

LIPSCHITZ REGULARITY OF ALMOST-MINIMIZERS IN ONE-PHASE PROBLEMS WITH GENERALIZED ORLICZ GROWTH

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ABSTRACT. Optimal local Lipschitz regularity for scalar almost minimizers of Alt-Caffarelli-type functionals

$$\mathcal{F}(v; \Omega) = \int_{\Omega} \varphi(x, |\nabla v(x)|) + \lambda \chi_{\{v>0\}}(x) \, dx,$$

with growth function φ a generalized Orlicz function, is established.

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1. INTRODUCTION AND THE MAIN RESULTS

Our paper is concerned with giving a unifying perspective on the regularity theory of scalar-valued almost minimizers of an Alt–Caffarelli type functional with generalized Orlicz growth in the energy:

$$\int_{\Omega} \varphi(x, |\nabla u|) + \lambda \chi_{\{u>0\}} \, dx, \tag{1.1}$$

where φ models non-standard growth conditions, encompassing classical p -growth and Orlicz growth settings and allowing for inhomogeneities in the space variable.

The Alt–Caffarelli functional arises naturally in a variety of contexts involving phase separation phenomena, such as fluid interfaces, combustion, and optimal material design. Its central feature is the appearance of a free boundary – i.e., the boundary of the region where the solution is positive – which is not known a priori and must be determined as part of the problem. From a mathematical perspective, this functional serves as a canonical model in the study of free boundary problems. It brings together variational methods,

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geometric measure theory, and PDE analysis. Understanding the regularity of minimizers and their associated free boundaries is crucial not only for theoretical reasons but also for the stability and accuracy of computational methods in applications.

Nonlinear growth conditions induce additional complexities: most nonlinear regularity approaches must distinguish between subquadratic ($1 < p < 2$) and superquadratic ($p \geq 2$) growth regimes. Structural assumptions on φ , such as uniform convexity and appropriate control of the growth exponents, are essential to obtain Lipschitz regularity results.

The Lipschitz continuity of minimizers to the Alt–Caffarelli functional is a fundamental property that guarantees the well-posedness of the problem and serves as a stepping stone toward free boundary regularity.

In the variational framework, minimizers (and more generally, almost minimizers) arise as solutions to energy minimization problems. The pioneering work of Alt and Caffarelli [1] established Lipschitz regularity for scalar minimizers, leveraging a delicate blow-up analysis and monotonicity formulas. Caffarelli later refined these techniques [5, 6, 7], introducing tools like Harnack-type inequalities and improvement of flatness. In its most general version, developed in an Orlicz-Sobolev setting in [29], this approach reformulates the minimization problem for $\mathcal{J}_{\varphi,\lambda}(u, \Omega) := \int_{\Omega} (\varphi(|\nabla u|) + \lambda \chi_{\{u>0\}}) dx$ as a nonlinear PDE

$$\begin{cases} \operatorname{div} \left(\varphi'(|\nabla u|) \frac{\nabla u}{|\nabla u|} \right) = 0 & \text{in } \{u > 0\} \cap \Omega \\ u = 0, \quad |\nabla u| = \lambda^* & \text{on } \partial\{u > 0\} \cap \Omega \\ \varphi'(\lambda^*)\lambda^* - \varphi(\lambda^*) = \lambda \end{cases}$$

with suitable boundary conditions. This weak formulation, and a sub- and supersolution method (in a suitable viscosity sense) allow the authors to prove that solutions to the optimization problem are locally Lipschitz continuous. This extends the Alt-Caffarelli's results for the scenario of the Orlicz-Sobolev framework. Additionally, they address Caffarelli's classification scheme: flat and Lipschitz free boundaries are locally $C^{1,\alpha}$ for some $\alpha(\text{universal}) \in (0, 1)$. Although not directly addressed in our contribution, let us also remark that a flourishing literature on the regularity of the free boundary has developed from these results, particularly in the scalar case. Key contributions include [1, 5, 6, 7, 16, 31] for the scalar case, and [3, 19, 12, 13, 14] for the vectorial setting.

The above tools are however not available when dealing with almost minimizers. This concept, introduced in works such as De Silva and Savin [15], allows for perturbative and non-exact minimization, providing a robust framework to study solutions in inhomogeneous or approximate settings. Related contributions by David, Engelstein, Smit Vega and Toro [9, 10, 11] use compactness and approximation arguments to extend regularity results beyond the exact minimizer class.

A remarkable approach to regularity of almost minimizers in a nonlinear p -Laplace setting has been later devised in [18]. Their approach is based on local regularity estimates on the p -harmonic replacement of an almost minimizer and a dichotomy theory according to which, roughly speaking, the average of the energy of an almost minimizer decreases in a smaller ball, unless one is arbitrarily close to the case of linear functions. This analysis, concerning interior regularity, introduces some restrictions on the exponent p , and is essentially carried out with different techniques and estimates for the sub- and superquadratic case.

In the vectorial setting, Lipschitz regularity becomes even more subtle due to the interaction between components. Recent results by Bayrami, Fotouhi, and Shahgholian [4], and by De Silva, Jeon, and Shahgholian [12, 13], address weakly coupled systems and singular behaviors, obtaining Lipschitz bounds under structural and coupling assumptions. In

particular, the paper [4] handles regularity (up to the boundary), obtained via a blow-up approach which will be also expedient for our analysis, as described below.

The analysis has been further extended to the case of non-standard growth (e.g., Orlicz spaces) further enriching the theory of Lipschitz regularity of almost minimizers in non-classical environments. This has been mostly done for autonomous integrands (i.e., the case where $\varphi(x, t) = \varphi(t)$). The scalar case has been addressed in [8], and later extended to the vector valued case in the recent contribution [30], always dealing with a weakly coupled system. We may also refer the reader to the introduction in [30] for a thorough analysis of the state-of-the-art for this kind of problems, as well as for a rich bibliography.

Description of our result. We now come to the description of our results. In this paper, we prove Lipschitz regularity for scalar-valued almost minimizers of (1.1) by exploiting the very general theory for non-autonomous functionals with Uhlenbeck structure introduced in [26]. It is based on a single condition involving both the x and t directions (see Definition 2.4 below), which allows one for comparison estimates with minimizers of a locally well-defined autonomous integrand $\tilde{\varphi}$ whose properties are recalled in Section 2.2.

This approach can be successfully combined with the scheme pursued in [4]. The path we follow runs along these lines:

- at first, higher integrability estimates, and a reverse Hölder-type inequality for almost minimizers are established;
- a local comparison estimate with the $\tilde{\varphi}$ -harmonic replacement allows one to recover a Morrey-type estimate (see (3.23)) and eventually C^α continuity of almost minimizers for any α in $(0, 1)$;
- a $C^{1,\alpha}$ regularity result *away from the free boundary* for suitable α is established by a compactness and lower semicontinuity argument on blown-up sequences of almost minimizers, which are shown to tend to a minimizer of an autonomous functional. Here, assumption (VA1) in Definition 2.4 plays a crucial role;
- a key step, as in [4], is finally to show that a bounded almost minimizer of (1.1) is sublinear in a neighborhood of a free-boundary point. For this, in particular, the C^α continuity of almost minimizers is exploited.

Observe that the combination of the estimates in [26] with blow-up arguments causes some additional nontrivial difficulties to our analysis, as we must ensure that some constants, which depend themselves on the chosen local almost minimizer u , can also be used for providing uniform estimates for the blown-up sequences: this is apparent, for instance, in the proofs of Propositions 3.11 and 3.12.

The unified treatment we develop encompasses all the relevant examples in literature, provided the bulk energy has a growth from below with exponent $p > 1$ and needs not distinguish between sub- and superquadratic energies. As relevant examples of energies undergoing non-standard growth we report here the perturbed *Orlicz*, the so-called *variable exponent*, and the *double-phase* case

$$a(x)\varphi(|\xi|), \quad |\xi|^{p(x)}, \quad \text{and} \quad |\xi|^p + a(x)|\xi|^q \quad \text{for } (x, \xi) \in \mathbb{R}^d \times \mathbb{R}^d,$$

while an exhaustive list of examples can be found in [26].

We remark that in this paper we limit ourselves to the case of scalar-valued almost minimizers but the same result can be extended to the vectorial problem of a weakly coupled system with some additional, but manageable, effort, using the same procedure of [4]. Extensions to both the autonomous and non-autonomous vectorial case for strongly coupled systems represents a challenging direction of research which we plan to address in future contributions.

In order to introduce the main result of our paper, we formulate our problem and specify the definition of almost-minimizer.

Given a constant $\lambda > 0$, and a bounded open set $\Omega \subset \mathbb{R}^d$ ($d \geq 2$), we will deal with local almost-minimizers of the functional

$$\mathcal{F}(v; \Omega) := \int_{\Omega} \varphi(x, |\nabla v(x)|) + \lambda \chi_{\{v>0\}}(x) \, dx, \quad (1.2)$$

for $v \in W^{1,\varphi}(\Omega)$ with $v \geq 0$. The precise notion of almost minimizers that we use is the following.

Definition 1.1. *We say that $u : \Omega \rightarrow \mathbb{R}$ is a (local) almost-minimizer for \mathcal{F} in Ω , with constant κ and exponent β , if*

$$\mathcal{F}(u; B_r(x_0)) \leq (1 + \kappa r^\beta) \mathcal{F}(w; B_r(x_0)),$$

for every ball $B_r(x_0)$ such that $\overline{B_r(x_0)} \subset \Omega$ and every $w \in W^{1,\varphi}(B_r(x_0))$ such that $u = w$ on $\partial B_r(x_0)$.

The main result of the paper is the following.

Theorem 1.1 (Interior regularity). *Let $\Omega \subset \mathbb{R}^d$ be a bounded open set. Let $\varphi \in \Phi_c(\Omega)$, $\varphi(x, \cdot) \in C^1([0, \infty))$ be satisfying (VA1), and such that φ_t comply with (A0), $(\text{inc})_{p-1}$ and $(\text{dec})_{q-1}$ for some $1 < p \leq q$. Let $u : \Omega \rightarrow \mathbb{R}$ be an almost-minimizer of \mathcal{F} in Ω . Then, u is locally Lipschitz continuous in Ω .*

We are investigating the boundary version of the previous result in a paper currently in preparation.

Outline of the paper. The rest of the paper is organized as follows. In Section 2 we fix the basic notation and recall some basic facts and technical results about Orlicz and generalized Orlicz functions, together with some technical lemmas. Section 3.1 contains some supporting regularity results for autonomous problems in divergence form, which are exploited in Section 3.2, where we obtain Caccioppoli type estimates and higher integrability results for almost minimizers, together with useful comparison estimates with the solution of a suitable autonomous problem. Section 3.3 collects two main ingredients in order to get the main result: the local Hölder continuity of almost minimizers, Theorem 3.7, and that of their gradients away from the free boundary, Theorem 3.8. Section 3.4 is entirely devoted to the proof of the main result: the main steps are Lemma 3.9, where an interior uniform bound for the gradient of an almost minimizer is provided; Proposition 3.11, showing that a suitable blow-up sequence of almost minimizers converges to the solution of a limit autonomous problem, and its consequence Proposition 3.12, where the sublinearity of a bounded almost minimizer in a neighborhood of a free-boundary point is shown. Finally, in Appendix A, we collect some technical results mainly employed in the proof of Lemma 3.10 and Proposition 3.11.

2. BASIC NOTATION AND PRELIMINARIES

We start with some basic notation. Let $\Omega \subset \mathbb{R}^d$ be open and bounded. For every $x \in \mathbb{R}^d$ and $r > 0$ we indicate by $B_r(x) \subset \mathbb{R}^d$ the open ball with center x and radius r . We will often use the shorthand B_r when either $x = 0$ or the center x is not relevant. For $x, y \in \mathbb{R}^d$, we use the notation $x \cdot y$ for the scalar product and $|x|$ for the Euclidean norm. The m -dimensional Lebesgue measure of the unit ball in \mathbb{R}^m is indicated by γ_m for every $m \in \mathbb{N}$. We denote by \mathcal{L}^d the d -dimensional Lebesgue measure. The closure of A is denoted by \overline{A} . The diameter of A is indicated by $\text{diam}(A)$. We write χ_A for the characteristic function of any $A \subset \mathbb{R}^d$, which is 1 on A and 0 otherwise.

Given two functions $f, g : [0, +\infty) \rightarrow \mathbb{R}$, we write $f \sim g$, and we say that f and g are equivalent, if there exist constants $c_1, c_2 > 0$ such that $c_1 g(t) \leq f(t) \leq c_2 g(t)$ for any $t \geq 0$. Similarly the symbol \lesssim stands for \leq up to a constant. $L^0(\Omega)$ denotes the set of the measurable functions on Ω .

2.1. Generalized Φ -functions and Orlicz spaces. We introduce some basic definitions and useful facts about generalized Φ -functions and Orlicz spaces, only considering concepts we will use. We refer the reader to [23] for a comprehensive treatment of the topic.

Definition 2.1. Let $\varphi : [0, +\infty) \rightarrow [0, +\infty]$ be increasing with $\varphi(0) = 0$, $\lim_{t \rightarrow 0^+} \varphi(t) = 0$ and $\lim_{t \rightarrow +\infty} \varphi(t) = +\infty$. Such φ is called a

- (i) weak Φ -function if $\frac{\varphi(t)}{t}$ is almost increasing, meaning that there exists $L \geq 1$ such that $\frac{\varphi(t)}{t} \leq L \frac{\varphi(s)}{s}$ for $0 < t \leq s$.
- (ii) convex Φ -function if φ is left-continuous and convex.

By virtue of Remark 2.3, each convex Φ -function is a weak Φ -function. If φ is a convex Φ -function, then there exists φ' the right derivative of φ , which is non-decreasing and right-continuous, and such that

$$\varphi(t) = \int_0^t \varphi'(s) \, ds.$$

A special subclass of convex Φ -functions is represented by the N -functions (see, e.g., [27, Ch.I]).

Definition 2.2. A function $\varphi : [0, \infty) \rightarrow [0, \infty)$ is said to be an N -function if it admits the representation

$$\varphi(t) = \int_0^t a(\tau) \, d\tau$$

where $a(s)$ is right-continuous, non-decreasing for $s > 0$, $a(s) > 0$ for $s > 0$ and satisfies the conditions

$$a(0) = 0, \quad \lim_{s \rightarrow +\infty} a(s) = +\infty. \quad (2.1)$$

The function $a(t)$ is nothing else than the right-derivative of $\varphi(t)$. As a straightforward consequence of the definition, we have that an N -function φ is continuous, $\varphi(0) = 0$ and φ is increasing. Moreover, φ is a convex function, and, in view of Remark 2.3, it satisfies $(inc)_1$. Conditions (2.1) imply

$$\lim_{t \rightarrow 0^+} \frac{\varphi(t)}{t} = 0, \quad \lim_{t \rightarrow +\infty} \frac{\varphi(t)}{t} = +\infty. \quad (2.2)$$

It can be shown that an equivalent definition of N -function is the following: a continuous convex function φ is called an N -function if it satisfies (2.2).

For our purposes, we need functions φ to depend also on the spatial variable x .

Definition 2.3. Let $\varphi : \Omega \times [0, \infty) \rightarrow [0, \infty]$. We call φ a generalized weak Φ -function (resp., convex Φ -function, N -function) if

- (1) $x \mapsto \varphi(x, |f(x)|)$ is measurable for every $f \in L^0(\Omega)$;
- (2) $t \mapsto \varphi(x, t)$ is a weak Φ -function (resp., a convex Φ -function, an N -function) for every $x \in \Omega$.

