GRADIENT REGULARITY FOR DOUBLE-PHASE ORTHOTROPIC FUNCTIONALS

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Abstract. We prove higher integrability for local minimizers of the double-phase orthotropic functional

$$\sum_{i=1}^{n} \int_{\Omega} (|u_{x_i}|^p + a(x)\,|u_{x_i}|^q)\,dx$$

when the weight function $a \geq 0$ is assumed to be α-Hölder continuous, while the exponents p, q are such that $2 \leq p \leq q$ and $\frac{q}{p} < 1 + \frac{\alpha}{n}$. Under natural Sobolev regularity of a, we further obtain explicit Lipschitz regularity estimates for local minimizers.

1. Introduction

The present paper deals with the gradient regularity for local minimizers of the functional

$$\mathcal{F}_0[u, \Omega] = \sum_{i=1}^{n} \int_{\Omega} (|u_{x_i}|^p + a(x)\,|u_{x_i}|^q)\,dx \tag{1}$$

for $u \in W^{1,p}(\Omega)$, $\Omega \subset \mathbb{R}^n$ a bounded open set, $2 \leq p \leq q$, and $a\colon \Omega \to [0, +\infty)$ a suitable continuous function. The energy (1) features both an *orthotropic structure* and a *double phase*

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structure and couples in a nontrivial way the difficulties coming from the two situations. In the simplest case, an orthotropic functional is of the form

$$\int_\Omega G(\nabla u)\,dx \qquad \text{with } G(z) := \frac{1}{p}\sum_{i=1}^{n} |z_i|^p\,. \tag{2}$$

This kind of functional first appeared in the framework of Optimal Transport with congestion effects [10, 11]. Key feature of (2) is the lack of ellipticity. By a direct computation we indeed get that

$$\left\langle D^2G(z)\xi,\xi\right\rangle = (p-1)\sum_{i=1}^{n} |z_i|^{p-2}|\xi_i|^2 \qquad \text{for every } z,\xi\in\mathbb{R}^n.$$

Hence, G presents a degenerate behavior whenever one component of z is zero, which happens on the union of n hyperplanes. For this reason, the regularity techniques for elliptic integrands are not applicable in the orthotropic setting, and the higher regularity for local minimizers of (2) is not fully understood. For example, the C^1 regularity has been tackled only in dimension $n = 2$ in [4, 5, 23, 29]. In order to avoid restrictions on the dimension, we focus here on Lipschitz regularity for local minimizers of the orthotropic double-phase functional (1), which has been studied in the pure orthotropic framework of (2) in [6, 7, 8, 9, 31]. We further refer to [16] for an alternative approach based on viscosity solutions and to [12, 13, 30] for higher differentiability results.

The second feature of the functional $\mathcal{F}_0$ is a double phase structure. The model case of a double phase energy is

$$\int_\Omega |\nabla u|^p + a(x)|\nabla u|^q\,dx \tag{3}$$

with $1 < p < q < \infty$ and $a(x) \geq 0$ almost everywhere. This functional pertains to the class of variational problems with nonstandard growth conditions, following the terminology of Marcellini in [25, 24]. These energies have been introduced by Zhikov in [32, 33, 34, 35] for modeling composite materials characterized by a point-dependent anisotropic behavior, ruled by the weight $a(\cdot)$ which may change point-by-point the rate of ellipticity of the integrand.

To obtain meaningful regularity results, suitable assumptions on p, q and $a(\cdot)$ are needed: in particular, p and q should not be too far apart, in order to avoid the Lavrentiev phenomenon [3, 19, 34], while a should be sufficiently regular. It was first shown in [14] that ∇u is locally Hölder continuous if the weight a is in $C^{0,\alpha}(\Omega)$ for some $\alpha \in (0, 1]$ and $\frac{q}{p} < 1 + \frac{\alpha}{n}$. This result has been then improved to include the borderline case in [1, 15, 21]. We refer the interested reader to the recent survey [26] and references therein concerning recent advances in regularity theory for minimizers of functionals under nonstandard growth.

In the attempt of understanding the role played by non-autonomous coefficients in the regularity theory for orthotropic functionals, it is rather natural to start from the double-phase type of energy introduced in (1).

Our first result is a higher integrability statement, which is valid assuming Hölder regularity for $a\colon \Omega \to [0, +\infty)$:

$$a \in C^{0,\alpha}(\Omega), \quad \alpha \in (0, 1], \tag{4}$$

as well as the bound $\frac{q}{p} < 1 + \frac{\alpha}{n}$. Before stating the result, we briefly recall the definition of local minimizers for $\mathcal{F}_0$.

