra is given by $\mathfrak{h}_n = V_1 \oplus V_2$, with $V_1 = \mathbb{C}^n$, $V_2 = \mathbb{R}$.

We assume that a scalar product $< \cdot, \cdot >$ is given on $\mathfrak{g}$ for which the $V_j's$ are mutually orthogonal. Let $\pi_j : \mathfrak{g} \to V_j$ denote the projection onto the j-th layer of $\mathfrak{g}$. Since the exponential map $exp : \mathfrak{g} \to \boldsymbol{G}$ is a global analytic diffeomorphism [V], we can define analytic maps $\xi_j : \boldsymbol{G} \to V_j$, $j = 1, ..., r$, by letting $\xi_j = \pi_j \circ \exp^{-1}$. As a rule, we will use letters g, g', g'', g_o for points in $\boldsymbol{G}$, whereas we will reserve the letters $\xi, \xi', \xi'', \xi_o, \eta$, for elements of the Lie algebra $\mathfrak{g}$. We let $m_j = dim\ V_j$, $j = 1, ..., r$, and denote by

$$n\ =\ m_1\ +\ ...\ +\ m_r$$

the topological dimension of $\boldsymbol{G}$. The notation $\{X_{j,1}, ..., X_{j,m_j}\}$, $j = 1, ..., r$, will indicate a fixed orthonormal basis of the $j-th$ layer V_j. For $g \in \boldsymbol{G}$, the projection of the *exponential coordinates* of g onto the layer V_j, $j = 1, ..., r$, are defined as follows

$$x_{j,s}(g)\ = < \xi_j(g), X_{j,s} >, \qquad s = 1, ..., m_j. \tag{2.3}$$

The vector $\xi_j(g) \in V_j$, $j = 1, ..., r$, will be routinely identified with the point

$$(x_{j,1}(g), ..., x_{j,m_j}(g))\ \in\ \mathbb{R}^{m_j}\ .$$

It will be easier to have a separate notation for the horizontal layer V_1. For simplicity, we set $m = m_1$, and let

$$X \ = \ \{X_1, \ \dots \ , \ X_m\} \ = \ \{X_{1,1}, \ \dots \ , \ X_{1,m_1}\} \ . \tag{2.4}$$

We indicate with

$$x_i(g) \ = < \xi_1(g), X_i >, \qquad i = 1, \dots, m \ , \tag{2.5}$$

the projections of the exponential coordinates of g onto V_1. Whenever convenient, we will identify $g \in \boldsymbol{G}$ with its exponential coordinates

$$x(g) \ \overset{def}{=} \ (x_1(g), \dots, x_m(g), x_{2,1}(g), \dots, x_{2,m_2}(g), \dots, x_{r,1}(g), \dots, x_{r,m_r}(g)) \ \in \ \mathbb{R}^n \ , \tag{2.6}$$

and we will ordinarily drop in the latter the dependence on g, i.e., we will write $g = (x_1, \dots, x_{r,m_r})$.

Each element of the layer V_j is assigned the formal degree j. Accordingly, one defines dilations on $\mathfrak{g}$ by the rule

$$\Delta_\lambda \xi \ = \ \lambda \ \xi_1 \ + \ \dots \ + \ \lambda^r \ \xi_r \ ,$$

provided that $\xi = \xi_1 + \dots + \xi_r \in \mathfrak{g}$. Using the exponential mapping $\exp : \mathfrak{g} \to \boldsymbol{G}$, these dilations are then tansferred to the group

$$\delta_\lambda(g) \ = \ \exp \ \circ \ \Delta_\lambda \ \circ \exp^{-1} \ g \ .$$

We will denote by

$$L_{g_o}(g) \ = \ g_o \ g \ , \qquad R_{g_o}(g) \ = \ g \ g_o \ , \tag{2.7}$$

respectively, the left- and right-translations on $\boldsymbol{G}$ by an element $g_o \in \boldsymbol{G}$. We continue to denote by X the corresponding system of left-invariant vector fields on $\boldsymbol{G}$ defined by

$$X_j(g) \ = \ (L_g)_*(X_j) \ , \qquad j = 1, \dots, m \ ,$$

where $(L_g)_*$ denotes the differential of L_g. The system X defines a basis for the so-called *horizontal subbundle* $\boldsymbol{HG}$ of the tangent bundle $\boldsymbol{TG}$. If we keep in mind that the integral curve of X_j passing through $g = exp(\xi)$ is given by $exp(\xi) \ exp(tX_j)$, then given a function $u : \boldsymbol{G} \to \mathbb{R}$, the action of X_j on u is specified by the equation

$$X_j u(g) \ = \ \lim_{t \to 0} \frac{u(g \ exp \ (tX_j)) \ - \ u(g)}{t} \ = \ \frac{d}{dt} \ u(g \ exp \ (tX_j))\Big|_{t=0} \ . \tag{2.8}$$

A similar formula holds for any left-invariant vector field. We now recall the Baker-Campbell-Hausdorff formula, see, e.g., sec.2.15 in [V],

$$exp(\xi) \ exp(\eta) \ = \ exp\bigg(\xi + \eta + \frac{1}{2} \ [\xi, \eta] + \frac{1}{12}\big\{[\xi, [\xi, \eta]] - [\eta, [\xi, \eta]]\big\} + \dots \bigg) \ , \tag{2.9}$$

where the dots indicate commutators of order four and higher. Using (2.9) we can express (2.8) using the coordinates (2.6), obtaining the following lemma.

Lemma 2.1. *For each $i = 1, ..., m$, and $g = (x_1, ..., x_{r,m_r})$, we have*

$$X_i = X_i(g) = \frac{\partial}{\partial x_i} + \sum_{j=2}^{r}\sum_{s=1}^{m_j} b^s_{j,i}(x_1, ..., x_{j-1,m_{(j-1)}}) \frac{\partial}{\partial x_{j,s}}$$
$$= \frac{\partial}{\partial x_i} + \sum_{j=2}^{r}\sum_{s=1}^{m_j} b^s_{j,i}(\xi_1, ..., \xi_{j-1}) \frac{\partial}{\partial x_{j,s}} ,$$

where each $b^s_{j,i}$ is a homogeneous polynomial of weighted degree $j-1$.

By weighted degree we mean that, as previously mentioned, the layer V_j, $j = 1, ..., r$, in the stratification of $\mathfrak{g}$ is assigned the formal degree j. Correspondingly, each homogeneous monomial $\xi_1^{\alpha_1}\xi_2^{\alpha_2}...\xi_r^{\alpha_r}$, with multi-indices $\alpha_j = (\alpha_{j,1}, ..., \alpha_{j,m_j})$, $j = 1, ..., r$, is said to have weighted degree k if

$$\sum_{j=1}^{r} j|\alpha_j| = \sum_{j=1}^{r} j \,(\sum_{s=1}^{m_j}\alpha_{j,s}) = k .$$

Throughout the paper we will indicate by dg the bi-invariant Haar measure on $\boldsymbol{G}$ obtained by lifting via the exponential map *exp* the Lebesgue measure on $\mathfrak{g}$. One easily checks that

$$(d\circ\delta_\lambda)(g) = \lambda^Q\, dg, \qquad \text{where} \quad Q = \sum_{j=1}^{r} j\, dim(V_j). \tag{2.10}$$

The number Q, called the *homogeneous dimension* of $\boldsymbol{G}$, plays an important role in the analysis of Carnot groups. In the non-abelian case $r > 1$, one clearly has $Q > n$.

We denote by $d(g, g')$ the *CC distance* on $\boldsymbol{G}$ associated with the system X. It is well-known that $d(g, g')$ is equivalent to the *gauge pseudo-metric* $\rho(g, g')$ on $\boldsymbol{G}$, i.e., there exists a constant $C = C(\boldsymbol{G}) > 0$ such that

$$C\ \rho(g, g') \le d(g, g') \le C^{-1}\ \rho(g, g'), \qquad g, g' \in \boldsymbol{G}, \tag{2.11}$$

see [NSW], [VSC]. The pseudo-distance $\rho(g, g')$ is defined as follows. Let $|\cdot|$ denote the Euclidean distance to the origin on $\mathfrak{g}$. For $\xi = \xi_1 + \cdots + \xi_r \in \mathfrak{g}$, $\xi_i \in V_i$, one lets

$$|\xi|_{\mathfrak{g}} = \left(\sum_{j=1}^{r} |\xi_j|^{2r!/j}\right)^{2r!}, \qquad |g|_{\boldsymbol{G}} = |\exp^{-1}\ g|_{\mathfrak{g}}, \qquad g \in \boldsymbol{G}, \tag{2.12}$$

and defines

$$\rho(g, g') = |g^{-1}\ g'|_{\boldsymbol{G}}. \tag{2.13}$$

Both d and ρ are invariant under left-translations

$$d(L_g(g'), L_g(g'')) = d(g', g'') , \qquad \rho(L_g(g'), L_g(g'')) = \rho(g', g'') . \tag{2.14}$$

and dilations

$$d(\delta_\lambda(g'), \delta_\lambda(g'')) = \lambda\ d(g', g'') , \qquad \rho(\delta_\lambda(g'), \delta_\lambda(g'')) = \lambda\ \rho(g', g'') . \tag{2.15}$$

Denoting with

$$B(g, R) = \{g' \in \boldsymbol{G} \mid d(g', g) < R\}, \qquad B_\rho(g, R) = \{g' \in \boldsymbol{G} \mid \rho(g', g) < R\}, \tag{2.16}$$

respectively the CC ball and the gauge pseudo-ball centered at g with radius R, by (2.15) and a rescaling one easily recognizes that there exist $\omega = \omega(\boldsymbol{G}) > 0$, and $\alpha = \alpha(\boldsymbol{G}) > 0$ such that

$$|B(g,R)| \;=\; \omega\; R^Q, \qquad |B_\rho(g,R)| \;=\; \alpha\; R^Q, \qquad g \in \boldsymbol{G}, R > 0. \tag{2.17}$$

The first equation in (2.17) shows, in particular, that for a Carnot group the Nagel-Stein-Wainger polynomial in (1.4) is simply the monomial ωR^Q.

2.2. **Free Lie algebras and groups.** In section four we will work with special systems $X = \{X_1, ..., X_m\}$ of vector fields of Hörmander type, for which both the X_j's and their commutators satisfy the minimal amount of relations. Such systems give rise to CC metrics for which the formula (1.4) is greatly simplified. More importantly, the corresponding CC geometry is locally well approximated by particular stratified Lie algebras.

Definition 2.2. *A free Lie algebra $\mathfrak{g}_{m,s}$ is a nilpotent Lie algebra of step s having m generators, but otherwise as few relations among the commutators as possible.*

The precise definition of such algebra, as quotients of the infinite dimensional free Lie algebra on m generators is given in detail in [RS], Example 4, page 256.

Definition 2.3. *Denote by $n_{m,s}$ the dimension (as a vector space) of the free nilpotent Lie algebra $\mathfrak{g}_{m,s}$. Let $X_1, ..., X_m$ be a set of smooth vector fields defined in an open neighborhood of a point $x_o \in M^n$, and let n_s be the dimension of the space generated by all commutators of the X_j's of length $\le s$ evaluated at the point x_o. We shall say that $X_1, ..., X_m$ are free up to step r if for any $1 \le s \le r$ we have $n_{m,s} = n_s$.*

Remark 2.4. *We observe that if the vector fields $X_1, ..., X_m$ are free up to step r in an open set $\Omega \subset M^n$, then commutators of different lengths are linearly independent, while commutators of the same length may be linearly dependent only because of anti-symmetry, or of the Jacobi identity. Consequently, any $n-$tuple $Y_{i_1}, ..., Y_{i_n}$ of commutators which is a basis for $\mathbb{R}^n$, must have the same cumulative degree*

$$Q \;=\; \sum_{k=1}^n d_{i_k} = \sum_{j=1}^r j(n_{m,j} - n_{m,j-1}) \ .$$

This simple observation implies that for any $K \subset\subset M^n$ there exists $R(K) > 0$, such that for any $x \in K$, and $0 < r < R(K)$, the polynomial in the right-hand side of (1.4) is actually a monomial

$$\Lambda(x,r) \;=\; r^Q \sum_I |a_I(x)| \ ,$$

and

$$C_1 \;\le\; \frac{|B(x,r)|}{r^Q \sum_I |a_I(x)|} \;\le\; C_2 \ . \tag{2.18}$$

From this point on, we will denote by $Y_1, ..., Y_m$ the generators of the Lie algebra $\mathfrak{g}_{m,s}$.

Consider $X_1, ..., X_m$ smooth vectors field in M^n which are free up to step r in the open set $\Omega \subset M^n$, and let $\xi \in \Omega$. For each $k \in \mathbb{N}$, $1 \le k \le r$, choose $\{X_{k,i}\}$, commutators of length k with $X_{1,i} = X_i$ such that the system $\{X_{k,i}\}$, $k = 1, ..., r$, $i = 1, ..., m_k$ evaluated at ξ is a basis of $\mathbb{R}^n$. Then we can define a system of coordinates (*canonical coordinates*) associated to $\{X_{k,i}\}$, based at the point ξ, as follows

$$(u_{k,j}) \leftrightarrow \exp(\Sigma u_{k,j} X_{k,j}) \cdot \xi \tag{2.19}$$

where $\exp(\cdot) \cdot \xi : T_\xi M^n \to M^n$ denotes the exponential map based at ξ.

Remark 2.5. *By virtue of Theorems 1-7 in* [NSW], *we know that there exist $R_o > 0$, and one particular collection of commutators $\{X_{k,i}\}$, the one corresponding to the largest of the monomials on the right hand side of (1.4), for which the box-like set, that in canonical coordinates (u_{ik}) is expressed by*

$$Box(\delta) \;=\; \{u_{k,i} \in \mathbb{R},\; k = 1, ..., r\; |u_{k,i}| \le \delta^i\} \;, \tag{2.20}$$

is equivalent to the metric ball $B(\xi, \delta)$ for any $0 < \delta < R_o$. Since we are considering vectors fields which are free up to step r at ξ, then all monomials in the right hand side of (1.4) are of the same degree, hence they are locally equivalent and give rise to equivalent sets of coordinates. Consequently we can state that for any compact set $K \subset\subset \Omega$, there exist constants $C_1, C_2 > 0$ such that

$$Box(C_1\delta) \;\subset\; B(\xi, \delta) \;\subset\; Box(C_2\delta) \;, \tag{2.21}$$

for any $\xi \in K$, and $0 < \delta < R_o$.

Following Rothschild and Stein [RS], pg. 273, we want to approximate the free vector fields $X_1, ..., X_m$ with left-invariant vector fields $\{Y_k\}$, $k = 1, ..., m$ generating the free nilpotent Lie algebra $\mathfrak{g}_{m,r}$. Let $\boldsymbol{G}_{m,r}$ denote the Lie group associated to $\mathfrak{g}_{m,r}$. For $k = 1, ..., r$ and $i = 1, ..., m_k$, denote by $\{Y_{k,i}\}$ a basis of the space V_k in the stratification $\mathfrak{g}_{m,r} = V_1 \oplus ... \oplus V_r$, and by $y_{k,i}$ the corresponding exponential coordinates in the group $\boldsymbol{G}_{m,r}$. We indicate by $Y_{1,i} = Y_i$, $i = 1, ..., m_1$ the algebra generators. If α denotes the multi-index $\{k, i\}$, then its degree is defined to be $|\alpha| = k$.

Our arguments will depend crucially upon the following fundamental result (see [RS], Theorem 5, page 273).

Theorem 2.6. *Let $X_1, ..., X_m$ be a system of smooth vector fields in M^n such that*

(i) $X_1, ..., X_m$ satisfy (1.2) with rank r.

(ii) $X_1, ..., X_m$ are free up to step r at $\xi \in M^n$.

There exists a neighborhood V of ξ, and a neighborhood U of the identity in $\boldsymbol{G}_{m,r}$, such that:

(A) *Let $\eta = \exp(\Sigma u_{jk} X_{jk}) \cdot \xi$, denote the canonical coordinate chart $\eta \to u_{jk}$ for V centered at ξ. The map $\theta : V \times V \to U \subset \boldsymbol{G}_{m,r}$ defined by*

$$\theta_\xi(\eta) = \theta(\xi, \eta) = \exp(\Sigma u_{jk} Y_{jk}) \tag{2.22}$$

is a diffeomorphism onto its image.