We write $\varphi \in \Phi_w(\Omega)$, $\varphi \in \Phi_c(\Omega)$ and $\varphi \in N(\Omega)$, respectively. If φ does not depend on x , we will adopt the shorthands $\varphi \in \Phi_w$, $\varphi \in \Phi_c$ and $\varphi \in N$, respectively. For the right-derivative of a generalized convex Φ -function, we will use the notation φ_t in place of φ' .

For a bounded function $\varphi : \Omega \times [0, +\infty) \rightarrow [0, +\infty)$ and a ball $B_r(x_0) \subset \Omega$ we define, for every $t \geq 0$,

$$\varphi_{r,x_0}^-(t) := \inf_{x \in B_r(x_0)} \varphi(x, t) \quad \text{and} \quad \varphi_{r,x_0}^+(t) := \sup_{x \in B_r(x_0)} \varphi(x, t). \quad (2.3)$$

Following the terminology of [23, 26], we give the following definitions. The first three ones concern with the regularity of φ with respect to the t - variable, (A1) imposes a bound on how much φ can change between nearby points, while the last one is a continuity assumption with respect to the spatial variable x .

Definition 2.4. Let $p, q > 0$. A function $\varphi : \Omega \times [0, +\infty) \rightarrow [0, +\infty)$ satisfies

- (inc) $_p$ if $t \in (0, +\infty) \mapsto \frac{\varphi(x,t)}{t^p}$ is increasing for every $x \in \Omega$
- (dec) $_q$ if $t \in (0, +\infty) \mapsto \frac{\varphi(x,t)}{t^q}$ is decreasing for every $x \in \Omega$
- (A0) if there exists $L \geq 1$ such that $\frac{1}{L} \leq \varphi(x, 1) \leq L$ for every $x \in \Omega$
- (A1) if there exists $L \geq 1$ such that, for any ball $B_r(x_0) \subset \Omega$,

$$\varphi_{r,x_0}^+(t) \leq L \varphi_{r,x_0}^-(t), \quad \forall t > 0 \text{ such that } \varphi_{r,x_0}^-(t) \in \left[1, \frac{1}{\mathcal{L}^d(B_r(x_0))}\right].$$

- (VA1) if there exists an increasing continuous function $\omega : [0, +\infty) \rightarrow [0, 1]$ with $\omega(0) = 0$ such that, for any ball $B_r(x_0) \subset \Omega$,

$$\varphi_{r,x_0}^+(t) \leq (1 + \omega(r)) \varphi_{r,x_0}^-(t), \quad \forall t > 0 \text{ such that } \varphi_{r,x_0}^-(t) \in \left[\omega(r), \frac{1}{\mathcal{L}^d(B_r(x_0))}\right].$$

Remark 2.1. Note that assumption (VA1) implies (A1), see [26, Remark 4.2]. By (inc) $_p$, condition (A1) implies

- (A1') there exists $\beta \in (0, 1)$ such that, for any ball $B_r(x_0) \subset \Omega$,

$$\varphi_{r,x_0}^+(\beta t) \leq \varphi_{r,x_0}^-(t), \quad \forall t > 0 \text{ such that } \varphi_{r,x_0}^-(t) \in \left[1, \frac{1}{\mathcal{L}^d(B_r(x_0))}\right],$$

which in its turn implies (A1) if (dec) $_q$, hence a doubling condition, holds. In this setting, conditions (A0) and (A1) are invariant under a notion of function equivalence (see [23, Lemma 4.1.3]), provided the constant L (or β) is suitably rescaled. In particular, if φ satisfies (inc) $_p$, (dec) $_q$, (A0), and (A1), so does $c\varphi$ for every $c \in \mathbb{R}$.

Remark 2.2. If φ satisfies (inc) $_p$ (resp., (dec) $_q$) for some $p > 0$ (resp., $q > 0$), then so do φ_{r,x_0}^+ and φ_{r,x_0}^- for any $B_r(x_0) \subset \Omega$.

Remark 2.3. If $\varphi : \Omega \times [0, +\infty) \rightarrow [0, +\infty)$ is convex and $\varphi(x, 0) = 0$ for every $x \in \Omega$, then φ satisfies (inc) $_1$. If φ satisfies (inc) $_{p_1}$, then it satisfies (inc) $_{p_2}$ for every $0 < p_2 \leq p_1$. If φ satisfies (dec) $_{q_1}$, then it satisfies (dec) $_{q_2}$ for every $q_2 \geq q_1$.

Next simple results can be found in [26, Section 3].

Proposition 2.4. Let $1 < p \leq q < +\infty$ and $\varphi \in \Phi_c(\Omega)$ with right derivative φ_t . Assume that φ_t satisfies (inc) $_{p-1}$ and (dec) $_{q-1}$. Then

- (i) φ satisfies (inc) $_p$ and (dec) $_q$, and the following estimate hold:

$$\varphi(x, s) \min\{t^p, t^q\} \leq \varphi(x, ts) \leq \max\{t^p, t^q\} \varphi(x, s), \quad \forall x \in \Omega, \forall s, t \in [0, +\infty). \quad (2.4)$$

- (ii) $\varphi(x, t)$ and $t\varphi_t(x, t)$ are equivalent, in the sense that

$$p \varphi(x, t) \leq t \varphi_t(x, t) \leq q \varphi(x, t), \quad \forall (x, t) \in \Omega \times [0, +\infty); \quad (2.5)$$

(iii) if, in addition, φ_t complies with (A0), then also φ does with constants depending on L, p, q . More precisely,

$$\frac{1}{Lq} \leq \varphi(x, 1) \leq \frac{L}{p}, \quad \forall x \in \Omega. \quad (2.6)$$

If, in addition, $\varphi(x, \cdot) \in C^1([0, +\infty))$ for every $x \in \Omega$, then $\varphi \in N(\Omega)$.

For $\varphi \in \Phi_w(\Omega)$, the generalized Orlicz space is defined by

$$L^\varphi(\Omega) := \left\{ f \in L^0(\Omega) : \|f\|_{L^\varphi(\Omega)} < \infty \right\}$$

with the (Luxemburg) norm

$$\|f\|_{L^\varphi(\Omega)} := \inf \left\{ \lambda > 0 : \varrho_\varphi\left(\frac{f}{\lambda}\right) \leq 1 \right\}, \quad \text{where } \varrho_\varphi(f) := \int_\Omega \varphi(x, |f(x)|) dx.$$

We denote by $W^{1,\varphi}(\Omega)$ the set of $f \in L^\varphi(\Omega)$ satisfying that $\partial_1 f, \dots, \partial_d f \in L^\varphi(\Omega)$, where $\partial_i f$ is the weak derivative of f in the x_i -direction, with the norm $\|f\|_{W^{1,\varphi}(\Omega)} := \|f\|_{L^\varphi(\Omega)} + \sum_i \|\partial_i f\|_{L^\varphi(\Omega)}$. Note that if φ satisfies $(\text{dec})_q$ for some $q \geq 1$, then $f \in L^\varphi(\Omega)$ if and only if $\varrho_\varphi(f) < \infty$, and if φ satisfies (A0), $(\text{inc})_p$ and $(\text{dec})_q$ for some $1 < p \leq q$, then $L^\varphi(\Omega)$ and $W^{1,\varphi}(\Omega)$ are reflexive Banach spaces. In addition we denote by $W_0^{1,\varphi}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{1,\varphi}(\Omega)$.

The following version of Sobolev-Poincarè inequality for weak Φ -functions can be deduced by [24, Proposition 3.6].

Proposition 2.5. *Let $B_r \subset \mathbb{R}^d$ be a ball and $\varphi \in \Phi_w(B_r)$ be complying with (A0), (A1), $(\text{inc})_p$, $(\text{dec})_q$, $1 \leq p < q$, and let $s \in [1, p]$ with $s < \frac{d}{d-1}$. Then there exists a constant $C_P = C_P(d, s, p, q, L)$ such that*

$$\int_{B_r} \varphi\left(x, \frac{|u - (u)_{B_r}|}{2r}\right) dx \leq C_P \left(\left(\int_{B_r} \varphi(x, |\nabla u|)^{\frac{1}{s}} dx \right)^s + 1 \right) \quad (2.7)$$

for any $u \in W^{1,1}(B_r)$ such that $\|\nabla u\|_{L^\varphi(B_r)} \leq 1$.

Proof. The role of the further assumption $(\text{dec})_q$ with respect to [24, Proposition 3.6] is that it allows to transfer the constant $\beta_3 = \beta_3(d, s, \varphi) > 0$ therein to the right-hand side of the inequality as $C_P := \max\{\beta_3^{-q}, 1\}$. \square

2.2. Regularized Orlicz function. Let $\varphi \in \Phi_c(\Omega)$, $\varphi(x, \cdot) \in C^1([0, \infty))$ satisfying (A1), and such that φ_t comply with (A0), $(\text{inc})_{p-1}$ and $(\text{dec})_{q-1}$ for some $1 < p \leq q$. Then, as proven in [26, Proposition 5.10], on each ball $B = B_{2r}(x_0)$ a regularized function $\tilde{\varphi} = \tilde{\varphi}_B \in C^1([0, \infty)) \cap C^2((0, \infty))$ can be constructed such that

(i) $\tilde{\varphi}$ satisfies (A0), $(\text{inc})_p$, $(\text{dec})_q$, while $\tilde{\varphi}'$ complies with (A0), $(\text{inc})_{p-1}$ and $(\text{dec})_{q-1}$. In particular,

$$\tilde{\varphi}'(t) \sim t\tilde{\varphi}''(t), \quad \text{uniformly for all } t > 0. \quad (2.8)$$

(ii)

$$\tilde{\varphi}(t) \leq c(\varphi(x, t) + 1) \quad \text{for all } (x, t) \in B \times [0, \infty). \quad (2.9)$$

Moreover, taking into account (2.8), it can be shown that (see [26, Lemma 3.8(2)-(3)])

$$\tilde{\varphi}''(|z_1| + |z_2|)|z_1 - z_2|^2 \lesssim \tilde{\varphi}(|z_1|) - \tilde{\varphi}(|z_2|) - \frac{\tilde{\varphi}'(|z_2|)}{|z_2|} z_2 \cdot (z_1 - z_2), \quad \text{for every } z_1, z_2 \in \mathbb{R}^d. \quad (2.10)$$

and

$$\tilde{\varphi}(|z_1 - z_2|) \lesssim \varepsilon [\tilde{\varphi}(|z_1|) + \tilde{\varphi}(|z_2|)] + \varepsilon^{-1} \tilde{\varphi}''(|z_1| + |z_2|)|z_1 - z_2|^2, \quad \text{for every } z_1, z_2 \in \mathbb{R}^d, \varepsilon > 0. \quad (2.11)$$

2.3. Some technical lemmas. The following lemma, useful in order to re-absorb certain terms, is a variant of the classical [22, Lemma 6.1].

Lemma 2.6. ([25, Lemma 4.3]) *Let Z be a bounded non-negative function in the interval $[r, R]$ and let X be an almost decreasing function on $[0, +\infty)$. Assume that there exists $\theta \in [0, 1)$ such that*

$$Z(s) \leq \theta Z(t) + X\left(\frac{1}{t-s}\right),$$

for all $r \leq s < t \leq R$. Then

$$Z(r) \lesssim X\left(\frac{1}{R-r}\right),$$

where the implicit constant depends on the constant of almost monotonicity and on θ .

In order to derive reverse Hölder estimates, we need a variant of the results by Gehring [20] and Giaquinta-Modica [22, Theorem 6.6].

Lemma 2.7. *Let $B_0 \subset \mathbb{R}^n$ be a ball, $f \in L^1(B_0)$, and $g \in L^{\sigma_0}(B_0)$ for some $\sigma_0 > 1$. Assume that for some $\theta \in (0, 1)$, $c_1 > 0$ and all balls B with $2B \subset B_0$*

$$\int_B |f| \, dx \leq c_1 \left(\int_{2B} |f|^\theta \, dx \right)^{1/\theta} + \int_{2B} |g| \, dx.$$

Then there exist $\sigma_1 > 1$ and $c_2 > 1$ such that for all $\sigma_2 \in [1, \sigma_1]$

$$\left(\int_B |f|^{\sigma_2} \, dx \right)^{1/\sigma_2} \leq c_2 \int_{2B} |f| \, dx + c_2 \left(\int_{2B} |g|^{\sigma_2} \, dx \right)^{1/\sigma_2}.$$

The following iteration lemma can be found, e.g., in [22, Lemma 7.3].

Lemma 2.8. *Let $f : [0, R] \rightarrow [0, \infty)$ be a non-decreasing function. Assume that*

$$f(\rho) \leq A \left(\left(\frac{\rho}{r} \right)^\delta + \varepsilon \right) f(r) + Br^\gamma \quad \text{for all } 0 < \rho \leq r \leq R,$$

for positive constants A and B , and $\delta \leq \gamma$. Then for any $\sigma \in (0, \delta)$, there exist $\varepsilon_0, c > 0$ depending only on γ, δ, A and σ such that if $\varepsilon < \varepsilon_0$, then

$$f(\rho) \leq c \left(\left(\frac{\rho}{r} \right)^{\delta-\sigma} f(r) + B\rho^{\delta-\sigma} \right).$$

3. LOCAL LIPSCHITZ REGULARITY OF ALMOST MINIMIZERS

3.1. Regularity estimates for autonomous problems. Let $\psi \in \Phi_c \cap C^1([0, \infty)) \cap C^2((0, \infty))$ with ψ' satisfying (inc) $_{p-1}$ and (dec) $_{q-1}$ for some $1 < p \leq q$. For a given ball $B_r(x_0) \Subset \Omega$ and $w_0 \in W^{1,\psi}(B_r(x_0))$ we consider a weak solution to the Dirichlet problem

$$\begin{cases} \operatorname{div} \left(\frac{\psi'(|\nabla w|)}{|\nabla w|} \nabla w \right) = 0 & \text{in } B_r(x_0), \\ w = w_0 & \text{on } \partial B_r(x_0). \end{cases} \quad (3.1)$$

As proven in [26, Lemma 4.12], the following Harnack-type inequality and excess decay estimate hold for any such w .