Definition 1.1. A function $u \in W^{1,p}(\Omega)$ is a local minimizer of the functional defined in (1) if $\mathcal{F}_0[u,\Omega] < +\infty$ and

$$\mathcal{F}_0[u,\Omega'] \leq \mathcal{F}_0[\varphi,\Omega'] \qquad \text{for every } u-\varphi \in W_0^{1,p}(\Omega') \text{ and every } \Omega' \Subset \Omega.$$

Theorem 1.1 (Higher integrability)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, the nonnegative function $a(\cdot)$ satisfy* (4)*, and $q \geq p \geq 2$ such that $\dfrac{q}{p} < 1 + \dfrac{\alpha}{n}$. Let $U \in W^{1,p}(\Omega)$ be a local minimizer of* (1)*. Then, for every $\tilde{q} < np/(n-2\alpha)$ there exists an exponent $\tilde{b} = \tilde{b}(n,p,q,\alpha)$ and a constant $c = c(n,p,q,a,\tilde{q}, diam(\Omega))$ such that for every ball $B_R \Subset \Omega$ it holds*

$$\left(\fint_{B_{R/2}} |\nabla U|_p^{\tilde{q}}\, dx \right)^{1/\tilde{q}} \leq c \left(\fint_{B_R} (|\nabla U|_p^p + 1)\, dx \right)^{\tilde{b}/p}. \tag{5}$$

In particular, (5) *holds for $\tilde{q} = q$ and $U \in W^{1,q}_{loc}(\Omega)$.*

At this stage, the orthotropic effect is rather mild, as higher integrability is obtained by a refined difference quotient technique (cf. [14, Theorem 1.5]) which only relies on the Euler-Lagrange equation of the functional, and thus is not influenced by the degenerate behavior of (1). The key step is to interpret the Hölder continuity of $a(\cdot)$ in terms of fractional differentiability, which then allows the use of embeddings in Nikol'skiĭ spaces. We refer to Section 3 for the details of the proof.

To guarantee Lipschitz continuity of local minimizers we are going to assume that $a(\cdot)$ belongs to a Sobolev function with a suitable high degree of integrability of ∇a, as it was done for instance in [2, 15, 17, 18]. More precisely, we consider

$$a \geq 0, \;\; a \in \begin{cases} W^{1,\frac{n}{1-\alpha}}(\Omega) & \text{if } \alpha \in (0,1) \\ W^{1,\infty}(\Omega) & \text{if } \alpha = 1. \end{cases} \tag{6}$$

Let us note that if Ω has Lipschitz boundary the embedding $W^{1,\frac{n}{1-\alpha}}(\Omega) \hookrightarrow C^{0,\alpha}(\overline{\Omega})$ holds.

Theorem 1.2 (Lipschitz regularity)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, the function $a(\cdot)$ satisfy* (6)*, and $q \geq p \geq 2$ such that $\dfrac{q}{p} < 1 + \dfrac{\alpha}{n}$. Then a local minimizer $U \in W^{1,p}(\Omega)$ of* (1) *is locally Lipschitz. In particular, there exist a constant C and an exponent Θ such that for every $B_{R/3} \subset B_R \Subset \Omega$ the estimate*

$$\|\nabla U\|_{L^\infty(B_{R/3})} \leq C \left(\int_{B_R} |\nabla U|^p + 1 \right)^{\Theta/p} \tag{7}$$

holds. The constant C depends on n,p,q,a,R while Θ on n,p,q,α.

As usual when dealing with higher order regularity, the initial difficulty lies in the fact that the minimizer U lacks the smoothness required to carry out all necessary computations directly. To overcome this issue, in Section 2.2 we approximate the local minimizer U by solutions u_ε of uniformly elliptic problems. We will prove in a direct way the absence of Lavrentiev phenomenon implied by $\frac{q}{p} < 1 + \frac{\alpha}{n}$. This ensures that the approximating solutions u_ε are as smooth as needed and converge to the original minimizer U. Therefore, it suffices to establish suitable a priori estimates for u_ε that remain stable as ε goes to 0.

