ON THE OPTIMIZATION OF THE ROBIN EIGENVALUES IN SOME CLASSES OF POLYGONS

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To the memory of Umberto Massari

ABSTRACT. Given the eigenvalue problem for the Laplacian with Robin boundary conditions, (with $\beta \in \mathbb{R} \setminus \{0\}$ the Robin parameter), we consider a shape minimization problem for a function of the first eigenvalues if $\beta > 0$ and a shape maximization problem if $\beta < 0$. Both problems are settled in a suitable class of generalized polygons with an upper bound on the number of sides, under either perimeter or volume constraint.

CONTENTS

1. INTRODUCTION

The classical Robin eigenvalue problem is formulated as follows: given a nonzero real parameter β, we look for which values of λ the boundary value problem

$$\begin{cases} \Delta u + \lambda u = 0 & \text{in } \Omega \\ \dfrac{\partial u}{\partial \nu} + \beta u = 0 & \text{on } \partial\Omega \end{cases} \tag{1}$$

admits a nonzero weak solution. Here Ω is a bounded set in $\mathbb{R}^d$ with sufficiently smooth boundary and ν is the outer normal. It is well known (see e.g. [8]) that this problem admits a weak formulation based on the bilinear form

$$\mathcal{E}_\beta(u,v) = \int_\Omega \nabla u \cdot \nabla v \, dx + \beta \int_{\partial\Omega} uv \, d\mathcal{H}^{d-1}, \quad u,v \in H^1(\Omega), \tag{2}$$

where $\mathcal{H}^{d-1}$ is the $(d-1)$-dimensional Hausdorff measure and u on $\partial\Omega$ is the boundary trace. By the trace inequality in the Sobolev space $H^1(\Omega)$ the bilinear form $\mathcal{E}_\beta$ is semibounded in $L^2(\Omega)$, i.e., there are constants $c_1, c_2 > 0$ such that $\mathcal{E}_\beta(u,u) + c_1\|u\|^2_{L^2} \geq \|u\|^2_{H^1}$, hence $\mathcal{E}_\beta$ defines a closed operator $(-\Delta_\beta)$ in $L^2(\Omega)$, which is self-adjoint and has compact resolvent, so that its spectrum is real and consists of an increasing sequence $(\lambda_{k,\beta})_{k\in\mathbb{N}}$ of eigenvalues such that $\lambda_{k,\beta} \to +\infty$ as

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$k \to +\infty$, with $\lambda_{1,\beta} > 0$ if $\beta > 0$. The corresponding variational formulation for the first eigenvalue leads to the minimization of the Rayleigh quotient

$$R_{\Omega,\beta}(u) := \frac{\displaystyle\int_\Omega |\nabla u|^2\,dx + \beta \int_{\partial\Omega} u^2\,d\mathcal{H}^{d-1}}{\displaystyle\int_\Omega u^2\,dx}, \tag{3}$$

$$\lambda_{1,\beta}(\Omega) = \min_{u \in H^1(\Omega)\setminus\{0\}} R_{\Omega,\beta}(u). \tag{4}$$

As in the more classical Dirichlet and Neumann problems (that can be formally obtained from (1) by letting β to $+\infty$ and to 0, respectively) one can look for the sets Ω that optimize suitable functions of the Robin spectrum. Such problems are different in nature according to the sign of the boundary parameter β. The simplest one is obviously that of finding the set Ω with minimal first eigenvalue with $\beta > 0$ among the sets with prescribed Lebesgue measure: the optimal set turns out to be the ball, as proved in two dimension by M.-H. Bossel [3] and in higher dimension by D. Daners [16]. In the negative boundary parameter regime $\beta < 0$, by using a constant test function in (4), it is easily checked that the first eigenvalue verifies

$$\lambda_{1,\beta}(\Omega) \le \beta\mathcal{H}^{d-1}(\partial\Omega)/|\Omega| < 0, \tag{5}$$

where $|\Omega|$ is the Lebesgue measure of Ω. As a consequence, the first eigenvalue is bounded from above but not from below neither for the perimeter constraint nor for the measure one. Therefore it is very natural to study the *maximization* of the first eigenvalue. Nevertheless, in this case the optimal set is not yet known. Generalizing the Courant-Fischer formula (4) to higher eigenvalues, we consider

$$\lambda_{k,\beta}(\Omega) = \min_{S \in \mathcal{S}_k} \max_{u \in S\setminus\{0\}} R_{\Omega,\beta}(u) \tag{6}$$

where $\mathcal{S}_k$ denotes the set of all k-dimensional subspaces of $H^1(\Omega)$. Notice that the presence of boundary integrals makes the study of the Robin problem deeply different from the Dirichlet and Neumann problems.

In this paper we study two shape optimization problems (according to the sign of β) for some functions of the Robin eigenvalues, in the planar case $d = 2$, where the class of competitors is that of polygons. In order to tackle these optimization problems, we fix a suitable topology on the polygons (the one induced by the H^c convergence, see Definition 2.3), we enlarge the class of simple polygons in Definition 2.1 to encompass their H^c limits, i.e., the *generalized polygons* introduced in Definition 2.4, and we extend the variational characterization of the eigenvalues to such class, see (8).

Though the formulation of spectral shape optimization problems in some subclasses of polygons is rather simple, there are several issues in this setting. Let us start with the *polygonal Faber-Krahn* inequality stated by Polya and Szëgo, see [25, Pag. 158]: among polygons with at most N sides and given area, does the regular N-agon minimizes the first Dirichlet eigenvalue? Even if the result seems natural and expected, a direct proof of it is available only $N = 3$ and $N = 4$. Indeed, in these two cases, the classic symmetrization techniques work and thus it is natural to obtain the equilateral triangle and the square as minimal polygons, respectively. On the other hand, if $N \ge 5$, the Steiner symmetrization of a polygon could increase the number of sides, in general. In [19, Section 3.3] it is proved that among polygons with *at most* N sides, if $\beta > 0$ optimizers exist and have *exactly* N sides. This result is proved by showing that a small cut near a convex corner produces a better competitor with more sides. This idea is exploited in Section 3 below. Anyway, the question of the precise shape of the optimal polygons remains open. Recently, it has been pointed out in [1, 2] that for some $N \geq 5$ the proof that the optimal set is the regular N-agon can be reduced to a finite number of certified numerical computations and it has been shown the local minimality of the regular pentagon and hexagon. Concerning other boundary conditions, the possibility to handle explicit eigenfunctions or to separate the variables plays an important role. For what concerns Neumann conditions, it is worth mentioning [26], where the authors address the problem of maximizing the Neumann eigenvalues on rectangles with a measure or perimeter constraint. Instead, concerning Robin eigenvalues with positive boundary parameter $\beta > 0$, in [17] the authors have proved that the square minimizes the first eigenvalue among all (unions of) rectangles of a given area. For the higher eigenvalues, they proved that the square (respectively, the union of k equal squares) minimizes $\lambda_{1,\beta}$ (respectively, $\lambda_{k,\beta}$) among rectangles (respectively, unions of rectangles) of given area if β is below a certain threshold; on the other hand, they showed that the optimizers are not the square or the union of k equal squares if β is large enough. It is worth mentioning also [22], where the author showed how the rectangular case supports some well-known conjectures about spectral shape optimization problems involving the Robin eigenvalues. For the case $\beta < 0$, we mention [21], where the authors proved that the equilateral triangle locally maximizes the first eigenvalue among all triangles of a given area, again provided that $|\beta|$ is below a certain threshold depending only upon the area constraint.

When dealing with the negative boundary parameter case, in some situations it turns out that the perimeter constraint is rather natural, see for instance [6, 15], where the optimality and the stability of the ball for the first eigenvalue in the convex case is addressed. Even in our framework, the (generalized) perimeter constraint is very helpful to obtain some additional properties of the optimal shapes, see Proposition 4.4.

As highlighted in the previous round-up of references, for the polygonal case we have very little information about the optimizers even for the first eigenvalue (except for special cases in which either the eigenfunctions are explicit or the symmetrization techniques work, see [23, 17]), or some restrictions on the admissible polygons. The main difficulty in tackling this problem is that it is not possible to transpose the same argument used in [3, 16] to prove an isoperimetric inequality for the first Robin eigenvalue. Indeed, such results are based on the radiality of the first eigenfunction of the disk and on a comparison between the (smooth) level sets of such function and the level sets of an eigenfunction of a generic domain. Anyway, the possible presence of parts of the boundary with multiplicity 2 is a challenging problem even in more general settings, see [14]. Moreover, in the Robin case, the optimality of the regular N-agon is ensured only if $N \in \{3, 4\}$ is within some range of the boundary parameter. For this reason, our intent is to get more general existence results for both cases $\beta > 0$ and $\beta < 0$, without any assumption on the magnitude of the parameter, the number of sides and the order of the eigenvalue. More precisely, we focus on a wider class of spectral functionals, whose prototype is the sum of the first k eigenvalues, we consider both the perimeter and volume constraint and we prove the existence of solutions and some qualitative properties, combining well established techniques holding in more general settings (generalization of the eigenvalues, existence in a weaker framework, continuity of the traces along moving boundaries, etc.) and peculiar features of the polygonal case.

The paper is organized as follows: in Section 2 we describe our framework, recall the necessary preliminary results and state our main results. Sections 3 and 4 are devoted to the proofs of our main results, namely Theorems 2.15 and 2.16, respectively. In Section 5 we discuss some further issues.

2. Preliminaries and Main results

In order to introduce the class of admissible polygons, we start from the definition of simple polygon. In the whole paper by *line segment* we always mean a *maximal* subset of a straight line in the plane belonging to the boundary of a polygon.

Following [7], we introduce the class of admissible polygonal sets.

Definition 2.1 (Simple polygons)**.** *A simple polygon is the open bounded planar region P delimited by a finite number of not self-intersecting line segments (called sides) which are pairwise joined at their endpoints (called vertices) to form a simple closed path.*

Let us denote by $\mathcal{P}_N$ the family of simple polygons with at most N sides. Notice that simple polygons are connected and simply connected.

In the following, we use as a key tool the H^c-convergence, as it preserves many topological properties of polygonal domains. Let us start from the Hausdorff distance in $\mathbb{R}^2$.

Definition 2.2 (Hausdorff distance)**.** *Let $A, B \subset \mathbb{R}^2$ be closed. We define the Hausdorff distance between A and B by*

$$d_H(A,B) := \max\left\{\sup_{x\in A}\mathrm{dist}(x,B), \sup_{x\in B}\mathrm{dist}(x,A)\right\}.$$

The Hausdorff convergence of a sequence of open sets is defined using the distance between their complements.

Definition 2.3 (Hausdorff convergence of open sets)**.** *Let $D \subset \mathbb{R}^2$ be compact and let $E, F \subset D$ be open. We define the Hausdorff complementary distance between A and B by*

$$d_{H^c}(E,F) = d_H(D\setminus E, D\setminus F)$$

and we say that the sequence (E_n) H^c-converges to E if $\lim_n d_{H^c}(E_n, E) = 0$. We say also that (E_n) locally H^c-converges to E if for any ball B the sequence $(E_n \cap B)$ H^c-converges to $E \cap B$.

It is easily seen that the definition of H^c-convergence is independent of the choice of the compact set D.

Notice that in general the H^c-limit of a sequence of simple polygons in $\mathcal{P}_N$ is not a simple polygon in $\mathcal{P}_N$, as shown in Figure 1.