Proposition 3.1. *Let $\psi \in \Phi_c \cap C^1([0, \infty)) \cap C^2((0, \infty))$ with ψ' satisfying $(\text{inc})_{p-1}$ and $(\text{dec})_{q-1}$ for some $1 < p \leq q$. Let $w \in W^{1,\psi}(B_r(x_0))$ be a weak solution to (3.1). Then there exists $\mu_0 = \mu_0(d, p, q) \in (0, 1)$ such that $\nabla w \in C_{\text{loc}}^{0,\mu_0}(B_r(x_0); \mathbb{R}^d)$ and the following estimates hold: there exists a constant $c = c(d, p, q) > 0$ such that, for every $B_\rho(y) \subset B_r(x_0)$,*

$$\sup_{B_{\rho/2}(y)} |\nabla w| \leq c \int_{B_\rho(y)} |\nabla w| \, dx, \quad (3.2)$$

and for any $\tau \in (0, 1)$,

$$\int_{B_{\tau\rho}(y)} |\nabla w - (\nabla w)_{B_{\tau\rho}(y)}| \, dx \leq c\tau^{\mu_0} \int_{B_\rho(y)} |\nabla w| \, dx. \quad (3.3)$$

For a given $u \in W_{\text{loc}}^{1,\varphi}(\Omega)$ local almost-minimizer of \mathcal{F} , and a ball $B_r(x_0) \Subset \Omega$, we consider $\tilde{\varphi}$ the N -function defined in Section 2.2 on $B_r(x_0)$, and the unique weak solution to the Dirichlet problem

$$\begin{cases} \operatorname{div} \left(\frac{\tilde{\varphi}'(|\nabla w|)}{|\nabla w|} \nabla w \right) = 0 & \text{in } B_r(x_0), \\ w = u & \text{on } \partial B_r(x_0), \end{cases} \quad (3.4)$$

or, equivalently, the solution to the minimization problem

$$\min_{w \in u + W_0^{1,\tilde{\varphi}}(B_r(x_0))} \int_{B_r(x_0)} \tilde{\varphi}(|\nabla w|) \, dx. \quad (3.5)$$

The existence and uniqueness in the minimization problem above follows from the fact that, by (2.9), $u \in W^{1,\tilde{\varphi}}(B_r(x_0))$, for which u is an admissible boundary-value function. This suggests the following definition.

Definition 3.1. *We define the $\tilde{\varphi}$ -harmonic replacement of u in $B_r(x_0)$, and we denote it by v_r , as the unique solution to the variational problem (3.5).*

Since we may choose $\psi = \tilde{\varphi}$ in Proposition 3.1, we have $\nabla v_r \in C_{\text{loc}}^{0,\mu_0}(B_r(x_0); \mathbb{R}^d)$ and the estimates (3.2) and (3.3) hold for v_r on every $B_\rho(y) \subset B_r(x_0)$.

3.2. Preliminary regularity results for almost-minimizers and comparison estimates. We start by proving a Caccioppoli-type inequality for almost minimizers of \mathcal{F} .

Lemma 3.2. *Let $\varphi \in \Phi_c(\Omega)$ be such that $(\text{dec})_q$ holds for some $q > 0$. Let u be an almost-minimizer of \mathcal{F} in Ω , with constant $\kappa \leq \kappa_0$ and exponent β , and let $x_0 \in \Omega$ and $B_{2r}(x_0) \Subset \Omega$, with $2r \leq 1$. Then there exists a constant $c = c(q, \kappa_0)$ such that*

$$\int_{B_r(x_0)} \varphi(x, |\nabla u|) \, dx \leq c \left(\int_{B_{2r}(x_0)} \varphi \left(x, \frac{|u - (u)_{x_0, 2r}|}{2r} \right) \, dx + \lambda \right). \quad (3.6)$$

Proof. Let $1 \leq s < t \leq 2$ and $\eta \in C_0^\infty(B_{tr}(x_0), [0, 1])$ be a cut-off function such that $\eta \equiv 1$ on $B_{sr}(x_0)$ and $|\nabla \eta| \leq \frac{2}{(t-s)r}$. Set $v := u - \eta(u - (u)_{x_0, 2r})$. Then we have $v = u$ on $\partial B_{tr}(x_0)$ and

$$\nabla v = (1 - \eta)\nabla u - \nabla \eta(u - (u)_{x_0, 2r}).$$

Then, since u is an almost minimizer of \mathcal{F} , using also the convexity of $\varphi(x, \cdot)$ and $(\text{dec})_q$ we get

$$\begin{aligned} \int_{B_{sr}(x_0)} \varphi(x, |\nabla u|) \, dx &\leq \mathcal{F}(u; B_{tr}(x_0)) \\ &\leq (1 + \kappa_0) \mathcal{F}(v; B_{tr}(x_0)) \\ &\leq c_1 \left(\int_{B_{tr}(x_0)} \varphi(x, |\nabla u|(1 - \eta)) \, dx + \int_{B_{tr}(x_0)} \varphi \left(x, \frac{|u - (u)_{x_0, 2r}|}{(t-s)r} \right) \, dx + \lambda \mathcal{L}^d(B_{2r}) \right) \\ &\leq c_1 \left(\int_{B_{tr}(x_0) \setminus B_{sr}(x_0)} \varphi(x, |\nabla u|) \, dx + \int_{B_{2r}(x_0)} \varphi \left(x, \frac{|u - (u)_{x_0, 2r}|}{(t-s)r} \right) \, dx + \lambda \mathcal{L}^d(B_{2r}) \right), \end{aligned}$$

where $c_1 = c_1(q, \kappa_0)$, whence adding $c_1 \int_{B_{sr}(x_0)} \varphi(x, |\nabla u|) \, dx$ to both the sides and then dividing by $1 + c_1$ we get

$$\int_{B_{sr}(x_0)} \varphi(x, |\nabla u|) \, dx \leq \frac{c_1}{1 + c_1} \left(\int_{B_{tr}(x_0)} \varphi(x, |\nabla u|) \, dx + \int_{B_{2r}(x_0)} \varphi \left(x, \frac{|u - (u)_{x_0, 2r}|}{(t-s)r} \right) \, dx + \lambda \mathcal{L}^d(B_{2r}) \right).$$

Now, an application of Lemma 2.6 with $\theta := \frac{c_1}{1+c_1}$, $Z(t) := \int_{B_{tr}(x_0)} \varphi(x, |\nabla u|) \, dx$ and $X(\tau) := \frac{c_1}{1+c_1} \int_{B_{2r}(x_0)} (\varphi(x, \tau|u - (u)_{x_0, 2r}|) + \lambda) \, dx$ gives

$$\int_{B_r(x_0)} \varphi(x, |\nabla u|) \, dx \leq c \int_{B_{2r}(x_0)} \varphi \left(x, \frac{|u - (u)_{x_0, 2r}|}{2r} \right) \, dx + c\lambda \mathcal{L}^d(B_{2r}).$$

Taking the average of both the sides we finally get the desired result. \square

The following lemma contains a higher integrability result and reverse Hölder type estimates for the gradient of an almost minimizer of \mathcal{F} .

Lemma 3.3. *Let $\varphi \in \Phi_w(\Omega)$ satisfy (A0), (A1), $(\text{inc})_p$, $(\text{dec})_q$ with constant $L \geq 1$ and $1 < p \leq q$. Let $u \in W_{\text{loc}}^{1,\varphi}(\Omega)$ be an almost-minimizer of \mathcal{F} in Ω with constant $\kappa \leq \kappa_0$ and exponent β , and let $x_0 \in \Omega$ and $B_{2r}(x_0) \Subset \Omega$, with $\|\nabla u\|_{L^\varphi(B_{2r}(x_0))} \leq 1$, and $2r \leq 1$. Then*

(i) (Higher integrability) *there exist $s_0 = s_0(d, p, q, L) > 0$ and $c = c(d, p, q, L, \kappa_0) \geq 1$ such that*

$$\left(\int_{B_r(x_0)} \varphi(x, |\nabla u|)^{1+s_0} \, dx \right)^{\frac{1}{1+s_0}} \leq c 2^{\frac{ds_0}{1+s_0}} \delta^{-\frac{ds_0}{1+s_0}} \left(\int_{B_{(1+\delta)r}(x_0)} \varphi(x, |\nabla u|) \, dx + \Lambda \right), \quad (3.7)$$

for any $\delta \in (0, 1]$, where $\Lambda := \lambda + 1$. In particular, this implies $\varphi(\cdot, |\nabla u|) \in L_{\text{loc}}^{1+s_0}(\Omega)$.

(ii) (Reverse Hölder type estimates) *for every $t \in (0, 1]$, there exist $c_t = c_t(d, p, q, L, \kappa_0, t) > 0$ such that*

$$\left(\int_{B_r(x_0)} \varphi(x, |\nabla u|)^{1+s_0} \, dx \right)^{\frac{1}{1+s_0}} \leq c_t \left(\left(\int_{B_{2r}(x_0)} \varphi(x, |\nabla u|)^t \, dx \right)^{\frac{1}{t}} + \Lambda \right), \quad (3.8)$$

and $c = c(d, p, q, L, \kappa_0) \geq 1$ such that

$$\int_{B_r(x_0)} \varphi(x, |\nabla u|) \, dx \leq \left(\int_{B_r(x_0)} \varphi(x, |\nabla u|)^{1+s_0} \, dx \right)^{\frac{1}{1+s_0}} \leq c \left(\varphi_{B_{2r}(x_0)}^- \left(\int_{B_{2r}(x_0)} |\nabla u| \, dx \right) + \Lambda \right). \quad (3.9)$$

Proof. The higher integrability result for ∇u in (i) can be obtained in a standard way combining the Caccioppoli inequality of Lemma 3.2 with the Sobolev-Poincarè inequality (Proposition 2.5), by Gehring's lemma (Lemma 2.7). The reverse Hölder type inequalities (ii) follow from (i) by a similar argument as for [26, Lemma 4.7]. We omit further details. \square

Note that, under our assumption on φ , it holds that

$$\|\nabla u\|_{L^\varphi(B_{2r}(x_0))} \leq 1 \iff \int_{B_{2r}(x_0)} \varphi(x, |\nabla u|) \, dx \leq 1$$

(it is sufficient that $\varphi \in \Phi_w(\Omega)$ and $\varphi(x, \cdot)$ be left-continuous; see, e.g., [23, Lemma 3.2.3]).

Remark 3.4 (Choice of small radii). (i) *Arguing as in [26], with fixed $\Omega' \Subset \Omega$ we can show that there exists $r_0 \in (0, 1)$, $r_0 = r_0(d, L, \omega(\cdot), \|\varphi(\cdot, |\nabla u|)^{1+s_0}\|_{L^1(\Omega')})$ satisfying*

$$r_0 \leq \frac{1}{2}, \quad \omega(2r_0) \leq \frac{1}{L}, \quad \mathcal{L}^d(B_{2r_0}) \leq \min \left\{ \frac{1}{2L}, 2^{-\frac{2(1+s_0)}{s_0}} \left(\int_{\Omega'} \varphi(x, |\nabla u|)^{1+s_0} \, dx \right)^{-\frac{2+s_0}{s_0}} \right\}, \quad (3.10)$$

where $L \geq 1$ is that of condition (A0) and s_0 is the exponent of Lemma 3.3, such that for any $B_{2r}(x_0) \subset \Omega'$ with $r \in (0, r_0]$, it holds that

$$\int_{B_{2r}(x_0)} \varphi(x, |\nabla u|) \, dx \leq 1.$$

Therefore, we can exploit the estimates of Lemma 3.3 in each of these balls.

(ii) If, in addition, $u \in L_{\text{loc}}^\infty(\Omega)$, the choice of s_0 and, accordingly, of r_0 can be done in such a way that the dependence of r_0 on u is through $\|u\|_{L^\infty(\Omega')}$. This observation will be crucial when dealing with some auxiliary sequences in our proofs. To show this, let $\Omega'' \Subset \Omega' \Subset \Omega$ be fixed, $y \in \overline{\Omega''}$ and $r < \frac{1}{4} \min\{1, \text{dist}(\Omega'', \partial\Omega')\}$. Then, by the Caccioppoli inequality of Lemma 3.2, (inc) $_p$, (dec) $_q$, (A0) we first find

$$\begin{aligned} \int_{B_{2r}(y)} \varphi(x, |\nabla u|) \, dx &\leq c(q, \kappa_0) \left(\int_{B_{4r}(y)} \varphi_{B_{4r}(y)}^+ \left(\frac{\|u\|_{L^\infty(B_{4r})}}{2r} \right) \, dx + \lambda \mathcal{L}^d(B_r) \right) \\ &\leq c(d, p, q, \kappa_0, L) r^d \left(\max \left\{ \left(\frac{\|u\|_{L^\infty(\Omega')}}{r} \right)^p, \left(\frac{\|u\|_{L^\infty(\Omega')}}{r} \right)^q \right\} + \lambda \right) \\ &=: \bar{c}(d, p, q, L, \kappa_0, r, \|u\|_{L^\infty(\Omega')}, \lambda), \end{aligned}$$

where \bar{c} depends increasingly on $\|u\|_{L^\infty(\Omega')}$. If $\bar{c} \leq 1$, then by (3.8) for $t = 1$ and a covering argument we get the bound

$$\begin{aligned} \int_{\Omega''} \varphi(x, |\nabla u|)^{1+s_0} \, dx &\leq [c_1(\bar{c} + \Lambda)]^{1+s_0} \\ &=: \tilde{c} = \tilde{c}(d, p, q, L, \kappa_0, \|u\|_{L^\infty(\Omega')}, \text{dist}(\Omega'', \partial\Omega'), \lambda), \end{aligned} \quad (3.11)$$

where \tilde{c} depends increasingly on $\|u\|_{L^\infty(\Omega')}$. Hence, in (3.10), it will suffice to require

$$r \leq \frac{1}{2}, \quad \omega(2r) \leq \frac{1}{L}, \quad \mathcal{L}^d(B_{2r}) \leq \min \left\{ \frac{1}{2L}, 2^{-\frac{2(1+s_0)}{s_0}} \tilde{c}^{-\frac{2+s_0}{s_0}} \right\}.$$

If $\tilde{c} > 1$, we can apply the same argument to $\bar{\varphi}(x, t) := \frac{1}{\tilde{c}} \varphi(x, t)$, observing that (A0), (inc) $_p$, (dec) $_q$ and (A1) still hold (see Remark 2.1). In this case, as the constants in (A0) may have changed, the value of $s_0 > 0$ could become possibly smaller.

With the following Lemma, we prove a further reverse Hölder type inequality for ∇u and a Calderón-Zygmund type estimate for the problem (3.4).

Lemma 3.5. *Let φ be as in Lemma 3.3. Let $u \in W_{\text{loc}}^{1,\varphi}(\Omega)$ be an almost minimizer of \mathcal{F} in Ω with constant $\kappa \leq \kappa_0$ and exponent β , and $v_r \in W^{1,\tilde{\varphi}}(B_r(x_0))$ be the $\tilde{\varphi}$ -harmonic replacement of u in $B_r(x_0)$, where $B_{2r}(x_0) \Subset \Omega$ with r satisfying (3.10), and $\tilde{\varphi}$ defined on $B_r(x_0)$ as in Section 2.2. Then there exists a constant $c = c(d, p, q, L, \kappa_0) \geq 1$ such that*

$$\left(\int_{B_r(x_0)} \varphi(x, |\nabla u|)^{1+s_0} dx \right)^{\frac{1}{1+s_0}} \leq c \left[\tilde{\varphi} \left(\int_{B_{2r}(x_0)} |\nabla u| dx \right) + \Lambda \right] \quad (3.12)$$

and

$$\begin{aligned} \int_{B_r(x_0)} \varphi(x, |\nabla v_r|) dx &\leq \left(\int_{B_r(x_0)} \varphi(x, |\nabla v_r|)^{1+\frac{s_0}{2}} dx \right)^{\frac{1}{1+\frac{s_0}{2}}} \\ &\leq c \left(\int_{B_r(x_0)} \varphi(x, |\nabla u|)^{1+\frac{s_0}{2}} dx + \Lambda \right)^{\frac{1}{1+\frac{s_0}{2}}} \end{aligned} \quad (3.13)$$

Moreover,

$$\int_{B_r(x_0)} |\nabla v_r| dx \leq c \left(\int_{B_{2r}(x_0)} |\nabla u| dx + \Lambda \right). \quad (3.14)$$

Proof. The proof can be obtained exactly as in [26, Lemma 5.15] by using (A0), Lemma 3.3(ii) and some general technical results (see [26, Proposition 5.12 and Lemma 4.15]). We omit further details. \square

With the following Proposition, we establish a comparison estimate between the gradient of an almost minimizer of \mathcal{F} and that of its $\tilde{\varphi}$ -harmonic replacement in a ball. From now on, the stronger condition (VA1) is needed in place of (A1).