(B) *In the coordinate system given by θ_ξ one can write*

$$X_i = Y_i + \mathcal{R}_i, \quad i = 1, ..., m \tag{2.23}$$

where $\mathcal{R}_i$ is a vector field of local degree less or equal than zero, depending smoothly on ξ, i.e. for any smooth f,

$$X_i\bigg(f(\theta_\xi(\cdot))\bigg) = (Y_i f + \mathcal{R}_i f)(\theta_\xi(\cdot)).$$

More in general, if α denotes the multi-index $\{k, i\}$, then we have

$$X_\alpha = Y_\alpha + \mathcal{R}_\alpha,$$

with $\mathcal{R}_\alpha$ a vector fields of degree less or equal than $|\alpha| - 1$.

Let us recall that a vector field on a Carnot group $\boldsymbol{G}$ has local degree less or equal than $d \in \mathbb{N}$ if, after taking the Taylor expansion at the origin of its coefficients, each term so obtained is an homogeneous operator of degree less or equal than d. More explicitly, denote by $\{y_\alpha\}$, $\alpha = (k, i)$, the exponential coordinates in $\boldsymbol{G}_{m,r}$ associated to the vector fields $Y_{k,i}$. We say that the vector field R_i has degree less or equal than $d \in \mathbb{N}$ if for any $N \in \mathbb{N}$, and any multi-index $\alpha = (k, i)$ one can find a function $g_{\alpha,i,N} \in C^\infty(\boldsymbol{G})$, with growth $g_{\alpha,i,N}(y) = O(||y||^N)$ such that

$$\mathcal{R}_i = \sum_{l=1}^{r} \sum_{|\alpha|=l} \bigg(p_{\alpha,i,N}(y)\partial_{y_\alpha} + g_{\alpha,i,N}(y)\partial_{y_\alpha} \bigg), \tag{2.24}$$

in a neighborhood of the origin. In (2.23), the functions $p_{\alpha,i,N}(y)$ depend on N and are homogeneous group polynomials (see [FS]) of degree less or equal than N and greater or equal than $|\alpha| - d$. The notation ∂_{y_α} indicates a first order derivative along one of the group coordinates whose formal degree is $|\alpha|$. In other words, modulo lower order terms, the operator R_i has order $|\alpha| - deg(p_{\alpha,i,N}) \leq |\alpha| - (|\alpha| - d) = d$.

2.3. **The lifting theorem of Rothschild and Stein.** Up to now we have seen how to locally approximate a system of free vector fields with its "tangent" Carnot group. Since not all systems of Hörmander type are free (for instance consider ∂_x, and $x\partial_y$ in $\mathbb{R}^2$), then there is need of some additional work in order to use the approximation scheme in the most general setting.

One of the main building blocks in the proof of Theorem 1.5 in section five is the Rothschild-Stein lifting theorem (see [RS], Theorem 4).

Theorem 2.7. *Let $X_1, ..., X_m$ be a system of smooth vector fields in M^n, satisfying (1.2) in an open set $U \subset M^n$. For any $\xi \in U$ there exists a connected open neighborhood of the origin $V \subset \mathbb{R}^{\tilde{n}-n}$, and smooth functions $\lambda_{kl}(x, t)$, with $x \in M^n$ and $t = (t_{n+1}, ..., t_{\tilde{n}}) \in V$, defined in a neighborhood $\tilde{U}$ of $\tilde{\xi} = (\xi, 0) \in U \times V$, such that the vector fields $\tilde{X}_1, ..., \tilde{X}_m$ given by*

$$\tilde{X}_k = X_k + \sum_{l=m+1}^{\tilde{m}} \lambda_{kl}(x, t)\partial_{t_l}$$

are free up to step r at every point in $\tilde{U}$.

Let us denote by $\tilde{B}((x,s),R)$ the Carnot-Carathéodory balls associated to the lifted vector fields $\tilde{X}_1,...,\tilde{X}_m$. Let π_1 and π_2 denote the projections of $U\times V$ onto U and V respectively,

$$\pi_1(x,t) \;=\; x\;, \qquad \pi_2(x,t) \;=\; t\;.$$

The following lemma sums up some basic results from [RS], Lemma 3.1.

Lemma 2.8. *One has that $\pi_1 : \tilde{B}((x,t),R) \to B(x,R)$ and moreover, this map is onto. If $x,y\in U$ and $t,s\in V$ then $d(x,y)\le \tilde{d}((x,s),(y,t))$*

The next estimate is crucial for our purposes, for its proof see [NSW], Lemma 3.2, and [SC], Theorem 4 and Lemma 7.

Lemma 2.9. *Let $E\subset\subset U$ be a compact set, and $v\in C_0^\infty(V)$. There is a constant $C = C(E,X,v)>0$ such that if $x\in E$ and $y\in B(x,R)$, then*

$$C^{-1}\frac{|\tilde{B}((x,0),R)|}{|B(x,R)|}\le\left|\int_V \chi_{\tilde{B}(x,0),R)}(y,s)\ v(s)ds\right|\le C\frac{|\tilde{B}((x,0),R)|}{|B(x,R)|}.$$

Essentially this lemma says that even if the sets $\tilde{B}((x,0),R)$ are not the product of balls in R^n and $\mathbb{R}^{\tilde{n}}$, in terms of volume of sections they behave like such. We remark explicitly that the integral in the above formula simply represents the Lebesgue measure of the set

$$\pi_2\Big(\tilde{B}((x,0),R)\cap(\{y\}\times V)\Big) \tag{2.25}$$

in the projection onto the second factor. Lemma 2.9 implies that if $y_1,y_2\in B(x,R)$, then

$$\begin{aligned} C^{-2}\left|\pi_2\Big(\tilde{B}((x,0),R)\cap(\{y_1\}\times V)\Big)\right| &\le \left|\pi_2\Big(\tilde{B}((x,0),R)\cap(\{y_2\}\times V)\Big)\right| \\ &\le C^2\left|\Big(\pi_2\Big(\tilde{B}((x,0),R)\cap(\{y_1\}\times V)\Big)\right|, \end{aligned} \tag{2.26}$$

i.e., the expression (2.25) is almost constant in y. Note that in (2.26) the symbol $|\cdot|$ denotes Lebesgue measure in different spaces.

3. Upper Ahlfors regularity of the perimeter measure in Carnot groups

In this section we prove the geometric estimates in Theorems 1.5 and 1.2 when the CC space is a Carnot group $\boldsymbol{G}$. These results will then be used as a fundamental step in the proof of the general case. We begin with the upper Ahlfors estimates, Theorems 3.2, and 3.3. Sharper versions of this theorem will be stated at the end of the section. In Section 5 we show that the hypothesis of the theorem are optimal. In the second part of the section we establish the lower Ahlfors estimates for the perimeter measure.

Definition 3.1. *Let $\boldsymbol{G}$ be a Carnot group of step r with homogeneous dimension Q. Consider a bounded, open set $\Omega = \{g \in \boldsymbol{G} \mid \phi(g) < 0\}$, where $\phi \in C^{1,1}(\boldsymbol{G})$ is a defining function for Ω. For any $g_o \in \partial\Omega$, we define the "type" of g_o to be the smallest $j = 1, ..., r$ such that there exists $s = 1, ..., m_j$ for which $X_{j,s}\phi(g_o) \neq 0$. We will denote by type(g_o) the type of g_o, and if for every $g_o \in \partial\Omega$ we have that type$(g_o) \leq k \in \mathbb{N}$ then we will say that Ω has type $\leq k$.*

Clearly, every domain with empty characteristic set has type 1. In a group of step two, every C^1 domain has type ≤ 2.

One of the main results of this section is a pointwise version of the upper Ahlfors estimates.

Theorem 3.2. *Let $\boldsymbol{G}$ be a Carnot group of step r with homogeneous dimension Q. Consider a bounded, open set $\Omega = \{g \in \boldsymbol{G} \mid \phi(g) < 0\}$, where $\phi \in C^{1,1}(\boldsymbol{G})$ is a defining function for Ω. For every $g_o \in \partial\Omega$, there exist $M = M(\boldsymbol{G}, X, \Omega, g_o) > 0$ and $R_o = R_o(\boldsymbol{G}, X, \Omega, g_o) > 0$ depending continuously on g_o, such that, for for any $0 < R < R_o$, one has*

$$\left(\sup_{B(g_o,R)\cap\partial\Omega} |X\phi| \right) \sigma(B(g_o,R)\cap\partial\Omega) \ \leq \ M \ R^{Q-\alpha} \ , \tag{3.1}$$

with

$$\alpha = \begin{cases} \text{type}(g_o) - 1, & \text{if } g_o \text{ is characteristic,} \\ 1, & \text{if } g_o \text{ is not characteristic.} \end{cases}$$

The local estimates (3.1) allow us to prove global type estimates corresponding to uniform choices of α: For domains of type ≤ 2 we can choose $\alpha = 1$, while for any domain we have the worst possible choice $\alpha = r - 1$.

Theorem 3.3. *Let $\boldsymbol{G}$ be a Carnot group of step r with homogeneous dimension Q. Consider a bounded, open set*

$$\Omega \ = \ \{g \in \boldsymbol{G} \mid \phi(g) < 0\} \ ,$$

where $\phi \in C^{1,1}(\boldsymbol{G})$ is a defining function for Ω. There exist $M = M(\boldsymbol{G}, X, \Omega) > 0$ and $R_o = R_o(\boldsymbol{G}, X, \Omega) > 0$ such that, for any $g_o \in \partial\Omega$, and every $0 < R < R_o$, one has with $\alpha = r - 1$

$$\left(\sup_{B(g_o,R)\cap\partial\Omega} |X\phi| \right) \sigma(B(g_o,R)\cap\partial\Omega) \ \leq \ M \ R^{Q-\alpha} \ , \tag{3.2}$$

$$P_X(\Omega; (B(g_o,R)) \ \leq \ M \ R^{Q-\alpha} \ . \tag{3.3}$$

Furthermore, if Ω is of type ≤ 2, then we can take $\alpha = 1$ in (3.2).

Remark 3.4. *As a corollary of the previous theorem, if Ω is of type ≤ 2, then the perimeter measure $P_X(\Omega; \cdot)$ is an upper 1-Ahlfors measure with respect to the CC balls, i.e., we have*

$$P_X(\Omega; B(g_o,R)) \ \leq \ M \ R^{Q-1} \ , \tag{3.4}$$

for every $g_o \in \partial\Omega$, and any $0 < R < R_o$.

Proof of Theorem 3.3. We begin by observing that we only need to prove (3.2), since (3.3) follows trivially from (3.2), (1.7), and from (1.8). One has in fact

$$P_X(g_o,R) \ \le\ C^{-1}\ \mu(B(g_o,R)) \ =\ C^{-1}\int_{B(g_o,R)\cap\partial\Omega}|X\phi|\ d\sigma$$
$$\le\ C^{-1}\left(\sup_{B(g_o,R)\cap\partial\Omega}|X\phi|\right)\ \sigma(B(g_o,R)\cap\partial\Omega)\ \le\ C^{-1}\ M\ R^{Q-\alpha}\ .$$

This being said, we stress that thanks to Theorem 3.2 we can assume that the following pointwise version of (3.2) hold: given any $g_o\in\partial\Omega$, there exist $C(\boldsymbol{G},X,\Omega,g_o), R(\boldsymbol{G},X,\Omega,g_o)>0$ such that for all $0<R<R(\boldsymbol{G},X,\Omega,g_o)$ one has

$$\left(\sup_{B(g_o,R)\cap\partial\Omega}|X\phi|\right)\ \sigma(B(g_o,R)\cap\partial\Omega)\ \le\ C(\boldsymbol{G},X,\Omega,g_o)\ R^{Q-\alpha}\ . \tag{3.5}$$

Here α is either $r-1$ or $\alpha=1$ if the domain is of type ≤ 2. Now we show how the global estimate can be proved starting from (3.5) in a standard fashion. By the compactness of $\partial\Omega$, we can find $g_j\in\partial\Omega$, $C_j=C(\boldsymbol{G},X,\Omega,g_j)>0$, and $R_j=R(\boldsymbol{G},X,\Omega,g_j)>0$, $j=1,...,p$, such that

$$\partial\Omega\ \subset\ \bigcup_{j=1}^{p} B(g_j,R_j)\cap\partial\Omega\ ,$$

and for which for any $j\in\{1,...,p\}$, (3.5) holds in $B(g_j,R_j)\cap\partial\Omega$ for $0<R<R_j$, with constant C_j. Let $R_o=\min\{R_1/2,...,R_p/2\}>0$, $C=\max\{C_1,...,C_p\}>0$. If $g_o\in\partial\Omega$, then $g_o\in B(g_j,R_j)$ for some $j\in\{1,...,p\}$, and one has

$$B(g_o,R)\ \cap\ \partial\Omega\ \subset\ B(g_j,2R)\ \cap\ \partial\Omega\ ,\qquad \text{for any}\quad 0<R<R_o\ .$$

We would conclude with $M=2^{Q-\alpha}C$

$$\left(\sup_{B(g_o,R)\cap\partial\Omega}|X\phi|\right)\ \sigma(B(g_o,R)\cap\partial\Omega)$$
$$\le\ \left(\sup_{B(g_j,2R)\partial\Omega}|X\phi|\right)\ \sigma(B(g_j,2R)\cap\partial\Omega)\ \le\ C_j\ (2R)^{Q-\alpha}\ \le\ M\ R^{Q-\alpha}\ ,$$

thus completing the proof. □

We now turn our attention to the proof of Theorem 3.2. We will accomplish it in two steps. In the first step, we reformulate the local estimate (3.1) in terms of the Lie algebra $\mathfrak{g}$, and show that the problem can be reduced to a model case in which $\partial\Omega$ is an arbitrary hyperplane in $\mathfrak{g}$. In the second step, we prove the estimate for the model case and for the remainder.

Proof of Theorem 3.2: Step One. Let $g_o\in\boldsymbol{G}$ and consider the group automorphism $L_{g_o^{-1}}:\boldsymbol{G}\to\boldsymbol{G}$, see (2.7). Since $L_{g_o^{-1}}$ is an isometry (2.14), we have $L_{g_o^{-1}}(B(g_o,R))=B(e,R)$, where e denotes the group identity. Moreover, $L_{g_o^{-1}}$ is a smooth map, hence in particular it is locally Lipschitz. Consequently, for every $R_o>0$ there exists $C=C(\boldsymbol{G},X,\Omega,g_o,R_o)>0$, depending on the Lipschitz norm of $L_{g_o^{-1}}$ in $B(g_o,R_o)\cap\partial\Omega$, such that for any $0<R<R_o$ one has

$$\sigma(B(g_o,R)\ \cap\ \partial\Omega)\ \le\ C\ \sigma(L_{g_o^{-1}}(B(g_o,R)\cap\partial\Omega))\ . \tag{3.6}$$

Since $\partial\Omega$ is compact, the constant C in (3.6) can be chosen independently of $g_o \in \partial\Omega$, and if we let $R_o \leq \min(\text{diam}(\Omega), \frac{1}{2})$, we can simply write $C = C(\boldsymbol{G}, X, \Omega)$. If we let $\tilde{\Omega} = L_{g_o^{-1}}(\Omega)$, then (3.6) can be rewritten for some constant $C = C(\boldsymbol{G}, X, \Omega) > 0$

$$\sigma(B(g_o, R) \cap \partial\Omega) \ \leq \ C \ \sigma(B(e, R) \cap \partial\tilde{\Omega}) \ . \tag{3.7}$$

Now, observe that

$$\tilde{\Omega} \ = \ \{g \in \boldsymbol{G} \mid \tilde{\phi}(g) < 0\} \ ,$$

where $\tilde{\phi} = \phi \circ L_{g_o^{-1}}$. By the left-invariance of the vector fields $X_1, ..., X_m$ we have $|X\tilde{\phi}| = |X\phi| \circ L_{g_o^{-1}}$. In particular,

$$\sup_{B(g_o,R)\cap\partial\Omega} |X\phi| \ = \ \sup_{B(e,R)\cap\partial\tilde{\Omega}} |X\tilde{\phi}| \ . \tag{3.8}$$

In view of (3.7) and (3.8) we obtain

$$\left(\sup_{B(g_o,R)\cap\partial\Omega} |X\phi| \right) \ \sigma(B(g_o, R) \cap \partial\Omega) \ \leq \ C \ \left(\sup_{B(e,R)\cap\partial\tilde{\Omega}} |X\tilde{\phi}| \right) \ \sigma(B(e, R) \cap \partial\tilde{\Omega}) \ . \tag{3.9}$$

Inequality (3.9) allows us to assume that $g_o = e$ in (3.1), which we will do hereafter. Moreover, with a slight abuse of notation we will denote $\tilde{\Omega}$ and $\tilde{\phi}$ by Ω and ϕ, respectively. In addition, thanks to (2.11), it is clear that in (3.9) we can replace the metric balls with gauge pseudo-balls defined in (2.16). Without further mention, we will work with the latter from this moment on.