The proof strategy builds upon the following steps: *(i)* differentiation of the Euler-Lagrange equation associated with the approximating problems; *(ii)* derivation of Caccioppoli-type inequalities for convex powers of the gradient components (see Section 4.1); *(iii)* construction of an iterative scheme based on reverse Hölder inequalities (cf. Propositions 4.1 and 4.2 and Corollary 4.3); and finally *(iv)* iteration to obtain the desired local L^∞ estimate on ∇u_ε (see Proposition 4.4). Despite appearing to be a rather standard work plan in regularity theory, the implementation of the above steps is quite delicate due to the degeneracy of the equation under consideration. In fact, we adopt the same general approach as in [9, Theorem 1.1], but the adaptation is far from straightforward. Our method ultimately yields an $L^\infty - L^m$ estimate for ∇u_ε, achieved after an infinite iteration process. One must carefully address a subtle issue characteristic of nonstandard growth problems: the exponent m in this preliminary estimate may turn out to be excessively large. Fortunately, this estimate can be "corrected" using again the reverse Hölder inequality together with an interpolation argument that effectively reduces the initial integrability requirement on ∇u_ε. This last step is contained in Proposition 5.1.

Outlook. In this work we have considered the restrictions $2 \leq p \leq q$ and $\frac{q}{p} < 1 + \frac{\alpha}{n}$. Future investigations will focus on the full range of exponents $1 < p \leq q$. Let us point out that in the pure orthotropic framework, Lipschitz regularity with subquadratic growth was considered in [8]. A further issue is the borderline case $\frac{q}{p} = 1 + \frac{\alpha}{n}$ (see [1, 15] in the double phase setting), where we lack the embedding in Nikol'skiĭ spaces and the final interpolation argument. Finally, in dimension $n = 2$ it would be of interest to extend the C^1-regularity results of [4, 5, 23, 29] to the present setting.

2. Preliminaries

In this Section we fix a bounded open set $\Omega \subset \mathbb{R}^n$, $n \geq 2$. For $1 \leq \gamma < \infty$ and $z \in \mathbb{R}^n$ denote $|z|_\gamma = \left(\sum_{i=1}^k |z_i|^\gamma\right)^{1/\gamma}$ and consider the functional

$$\mathcal{F}_0[u;\omega] = \int_\omega \left(|\nabla u(x)|_p^p + a(x)|\nabla u(x)|_q^q\right) dx, \tag{8}$$

for $u \in W^{1,p}(\Omega)$ and $\omega \subseteq \Omega$ open. We also denote $\mathcal{F}_0[u] = \mathcal{F}_0[u, \Omega]$.

The inequality

$$\sum_{i=1}^m |x_i|^\gamma \leq \left(\sum_{i=1}^m |x_i|\right)^\gamma \leq m^{\gamma-1} \sum_{i=1}^m |x_i|^\gamma \quad \forall x_1, \dots, x_m \in \mathbb{R}^k \tag{9}$$

holds for $\gamma \geq 1$ and $k, m \in \mathbb{N}$.

Given two functions $f, g\colon \mathbb{R}^n \to \mathbb{R}$, we write $f \approx g$ if there exists $c \geq 1$ such that $c^{-1} g(x) \leq f(x) \leq c\, g(x)$ for every $x \in \mathbb{R}^n$. Similarly the symbol $\lesssim$ stands for $\leq$ up to a constant.

For $\gamma > 1$, $z \in \mathbb{R}^n$, and $i = 1, \dots, n$ we define $W_\gamma^i(z) := |z_i|^{(\gamma-2)/2} z_i$ and $W_\gamma(z) := (W_\gamma^1(z), \dots, W_\gamma^i(z), \dots, W_\gamma^n(z))$. The following inequalities hold:

$$|W_\gamma(z) - W_\gamma(z')|^2 \leq c \sum_{i=1}^n \left(|z_i|^{\gamma-2} z_i - |z_i'|^{\gamma-2} z_i'\right)(z_i - z_i'), \tag{10}$$

$$|W_\gamma(z) - W_\gamma(z')| \approx \sum_{i=1}^n (|z_i| + |z_i'|)^{(\gamma-2)/2} |z_i - z_i'|\,, \tag{11}$$

for a constant $c>0$ only depending on n and on γ. In particular, (11) holds componentwise (cf. [20]):

$$|W^i_\gamma(z)-W^i_\gamma(z')|\approx(|z_i|+|z'_i|)^{(\gamma-2)/2}|z_i-z'_i|. \tag{12}$$

We also introduce the function $V_\gamma\colon\mathbb{R}^n\to\mathbb{R}^n$ as $V_\gamma(z)=|z|^{(\gamma-2)/2}z$. Then, V_γ satisfies the relations

$$|V_\gamma(z)-V_\gamma(z')|^2\le c\langle|z|^{\gamma-2}z-|z'|^{\gamma-2}z',z-z'\rangle\,, \tag{13}$$

$$|V_\gamma(z)-V_\gamma(z')|\approx(|z|+|z'|)^{(\gamma-2)/2}|z-z'|\,, \tag{14}$$

for every $z,z'\in\mathbb{R}^n$, for a positive constant c only depending on n and on γ.

