MIXED LOCAL AND NONLOCAL EIGENVALUES

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dedicated to Gioconda Moscariello on the occasion of her 70th birthday, with admiration

Abstract. We discuss some basic properties of the eigenfunctions of a class of mixed local and nonlocal operators whose prototype is

$$-\Delta_p u - (-\Delta_p)^s u = \lambda |u|^{p-2} u,$$

for any summability exponent $p \in (1, \infty)$ and any differentiability order $s \in (0, 1)$.

1. Introduction

In these notes we investigate some basic properties of the eigenfunctions related to a class of mixed local and nonlocal equations. Let $\Omega \subset \mathbb{R}^N$ be a bounded domain. For any $p \in (1, \infty)$, $s \in (0, 1)$, and any $\lambda > 0$, we consider the following equation,

$$-\operatorname{div} \nabla_z \mathcal{H}(x, \nabla u) - \mathcal{L}_K u = \lambda |u|^{p-2} u \quad \text{in } \Omega. \tag{1.1}$$

Above, the integro-differential operator $\mathcal{L}_K$ of order (s, p) given by

$$\mathcal{L}_K u(x) := \mathrm{P.\,V.} \int_{\mathbb{R}^N} K(x, y) |u(x) - u(y)|^{p-2} (u(x) - u(y)) \, \mathrm{d}y, \quad x \in \Omega, \tag{1.2}$$

is driven by its measurable kernel $K : \mathbb{R}^N \times \mathbb{R}^N \to [0, \infty)$ satisfying the following assumptions

$$\begin{cases} K(x, y) = K(y, x) \text{ for a. e. } x, y \in \mathbb{R}^N, \\ \dfrac{\kappa}{|x - y|^{N+sp} \Lambda} \leq K(x, y) \leq \dfrac{\kappa \Lambda}{|x - y|^{N+sp}} \text{ for a. e. } x, y \in \mathbb{R}^N, \end{cases} \tag{1.3}$$

where $\Lambda > 0$ and $\kappa \in (0, 1]$ are fixed constants.

The Carathéodory function $\mathcal{H} : \Omega \times \mathbb{R}^N \to \mathbb{R}^+$ satisfies the following assumptions

$$\begin{cases} z \mapsto \mathcal{H}(x, z) \text{ is } C^1 \text{ and convex;} \\ \mathcal{H}(x, tz) = |t|^p \mathcal{H}(x, z) \text{ for any } t \in \mathbb{R}, \, (x, z) \in \Omega \times \mathbb{R}^N, \end{cases} \tag{1.4}$$

for a fixed constant $\Lambda > 0$, as above.

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Mixed local and nonlocal equations are a subject of largely growing interest. Basically, the main object under investigation is an elliptic operator that combines two different orders of differentiation, the simplest model case being $-\Delta + (-\Delta)^s$. As expected, the simultaneous presence of a leading local operator and a lower order fractional one does constitute the essence of the matter. Despite its very recent history, the related literature is really too wide to attempt any comprehensive list of references here. We would like to mention only the very interesting results in the recent papers [2, 5, 7, 11–13, 16], since those are focused on the same topic we are dealing with; that is, properties of eigenvalues of some mixed-type operators. See also the references therein.

In view of the assumptions in (1.3)-(1.4), the natural setting is the family $\mathbb{X}_0^{1,p}$ of functions given by

$$\mathbb{X}_0^{1,p}(\Omega) := \Big\{ u \in W^{1,p}(\mathbb{R}^N) : u_{|\Omega} \in W_0^{1,p}(\Omega),\ u = 0 \text{ a.e. in } \mathbb{R}^N \setminus \Omega \Big\};$$

compare also with [4, Section 1]. We immediately notice that for any $u \in \mathbb{X}_0^{1,p}(\Omega)$ – as one can check in [10, Lemma 2.3] – the following inequality does hold,

$$[u]_{W^{s,p}(\mathbb{R}^N)} \lesssim \|\nabla u\|_{L^p(\Omega)}.$$

In the display above we denoted by $[\,\cdot\,]_{W^{s,p}(\mathbb{R}^N)}$ the standard Gagliardo seminorm; see [8] and the references therein for the basics on fractional Sobolev spaces.

We now need to recall the natural definition of weak solutions to (1.1), by also providing the related definition of eigenvalues and eigenfunctions.