Proposition 3.6. *Let $\varphi \in \Phi_c(\Omega)$, $\varphi(x, \cdot) \in C^1([0, \infty))$ be satisfying (VA1), such that φ_t satisfies (A0), $(\text{inc})_{p-1}$ and $(\text{dec})_{q-1}$ for some $1 < p \leq q$. Let $u \in W_{\text{loc}}^{1,\varphi}(\Omega)$ be an almost-minimizer of \mathcal{F} in Ω , with constant $\kappa \leq \kappa_0$ and exponent β , let $x_0 \in \Omega$ and $r > 0$ be such that $B_{2r}(x_0) \Subset \Omega$ and complying with (3.10). Let $\tilde{\varphi}$ be defined on $B_r(x_0)$ as in Section 2.2, and $v_r \in W^{1,\tilde{\varphi}}(B_r(x_0))$ be the $\tilde{\varphi}$ -harmonic replacement of u in $B_r(x_0)$. Then, there exists a constant $c = c(d, p, q, L, \kappa_0) \geq 1$ such that*

$$\int_{B_r(x_0)} \tilde{\varphi}'(|\nabla u| + |\nabla v_r|) |\nabla u - \nabla v_r|^2 dx \lesssim \left(\omega(2r)^{\frac{p}{q}} + r^{\min\{\beta, \gamma\}} \right) \tilde{\varphi} \left(\int_{B_{2r}(x_0)} |\nabla u| dx \right) + \Lambda. \quad (3.15)$$

where $\gamma := \min \left\{ 1, \frac{ds_0^2}{4(2+s_0)} \right\}$ and s_0 is the exponent of Lemma 3.3.

Proof. Here we follow the argument of [26, Lemma 6.2]. First, since v_r is a weak solution to (3.4) and $u - v_r \in W_0^{1,\tilde{\varphi}}(B_r(x_0))$ is an admissible test function for the weak formulation of (3.4), we obtain

$$\int_{B_r(x_0)} \frac{\tilde{\varphi}'(|\nabla v_r|)}{|\nabla v_r|} \nabla v_r \cdot \nabla(u - v_r) dx = 0.$$

Now, with this and (2.10), we get

$$\begin{aligned}
 \int_{B_r(x_0)} \widetilde{\varphi}''(|\nabla u| + |\nabla v_r|)|\nabla u - \nabla v_r|^2 dx &\lesssim \int_{B_r(x_0)} \widetilde{\varphi}(|\nabla u|) dx - \int_{B_r(x_0)} \widetilde{\varphi}(|\nabla v_r|) dx \\
 &\quad - \int_{B_r(x_0)} \widetilde{\varphi}'(|\nabla v_r|) \frac{\nabla v_r}{|\nabla v_r|} \cdot (\nabla u - \nabla v_r) dx \\
 &= \int_{B_r(x_0)} \widetilde{\varphi}(|\nabla u|) dx - \int_{B_r(x_0)} \widetilde{\varphi}(|\nabla v_r|) dx \\
 &= \int_{B_r(x_0)} \widetilde{\varphi}(|\nabla u|) - \varphi(x, |\nabla u|) dx + \int_{B_r(x_0)} \varphi(x, |\nabla u|) - \varphi(x, |\nabla v_r|) dx \\
 &\quad + \int_{B_r(x_0)} \varphi(x, |\nabla v_r|) - \widetilde{\varphi}(|\nabla v_r|) dx =: I_1 + I_2 + I_3.
 \end{aligned} \tag{3.16}$$

We proceed to estimate each integral separately.

Both the terms I_1 and I_3 can be estimated as

$$\int_{B_r(x_0)} |\varphi(x, |\nabla u|) - \widetilde{\varphi}(|\nabla u|)| dx, \quad \int_{B_r(x_0)} |\varphi(x, |\nabla v_r|) - \widetilde{\varphi}(|\nabla v_r|)| dx \leq c \left(\omega(2r)^{\frac{p}{q}} + r^\gamma \right) \left[\widetilde{\varphi} \left(\int_{B_{2r}(x_0)} |\nabla u| dx \right) + \Lambda \right] \tag{3.17}$$

by the very same argument of [26, Lemma 6.2] exploiting (A0), (inc) $_p$, (dec) $_q$, (2.9), (3.12) and (3.13), where the constant c depends on d, p, q, L .

As for I_2 , using v_r in the definition of u as almost minimizer of \mathcal{F} , and taking into account (3.13) and (3.12), we get

$$\begin{aligned}
 I_2 &\leq \mathcal{L}^d(B_r) \kappa r^\beta \left(\int_{B_r(x_0)} \varphi(x, |\nabla v_r|) dx + \lambda \right) + \mathcal{L}^d(B_r) \lambda \\
 &\leq c \kappa r^\beta \mathcal{L}^d(B_r) \widetilde{\varphi} \left(\int_{B_{2r}(x_0)} |\nabla u| dx \right) + c \mathcal{L}^d(B_r) \Lambda.
 \end{aligned}$$

Plugging this estimate together with (3.17) into (3.16) we obtain (3.15). \square

3.3. Local Hölder continuity. In this section, we will establish two main regularity results for almost minimizers, which will be instrumental in the proof of the local Lipschitz continuity result. First, we will prove the $C^{0,\alpha}$ -regularity of any almost-minimizer u for \mathcal{F} , locally within Ω , for *any* exponent $\alpha \in (0, 1)$. Then, we will show that ∇u is locally $C^{0,\alpha}$ for *some* exponent $\alpha \in (0, 1)$ away from the free-boundary $\partial\{u > 0\}$.

We are now in position to prove the first regularity result.

Theorem 3.7 ($C^{0,\alpha}$ -regularity). *Let $\varphi \in \Phi_c(\Omega)$, $\varphi(x, \cdot) \in C^1([0, \infty))$ be satisfying (VA1), and such that φ_t comply with (A0), (inc) $_{p-1}$ and (dec) $_{q-1}$ for some $1 < p \leq q$. If $u \in W_{\text{loc}}^{1,\varphi}(\Omega)$ is an almost minimizer of \mathcal{F} with constant $\kappa \leq \kappa_0$ and exponent β , then $u \in C_{\text{loc}}^{0,\alpha}(\Omega)$ for any $\alpha \in (0, 1)$. More precisely, if $\Omega' \Subset \Omega$ is fixed, there exists $0 < R_0 < \frac{1}{2} \text{dist}(\Omega', \partial\Omega)$ complying with (3.10) (thus depending on u) and a constant $c = c(d, p, q, \alpha, \kappa_0, R_0, \Omega')$ such that*

$$[u]_{C^\alpha(\Omega')} \leq c \left(\int_{\Omega} |\nabla u| dx + \Lambda \right). \tag{3.18}$$

Proof. Let $R_0 \in (0, 1)$ be sufficiently small to be determined later. Let $\Omega' \Subset \Omega$ be fixed, and assume that (3.10) holds for R_0 , $R_0 < \text{dist}(\Omega', \partial\Omega)/2$. For any $B_{2r} \Subset \Omega'$, with $2r \leq R_0$, denote by v_r the $\tilde{\varphi}$ -harmonic replacement of u in B_r . Set $\eta(r) := \omega(2r)^{\frac{p}{q}} + r^{\min\{\beta, \gamma\}}$, and note that $\eta(\cdot) \leq 2$.

Let $\tau \in (0, \frac{1}{2})$. Then, by (2.11) with $\varepsilon = \frac{1}{2}$, $z_1 = \nabla u$ and $z_2 = \nabla v_r$, Proposition 3.6, (3.2) for $w = v_r$, (2.9) and (3.13) we get

$$\begin{aligned}
\int_{B_{\tau r}} \tilde{\varphi}(|\nabla u|) \, dx &\lesssim \int_{B_r} \tilde{\varphi}''(|\nabla u| + |\nabla v_r|) |\nabla u - \nabla v_r|^2 \, dx + \int_{B_{\tau r}} \tilde{\varphi}(|\nabla v_r|) \, dx \\
&\leq c r^d \eta(2r) \tilde{\varphi} \left(\int_{B_{2r}(x_0)} |\nabla u| \, dx \right) + c \Lambda r^d + c(\tau r)^d \sup_{B_{r/2}} \tilde{\varphi}(|\nabla v_r|) \\
&\leq c r^d \eta(R_0) \tilde{\varphi} \left(\int_{B_{2r}(x_0)} |\nabla u| \, dx \right) + c \Lambda r^d + c(\tau r)^d \int_{B_r} \tilde{\varphi}(|\nabla v_r|) \, dx \\
&\leq c r^d \eta(R_0) \tilde{\varphi} \left(\int_{B_{2r}} |\nabla u| \, dx \right) + c \Lambda r^d + c(\tau r)^d \int_{B_r} (\varphi(x, |\nabla v_r|) + 1) \, dx \\
&\leq c r^d \eta(R_0) \tilde{\varphi} \left(\int_{B_{2r}} |\nabla u| \, dx \right) + c \Lambda r^d + c(\tau r)^d \tilde{\varphi} \left(\int_{B_{2r}} |\nabla u| \, dx \right) + c \Lambda (\tau r)^d \\
&\leq c r^d (\eta(R_0) + \tau^d) \tilde{\varphi} \left(\int_{B_{2r}} |\nabla u| \, dx \right) + c \Lambda r^d,
\end{aligned} \tag{3.19}$$

whence, by Jensen's inequality,

$$\begin{aligned}
\tilde{\varphi} \left(\int_{B_{\tau r}} |\nabla u| \, dx \right) &\leq \int_{B_{\tau r}} \tilde{\varphi}(|\nabla u|) \, dx \leq c \left(\frac{\eta(R_0)}{\tau^d} + 1 \right) \tilde{\varphi} \left(\int_{B_{2r}} |\nabla u| \, dx \right) + c \frac{\Lambda}{\tau^d} \\
&\leq c \tilde{\varphi} \left(\left(\frac{\eta(R_0)}{\tau^d} + 1 \right) \int_{B_{2r}} |\nabla u| \, dx + \frac{\Lambda}{\tau^d} \right),
\end{aligned} \tag{3.20}$$

so that

$$\int_{B_{\tau r}} |\nabla u| \, dx \lesssim (\eta(R_0) + \tau^d) \int_{B_{2r}} |\nabla u| \, dx + \Lambda r^d. \tag{3.21}$$

The previous estimate trivially holds also for $\tau \in (\frac{1}{2}, 2)$. Let $\sigma \in (0, d)$ and choose R_0 small enough such that

$$\eta(R_0) \leq \varepsilon_0, \tag{3.22}$$

where ε_0 is that of Lemma 2.8, applied to the function $f(\tau) := \int_{B_{\tau r}} |\nabla u| dx$. We then obtain the Morrey-type estimate

$$\int_{B_\rho} |\nabla u| dx \leq c \left(\left(\frac{\rho}{R_0} \right)^{-\sigma} \int_{B_{R_0}} |\nabla u| dx + \Lambda \rho^{-\sigma} \right) \quad \text{for all balls } B_\rho \subset \Omega' \text{ with } \rho \in (0, R_0], \quad (3.23)$$

where the constant $c \geq 1$ depends on d, p, q, L, κ_0 and σ .

Now, let $\alpha \in (0, 1)$ be arbitrarily fixed. Then choosing $\sigma = 1 - \alpha$ in (3.23), by Morrey's Theorem (see, e.g., [21, Chapter III, Theorem 1.1]) we infer that $u \in C^{0,\alpha}(\overline{B_\rho})$, for every $B_\rho \subset B_{R_0}$ and

$$[u]_{C^\alpha(B_\rho)} \leq c \left[R_0^{1-\alpha} \int_{B_{R_0}} |\nabla u| dx + \Lambda \right]. \quad (3.24)$$

Therefore, $u \in C_{\text{loc}}^{0,\alpha}(\Omega)$ for every $\alpha \in (0, 1)$ fixed. Since $\overline{\Omega'}$ is compact, (3.18) follows from (3.24) by a standard covering argument. \square

With the following result, we establish the local Hölder continuity of the gradient of almost-minimizers, away from the free boundary.

Theorem 3.8 ($C^{1,\alpha}$ -regularity in $\{u > 0\}$). *Let $\varphi \in \Phi_c(\Omega)$, $\varphi(x, \cdot) \in C^1([0, \infty))$ be such that φ_t comply with (A0), (inc) $_{p-1}$ and (dec) $_{q-1}$ for some $1 < p \leq q$. If $u \in W_{\text{loc}}^{1,\varphi}(\Omega)$ is an almost minimizer of \mathcal{F} with constant $\kappa \leq \kappa_0$ and exponent β and φ satisfies (VA1) with*

$$\omega(r) \leq cr^\theta, \quad \text{for all } r \in (0, 1] \text{ and for some } \theta \in (0, 1), \quad (3.25)$$

then $u \in C_{\text{loc}}^{1,\bar{\alpha}}(\{u > 0\})$ for some $\bar{\alpha} \in (0, 1)$ depending on $d, p, q, L, \beta, \theta$. More precisely, for any $\widetilde{\Omega} \Subset \{u > 0\} \cap \Omega$, there exists an exponent $\bar{\alpha} = \bar{\alpha}(d, p, q, L, \beta, \theta)$ and a constant $C = C(\widetilde{\Omega}, d, p, q, L, \kappa_0, \beta, \theta)$ such that

$$[\nabla u]_{C^{\bar{\alpha}}(\widetilde{\Omega})} \leq C \left(\int_{\Omega} |\nabla u| dx + \Lambda \right). \quad (3.26)$$

Proof. Let $\widetilde{\Omega} \Subset \{u > 0\} \cap \Omega$. Then, from (3.23) and a standard covering argument we deduce that, for every $\sigma \in (0, d)$,

$$\int_{B_{2r}} |\nabla u| dx \leq c_\sigma \left(\int_{\Omega} |\nabla u| dx + \Lambda \right) r^{-\sigma} \quad \text{for all balls } B_{2r} \subset \widetilde{\Omega} \text{ with } 2r \in (0, R_0], \quad (3.27)$$

where R_0 is that of Theorem 3.7 and $c_\sigma = c_\sigma(d, p, q, L, R_0, \sigma)$.