At this point it is convenient to work with the exponential coordinates (2.6) in the Lie algebra $\mathfrak{g}$, rather than dealing directly with the group $\boldsymbol{G}$. We thus set $D = \exp^{-1}(\Omega) \subset \mathfrak{g}$. Observing that the Riemannian Hausdorff measure H_{n-1} in $\boldsymbol{G}$, the Haar measure dg in $\boldsymbol{G}$, and the gauge pseudo-metric, are all obtained by pushing forward via the exponential mapping corresponding measures and pseudo-metric in $\mathfrak{g}$, with another slight abuse of notation we will denote by H_{n-1} the (Euclidean) $(n-1)$-dimensional Hausdorff measure in $\mathfrak{g}$ and set $\sigma = H_{n-1} \lfloor \partial D$. We will continue to indicate with ϕ the pull-back $(\phi \circ \exp)$. In this notation ϕ is a defining function for D. The notation $B(\xi, R) \subset \mathfrak{g}$ will indicate the Lie algebra gauge pseudo-balls of radius R and center $\xi \in \mathfrak{g}$ defined by means of $|\cdot|_{\mathfrak{g}}$ in (2.12).

With these reductions, we have converted the proof of (3.1) into the task of establishing the existence of $C_o = C_o(\mathfrak{g}, X, D, 0) > 0$ and $R_o = R_o(\mathfrak{g}, X, D, 0) > 0$, such that for $0 < R < R_o$

$$\left(\sup_{B(0,R)} |X\phi| \right) \sigma(B(0, R) \cap \partial D) \ \leq \ C \ R^{Q-\alpha} \ . \tag{3.10}$$

Our next reduction consists in substituting the quantity $\sigma(B(0, R) \cap \partial D)$ in the left-hand side of (3.10), with $\sigma(B(0, R) \cap \Pi)$, where $\Pi = T_0\partial D \subset \mathfrak{g}$ denotes the tangent plane at the origin $0 \in \partial D$. We note that when we write $\sigma(B(0, R) \cap \Pi)$, the measure σ denotes the restriction of H_{n-1} to Π, while for $\sigma(B(0, R) \cap \partial D)$, σ denotes the restriction of H_{n-1} to ∂D. It seems preferable to allow for this slight ambiguity rather than adopting a more cumbersome notation. We observe next that the defining function of Π is given by $\pi(\xi) =< \nabla\phi(0), \xi >$, $\xi \in \mathfrak{g}$. By the hypothesis $\phi \in C^{1,1}$, and by Taylor's theorem, one can write

$$\phi(\xi) = \pi(\xi) + H(\xi) \ , \tag{3.11}$$

with $H = O(|\xi|^2)$ (we recall here that $|\cdot|$ denotes the Euclidean norm on $\mathfrak{g}$), with $\xi \in B(0, R_o)$, and for sufficiently small R_o. Consequently, we obtain

$$\nabla\phi(\xi) = \nabla\phi(0) + \vec{O}(\xi), \tag{3.12}$$

where $\vec{O}(\xi) = \{O_{j,s}(\xi)\}$, $j = 1, ..., r$, $s = 1, ..., m_j$ and $O_{j,s}(\xi) = O(|\xi|)$. In view of (3.12) and Lemma 2.1 we can compute the horizontal gradient of ϕ as follows

$$\begin{aligned} X_i\phi(\xi) =< X_i,\ \nabla\phi(\xi) >= & \quad < X_i, \nabla\phi(0) + \vec{O} > \\ = & \quad \sum_{j=1}^{r}\sum_{s=1}^{m_j} b^s_{j,i}(\xi_1, ..., \xi_{j-1})\Big(\frac{\partial\pi}{\partial x_{j,s}} + O_{j,s}(\xi)\Big). \end{aligned} \tag{3.13}$$

Here, for simplicity, we have let $b^s_{1,i} = \delta_{is}$.

Observe that the Euclidean metric on $\mathfrak{g}$, the gauge pseudo-metric and the Hausdorff measure are all invariant with respect to the action of the orthogonal group $O(\mathbb{R}^{m_i})$ on V_i. By a change of coordinates, performing a rotation inside each layer V_i, we can assume without loss of generality that there exist real numbers $a_1, ..., a_r$, such that the equation of the hyperplane Π is given by

$$\pi(\xi) \ = \ \sum_{j=1}^{r} a_j\ x_{j,1} \ = \ 0\ . \tag{3.14}$$

We will denote by $\mathcal{N}$ the set of indices $j = 1, ..., r$ such that $a_j \neq 0$, and by $\mathcal{N}^C$ the set of indices for which $a_j = 0$. Note that if the point g_o is non-characteristic, then $a_1 \neq 0$, while in general, the smallest index in $\mathcal{N}$ is simply the type of $g_o \in \partial\Omega$. We choose R_o small enough such that

$$|O_{j,s}(\xi)| \ \leq \ \min_{j\in\mathcal{N}} |a_j|\ , \qquad |\xi| \ \leq \ R_o\ , \tag{3.15}$$

and note that R_o will depend on the choice of the base point g_o in the statement of the theorem. In view of (3.13) and (3.15) we obtain

$$\begin{aligned} |X_i\phi(\xi)| \ \leq & \quad \sum_{j=1}^{r}\sum_{s=1}^{m_j} |b^i_{j,s}(\xi_1, ..., \xi_{j-1})|\Big(a_j\delta_{s1} \ + \ O_{j,s}(\xi)\Big) \\ \leq & \quad 2\sum_{j\in\mathcal{N}}\sum_{s=1}^{m_j} |b^i_{j,s}(\xi_1, ..., \xi_{j-1})|\ |a_j| \ + \ \sum_{j\in\mathcal{N}^C}\sum_{s=1}^{m_j} |b^i_{j,s}(\xi_1, ..., \xi_{j-1})|\ |O_{j,s}(\xi)| \\ \leq & \quad I(\xi) \ + \ II(\xi)\ . \end{aligned} \tag{3.16}$$

Remark 3.5. *We want to point out an incorrect statement in* [CGN2], *namely equation (3.3) in that paper, in which only part I of (3.16) appears, and not the remainder. To see that this is not true in general, we consider the Heisenberg group $\mathbb{H}^1$ and the function $\phi(x, y, t) = t + xt$. In this particular example we can compute explicitly*

$$|X\phi|^2 \ = t^2 + \frac{1}{4}(x^2 + y^2)(1 + x)^2 + ty(1 + x)\ ,$$

while

$$|X\pi|^2 \ = \ \frac{1}{4}(x^2 + y^2)\ .$$

Hence, it is not true that $|X\phi| \le C|X\pi|$ in the ball $B(0,R)$. Since in [CGN2] *the setting is that of a step two Carnot group, then the additional term is easily estimated and the results in that paper continue to hold.*

If we parametrize ∂D near the origin as the graph of a $C^{1,1}$ function defined on the tangent plane $\Pi = T_0 \partial D$, then it is easy to see that there exist constants $C_1 = C_1(\mathfrak{g}, X, D) > 0$, $R_1 = R_1(\mathfrak{g}, X, D) > 0$, such that for all $0 < R < R_1$ we have

$$\sigma(B(0,R)\cap\Pi) \ \le\ \sigma(B(0,R)\cap\partial D) \ \le\ C_1\ \sigma(B(0,R)\cap\Pi)\ . \tag{3.17}$$

From (3.16), (3.17), we obtain the existence of constants $C_2 = C_2(\mathfrak{g}, X, D) > 0$, $R_2 = R_2(\mathfrak{g}, X, D) > 0$ (these constants also depend on the choice of the base point g_o in the statement of the theorem), such that

$$\left(\sup_{B(0,R)\cap\partial D} |X\phi|\right)\ \sigma(B(0,R)\cap\partial D)\ \le C_2\ \left(\sup_{B(0,R)}\ [I+II](\xi)\right)\ \sigma(B(0,R)\cap\Pi), \tag{3.18}$$

for any $0 < R < R_2$.

Our final reduction consists in replacing the gauge pseudo-ball $B(0,R)\subset\mathfrak{g}$ with a simpler set, the anisotropic "box" centered at $0\in\mathfrak{g}$ of radius R, see (2.20). The latter is given by

$$\text{Box}(R) = \{\xi = \xi_1 + ... + \xi_r \in V_1 \oplus ... \oplus V_r = \mathfrak{g}\ |\ \ |\xi_j| \le R^j,\ j = 1, ...r\}\ . \tag{3.19}$$

In view of the "ball-box" theorem quoted earlier in (2.21) (Thrm. 3 in [NSW]), it is possible to find a constant $C(\boldsymbol{G}) > 1$ such that for any $R > 0$

$$\text{Box}(C^{-1}R)\ \subset B(0,R)\ \subset\ \text{Box}(CR)\ . \tag{3.20}$$

The inclusion (3.20) allows to further simplify the right-hand side of (3.18) by replacing the gauge pseudo-ball $B(0,R)$ with $\text{Box}(CR)$. This substitution yields constants $C = C(\boldsymbol{G}, X, \Omega, g_o) > 0$, and $R(\boldsymbol{G}, X, \Omega, g_o) > 0$ such that

$$\left(\mathop{sup}_{B(g_o,R)\cap\partial\Omega} |X\phi|\right)\ \sigma(B(g_o,R)\cap\partial\Omega)\ \le\ C\ \left(\sup_{\text{Box}(CR)}\ [I+II]\right)\ \sigma(\text{Box}(CR)\cap\Pi), \tag{3.21}$$

for any $0 < R < R(\boldsymbol{G}, X, \Omega, g_o)$.

Proof of Theorem 3.2: Step Two. The rest of the proof is dedicated to proving the estimate

$$\left(\sup_{\text{Box}(R)}\ [I+II]\right)\ \sigma(\text{Box}(R)\cap\Pi) \le\ C\ R^{Q-\alpha}\ , \tag{3.22}$$

for R suitably small. Estimate (3.22), coupled with (3.21), will yield (3.10), and therefore (3.1), thus establishing the theorem. We will obtain (3.22) as a corollary of the following two lemmas.

Lemma 3.6. *With the notation established above, we have*

$$\left(\sup_{Box(R)}\ I\right)\ \sigma(Box(R)\cap\Pi) \le\ C\ R^{Q-1}\ ,$$

for R suitably small.

Proof. Using (3.16) and Lemma 2.1 we obtain

$$\sup_{\mathrm{Box}(R)} I \ \leq\ C_G \left(\sum_{j\in\mathcal{N}} |a_j|\ R^{j-1} \right) . \tag{3.23}$$

By an isometric linear map, we can transform

$$\mathrm{Box}(R)\cap\Pi \ =\ \{(x_{1,1},...,x_{r,m_r})\in\mathfrak{g}\approx\mathbb{R}^{\sum_{j=1}^r m_j}\ \mid\ \sum_{j=1}^r a_j x_{j,1}=0,\ |\xi_j|\leq R^j, j=1,...,r\}\ ,$$

into the set

$$\left(\Pi_{j\in\mathcal{N}^C}(-R^j,R^j)^{m_j}\right)\times\left(\Pi_{k\in\mathcal{N}}(-R^k,R^k)^{m_k-1}\right)\times S\ .$$

Here, we have denoted by $|\mathcal{N}|$ the number of elements in $\mathcal{N}$, and by $\{\mathcal{N}_1,...,\mathcal{N}_{|\mathcal{N}|}\}$ the elements themselves. We have let

$$S\ =\ \{s=(s_1,...,s_{|\mathcal{N}|})\in\mathbb{R}^{|\mathcal{N}|}\ \mid\ \sum_{\mathcal{N}_j\in\mathcal{N}} a_{\mathcal{N}_j}s_j=0,\ |s_j|\leq R^{\mathcal{N}_j}, j=1,...,|\mathcal{N}|\}\ .$$

Consequently,

$$\begin{aligned}\sigma(\mathrm{Box}(R)\cap\Pi)\ &\leq\ CR^{\sum_{j\in\mathcal{N}^C} jm_j}\ R^{\sum_{j\in\mathcal{N}} j(m_j-1)}\ H_{|\mathcal{N}|-1}(S)\\ &=\ C\ R^Q\ R^{-\sum_{j\in\mathcal{N}} j}\ H_{|\mathcal{N}|-1}(S)\ .\end{aligned} \tag{3.24}$$

Next, we estimate from above the quantity in the right-hand side of (3.24). Now, it is not easy to compute $H_{|\mathcal{N}|-1}(S)$ exactly. However the following simple argument produces the bound (3.26), which will suffice for our purposes. We recall that if

$$\Sigma\ =\ \{s=(s_1,...,s_{|\mathcal{N}|})\mid\sum_{j\in\mathcal{N}} a_{\mathcal{N}_j}s_j=0\}$$

is a hyperplane in $\mathbb{R}^{|\mathcal{N}|}$, and U represents the projection of a portion $\Delta\subset\Sigma$ onto the coordinate hyperplane $\{s_i=0\}$, then the $(|\mathcal{N}|-1)$-dimensional measure of Δ is given by

$$H_{|\mathcal{N}|-1}(\Delta)\ =\ \frac{\sqrt{\sum_{j\in\mathcal{N}} a_j^2}}{|a_{\mathcal{N}_i}|}\ H_{|\mathcal{N}|-1}(U)\ . \tag{3.25}$$

We now apply (3.25) with $\Delta=S$ to reach the crucial conclusion that $H_{|\mathcal{N}|-1}(S)$ is bounded from above by any of the quantities

$$\frac{\sqrt{\sum_{j\in\mathcal{N}} a_j^2}}{|a_{\mathcal{N}_i}|}\ R^{\sum_{\{j\neq\mathcal{N}_i,j\in\mathcal{N}\}} j}\ ,\qquad \mathcal{N}_i\in\mathcal{N}\ .$$

In fact, $R^{\sum_{\{j\neq\mathcal{N}_i,j\in\mathcal{N}\}} j}$ is the $H_{|\mathcal{N}|-1}$ measure of the projection onto $\{s_i=0\}$ of the box $\{|s_j|\leq R^{\mathcal{N}_j}, j=1,...,|\mathcal{N}|\}\subset\mathbb{R}^{|\mathcal{N}|}$. Consequently, we have

$$H_{|\mathcal{N}|-1}(S)\ \leq\ \min\left\{\frac{\sqrt{\sum_{j\in\mathcal{N}} a_j^2}}{|a_{\mathcal{N}_i}|}\ R^{\sum_{\{j\neq\mathcal{N}_i,j\in\mathcal{N}\}} j}\ ,\ \mathcal{N}_i\in\mathcal{N}\right\}\ . \tag{3.26}$$

Using (3.26) we can now complete the first part of the estimate (3.22). Let i_o be the index of the minimum element in the above expression, so that

$$\frac{|a_i|}{|a_{i_o}|} \leq \frac{R^{i_o}}{R^i}, \qquad i \in \mathcal{N} . \tag{3.27}$$

Note that i_o depends on R, however this dependence will have no effect on the constant C in the estimate (3.22). By (3.23), (3.24), (3.26), and (3.27) we conclude

$$\begin{aligned}
&\left(\sup_{\mathrm{Box}(R)} I\right) \sigma(\mathrm{Box}(R)\cap\Pi) \\
&\leq C(\boldsymbol{G}) \sqrt{\sum_{j\in\mathcal{N}} a_j^2}\; R^Q \left\{\sum_{j\in\mathcal{N}} \frac{|a_j|}{|a_{i_o}|}\, \frac{R^{j-1}}{R^{i_o}}\right\} \\
&\leq C(\boldsymbol{G}) \sqrt{\sum_{j\in\mathcal{N}} a_j^2}\; R^Q \left\{\frac{1}{R} + \sum_{j\in\mathcal{N}} \frac{1}{R}\right\} \\
&\leq C(\boldsymbol{G}) \sqrt{\sum_{j\in\mathcal{N}} a_j^2}\; R^{Q-1} .
\end{aligned} \tag{3.28}$$

□

This establishes the first part of (3.22). We now turn our attention to the proof of the second part of the estimate.

Lemma 3.7. *With the notation established above, we have for R suitably small,*

$$\left(\sup_{Box(R)} II\right) \sigma(Box(R)\cap\Pi) \leq C\, R^{Q-1} \begin{cases} R^{-k_o+2} & \text{if } k_o > 1 \ , \ \textit{(characteristic point)} \\ R^{j_o} & \text{if } k_o = 1 \ , \ \textit{(non-characteristic point)} \end{cases}$$

where $k_o = \min\, \mathcal{N}$ and $j_o = \min\, \mathcal{N}^C$.

Once again we point out that k_o is nothing else but the type of the point $g_o \in \partial\Omega$ in the statement of the theorem.