Definition 1.1. *For any $p \in (1, \infty)$, for any $s \in (0, 1)$, and for $\lambda > 0$, we say that $u \in \mathbb{X}_0^{1,p}(\Omega)$ is a weak solution to* (1.1) *if it satisfies*

$$\begin{aligned} &\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} K(x,y)|u(x)-u(y)|^{p-2}\Big(u(x)-u(y)\Big)\Big(\phi(x)-\phi(y)\Big)\,\mathrm{d}x\mathrm{d}y \\ &\quad + \frac{1}{p}\int_\Omega \langle \nabla_z \mathcal{H}(x,\nabla u), \nabla\phi(x)\rangle\,\mathrm{d}x = \lambda\int_\Omega |u(x)|^{p-2}u(x)\phi(x)\,\mathrm{d}x, \end{aligned} \tag{1.5}$$

for any $\phi \in \mathbb{X}_0^{1,p}(\Omega)$.

A number $\lambda > 0$ is called mixed eigenvalue *if there exists a non-trivial weak solution $u \in \mathbb{X}_0^{1,p}(\Omega)$ to* (1.1). *In such a case, the function u is called* mixed eigenfunction *associated with λ.*

As in the classical framework when the leading operator is the p-Laplacian one (the fractional p-Laplacian operator, respectively), mixed eigenfunctions are related to the problem of minimizing, among all functions $\phi \in \mathbb{X}_0^{1,p}(\Omega)$, the related (*mixed*) *Rayleigh quotient* $\mathcal{R}_{\rm mix}$,

$$\mathcal{R}_{\rm mix}(\phi) := \frac{\mathcal{E}_K(\phi) + \mathcal{E}_{\mathcal{H}}(\phi)}{\|\phi\|^p_{L^p(\mathbb{R}^N)}}, \tag{1.6}$$

where the energy functionals $\mathcal{E}_K(\cdot)$ and $\mathcal{E}_{\mathcal{H}}(\cdot)$ are defined as follows

$$\mathcal{E}_K(\phi) := \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} K(x,y)|\phi(x)-\phi(y)|^p\,\mathrm{d}x\mathrm{d}y, \tag{1.7}$$

and

$$\mathcal{E}_{\mathcal{H}}(\phi) := \int_\Omega \mathcal{H}(x,\nabla\phi)\,\mathrm{d}x. \tag{1.8}$$

Our main result stated below responds to a natural question and it is in clear accordance with both the classical nonlinear counterpart and the nonlocal counterpart.

Theorem 1.2. *Let $p \in (1,\infty)$, $s \in (0,1)$. Assume that the variational problem*

$$\lambda^s_{1,p}(\Omega) := \min_{\phi \in \mathbb{X}^{1,p}_0(\Omega)} \mathcal{R}_{\rm mix}(\phi) \tag{1.9}$$

admits at least one solution, with $\mathcal{R}_{\rm mix}$ being defined in (1.6)*, and let $u \in \mathbb{X}^{1,p}_0(\Omega)$ be a mixed eigenfunction associated with λ such that $u > 0$ in Ω. Then, under* (1.3) *and* (1.4)*, it holds*

$$\lambda = \lambda^s_{1,p}(\Omega). \tag{1.10}$$

The proof will extend to the mixed framework the one presented in the purely fractional framework in [9] and the one firstly described in the purely local framework in [3], which relies on the convexity of the involved (semi)norms along suitable curves connecting pairs of functions. For pioneering important results in this direction, it is worth mentioning the relevant paper [1]. We notice that, because of the chosen strategy, no regularity assumptions are needed on the weak solutions to (1.1), whose boundedness is proven via a suitable iteration based on a (mixed) energy estimates.

We conclude this section by noticing that in the pure fractional case one can also deduce that the positive fractional eigenfunctions corresponding to $\lambda^s_{1,p}(\Omega)$ are proportional, via the convexity mentioned above. This is done in Theorem 4.2 in [9] by using the very definition of the Gagliardo seminorms. The same strategy cannot be plainly applied in the mixed framework here. For this, we refer the reader to the very interesting paper [11], where it has been pointed out that a related result for sequences can be achieved via the inverse iteration method; see in particular Theorem 2.7 there.