Let us fix any of such balls, say $B_r \Subset \widetilde{\Omega}$, with $r \leq R_0/2$ small enough and to be determined later, and denote by v_r the $\widetilde{\varphi}$ -harmonic replacement of u in B_r . Arguing as in the estimate (3.16) we get

$$\begin{aligned} \int_{B_r} \widetilde{\varphi}''(|\nabla u| + |\nabla v_r|) |\nabla u - \nabla v_r|^2 dx &\leq \int_{B_r} \widetilde{\varphi}(|\nabla u|) - \varphi(x, |\nabla u|) dx + \int_{B_r} \varphi(x, |\nabla u|) - \varphi(x, |\nabla v_r|) dx \\ &\quad + \int_{B_r} \varphi(x, |\nabla v_r|) - \widetilde{\varphi}(|\nabla v_r|) dx. \end{aligned} \quad (3.28)$$

Since, by the convex-hull property, $v_r(B_r)$ is contained in the convex hull of $v_r(\partial B_r) = u(\partial B_r)$ and so in particular $\chi_{\{u>0\}} = \chi_{\{v_r>0\}}$ on B_r , taking into account also (3.17), (3.25), (2.9) and (3.12) we get

$$\begin{aligned}
\int_{B_r} \varphi(x, |\nabla u|) - \varphi(x, |\nabla v_r|) \, dx &\leq \kappa r^\beta \int_{B_r} \varphi(x, |\nabla v_r|) \, dx + \kappa r^\beta \lambda \int_{B_r} \chi_{\{v_r>0\}}(x) \, dx \\
&\leq \kappa r^\beta \int_{B_r} (\varphi(x, |\nabla v_r|) - \widetilde{\varphi}(|\nabla v_r|)) \, dx + \kappa r^\beta \int_{B_r} \widetilde{\varphi}(|\nabla u|) \, dx \\
&\quad + \lambda \kappa r^\beta \\
&\leq \kappa r^\beta \left(r^{\frac{\theta p}{q}} + r^\gamma \right) \left[\widetilde{\varphi} \left(\int_{B_{2r}} |\nabla u| \, dx \right) + \Lambda \right] + \kappa r^\beta \int_{B_r} \varphi(x, |\nabla u|) \, dx \quad (3.29) \\
&\quad + \Lambda \kappa r^\beta \\
&= \kappa r^\beta \left(r^{\frac{\theta p}{q}} + r^\gamma + 1 \right) \left[\widetilde{\varphi} \left(\int_{B_{2r}} |\nabla u| \, dx \right) + \Lambda \right] \\
&\leq c \kappa r^\beta \left[\widetilde{\varphi} \left(\int_{B_{2r}} |\nabla u| \, dx \right) + \Lambda \right].
\end{aligned}$$

Now, plugging (3.29) into (3.28), and estimating the other two integrals again as in (3.17), we obtain

$$\int_{B_r} \widetilde{\varphi}''(|\nabla u| + |\nabla v_r|) |\nabla u - \nabla v_r|^2 \, dx \lesssim (\kappa_0 r^\beta + r^{\gamma_1}) \left[\widetilde{\varphi} \left(\int_{B_{2r}} |\nabla u| \, dx \right) + \Lambda \right], \quad (3.30)$$

where $\gamma_1 := \min\{\frac{\theta p}{q}, \gamma\} < 1$. Following the argument in [26, Lemma 6.3], we set $\eta(r) := r^\beta + r^{\gamma_1} \leq 2$, and apply (2.11) with $\varepsilon = \sqrt{\eta(r)}$, (2.9), Lemma 3.5, (3.30) and (3.15). We find that

$$\begin{aligned}
&\int_{B_r} \widetilde{\varphi}(|\nabla u - \nabla v_r|) \, dx \\
&\lesssim \sqrt{\eta(r)} \int_{B_r} [\widetilde{\varphi}(|\nabla u|) + \widetilde{\varphi}(|\nabla v_r|)] \, dx + \frac{1}{\sqrt{\eta(r)}} \int_{B_r} \widetilde{\varphi}''(|\nabla u| + |\nabla v_r|) |\nabla u - \nabla v_r|^2 \, dx \\
&\lesssim \sqrt{\eta(r)} \int_{B_r} [\varphi(x, |\nabla u|) + \varphi(x, |\nabla v_r|) + 1] \, dx + \sqrt{\eta(r)} \left(\widetilde{\varphi} \left(\int_{B_{2r}} |\nabla u| \, dx \right) + \Lambda \right) \\
&\lesssim \sqrt{\eta(r)} \left[\widetilde{\varphi} \left(\int_{B_{2r}} |\nabla u| \, dx \right) + \Lambda \right].
\end{aligned}$$

Therefore, by Jensen's inequality and $(\text{dec})_q$ of $\tilde{\varphi}$, we have

$$\tilde{\varphi} \left(\int_{B_r} |\nabla u - \nabla v_r| \, dx \right) \leq \int_{B_r} \tilde{\varphi}(|\nabla u - \nabla v_r|) \, dx \lesssim \tilde{\varphi} \left(\eta(r)^{\frac{1}{2q}} \left(\int_{B_{2r}} |\nabla u| \, dx + \Lambda \right) \right),$$

whence, since $\tilde{\varphi}$ is strictly increasing, we finally obtain

$$\int_{B_r} |\nabla u - \nabla v_r| \, dx \lesssim r^{\beta_1} \left(\int_{B_{2r}} |\nabla u| \, dx + \Lambda \right), \quad (3.31)$$

where $\beta_1 := \min\{\frac{\beta}{2q}, \frac{\gamma_1}{2q}\} < 1$ and the implicit constant depends also on κ_0 .

On the other hand, with $\tau \in (0, \frac{1}{2})$,

$$\begin{aligned} \int_{B_{\tau r}} |\nabla u - (\nabla u)_{B_{\tau r}}| \, dx &\leq c \int_{B_{\tau r}} |\nabla u - (\nabla v_r)_{B_{\tau r}}| \, dx \\ &\leq c\tau^{-d} \int_{B_r} |\nabla u - \nabla v_r| \, dx + c \int_{B_{\tau r}} |\nabla v_r - (\nabla v_r)_{B_{\tau r}}| \, dx. \end{aligned} \quad (3.32)$$

Note that using (3.3) and (3.14) we obtain

$$\begin{aligned} \int_{B_{\tau r}} |\nabla v_r - (\nabla v_r)_{B_{\tau r}}| \, dx &\leq c\tau^{\mu_0} \int_{B_{r/2}} |\nabla v_r| \, dx \\ &\leq c\tau^{\mu_0} \left(\int_{B_r} |\nabla u| \, dx + \Lambda \right). \end{aligned} \quad (3.33)$$

Inserting the estimate (3.33) into (3.32), and taking into account (3.31), we get

$$\begin{aligned} \int_{B_{\tau r}} |\nabla u - (\nabla u)_{B_{\tau r}}| \, dx &\leq c\tau^{-d} \int_{B_r} |\nabla u - \nabla v_r| \, dx + c\tau^{\mu_0} \left(\int_{B_{2r}} |\nabla u| \, dx + \Lambda \right) \\ &\leq c \left[\tau^{-d} r^{\beta_1} + \tau^{\mu_0} \right] \left(\int_{B_{2r}} |\nabla u| \, dx + \Lambda \right). \end{aligned} \quad (3.34)$$

Now, with (3.27), we get

$$\int_{B_{\tau r}} |\nabla u - (\nabla u)_{B_{\tau r}}| \, dx \leq c \left[\tau^{-d} r^{\beta_1 - \sigma} + \tau^{\mu_0} r^{-\sigma} \right] \left(\int_{\Omega} |\nabla u| \, dx + \Lambda \right). \quad (3.35)$$

Choosing $\tau := r^{\frac{\beta_1}{\mu_0 + d}}$, we have $\tau^{-d} r^{\beta_1 - \sigma} = \tau^{\mu_0} r^{-\sigma} = r^{\frac{\beta_1 \mu_0}{\mu_0 + d} - \sigma}$, so that

$$\begin{aligned} \int_{B_{\tau r}} |\nabla u - (\nabla u)_{B_{\tau r}}| \, dx &\leq c r^{\frac{\beta_1 \mu_0}{\mu_0 + d} - \sigma} \left(\int_{\Omega} |\nabla u| \, dx + \Lambda \right) \\ &= c(\tau r)^{\frac{\beta_1 \mu_0 - \sigma(\mu_0 + d)}{\mu_0 + d + \beta_1}} \left(\int_{\Omega} |\nabla u| \, dx + \Lambda \right), \end{aligned} \quad (3.36)$$

and if $r \leq \frac{1}{2} \min\{R_0, 4^{-\frac{\mu_0+d}{\beta_1}}\}$, then $\rho := \tau r < \frac{r}{2}$. Fixed $\bar{\sigma} := \frac{\mu_0\beta_1}{2(\mu_0+d)}$ and setting $\bar{\alpha} := \frac{\beta_1\mu_0 - \bar{\sigma}(\mu+d)}{\mu_0+d+\beta_1} = \frac{\beta_1\mu_0}{2(\mu_0+d+\beta_1)} < 1$, the previous estimate (3.36) can be rewritten as

$$\int_{B_\rho} |\nabla u - (\nabla u)_{B_\rho}| \, dx \leq c \left(\int_{\Omega} |\nabla u| \, dx + \Lambda \right) \rho^{\bar{\alpha}}, \quad (3.37)$$

where $B_\rho \subset B_{2r} \subset \widetilde{\Omega}$. From the Campanato-type embedding (see, e.g., [21, Chapter III, Theorem 1.3]), this implies $\nabla u \in C_{\text{loc}}^{0,\bar{\alpha}}(\widetilde{\Omega})$ and, since $\widetilde{\Omega}$ was arbitrary, $\nabla u \in C_{\text{loc}}^{0,\bar{\alpha}}(\{u > 0\})$. Moreover, (3.26) follows by a covering argument as in the end of the proof of Theorem 3.8. \square

3.4. Proof of the local Lipschitz continuity.

Lemma 3.9. *Under the assumptions of Theorem 3.8, let u be a bounded almost minimizer of \mathcal{F} in $B_1(0)$ with constant $\kappa \leq \kappa_0$ and exponent β . Assume that $B_1(0) = \{u > 0\}$. Then*

$$|\nabla u(0)| \leq C, \quad (3.38)$$

where the constant C depends on $p, q, d, L, \kappa_0, \beta, \theta, \Lambda, \|u\|_{L^\infty(B_1(0))}$.

Proof. We have

$$|\nabla u(0)| \leq |\nabla u(0) - (\nabla u)_{B_{\frac{1}{4}}(0)}| + |(\nabla u)_{B_{\frac{1}{4}}(0)}|. \quad (3.39)$$

Now, the first term on the right hand side above can be estimated by Theorem 3.8: observe that, since we are assuming u to be bounded, by Remark 3.4, (ii) the constant appearing there depends on $p, q, d, L, \kappa_0, \beta, \theta, \Lambda, \|u\|_{L^\infty(B_1(0))}$. We then have

$$\begin{aligned} |\nabla u(0) - (\nabla u)_{B_{\frac{1}{4}}(0)}| &\leq \int_{B_{\frac{1}{4}}(0)} |\nabla u(0) - \nabla u(x)| \, dx \\ &\leq 4^{-\alpha} [\nabla u]_{C^\alpha(B_{\frac{1}{4}}(0))} \\ &\leq C \left(\int_{B_{\frac{1}{2}}(0)} |\nabla u| \, dx + \Lambda \right), \end{aligned} \quad (3.40)$$

while

$$|(\nabla u)_{B_{\frac{1}{4}}(0)}| \leq \mathcal{L}^d(B_{\frac{1}{4}}) \left(\int_{B_{\frac{1}{2}}(0)} |\nabla u| \, dx + \Lambda \right). \quad (3.41)$$

Further, from the Caccioppoli inequality Lemma 3.2, (2.6), (inc)₁ and the boundedness of u on $B_1(0)$, we get

$$\begin{aligned} \int_{B_{\frac{1}{2}}(0)} |\nabla u| \, dx &\leq Lq \int_{B_{\frac{1}{2}}(0)} (\varphi(x, |\nabla u|) + 1) \, dx \leq c \int_{B_1(0)} (\varphi(x, 2\|u\|_{L^\infty(B_1(0))}) + \Lambda) \, dx \\ &\leq c \left(\max\{\|u\|_{L^\infty(B_1(0))}^p, \|u\|_{L^\infty(B_1(0))}^q\} + \Lambda \right). \end{aligned} \quad (3.42)$$

Combining (3.39)–(3.42) we obtain (3.38), and this concludes the proof. \square

Lemma 3.10. *Let $R > 0$ be such that $B_{2R}(0) \Subset \Omega$, and $(r_j)_{j \in \mathbb{N}}, (\sigma_j)_{j \in \mathbb{N}}$ be sequences of nonnegative numbers, with $R < \frac{1}{2r_j}$ for every j , $r_j \rightarrow 0$ as $j \rightarrow +\infty$, and*

$$\sigma_j \rightarrow +\infty \quad \text{and} \quad \varphi(0, \sigma_j)r_j \rightarrow 0 \quad \text{as } j \rightarrow +\infty. \quad (3.43)$$

We define, for every j ,

$$\varphi_j(x, t) := \frac{\varphi(r_j x, \sigma_j t)}{\varphi(0, \sigma_j)}, \quad x \in B_{2R}(0), \quad t > 0. \quad (3.44)$$

Then,

(i) the functions

$$\varphi_j^-(t) := \inf_{y \in B_{2R}(0)} \varphi_j(y, t), \quad \text{and} \quad \varphi_j^+(t) := \sup_{y \in B_{2R}(0)} \varphi_j(y, t), \quad (3.45)$$

are weak Φ functions satisfying $(\text{inc})_p$ and $(\text{dec})_q$. Moreover, for j large enough,

$$\min\{t^p, t^q\} \lesssim \varphi_j^-(t) \leq \max\{t^p, t^q\}, \quad \min\{t^p, t^q\} \leq \varphi_j^+(t) \lesssim \max\{t^p, t^q\}, \quad (3.46)$$

where the hidden constants are independent of j ;

(ii) there exists $j_0 \in \mathbb{N}$ such that φ_j complies with (A0) for $j \geq j_0$ with $L = 2$;

(iii) there exists $j_0 \in \mathbb{N}$ such that φ_j complies with (VA1) for $j \geq j_0$ with the same ω ;

(iv) there exists a convex function $\varphi_\infty \in C^1([0, +\infty))$, whose derivative φ'_∞ complies with $(\text{inc})_{p-1}$ and $(\text{dec})_{q-1}$ such that

$$\varphi_j(x, t) \rightarrow \varphi_\infty(t) \quad \text{uniformly on } B_{2R}(0) \times K, \quad \text{where } K \subset [0, +\infty) \text{ is compact.} \quad (3.47)$$

Proof. For the proof of (i) we can argue as in [28, pp. 13–14]. Concerning the proof of (ii), let $j_0 \in \mathbb{N}$ be such that

$$r_j < 1, \quad \sigma_j > L^{\frac{1}{p}}, \quad \varphi(0, \sigma_j) > 1, \quad \text{and} \quad \varphi(0, \sigma_j)r_j < \min\{1, 1/\text{diam}(\Omega)\gamma_d\} \quad \text{for } j \geq j_0. \quad (3.48)$$

Thanks to (3.48) it is easy to show that $\varphi_{2r_j R}^-(\sigma_j) \in [\omega(2r_j R), \frac{1}{\mathcal{L}^d(B_{2r_j R})}]$. Then applying (VA1) for φ and using that $\omega(2r_j R) \leq 1$, we deduce (ii).