Proof of Lemma 3.7. Note that for $\xi \in \mathrm{Box}(R)$, the Euclidean norm of ξ is less than R, see (3.19). From Lemma 2.1 and (3.16), we have

$$II(\xi) = \sum_{j\in\mathcal{N}^C} \sum_{s=1}^{m_j} |b^s_{j,i}(\xi_1, ..., \xi_{j-1})|\; |O_{j,s}(\xi)| \;\leq\; C \sum_{j\in\mathcal{N}^C} R^{j-1} R \, . \tag{3.29}$$

In view of the latter and of (3.24) and (3.26) we obtain

$$\begin{aligned}
&\left(\sup_{\mathrm{Box}(R)} II\right) \sigma(\mathrm{Box}(R)\cap\Pi) \;\leq\; C \left(\textstyle\sum_{k\in\mathcal{N}^C} R^{k-1} R\right) R^{\sum_{j\in\mathcal{N}^C} j m_j}\; R^{\sum_{j\in\mathcal{N}} j(m_j-1)}\; H_{|\mathcal{N}|-1}(S) \\
&\leq\; C\left(\textstyle\sum_{k\in\mathcal{N}^C} R^{k-1} R\right) R^Q\; R^{-\sum_{j\in\mathcal{N}} j} \min\left\{\frac{\sqrt{\sum_{j\in\mathcal{N}} a_j^2}}{|a_{\mathcal{N}_i}|}\; R^{\sum_{\{j\neq\mathcal{N}_i, j\in\mathcal{N}\}} j} \ , \ \mathcal{N}_i \in \mathcal{N}\right\} \\
&\leq\; C R^{Q-1}\left(\textstyle\sum_{k\in\mathcal{N}^C} R^{k+1}\right) \min\left\{\frac{\sqrt{\sum_{j\in\mathcal{N}} a_j^2}}{|a_{\mathcal{N}_i}| R^{\mathcal{N}_i}} \ , \ \mathcal{N}_i \in \mathcal{N}\right\}.
\end{aligned} \tag{3.30}$$

If $k_o = 1$, then the point g_o is not characteristic, and consequently $j_o > 1$. In this case, from (3.29) we infer

$$\left(\sup_{\text{Box}(R)} II\right)\sigma(\text{Box}(R)\cap\Pi) \leq CR^{Q-1}R^{j_o+1}\min\left\{\frac{\sqrt{\sum_{j\in\mathcal{N}}a_j^2}}{|a_1|R},\ldots,\frac{\sqrt{\sum_{j\in\mathcal{N}}a_j^2}}{|a_{\mathcal{N}_{|\mathcal{N}|}}|R^{|\mathcal{N}|}}\right\}$$

$$\leq C\frac{\sqrt{\sum_{j\in\mathcal{N}}a_j^2}}{|a_1|}R^{Q-1}R^{j_o} \tag{3.31}$$

for R sufficiently small (depending on the a_i's). If $k_o > 1$, then g_o is a characteristic point and (3.29) implies

$$\begin{aligned}\left(\sup_{\text{Box}(R)} II\right) & \quad \sigma(\text{Box}(R)\cap\Pi)\\ \leq & \quad CR^{Q-1}\left(R^2+\ldots+R^{k_o+1}\right)\min\left\{\frac{\sqrt{\sum_{j\in\mathcal{N}}a_j^2}}{|a_{k_o}|R^{k_o}},\ldots,\frac{\sqrt{\sum_{j\in\mathcal{N}}a_j^2}}{|a_{\mathcal{N}_{|\mathcal{N}|}}|R^{|\mathcal{N}|}}\right\}\\ \leq & \quad C\frac{\sqrt{\sum_{j\in\mathcal{N}}a_j^2}}{|a_{k_o}|}R^{Q-1}R^{-k_o+2},\end{aligned}$$

thus completing the proof of the lemma.

□

As we already mentioned, Lemmas 3.6 and 3.7 imply the estimate (3.22), thus concluding the proof of Theorem 3.2.

The argument in the previous proof yields a sharper geometric estimates than the one stated in Theorem 3.3 in the case when the domain is an hyperplane.

Theorem 3.8. *Let $\boldsymbol{G}$ be a Carnot group of step r with homogeneous dimension Q. Consider a bounded, open set $\Omega = \{g \in \boldsymbol{G} \mid \phi(g) < 0\}$, where $\phi \in C^{r,1}(\boldsymbol{G})$ is a defining function for Ω, such that*

$$\phi(\xi) =< \nabla\phi(0), \xi > +O(|\xi|^r).$$

There exist $M = M(\boldsymbol{G}, X, \Omega) > 0$ and $R_o = R_o(\boldsymbol{G}, X, \Omega) > 0$ such that, for any $0 < R < R_o$, one has

$$\left(\sup_{B(0,R)\cap\partial\Omega} |X\phi|\right)\ \sigma(B(0,R)\cap\partial\Omega)\ \leq\ M\ R^{Q-1}\ . \tag{3.32}$$

To conclude the section, let us observe that Theorem 3.2 yields pointwise estimates in the set $U = \{$ set of points of type $\leq 2\}$. Such set is open and has full Riemannian $(n-1)-$dimensional Hausdorff measure in $\partial\Omega$. The open condition follows trivially from Ω being C^1 and observing that

$$U \ = \ \{g \in \partial\Omega \text{ such that } \sum_{k=1}^{2}\sum_{j=1}^{m_k}|X_{k,j}\phi(g)| \neq 0\}\ .$$

Since U contains the set of non-characteristic points, then the fact that $\partial\Omega\backslash U$ has zero surface measure is a consequence of a result of Derridj [De]. Note that even if Derridj's original argument was stated for smooth domains, it extends with no changes, to the case where the domain is only C^2. In fact, once we use a C^2 diffeomorphism to "flatten" a portion of the boundary, the push forward of the smooth vector fields in the Lie algebra become C^1 vector fields and using Lemmata 1 and 2 in [De] we arrive at the desired estimate.

4. **Lower Ahlfors regularity of the perimeter measure in Carnot groups**

Having proved that the measure μ of a $C^{1,1}$ hypersurface of type less or equal than 2 is an upper 1-Ahlfors measure, we now turn our attention to the more delicate question of the lower 1-Ahlfors regularity.

Theorem 4.1. *Let $\boldsymbol{G}$ be a Carnot group of step r with homogeneous dimension Q. Consider a bounded, open set $\Omega = \{g \in \boldsymbol{G} \mid \phi(g) < 0\}$, where ϕ is a defining function for Ω.*

(i) *If $\phi \in C^{1,1}(\boldsymbol{G})$ and $g_o \in \partial\Omega$ is non-characteristic then there exist $M = M(\boldsymbol{G}, X, \Omega, g_o) > 0$ and $R_o = R_o(\boldsymbol{G}, X, \Omega, g_o) > 0$ depending continuously on g_o, such that, for any $0 < R < R_o$, one has*

$$\mu(B(g_o, R)) \ \geq \ M^{-1} \ R^{Q-1} \ . \tag{4.1}$$

(ii) *If $\phi \in C^2(\boldsymbol{G})$ and $g_o \in \partial\Omega$ is of type 2, then there exist $M = M(\boldsymbol{G}, X, \Omega, g_o) > 0$ and $R_o = R_o(\boldsymbol{G}, X, \Omega, g_o) > 0$ depending continuously on g_o, such that, for any $0 < R < R_o$, estimate* (4.1) *still holds.*

A standard compactness theorem, like the one presented in the previous section yields the global estimates

Theorem 4.2. *Let $\boldsymbol{G}$ be a Carnot group of step r with homogeneous dimension Q. Consider a bounded, open set $\Omega = \{g \in \boldsymbol{G} \mid \phi(g) < 0\}$, where $\phi \in C^2(\boldsymbol{G})$ is a defining function for Ω, and assume that Ω is of type less or equal than two (if Ω is of type one, then the $C^{1,1}$ regularity suffices). The Borel measure μ defined in* (1.7) *is a lower* 1*-Ahlfors measure with respect to the CC balls, i.e., there exist $M = M(\boldsymbol{G}, X, \Omega) > 0$ and $R_o = R_o(\boldsymbol{G}, X, \Omega) > 0$ such that, for any $g_o \in \partial\Omega$, and $0 < R < R_o$, one has*

$$\mu(B(g_o, R)) \ \geq \ M^{-1} \ R^{Q-1} \ . \tag{4.2}$$

In particular, combining (3.4) *with* (4.2) *we conclude that μ is a* 1*-Ahlfors measure.*

In order to prove the theorem we need two preliminary results,

Lemma 4.3. *Let $n, r \in \mathbb{N}$, $n \leq r$, and consider a multi-index*

$$I \ = \ \{d_1, ..., d_n\} \ \in \ \mathbb{N}^n \ ,$$

with $1 \le d_i \le r,\quad d_i < d_{i+1}$. *Set* $N = \sum_{i=1}^n d_i$. *Cosider a* $n-$*tuple* $(a_1, ..., a_n)$ *of non-zero real numbers and for* $R > 0$ *define the portion of hyperplane*

$$S_R = \ \{(s_1, ..., s_n) \in \mathbb{R}^n \text{ such that } \sum_{i=1}^n a_i s_i = 0, \text{ and } |s_i| < R^{d_i}, \text{ for } i = 1, ..., n\ \}.$$

There exists $R_o = R_o(a_i, d_i, n, r)$ *such that if* $0 < R < R_o$, *then*

$$H_{n-1}(S_R) \ \ge \ \frac{\sqrt{\sum_{j=1}^n a_j^2}}{|a_i|}\ R^{N-d_1}\ , \tag{4.3}$$

where H_{n-1} *denotes the* $n-1$ *dimensional Hausdorff measure in* R^n.

Proof of Lemma 4.3. As in the proof of Theorem 3.3, from (3.25) we have that for any $i = 1, ..., n$,

$$H_{n-1}(S_R) \ = \ \frac{\sqrt{\sum_{j=1}^n a_j^2}}{|a_i|}\ H_{n-1}(\pi_i(S_R))\ , \tag{4.4}$$

where $\pi_i(S_R)$ represents the projection of $S_R \subset \mathbb{R}^n$ onto the coordinate hyperplane $\{s_i = 0\}$.

We claim that there exists R_o as in the statement of the theorem, such that for $0 < R < R_o$, the projection $\pi_1(S_R)$ is as large as possible, i.e.

$$H_{n-1}(\pi_1(S_R)) \ = \ R^{N-d_1}. \tag{4.5}$$

To verify this statement we choose any point

$$(s_2, ..., s_n) \in \mathbb{R}^{n-1} \text{ with } |s_i| < R^{d_i}, \text{ for } i = 2, ..., n.$$

Define

$$s_1 = -\sum_{i=2}^n \frac{a_i}{a_1} s_i.$$

A simple computation shows that

$$|s_1| \le \sum_{i=2}^n \frac{|a_i|}{|a_1|} R^{d_i} \le R^{d_1} \sum_{i=2}^n \frac{|a_i|}{|a_1|} R^{d_i - d_1}.$$

Hence, for R_o small enough and $0 < R < R_o$ we have $|s_1| < R^{d_1}$, and consequently $(s_1, ..., s_n) \in S_R$. This shows that $(s_2, ..., s_n) \in \pi_1(S_R)$ and proves (4.5). □

The lemma above is a key step in the proof of the following

Theorem 4.4. *Let* $\boldsymbol{G}$ *be a Carnot group of step* r *with homogeneous dimension* Q. *Consider a bounded, open set*

$$\Omega \ = \ \{g \in \boldsymbol{G} \ \ |\ \ \phi(g) < 0\}\ ,$$

where $\phi \in C^{1,1}(\boldsymbol{G})$ *is a defining function for* Ω. *Let* $g_o \in \partial\Omega$ *be of type* k_0. *There exists* $R_o = R_o(g_o, \boldsymbol{G}, X, \Omega) > 0$, *and* $C = C(g_o, \boldsymbol{G}, X, \Omega) > 0$, *such that for* $0 < R < R_o$ *we have*

$$\sigma(B(g_o, R) \cap \partial\Omega) \ge C R^{Q-k_0}. \tag{4.6}$$

The constant can be chosen to be $C = \frac{\sqrt{\sum_{j\in\mathcal{N}} a_j^2}}{|a_{k_0}|}$.

Remark 4.5. *Notice that this theorem, together with the argument in* (3.29) *implies that in any Carnot group* $\boldsymbol{G}$ *and for any* $C^{1,1}$ *domain* $\Omega \subset \boldsymbol{G}$, *there exists a constant* $C = C(\Omega, \boldsymbol{G}) > 0$, *such that*

$$C^{-1} R^{Q-type(g_o)} \leq \sigma(B(g_o,R)\cap\partial\Omega) \leq C R^{Q-type(g_o)}, \tag{4.7}$$

for R *sufficiently small depending on* g_o, ϕ *and* $\boldsymbol{G}$.

Proof of Theorem 4.4. We will use the notations and the reductions introduced in the proof of Theorem 3.3. In particular, it is clear that in order to prove (4.6) it suffices to show that there exist $C = C(g_o, \mathfrak{g}, X, D, 0) > 0$, a nd $R_o = R_o(g_o, \mathfrak{g}, X, D, 0) > 0$, such that for any $0 < R < R_o$, one has

$$\sigma(\text{Box}(R)\cap\Pi) \geq C\ R^{Q-k_0}\ . \tag{4.8}$$

In view of the argument between (3.23) and (3.24), the estimate (4.8) will immediately follow from

$$\begin{aligned} \sigma(\{(-R,R)^{m_1-1}\times ... \times(-R^r,R^r)^{m_r-1}\times S\}) = \quad & R^{\sum_{j\in\mathcal{N}^C} jm_j} R^{\sum_{l\in\mathcal{N}} l(m_l-1)} H_{|\mathcal{N}|-1}(S) \\ \geq \quad & C\ R^{Q-k_0}\ , \end{aligned} \tag{4.9}$$

where S is defined as in the proof of Theorem 3.3.

At this point we use Lemma 4.3 substituting $n = |\mathcal{N}|$, $\mathcal{N} = I$, and $d_i = \mathcal{N}_i$. Estimate (4.3) yields

$$H_{|\mathcal{N}|-1}(S) \geq \frac{\sqrt{\sum_{j\in\mathcal{N}} a_j^2}}{|a_{k_0}|}\ R^{\sum_{j\in\mathcal{N}, j\neq k_0} j}. \tag{4.10}$$

The conclusion now follows immediately from (4.9) and (4.10), in fact

$$\begin{aligned} H_{r-1}(B(g_o,R)\cap\partial\Omega) = \quad & H_{r-1}(\text{Box}(R)\cap\Pi)(1+|O(R)|) \\ \geq \quad & H_{r-1}(\text{Box}(R)\cap\Pi) \\ \geq \quad & R^{\sum_{j\in\mathcal{N}^C} jm_j} R^{\sum_{l\in\mathcal{N}} l(m_l-1)} H_{|\mathcal{N}|-1}(S) \\ \geq \quad & \frac{\sqrt{\sum_{j\in\mathcal{N}} a_j^2}}{|a_{k_0}|}\ R^{\sum_{j\in\mathcal{N}^C} jm_j}\ R^{\sum_{l\in\mathcal{N}} l(m_l-1)}\ R^{\sum_{j\in\mathcal{N}, j\neq k_0} j}. \\ \geq \quad & \frac{\sqrt{\sum_{j\in\mathcal{N}} a_j^2}}{|a_{k_0}|}\ R^{Q-k_0}. \end{aligned} \tag{4.11}$$

□

The proof of Theorem 4.4 and estimate (4.3) allow us to prove easily the following

Corollary 4.6. *In the hypothesis of Theorem 4.4, with $k_0 = 2$ we have that*

(4.12)
$$H_{dim(\mathfrak{g})-m_1-1}\Big[\pi_1(Box(R)\cap\Pi)\Big] \geq CR^{\sum_{j\in\mathcal{N}^C, j\neq 1} jm_j} R^{\sum_{l\in\mathcal{N}} l(m_l-1)} R^{\sum_{l\in\mathcal{N}, l\neq 2} l} \;=\; CR^{Q-m_1-2},$$

where $\pi_1 : \mathfrak{g} \to V_2 \oplus ... \oplus V_r$ denotes the orthogonal projection onto the complement of V_1.