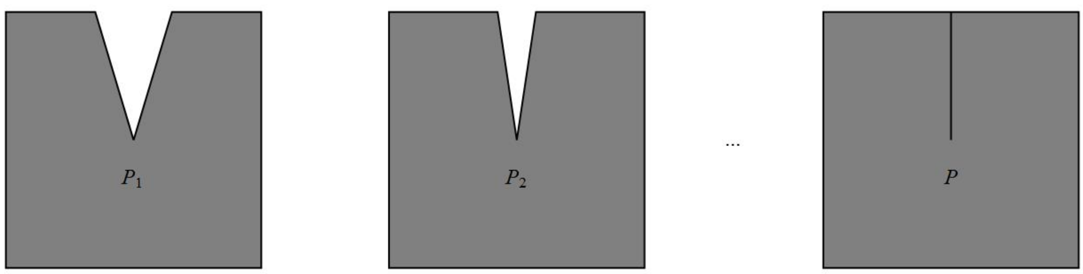


Figure 1. The sequence $(P_n) \subset \mathcal{P}_7$ H^c-converges to the "degenerate polygon" P, which has a boundary given by line segments, but is not a simple polygon.

To overcome this problem, we follow the approach in [7] and set our shape optimization problems in a class of sets that contains the H^c-limits of simple polygons.

Definition 2.4 (Generalized polygons)**.** *We say that an open set $P \subset \mathbb{R}^2$ is a generalized polygon with at most N sides if there exists a sequence (P_n) of simple polygons in $\mathcal{P}_N$ such that (P_n) locally H^c-converges to P. We denote by $\overline{\mathcal{P}_N}$ the class of generalized polygons with at most N sides.*

Notice that if P is a generalized polygon, any side having double multiplicity has at least one vertex on the topological boundary of $\overline{P}$.

The following compactness result is contained in [4, Proposition 4.6.1]

Proposition 2.5. *Let $D \subset \mathbb{R}^d$ a fixed compact set. Then, the class of the open sets contained in D is compact in the Hausdorff-complementary topology.*

Remark 2.6 (see Remark 2.2.18 in [20])**.** *If $\Omega_n \xrightarrow{H^c} \Omega$ into a fixed compact B, then, denoting by $\#E$ the number of connected components of E, $\#(\Omega^c \cap B) \leq \liminf_n \#(\Omega_n^c \cap B)$.*

In dimension $d = 2$ this allows us to obtain further topological information: if a bounded open set in $\mathbb{R}^2$ is a disjoint union of simply connected open sets, then its complement (in the compact B) is a compact connected set. This implies that the H^c-limit of unions of simply connected set is union of simply connected set. Indeed, let (Ω_n) be a sequence of open bounded subsets of $\mathbb{R}^2$ such that each Ω_n is a bounded disjoint union of simply connected open sets; if $\Omega_n \xrightarrow{H^c} \Omega$, then

$$1 \leq \#(\Omega^c \cap B) \leq \liminf_n \#(\Omega_n^c \cap B) = 1$$

and so Ω is union of simply connected open sets.

Remark 2.7. *The following facts hold true for the family $\overline{\mathcal{P}_N}$ (see [7] for details).*

(i) $\overline{\mathcal{P}_N}$ is closed with respect to the local H^c-convergence since the number of connected components of the complement of each generalized polygon is uniformly bounded.

(ii) Every $P \in \overline{\mathcal{P}_N}$ is union of simply connected generalized polygons, since P^c is connected.

(iii) $P \in \overline{\mathcal{P}_N}$ may be disconnected; each connected component of P is delimited by a finite number of line segments (still called the sides of P), which are pairwise joined at their endpoints (still called vertices of P) to form a closed path, possibly containing self-intersections; in particular, P has at most N sides, counted with their multiplicity. P has at most $\lfloor \frac{N-1}{2} \rfloor$ connected components. If N is odd, this upper bound is obtained for instance if P is the union of triangles with consecutive bases lying on the same line (so their union is considered as one side, according to our definition of line segment). If N is even, the upper bound is obtained by replacing one of the triangles in the previous construction with a quadrilateral.

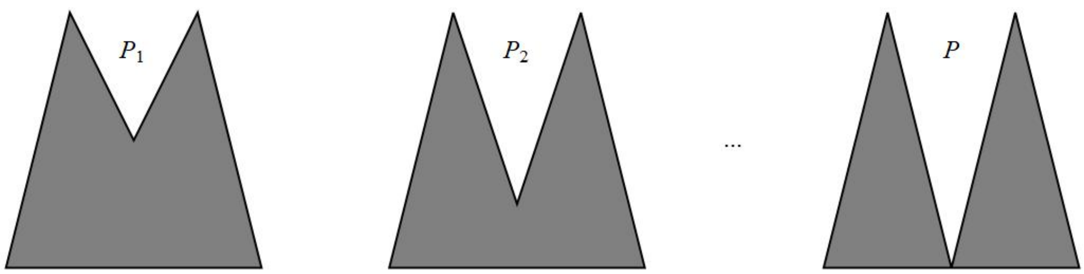


FIGURE 2. The sequence $(P_n) \subset \mathcal{P}_5$ H^c-converges to $P \in \overline{\mathcal{P}_5}$, that has $2 = \lfloor \frac{5-1}{2} \rfloor$ connected components.

(iv) *Since the topological boundary of the closure of any $P \in \overline{\mathcal{P}_N}$ is a closed curve, P is bounded and thus has finite Lebesgue measure. Conversely, since any $P \in \overline{\mathcal{P}_N}$ has finite Lebesgue measure, it is bounded (otherwise, in view of the bound on the number of sides, necessarily P would have two parallel sides with infinite length, contradicting the fact that $|P| < +\infty$.)*

Remark 2.8. *Let us observe that the number of sides is lower semicontinuous for locally H^c-converging sequences $(P_n) \subset \overline{\mathcal{P}_N}$.*

Indeed, let (P_n) be a sequence of generalized polygons H^c converging to $P \in \overline{\mathcal{P}_N}$ and let $M \leq N$ be the biggest integer such that there are infinitely many P_n in $\overline{\mathcal{P}_M}$. Then, by Remark 2.7(i) the limit P belongs to $\overline{\mathcal{P}_M} \subseteq \overline{\mathcal{P}_N}$. Notice that this fact does not hold if the number of sides is not bounded a priori (the sequence (R_n) of regular n-agons of measure m centered at a point $x_0 \in \mathbb{R}^2$ H^c-converges to the disk of measure m centered at x_0).

The definition of the trace of $u \in H^1(P)$, where P is a simple polygon, is defined in the usual way. Following [18, Section 1.1.7], we can define the trace of u also when P is a generalized polygon, that may lie on both sides of an inner boundary segment. To this aim, we can assume that there is only one inner boundary segment S (otherwise, divide P in smaller polygons with such a property). Then adding a further segment S' starting from the inner endpoint of S, we divide P as the union of two simple polygons and in each of them the trace of u is well defined. Of course, the traces of u on both sides of S' coincide, whereas on S they can be different. Henceforth, we denote by u^+ and u^- these traces. In the sequel, it is not important to make explicit a criterion to distinguish the two sides, which we call *right* and *left* only to simplify the presentation. Moreover, to deal with simple and generalized polygons at the same time, we agree that all functions defined in P are extended as 0 out of $\overline{P}$ and denote by u^+ the interior trace.

In order to take into account inner boundary segments, we relax the definition of the Rayleigh quotients (6) and we define the *generalized Rayleigh quotient* by

$$\overline{R}_{P,\beta}(u) := \frac{\displaystyle\int_P |\nabla u|^2\, dx + \beta \int_{\partial P} \left[(u^+)^2 + (u^-)^2\right]\, d\mathcal{H}^1}{\displaystyle\int_P u^2\, dx} \tag{7}$$

and the *generlized eigenvalues* on a generalized polygon P by

$$\overline{\lambda}_{k,\beta}(P) := \inf_{S \in \mathcal{S}_k} \max_{u \in S \setminus \{0\}} \overline{R}_{P,\beta}(u), \tag{8}$$

where as above $\mathcal{S}_k$ denotes the set of all k-dimensional subspaces of $H^1(P)$. This definition is well posed, since it does not depend on the orientation of ∂P. Moreover, if P is a simple polygon, then $\overline{\lambda}_{k,\beta}(P) = \lambda_{k,\beta}(P)$, as on the boundary $u^+ = u$ and $u^- = 0$.

We recall two useful properties of the classical eigenvalues $\lambda_{k,\beta}$ (see [8]) that are generalized to $\overline{\lambda}_{k,\beta}$ in a standard way.

Remark 2.9 (**Some properties of generalized eigenvalues).** *Let $P \in \overline{\mathcal{P}_N}$. The following properties hold:*

- *$\beta \mapsto \overline{\lambda}_{k,\beta}(P)$ is strictly increasing.*
- *For $t > 0$ we have*
$$\overline{\lambda}_{k,\beta}(tP) = \frac{1}{t^2}\overline{\lambda}_{k,t\beta}(P).$$
In particular, when $\beta > 0$, we have
$$\overline{\lambda}_{k,\beta}(tP) < \overline{\lambda}_{k,\beta}(P) \tag{9}$$
for every $t > 1$.

In particular, inequality (9) entails the monotonicity under dilation of the generalized eigenvalues with positive boundary parameter.

As we are dealing with convergent sequences of (generalized) polygons, the corresponding Rayleigh quotients are settled on different function spaces whose convergence in turn must be defined. The right notion of convergence is the *Mosco convergence* (see e.g. [4, Chapter 4]), as it combines a geometric notion of convergence of the domains with a functional-analytic convergence of the Sobolev spaces.

Definition 2.10 (Convergence in the sense of Mosco)**.** *Let X be a Banach space and let (G_n) be a sequence of closed subsets of X. We define weak upper and strong lower limits in the sense of Kuratowski the spaces*

$$w-\limsup_{n\to+\infty} G_n = \{u \in X : \exists (n_k)_k,\ \exists u_{n_k} \in G_{n_k} \text{ s.t. } u_{n_k} \to u \text{ weakly in } X\},$$

$$s-\liminf_{n\to+\infty} G_n = \{u \in X :\ \exists u_n \in G_n \ s.t.\ u_n \to u \text{ strongly in } X\}.$$

We say that (G_n) Mosco-converges to G if

$$G = w-\limsup_{n\to+\infty} G_n = s-\liminf_{n\to+\infty} G_n.$$

Under suitable topological constraints, Mosco convergence is equivalent to convergence in measure and Hausdoff convergence. An important result is the following theorem, see [4, Theorem 7.2.1].

Theorem 2.11. *Let us denote by $\#E$ the number of connected components of the open set $E \subset \mathbb{R}^2$. Let $\ell \in \mathbb{N}$ and let (Ω_n) be a sequence of open domains in $\mathbb{R}^2$ such that (Ω_n) is H^c-convergent to some Ω, with $\#(\mathbb{R}^2 \setminus \Omega_n) \le \ell$ for every $n \in \mathbb{N}$. Then $H^1(\Omega_n)$ converges to $H^1(\Omega)$ in the sense of Mosco if and only if $|\Omega_n|$ converges to $|\Omega|$.*

In order to handle the possible fractures of a generalized polygon, we define the generalized perimeter of a polygon, recalling the approach in [11, 14]. Roughly speaking, we quantify the boundary length with the natural multiplicity.

Definition 2.12 (Generalized perimeter)**.** *Let $N \in \mathbb{N}$, $P \in \overline{\mathcal{P}_N}$. We set*

$$\partial^\star P := \partial \overline{P}, \qquad \Gamma := \partial P \setminus \partial^\star P$$

and we call generalized perimeter of P the quantity

$$\widetilde{Per}(P) := \mathcal{H}^1(\partial^* P) + 2\mathcal{H}^1(\Gamma).$$

Let us recall a condition that entails the compactness of the minimizing sequences. It is a reformulation of [9, Lemma 4] adapted to our framework.