We turn to the proof of (iii). Let $j_0 \in \mathbb{N}$ be such that (3.48) holds. We show that φ_j satisfies (VA1) in $B_{2R}(0)$ for $j \geq j_0$. Let j as above be fixed and let $\tau \in (0, 1)$. We have to prove that

$$\sup_{x \in B_{2\tau R}(0)} \varphi_j(x, t) \leq (1 + \omega(2\tau R)) \inf_{x \in B_{2\tau R}(0)} \varphi_j(x, t), \quad \forall t > 0 \text{ s. t. } \inf_{x \in B_{2\tau R}(0)} \varphi_j(x, t) \in \left[\omega(2\tau R), \frac{1}{\mathcal{L}^d(B_{2\tau R})} \right]. \quad (3.49)$$

Note that

$$\inf_{x \in B_{2\tau R}(0)} \varphi_j(x, t) \in \left[\omega(2\tau R), \frac{1}{\mathcal{L}^d(B_{2\tau R})} \right] \iff \varphi_{B_{2\tau R r_j}(0)}^-(\sigma_j t) \in \left[\varphi(0, \sigma_j)\omega(2\tau R), \frac{\varphi(0, \sigma_j)}{\mathcal{L}^d(B_{2\tau R})} \right]. \quad (3.50)$$

Now, from (3.48),

$$\left[\varphi(0, \sigma_j)\omega(2\tau R), \frac{\varphi(0, \sigma_j)}{\mathcal{L}^d(B_{2\tau R})} \right] \subseteq \left[\omega(2\tau R r_j), \frac{\varphi(0, \sigma_j)r_j^d}{\mathcal{L}^d(B_{2\tau R r_j})} \right] \subseteq \left[\omega(2\tau R r_j), \frac{1}{\mathcal{L}^d(B_{2\tau R r_j})} \right]. \quad (3.51)$$

Then, by (VA1) for φ , $r_j < 1$ and the fact that ω is increasing, we get

$$\varphi_{B_{2\tau R r_j}(0)}^+(\sigma_j t) \leq (1 + \omega(2\tau R))\varphi_{B_{2\tau R r_j}(0)}^-(\sigma_j t), \quad \forall t > 0 \text{ s. t. } \inf_{x \in B_{2\tau R}(0)} \varphi_j(x, t) \in \left[\omega(2\tau R), \frac{1}{\mathcal{L}^d(B_{2\tau R})} \right], \quad (3.52)$$

whence (3.49) follows up to dividing both the sides by $\varphi(0, \sigma_j)$. The proof of assertion (3.47) is postponed to the Appendix, see Lemma A.2. \square

Let $(\varphi_j)_{j \in \mathbb{N}}$ be the sequence defined in (3.44) and, correspondingly, consider the scaled functional

$$\hat{\mathcal{F}}_j(v, \Omega) := \int_{\Omega} \varphi_j(x, |\nabla v|) dx + \frac{\lambda}{\varphi(0, \sigma_j)} \int_{\Omega} \chi_{\{v > 0\}}(x) dx. \quad (3.53)$$

With given u , we also consider for every j the blow-up function

$$v_j(x) := \frac{u(r_j x)}{\sigma_j r_j}, \quad x \in B_{2R}(0). \quad (3.54)$$

We then have the following result about the asymptotic behavior of a blow-up sequence defined by scaling an almost minimizer of \mathcal{F} .

Proposition 3.11. *Let $R, (r_j)_{j \in \mathbb{N}}, (\sigma_j)_{j \in \mathbb{N}}, (\varphi_j)_{j \in \mathbb{N}}$ and φ_{∞} be as in Lemma 3.10. Let u be a bounded almost minimizer of \mathcal{F} in $B_2(0)$ with constant $\kappa \leq \kappa_0$ and exponent β . Then, for every j , the function v_j defined in (3.54) is an almost minimizer of the scaled functional $\hat{\mathcal{F}}_j$ (3.53) in $B_{2R}(0)$, with constant $\hat{\kappa} := \kappa r_j^{\beta}$ and the same exponent β . Moreover, if $\|v_j\|_{L^{\infty}(B_{2R}(0))} \leq M$, there exists $v_{\infty} \in W^{1,1}(B_R(0))$ such that, up to a subsequence, $v_j \rightharpoonup v_{\infty}$ weakly in $W^{1,p}(B_R(0))$, and uniformly in $B_R(0)$, and v_{∞} is φ_{∞} -harmonic in $B_R(0)$.*

Proof. Let $B_{\rho}(x_0)$ be a ball such that $\overline{B_{\rho}(x_0)} \subset B_{\frac{1}{r_j}}(0)$, and $w \in W^{1,p}(B_{\rho}(x_0))$ such that $w = v_j$ on $\partial B_{\rho}(x_0)$. Setting $y_0 := r_j x_0$, we then have

$$u(y) = \sigma_j r_j w\left(\frac{y}{r_j}\right) =: \tilde{w}_j(y), \quad \text{on } \partial B_{r_j \rho}(y_0)$$

and, by the almost minimality of u , we get

$$\int_{B_{r_j \rho}(y_0)} \varphi(y, |\nabla u(y)|) + \lambda \chi_{\{u > 0\}}(y) dy \leq (1 + \kappa(r_j \rho)^{\beta}) \int_{B_{r_j \rho}(y_0)} \varphi(y, |\nabla \tilde{w}_j(y)|) + \lambda \chi_{\{\tilde{w}_j > 0\}}(y) dy. \quad (3.55)$$

Now, with the change of variables $x = \frac{y}{r_j}$, we have

$$\begin{aligned} \int_{B_{r_j \rho}(y_0)} \varphi(y, |\nabla \tilde{w}_j(y)|) + \lambda \chi_{\{\tilde{w}_j > 0\}}(y) dy &= r_j^d \int_{B_{\rho}(x_0)} \varphi(r_j x, |\nabla \tilde{w}_j(r_j x)|) + \lambda \chi_{\{\tilde{w}_j > 0\}}(r_j x) dx \\ &= r_j^d \int_{B_{\rho}(x_0)} \varphi(r_j x, \sigma_j |\nabla w(x)|) + \lambda \chi_{\{w > 0\}}(x) dx \end{aligned} \quad (3.56)$$

and, in a similar way,

$$\int_{B_{r_j \rho}(y_0)} \varphi(y, |\nabla u(y)|) + \lambda \chi_{\{u > 0\}}(y) dy = r_j^d \int_{B_{\rho}(x_0)} \varphi(r_j x, \sigma_j |\nabla v_j(x)|) + \lambda \chi_{\{v_j > 0\}}(x) dx. \quad (3.57)$$

Plugging (3.57) and (3.56) into (3.55), and multiplying both the sides of the inequality by $\frac{1}{\varphi(0, \sigma_j)}$, and recalling the definition of φ_j , we then infer that v_j is an almost minimizer of the functional $\hat{\mathcal{F}}_j$ defined in (3.53).

Now, we notice that applying Lemma 3.2 to φ_j we obtain for v_j the Caccioppoli-type estimate

$$\int_{B_{\rho}(y)} \varphi_j(x, |\nabla v_j|) dx \leq c \left(\int_{B_{2\rho}(y)} \varphi_j \left(x, \frac{|v_j - (v_j)_{y, 2\rho}|}{2\rho} \right) dx + \frac{\lambda}{\varphi(0, \sigma_j)} \rho^d \right) \quad (3.58)$$

for any $B_{2\rho}(y) \Subset B_{2R}(0)$, where the constant c only depends on d, p, q, κ, β , and is a uniform constant with respect to j .

Recall that $\|v_j\|_{L^\infty(B_{2R}(0))} \leq M$. By (3.46), φ_j satisfy $(\text{inc})_p$ and $(\text{dec})_q$, with constants independent of j . They also satisfy (A0) and (VA1) with L and ω independent of j , by Lemma 3.10. Then, the radius r_0 of Remark 3.4(ii) can be chosen independently of j . It follows that, applying Proposition 3.7 to each v_j , each of them is locally α -Hölder continuous on $B_{2R}(0)$, and the $C^{0,\alpha}$ -estimate (3.18) on $B_R(0)$, holds with a uniform bound not depending on j .

Since φ_j^- is $(\text{inc})_p$, combining the bounds $\|v_j\|_{L^\infty(B_{2R}(0))} \leq M$, (3.58) and (3.46) we obtain

$$\sup_{j \geq j_0} \int_{B_\rho(0)} |\nabla v_j|^p dx \leq \sup_{j \geq j_0} \int_{B_\rho(0)} \left(\frac{\varphi_j^-(|\nabla v_j|)}{\varphi_j^-(1)} + 1 \right) dx \leq C, \quad (3.59)$$

for a constant C depending on $d, p, q, L, \kappa_0, M, R, \lambda$. Hence the above inequality also holds for $\rho = R$, whence we infer the existence of a function $v_\infty \in W^{1,p}(B_R(0))$ such that, up to a subsequence,

$$v_j \rightharpoonup v_\infty \quad \text{weakly in } W^{1,p}(B_R(0)). \quad (3.60)$$

By (3.18) on $B_R(0)$, 3.60 also gives

$$v_j \rightarrow v_\infty \quad \text{uniformly in } B_R(0)$$

since the sequence (v_j) is equibounded by M on $B_R(0)$.

So we are left to prove that v_∞ is φ_∞ -harmonic in $B_R(0)$. We first notice that using (3.58) for $\frac{R}{2} < \rho < R$, exploiting the uniform bound $\|v_j\|_{L^\infty(B_{2R}(0))} \leq M$ and letting $\rho \rightarrow R$, we also get that the sequence of positive measures $\mu_j := \varphi_j(\cdot, |\nabla v_j|) \mathcal{L}^d$ is equibounded on $B_R(0)$. Thus, we can find a Radon measure μ on $B_R(0)$ such that

$$\mu_j \rightharpoonup^* \mu \quad \text{on } B_R(0)$$

up to a subsequence (not relabeled). Now, let us fix $w \in W^{1,\varphi_\infty}(B_R(0))$ be such that $\{w \neq v_\infty\} \Subset B_R(0)$. Since φ_∞ satisfies $(\text{dec})_q$, we can find a sequence $(w^\varepsilon)_{\varepsilon>0} \subset W^{1,\varphi_\infty}(B_R(0))$ of regularizations of w , strongly converging to w in $W^{1,\varphi_\infty}(B_R(0))$ as $\varepsilon \rightarrow 0$ (see, e.g., [23, Lemma 6.4.5]).

Let $\rho < \rho' \in (0, R)$, with $\mu(\partial B_{\rho'}) = \mu(\partial B_\rho) = 0$ and $\{w \neq v_\infty\} \Subset B_\rho$. Let $\eta \in C_c^\infty(B_{\rho'})$ be such that $\eta = 1$ on B_ρ , $0 \leq \eta \leq 1$, $|\nabla \eta| \leq \frac{2}{\rho' - \rho}$, and define $\zeta_j = \eta w^\varepsilon + (1 - \eta)v_j$. Since $\{\zeta_j \neq v_j\} \Subset B_{\rho'}$, using the almost minimality of v_j , straightforward computations lead to

$$\begin{aligned} & \int_{B_{\rho'}} \varphi_j(x, |\nabla v_j|) dx \leq (1 + \kappa(r_j \rho')^\beta) \hat{\mathcal{F}}_j(\zeta_j, B_{\rho'}) \\ & \leq \int_{B_\rho} \varphi_j(x, |\nabla w^\varepsilon|) dx + c(1 + \kappa(r_j \rho')^\beta) \int_{B_{\rho'} \setminus B_\rho} \left(\varphi_j(x, |\nabla v_j|) + \varphi_j(x, |\nabla w^\varepsilon|) + \varphi_j \left(x, \frac{|w^\varepsilon - v_j|}{\rho' - \rho} \right) \right) dx \\ & \quad + \frac{\lambda}{\varphi(0, \sigma_j)} \mathcal{L}^d(B_\rho) + \kappa(r_j \rho')^\beta \hat{\mathcal{F}}_j(w^\varepsilon, B_\rho) + (1 + \kappa(r_j \rho')^\beta) \frac{\lambda}{\varphi(0, \sigma_j)} \mathcal{L}^d(B_{\rho'} \setminus B_\rho) \end{aligned} \quad (3.61)$$

for a suitable constant $c \geq 1$ depending only on L and p, q . First, we note that

$$\limsup_{j \rightarrow +\infty} \int_{B_{\rho'} \setminus B_\rho} \varphi_j(x, |\nabla v_j|) dx \leq \mu(B_{\rho'} \setminus B_\rho) \quad \text{and} \quad \hat{\mathcal{F}}_j(w^\varepsilon, B_\rho) \leq C \mathcal{L}^d(B_\rho)$$

for j sufficiently large, and

$$\frac{\lambda}{\varphi(0, \sigma_j)} \mathcal{L}^d(B_{\rho'}) + \kappa(r_j \rho')^\beta \hat{\mathcal{F}}_j(w^\varepsilon, B_\rho) + (1 + \kappa(r_j \rho')^\beta) \frac{\lambda}{\varphi(0, \sigma_j)} \mathcal{L}^d(B_{\rho'} \setminus B_\rho) \rightarrow 0 \quad (3.62)$$

as $j \rightarrow +\infty$, for fixed ρ, ρ', ε .

Now we deal with the convergence of the integral terms above. Using the uniform convergence (3.47) we have that

$$\lim_{j \rightarrow +\infty} \int_{B_{\rho'} \setminus B_\rho} \varphi_j(x, |\nabla w^\varepsilon|) dx = \int_{B_{\rho'} \setminus B_\rho} \varphi_\infty(|\nabla w^\varepsilon|) dx,$$

since $|\nabla w^\varepsilon|$ is bounded. Likewise, we have

$$\lim_{j \rightarrow +\infty} \int_{B_{\rho'} \setminus B_\rho} \varphi_j \left(x, \frac{|w^\varepsilon - v_j|}{\rho' - \rho} \right) dx = \int_{B_{\rho'} \setminus B_\rho} \varphi_\infty \left(\frac{|w^\varepsilon - v_\infty|}{\rho' - \rho} \right) dx.$$

Therefore, passing to the liminf as $j \rightarrow +\infty$ in (3.61), we have

$$\begin{aligned} & \liminf_{j \rightarrow +\infty} \int_{B_{\rho'}} \varphi_j(x, |\nabla v_j|) dx \\ & \leq \int_{B_\rho} \varphi_\infty(|\nabla w^\varepsilon|) dx + c \left[\int_{B_{\rho'} \setminus B_\rho} \left(\varphi_\infty(|\nabla w^\varepsilon|) + \varphi_\infty \left(\frac{|w^\varepsilon - v_\infty|}{\rho' - \rho} \right) \right) dx \right] + \mu(B_{\rho'} \setminus B_\rho). \end{aligned}$$

Now we let $\varepsilon \rightarrow 0$ and, recalling that $w = v_\infty$ outside B_ρ , we easily obtain

$$\liminf_{j \rightarrow +\infty} \int_{B_\rho} \varphi_j(x, |\nabla v_j|) dx \leq \int_{B_\rho} \varphi_\infty(|\nabla w|) dx + c \int_{B_{\rho'} \setminus B_\rho} \varphi_\infty(|\nabla w|) dx + \mu(B_{\rho'} \setminus B_\rho).$$

Therefore, with the lower semicontinuity result (A.4), letting ρ' tend to ρ we finally get that for every $\rho \in (0, R)$ and any $w \in W^{1, \varphi_\infty}(B_R)$ such that $\{w \neq v_\infty\} \Subset B_\rho$ we have

$$\int_{B_\rho} \varphi_\infty(|\nabla v_\infty|) dx \leq \int_{B_\rho} \varphi_\infty(|\nabla w|) dx,$$

as desired. \square

In order to prove the Lipschitz continuity of an almost minimizer, a tool will be the following Proposition, where we show that a bounded almost minimizer of \mathcal{F} is sublinear in a neighborhood of a free-boundary point.