We are now ready for the proof of the main result of this section

Proof of Theorem 4.1. We will use the notations and the reductions introduced in the proof of Theorem 3.3. In particular, it is clear that in order to prove (4.1) it suffices to show that if the origin is in ∂D and we denote by Π the tangent space to the boundary at the origin, then there exist $C = C(\mathfrak{g}, X, \Omega) > 0$, and $R_o = R_o(\mathfrak{g}, X, \Omega) > 0$ which will depend continuously on the choice of g_o in the statement of the theorem, and such that for any $0 < R < R_o$, one has

$$\int_{\text{Box}(R)\cap\Pi} |X\phi|\, d\sigma \;\geq C\; R^{Q-1}\;. \tag{4.13}$$

and consequently,

$$\begin{aligned}\int_{\text{Box}(R)\cap\partial D} |X\phi|\, d\sigma = &\quad \int_{\text{Box}(R)\cap\Pi} |X\phi|(1+O(|\xi|))d\sigma\\ \geq &\quad C_1 \int_{\text{Box}(R)\cap\Pi} |X\phi|\, d\sigma\\ \geq &\quad C_2\; R^{Q-1}.\end{aligned}$$

The proof is divided in two parts: In the first part we will study the case when the origin is of type one, i.e. it is a non-characteristic point.

Type one: If the origin is not characteristic then $a_1 \neq 0$, and $k_0 = 1$. Since the non-characteristic hypothesis is an open condition, then it holds in a neighborhood of the point in question, giving us local uniform control on $|a_1|$ (from here on a_1 will be part of the constants that we use in the estimates). The key (elementary) observation is that

$$0 < C^{-1} \leq \sup_{\Pi\cap Box(R)} |X\pi| \leq C, \tag{4.14}$$

where $C = C(a_1, \boldsymbol{G}) > 0$.

In view of (3.23), (3.24), and (4.14) the estimate (4.13) will immediately follow from

$$\sigma(\text{Box}(R)\cap\Pi) \geq \; C\; R^{Q-1}\;. \tag{4.15}$$

This is an immediate consequence of Theorem 4.4.

Type two: Here we need to assume that $\phi \in C^2$ in a neighborhood of $\partial\Omega$. Repeating the argument in (3.11)-(3.13) we obtain

$$\phi(\xi) = \pi(\xi) + H, \tag{4.16}$$

with $H = o(|\xi|^2)$, and for $\xi \in B(0, R_o)$, and for sufficiently small R_o. Consequently we obtain

$$\nabla\phi(\xi) = \nabla\phi(0) + \vec{o}(\xi), \tag{4.17}$$

where $\vec{o}(\xi) = \{o_{k,j}(\xi)\}$, $k = 1, ..., r$, $j = 1, ..., m_k$ and $o_{k,j}(\xi) = o(|\xi|)$. In view of (4.17) and Lemma 2.1, we can compute the horizontal gradient of ϕ as follows

$$\begin{aligned} X_i\phi(\xi) = < X_i,\ \nabla\phi(\xi) >= & \quad < X_i, \nabla\phi(0) + \vec{o} > \\ = & \quad \sum_{k=1}^{r}\sum_{l=1}^{m_k} b^i_{kl}(\xi_1, ..., \xi_{k-1})\Big(\frac{\partial\pi}{\partial x_{k,l}} + o_{kl}(\xi)\Big). \end{aligned} \tag{4.18}$$

Here, for simplicity, we have let $b^i_{1,l} = \delta_{il}$.

Repeating the argument in Lemma 3.7, with the new regularity hypothesis $\phi \in C^2$, and knowing that the origin is of type two, we obtain

Lemma 4.7. *In the notation established above, for every $\epsilon > 0$ we can choose $R_o = R_o(\epsilon, \mathfrak{g}, X, \Omega) > 0$ such that if $0 < R < R_o$, one has*

$$\left(\sup_{Box(R)} |\sum_{k=1}^{r}\sum_{l=1}^{m_k} b^i_{kl}(\xi_1, ..., \xi_{k-1}) o_{kl}(\xi)|\right)\ \sigma(Box(R)\cap\Pi) \le\ \epsilon\ R^{Q-1} \tag{4.19}$$

Consequently, the estimate (4.13) is reduced to the proof of

$$\int_{\mathrm{Box}(R)\cap\Pi} |X\pi| d\sigma \ge C\ R^{Q-1} \tag{4.20}$$

In fact, if (4.20) holds, then from (4.18), and (4.19) we would obtain

$$\int_{\mathrm{Box}(R)\cap\Pi} |X\phi|\ d\sigma \ge C\int_{\mathrm{Box}(R)\cap\Pi} |X\pi| d\sigma - \epsilon R^{Q-1} \ge\ CR^{Q-1}. \tag{4.21}$$

We now proceed with the proof of (4.20). Since the origin is of type two, then we have

$$\pi(\xi) = \pi(\xi_2, ..., \xi_r) = \sum_{k\in\mathcal{N}} a_k \xi_{k,1},$$

and for $\xi \in \mathrm{Box}(R)\cap\Pi$, it follows that

$$\begin{aligned} |X_i\pi(\xi)| = & \quad |\sum_{k\in\mathcal{N}} b^i_{k,1} a_k| \\ \ge & \quad |b^i_{2,1}a_2| - |\sum_{k\in\mathcal{N},\ k>2} b^i_{k,1} a_k| \\ \ge & \quad |b^i_{2,1}a_2| - \sum_{k\in\mathcal{N},\ k>2} CR^{k-1}. \end{aligned} \tag{4.22}$$

The crucial observation now is that $b^i_{2,1}$ depends only on ξ_1, and not on the higher order coordinates.

Remark 4.8. *Because of the definition of Carnot group, we know that there exists at least one* $i = 1, ..., m_1$, *and one* $l = 1, ..., m_2$ *such that* $b^l_{2,i} \neq 0$. *Without loss of generality we can assume that* $b^1_{2,i} \neq 0$ *for some* i. *In fact, if that is not the case and* $b^{l_0}_{2,i} \neq 0$ *for some* $l_0 \neq 1$, *then we will change the definition of* a_2 *by rotating the* V_2 *component of* Π *onto the direction* l_0. *The rest of the proof will follow with trivial changes.*

On the other hand, since π does not depend on ξ_1, we have that

$$\text{Box}(R) \cap \Pi = (-R, R)^{m_1} \times \pi_1(\text{Box}(R) \cap \Pi),$$

where $\pi_1 : \mathfrak{g} \to V_2 \oplus ... \oplus V_r$ denotes the orthogonal projection onto the complement of V_1. Corollary 4.6 yields

$$H_{\dim(\mathfrak{g})-m_1-1}\Big[\pi_1(\text{Box}(R) \cap \Pi)\Big] \geq C R^{Q-m_1-2}. \tag{4.23}$$

Estimate (4.22) allows us to infer

$$\int_{\text{Box}(R)\cap\Pi} |X\pi| d\sigma \geq \sum_{i=1}^{m_1} \int_{\text{Box}(R)\cap\Pi} \Big(|b^i_{2,1} a_2| - \sum_{k\in\mathcal{N},\ k>2} C R^{k-1} \Big) d\sigma$$

$$\text{arguing as in (3.29)} \quad \geq \Big(\sum_{i=1}^{m_1} \int_{\text{Box}(R)\cap\Pi} |b^1_{2,i} a_2| d\sigma \Big) - \frac{\sqrt{\sum_{j\in\mathcal{N}} a_j^2}}{|a_2|} \Big(\sum_{k\in\mathcal{N},\ k>2} C R^{k-1} \Big) R^{Q-2}$$

From (4.23) we infer

$$\sum_{i=1}^{m_1} \int_{\text{Box}(R)\cap\Pi} |b^i_{2,1} a_2| d\sigma$$

$$\geq C \left(R^{\sum_{j\in\mathcal{N}^C,\ j\neq 1} j m_j} R^{\sum_{l\in\mathcal{N}} l(m_l-1)} R^{\sum_{l\in\mathcal{N}, l\neq 2} l} \right) \sum_{i=1}^{m_1} \int_{(-R,R)^{m_1}} |b^i_{2,1} a_2| d\xi_{1,1} ... d\xi_{1,m_1}$$

$$\geq C R^{Q-m_1-2} \sum_{i=1}^{m_1} \int_{(-R,R)^{m_1}} |b^i_{2,1} a_2| d\xi_{1,1} ... d\xi_{1,m_1} \tag{4.24}$$

Since $b^i_{2,1}$ is a non-zero polynomial of order one in $\xi_{1,i}$, $i = 1, ..., m_1$, we can write $p(\xi) = b^i_{2,1} a_2$, and set

$$C_0 = \int_{(-1,1)^{m_1}} |p(\xi)| d\xi > 0.$$

Now make a change of variables $\xi' = R\xi$ and obtain

$$\int_{(-R,R)^{m_1}} |p(\xi)| d\xi = C_0 \ R^{m_1+1}.$$

Consequently

$$\sum_{i=1}^{m_1} \int_{(-R,R)^{m_1}} |b^i_{2,1} a_2| d\xi_{1,1} ... d\xi_{1,m_1} \geq \ C_o \ R^{m_1+1}. \tag{4.25}$$

At this point the conclusion follows from (4.22), (4.24), (4.24), and (4.25).

□

Remark 4.9. *An elementary modification of the argument above yields that* (4.20) *holds for arbitrary points, regardless of the type of the center of the surface ball. In particular, we have that for every hyperplane* $\pi \subset \mathfrak{g}$ *passing through the origin and for every* $0 < R < R_0$,

$$C_1 R^{Q-1} \geq \int_{Box(R)\cap\Pi} |X\pi| d\sigma \geq C_2 \ R^{Q-1} \tag{4.26}$$

for some choice of the positive constants R_0, C_1, *and* C_2 *depending on* π. *Note that this does not implies that the perimeter measure of hyperplanes is* $1-$*Ahlfors-regular, at least for Carnot groups of step higher than three. In fact, when we translate an hyperplane so that a fixed boundary point goes to the origin, unless the group law is an affine transformation of* $\mathfrak{g}$ *the defining function of the translated domain will not be linear anymore. For Carnot groups of step two, the estimate above indeed implies that the perimeter measure of hyperplanes are* $1-$*Ahlfors regular. In a certain sense, estimate* (4.26) *shows the effect of higher order data (like for instance curvature) on the Ahlfors estimates in the Carnot group setting.*

We conclude this section by giving an alternative, more indirect approach to the lower Ahlfors regularity in a Carnot group of step 2. Using the relative isoperimetric inequality established in [GN1] and the geometric properties for minimally smooth domains obtained in [CG], we are able to establish the lower 1-Ahlfors reagularity for $C^{1,1}$ domains. We emphasize that such smoothness is sharp, since in Section 7.2 of [DGN2] it was proved that in the Heisenberg group $\mathbb{H}^n$ for every $\alpha \in (0,1)$ there exists a bounded $C^{1,\alpha}$ domain whose perimeter measure fails to be lower 1-Ahlfors regular.

Theorem 4.10. *Let* $\boldsymbol{G}$ *be a Carnot group of step two, and let* $\Omega \subset \boldsymbol{G}$ *be a* $C^{1,1}$ *domain. There exist constants* $C = C(\Omega, \boldsymbol{G}) > 0$ *and* $R_o = R_o(\Omega, \boldsymbol{G}) > 0$ *such that*

$$P_X(\Omega, B(g,R)) \ \geq \ C \ R^{Q-1} \ ,$$

for every $0 < R < R_o$ *and for all* $g \in \partial\Omega$.

The proof of Theorem 4.10 will follow immediately from the following two results. The former is a special case of the general result in Theorem 1.17 in [GN1]. We recall that in every CC space the metric balls are Poincaré-Sobolev domains.

Theorem 4.11. *Let* $\boldsymbol{G}$ *be a Carnot group with homogeneous dimension* Q. *There exists a positive constant* $C_1 = C_1(\boldsymbol{G})$ *such that for every set of locally finite* X*-perimeter* $\Omega \subset \boldsymbol{G}$ *one has*

$$\min(|\Omega \cap B(g,R)|, |\Omega^c \cap B(g,R)|)^{\frac{Q-1}{Q}} \ \leq \ C_1 \ P_X(\Omega, B(g,R)) \ ,$$

for all $R > 0$.

The next result was established in [CG].

Theorem 4.12. *A bounded $C^{1,1}$ domain Ω in a Carnot group of step two satisfies the exterior and the interior corkscrew conditions. In particular, there exist positive constants C_2 and R_2 such that*

$$\min(|\Omega \cap B(g,R)|, |\Omega^c \cap B(g,R)|) \ \geq \ C_2 \ R^Q \ ,$$

for all $g \in \partial\Omega$, and every $0 < R < R_2$.

It is obvious that Theorem 4.10 follows from Theorems 4.11 and 4.12, and from the fact that if $\Omega \subset \boldsymbol{G}$ is bounded and $C^{1,1}$, then it has locally finite X-perimeter. Using (2.11), the lower Ahlfors estimate in Theorem 4.10 is easily seen to hold also if we substitute metric balls with the gauge balls.

5. Counterexample to the $1-$Ahlfors estimate for groups of step $r \geq 3$

In Theorem 1.2 we have established the 1-Ahlfors regularity of the X-perimeter measure under the assumption that Ω be a C^2 domain of type ≤ 2. We recall that such hypothesis is automatically fulfilled when the step of the group is $r = 2$. In this section we show that the type assumption is optimal, in the sense that we prove the existence of a group $\boldsymbol{G}$ of step 3, and of a domain $\Omega \subset \boldsymbol{G}$ of type 3 for which the X-perimeter measure fails to be $1-$Ahlfors regular. We remark that additional smoothness does not suffice since in our example the defining function ϕ of Ω is of class C^∞.

We consider the cycle group $\boldsymbol{G} = K_3$, see ex. 1.1.3 in [CGr], whose Lie algebra is given by the stratification,

$$\boldsymbol{G} \ = \ V_1 \oplus V_2 \oplus V_3 \ ,$$

where $V_1 = span\{X_1, X_2\}$, $V_2 = span\{X_3\}$, and $V_3 = span\{X_4\}$, so that $m_1 = 2$ and $m_2 = m_3 = 1$. We assign the commutators

$$[X_1, X_2] \ = \ X_3 \qquad [X_1, X_3] \ = \ X_4 \ , \tag{5.1}$$

all other commutators being assumed trivial. We observe right-away that the homogeneous dimension of $\boldsymbol{G}$ is

$$Q \ = \ m_1 \ + \ 2\, m_2 \ + \ 3\, m_3 \ = \ 7 \ .$$

The group law in $\boldsymbol{G}$ is given by the Baker-Campbell-Hausdorff formula (2.9). In exponential coordinates, if $g = \exp(X)$, $g' = \exp(X')$, where $X = \sum_{i=1}^4 x_i X_i$, $X' = \sum_{i=1}^4 y_i X_i$, we have

$$g \ \circ \ g' \ = \ X \ + \ X' \ + \frac{1}{2}\, [X, X'] \ + \ \frac{1}{12}\, \{[X,[X,X']] \ - \ [X',[X,X']]\}.$$

A computation based on (5.1) gives (see also ex. 1.2.5 in [CGr])

$$g \ \circ \ g' \ = \ \Big(x_1 + y_1, x_2 + y_2, x_3 + y_3 + P_3, x_4 + y_4 + P_4\Big) \ ,$$

where

$$P_3 \;=\; \frac{1}{2}\,(x_1 y_2 - x_2 y_1)\ ,$$

$$P_4 \;=\; \frac{1}{2}(x_1 y_3 - x_3 y_1) + \frac{1}{12}\left(x_1^2 y_2 - x_1 y_1 (x_2 + y_2) + x_2 y_1^2\right)\ .$$

Using (2.8), (2.9) we find that a left invariant basis of the Lie algebra $\mathfrak{g}$ is given by the vector fields

$$\begin{aligned} X_1 &= \frac{\partial}{\partial x_1} - \frac{x_2}{2}\frac{\partial}{\partial x_3} - \left(\frac{x_3}{2} + \frac{x_1 x_2}{12}\right)\frac{\partial}{\partial x_4}\ ,\\ X_2 &= \frac{\partial}{\partial x_2} + \frac{x_1}{2}\frac{\partial}{\partial x_3} + \frac{x_1^2}{12}\frac{\partial}{\partial x_4}\ ,\\ X_3 &= \frac{\partial}{\partial x_3} + \frac{x_1}{2}\frac{\partial}{\partial x_4}\ ,\\ X_4 &= \frac{\partial}{\partial x_4}\ . \end{aligned} \tag{5.2}$$

We now consider the smooth function

$$\phi(g) \;=\; x_1 x_2 \;+\; x_4\ ,$$

for which we obviously have

$$\nabla\phi(g) \;=\; \left(x_2, x_1, 0, 1\right)\ \neq\ 0\ ,$$

and the C^∞ domain $\Omega = \{g \in \boldsymbol{G} \mid \phi(g) < 0\}$. Using this formula and (5.2) we easily obtain

$$X\phi(g) \;=\; (X_1\phi(g), X_2\phi(g)) \;=\; \left(x_2 - \big(\frac{x_3}{2} + \frac{x_1 x_2}{12}\big),\ x_1 + \frac{x_1^2}{12}\right)\ . \tag{5.3}$$

We note explicitly that the characteristic set Σ of Ω is non-empty, therefore Ω is at least of type 2. Using (5.3) we see that $\Sigma = \Sigma_1 \cup \Sigma_2$, where Σ_1 and Σ_2 are identified by the equations

$$\Sigma_1\ :\qquad x_1 \;=\; 0\ ,\qquad x_3 \;=\; 2\,x_2\ ,\qquad x_4 \;=\; 0\ ,$$

$$\Sigma_2\ :\qquad x_1 \;=\; -\,12\ ,\qquad x_3 \;=\; 2\,x_4\ .$$

However, (5.1) and the equation for X_3 in (5.2) give

$$[X_1, X_2]\phi(g) \;=\; X_3\phi(g) \;=\; \frac{x_1}{2}\ ,$$

and the latter function vanishes on Σ_1. This shows that all points in Σ_1 are of type 3, and therefore so is Ω. Since the tangent hyperplane at the origin Π is given by $\{x_4 = 0\}$, we easily find

$$\sigma(\mathrm{Box}(R)\ \cap\ \Pi) \;=\; 2\pi R^4\ . \tag{5.4}$$

If we choose $g_R = (R/2, 0, 0, 0) \in \{\phi = 0\} \cap \mathrm{Box}(R)$, then

$$|X\phi(g_R)| \;=\; \frac{R}{2}\left(1 + \frac{R}{6}\right)\ . \tag{5.5}$$

In view of (5.3)-(5.5), we conclude that for R sufficiently small the following estimate holds

$$\left(\sup_{\mathrm{Box}(R)\ \cap\ \{\phi=0\}} |X\phi|\right)\sigma(\mathrm{Box}(R) \cap \Pi)\ \geq\ C\ R^5\ , \tag{5.6}$$

for some constant $C > 0$ independent of R. Since $Q = 7$, it follows that (3.2) cannot hold with $\alpha = 1$.