In what follows c or C will be often general positive constants, possibly varying from line to line, but depending on only the structural parameters n,p,q, on the function a in terms of $\alpha,[a]_{0,\alpha},\|a\|_{L^\infty(\Omega)},\|\nabla a\|_{W^{1,\frac{n}{1-\alpha}}(\Omega)}$ and on the energy controlled by the coercivity of the functional, that is $\|\nabla u\|_{L^p(\Omega)}$. The dependences on some of $\alpha,[a]_{0,\alpha},\|a\|_{L^\infty(\Omega)},\|\nabla a\|_{W^{1,\frac{n}{1-\alpha}}(\Omega)}$ or all of these, and on $\|\nabla u\|_{L^p(\Omega)}$ will be simply denoted with a, and u, respectively.

2.1. **Auxiliary lemmas.** The following technical result is classical in the Regularity Theory. We state it here for the reader's convenience (see [20, Lemma 6.1]).

Lemma 2.1. *Let $0<r\le R<1$ and let $Z\colon y\in[r,R]\mapsto Z(y)\in[0,+\infty)$ be a bounded function. Assume that there exist $\mathcal{A},\mathcal{B},\mathcal{C}\ge0$, $\alpha_0\ge\beta_0>0$ and $\theta\in[0,1)$ such that*

$$Z(s)\le\frac{\mathcal{A}}{(t-s)^{\alpha_0}}+\frac{\mathcal{B}}{(t-s)^{\beta_0}}+\mathcal{C}+\theta Z(t)$$

holds for all s,t such that $r\le s<t\le R$. Then, for any $\lambda\in\left(\theta^{\frac{1}{\alpha_0}},1\right)$ it holds

$$Z(r)\le\frac{1}{(1-\lambda)^{\alpha_0}}\frac{\lambda^{\alpha_0}}{\lambda^{\alpha_0}-\theta}\left(\frac{\mathcal{A}}{(R-r)^{\alpha_0}}+\frac{\mathcal{B}}{(R-r)^{\beta_0}}+\mathcal{C}\right).$$

2.2. **Approximation.** We will use an approximation scheme. Let $U\in W^{1,p}(\Omega)$ be a local minimizer of $\mathcal{F}_0$. We fix a ball

$$B_R\Subset\Omega\quad\text{ such that }\quad 2B_R\Subset\Omega\text{ as well.}$$

Here by $\lambda\,B_R$ we denote the ball concentric with B_R, scaled by a factor $\lambda>0$. We set

$$\varepsilon_0=\min\left\{1,\frac{R}{2}\right\}>0.$$

For every $0<\varepsilon\le\varepsilon_0$, we denote by ϱ_ε a standard mollifier, supported in a ball of radius ε centered at the origin. For every $x\in\overline{B}_R$, we then define

$$U^\varepsilon(x)=U*\varrho_\varepsilon(x).$$

We notice that

$$|\nabla U^\varepsilon|(x)\le\|\nabla U\|_{L^p(\Omega)}C\varepsilon^{-\frac{n}{p}}=C_1\varepsilon^{-\frac{n}{p}}. \tag{15}$$

For $v\in W^{1,p}(\Omega)$, and $\omega\subseteq\Omega$ we further define the approximate functional

$$\mathcal{F}_\varepsilon[v,\omega]:=\mathcal{F}_0[v,\omega]+\frac{\varepsilon}{1+\|\nabla U^\varepsilon\|^q_{L^q(B_R)}}\int_\omega|\nabla v(x)|^q\,dx. \tag{16}$$

In the sequel we shall use the following lemma based on some arguments due to Zhikov [34] (see also [19, Lemma 13]).

Lemma 2.2. *Let $a\colon \Omega \to [0,+\infty)$ be a function satisfying* (4) *and let $1 < p \le q < +\infty$, be such that $\frac{q}{p} \le 1 + \frac{\alpha}{n}$. Then*

$$\lim_{\varepsilon \to 0} \mathcal{F}_0[U^\varepsilon, B_R] = \mathcal{F}_0[U, B_R]. \tag{17}$$

Proof. Define

$$\begin{aligned}
&F(x,z) = |\xi|_p^p + a(x)|z|_q^q,\\
&a_\varepsilon(x) = \inf\{a(y) : y \in B_{R+\varepsilon}, |x-y| \le \varepsilon\},\\
&F_\varepsilon(x,z) = |\xi|_p^p + a_\varepsilon(x)|z|_q^q.
\end{aligned}$$