Proposition 3.12. *Let u be an almost minimizer of \mathcal{F} in $B_1(x_0)$, where $x_0 \in \partial\{u > 0\} \cap \Omega$, such that*

$$\sup_{x \in B_1(x_0)} u(x) \leq M. \quad (3.63)$$

Then there exists a constant $C_0 = C_0(d, p, q, L, \kappa_0, \beta) \geq 1$ such that

$$0 \leq u(x) \leq C_0 M |x - x_0| \quad (3.64)$$

for all $x \in B_r(x_0)$ and any $0 < r < 1$.

Proof. We may assume, without loss of generality, that $x_0 = 0$, and, throughout the proof, we will omit the center in the notation for a ball centered at x_0 . We set

$$S(k, u) := \sup_{x \in B_{r_k}} |u(x)|, \quad r_k := 2^{-k}, \quad k \geq 0, \quad (3.65)$$

and our aim is to prove that there exists a constant $C \geq 1$ such that

$$S(k+1, u) \leq \max \left\{ CM r_{k+1}, \frac{S(k, u)}{2} \right\}, \quad \text{for every } k \geq 0. \quad (3.66)$$

Indeed, once (3.66) has been established, arguing by induction we can prove that

$$S(k, u) \leq CM r_k, \quad \text{for every } k \geq 0. \quad (3.67)$$

From this, given $r \in (0, 1]$ and chosen $k \geq 0$ such that $r_{k+1} < r \leq r_k$, we obtain

$$\|u\|_{L^\infty(B_r)} \leq \|u\|_{L^\infty(B_{r_k})} = S(k, u) \leq CMr_k = 2CMr_{k+1} \leq 2CMr, \quad (3.68)$$

and then (3.64), with $C_0 := 2C \geq 1$.

In order to prove (3.66), we argue by contradiction, and, for every $j \geq 1$, we assume the existence of u_j almost minimizer of \mathcal{F} in B_1 , with constant κ and exponent β , and of an integer k_j such that

$$S(k_j + 1, u_j) > \max \left\{ jMr_{k_j+1}, \frac{S(k_j, u_j)}{2} \right\}, \quad (3.69)$$

Note that $\|u_j\|_{L^\infty(B_1)} \leq M$ implies $k_j \rightarrow +\infty$, since by (3.69) we infer $k_j > \log_2(j) - 1$ for every j . Furthermore, with the uniform bound $\|u_j\|_{L^\infty(B_1)} \leq M$ and the same argument used in Proposition 3.11, we can show that for any $\eta \in (0, 1)$ the u_j are uniformly locally η -Hölder continuous in B_1 . Since $u_j(0) = 0$, we obtain

$$\sup_{x \in B_{r_{k_j+1}}} |u_j(x)| = \sup_{x \in B_{r_{k_j+1}}} |u_j(x) - u_j(0)| \leq C_\eta r_{k_j+1}^\eta < C_\eta r_{k_j}^\eta, \quad (3.70)$$

where C_η is independent of j .

Now, we set

$$\sigma_j := \frac{S(k_j + 1, u_j)}{r_{k_j}}, \quad (3.71)$$

and we consider the scaled function

$$v_j(x) := \frac{u_j(r_{k_j}x)}{\sigma_j r_{k_j}} = \frac{u_j(r_{k_j}x)}{S(k_j + 1, u_j)}, \quad x \in B_{\frac{1}{r_{k_j}}}. \quad (3.72)$$

Note that, by (3.69),

$$\sigma_j \geq j^{\frac{M}{2}} \rightarrow +\infty \quad \text{as } j \rightarrow +\infty. \quad (3.73)$$

Setting

$$\varphi_j(x, t) := \frac{\varphi(r_{k_j}x, \sigma_j t)}{\varphi(0, \sigma_j)}, \quad (3.74)$$

with this choice of σ_j and φ_j we introduce the scaled functional $\hat{\mathcal{F}}_j$ defined as in (3.53). Since, by (3.73), $\sigma_j > 1$ for j large enough, and $\varphi(0, t)$ is $(\text{dec})_q$, we have, in view of (3.70) for $\eta = 1 - \frac{1}{2q}$

$$\varphi(0, \sigma_j)r_{k_j} \leq \varphi(0, 1)\sigma_j^q r_{k_j} = \varphi(0, 1) \left(\frac{S(k_j + 1, u_j)}{r_{k_j}^{1-\frac{1}{2q}}} \right)^q r_{k_j}^{\frac{1}{2}} \leq \varphi(0, 1)C_{1-\frac{1}{2q}}^q r_{k_j}^{\frac{1}{2}}, \quad \text{for } j \text{ large enough,}$$

whence (3.43) follows.

Taking into account (3.69), for $x \in B_1$, we have

$$v_j(x) \leq \frac{S(k_j, u_j)}{S(k_j + 1, u_j)} \leq 2 \frac{S(k_j, u_j)}{S(k_j, u_j)} = 2. \quad (3.75)$$

By Proposition 3.11 v_j is an almost minimizer of $\hat{\mathcal{F}}_j$ in $B_1(0)$ with constant $\kappa r_{k_j}^\beta \leq \kappa_0$ and exponent β , there exist a $C^1([0, +\infty))$ convex function φ_∞ whose derivative φ'_∞ complies with $(\text{inc})_{p-1}$ and $(\text{dec})_{q-1}$, a function $v_\infty \in W^{1,1}(B_1(0))$ such that, up to a subsequence, $v_j \rightarrow v_\infty$ uniformly in $B_1(0)$, and v_∞ is φ_∞ -harmonic in $B_{\frac{1}{2}}(0)$. Since, by (3.75), it holds that $0 \leq v_\infty \leq 2$ in $B_{\frac{1}{2}}(0)$, and $v_\infty(0) = 0$ being $v_j(0) \equiv 0$, from the strong minimum principle we

must have $v_\infty \equiv 0$ in $B_{\frac{1}{2}}(0)$. However, from (3.72) we deduce that $\sup_{x \in B_{\frac{1}{2}}(0)} v_\infty(x) = 1$ and this gives a contradiction. The proof is concluded. \square

We are now in position to prove the main result, Theorem 1.1.

Proof of Theorem 1.1. Let u be an almost minimizer of \mathcal{F} in Ω , with constant $\kappa \leq \kappa_0$ and exponent β . Let $\tilde{\Omega} \Subset \Omega$, define r_0 as in (3.10) and set

$$r_1 := \frac{1}{4} \min \{2r_0, \text{dist}(\tilde{\Omega}, \partial\Omega)\} \quad \text{and} \quad \Omega_{r_1} := \{x \in \Omega : \text{dist}(x, \partial\Omega) \geq r_1\}.$$

We recall that, by virtue of Theorem 3.7, $u \in C^{0,\alpha}(\Omega_{r_1})$ for any fixed $\alpha \in (0, 1)$, and set $M := \|u\|_{L^\infty(\Omega_{r_1})}$. Observe that this value M depends on r_0 , and hence on u and $\tilde{\Omega}$ via the integral

$$\int_{\tilde{\Omega}} \varphi(x, |\nabla u|)^{1+s_0} dx.$$

Now, let $x_0 \in \tilde{\Omega} \cap \{u > 0\}$ be arbitrarily fixed and, in order to estimate $|\nabla u(x_0)|$ we distinguish between two cases, according to $\tau := \text{dist}(x_0, \partial\{u > 0\} \cap \Omega)$.

Let $\tau \leq r_1$ first, and choose $y_0 \in \partial\{u > 0\} \cap \Omega$ such that $|y_0 - x_0| = \tau$. Since $B_{2\tau}(y_0) \subset \Omega_{r_1}$, we have $|u| \leq M$ in $B_{2\tau}(y_0)$. Then, by virtue of Proposition 3.12, for every $x \in B_\tau(x_0) \subset B_{2\tau}(y_0)$ we have

$$u(x) \leq CM|x - y_0| \leq 2CM\tau. \quad (3.76)$$

Now, let us consider the scaled function $u_\tau(x) := \frac{u(x_0 + \tau x)}{\tau}$, $x \in B_1(0)$. Since u is an almost minimizer of \mathcal{F} in $B_\tau(x_0)$ with constant κ and exponent β , a simple computation shows that u_τ is an almost minimizer in $B_1(0)$, with constant $\kappa\tau^\beta$ and exponent β , of the functional \mathcal{F}_τ defined as

$$\mathcal{F}_\tau(w, \Omega) := \int_{\Omega} \varphi_\tau(x, |\nabla w|) dx + \lambda \int_{\Omega} \chi_{\{w > 0\}}(x) dx,$$

where $\varphi_\tau(x, t) := \varphi(x_0 + \tau x, t)$. It is easy to check that $\varphi_\tau \in \Phi_c(B_1(0))$, $\varphi_\tau(x, \cdot) \in C^1([0, \infty))$ and that $(\varphi_\tau)_t$ complies with (A0), (inc) $_{p-1}$ and (dec) $_{q-1}$. We only have to remark that also (VA1) holds. For this, let $\rho \in (0, 1)$, $B_\rho(y) \subset B_1(0)$ and recall that φ satisfies (VA1) on $B_{\tau\rho}(x_0 + \tau y)$, so that

$$\varphi_{B_{\tau\rho}(x_0 + \tau y)}^+(t) \leq (1 + \omega(\tau\rho))\varphi_{B_{\tau\rho}(x_0 + \tau y)}^-(t), \quad \text{if } \varphi_{B_{\tau\rho}(x_0 + \tau y)}^-(t) \in [\omega(\tau\rho), 1/\mathcal{L}^d(B_{\tau\rho})].$$

Now, since $\varphi_\tau^\pm(t) = \varphi_{B_{\tau\rho}(x_0 + \tau y)}^\pm(t)$, where $\varphi_\tau^\pm(t)$ are computed on $B_\rho(y)$, and $\tau \leq 1$, from the previous estimate we infer

$$\varphi_\tau^+(t) \leq (1 + \rho^\beta)\varphi_\tau^-(t), \quad \text{if } \varphi_\tau^-(t) \in [\rho^\beta, 1/\mathcal{L}^d(B_\rho)].$$

Moreover, by (3.76), $|u_\tau| \leq 2CM$ in $B_1(0)$. Therefore, by Lemma 3.9, we deduce that

$$|\nabla u(x_0)| = |\nabla u_\tau(0)| \leq \tilde{C},$$

where the constant \tilde{C} depends on $p, q, d, L, \kappa_0, \beta, \Lambda, u, \tilde{\Omega}$.

If, instead, $\tau \geq r_1$, we can perform an analogous argument as before with $u_{r_1}(x) := \frac{u(x_0 + r_1 x)}{r_1}$, $x \in B_1(0)$ in place of u_τ , which satisfies $\|u_{r_1}\|_{L^\infty(B_1(0))} \leq \frac{M}{r_1}$, and \mathcal{F}_{r_1} in place of \mathcal{F}_τ . This concludes the proof. \square

APPENDIX A.

We recall a technical result (see [28, Lemma 2.19]) about the φ -recession function associated to a sequence of convex functions φ_j , capturing the behaviour at infinity of φ_j .

Lemma A.1. *Let $(\varphi_j)_{j \in \mathbb{N}}$, $\varphi_j : [0, +\infty) \rightarrow [0, +\infty)$, be a sequence of C^1 convex functions satisfying $\varphi_j(0) = 0$ and assume that φ'_j satisfies $(\text{inc})_{p-1}$ and $(\text{dec})_{q-1}$, where $1 < p \leq q < +\infty$. Let $(\beta_j) \subset (0, \infty)$ be a sequence such that $\lim_j \beta_j = +\infty$. Then, setting*

$$\bar{\varphi}_j(t) := \frac{\varphi_j(t\beta_j)}{\varphi_j(\beta_j)}, \quad t \in [0, +\infty), \quad j \in \mathbb{N},$$

there exists a subsequence (β_{j_k}) such that $\bar{\varphi}_{j_k}$ converge to a C^1 convex function φ_∞ uniformly on compact subsets of $[0, +\infty)$. Moreover, φ'_∞ satisfies $(\text{inc})_{p-1}$ and $(\text{dec})_{q-1}$.

We are now in position to prove Lemma 3.10(iv).

Lemma A.2. *Let φ_j be the sequence defined as in (3.44). Then (3.47) holds. Moreover, φ_∞ is a $C^1([0, +\infty))$ function such that φ'_∞ satisfies $(\text{inc})_{p-1}$ and $(\text{dec})_{q-1}$.*

Proof. We can apply Lemma A.1 to the constant sequence $\varphi_j(t) \equiv \varphi(0, t)$, with $\beta_j := \sigma_j$. Setting for brevity $\psi_j(t) := \varphi(0, t\sigma_j)$, and defining

$$\bar{\varphi}_j(t) := \frac{\psi_j(t)}{\psi_j(1)} = \frac{\varphi(0, t\sigma_j)}{\varphi(0, \sigma_j)},$$

we then obtain that, up to a subsequence, $\bar{\varphi}_j$ converges to a C^1 convex function φ_∞ uniformly on compact subsets of $[0, +\infty)$, with φ'_∞ satisfying $(\text{inc})_{p-1}$ and $(\text{dec})_{q-1}$.

Now, in order to prove (3.47), it will suffice to show that, with fixed $\tau > 0$, there exists $j_1 \geq 1$ such that

$$|\varphi_j(y, t) - \bar{\varphi}_j(t)| \leq \eta_{j,\tau}, \quad \text{for every } (y, t) \in B_R \times [0, \tau], \text{ for } j \geq j_1, \quad (\text{A.1})$$

for some $\eta_{j,\tau}$ which is infinitesimal as $j \rightarrow +\infty$.

The proof of the estimate (A.1) follows by minor adaptations of the argument for the Step 3 of the proof of [28, Lemma 3.1]. For the reader's convenience, we provide the details.

First, we observe that

$$|\varphi_j(y, t) - \bar{\varphi}_j(t)| = \frac{1}{\psi_j(1)} |\varphi(r_j y, t\sigma_j) - \psi_j(t)| \leq \omega(r_j) \frac{\varphi_{r_j}^-(t\sigma_j)}{\psi_j(1)},$$

if $\varphi_{r_j}^-(t\sigma_j) \in [\omega(r_j), \frac{1}{\mathcal{L}^d(B_{r_j})}]$, thanks to (VA1). Recalling the definitions of $\varphi_{r_j}^-$ and ψ_j , together with (2.4), the last term can be estimated as

$$\omega(r_j) \frac{\varphi_{r_j}^-(t\sigma_j)}{\psi_j(1)} \leq \omega(r_j) \bar{\varphi}_j(t) \leq \max\{\tau^p, \tau^q\} \omega(r_j).$$

On the other hand, if $\varphi_{r_j}^-(t\sigma_j) < \omega(r_j)$, from (2.4) and (2.6), we deduce

$$\min\{(t\sigma_j)^p, (t\sigma_j)^q\} \leq Lq\omega(r_j) \leq 1, \quad \text{for } j \text{ large enough,}$$

entailing

$$t\sigma_j \leq (Lq\omega(r_j))^{\frac{1}{q}}.$$

Then

$$|\varphi_j(y, t) - \bar{\varphi}_j(t)| \leq \frac{2}{\psi_j(1)} (Lq\omega(r_j))^{\frac{p}{q}} \frac{L}{p} \lesssim \frac{1}{\psi_j(1)}.$$

Finally, case $\varphi_{r_j}^-(t\sigma_j) > \frac{1}{\mathcal{L}^d(B_{r_j})}$ cannot occur for j large enough since, taking into account (2.4) and (3.43), it would lead to

$$\frac{1}{\mathcal{L}^d(B_{r_j})} < \psi_j(t) \leq \max\{\tau^p, \tau^q\} \psi_j(1),$$

that is

$$\frac{1}{r_j^{d-1}} < \max\{\tau^p, \tau^q\} \gamma_d \varphi(0, \sigma_j) r_j,$$

which clearly would give a contradiction for j large.