Next, we turn our attention to (3.4), and show that also the estimate

$$P_X(\Omega, B(0,R)) \cong \int_{\text{Box}(R)\, \cap\, \{\phi=0\}} |X\phi|(g)d\sigma(g) \le C\ R^{Q-1} \tag{5.7}$$

fails for our choice of ϕ. In fact, we observe that we can parametrize $\partial\Omega = \{\phi = 0\}$ as a graph $x_4 = -x_1x_2$. If we write $X\phi = (x_2 - p_2, x_1 + p_1)$, with $p_2 = \left(\frac{x_3}{2} + \frac{x_1x_2}{12}\right)$, $p_1 = \frac{x_1^2}{12}$, we can estimate

$$\begin{aligned}
&\int_{\text{Box}(R)\cap\{\phi=0\}} |X\phi|(g)d\sigma(g)\\
&\ge \int_{-R^2}^{R^2} \int_{x_1^2+x_2^2<R^2} \left\{\sqrt{x_1^2+x_2^2} - 2\sqrt{|x_1p_1|+|x_2p_2|}\right\} \sqrt{1+x_1^2+x_2^2}dx_1dx_2dx_3\\
&\ge 2\ R^2 \int_{x_1^2+x_2^2<R^2} \sqrt{x_1^2+x_2^2}dx_1dx_2\\
&- 2 \sup_{(-R,R)^2\times(-R^2,R^2)} \sqrt{|x_1p_1|+|x_2p_2|} \int_{-R^2}^{R^2} \int_{x_1^2+x_2^2<R^2} \sqrt{1+x_1^2+x_2^2}\ dx_1dx_2dx_3\\
&\ge C_1\ R^5 - C_2\ R^{3/2}\ R^4 \ge C\ R^5\ .
\end{aligned}$$

Since $Q = 7$ the latter estimate contradicts (5.7). To conclude the section we want to point out that we also have the estimate

$$\begin{aligned}
\int_{\text{Box}(R)\cap\{\phi=0\}} & |X\phi|(g)d\sigma(g)\\
\le\ & C\left(R^2\int_{(-R,R)^2}\sqrt{x_1^2+x_2^2}dx_1dx_2 + R^{\frac{3}{2}}R^4\right)\\
\le\ & C\ R^5\ . \qquad (5.8)
\end{aligned}$$

6. Geometric estimates for a system of free vector fields of Hörmander type

Our next goal consists in extending the results in the previous section and prove area estimates for surface Carnot-Caratheodory balls associated to a free system of Hörmander vector fields. Our proof rests principally on the computations in the group case, and on the Rothschild-Stein approximation Theorem 2.6, which allows to locally approximate the vector fields $X_1, ..., X_m$ with left invariant generators $Y_1, ..., Y_m$ of a free Lie algebra.

We will use the notation introduced in Theorem 2.6, and in the preceding paragraphs. Moreover, we will denote by $B(x,R)$ the solid Carnot-Caratheodory balls in $\mathbb{R}^n$, and with $B_R(y)$ the Carnot-Caratheodory balls in Lie groups or in their Lie algebras.

Let $\Omega = \{x \in M^n \mid \phi(x) < 0\} \subset M^n$ be a bounded $C^{1,1}$ domain, and $X_1, ..., X_m$ smooth vector fields, free up to step r in a neighborhood of $\partial\Omega$, and satisfying the Hörmander condition

(1.2). Choose a positive $R_1 = R_1(X, \Omega, x_o) < 1$, small enough such that $B(x_o, 2R_1) \subset V$, where V is the neighborhood of Q, which is the domain for the coordinate chart

$$\theta_{x_o}(\cdot) : V \to U \subset \boldsymbol{G}_{m,r},$$

as in Theorem 2.6 (**A**). For $0 < R < R_1$, set $\Delta = \Delta(x_o, R) = B(x_o, R) \cap \partial\Omega$. Let us adopt the following notation

$$\begin{aligned} \Omega' &= \theta_{x_o}(\Omega) \subset \boldsymbol{G}_{m,r} \\ \phi'(x) &= \phi(\theta_Q^{-1}(x)), \text{ so that } \Omega' = \{y \in \boldsymbol{G}_{m,r} \mid \phi'(y) < 0\} \end{aligned} \tag{6.1}$$

Definition 6.1. *Let $X_1, ..., X_m$ smooth vector fields in M^n, free up to step r and satisfying the Hörmander condition (1.2). Let $\Omega = \{x \in M^n \mid \phi(x) < 0\} \subset M^n$ be a bounded $C^{1,1}$ domain. Choose any collection $\{X_{ik}\}$, of commutators of length k with $X_{i1} = X_i$ such that the system $\{X_{ik}\}$, $k = 1, ..., r$ evaluated at $x_o \in \partial\Omega$ is a basis of $\mathbb{R}^n$. We define the "type" of x_o to be the smallest $k = 1, ..., r$ such that there exists $l = 1, ..., m_k$ for which $X_{k,l}\phi(g_0) \neq 0$. We will denote by*

$$type(x_o)$$

the type of x_o, and if for every $x_o \in \partial\Omega$ we have that $type(x_o) \leq s \in \mathbb{N}$ then we will say that Ω has type less or equal than s.

Remark 6.2. *Note that the definition of "type" of x_o is independent of the choice of the collection $\{X_{ik}\}$, in view of the definition of free vector fields. An equivalent definition is the following: We define the "type" of x_o to be the smallest $k = 1, ..., r$ such that there exists a commutator $Z = [X_{i_1}, [X_{i_2}, ..., [X_{i_{k-1}}, X_{i_k}]...]$ of order k such that $Z\phi(x_o) \neq 0$.*

We want to rephrase the notion of type in terms of the osculating group $\boldsymbol{G}_{m,r}$.

Lemma 6.3. *Let $\Omega = \{x \in M^n \mid \phi(x) < 0\} \subset M^n$ be a bounded $C^{1,1}$ domain, and $X_1, ..., X_m$ smooth vector fields, free up to step r in a neighborhood of $\partial\Omega$, and satisfying the Hörmander condition (1.2). For $x_o \in \partial\Omega$ denote by $\boldsymbol{G}_{m,r}$ the osculating free group from Theorem 2.6 and by $Y_1, ..., Y_m$ a left invariant basis of $\mathfrak{g}_{m,r}$. If x_o is of type less or equal than two in M^n (according to Definition 6.1), then the origin is of type less or equal then two in Ω' (according to Definition 3.1).*

Proof. Notice that for any $N > r$, and $i = 1, ..., m$, we have

$$\mathcal{R}_i\phi'(0) = \sum_{l=1}^{r} \sum_{|\alpha|=l} p_{\alpha,i,N}(0)\partial_{y_\alpha}\phi'(0) \tag{6.2}$$

where $p_{\alpha,i,N}$ are homogeneous group polynomials of order greater or equal than $|\alpha| \geq 1$. In particular, $p_{\alpha,i,N}(0) = 0$, and we obtain

$$\mathcal{R}_i\phi'(0) = 0, \quad i = 1, ..., m. \tag{6.3}$$

In case,

$$\text{type}(x_o) = 1$$

then (2.23) implies

$$X\phi(x_o) = Y\phi'(0). \tag{6.4}$$

Hence the origin is of type one in $\Omega' \subset \boldsymbol{G}_{m,r}$. Next, we assume

$$\text{type}(x_o) = 2$$

and consider any collection $\{X_{k,i}\}$ as in Definition 6.1. Recall from Theorem 2.6 that for any multi-index $\beta = (2, i)$, $i = 1, ..., m_2$ we have

$$X_\beta\phi(x_o) = Y_\beta\phi'(0) + \mathcal{R}_\beta\phi'(0),$$

where $\mathcal{R}_\beta$ is a vector field of order less or equal than 1. At this point we observe that modulo higher order terms (which will vanish at the origin) we must have

$$\mathcal{R}_{(2,i)}\phi'(0) = \sum_{l=1}^{r}\sum_{|\alpha|=l} p_{\alpha,\beta,N}(0)\partial_{y_\alpha}\phi'(0) \tag{6.5}$$

with the degree of $p_{\alpha,\beta,N}$ greater or equal than $|\alpha| - 1$. The only term which will not vanish in this expression are those corresponding to $|\alpha| = 1$, which lead us to

$$\mathcal{R}_{2,i}\phi'(0) = \sum_{|\alpha|=1} p_{\alpha,\beta,N}(0)\partial_{y_\alpha}\phi'(0) = \sum_{j=1}^{m} c_j Y_j\phi'(0), \tag{6.6}$$

for the choice of the coefficients $c_j = p_{(1,j),\beta,N}(0)$. Since we are assuming that $X\phi(x_o) = 0$ then in view of (6.4), we also have $Y\phi'(0) = 0$, and consequently $\mathcal{R}_\beta\phi'(0) = 0$. At this point Theorem 2.6 gives us the equality

$$X_\beta\phi(x_o) = Y_\beta\phi'(0),$$

for any $\beta = (2, i)$, $i = 1, ..., m_2$. This implies that the type of the origin in Ω' is two. □

The argument in the previous proof can be easily generalized to yield

Corollary 6.4. *Using the notation in the previous lemma: If x_o is of type less or equal than k in M^n (according to Definition 6.1), then the origin is of type less or equal then k in Ω' (according to Definition 3.1).*

One of our main results in this section is the following

Theorem 6.5. *Let $X_1, ..., X_m$ be smooth vector fields in M^n, free up to step r and satisfying the Hörmander condition (1.2). Let $\Omega = \{x \in M^n \mid \phi(x) < 0\} \subset M^n$ be a bounded $C^{1,1}$ domain. For each point x_o in $\partial\Omega$ of type less or equal than two, there exists constants $C = C(\Omega, X, x_o) > 0$ and $R = R(\Omega, X, x_o) > 0$ depending continuously on x_o, such that if $0 < R < R_o$ then*

$$\left(\sup_{x\in\Delta(x_o,R)}|X\phi(x)|\right)\ \sigma(\Delta(x_o,R))\ \le\ C(\Omega,X,x_o)\frac{|B(x_o,R)|}{R}. \tag{6.7}$$

Here we recall that $\Delta(x_o,R)=B(x_o,R)\cap\partial\Omega$, and $|\cdot|$ denotes the Lebesgue-Hausdorff measure of the set.

Theorem 6.6. *Let $X_1,...,X_m$ be smooth vector fields in M^n, free up to step r and satisfying the Hörmander condition (1.2). Let $\Omega=\{x\in M^n \mid \phi(x)<0\}\ \subset M^n$ be a bounded $C^{1,1}$ domain, of type less or equal than two. There exist constants $C=C(\Omega,X)>0$, and $R_o=R_o(\Omega,X)>0$ such that for any $x_o\in\partial\Omega$ and $0<R<R_0$, one has that (6.7) holds with the uniform constants C and R_o.*

Proof. The idea of the proof is very simple: We rephrase the estimate (6.7) in terms of the "tangent" free algebra $\mathfrak{g}_{m,r}$ via the map $\theta(\cdot,\cdot)$ defined in Theorem 2.6 (B) and the exponential coordinates. At this point, formula (2.23) allows us to divide the problem in two steps. First we estimate the part corresponding to Y_i, using the results from the previous section, and then we deal with the error term $\mathcal{R}_i$ in (2.24). This error term is an operator of order less or equal than zero, hence it does not contribute (modulo higher order perturbations) to the final estimate. In the following we describe in detail this general idea.

Choose a positive $R_1=R_1(X,\Omega,x_o)<1$, small enough such that $B(x_o,2R_1)\subset V$, where V is the neighborhood of Q, which is the domain for the coordinate chart

$$\theta_{x_o}(\cdot):V\to U\subset \boldsymbol{G}_{m,r},$$

as in Theorem 2.6 (**A**). For $0<R<R_1$, set $\Delta=\Delta(x_o,R)=B(x_o,R)\cap\partial\Omega$, and let Ω' and ϕ' be as in (6.1).

Since $\theta_{x_o}:V\to U$ is a smooth map, then there exists a constant $C=C(X,\Omega,x_o)>0$ depending also on the Lipschitz norm of θ_{x_o} in $B(0,2R_1)$ such that

$$\begin{aligned}\sigma(\Delta)\le&\quad C\sigma(\theta_{x_o}(\Delta))\\ \le&\quad C\sigma(\theta_{x_o}(B(x_o,R))\cap\partial\Omega').\end{aligned} \tag{6.8}$$

Here we are introducing a slight ambiguity in the notation, in fact in the previous formula, we have used the same symbol σ to denote the surface measure in M^n (on the left hand side) and the surface measure of $\boldsymbol{G}_{m,r}$ (on the right hand side). Since it is clear which is which, and the two measures are bi-lipschitz equivalent, we will continue to use this notation in the interest of clarity.

Let us observe that since θ is a diffeomorphism we can choose a constant $C>0$ such that we also have the estimate

$$\sigma(\Delta)\ge C^{-1}\sigma(\theta_{x_o}(B(x_o,R))\cap\partial\Omega'). \tag{6.9}$$

Since $X_1,...,X_m$ are free up to step $r\in\mathbb{N}$ in a neighborhood of x_o, then the argument in Remark 2.5 allows us to write that for some constants $C_1,C_2>0$ depending only on Ω we have

(6.10) $$\text{Box}_{\,C_1 R} \subset B(x_o, R) \subset \text{Box}_{\,C_2 R},$$

where Box_R denotes the box-like sets defined in Remark 2.5, and $0 < R < R_1$, with a smaller R_1, if needed. Consequently, $\theta_Q(B(x_o, R)) \subset \boldsymbol{G}_{m,r}$ will be contained in one of the group "boxes" Box_{CR} of size comparable to R defined in (3.19). By virtue of the box-ball theorem (2.21) (see Theorem 3, [NSW]) we have

(6.11) $$\theta_{x_o}(B(x_o, R)) \subset B_{C_3 R}(0) \subset \boldsymbol{G}_{m,r},$$

for some positive constant C_3 depending only on X, and Ω. From (6.8)–(6.11) we obtain

(6.12) $$\sigma(\Delta) \le C\ \sigma(\text{Box}_R(0) \cap \partial\Omega) \le C\ \sigma(B_{C_3 R}(0) \cap \partial\Omega') = C\ \sigma(\Delta'),$$

where we have let $\Delta' = B_{C_3 R}(0) \cap \partial\Omega'$. Once again, we also have the reverse inequality

$$\sigma(\Delta') \le C\sigma(\Delta).$$

In view of Lemma 6.3, we know that $\Omega' \subset \boldsymbol{G}_{m,r}$ is a $C^{1,1}$ domain of type less or equal than two. This observation allows us to invoke Theorem 3.3, proved in the previous section, and infer that there exists $R_2 = R_2(\Omega', \boldsymbol{G}_{m,r}) > 0$, such that for any $0 < R < R_2$ one has

(6.13) $$\sigma(\Delta') \sup_{B_R(0)} |Y\phi'| \le C R^{Q-1},$$

for some positive constant $C = C(\Omega', \boldsymbol{G}_{m,r})$. Choose $R_0 = \min\{R_1, R_2\}$. In order to prove Theorem 6.6, we need to estimate the quantity

(6.14) $$\sigma(\Delta) \sup_{B(x_o,R)} |X\phi| \le C\sigma(\Delta') \Big(\sup_{B_R(0)} |Y\phi'| + \sup_{B_R(0)} \sum_{i=1}^{m} |\mathcal{R}_i \phi'| \Big)$$

in the range $0 < R < R_0$.