Outline of the paper. The paper is organized as follows. In Section 2 we prove that weak solutions to (1.1) are globally bounded. Section 3 is devoted to the proof of Theorem 1.2.

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2. Caccioppoli and boundedness estimates

This section is devoted to establish some local and global estimates for mixed eigenvalues. In particular, we state and prove a mixed Caccioppoli-type inequality together with a global boundedness for weak solutions to (1.1). For the sake of the reader, notice that the tailored Caccioppoli-type estimate presented in Theorem 2.1 below is not needed in the proof of our main result in Theorem 1.2. However, we believe that it could be valuable for future developments especially in the nonlinear case. For more general Caccioppoli-type inequalities in the mixed local nonlocal scenario, we refer to [4, Sections 4-5].

Before going into the proofs, we need some further notation. Indeed, together with the regularity and homogeneity properties in (1.4), a natural p-growth assumption will be required; that is,

$$\begin{cases} \Lambda^{-1}|z|^p \leq \mathcal{H}(x,z) \leq \Lambda |z|^p \text{ for } (x,z) \in \Omega \times \mathbb{R}^N; \\ |\nabla_z \mathcal{H}(x,z)| \leq \Lambda |z|^{p-1} \text{ for } (x,z) \in \Omega \times \mathbb{R}^N. \end{cases} \tag{2.1}$$

Moreover, in order to lightening a bit the notation, we will denote with

$$\mathcal{A}(u(x),u(y)) := |u(x)-u(y)|^{p-2}\Big(u(x)-u(y)\Big)\,;$$

also, in order to control the growth of a function at infinity, we recall the definition of the nonlocal tail of a function u with respect to a ball $B_r(x_0)$,

$$\mathrm{Tail}(u;x_0,r) := \left(r^{sp} \int_{\mathbb{R}^N \setminus B_r(x_0)} \frac{|u(y)|^{p-1}}{|x_0-y|^{N+sp}}\,\mathrm{d}y \right)^{\frac{1}{p-1}}\,;$$

as firstly introduced in [6].

The first main result in the present section is the following Caccioppoli-type estimate for weak solutions to (1.1).

Theorem 2.1. *Let $p \in (1,\infty)$, $s \in (0,1)$, and let $v \in \mathbb{X}^{1,p}_0(\Omega)$ be a weak solution to (1.1). Then, for any $m>0$ and any $B_\varrho(x_0) \Subset B_r(x_0) \Subset \Omega$, it holds*

$$\begin{aligned} &\int_{\{v>m\}\cap B_\varrho(x_0)} \int_{\{v>m\}\cap B_\varrho(x_0)} K(x,y)|v(x)-v(y)|^p \,\mathrm{d}x\mathrm{d}y \\ &\qquad + \int_{\{v>m\}\cap B_\varrho(x_0)} |\nabla v|^p \,\mathrm{d}x \\ &\leq \frac{r^N \mathrm{Tail}((v-m)_+;x_0,r)^{p-1}}{(r-\varrho)^{N+sp}} \int_{\{v>m\}\cap B_r(x_0)} (v-m)_+ \,\mathrm{d}x \\ &\qquad + \frac{c}{(r-\varrho)^p} \int_{B_r(x_0)\setminus B_\varrho(x_0)} (v-m)_+^p \,\mathrm{d}x + c\lambda \int_{\{v>m\}\cap B_r(x_0)} |v|^p \,\mathrm{d}x\,, \end{aligned} \tag{2.2}$$

where $c>0$ depends only on n,p,s,Ω,Λ and κ.

Proof. Fix a smooth function $\psi \in C^\infty_0(B_\tau(x_0))$, with $\tau := \varrho + 2^{-2}(r-\varrho)$ such that $\psi \equiv 1$ on $B_\varrho(x_0)$ and $|\nabla \psi| \lesssim 1/(r-\varrho)$. For any $m>0$, choose in the weak formulation (1.5) the test function ϕ defined as $\phi \equiv (v-m)_+\psi^p =: v_m \psi^p$.