Lemma 2.13. *Let $M > 0$, $P \in \overline{\mathcal{P}_N}$ and $u \in H^1(P)$ such that $\|u\|_{L^2(P)} = L > 0$ and assume that*

$$\int_P |\nabla u|^2\,dx + \int_{\partial P} \left[(u^+)^2 + (u^-)^2\right]\,d\mathcal{H}^1 \le M.$$

Then, there exist $y \in \mathbb{R}^2$ and a positive constant $C = C(|P|, M, L)$ such that

$$|\mathrm{supp}(u) \cap Q_1(y)| \ge C(|P|, M, L),$$

where $Q_1(y)$ is the square with center y and sidelength 1.

The lower semicontinuity of the boundary integral is proved in [5, Lemma 19].

Lemma 2.14. *Let $\Omega \subseteq \mathbb{R}^2$ be open, $k \in \mathbb{N}$, and let $(K_n), K \subset \Omega$ be compact sets with at most k connected components, such that $\limsup_n \mathcal{H}^1(K_n) < +\infty$ and $K_n \to K$ in the Hausdorff metric. Let $u_n \in H^1(\Omega \setminus K_n)$ be such that*

$$\limsup_{n\to+\infty} \|u_n\|_{H^1(\Omega\setminus K_n)} + \int_{K_n} \left[(u_n^+)^2 + (u_n^-)^2\right] \, d\mathcal{H}^1 < +\infty. \tag{10}$$

Then, there exists $u \in H^1(\Omega \setminus K)$ such that, up to subsequences, we have

$$u_n \to u \quad \text{strongly in } L^2_{loc}(\Omega),$$

$$\nabla u_n \rightharpoonup \nabla u \quad \text{weakly in } L^2(\Omega;\mathbb{R}^2),$$

and

$$\int_K \left[(u^+)^2 + (u^-)^2\right] \, d\mathcal{H}^1 \le \liminf_{n\to+\infty} \int_{K_n} \left[(u_n^+)^2 + (u_n^-)^2\right] \, d\mathcal{H}^1. \tag{11}$$

Main results. Since the kind of optimization depends on the sign of the boundary parameter β, we split the discussion into two parts. Let $N \ge 3$ and $m, p > 0$.

Let us fix $\beta > 0$ and let us study the problems

$$\min\left\{F(\overline{\lambda}_{1,\beta}(P), \ldots, \overline{\lambda}_{k,\beta}(P)) : P \in \overline{\mathcal{P}_N},\ |P| \le m\right\}, \tag{12}$$

and

$$\min\left\{F(\overline{\lambda}_{1,\beta}(P), \ldots, \overline{\lambda}_{k,\beta}(P)) : P \in \overline{\mathcal{P}_N},\ \widetilde{Per}(P) \le p\right\}, \tag{13}$$

where $F \in C^1(\mathbb{R}^k)$ is nondecreasing in each variable with strictly positive derivative with respect to the first variable and such that

$$\lim_{|\xi|\to+\infty} F(\xi) = +\infty. \tag{14}$$

Our main result for $\beta > 0$ is the following.

Theorem 2.15. *Problems (12) and (13) admit a solution in the class of generalized polygons. Any minimizer P of (12) verifies $|P| = m$ and any minimizer P of (13) verifies $\widetilde{Per}(P) = p$. Moreover, any solution of both problems has exactly N sides counted with their multiplicity.*

Let us consider now the case of negative boundary parameter. For the sake of ease we set $\eta := -\beta > 0$. As explained in Section 1, is it natural to consider the following maximization problems

$$\max\left\{F(\overline{\lambda}_{1,-\eta}(P), \ldots, \overline{\lambda}_{k,-\eta}(P)) : P \in \overline{\mathcal{P}_N},\ |P| \le m\right\}, \tag{15}$$

$$\max\left\{F(\overline{\lambda}_{1,-\eta}(P), \ldots, \overline{\lambda}_{k,-\eta}(P)) : P \in \overline{\mathcal{P}_N},\ \widetilde{Per}(P) \le p\right\}, \tag{16}$$

where $F : \mathbb{R}^k \to \mathbb{R}$ is nondecreasing and upper semicontinuous in each variable and such that, for any variable ξ_j, it holds

$$\lim_{\xi_j\to-\infty} F(\xi_1, \ldots, \xi_k) = -\infty.$$

Our main result for $\beta < 0$ is the following.

Theorem 2.16. *Problems (15) and (16) admit a solution in the class of generalized polygons. If $\overline{\lambda}_{k,-\eta}(P) < 0$ for an optimal polygon P in (16), then P is union of simple polygons.*

3. Positive boundary parameter

We start this section with an important remark.

Remark 3.1 (Generalized eigenvalues of a cracked set are actually eigenvalues)**.** *Following the approach of* [11] *(see also* [14] *for analogous results in any dimension), we get that the* $\overline{\lambda}_{k,\beta}(P)$ *are actually eigenvalues of the elliptic operator defined by the bilinear form*

$$\overline{\mathcal{E}}_\beta(u,v) := \int_P \nabla u \cdot \nabla v \, dx + \beta \int_{\partial P} \left[u^+ v^+ + u^- v^-\right] \, d\mathcal{H}^1, \quad u, v \in H^1(P).$$

Clearly, the latter definition coincides with (2) *when* P *is a simple polygon. The minimum in the Courant-Fischer formula is thus attained, namely*

$$\overline{\lambda}_{k,\beta}(P) = \min_{S \in \mathcal{S}_k} \max_{u \in S \setminus \{0\}} \overline{R}_{P,\beta}(u), \tag{17}$$

where $\overline{R}_{P,\beta}(u)$ *is defined in* (7)*. This fact allows us to test the Rayleigh quotient with actual eigenfunctions.*

Remark 3.2. *If* P *is a connected generalized polygon and* $u \in H^1(P)$ *is a positive eigenfunction for* $\overline{\lambda}_{1,\beta}(P)$ *for some* $\beta > 0$*, then there exists* $\alpha > 0$ *such that* $u \geq \alpha$*, see* [10, Theorem 1].

In order to count the sides of the optimal polygons, we adapt a cutting technique used in [13]. There, the author proved that optimal convex shapes for a class of Robin spectral functionals are C^1; the technique exploits an argument by contradiction addressed to remove possible corners. In the present framework, we cut a generalized polygon near the vertex of a convex corner to show that increasing the number of sides (within the prescribed constraint) decreases the value of the functional. Aiming at cutting a convex corner of the polygon P, if there is such a corner with both sides of multiplicity one, we argue on that corner. If all the convex corners have a side of multiplicity two, we cut the corner only on one side as follows.

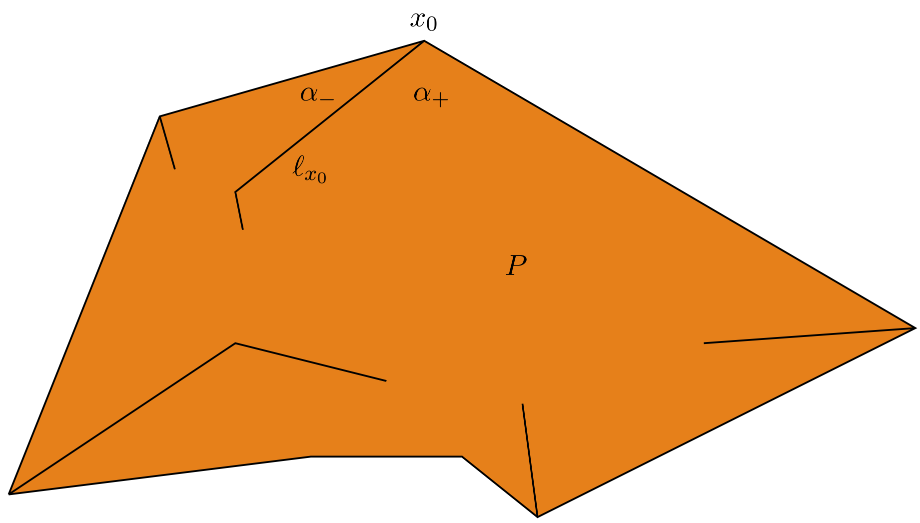


Figure 3. In the polygon P all convex corners are determined by self-intersected sides; choosing the vertex x_0 as above, without loss of generality, we can argue only on $P \cap \alpha_+$ or $P \cap \alpha_-$.

For definiteness, assume that $x_0 = 0$ is the vertex of a convex corner of the polygon P and that the segment ℓ_{x_0} vertexed at x_0 has double multiplicity. Assume also that P lies in the half-plane

$\{x_2 < 0\}$ and apply the technique only on one side of the segment, say on

$$P \cap \alpha_+ = P \cap \{x \cdot \nu_{\ell_{x_0}} > 0\},$$

where $\nu_{\ell_{x_0}}$ is one of the normal unit vectors to ℓ_{x_0}. Accordingly, we define the following sets:

$$m_\varepsilon := P \cap \alpha_+ \cap \{x_2 > -\varepsilon\}\,,\ P_\varepsilon := P \setminus m_\varepsilon,\ \sigma_\varepsilon := P \cap \alpha_+ \cap \{x_2 = -\varepsilon\}\,,\ s_\varepsilon := \partial m_\varepsilon \setminus \sigma_\varepsilon. \tag{18}$$

Notice that

$$\max_{x \in \overline{m_\varepsilon}} \mathrm{dist}(x, \sigma_\varepsilon) = \mathrm{dist}(0, \sigma_\varepsilon) = \varepsilon.$$

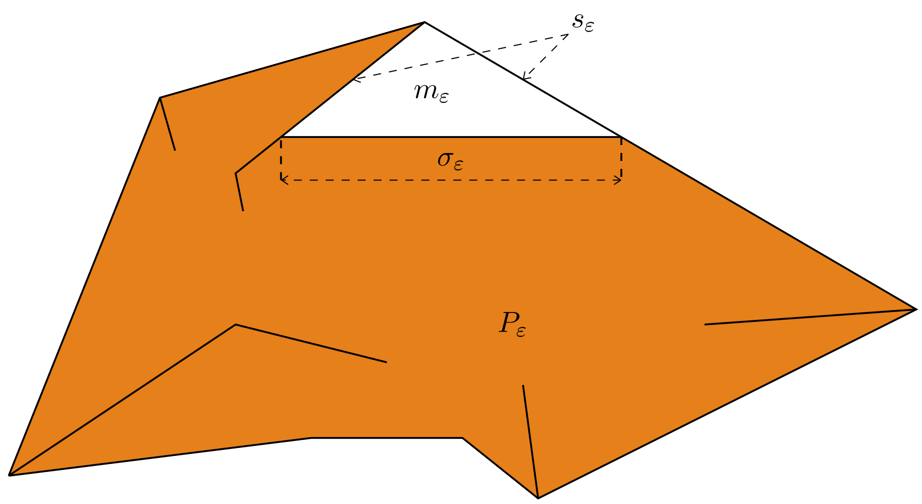


FIGURE 4. The cutting procedure of P

Lemma 3.3. *Let $P \in \overline{\mathcal{P}_N}$. Then, there exist $\varepsilon_0 > 0$ and $C = C(P, \beta) > 0$ such that for every $0 < \varepsilon < \varepsilon_0$, we have*

$$\overline{\lambda}_{1,\beta}(P_\varepsilon) \le \overline{\lambda}_{1,\beta}(P) - C\varepsilon. \tag{19}$$