Therefore, (A.1) is proven with $\eta_{j,\tau} := \max\left\{\max\{\tau^p, \tau^q\} \omega(r_j), \frac{2}{\psi_j(1)} (Lq\omega(r_j))^{\frac{p}{q}} \frac{L}{p}\right\}$ for j large enough. \square

In order to prove the lower semicontinuity result of Lemma A.5, we also need the following definitions and results about the maximal operator in Orlicz spaces (see [23, Section 4.3]).

Definition A.1. Given an open set $\Omega \subseteq \mathbb{R}^d$ and $f \in L_{\text{loc}}^1(\Omega)$, the (centered) Hardy-Littlewood maximal operator is $Mf : \Omega \rightarrow [0, \infty]$ defined as

$$Mf(x) := \sup_{\rho>0} \frac{1}{\mathcal{L}^d(B_\rho(x))} \int_{B_\rho(x) \cap \Omega} |f(y)| \, dy. \quad (\text{A.2})$$

The following result of boundedness for the local maximal operator can be found, e.g., in [28, Corollary 1.9].

Lemma A.3. Let $\varphi \in \Phi_w$ satisfy $(\text{inc})_p$ and $(\text{dec})_q$, with $1 < p \leq q < +\infty$. Then there exists $C = C(\varphi^{-1}(1), d, p, q)$ such that

$$\int_{\Omega} \varphi(Mf) \, dx \leq C \int_{\Omega} \varphi(|f|) \, dx$$

for every $f \in L^\varphi(\Omega)$ satisfying $\int_{\Omega} \varphi(|f|) \, dx \leq 1$.

We conclude the list of auxiliary results with the following Lusin-type approximation result in $W^{1,\varphi}$, which can be inferred from [17, Theorem 3.3]. Indeed, for a fixed ball B , the argument therein can be applied to φ_B^- without requiring $u \in W_0^{1,\varphi}(B)$, as the null extension outside B is not needed taking into account the boundedness of the restricted maximal operator, Lemma A.3.

Theorem A.4. Let $\varphi \in \Phi_w$ satisfy $(\text{inc})_p$ and $(\text{dec})_q$, with $1 < p \leq q < +\infty$, and let $B \subset \mathbb{R}^d$ be a ball. For every $u \in W^{1,\varphi}(B)$ and every $\tau > 0$ there exists a function $u_\tau : B \rightarrow \mathbb{R}$, $u_\tau \in W^{1,\infty}(B)$ satisfying $\text{Lip}(u_\tau) \leq c\tau$ with $c = c(d)$, such that $u_\tau = u$ in $\{M|\nabla u| \leq \tau\}$ and

$$\mathcal{L}^d(\{M|\nabla u| > \tau\}) \leq \frac{1}{\varphi_B^-(\tau)} \int_{\{M|\nabla u| > \tau\}} \varphi_B^-(M|\nabla u) \, dx,$$

where M is introduced in Definition A.1.

Lemma A.5. Let φ_j be the sequence defined as in (3.44), and φ_∞ be the function in (3.47). Let $(v_j)_{j \in \mathbb{N}} \subset W^{1,1}(B_R(0))$ be such that

$$\sup_{j \in \mathbb{N}} \int_{B_R(0)} \varphi_j(y, |\nabla v_j|) \, dy \leq C, \quad (\text{A.3})$$

and $v_j \rightarrow v_0$ a.e. in $B_R(0)$. Then

$$\int_{B_R} \varphi_\infty(|\nabla v_0|) \, dy \leq \liminf_{j \rightarrow +\infty} \int_{B_R} \varphi_j(y, |\nabla v_j|) \, dy. \quad (\text{A.4})$$

Proof. Thanks to the bound in (A.3), we apply Lemma A.3 to φ_j^- , which is a weak Φ -function satisfying $(\text{inc})_p$ and $(\text{dec})_q$, obtaining

$$\int_{B_R(0)} \varphi_j^-(M|\nabla v_j|) \, dy \leq C, \quad (\text{A.5})$$

having taken into account that, thanks to (3.46), $(\varphi_j^-)^{-1}(1) \simeq 1$, and the hidden constants do not depend on j . By Chacon's Biting Lemma (see, e.g., [2, Lemma 5.32]) there exist a sequence of Borel subsets A_h of $B_R(0)$ such that $\mathcal{L}^d(A_h) \rightarrow 0$ as $h \rightarrow +\infty$, and a (not relabelled) subsequence such that $(\varphi_j^-(M|\nabla v_j|)\chi_{B_R(0)\setminus A_h})_j$ is equintegrable for every $h \geq 1$.

Let $\tau > 1$. Then, applying Theorem A.4 to v_j , we find $v_j^\tau : B_R(0) \rightarrow \mathbb{R}$ such that

$$\text{Lip}(v_j^\tau) \leq c\tau \quad \text{and} \quad v_j^\tau = v_j \text{ in } B_R(0) \setminus E_j^\tau, \quad (\text{A.6})$$

where $E_j^\tau := \{M|\nabla v_j| > \tau\}$ and, by Chebychev's inequality,

$$\mathcal{L}^d(E_j^\tau \setminus A) \leq \frac{1}{\varphi_j^-(\tau)} \int_{E_j^\tau \setminus A} \varphi_j^-(M|\nabla v_j|) \, dy, \quad (\text{A.7})$$

for any Borel set $A \subset B_R(0)$. Moreover, from (A.7) with $A = \emptyset$, (A.5), $(\text{inc})_p$ for φ_j^- , and the fact that by (3.46), $\varphi_j^-(1) \gtrsim 1$ for j large enough, we deduce

$$\mathcal{L}^d(E_j^\tau) \leq \frac{1}{\varphi_j^-(1)\tau^p} \int_{E_j^\tau} \varphi_j^-(M|\nabla v_j|) \, dy \leq \frac{C}{\tau^p}, \quad (\text{A.8})$$

for j large enough.

We compute

$$\begin{aligned} \int_{B_R(0)} \varphi_j(y, |\nabla v_j|) \, dy &\geq \int_{B_R(0) \setminus (A_h \cup E_j^\tau)} \varphi_j(y, |\nabla v_j^\tau|) \, dy = \int_{B_R(0) \setminus A_h} \varphi_j(y, |\nabla v_j^\tau|) \, dy \\ &\quad - \int_{E_j^\tau \setminus A_h} \varphi_j(y, |\nabla v_j^\tau|) \, dy = \int_{B_R(0) \setminus A_h} [\varphi_j(y, |\nabla v_j^\tau|) - \varphi_\infty(|\nabla v_j^\tau|)] \, dy \\ &\quad + \int_{B_R(0) \setminus A_h} \varphi_\infty(|\nabla v_j^\tau|) \, dy - \int_{E_j^\tau \setminus A_h} \varphi_j(y, |\nabla v_j^\tau|) \, dy. \end{aligned}$$

Since the convergence (3.47) implies

$$\lim_{j \rightarrow +\infty} \int_{B_R(0) \setminus A_h} [\varphi_j(y, |\nabla v_j^\tau|) - \varphi_\infty(|\nabla v_j^\tau|)] \, dy = 0,$$

passing to the liminf in the previous inequality we obtain

$$\liminf_{j \rightarrow +\infty} \int_{B_R(0)} \varphi_j(y, |\nabla v_j|) \, dy \geq \liminf_{j \rightarrow +\infty} \int_{B_R(0) \setminus A_h} \varphi_\infty(|\nabla v_j^\tau|) \, dy - \limsup_{j \rightarrow +\infty} \int_{E_j^\tau \setminus A_h} \varphi_j(y, |\nabla v_j^\tau|) \, dy. \quad (\text{A.9})$$

We are first dealing with the second term. We have

$$\int_{E_j^\tau \setminus A_h} \varphi_j(y, |\nabla v_j^\tau|) \, dy \leq \int_{E_j^\tau \setminus A_h} \varphi_j^+(|\nabla v_j^\tau|) \, dy.$$

In $E_j^\tau \setminus A_h$ we distinguish between the points of $B_R(0)$ where $\varphi_{r_j}^-(|\nabla v_j^\tau| \sigma_j) \in [\omega(r_j), 1/\mathcal{L}^d(B_{r_j})]$, denoting the corresponding set by $S_{j,\tau}^1$, and the points where that condition does not hold. We then define

$$S_{j,\tau}^2 := \{\varphi_{r_j}^-(|\nabla v_j^\tau| \sigma_j) < \omega(r_j)\} \cap B_R(0) \quad \text{and} \quad S_{j,\tau}^3 := \{\varphi_{r_j}^-(|\nabla v_j^\tau| \sigma_j) > 1/\mathcal{L}^d(B_{r_j})\} \cap B_R(0).$$

The set $S_{j,\tau}^3$ has to be empty for j sufficiently large, as otherwise, using (2.4) for any fixed point therein, the resulting inequality $\varphi(0, \sigma_j) \tau^q > \frac{1}{\gamma_d r_j^d}$ would imply $\frac{1}{\varphi(0, \sigma_j) r_j}$ uniformly bounded with respect to j .

In $S_{j,\tau}^2$, thanks to (2.4) and (2.6), $\min\{(|\nabla v_j^\tau| \sigma_j)^p, (|\nabla v_j^\tau| \sigma_j)^q\} \leq Lq\omega(r_j) \leq 1$ for j large enough, then

$$\begin{aligned} & \int_{(E_j^\tau \setminus A_h) \cap S_{j,\tau}^2} \varphi_j^+(|\nabla v_j^\tau|) \, dy \\ & \leq \frac{1}{\varphi(0, \sigma_j)} \int_{(E_j^\tau \setminus A_h) \cap S_{j,\tau}^2} \max\{(|\nabla v_j^\tau| \sigma_j)^p, (|\nabla v_j^\tau| \sigma_j)^q\} \varphi_{r_j}^+(1) \, dy \\ & \leq \gamma_d (Lq\omega(r_j))^{\frac{p}{q}} \frac{L}{p} \frac{1}{\varphi(0, \sigma_j)} \xrightarrow{j \rightarrow +\infty} 0. \end{aligned}$$

In $S_{j,\tau}^1$ condition (VA1) holds, then

$$\begin{aligned} \int_{(E_j^\tau \setminus A_h) \cap S_{j,\tau}^1} \varphi_j^+(|\nabla v_j^\tau|) \, dy & \leq 2 \int_{(E_j^\tau \setminus A_h) \cap S_{j,\tau}^1} \varphi_j^-(|\nabla v_j^\tau|) \, dy \leq c \varphi_j^-(\tau) \mathcal{L}^d(E_j^\tau \setminus A_h) \\ & \leq c \int_{\{M|\nabla v_j| > \tau\} \setminus A_h} \varphi_j^-(M|\nabla v_j|) \, dy, \end{aligned}$$

where we used (A.6), (A.7), (3.46). From the equiintegrability of the functions $\varphi_j^-(M|\nabla \hat{v}_j|)$ in $B_R(0) \setminus A_h$ and from (A.8), given $\eta > 0$, we fix $\tau = \lambda(\eta)$ sufficiently large in order that

$$c \int_{\{M|\nabla v_j| > \tau\} \setminus A_h} \varphi_j^-(M|\nabla v_j|) \, dy < \eta. \quad (\text{A.10})$$

Therefore we can state that

$$\limsup_{j \rightarrow +\infty} \int_{E_j^\tau \setminus A_h} \varphi_j(y, |\nabla v_j^\tau|) \, dy < \eta.$$

Concerning the first term in (A.9), for the above fixed $\tau = \tau(\eta)$, the sequence $(v_j^\tau)_j$ is equibounded in $W^{1,\infty}(B_R(0))$, therefore, up to a subsequence, it converges to a function v^τ weakly* in $W^{1,\infty}(B_R(0))$ and in measure. Moreover, by the lower semicontinuity under convergence in measure of the map

$$w \mapsto \mathcal{L}^d(\{x \in B_R(0) \setminus A_h : w(x) \neq 0\}),$$

then

$$\begin{aligned}
 \tau^p \mathcal{L}^d(\{x \in B_1 \setminus A_h : v^\tau \neq v_0\}) &\leq \liminf_{j \rightarrow +\infty} \tau^p \mathcal{L}^d(\{x \in B_1 \setminus A_h : v_j^\tau \neq v_j\}) \\
 &\leq \liminf_{j \rightarrow +\infty} \tau^p \mathcal{L}^d(E_j^\tau \setminus A_h) \\
 &\leq \liminf_{j \rightarrow +\infty} \frac{\tau^p}{\varphi_j^-(\tau)} \int_{\{|M|\nabla v_j|>\tau\} \setminus A_h} \varphi_j^-(M|\nabla v_j|) \, dy \\
 &\leq \liminf_{j \rightarrow +\infty} \frac{1}{\varphi_j^-(1)} \int_{\{|M|\nabla v_j|>\tau\} \setminus A_h} \varphi_j^-(M|\nabla v_j|) \, dy \\
 &\leq c \liminf_{j \rightarrow +\infty} \int_{\{|M|\nabla v_j|>\tau\} \setminus A_h} \varphi_j^-(M|\nabla v_j|) \, dy \leq c \eta,
 \end{aligned} \tag{A.11}$$

using that φ_j^- satisfies $(\text{inc})_p$, the bound from above of $\varphi_j^-(1)$, and (A.10). All things considered, setting $C_s := \{x \in B_R(0) : |\nabla v_0(x)| \leq s\}$, from (A.9) we derive

$$\begin{aligned}
 \liminf_{j \rightarrow +\infty} \int_{B_R(0)} \varphi_j(y, |\nabla v_j|) \, dy &\geq \int_{B_R(0) \setminus A_h} \varphi_\infty(|\nabla v^\tau|) \, dy - \eta \\
 &\geq \int_{(B_R(0) \setminus A_h) \cap \{v^\tau = v_0\} \cap C_s} \varphi_\infty(|\nabla v_0|) \, dy - \eta \\
 &= \int_{(B_R(0) \setminus A_h) \cap C_s} \varphi_\infty(|\nabla v_0|) \, dy - \int_{(B_R(0) \setminus A_h) \cap \{v^\tau \neq v_0\} \cap C_s} \varphi_\infty(|\nabla v_0|) \, dy - \eta \\
 &\geq \int_{(B_R(0) \setminus A_h) \cap C_s} \varphi_\infty(|\nabla v_0|) \, dy - \varphi_\infty(s) \mathcal{L}^d(\{x \in B_R(0) \setminus A_h : v^\tau \neq v_0\}) - \eta \\
 &\geq \int_{(B_R(0) \setminus A_h) \cap C_s} \varphi_\infty(|\nabla v_0|) \, dy - \varphi_\infty(s) c \eta - \eta,
 \end{aligned}$$

where we used (A.11) in the last inequality. Thus, letting first η tend to zero, then h and finally s tend to infinity, we proved (A.4). \square

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