This yields

$$
\begin{aligned}
&\lambda \int_\Omega |v(x)|^{p-2} v(x) v_m \psi^p(x)\,\mathrm{d}x \\
&\quad = \frac{1}{p}\int_\Omega \langle \nabla_z \mathcal{H}(x,\nabla v), \nabla(v_m \psi^p(x))\rangle \,\mathrm{d}x \\
&\qquad + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \mathcal{A}(v(x),v(y))(v_m\psi^p(x) - v_m\psi^p(y))K(x,y)\,\mathrm{d}x\mathrm{d}y \\
&\quad =: \mathcal{I}_1 + \mathcal{I}_2. \qquad (2.3)
\end{aligned}
$$

We separately estimate the integrals on the right-hand side in (2.3). We begin by estimating the local contribution given by the integral $\mathcal{I}_1$; we get, by the homogeneity of $\mathcal{H}(\cdot)$ and Young's Inequality that

$$
\begin{aligned}
\mathcal{I}_1 \quad &= \quad \frac{1}{p}\int_\Omega \langle \nabla_z \mathcal{H}(x,\nabla v, \nabla v_m\rangle \psi^p \,\mathrm{d}x \\
&\qquad + \int_{B_r(x_0)\setminus B_\varrho(x_0)} \langle \nabla_z \mathcal{H}(x,\nabla v), \nabla\psi\rangle v_m \psi^{p-1}\,\mathrm{d}x \\
&\overset{(1.4)_2,(2.1)_2}{\gtrsim} \quad \int_\Omega \mathcal{H}(x,\nabla v_m)\psi^p\,\mathrm{d}x \\
&\qquad - \int_{B_r(x_0)\setminus B_\varrho(x_0)} \left(\frac{1}{2}|\nabla v_m|^p \psi^p - \frac{c}{(r-\varrho)^p} v_m^p\right)\mathrm{d}x \\
&\overset{(2.1)_1}{\gtrsim} \quad \frac{1}{2}\int_{\{v>m\}\cap B_r(x_0)} |\nabla v|^p\,\mathrm{d}x \\
&\qquad - \frac{c}{(r-\varrho)^p}\int_{B_r(x_0)\setminus B_\varrho(x_0)} (v-m)_+^p\,\mathrm{d}x\,,
\end{aligned}
$$

where we have made the substitution below (compare with [14, Lemma 1.19]),

$$
\nabla v_m = \begin{cases} \nabla v & \text{on } \{v>m\}, \\ 0 & \text{on } \{v \le m\}. \end{cases}
$$

Let us focus now on $\mathcal{I}_2$. We have that

$$
\begin{aligned}
\mathcal{I}_2 \quad &= \quad \int_{B_r(x_0)}\int_{B_r(x_0)} \mathcal{A}(v(x),v(y))\Big(v_m\psi^p(x) - v_m\psi^p(y)\Big)K(x,y)\,\mathrm{d}x\mathrm{d}y \\
&\qquad + 2\int_{\mathbb{R}^N\setminus B_r(x_0)}\int_{B_r(x_0)} \mathcal{A}(v(x),v(y))(v_m\psi^p)(x)K(x,y)\,\mathrm{d}x\mathrm{d}y \\
&\overset{(1.3)}{\gtrsim} \quad \int_{\{v>m\}\cap B_\varrho(x_0)}\int_{\{v>m\}\cap B_\varrho(x_0)} K(x,y)|v(x)-v(y)|^p\,\mathrm{d}x\mathrm{d}y \\
&\qquad - \frac{r^N \mathrm{Tail}((v-m)_+;x_0,r)^{p-1}}{(r-\varrho)^{N+sp}}\int_{\{v>m\}\cap B_r(x_0)} (v-m)_+\,\mathrm{d}x \\
&\qquad - \frac{1}{(r-\varrho)^p}\int_{B_r(x_0)\setminus B_\varrho(x_0)} (v-m)_+^p\,\mathrm{d}x,
\end{aligned}
$$

where we have estimated the nonlocal contribution by following the same approach used in the purely nonlinear fractional framework in [6, Theorem 1.4] and recalling that

$$\sup_{x\in \operatorname{supp}(\psi)} \frac{|x_0-y|}{|x-y|} \;\leq\; 1+\sup_{y\in\mathbb{R}^N\setminus B_r(x_0)}\,\sup_{x\in B_\tau(x_0)} \frac{|x-x_0|}{|y-x_0|-|x_0-x|} \;\lesssim\; \frac{r}{r-\varrho}.$$

As for the left-hand side in (2.3), we have that, on the level set $\{v>m\}\cap B_r(x_0)$, it holds

$$\int_{\{v>m\}\cap B_r(x_0)} |v(x)|^{p-2}v(x)v_m\psi^p(x)\,\mathrm{d}x \;\leq\; \lambda\|v\|^p_{L^p(\{v>m\}\cap B_r(x_0))}.$$

Combining all the previous estimates in (2.3), we finally arrive to the desired result in (2.2). □

We now turn to the proof of the global boundedness of the mixed eigenfunctions. The proof below follows the strategy in the standard elliptic regularity theory and it involves a classical iterative scheme together with a (fractional) Sobolev embedding theorem.