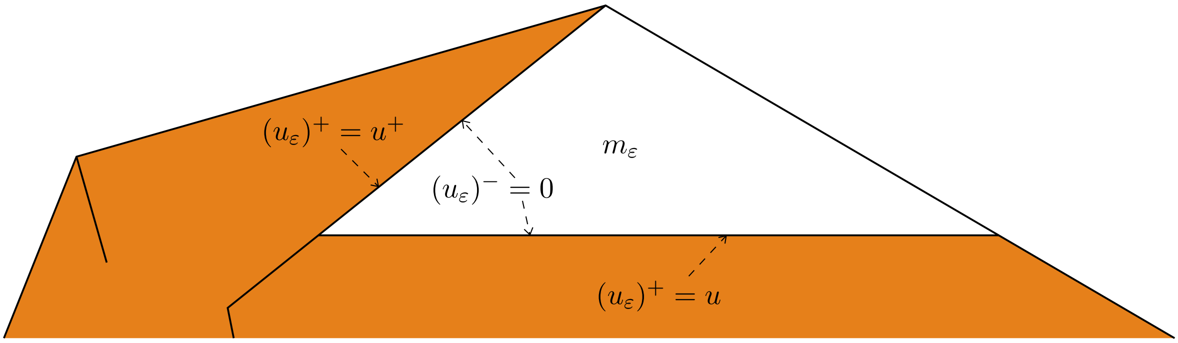
Proof. We start observing that both $\mathcal{H}^1(\sigma_\varepsilon)$ and $\mathcal{H}^1(s_\varepsilon)$ are infinitesimal of order ε, hence $|m_\varepsilon| = \varepsilon\mathcal{H}^1(\sigma_\varepsilon)/2$ is infinitesimal of order ε^2 as $\varepsilon \to 0$. Moreover, since the m_ε are triangles and the σ_ε are parallel, there exists a constant $C_1 > 1$, depending only on P, such that

$$\mathcal{H}^1(s_\varepsilon) = C_1 \mathcal{H}^1(\sigma_\varepsilon). \tag{20}$$

Let us compare $\overline{\lambda}_{1,\beta}(P_\varepsilon)$ with $\overline{\lambda}_{1,\beta}(P)$. Let us consider $u \in H^1(P)$ an $L^2(P)$-normalized eigenfunction for $\overline{\lambda}_{1,\beta}(P)$ positively bounded away from zero (see Remark 3.2) and denote by u_ε its restriction to P_ε extended by zero outside P_ε. We have that $u_\varepsilon \in H^1(P_\varepsilon)$ is a test function for $\overline{\lambda}_{1,\beta}(P_\varepsilon)$ and it holds

$$(u_\varepsilon)^+ = u,\ (u_\varepsilon)^- = 0 \quad \text{on } \sigma_\varepsilon$$

$$(u_\varepsilon)^+ = u^+,\ (u_\varepsilon)^- = 0 \quad \text{on } \partial P_\varepsilon \setminus \sigma_\varepsilon.$$



We thus have

$$
\begin{aligned}
\overline{\lambda}_{1,\beta}(P_\varepsilon) &\le \frac{\displaystyle\int_{P_\varepsilon} |\nabla u_\varepsilon|^2\,dx + \beta\int_{\partial P_\varepsilon}\left((u_\varepsilon^+)^2 + (u_\varepsilon^-)^2\right)\,d\mathcal{H}^1}{\displaystyle\int_{P_\varepsilon} u_\varepsilon^2\,dx} \\
&\le \frac{\displaystyle\int_P |\nabla u|^2\,dx + \beta\int_{\partial P}\left((u^+)^2 + (u^-)^2\right)\,d\mathcal{H}^1 + \beta\int_{\sigma_\varepsilon} u^2\,d\mathcal{H}^1 - \beta\int_{s_\varepsilon}(u^-)^2\,d\mathcal{H}^1}{\displaystyle 1 - \int_{m_\varepsilon} u^2\,dx} \qquad (21) \\
&\le \left[\overline{\lambda}_{1,\beta}(P) + \beta\int_{\sigma_\varepsilon} u^2\,d\mathcal{H}^1 - \beta\int_{s_\varepsilon}(u^-)^2\,d\mathcal{H}^1\right]\left(1 + 2\int_{m_\varepsilon} u^2\,dx\right) \\
&\le \overline{\lambda}_{1,\beta}(P) + \beta\left(\int_{\sigma_\varepsilon} u^2\,d\mathcal{H}^1 - \int_{s_\varepsilon}(u^-)^2\,d\mathcal{H}^1\right) + C_2\varepsilon^2
\end{aligned}
$$

for ε small enough. We can consider the restriction of u on m_ε continuous on $\overline{m_\varepsilon}$, since it solves the mixed boundary value problem

$$
\begin{cases}
\Delta v + \overline{\lambda}_{1,\beta}(P)u = 0 & \text{in } m_\varepsilon \\
\dfrac{\partial v}{\partial \nu} + \beta v = 0 & \text{on } s_\varepsilon \\
v = u & \text{on } \sigma_\varepsilon
\end{cases}
$$

(see [24]), so we can argue by continuity as in the Lipschitz case. To ease the readibility, we denote such solution on m_ε still by u. Let us fix $\delta > 0$, with $(u(0)+\delta)^2 \le C_1(u(0)-\delta)^2$. There exists $\varepsilon_0 > 0$ such that

$$0 < u(0) - \delta < u(x) < u(0) + \delta$$

for every $x \in \overline{m_{\varepsilon_0}}$, since $u(0) > 0$. In particular, the trace $u^-(x)$ of u for $x \in s_\varepsilon$ satisfies the above estimate as well.

Now, m_ε is decreasing in ε with respect to inclusions, then we can choose ε_0 small enough so that P_ε satisfies (21). Combining the latter with (20) we get

$$
\begin{aligned}
\overline{\lambda}_{1,\beta}(P_\varepsilon) &\le \overline{\lambda}_{1,\beta}(P) + \beta\left(\mathcal{H}^1(\sigma_\varepsilon)(u(0)+\delta)^2 - \mathcal{H}^1(s_\varepsilon)(u(0)-\delta)^2\right) + C_2\varepsilon^2 \\
&\le \overline{\lambda}_{1,\beta}(P) + \beta\mathcal{H}^1(\sigma_\varepsilon)\left((u(0)+\delta)^2 - C_1(u(0)-\delta)^2\right) + C_2\varepsilon^2 \\
&= \overline{\lambda}_{1,\beta}(P) - \beta C_3\varepsilon + C_2\varepsilon^2 \le \overline{\lambda}_{1,\beta}(P) - C\varepsilon,
\end{aligned}
$$

where the last constant C takes into account all the previous constants and depends only on the domain Ω and on β. □

Higher order eigenvalues are considered in the following result.

Lemma 3.4. *Let $P \in \overline{\mathcal{P}_N}$. Then, for every $h \in \mathbb{N}, h \geq 2$,*

$$\overline{\lambda}_{h,\beta}(P_\varepsilon) \leq \overline{\lambda}_{h,\beta}(P) + o(\varepsilon). \tag{22}$$

Proof. Let $\{u_1, \dots, u_h\} \subset H^1(P)$ be a L^2-orthonormal basis of h eigenfunctions associated with $\overline{\lambda}_{1,\beta}(P), \dots, \overline{\lambda}_{h,\beta}(P)$ respectively and define $S := \text{span}\,\{u_1, \dots, u_h\}$. Then, for ε sufficiently small, $S_\varepsilon := \text{span}\,\{u_1|_{P_\varepsilon}, \dots, u_h|_{P_\varepsilon}\}$ is a test space for the variational characterization (8) of $\overline{\lambda}_{h,\beta}(P_\varepsilon)$. In particular, we can consider on P_ε test functions of the form $\sum_{i=1}^h \alpha_i^\varepsilon u_i$ with $\sum_{i=1}^h (\alpha_i^\varepsilon)^2 = 1$. This in turn implies that, up to subsequences, $\alpha_i^\varepsilon \to \alpha_i \in [-1, 1]$ and

$$\sum_{i=1}^h \alpha_i^\varepsilon u_i \longrightarrow \sum_{i=1}^h \alpha_i u_i$$

strongly in $H^1(P)$. Let us denote by $(\overline{\alpha}_1^\varepsilon, \dots, \overline{\alpha}_h^\varepsilon)$ the h-tuple of coefficients that maximizes $\overline{R}_{P_\varepsilon,\beta}$ in S_ε with $\sum_{i=1}^h (\overline{\alpha}_i^\varepsilon)^2 = 1$ and by $(\overline{\alpha}_1, \dots, \overline{\alpha}_h)$ the h-tuple of coefficients such that, for every $i = 1, \dots, h$, $\overline{\alpha}_i^\varepsilon \to \overline{\alpha}_i$ (up to subsequences). For any $\varepsilon > 0$ sufficiently small, we can estimate $\overline{\lambda}_{h,\beta}(P_\varepsilon)$ using S_ε as a test space:

$$\begin{aligned}
\overline{\lambda}_{h,\beta}(P_\varepsilon) \leq & \max_{\substack{\alpha_1^\varepsilon,\dots,\alpha_h^\varepsilon \in \mathbb{R} \\ \sum_i (\alpha_i^\varepsilon)^2 = 1}} \frac{\displaystyle\int_{P_\varepsilon} \Big|\sum_i \alpha_i^\varepsilon \nabla u_i\Big|^2 dx + \beta \int_{\partial P_\varepsilon} \left[\Big(\sum_i \alpha_i^\varepsilon u_i^-\Big)^2 + \Big(\sum_i \alpha_i^\varepsilon u_i^+\Big)^2\right] d\mathcal{H}^1}{\displaystyle\int_{P_\varepsilon} \Big(\sum_i \alpha_i^\varepsilon u_i\Big)^2 dx} \\
\leq & \frac{\displaystyle\int_P \Big|\sum_i \overline{\alpha}_i^\varepsilon \nabla u_i\Big|^2 dx + \beta \int_{\partial P} \left[\Big(\sum_i \overline{\alpha}_i^\varepsilon u_i^-\Big)^2 + \Big(\sum_i \overline{\alpha}_i^\varepsilon u_i^+\Big)^2\right] d\mathcal{H}^1}{\displaystyle 1 - \int_{m_\varepsilon} \Big(\sum_i \overline{\alpha}_i^\varepsilon u_i\Big)^2 dx} \\
& + \frac{\displaystyle\beta \int_{\sigma_\varepsilon} \Big(\sum_i \overline{\alpha}_i^\varepsilon u_i\Big)^2 d\mathcal{H}^1 - \beta \int_{s_\varepsilon} \Big(\sum_i \overline{\alpha}_i^\varepsilon u_i^-\Big)^2 d\mathcal{H}^1}{\displaystyle 1 - \int_{m_\varepsilon} \Big(\sum_i \overline{\alpha}_i^\varepsilon u_i\Big)^2 dx} \\
\leq & \overline{\lambda}_{h,\beta}(P) + \beta \int_{\sigma_\varepsilon} \Big(\sum_i \overline{\alpha}_i^\varepsilon u_i\Big)^2 d\mathcal{H}^1 - \beta \int_{s_\varepsilon} \Big(\sum_i \overline{\alpha}_i^\varepsilon u_i^-\Big)^2 d\mathcal{H}^1 + C|m_\varepsilon|
\end{aligned} \tag{23}$$

where we have used that

$$\overline{R}_{P,\beta}\left(\sum_i \overline{\alpha}_i^\varepsilon u_i\right) \leq \max_{v \in S} \overline{R}_{P,\beta}(v) = \lambda_{h,\beta}(P).$$