We will prove the following

Lemma 6.7. *In the notation established above, there exists $R(\Omega', \boldsymbol{G}_{m,r}) > 0$ such that for any $0 < R < R(\Omega', \boldsymbol{G}_{m,r})$ and for every $i = 1, ..., m$*

(6.15) $$|\Delta'| \sup_{B_R(0)} |\mathcal{R}_i \phi'| \le C R^Q,$$

for some positive $C = C(\Omega, \boldsymbol{G}_{m,r})$.

The proof of the Theorem follows immediately from Lemma 6.7, (6.13) and (6.14). □

Proof of Lemma 6.7. Following the arguments in (3.9)-(3.21), we make a number of reductions on the problem. In particular we will assume without loss of generality that the surface portion Δ' is a portion of the tangent hyper-plane

$$\Pi = \{(y_{ik}) \in \mathfrak{g}_{m,r} \mid \pi(y) \ = 0 \ \},$$

with $\pi(y) = \sum_{j=1}^r a_j y_{1,j}$, and we will substitute the gauge ball $B_R(0)$ with the box-like set $\text{Box}_R(0)$. As in (3.11) we write

$$\phi(y) = \pi(y) + H, \tag{6.16}$$

with $H = O(|y|^2)$. After such reductions we have that for some positive $R(\Omega', \boldsymbol{G}_{m,r}) < 1$, and $C = C(\Omega', \boldsymbol{G}_{m,r}) > 0$, if we choose $0 < R < R(\Omega', \boldsymbol{G}_{m,r})$ than

$$\begin{aligned} \sigma(\Delta') \sup_{B_R(0)} |\mathcal{R}_i \phi'| \leq & \quad C\sigma(\text{Box}_R(0) \cap \Pi) \Big\{ \sup_{\text{Box}_R(0)} |\mathcal{R}_i\Big(\sum_{j=1}^r a_j y_{1,j}\Big)| \ + \ \Big|\mathcal{R}_i(\ H\ )\Big|\Big\}, \\ \leq & \quad C\Big\{I \ + \ II\Big\}, \end{aligned} \tag{6.17}$$

where we have let

$$I \ = \ \sigma(\text{Box}_R(0) \cap \Pi) \ \sup_{\text{Box}_R(0)} |\mathcal{R}_i\Big(\sum_{j=1}^r a_j y_{1,j}\Big)|, \tag{6.18}$$

and

$$II \ = \ \sigma(\text{Box}_R(0) \cap \Pi) \ \sup_{\text{Box}_R(0)} \Big|\mathcal{R}_i(\ H\ )\Big|. \tag{6.19}$$

Now, we choose any integer N larger than the homogeneous dimension $Q = \sum_{i=1}^r i m_i$ of the group $\boldsymbol{G}_{m,r}$, and let $g_{\alpha,i,N}$ denote the higher order terms in the $N-$th order Taylor expansion of R_i (see 2.24).

Estimate of I: A direct computation yields

$$\begin{aligned} \mathcal{R}_i \quad & \Big(\sum_{j=1}^r a_j y_{1,j}\Big) = \sum_{l=1}^r \sum_{|\alpha|=l} [p_{\alpha,i,N}(y) + g_{\alpha,i,N}(y)] \partial_{y_\alpha} \Big(\sum_{j=1}^r a_j y_{1,j}\Big) \\ & = \sum_{l=1}^r \Big(\sum_{|\alpha|=l} [p_{\alpha,i,N} + g_{\alpha,i,N}] \Big) a_l. \end{aligned} \tag{6.20}$$

Because of the homogeneity of the polynomials $p_{\alpha,i,N}$, and the growth condition $g_{\alpha,i,N}(y) = O(|y|_{\mathfrak{g}}^N)$, we have for any $R > 0$ suitably small

$$\sup_{\text{Box}_R(0)} \sum_{|\alpha|=l} \Big[|p_{\alpha,i,N}| + |g_{\alpha,i,N}| \Big] \leq C_{\mathfrak{g}_{m,r}} R^l, \ \ l = 1, ..., r. \tag{6.21}$$

We also observe that

$$\sigma(\mathrm{Box}_R(0)\cap\Pi) = \sigma\Big(\{(y_{ik})\in\mathfrak{g}_{m,r} \mid \sum_{j=1}^r a_j y_{1,j}=0, \text{ and } \sum_{|\alpha|=l}|y_\alpha|^2\le R^{2l},\ l=1,...,r\}\Big)$$

$$= C_{m,r}R^{\sum_{l=1}^r l(m_l-1)}\sigma\Big(\{h\in\mathbb{R}^r \mid \sum_{l=1}^r a_l h_l = 0, \text{ and } |h_l|\le R^l,\ l=1,...,r,\}\Big)$$

$$(6.22) \le C_{m,r}R^{\sum_{l=1}^r l(m_l-1)} \min\Big\{\frac{R^{\sum_{l\neq 1} l}}{a_1},...,\frac{R^{\sum_{l\neq r} l}}{a_r}\Big\}.$$

Let j be the index corresponding to the minimum element in the list appearing in the last formula, so that we have

$$\frac{a_k}{a_j}\le\frac{R^j}{R^k}, \text{ for } k=1,...,r. \tag{6.23}$$

From (6.20)-(6.23) it is easy to deduce the following estimate

$$|\mathrm{Box}_r(0)\cap\Pi| \quad \sup_{gBux_r(0)}|R_i\Big(\sum_{j=1}^r a_j y_{1,j}\Big)|$$

$$\le C_{m,r}R^{\sum_{l=1}^r l(m_l-1)}\sup_{\mathrm{Box}_r(0)}\sum_{l=1}^r\Big[\Big(\sum_{|\alpha|=l}[|p_{\alpha,i,N}|+|g_{\alpha,i,N}|]\Big)\frac{a_l}{a_j}R^{\sum_{l\neq j} l}\Big]$$

$$\le C_{m,r}R^{\sum_{l=1}^r l(m_l-1)}\sup_{\mathrm{Box}_r(0)}\sum_{l=1}^r\Big[\Big(\sum_{|\alpha|=l}[|p_{\alpha,i,N}|+|g_{\alpha,i,N}|]\Big)\frac{R^{\sum_{l=1}^r l}}{R^l}\Big] \tag{6.24}$$

$$\text{by (6.21)} \quad \le C_{m,r}R^{\sum_{l=1}^r l(m_l-1)}R^{\sum_{l=1}^r l}$$

$$\le C_{m,r}R^{\sum_{l=1}^r l m_l} = C_{m,r}R^Q.$$

Estimate of II: Let $N>r$ and recall that $g_{\alpha,i,N}(y) = O(|y|_{\mathfrak{g}}^N)$. For every multi-index $\alpha=(k,j)$, denote by $O_\alpha = \partial_{y_\alpha}H$, and observe that since $\phi\in C^{1,1}$ then $O_\alpha = O(|y|)$. In view of the proof of Theorem 4.4 and Remark 4.5 we have

$$II \le CR^{Q-k_0}\Big(|\mathcal{R}_i(H)|\Big)$$

$$\le CR^{Q-k_0}\Big(|\sum_{l=1}^r\sum_{|\alpha|=l}p_{\alpha,i,N}\partial_{y_\alpha}H| + R^N\Big)$$

$$\le CR^{Q-k_0}\Big(\sum_{l=1}^r R^l\sum_{|\alpha|=l}|O_\alpha| + R^N\Big)$$

$$\le CR^{Q-k_0}\Big(\sum_{l=1}^r R^{l+1}\Big)$$

$$\le CR^{Q-k_0+2} \tag{6.25}$$

Since the type of Ω' is less or equal than two, then $k_0\le 2$ and the lemma is proved. □

We are now in a position to prove the lower Ahlfors regularity estimates.

Theorem 6.8. *Let $X_1, ..., X_m$ be smooth vector fields in M^n, free up to step r and satisfying the Hörmander condition (1.2). Let $\Omega = \{x \in M^n \mid \phi(x) < 0\} \subset M^n$ be a bounded C^2 domain. For each point x_o in $\partial\Omega$ of type less or equal than two, there exists constants $C = C(\Omega, X, x_o) > 0$ and $R = R(\Omega, X, x_o) > 0$ depending continuously on x_o, such that if $0 < R < R_o$ then*

$$\int_{\Delta(x_o,R)} |X\ \phi(x)| d\sigma(x) \ \geq\ C(\Omega, X, x_o) \frac{|B(x_o, R)|}{R}. \tag{6.26}$$

Here we recall that $\Delta(x_o, R) = B(x_o, R) \cap \partial\Omega$, and $|\cdot|$ denotes the Lebesgue-Hausdorff measure of the set.

The global estimates follow from the previous theorem through a standard compactness argument.

Theorem 6.9. *Let $X_1, ..., X_m$ be smooth vector fields in M^n, free up to step r and satisfying the Hörmander condition (1.2). Let $\Omega = \{x \in M^n \mid \phi(x) < 0\} \subset M^n$ be a bounded C^2 domain, of type less or equal than two. There exist constants $C = C(\Omega, X) > 0$, and $R_o = R_o(\Omega, X) > 0$ such that for any $x_o \in \partial\Omega$ and $0 < R < R_o$, one has that* (6.26) *holds with the uniform constants C and R_o.*

Proof of Theorem 6.8. We adopt the same notation as in the previous theorem. Using the same arguments as those in (6.8)–(6.12), we find that there exist constants $C = C(X, \Omega, x_o) > 0$ and $R_0 = R_0(X, \Omega, x_o) > 0$ such that

$$\int_{\Delta(x_o,R)} |X\ \phi(x)| d\sigma(x) \geq C \int_{\Delta'} \sum_{i=1}^{m} \Big(|Y_i \phi'| + |\mathcal{R}_i \phi'| \Big) d\sigma. \tag{6.27}$$

In view of Theorem 4.2 and Lemma 6.7 we obtain

$$\begin{aligned} \int_{\Delta(x_o,R)} |X\ \phi(x)| d\sigma(x) \quad & \geq C \int_{\Delta'} \sum_{i=1}^{m} \Big(|Y_i \phi'| + |\mathcal{R}_i \phi'| \Big) d\sigma \\ & \geq C \Big(\int_{\Delta'} \sum_{i=1}^{m} |Y_i \phi'| d\sigma - |\Delta'| \sup_{B_R(0)} |\mathcal{R}_i \phi'| \Big) \\ & \geq C \Big(R^{Q-1} - R^Q \Big) \\ & \geq C R^{Q-1}. \end{aligned} \tag{6.28}$$

And the proof of the theorem is complete.

□

Note that simple modifications of the arguments in this section's proofs allow to extend the estimates in Theorem 1.2 to $C^{1,1}$ domains in CC spaces corresponding to systems of free vector fields.

Theorem 6.10. *Let $X_1, ..., X_m$ be smooth vector fields in M^n, free up to step r and satisfying the Hörmander condition (1.2). Let $\Omega = \{x \in M^n \mid \phi(x) < 0\} \subset M^n$ be a bounded $C^{1,1}$ domain. For each point x_o in $\partial\Omega$, there exists constants $C = C(\Omega, X, x_o) > 0$ and $R = R(\Omega, X, x_o) > 0$ depending continuously on x_o, such that if $0 < R < R_o$ then*

$$\left(\sup_{x\in\Delta(x_o,R)} |X\phi(x)|\right) \sigma(\Delta(x_o, R)) \le C(\Omega, X, x_o)\frac{|B(x_o, R)|}{R^{s(x_o)}}. \tag{6.29}$$

and consequently

$$P_X(\Omega; B(x_o, R)) \le M \frac{|B(x_o, R)|}{R^{s(x_o)}}, \tag{6.30}$$

with

$$s(x_o) = \begin{cases} type(x_o) - 1, & \text{if } x_o \text{ is characteristic,} \\ 1, & \text{if } x_o \text{ is not characteristic.} \end{cases} \tag{6.31}$$

7. The case of a general CC space : Proof of Theorems 1.4 and 1.6

In this section we establish the Ahlfors regularity for C^2 domains of type less or equal than two, in a general CC space (M^n, d). We will assume that $X_1, ..., X_m$ are smooth vector fields which satisfy (1.2) with step $r \in \mathbb{N}$ at every point of an open subset $U \subset M^n$. The proof rests on the Rothschild-Stein Lifting Theorem 2.7 and on Lemma 2.9.

Consider an open, bounded C^2 set $\Omega \subset U \subset M^n$, and assume that there exists a (C^2) function $\phi : U \to \mathbb{R}$ such that $\Omega = \{x \in M^n \mid \phi(x) < 0\}$. Let $x_o \in \partial\Omega$. The condition that the type of Ω is less or equal than two means that either x_o is not characteristic or there exist indices $i_0, j_0 = 1, ..., m$ such that

$$[X_i, X_j]\phi(x_o) \neq 0 \tag{7.1}$$

Proof of Theorems 1.4 and 1.6. The strategy behind the proof of this result is straightforward. We lift the vector fields, the domain and the metric. Then we use the results from Theorem 6.6, and Lemma 2.9 to establish the estimate (1.16). Let $E \subset U$ be a compact set which contains Ω. We will use E as in Lemma 2.9. At this point we refer to Theorem 2.7, which gives a neighborhood V of the origin in $\mathbb{R}^{\tilde{n}-n}$, and the "lifted" vector fields $\tilde{X}_1, ..., \tilde{X}_m$ in $U \times V$. Denote by $\tilde{\Omega} = \Omega \times V$, and $\tilde{x_o} = (x_o, 0) \in \partial\tilde{\Omega}$. Choose $R_1 = R_1(X, \Omega, x_o) > 0$ small enough so that $\tilde{B}((x_o, 0), R) \subset E \times V$. For $0 < R < R_1$, set $\tilde{\Delta} = \tilde{\Delta}((x_o, 0), R) = \tilde{B}((x_o, 0), R) \cap \tilde{\Omega}$. Let $\tilde{\phi}(x, t) = \phi(x)$, so that $\tilde{\Omega} = \{\tilde{\phi} < 0\}$, and $|\tilde{X}\tilde{\phi}| = |X\phi|$. Notice that since $\tilde{X}_i\tilde{X}_j\tilde{\phi} = X_iX_j\phi$, then $\tilde{\Omega}$ is of type less or equal than two according to Definition 6.1.