Theorem 2.2. *Let $p\in(1,\infty)$, $s\in(0,1)$, and let $v\in\mathbb{X}^{1,p}_0(\Omega)$ be a weak solution to (1.1). Then, $v\in L^\infty(\mathbb{R}^N)$.*

Proof. With no loss of generality we assume $sp\leq n$. If the reverse inequality holds true then by Morrey-Sobolev embedding [8, Theorem 8.2] the theorem plainly follows. Moreover, once we get the result in the case when $sp<n$, the borderline case when $sp=n$ will follows in a similar fashion by rearranging the exponents; see, e. g., the last part of the proof of Theorem 1.1 in [15].

We first remark that the desired result will follow once proved that

$$\|v_+\|_{L^\infty(\mathbb{R}^N)}\leq 1 \quad \text{if } \|v_+\|_{L^p(\mathbb{R}^N)}\leq\delta, \tag{2.4}$$

where $\delta>0$ will be fixed later on. Fix now $m\in\mathbb{N}$ and define

$$\omega_m := \Big(v-(1-2^{-m})\Big)_+$$

Note that the following inequalities hold true:

$$\omega_{m+1}(x)\leq\omega_m(x), \quad \text{a. e. in } \mathbb{R}^N, \tag{2.5}$$

and

$$v(x)\leq(2^{m+1}-1)\omega_m(x), \quad \text{for } x\in\{\omega_{m+1}>0\}, \tag{2.6}$$

as well as the following inclusions

$$\{\omega_{m+1}>0\}\subseteq\{\omega_m>2^{-(m+1)}\}, \quad \text{for all } m\in\mathbb{N}. \tag{2.7}$$

Now, observe that the desired global boundedness will follow once we estimate $Y_m := \|\omega_m\|^p_{L^p(\mathbb{R}^N)}$. For this, we firstly apply the fractional Sobolev inequality [8, Theorem 6.5] – recall that we denoted by $p^*:=np/(n-sp)$ the fractional critical Sobolev exponent – and then we can estimate as follows via (1.3), (1.4) and (2.1),

$$\left(\int_{\{\omega_{m+1}>0\}} \omega_{m+1}^{p^*}\,\mathrm{d}x\right)^{\frac{p}{p^*}}$$

$$\begin{aligned}
&\le \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\mathcal{A}(v(x),v(x))\Big(\omega_{m+1}(x)-\omega_{m+1}(y)\Big)}{|x-y|^{N+sp}}\,\mathrm{d}x\mathrm{d}y\\
&\le c\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \mathcal{A}(v(x),v(x))\Big(\omega_{m+1}(x)-\omega_{m+1}(y)\Big)K(x,y)\,\mathrm{d}x\mathrm{d}y\\
&\quad+\int_{\{\omega_{m+1}>0\}\cap\Omega}|\nabla v|^p\,\mathrm{d}x\\
&\le c\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \mathcal{A}(v(x),v(x))\Big(\omega_{m+1}(x)-\omega_{m+1}(y)\Big)K(x,y)\,\mathrm{d}x\mathrm{d}y\\
&\quad+\frac{c}{p}\int_{\{\omega_{m+1}>0\}\cap\Omega}\langle\nabla_z\mathcal{H}(x,\nabla v),\nabla v\rangle\,\mathrm{d}x\\
&\overset{(1.5)}{=} c\int_{\{\omega_{m+1}>0\}}|v(x)|^{p-2}v(x)\omega_{m+1}(x)\,\mathrm{d}x\\
&\overset{(2.5)-(2.6)}{\le} c(2^{m+1}-1)^{p-1}Y_m,
\end{aligned}$$

where we have also used that

$$|v(x)-v(y)|^{p-2}\Big(v(x)-v(y)\Big)\Big(\omega_{m+1}(x)-\omega_{m+1}(y)\Big)\;\ge\;|\omega_{m+1}(x)-\omega_{m+1}(y)|^p;$$

see for example [9, Formula (14)].