Now, as higlighted at the beginning of Lemma 3.3, $\mathcal{H}^1(\sigma_\varepsilon)$ and $\mathcal{H}^1(s_\varepsilon)$ are both infinitesimal of order ε and $|m_\varepsilon|$ is infinitesimal of order ε^2 as $\varepsilon \to 0$; moreover, $\overline{\alpha}_i^\varepsilon - \overline{\alpha}_i \to 0$. Then, summing and subtracting the contribution of $\sum_i \overline{\alpha}_i u_i$ in the boundary integrals in (23), we get

$$\begin{aligned}\overline{\lambda}_{h,\beta}(P_\varepsilon) \leq & \overline{\lambda}_{h,\beta}(P) + \beta\Bigg[\int_{\sigma_\varepsilon}\Big(\sum_i (\overline{\alpha}_i^\varepsilon - \overline{\alpha}_i)u_i + \overline{\alpha}_i u_i\Big)^2 d\mathcal{H}^1 \\ & - \int_{s_\varepsilon}\Big(\sum_i (\overline{\alpha}_i^\varepsilon - \overline{\alpha}_i)u_i^- + \overline{\alpha}_i u_i^-\Big)^2 d\mathcal{H}^1\Bigg] + C\varepsilon^2 \\ \leq & \overline{\lambda}_{h,\beta}(P) + \beta\left(\int_{\sigma_\varepsilon}\Big(\sum_i \overline{\alpha}_i u_i\Big)^2 d\mathcal{H}^1 - \int_{s_\varepsilon}\Big(\sum_i \overline{\alpha}_i u_i^-\Big)^2 d\mathcal{H}^1\right) + o(\varepsilon).\end{aligned} \tag{24}$$

To conclude, we now consider two possible situations. If

$$\Big(\sum_i \overline{\alpha}_i u_i(0)\Big)^2 \neq 0,$$

(where the pointwise value $u(0)$ is meant in the same sense as in Lemma 3.3) then, for any sufficiently small ε, we can proceed as in Lemma 3.3 and conclude that

$$\int_{\sigma_\varepsilon}\Big(\sum_i \overline{\alpha}_i u_i\Big)^2 d\mathcal{H}^1 - \int_{s_\varepsilon}\Big(\sum_i \overline{\alpha}_i u_i^-\Big)^2 d\mathcal{H}^1 \leq 0.$$

On the other hand, if

$$\Big(\sum_i \overline{\alpha}_i u_i(0)\Big)^2 = 0,$$

the uniform continuity of the eigenfunctions u_i on m_ε implies that $\sum_i \overline{\alpha}_i u_i$ has values close to $\sum_i \overline{\alpha}_i u_i(0) = 0$ in m_ε, namely that

$$\Big|\sum_i \overline{\alpha}_i u_i\Big| \leq \delta(\varepsilon)$$

in m_ε, where $\delta(\varepsilon) \to 0$. Then, both boundary integrals are $o(\varepsilon)$ as $\varepsilon \to 0$. In both the possible situations, (24) gives

$$\overline{\lambda}_{h,\beta}(P_\varepsilon) \leq \overline{\lambda}_{h,\beta}(P) + o(\varepsilon).$$

□

Remark 3.5. *Let us compare the results of the previous lemmas. In Lemma 3.3 we proved that, after a small cut, the first eigenvalue decreases by a term of the same order as the perimeter. In Lemma 3.4, we proved that a small cut could increase $\lambda_{h,\beta}$ ($h \geq 2$) at most by a term infinitesimal of higher order than the perimeter. In other words, the possible increase of $\lambda_{h,\beta}$ ($h \geq 2$) is infinitesimal of higher order than the decrease of $\lambda_{1,\beta}$.*

In the following proposition we prove that the number of sides of possible optimal generalized polygons is maximal.

Proposition 3.6. *Let $F : \mathbb{R}^k \to \mathbb{R}$ satisfy hypotheses (12) and (13). For every $P \in \overline{\mathcal{P}_N}$, there exists a generalized polygon $P' \in \overline{\mathcal{P}_{N+1}}$ such that $|P'| \leq |P|$, $\widetilde{Per}(P') \leq \widetilde{Per}(P)$ and*

$$F(\overline{\lambda}_{1,\beta}(P'), \dots, \overline{\lambda}_{k,\beta}(P')) < F(\overline{\lambda}_{1,\beta}(P), \dots, \overline{\lambda}_{k,\beta}(P)).$$

In particular, no $P \in \overline{\mathcal{P}_N}$ can be a minimizer for (12) or (13) in $\overline{\mathcal{P}_{N+1}}$.

Proof. The proof is based on an argument similar to that in Theorem [13, Theorem 5.3]. Consider $P \in \overline{\mathcal{P}_N}$. For every $\varepsilon > 0$ sufficiently small, the polygon P_ε defined in (18) has $N+1$ sides; moreover, Lemma 3.3 and Lemma 3.4 imply that

$$\overline{\lambda}_{1,\beta}(P_\varepsilon) - \overline{\lambda}_{1,\beta}(P) \leq -C\varepsilon \quad \text{and} \quad \overline{\lambda}_{k,\beta}(P_\varepsilon) - \overline{\lambda}_{k,\beta}(P) = o(\varepsilon). \tag{25}$$

The hypotheses on F lead to the first assertion, once we set $P' := P_\varepsilon \in \overline{\mathcal{P}_{N+1}}$ for a suitable value of $\varepsilon > 0$. Indeed, a Taylor expansion gives

$$\begin{aligned} F(\overline{\lambda}_{1,\beta}(P_\varepsilon), \dots, \overline{\lambda}_{k,\beta}(P_\varepsilon)) = & F(\overline{\lambda}_{1,\beta}(P), \dots, \overline{\lambda}_{k,\beta}(P)) \\ & + \sum_{h=1}^{k} \frac{\partial F}{\partial x_h}(\overline{\lambda}_{1,\beta}(P), \dots, \overline{\lambda}_{k,\beta}(P)) \cdot (\lambda_{h,\beta}(P_\varepsilon) - \overline{\lambda}_{h,\beta}(P)) \\ & + o\left(\left|\left(\lambda_{h,\beta}(P_\varepsilon) - \overline{\lambda}_{h,\beta}(P)\right)_{h=1,\dots,k}\right|\right). \end{aligned} \tag{26}$$

Now, using (25) yields

$$\begin{aligned} \left|\left(\lambda_{h,\beta}(P_\varepsilon) - \overline{\lambda}_{h,\beta}(P)\right)_{h=1,\dots,k}\right| &= \sqrt{\left(\lambda_{1,\beta}(P_\varepsilon) - \overline{\lambda}_{1,\beta}(P)\right)^2 + \dots + \left(\lambda_{k,\beta}(P_\varepsilon) - \overline{\lambda}_{k,\beta}(P)\right)^2} \\ &= \sqrt{\varepsilon^2 + o(\varepsilon)^2} = \varepsilon + o(\varepsilon) \end{aligned}$$

Plugging the latter in (26) and using again (25) on each term in the sum at the third line we finally get

$$\begin{aligned} & F(\overline{\lambda}_{1,\beta}(P_\varepsilon), \dots, \overline{\lambda}_{k,\beta}(P_\varepsilon)) \\ & \quad \leq F(\overline{\lambda}_{1,\beta}(P), \dots, \overline{\lambda}_{k,\beta}(P)) - \frac{\partial F}{\partial x_1}(\overline{\lambda}_{1,\beta}(P), \dots, \overline{\lambda}_{k,\beta}(P)) \cdot (C\varepsilon) + o(\varepsilon) \\ & \quad = F(\overline{\lambda}_{1,\beta}(P), \dots, \overline{\lambda}_{k,\beta}(P)) - C'\varepsilon + o(\varepsilon) < F(\overline{\lambda}_{1,\beta}(P), \dots, \overline{\lambda}_{k,\beta}(P)). \end{aligned}$$

In particular, if we consider a generalized polygon $P \in \overline{\mathcal{P}_N} \subset \overline{\mathcal{P}_{N+1}}$, the corresponding generalized polygon P' (built as above) gives us a strictly lower value for Problem (12) in $\overline{\mathcal{P}_{N+1}}$, then P cannot be a minimizer in $\overline{\mathcal{P}_{N+1}}$. □

We are now in a position to prove our Main Theorem 2.15.

Proof. Let us start with the measure-constrained problem and consider a minimizing sequence (P_n) for (12). Without loss of generality, we assume that

$$F(\overline{\lambda}_{1,\beta}(P_n), \dots, \overline{\lambda}_{k,\beta}(P_n)) \leq 2 \inf_{P \in \overline{\mathcal{P}_N}, |P| \leq m} F(\overline{\lambda}_{1,\beta}(P), \dots, \overline{\lambda}_{k,\beta}(P)). \tag{27}$$

Let us suppose that the diameters of the polygons P_n are not uniformly bounded, otherwise the existence of a limit polygon $P \neq \emptyset$ is trivial. Since $\mathbb{R}^2 \setminus P_n$ has a uniformly bounded number of connected components, we deduce from Remark 2.7(i) that there exists $P \in \overline{\mathcal{P}_N}$ such that

$$P_n \xrightarrow{H^c_{loc}} P$$

and locally in measure as well. We have two possibilities. If $P \neq \emptyset$ we are done, as $0 < |P| \leq m$ and thus it is a bounded generalized polygon. If $P = \emptyset$, in view of the local convergence in measure, for any $x \in \mathbb{R}^2$ we have

$$\lim_{n \to +\infty} |P_n \cap Q_1(x)| = 0. \tag{28}$$

This limit gives a contradiction against the minimality of (P_n). Indeed, in view of the assumption (14) on F and of the bound (27), the modulus of the k-tuple $(\overline{\lambda}_{1,\beta}(P_n), \dots, \overline{\lambda}_{k,\beta}(P_n))$ is uniformly bounded from above, so, in particular, the largest component $\overline{\lambda}_{k,\beta}(P_n)$ is bounded above by a uniform constant $\Lambda > 0$. Now, taking for every $n \in \mathbb{N}$ an L^2-normalized eigenfunction u_n of $\overline{\lambda}_{k,\beta}(P_n)$, the uniform estimate

$$\int_{P_n} |\nabla u_n|^2 \, dx + \beta \int_{\partial P_n} \left[(u_n^-)^2 + (u_n^+)^2 \right] \, d\mathcal{H}^1 \leq \Lambda,$$

induces the following

$$\int_{P_n} |\nabla u_n|^2 \, dx + \int_{\partial P_n} \left[(u_n^-)^2 + (u_n^+)^2 \right] \, d\mathcal{H}^1 \leq \Lambda \left(1 + \frac{1}{\beta} \right).$$

So, in view of Lemma 2.13, there exists $y \in \mathbb{R}^2$ such that

$$|P_n \cap Q_1(y)| \geq C$$

for some uniform constant $C > 0$, in contradiction with (28).

In conclusion, the limit polygon P is nonempty and bounded; then, the H^c_{loc}-convergence of (P_n) to P is actually H^c-convergence of (P_n) to P. This in turn implies that $H^1(P_n) \to H^1(P)$ in the sense of Mosco, see Theorem 2.11.