Let us start by proving the estimates from above (1.16). For this part of the proof we only require $\phi \in C^{1,1}$. Notice that $|\tilde{X}\tilde{\phi}|$ is only a function of x, hence

$$\left(\sup_{P\in\Delta(x_o,R)}|X\phi(P)|\right) = \left(\sup_{(x,s)\in\tilde{\Delta}((x_o,0),R)}|\tilde{X}\tilde{\phi}(x,s)|\right). \tag{7.2}$$

It is convenient to rewrite $\tilde{\Delta}$ and $\tilde{B}((x_o,0),R)$ in the following (completely obvious) way

$$\begin{aligned}\tilde{B}((x_o,0),R) = &\left(B(x_o,R)\times V\right)\cap\tilde{B}((x_o,0),R),\\ \tilde{\Delta} = &\left(\Delta(x_o,R)\times V\right)\cap\tilde{\Delta} = \left(\Delta(x_o,R)\times V\right)\cap\tilde{B}((x_o,0),R).\end{aligned} \tag{7.3}$$

At this point we recall from Theorem 6.6 that there exist $C(\Omega,X)>0$ and $R_2=R_2(X,\Omega)>0$ such that

$$\left(\sup_{(x,s)\in\tilde{\Delta}((x_o,0),R)}|\tilde{X}\tilde{\phi}(x,s)|\right)\sigma(\tilde{\Delta}) \le\ C(\Omega,X)\frac{|\tilde{B}((x_o,0),R)|}{R}, \tag{7.4}$$

for any $0<R<R_2$. Set $R_0=\min\{R_1,R_2\}$. In view of (7.3), and (2.26) one has for $0<R<R_0$, and $v\in C_0^\infty(V)$,

$$\begin{aligned}\sigma(B(x_o,R)\cap\partial\Omega)\left|\int_V\chi_{\tilde{B}(x_o,0),R)}(y,s)v(s)ds\right| &\le C\left|\int_{\Delta(x_o,R)}\int_V\chi_{\tilde{B}(x_o,0),R)}(y,s)v(s)ds\ d\sigma(y)\right|\\ &= C\left|\int_{(\Delta(x_o,R)\times V)}\chi_{\tilde{B}((x_o,0),R)}(y,s)v(s)ds\ d\sigma(y)\right|\\ &= C\left|\int_{(\Delta(x_o,R)\times V)\cap\tilde{B}((x_o,0),R)}v(s)ds\ d\sigma(y)\right|\\ \text{from (7.3)}\qquad &\le C(v,X)\sigma(\tilde{\Delta}).\end{aligned} \tag{7.5}$$

The conclusion now follows from (7.2), (7.4) and (7.5). In fact, for R suitably small one has

$$\begin{aligned}\left(\sup_{P\in\Delta(x_o,R)}|X\phi(P)|\right)\quad &\sigma(\Delta(x_o,R))\\ &= \left(\sup_{(x,s)\in\tilde{\Delta}((x_o,0),R)}|\tilde{X}\tilde{\phi}(x,s)|\right)\sigma(\Delta(x_o,R))\\ \text{by (7.5)}\qquad &\le C(v,X)\frac{\sigma(\tilde{\Delta})}{\left|\int_V\chi_{\tilde{B}(x_o,0),R)}(y,s)\ v(s)ds\right|}\left(\sup_{(x,s)\in\tilde{\Delta}((x_o,0),R)}|\tilde{X}\tilde{\phi}(x,s)|\right)\\ \text{by (7.4)}\qquad &\le C(v,X)\frac{1}{\left|\int_V\chi_{\tilde{B}(x_o,0),R)}(y,s)\ v(s)ds\right|}\frac{|\tilde{B}((x_o,0),R)|}{R}\\ \text{from Lemma 2.9}\qquad &\le C(v,X,\Omega)\frac{|B(x_o,R)|}{|\tilde{B}(x_o,0),R)|}\frac{|\tilde{B}((x_o,0),R)|}{R}\\ &= C(v,X,\Omega)\frac{|B(x_o,R)|}{R}.\end{aligned} \tag{7.6}$$

We now turn our attention to the lower Ahlfors regularity.

Let $\tilde{B} = \tilde{B}(x_o, 0), R)$, and $B = B(x_o, R)$. Since $\tilde{B} \subset \bar{\tilde{B}} \subset U \times V$, then for every $x \in \partial\Omega$ and suitably small $R_0 = R_0(X, \Omega) > 0$, we can find $v \in C_0^\infty(V)$ such that

$$v(s)\chi_{\tilde{B}}(x,s) = \chi_{\tilde{B}}(x,s), \tag{7.7}$$

for any $0 < R < R_0$ and $s \in V$.

Recalling that $\tilde{X}\tilde{\phi}(x,s) = X\phi(x)$, and in view of (7.3), and Lemma 2.9 we have

$$\begin{aligned}
\int_{\tilde{\Delta}} |\tilde{X}\tilde{\phi}(x,s)| d\sigma(s,t) &= \int_{\tilde{\Delta}} |\tilde{X}\tilde{\phi}(x,s)| \; v(s)\chi_{\tilde{B}}(x,s) d\sigma(s,t) \\
&= \int_{\Delta \times V} |\tilde{X}\tilde{\phi}(x,s)| \; v(s)\chi_{\tilde{B}}(x,s) d\sigma(s,t) \\
&= \int_{\Delta} |X\phi|(x) \int_V v(s)\chi_{\tilde{B}}(x,s) ds \; d\sigma(x) \\
&\leq C(X,\Omega) \int_{\Delta} |X\phi|(x) d\sigma(x) \frac{|\tilde{B}|}{|B|}. \qquad (7.8)
\end{aligned}$$

The proof now follows immediately from (7.8) and from Theorem 6.9 □

8. 1-Ahlfors regularity of the X-perimeter and the Dirichlet problem

In this section we bring up an interesting connection between 1-Ahlfors regularity of the X-perimeter $P_X(\Omega; \cdot)$ and the Dirichlet problem for the sub-Laplacian $\mathcal{L} = \sum_{i=1}^m X_i^* X_i$ associated with the system X. We recall that the latter consists in finding, for a given $\phi \in C(\partial\Omega)$, a $\mathcal{L}$-harmonic function u in Ω, i.e., a solution to $\mathcal{L}u = \sum_{i=1}^m X_i^* X_i u = 0$, such that $u = \phi$ on $\partial\Omega$ continuously. For any bounded open set $\Omega \subset M^n$ there exists a unique (generalized) solution H_ϕ^Ω to the Dirichlet problem, see [CG]. A point $x_o \in \partial\Omega$ is called *regular* if for any $\phi \in C(\partial\Omega)$ one has

$$\lim_{x \to x_o} H_\phi^\Omega(x) \;=\; \phi(x_o) \; .$$

We will prove the following result.

Theorem 8.1. *Let Ω be a bounded domain in a Carnot group $\boldsymbol{G}$. If the perimeter measure $P_X(\Omega; \cdot)$ is 1-Ahlfors regular, then every $g_o \in \Omega$ is regular for the Dirichlet problem.*

The full proof of this result will be accomplished in several steps, and it is ultimately based on an important generalization to sub-Laplacians of the classical criterion of Wiener. We begin by introducing the relevant definitions. A couple (K, Ω), $K \subset \Omega \subset M^n$, with K compact and Ω open, is called a condenser. For a given condenser (K, Ω), we let

$$\mathcal{F}(K,\Omega) \;=\; \{\phi \in C_o^\infty(\Omega) \mid \phi \geq 1 \text{ on } K\} \, .$$

The X-*capacity* of the condenser (K,Ω) is defined as follows, see [CDG2],

$$cap_X\ (K,\Omega)\ =\ \inf_{\phi\in\mathcal{F}(K,\Omega)}\int_\Omega |X\phi(g)|^2\ dg.$$

When $\Omega=\boldsymbol{G}$, then we simply write $cap_X\ K$, instead of $cap_X\ (K,\boldsymbol{G})$. The following properties of the capacity are simple consequences of its definition, and we list them without proof.

Proposition 8.2. *Let $K\subset\Omega_1\subset\Omega_2$, then*

$$cap_X\ (K,\Omega_2)\ \le\ cap_X\ (K,\Omega_1).$$

If, instead, one has $K_1\subset K_2\subset\Omega$, then

$$cap_X\ (K_1,\Omega)\ \le\ cap_X(K_2,\Omega).$$

According to the capacitary estimates in [D], [CDG2], when M^n is a Carnot group $\boldsymbol{G}$ with homogeneous dimension Q, then there exists $C=C(\boldsymbol{G})>0$ such that for every $g_o\in\boldsymbol{G}$ and $r>0$

$$cap_X\ (\overline{B}(g_o,r),B(g_o,2r))\ =\ C\ r^{Q-2}\ . \tag{8.1}$$

The following basic criterion of Wiener type was proved in [NS], see also [D] for a generalization to quasilinear equations.

Theorem 8.3. *Given a bounded open set $\Omega\subset M^n$, a point $x_o\in\partial\Omega$ is regular if and only if for some small $\delta>0$*

$$\int_0^\delta \frac{cap_X\ (\Omega^c\cap\overline{B}(g_o,t),B(g_o,2t))}{cap_X\ (\overline{B}(g_o,t),B(g_o,2t))}\ \frac{dt}{t}\ =\ \infty.$$

We will also need the following result.

Let $\mathcal{H}^{Q-1}(\cdot)$ denote the $(Q-1)-$Hausdorff measure in $\boldsymbol{G}$ with respect to the Carnot-Carathéodory metric. The next simple theorem is a particular instance of a potential theoretic result which holds in any metric space with controlled geometry, See Theorem 5.9 in [HK].

Theorem 8.4. *In a Carnot group $\boldsymbol{G}$ consider a compact subset F of a ball $B=B(g_0,R)$. If for some $0<\lambda\le 1$ we have*

$$\mathcal{H}^{Q-1}(F)\ \ge\ \lambda\ \frac{|B(g_o,R)|}{R}\ , \tag{8.2}$$

then there exists $C=C(Q)\ge 1$ such that for every $u\in C_o^\infty(B(g_o,CR))$ satisfying $u\ge 1$ on F, one has

$$\int_{B(g_o,CR)}|Xu(g)|^2\ dg\ \ge\ \frac{\lambda}{C}\ \frac{|B(g_o,R)|}{R^2}\ . \tag{8.3}$$

Corollary 8.5. *In a Carnot group $\boldsymbol{G}$ consider a bounded open set $\Omega\subset\boldsymbol{G}$. If the Hausdorff measure $\mathcal{H}^{Q-1}$ is lower $1-$Ahlfors regular, then there exists $C=C(\boldsymbol{G})>0$, such that for any $g_o\in\partial\Omega$ and any $r>0$*

$$cap_X\ (\overline{B}(g_o,R)\cap\partial\Omega,B(g_o,2R))\ \ge\ C\ R^{Q-2}\ .$$

Proof. Let $F = \overline{B}(g_o, R)$ be the compact subset of $B(g_o, 2R)$ in the statement of Theorem 8.4. Since we are assuming the lower 1−Ahlfors regularity, then hypothesis (8.2) is automatically satisfied. By virtue of Theorem 8.4 and of the definition of capacity we obtain that there exists $C_1 \geq 1$ and $C_2 > 0$ such that

$$cap_X\ (\overline{B}(g_o, R) \cap \partial\Omega, B(g_o, 2C_1 R))\ \geq\ C_2\ R^{Q-2}\ .$$

The conclusion now follows from the latter inequality and from Proposition 8.2. □

We next establish, in the special setting of Carnot groups, a general property of the Hausdorff measure in a space of homogeneous type.

Proposition 8.6. *Let μ be a Borel measure in a Carnot group $\boldsymbol{G}$ with homogeneous dimension Q, and suppose that for a bounded open set $\Omega \subset \boldsymbol{G}$ the measure $P_X(\Omega; \cdot)$ is upper 1-Ahlfors regular, i.e., for some $M > 0$ one has for every $g \in \partial\Omega$ and $R > 0$*

$$P_X(\Omega; B(g, R))\ \leq\ M\ R^{Q-1}\ . \tag{8.4}$$

There exists a constant $C = C(M) > 0$ such that for every

$$\mathcal{H}^{Q-1}(\partial\Omega \cap B(g, R))\ \geq\ C\ P_X(\Omega; B(g, R))\ . \tag{8.5}$$

Proof. Consider the compact set

$$K\ =\ \partial\Omega\ \cap \overline{B}(g_o, R)\ .$$

For each $\epsilon > 0$ we can find a covering $\{B(x_i, r_i)\}_{i\in N}$ of K such that $0 < r_i < \epsilon$. Using the hypothesis (8.4) we obtain

$$P_X(\Omega; \overline{B}(g, R))\ \leq\ \sum_{i=1}^{\infty} P_X(\Omega; B(g_i, r_i))\ \leq\ M\ \sum_{i=1}^{\infty} r_i^{Q-1}.$$

Since

$$\mathcal{H}^{Q-1}(\partial\Omega \cap \overline{B}(g, R)) = \liminf_{\epsilon\to 0}\ \{\sum_{i=1}^{\infty} {r_i}^{Q-1}\ |\ \partial\Omega \cap \overline{B}(g, R) \subset \bigcup_{i=1}^{\infty} B(g_i, r_i), r_i < \epsilon\},$$

we reach the conclusion.

□

Notice that there is nothing special about the role of the perimeter measure in the last proposition. The same result still holds when $P_X(\Omega, \cdot)$ is substituted by any Borel measure on $\partial\Omega$. We are now ready to give the

Proof of Theorem 8.1. Assume that $P_X(\Omega; \cdot)$ is 1-Ahlfors regular, i.e., we have for some constant $M > 0$

$$M^{-1}\ R^{Q-1}\ \leq\ P_X(\Omega; B(g, R))\ \leq\ M\ R^{Q-1}\ , \tag{8.6}$$

for every $g \in \partial\Omega$ and every $R > 0$. Using the upper bound in (8.6) in view of Proposition 8.6 we conclude that (8.5) holds. The lower bound in (8.6) yields the lower 1−Ahlfors regularity of the Hausdorff measure $\mathcal{H}^{Q-1}$. Thanks to Theorem 8.4 and Corollary 8.5, this estimate implies

$$cap_X\ (\partial\Omega \cap \overline{B}(g_o, R), B(g_o, 2R))\ \geq\ C''\ R^{Q-2}\ .$$

Due to (8.1) the latter estimate implies

$$\frac{cap_X\ (\partial\Omega \cap \overline{B}(g_o,R), B(g_o,2R))}{cap_X\ (\overline{B}(g_o,R), B(g_o,2R))}\ \geq\ C'''\ >\ 0\ . \tag{8.7}$$

We now apply the second part of Proposition 8.2 with $K_2 = \Omega^c \cap \overline{B}(g_o,R)$, $K_1 = \partial\Omega \cap \overline{B}(g_o,R)$, $\Omega = \boldsymbol{G}$, obtaining

$$cap_X\ (\partial\Omega \cap \overline{B}(g_o,R), B(g_o,2R))\ \leq\ cap_X(\Omega^c \cap \overline{B}(g_o,R), B(g_o,2R))\ .$$

This estimate, combined with (8.7), and with Theorem 8.3, finally implies the regularity of the point $g \in \partial\Omega$. By the arbitrariness of $g \in \partial\Omega$ we reach the conclusion.

□

Interestingly, using Theorem 8.1 in combination with some examples due to Hansen and Hueber [HH], it is possible to provide a proof of the optimal character of the type assumption in Theorems 1.5 and 1.7, which is alternative to that in section 5. Let us start by recalling Theorem 3.6, [HH].

Theorem 8.7. *Let $\boldsymbol{G}$ be a Carnot group of step $r \in \mathbb{N}$, and denote by m_1 the dimension of the first layer of the stratification V_1. If $r \leq 2$, or if $m_1 = 2$ and $r \leq 4$, then every bounded domain $\Omega \subset \boldsymbol{G}$ is regular for the Dirichlet problem, provided it satisfies a pointwise exterior ball condition (with respect to the underlying Euclidean metric). In all other cases there exist bounded domains with smooth boundary which are not regular.*

Let us describe more explicitly the smooth domains with irregular boundary points mentioned in the theorem. If $r \geq 3$ and $m_1 \geq 3$, or $m_1 = 2$ and $r \geq 4$, then for every $\gamma > 0$ we set $y = \{y_{k,j}\}$ the be the point on the $y_{r,m_r}-$axis at distance γ from the origin, i.e. $y_{j,k} = 0$ if $k = 1, ..., r$, $j \neq m_r$, and $y_{r,m_r} = \gamma$. Consider the Euclidean ball

$$B_E = \left\{ x = \{x_{k,j}\} \in \boldsymbol{G} \ \text{ such that } \sum_{k=1}^{r}\sum_{j=1}^{m_j}(x_{k,j} - y_{k,j})^2 < \gamma^2 \right\} \subset \boldsymbol{G}.$$

In Theorem 3.4 and 3.5, in [HH], it is proved that B_E is thin at the origin, and consequently the origin is an irregular boundary point for B_E^C. This proves that there exist bounded smooth domains with irregular points in Carnot groups of step $r \geq 3$ (for instance consider $B(0, 100\gamma) \setminus B_E$). Consequently, in view of Theorem 8.1, the perimeter measure of such domains cannot be $1-$Ahlfors regular. Note that the origin is always of type r.

Corollary 8.8. *If $r \geq 3$ and $m_1 \geq 3$, or $m_1 = 2$ and $r \geq 4$, then there exist Carnot groups $\boldsymbol{G}$ of step $r \in \mathbb{N}$, with dim $V_1 = m_1$, and bounded, smooth domains $\Omega \subset \boldsymbol{G}$, whose perimeter measure $P_X(\Omega; \cdot)$ is not $1-$Ahlfors regular.*

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Department of Mathematics, University of Arkansas, Fayetteville, AR 72701
E-mail address, Luca Capogna: `lcapogna@uark.edu`

Department of Mathematics, Purdue University, West Lafayette, IN 47907
E-mail address, Nicola Garofalo: `garofalo@math.purdue.edu`