Moreover, an application of Hölder's Inequality (with p^*/p and n/sp) yields

$$\|\omega_{m+1}\|^p_{L^p(\{\omega_{m+1}>0\})}\;\le\;\|\omega_{m+1}\|^p_{L^{p^*}(\{\omega_{m+1}>0\})}|\{\omega_{m+1}>0\}|^{\frac{sp}{n}}.$$

Finally, by Chebyshev's Inequality we get

$$|\{\omega_{m+1}>0\}|\overset{(2.6),(2.7)}{\le}|\{\omega_m>2^{-(m+1)}\}|\;\le\;2^{p(m+1)}Y_m.$$

All in all, we obtain

$$Y_{m+1}\;\le\;c\lambda(2^{p(m+1)}Y_m)^{1+\frac{sp}{n}}\;\le\;C^mY_m^{1+\beta}, \tag{2.8}$$

which will permit us to apply the classical iteration scheme in order to get

$$\lim_{m\to\infty}Y_m=0, \tag{2.9}$$

once we note that

$$Y_0=\|v_+\|^p_{L^p}\le C^{-\frac{1}{\beta^2}}=:\delta^p.$$

Finally, since Y_m converges to $(u-1)_+$ pointwise a. e., we can deduce that condition (2.9) implies the desired boundedness estimate in (2.4). □

3. Uniqueness of mixed eigenfunctions

This section is devoted to the proof of Theorem 1.2. The proof combines together the convexity property along geodesics of both the local part, see [3], and the nonlocal one, see [9]. It is worth noticing that a different proof of the result presented here in a general linear setting under mixed Dirichlet/Neumann boundary conditions can be found, amongst other related interesting results, in the very recent paper [12]; see, in particular, Section 3 there.

Before going straight into the proof of our main results we can see that minimizers of the Rayleigh quotient $\mathcal{R}_{\rm mix}$ in (1.6) are nonnegative functions. Indeed, if one considers a minimizer u, it is sufficient to observe that the following inequality,

$$|u(y)-u(x)| \,\geq\, \Big||u(y)|-|u(x)|\Big|,$$

is strict at almost all points x, y such where $u(x)u(y)<0$, and that when considering $\tilde{u}:=|u|$, in view the homogeneity in the z-variable of $\mathcal{H}$, one has that $\mathcal{H}(x,\nabla\tilde{u})=\mathcal{H}(x,\nabla u)$ almost everywhere.

We now focus on the proof of our main result. Let us introduce, for any function ϕ_0,ϕ_1 and for any $t\in[0,1]$, the quantity σ_t given by

$$\sigma_t(x) := \Big((1-t)\phi_0(x)^p + t\phi_1(x)^p\Big)^{\frac{1}{p}}. \tag{3.1}$$

We would need the following helpful lemmata.

Lemma 3.1 (Lemma 2.1 – [3])**.** *Let $\Omega\subset\mathbb{R}^N$ be an open set, $p>1$, and let $\mathcal{E}_{\mathcal{H}}$ be the functional defined in* (1.8) *with $\mathcal{H}:\Omega\times\mathbb{R}^N\to\mathbb{R}_+$ being a measurable function such that*

$z\mapsto\mathcal{H}(x,z)$ is convex and positively homogeneous of degree p, i. e.
$\mathcal{H}(x,tz)=t^p\mathcal{H}(x,z)$ for any $t\geq0,(x,z)\in\Omega\times\mathbb{R}^N$.