To prove that P is a minimizer for (12), let us consider an admissible h-dimensional test space V_n for $\overline{\lambda}_{h,\beta}(P_n)$ such that

$$\max_{w \in V_n} \overline{R}_{P_n,\beta}(w) = \overline{\lambda}_{h,\beta}(P_n).$$

Let us consider an $L^2(P_n)$-orthonormal basis of V_n, say $\{u_1^n, \ldots, u_h^n\}$. In view of Mosco convergence, for every $j = 1, \ldots, h$ there exist $u_j \in H^1(P)$ such that $u_j^n \to u_j$ strongly in $L^2(\mathbb{R}^2)$ and $\nabla u_j^n \rightharpoonup \nabla u_j$ weakly in $L^2(\mathbb{R}^2; \mathbb{R}^2)$ (here we denote with the same symbol the zero extension of the functions outside their domains). Let V be the h-dimensional vector space spanned by $\{u_1, \ldots, u_h\}$ (in view of the L^2-convergence we can suppose the u_j functions linearly independent) and let us consider $v := \sum_{j=1}^h \alpha_j u_j$ such that

$$\overline{R}_{P,\beta}(v) = \max_{w \in V} \overline{R}_{P,\beta}(w).$$

Let us consider $v_n := \sum_{j=1}^h \alpha_j u_j^n \in V_n$ and observe that $v_n \to v$ strongly in $L^2(\mathbb{R}^2)$ and $\nabla v_n \rightharpoonup \nabla v$ weakly in $L^2(\mathbb{R}^2; \mathbb{R}^2)$. Thanks to the continuity of the volume integrals at the denominator and to the lower semicontinuity of the gradient integral and of the boundary integral (see [5, Lemma 19]), we obtain

$$\begin{aligned} \lambda_{k,\beta}(P) &\leq \max_{w \in V} \overline{R}_{P,\beta}(w) = \overline{R}_{P,\beta}(v) \leq \liminf_{n \to +\infty} \overline{R}_{P_n,\beta}(v_n) \\ &\leq \liminf_{n \to +\infty} \max_{w \in V_n} \overline{R}_{P_n\beta}(w) = \liminf_{n \to +\infty} \lambda_{h,\beta}(P_n). \end{aligned}$$

Letting $n \to +\infty$, we obtain that P is a minimizer for Problem (12).

Moreover, P has exactly N sides and $|P| = m$. Indeed, if it had less than N sides, say $N - K$ sides, we could apply K times Lemma 3.6 to obtain a polygon P' with exactly N sides, $|P'| \leq m$ and

$$F(\overline{\lambda}_{1,\beta}(P'), \ldots, \overline{\lambda}_{k,\beta}(P')) < F(\overline{\lambda}_{1,\beta}(P), \ldots, \overline{\lambda}_{k,\beta}(P)),$$

contradicting the minimality of P; the saturation of the constraint is assured thanks to the decreasing monotonicity under dilations (9). We also deduce that no minimizer in $\overline{\mathcal{P}_N}$ can be a minimizer in $\overline{\mathcal{P}_{N+1}}$ and this implies that the sequence of minima (m_N) is strictly decreasing.

Concerning the case of problems (13) and (30) with perimeter constraint, the proof follows the same pattern. It only differs since the generalized perimeter constraint entails a uniform bound on the diameters, so we get the existence of an optimal polygon in a standard way and the limit polygon cannot be empty by Lemma 2.13. □

As a corollary of the results of the section, the following problems

$$\min\left\{F(\lambda_{1,\beta}(P),\dots,\lambda_{k,\beta}(P)) : P\in\mathcal{P}_N, |P|\le m,\ P \text{ convex}\right\}, \tag{29}$$

$$\min\left\{F(\lambda_{1,\beta}(P),\dots,\lambda_{k,\beta}(P)) : P\in\mathcal{P}_N, \mathcal{H}^1(\partial P)\le p,\ P \text{ convex}\right\} \tag{30}$$

admit a solution. Notice that, in view of the convexity hypotheses, it is not necessary to consider also degenerate polygons. An existence result is obtainable as a corollary to Theorem 2.15.

Corollary 3.7. *Problem (29) (respectively (30)) admits a solution with exactly N sides and with maximal measure (respectively with maximal boundary length).*

Proof. The proof is a combination of Theorem 2.15 and of the preservation of the convexity constraint under H^c-convergence. □

We conclude this section with some remarks about the classic problems without any geometric assumptions

$$\min\Big\{F(\lambda_{1,\beta}(P),\dots,\lambda_{k,\beta}(P)) : P \text{ union of simple polygons, } P \text{ has at most } N \text{ sides}, |P|\le m,\Big\} \tag{31}$$

and

$$\min\Big\{F(\lambda_{1,\beta}(P),\dots,\lambda_{k,\beta}(P)) : P \text{ union of simple polygons, } P \text{ has at most } N \text{ sides}, \mathcal{H}^1(\partial P)\le p,\Big\}. \tag{32}$$

Indeed, in these case we are not allowed to apply the direct methods of the calculus of variation. That is the reason why we focused our analysis of their generalized versions (12) and (13), respectively. Nevertheless, an easy appoximation argument shows that (12) and (13) can be seen as a sort of relaxation of (31) and (32).

Proposition 3.8. *For any $P\in\overline{\mathcal{P}_N}$ admissible for Problem (12) (resp. (13)), there exists a sequence of (P_n) admissible polygons for (31) (resp. (32)) such that*

$$\lim_{n\to+\infty} F(\lambda_{1,\beta}(P_n),\dots,\lambda_{k,\beta}(P_n)) = F(\overline{\lambda}_{1,\beta}(P),\dots,\overline{\lambda}_{k,\beta}(P)),$$

$$\lim_{n\to+\infty}|P_n| = |P| \quad \textit{and} \quad \lim_{n\to+\infty}\mathcal{H}^1(P_n) = \widetilde{Per}(P).$$

In particular the infima of problems (31) and (32) respectively coincide with the minima of problems (12) and (13).

Proof. The proof is based on some simple observations. First of all, it is always possible to "detach the cracks", i.e. to split sides with multiplicity two and obtain two sides of multiplicity one; this new sides have Hausdorff distance as small as we wish from the original crack. In fact, if the crack consists of only one segment $[A,B]$ of multiplicity two, two situations can occur. If one vertex, say B, belongs to the interior of $\overline{P}$ then it is sufficient to take as a new endpoint a point $A_\varepsilon\in\partial P$ not lying on the same line as $[A,B]$ and having exactly distance ε from A. The new polygon is obtained replacing one of the two versions of $[A,B]$ with $[A_\varepsilon,B]$. Notice that we increase neither the area (since we are taking a subset of P) nor the generalized perimeter (since in the triangle of vertices A,B,A_ε we replace the two sides $[A,B]$ and $[A_\varepsilon,A]$ with the third side $[A_\varepsilon,B]$). If both vertices A and B do not belong to the interior of $\overline{P}$, we can still replace the side where $[A,B]$ lies without increasing either the volume or the generalized perimeter. In fact, if one of the intersecting sides is a subset of the other, it is sufficient to replace the shortest one with a segment $[A_\varepsilon,B_\varepsilon]$ obtained intersecting P with a parallel line to $[A, B]$ at distance ε from the original segment; otherwise, it is enough to move one vertex, A or B, on another side it belongs and away from the intersection, giving a rotation around the fixed vertex that provides the required detachment. All the possible situations are shown in Figure 5.

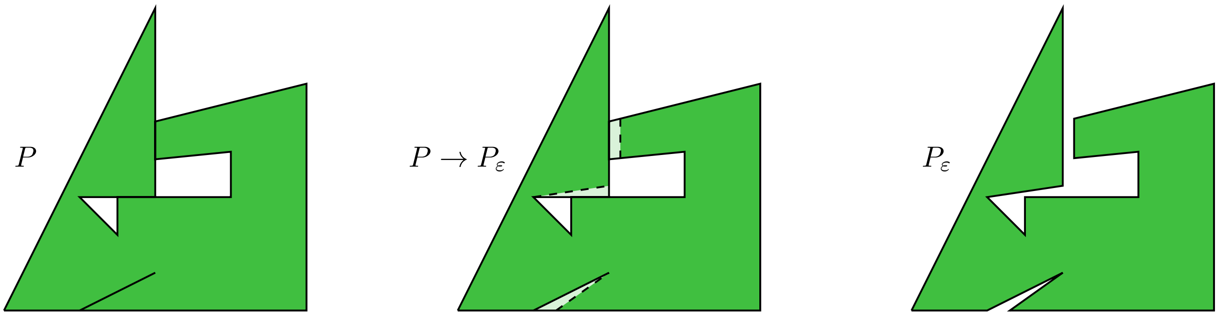


FIGURE 5. Building P_ε starting from P

On the other hand, if a subset of ∂P with multiplicity two consists of more consecutive segments, it is enough to apply the previous argument a finite number of times, starting from the external vertices and detaching one segment per time. Notice that in all cases, we do not increase the total number of sides, see Figure 6.

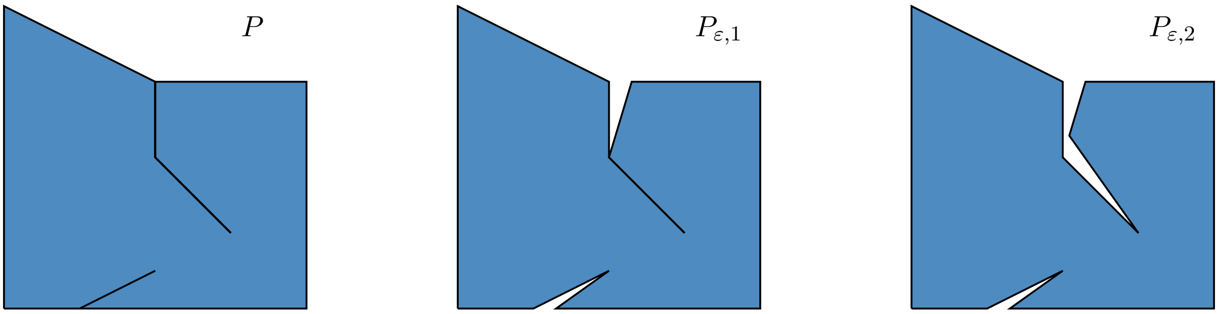


FIGURE 6. When a crack of P consists of more consecutive segments, we apply the procedure several times

Considering now $\varepsilon = 1/n$, we build a sequence of unions of simple polygons with at most N sides such that

$$P_n \xrightarrow{H^c} P, \quad \lim_{n\to+\infty} |P_n| = |P| \quad \text{and} \quad \lim_{n\to+\infty} \mathcal{H}^1(P_n) = \widetilde{Per}(P);$$

The claim is thus an immediate consequence of the continuity of the eigenvalues proved in [11, Theorem 6.1, Theorem 6.2]. □

4. NEGATIVE BOUNDARY PARAMETER

In this section we consider the case $\beta < 0$ and set $\eta := -\beta > 0$. We deal with the problem

$$\begin{cases} \Delta u + \lambda u = 0 & \text{in } \Omega \\ \eta u = \dfrac{\partial u}{\partial \nu} & \text{on } \partial\Omega. \end{cases} \tag{33}$$

The existence result follows the same scheme as in [5], with the main difference that here we also consider admissible shapes that do not saturate the area (or generalized perimeter) constraint. We first show that the maximizing sequences do not collapse to the empty set.

Remark 4.1 (Vanishing is not allowed)**.** *A maximizing sequence cannot vanish, either for Problem* (15) *or for Problem* (16)*. Indeed, if* $|P_n| \to 0$*, then by* (5) *we have*

$$\overline{\lambda}_{1,-\eta}(P_n) \leq -\eta \frac{\widetilde{Per}(P_n)}{|P_n|} \leq -\eta \frac{\mathcal{H}^1(\partial^* P_n)}{|P_n|} \leq -\eta \frac{2\sqrt{\pi}}{\sqrt{|P_n|}} \xrightarrow{n\to+\infty} -\infty,$$

that gives a contradiction with the maximality of the sequence in view of the assumptions on F*.*

We now show that if $\overline{\lambda}_{k,-\eta}(P)$ is not too small then the number of the connected components of P and their diameter are bounded according to the size of $\overline{\lambda}_{k,\beta}(P)$, see [5, Proposition 14].