Then, for every nonnegative $\phi_0,\phi_1\in W^{1,p}(\Omega)$, such that $\mathcal{E}_{\mathcal{H}}(\phi_i)<\infty$, for $i=0,1$, for σ_t defined as in (3.1)*, the function $\mathcal{E}_{\mathcal{H}}$ satisfies*

$$\mathcal{E}_{\mathcal{H}}(\sigma_t(x)) \leq (1-t)\mathcal{E}_{\mathcal{H}}(\phi_0) + t\mathcal{E}_{\mathcal{H}}(\phi_1), \quad \forall t\in[0,1]. \tag{3.2}$$

Lemma 3.2 (Lemma 4.1 – [9])**.** *Let $\Omega\subset\mathbb{R}^N$ be an open set, $p\in(1,\infty)$, $s\in(0,1)$, and let $\mathcal{E}_K$ be the functional defined in* (1.7)*. Then, for every nonnegative $\phi_0,\phi_1\in\mathbb{X}^{1,p}_0(\Omega)$ and for σ_t defined as in* (3.1)*, the function $\mathcal{E}_K$ satisfies*

$$\mathcal{E}_K(\sigma_t(x)) \leq (1-t)\mathcal{E}_K(\phi_0) + t\mathcal{E}_K(\phi_1), \quad \forall t\in[0,1]. \tag{3.3}$$

We are now in the position to prove Theorem 1.2.

Proof of Theorem 1.2**.** Let $v\in\mathbb{X}^{1,p}_0(\Omega)$ be a strictly positive weak solution to (1.1). Without loss of generality let us assume that $\|v\|_{L^p(\mathbb{R}^N)}\equiv1$. Consider a solution u to the minimization problem (1.9) with $\|u\|_{L^p(\mathbb{R}^N)}\equiv1$. Denote with

$$u_\varepsilon := u+\varepsilon, \qquad v_\varepsilon := v+\varepsilon, \quad \forall\varepsilon\ll1,$$

and with σ^ε_t the function (3.1) with $\phi_0\equiv v_\varepsilon$ and $\phi_1\equiv u_\varepsilon$.

By Lemma 3.1 and Lemma 3.2 $t\mapsto\sigma^\varepsilon_t$ is a curve belonging to $\mathbb{X}^{1,p}_0(\Omega)$ along which the energy $\mathcal{E}_{\mathcal{H}}$ in (1.8) and $\mathcal{E}_K$ in (1.7) is convex. Thus, by testing (1.5) with $\phi \equiv v$, we get

$$\begin{aligned}
&\mathcal{E}_K(\sigma_t^\varepsilon) - \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} K(x,y)|v(x)-v(y)|^p \,\mathrm{d}x\mathrm{d}y \\
&\qquad + \mathcal{E}_{\mathcal{H}}(\sigma_t^\varepsilon) - \int_\Omega \mathcal{H}(x,\nabla v)\,\mathrm{d}x \\
&\qquad = \mathcal{E}_K(\sigma_t^\varepsilon) - \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} K(x,y)|v(x)-v(y)|^p \,\mathrm{d}x\mathrm{d}y \\
&\qquad\quad + \mathcal{E}_{\mathcal{H}}(\sigma_t^\varepsilon) - \frac{1}{p}\int_\Omega \langle \nabla_z \mathcal{H}(x,\nabla v), \nabla v\rangle\,\mathrm{d}x \\
&\qquad \le t\Big(\mathcal{E}_K(u) + \mathcal{E}_{\mathcal{H}}(u) - \mathcal{E}_K(v) - \mathcal{E}_{\mathcal{H}}(v)\Big) \\
&\qquad = t\Big(\lambda_{1,p}^s(\Omega) - \lambda\Big), \qquad \forall t\in[0,1],\ \varepsilon \ll 1; \qquad (3.4)
\end{aligned}$$

above we have used respectively (3.2), (3.3), and the fact that v is a weak solution to (1.1) and u a minimizer of (1.9).

Now, in virtue of the convexity of $\tau \mapsto |\tau|^p$, we obtain that

$$\begin{aligned}
&|\sigma_t^\varepsilon(x) - \sigma_t^\varepsilon(y)|^p - |\sigma_0^\varepsilon(x) - \sigma_0^\varepsilon(y)|^p \\
&\quad \ge p|\sigma_0^\varepsilon(x) - \sigma_0^\varepsilon(y)|^{p-2}\Big(\sigma_0^\varepsilon(x) - \sigma_0^\varepsilon(y)\Big) \qquad (3.5)\\
&\quad \times\Big(\sigma_t^\varepsilon(x) - \sigma_t^\varepsilon(y) - \sigma_0^\varepsilon(x) - \sigma_0^\varepsilon(y)\Big),
\end{aligned}$$

where we have denoted by $\sigma_0^\varepsilon \equiv v_\varepsilon$. As for the local term, recalling the convexity of $\mathcal{H}(x,\cdot)$, we get

$$\begin{aligned}
&\mathcal{E}_{\mathcal{H}}(\sigma_t^\varepsilon) - \int_\Omega \mathcal{H}(x,\nabla v)\,\mathrm{d}x \\
&\qquad = \int_\Omega \mathcal{H}(x,\nabla\sigma_t^\varepsilon)\,\mathrm{d}x - \int_\Omega \mathcal{H}(x,\nabla\sigma_0^\varepsilon)\,\mathrm{d}x \\
&\qquad \ge \int_\Omega \langle \nabla_z\mathcal{H}(x,\nabla\sigma_0^\varepsilon), \nabla(\sigma_t^\varepsilon - \sigma_0^\varepsilon)\rangle\,\mathrm{d}x. \qquad (3.6)
\end{aligned}$$

Then, inserting (3.5) and (3.6) in (3.4), it yields

$$\begin{aligned}
\lambda_{1,p}^s(\Omega) - \lambda \ \ge\ & \int_\Omega \left\langle \nabla_z\mathcal{H}(x,\nabla\sigma_0^\varepsilon), \frac{\nabla\sigma_t^\varepsilon - \nabla v}{t}\right\rangle \mathrm{d}x \qquad (3.7)\\
& + p\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} K(x,y)|v(x)-v(y)|^{p-2}(v(x)-v(y)) \\
& \times \frac{\Big(\sigma_t^\varepsilon(x) - \sigma_t^\varepsilon(y) - v(x) - v(y)\Big)}{t}\,\mathrm{d}x\mathrm{d}y.
\end{aligned}$$

Since the function $\sigma_t^\varepsilon - v_\varepsilon$ belongs to $\mathbb{X}_0^{1,p}(\Omega)$, we can plug it as test function in (1.5), so that from (3.7) we get

$$\lambda_{1,p}^s(\Omega) - \lambda \,\geq\, p\lambda \int_\Omega v(x)^{p-1} \frac{\Big(\sigma_t^\varepsilon(x) - v_\varepsilon(x)\Big)}{t}\,\mathrm{d}x. \tag{3.8}$$

Note also that by the concavity of the map $\tau \mapsto |\tau|^{\frac{1}{p}}$, we have that

$$v(x)^{p-1} \frac{\Big(\sigma_t^\varepsilon(x) - v_\varepsilon(x)\Big)}{t} \,\geq\, v(x)^{p-1}\Big(u(x) - v(x)\Big),$$

which belongs to $L^1(\Omega)$. Then, we can apply Fatou's Lemma on the left-hand side in (3.8) to get

$$\begin{aligned}\lambda \int_\Omega \left(\frac{v(x)}{v(x)+\varepsilon}\right)^{p-1} &\Big(u_\varepsilon(x)^p - v_\varepsilon(x)^p\Big)\,\mathrm{d}x \\ &\leq p\lambda \liminf_{t\to 0^+} \int_\Omega v(x)^{p-1} \frac{\Big(\sigma_t^\varepsilon(x) - v(x)\Big)}{t}\,\mathrm{d}x \,\leq\, \lambda_{1,p}^s(\Omega) - \lambda,\end{aligned}$$

for any $\varepsilon \ll 1$ since

$$\frac{\mathrm{d}}{\mathrm{dt}}\Big|_{t=0} \sigma_t^\varepsilon(x) = \frac{1}{p} v_\varepsilon^{1-p}(x)\Big(u_\varepsilon(x)^p - v_\varepsilon(x)^p\Big).$$

Now, since $v > 0$ we have that $\operatorname{supp}(v) \equiv \Omega$, by the Dominated Convergence theorem, for $\varepsilon \to 0^+$ we get

$$0 \,=\, \lambda \int_\Omega \Big(u(x)^p - v(x)^p\Big)\,\mathrm{d}x \,\leq\, \lambda_{1,p}^s(\Omega) - \lambda,$$

by the normalization of the L^p-norm of u and v. Thus,

$$\lambda \,\leq\, \lambda_{1,p}^s(\Omega),$$

and the reverse inequality holds true being $\lambda_{1,p}^s(\Omega)$ the least possible mixed eigenvalue. □

Conflict of interest. The authors declare to have no conflict of interests. No data are attached to this paper.

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