Proposition 4.2 (A priori bound on diameter and on the number of connected components)**.** *Let* $N \in \mathbb{N}$*,* $P \in \overline{\mathcal{P}_N}$*,* $|P| = m$ *and let* $A > 0$ *be such that* $\overline{\lambda}_{k,\beta}(P) > -A$*. Then* P *is union of* M *equibounded connected components*

$$P = P_1 \cup \ldots \cup P_M,$$

with $M < \frac{mA^2}{4\pi\eta^2} + k$ *and* $\operatorname{diam}(P_j) \leq D(m, \beta, k, A)$*, i.e., the diameters of the connected components are uniformly bounded.*

We are now in a position to prove the main result of the section.

Theorem 4.3 (Existence of a maximal generalized polygon)**.** *Problems* (15) *and* (16) *admit solutions in* $\overline{\mathcal{P}_N}$*. Each optimal polygon* P *is bounded and can be written as the union of at most*

$$\min\left\{\left\lfloor \frac{N-1}{2} \right\rfloor, \frac{mA_*^2}{4\pi\eta^2} + k\right\}$$

equibounded connected components, where $A_* > 0$ *is such that*

$$F(\lambda_{1,-\eta}(E_N), \ldots, \lambda_{k,-\eta}(E_N)) > F(-A_*, \ldots, -A_*)$$

with E_N *the regular* N*-agon saturating the constraint of the problem.*

Proof. Let $(P_n) \subset \overline{\mathcal{P}_N}$ be a maximizing sequence for $F(\overline{\lambda}_{1,-\eta}(\cdot), \ldots, \overline{\lambda}_{k,-\eta}(\cdot))$. In view of the hypotheses on F, we observe that any admissible polygon E such that $\overline{\lambda}_{h,-\eta}(E) \leq -A_*$ for every $h = 1, \ldots, k$ cannot be optimal. Then, it is not restrictive to assume that $\overline{\lambda}_{h,-\eta}(P_n) > -A_*$ for every $h = 1, \ldots, k$. By Proposition 4.2 we have that $\operatorname{diam}(P_n) < D$ for some D independent of n. As a consequence, since we are in the polygonal framework, we also have $\sup_{n\in\mathbb{N}} \mathcal{H}^1(\partial P_n) < +\infty$. Thanks again to Proposition 4.2, we can write P_n as union of M_n equibounded connected components $P_n^1, \ldots, P_n^{M_n}$ as follows:

$$P_n = P_n^1 \cup \ldots \cup P_n^{M_n}, \quad M_n \leq \min\left\{\left\lfloor \frac{N-1}{2} \right\rfloor, \frac{mA_*^2}{4\pi\eta^2} + k\right\}.$$

The bound on M_n is given both by Proposition 4.2 and by the fact that we are dealing with polygons with at most N sides, see Remark 2.7(iii). These facts entail the existence of $P \in \overline{\mathcal{P}_N}$ such that $P_n \xrightarrow{H^c} P$ (up to subsequences).

Now, it remains to prove that $F(\overline{\lambda}_{1,-\eta}(\cdot), \ldots, \overline{\lambda}_{k,-\eta}(\cdot))$ is upper semicontinuous in $\overline{\mathcal{P}_N}$ with respect to the H^c-convergence. This has already been proved in [5, Proposition 18]; we report the highlights of the proof for the convenience of the reader.

Let us show that, for any $h = 1, \ldots, k$, we have

$$\overline{\lambda}_{h,-\eta}(P) \geq \limsup_{n\to+\infty} \overline{\lambda}_{h,-\eta}(P_n); \tag{34}$$

the upper semicontinuity of F in each variable will give the thesis. Let us fix $\varepsilon > 0$ and let S be an admissible vector space in the min-max formula (8) for $\overline{\lambda}_{h,-\eta}(P)$ such that

$$\overline{\lambda}_{h,-\eta}(P) \geq \max_{u\in S\setminus\{0\}} \overline{R}_{P,-\eta}(u) - \varepsilon. \tag{35}$$

Let $\{u_j : j = 1, \dots, h\}$ be an $L^2(P)$-orthonormal basis for S. Then, for every $j = 1, \dots, h$, there exists $v_j^n \in H^1(P_n)$ such that, denoting by the same symbol the extension by zero of a function outside its support, $v_j^n \to u_j$ strongly in $L^2(\mathbb{R}^2)$ and $\nabla v_j^n \to \nabla u_j$ strongly in $L^2(\mathbb{R}^2;\mathbb{R}^2)$. Now, since $\{u_1, \dots, u_h\}$ is $L^2(P)$-orthonormal and $P_n \to P$ in $L^1(\mathbb{R}^2)$, we deduce that, for $n \in \mathbb{N}$ sufficiently large, $\{v_1^n, \dots, v_h^n\}$ can be chosen linearly independent in $L^2(P)$. Let $S_n := \operatorname{span}\{v_1^n, \dots, v_h^n\}$; it is an admissible subspace for the computation of $\overline{\lambda}_{h,-\eta}(P)$. Let

$$v^n = \sum_{j=1}^h \alpha_j^n v_j^n \in S_n$$

realize the maximum for the generalized Rayleigh quotient $\overline{R}$ on S_n:

$$\max_{w\in S_n} \overline{R}(w) = \overline{R}(v^n).$$

Without loss of generality, we can assume

$$\sum_{j=1}^h (\alpha_j^n)^2 = 1.$$

Then, up to subsequences, $\alpha_j^n \to \alpha_j$ in $\mathbb{R}$, with

$$\sum_{j=1}^h (\alpha_j)^2 = 1.$$

Setting

$$v = \sum_{j=1}^h \alpha_j u_j,$$

we have that $v \in S \setminus \{0\}$, $v^n \to v$ strongly in $L^2(\mathbb{R}^2)$ and $\nabla v^n \to \nabla v$ in $L^2(\mathbb{R}^2;\mathbb{R}^2)$. Using (35), the continuity of the volume integrals and the lower semicontinuity of the boundary integral (see Lemma 2.14), we have

$$\begin{aligned}\limsup_{n\to+\infty} \overline{\lambda}_{h,-\eta}(P_n) &\leq \limsup_{n\to+\infty} \sup_{w\in S_n} \overline{R}_{P_n,-\eta}(w) \leq \limsup_{n\to+\infty} \overline{R}_{P,-\eta}(v^n) + \varepsilon \\ &\leq \overline{R}_{P,-\eta}(v) + \varepsilon \leq \max_{u\in S\setminus\{0\}} \overline{R}_{P,-\eta}(u) \leq \overline{\lambda}_{h,-\eta}(P) + \varepsilon.\end{aligned}$$

Letting $\varepsilon \to 0^+$ we obtain (34); this concludes the proof. □

When we deal with the negative boundary parameter case, it has been seen in several situations that the perimeter constraint turns out to be rather natural; see, for instance, [6, 15] where the optimality and the stability of the ball for the first eigenvalue in the convex case is addressed. Even in our framework, the (generalized) perimeter constraint gives some additional properties of the optimal shapes. The following result gives a further property of the solutions of (16), whenever all the eigenvalues involved in the functional are negative. This happens, for instance, if

$$F(x_1, \dots, x_k) = x_1$$

and the k-th eigenvalue of an optimal polygon P is negative. The condition $\overline{\lambda}_{k,-\eta}(P) < 0$ is satisfied if η is sufficiently large, more precisely if $\eta > \overline{\sigma}_k(P)$, were $\overline{\sigma}_k(P)$ is the k-th generalized Steklov eigenvalue of P, see e.g. [8]. Notice that $\overline{\sigma}_1(P) = 0$ with eigenfunction given by the characteristic function of P, and so the strict negativity of the first Robin eigenvalue for all $\eta > 0 = \overline{\sigma}_1(P)$ is coherent with the previous consideration.

Proposition 4.4. *Let $F : \mathbb{R}^k \to \mathbb{R}$ satisfy Hypotheses (16). For every polygon $P \in \overline{\mathcal{P}_N}$ with $\overline{\lambda}_{k,-\eta}(P) < 0$, there exists a polygon $P^\bullet \in \overline{\mathcal{P}_N}$ which is a union of simple polygons, such that $\widetilde{Per}(P^\bullet) \le \widetilde{Per}(P)$ and*

$$F(\overline{\lambda}_{1,\beta}(P^\bullet), \dots, \overline{\lambda}_{k,\beta}(P^\bullet)) \ge F(\overline{\lambda}_{1,\beta}(P), \dots, \overline{\lambda}_{k,\beta}(P)).$$

In particular, every solution P_0 of

$$\max\left\{\overline{\lambda}_{1,-\eta}(P) : P \in \overline{\mathcal{P}_N},\ \widetilde{Per}(P) \le p\right\}$$

is union of simple polygons and thus $\overline{\lambda}_{1,-\eta}(P_0) = \lambda_{1,-\eta}(P_0)$.

Proof. Let P be as in the statement. Let us denote by U the unbounded connected component of $\mathbb{R}^2 \setminus \overline{P}$ and consider

$$P^\bullet := \mathbb{R}^2 \setminus \overline{U}.$$

The set $P^\bullet$ is obtained from P by filling the holes and the fractures and this construction makes $P^\bullet$ a finite union of simple polygons; it is clear that $P^\bullet$ has at most N sides. Moreover, $P^\bullet$ is an admissible polygon for (16) since $\widetilde{Per}(P^\bullet) \le \widetilde{Per}(P)$.

Let us prove that for any $h \in \{1, \dots, k\}$ it holds

$$\overline{\lambda}_{h,-\eta}(P^\bullet) \ge \overline{\lambda}_{h,-\eta}(P). \tag{36}$$

If $\overline{\lambda}_{h,-\eta}(P^\bullet) \ge 0$, then (36) is immediate since $\overline{\lambda}_{h,-\eta}(P) < 0$. Let us assume now that $\overline{\lambda}_{h,-\eta}(P^\bullet) < 0$. Now, we fix $\varepsilon < |\overline{\lambda}_{h,-\eta}(P^\bullet)|$ and consider an h-dimensional subspace $S = \operatorname{span}\{u_1, \dots, u_h\}$ of $H^1(P^\bullet)$ such that

$$\overline{\lambda}_{h,-\eta}(P^\bullet) + \varepsilon \ge \max_{\alpha_1,\dots,\alpha_h \in \mathbb{R}} \overline{R}_{P^\bullet,-\eta}.$$

Clearly, the space generated by the restrictions of $u_1, \dots, u_h$ to P, still denoted by S, is also an h-dimensional subspace of $H^1(P)$, so it is admissible to compute $\overline{\lambda}_{h,-\eta}(P)$. Let us denote by $\overline{\alpha_1}, \dots, \overline{\alpha_h} \in \mathbb{R}$ an h-tuple of coefficients realizing the maximum in S of the Rayleigh quotient relative to P. Notice that, since $P^\bullet \supset P$ and $\partial P \supset \partial P^\bullet$, for any $\varphi \in H^1(P^\bullet)$ it holds

$$\int_{P^\bullet} |\nabla\varphi|^2\, dx \ge \int_P |\nabla\varphi|^2\, dx,\quad \int_{P^\bullet} \varphi^2\, dx \ge \int_P \varphi^2\, dx,$$

$$\int_{\partial P^\bullet} \left[(\varphi^+)^2 + (\varphi^-)^2\right]\, d\mathcal{H}^1 \le \int_{\partial P} \left[(\varphi^+)^2 + (\varphi^-)^2\right]\, d\mathcal{H}^1.$$

Moreover, since the involved Rayleigh quotients are negative, they are monotonically increasing with respect to the volume integrals and monotonically decreasing with respect to the boundary integral. Taking all these considerations into account we get

$$\overline{\lambda}_{h,-\eta}(P^\bullet) + \varepsilon \ge \max_{\alpha_1,\dots,\alpha_h \in \mathbb{R}} \frac{\displaystyle\int_{P^\bullet} \Big|\sum_i \alpha_i \nabla u_i\Big|^2 dx - \eta \int_{\partial P^\bullet} \left[\Big(\sum_i \alpha_i u_i^-\Big)^2 + \Big(\sum_i \alpha_i u_i^+\Big)^2\right] d\mathcal{H}^1}{\displaystyle\int_{P^\bullet} \Big(\sum_i \alpha_i u_i\Big)^2 dx}$$

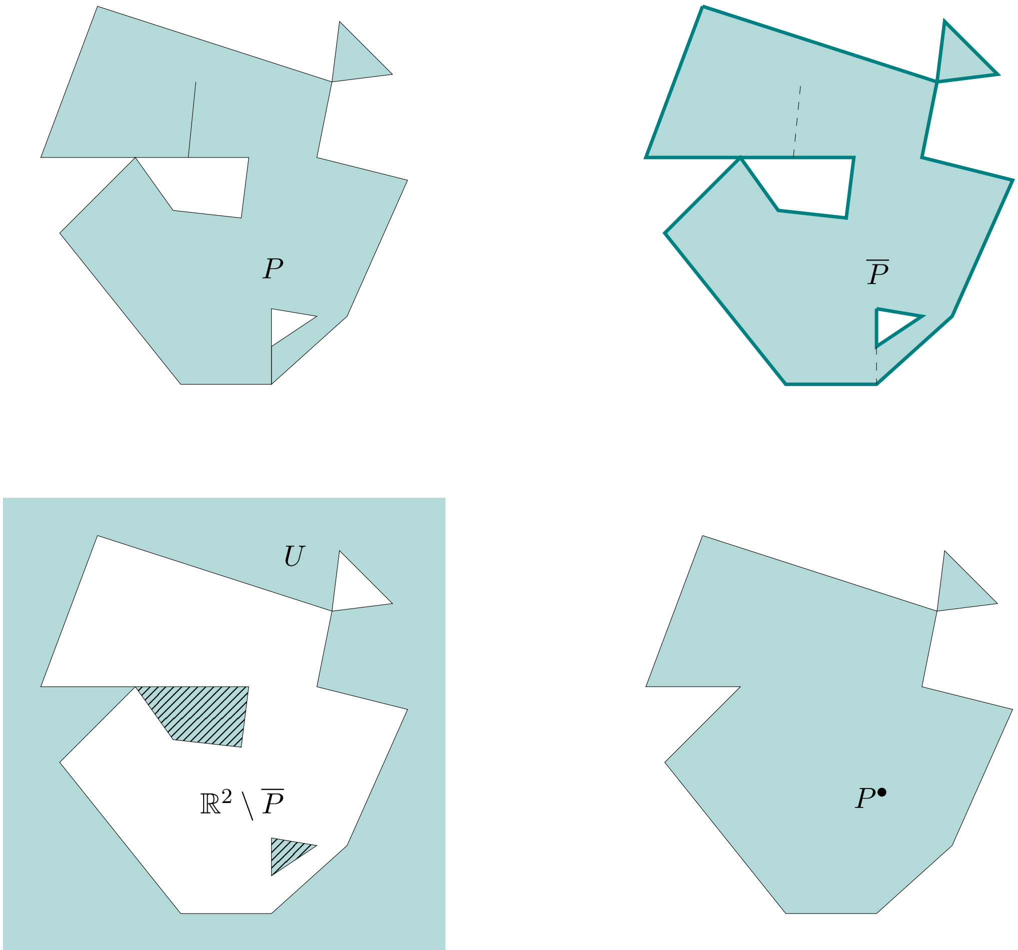


FIGURE 7. Construction of $P^\bullet$

$$\geq \frac{\displaystyle\int_{P^\bullet} \Big|\sum_i \overline{\alpha_i} \nabla u_i\Big|^2 dx - \eta \int_{\partial P^\bullet} \left[\Big(\sum_i \overline{\alpha_i} u_i^-\Big)^2 + \Big(\sum_i \alpha_i u_i^+\Big)^2\right] d\mathcal{H}^1}{\displaystyle\int_{P^\bullet} \Big(\sum_i \overline{\alpha_i} u_i\Big)^2 dx}$$

$$\geq \frac{\displaystyle\int_{P} \Big|\sum_i \overline{\alpha_i} \nabla u_i\Big|^2 dx - \eta \int_{\partial P} \left[\Big(\sum_i \overline{\alpha_i} u_i^-\Big)^2 + \Big(\sum_i \alpha_i u_i^+\Big)^2\right] d\mathcal{H}^1}{\displaystyle\int_{P} \Big(\sum_i \overline{\alpha_i} u_i\Big)^2 dx}$$

$$= \max_{\alpha_1,\dots,\alpha_h \in \mathbb{R}} \frac{\displaystyle\int_{P} \Big|\sum_i \alpha_i \nabla u_i\Big|^2 dx - \eta \int_{\partial P} \left[\Big(\sum_i \alpha_i u_i^-\Big)^2 + \Big(\sum_i \alpha_i u_i^+\Big)^2\right] d\mathcal{H}^1}{\displaystyle\int_{P} \Big(\sum_i \alpha_i u_i\Big)^2 dx}$$

$$\geq \overline{\lambda}_{h,-\eta}(P).$$

In view of the arbitrariness of $\varepsilon > 0$ we get (36) as required.

We conclude the proof in view of the monotonicity of F in each variable. □

5. Final remarks

It is interesting to see how the qualitative properties proved in Sections 3 and 4 are somehow dual. Indeed, for $\beta > 0$, we are able to count the sides of an optimal generalized polygon, but not to show that it is actually a polygon. Unfortunately, removing the possible fractures seems hard in this framework, as we do not have any monotonicity with respect to inclusion, as in the case of Dirichlet boundary conditions. Even with the perimeter constraint we cannot infer anything about the convexity of the minimal polygons, differently to the cases in which a monotonicity holds; see [4] for several examples or [12] for a recent application to a fourth order problem. On the other hand, the case $\beta < 0$ allows to remove fractures or holes but not to count the size of the maximal polygon.

References

[1] B. Bogosel and D. Bucur. On the polygonal Faber-Krahn inequality. *J. Éc. polytech. Math.*, 11:19–105, 2024.

[2] B. Bogosel and D. Bucur. Polygonal faber-krahn inequality: Local minimality via validated computing, 2024.

[3] M.-H. Bossel. Membranes élastiquement liées: extension du théorème de Rayleigh-Faber-Krahn et de l'inégalité de Cheeger. *C. R. Acad. Sci. Paris Sér. I Math.*, 302(1):47–50, 1986.

[4] D. Bucur and G. Buttazzo. *Variational methods in shape optimization problems*. Springer-Progress in Nonlinear Differential Equations and Their Applications, 2004.

[5] D. Bucur and S. Cito. Geometric control of the Robin Laplacian eigenvalues: the case of negative boundary parameter. *J. Geom. Anal.*, 30(4):4356–4385, 2020.

[6] D. Bucur, V. Ferone, C. Nitsch, and C. Trombetti. A sharp estimate for the first Robin-Laplacian eigenvalue with negative boundary parameter. *Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl.*, 30(4):665–676, 2019.

[7] D. Bucur and I. Fragalà. A Faber-Krahn inequality for the Cheeger constant of N-gons. *J. Geom. Anal.*, 26(1):88–117, 2016.

[8] D. Bucur, P. Freitas, and J. Kennedy. The Robin problem. In *Shape optimization and spectral theory*, pages 78–119. De Gruyter Open, Warsaw, 2017.

[9] D. Bucur and A. Giacomini. A variational approach to the isoperimetric inequality for the Robin eigenvalue problem. *Arch. Ration. Mech. Anal.*, 198(3):927–961, 2010.

[10] D. Bucur, A. Giacomini, and M. Nahon. Boundary behavior of robin problems in non-smooth domains. *Preprint available at https://arxiv.org/abs/2206.09771*, page 18, 2022.

[11] D. Bucur, A. Giacomini, and P. Trebeschi. Stability results for the robin-laplacian on nonsmooth domains. *SIAM Journal on Mathematical Analysis*, 54(4):4591–4624, 2022.

[12] M. Carriero, S. Cito, and A. Leaci. Minimization of the buckling load of a clamped plate with perimeter constraint. *Applied Mathematics & Optimization*, 89(1):25, 2024.

[13] S. Cito. Existence and Regularity of Optimal Convex Shapes for Functionals Involving the Robin Eigenvalues. *Journal of Convex Analysis*, 26(3):925–943, 2019.

[14] S. Cito and A. Giacomini. Minimization of the k-th eigenvalue of the Robin-Laplacian with perimeter constraint. *Calc. Var. Partial Differential Equations*, 63(9):Paper No. 244, 38, 2024.

[15] S. Cito and D. A. La Manna. A quantitative reverse Faber-Krahn inequality for the first Robin eigenvalue with negative boundary parameter. *ESAIM: Control, Optimisation and Calculus of Variations*, 27:S23, 2021.

[16] D. Daners. A Faber-Krahn inequality for Robin problems in any space dimension. *Math. Ann.*, 335(4):767–785, 2006.

[17] P. Freitas and J. B. Kennedy. Extremal domains and Pólya-type inequalities for the Robin Laplacian on rectangles and unions of rectangles. *Int. Math. Res. Not. IMRN*, (18):13730–13782, 2021.

[18] P. Grisvard. *Elliptic problems in nonsmooth domains*, volume 24 of *Monographs and Studies in Mathematics*. Pitman (Advanced Publishing Program), Boston, MA, 1985.

[19] A. Henrot. *Extremum problems for eigenvalues of elliptic operators*. Frontiers in Mathematics. Birkhäuser Verlag, Basel, 2006.

[20] A. Henrot and M. Pierre. *Shape variation and optimization*, volume 28 of *EMS Tracts in Mathematics*. European Mathematical Society (EMS), Zürich, 2018.

[21] D. Krejčiřík, V. Lotoreichik, and T. Vu. Reverse isoperimetric inequality for the lowest Robin eigenvalue of a triangle. *Appl. Math. Optim.*, 88(2):Paper No. 63, 33, 2023.

[22] R. S. Laugesen. The Robin Laplacian—Spectral conjectures, rectangular theorems. *J. Math. Phys.*, 60(12):121507, 31, 2019.

[23] B. J. McCartin. *Laplacian eigenstructure of the equilateral triangle*. Hikari Ltd., Ruse, 2011.

[24] Z. Mghazli. Regularity of an elliptic problem with mixed Dirichlet-Robin boundary conditions in a polygonal domain. *Calcolo*, 29(3-4):241–267, 1992.

[25] G. Pólya and G. Szegö. *Isoperimetric Inequalities in Mathematical Physics*, volume No. 27 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 1951.

[26] M. van den Berg, D. Bucur, and K. Gittins. Maximising neumann eigenvalues on rectangles. *Bulletin of the London Mathematical Society*, 48(5):877–894, 2016.

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