Then, by the Hölder continuity of $a(x)$, $a(x) \ge a_\varepsilon(x) \ge a(x) - \varepsilon^\alpha [a]_{0,\alpha}$. Moreover, by the definition of a_ε we easily deduce that

$$F_\varepsilon(x,z) \le F(y,z) \quad \forall x, y \in B_{R+\varepsilon}, |x-y| \le \varepsilon. \tag{18}$$

On the other hand, given $\delta \in (0,1)$, if $|z| \le C_1 \varepsilon^{-\frac{n}{p}}$, we have

$$\begin{aligned}
F_\varepsilon(x,z) &= \delta F(x,z) + \delta(a_\varepsilon(x) - a(x))|z|_q^q + (1-\delta)F_\varepsilon(x,z)\\
&\ge \delta F(x,z) - \delta\varepsilon^\alpha [a]_{0,\alpha}|z|_q^q + (1-\delta)F_\varepsilon(x,z)\\
&\ge \delta F(x,z) - \delta\varepsilon^\alpha [a]_{0,\alpha} C_1^{q-p} \varepsilon^{-\frac{n}{p}(q-p)}|z|_p^p + (1-\delta)|z|_p^p\\
&\ge \delta F(x,z) + (1 - \delta - \delta[a]_{0,\alpha} C_1^{q-p})|z|_p^p.
\end{aligned}$$

Let us remark that the last estimate relies on the fact that $\frac{q}{p} \le 1 + \frac{\alpha}{n}$. Therefore, choosing δ small enough, we infer that

$$F(x,z) \le \frac{1}{\delta} F_\varepsilon(x,z) \qquad \text{if } |z| \le C_1 \varepsilon^{-\frac{n}{p}}. \tag{19}$$

The convexity of the function $z \mapsto F_\varepsilon(\cdot, z)$ allows to apply Jensen's inequality in connection with the fact that U^ε is defined by a convolution. This gives, together with (18),

$$\begin{aligned}
F_\varepsilon(x, \nabla U^\varepsilon(x)) &\le \int_{B_{R+\varepsilon}} F_\varepsilon(x, \nabla U(y))\rho_\varepsilon(x-y)\,dy\\
&\le \int_{B_{R+\varepsilon}} F(y, \nabla U(y))\rho_\varepsilon(x-y)\,dy = F(\cdot, \nabla U) * \rho_\varepsilon(x).
\end{aligned} \tag{20}$$

Then using (15) and (19)

$$F(x, \nabla U^\varepsilon(x)) \le \frac{1}{\delta} F(\cdot, \nabla U) * \rho_\varepsilon(x).$$

Recalling that $F(\cdot, \nabla U) * \rho_\varepsilon(x) \to F(x, \nabla U(x))$ strongly in $L^1(B_R)$, the assertion easily follows by recalling that $U^\varepsilon \to U$ in $W^{1,p}(B_R)$, using a well-known variant of Lebesgue's dominated convergence theorem. $\square$

Proposition 2.3. *For every $\varepsilon \in (0, \varepsilon_0]$, there exists a unique solution u^ε to*

$$\min\left\{\mathcal{F}_\varepsilon[v, B_R] :\, v \in U^\varepsilon + W_0^{1,p}(B_R)\right\}. \tag{21}$$

Moreover, this minimizer satisfies the estimate

$$\int_{B_R} |\nabla u^\varepsilon|^p \, dx \le C \tag{22}$$

for some $C = C(n, p, q, a, U)$.

Proof. The existence of u^ε follows from a simple application of the direct method, while uniqueness follows by strict convexity. By minimality, we have $\mathcal{F}_\varepsilon[u^\varepsilon, B_R] \le \mathcal{F}_\varepsilon[U^\varepsilon, B_R]$, so that

$$\begin{aligned}\int_{B_R} |\nabla u^\varepsilon|^p \, dx &\le n^{(p-2)/2}\, \mathcal{F}_\varepsilon[u^\varepsilon, B_R] \\ &\le n^{(p-2)/2}\, \mathcal{F}_0[U^\varepsilon, B_R] + n^{(p-2)/2} \frac{\varepsilon}{1 + \|\nabla U^\varepsilon\|^q_{L^q(B_R)}} \int_{B_R} |\nabla U^\varepsilon(x)|^q \, dx \\ &\le n^{(p-2)/2}\, \mathcal{F}_0[U^\varepsilon, B_R] + n^{(p-2)/2}\varepsilon \le C,\end{aligned}$$

where the uniform bound follows by Lemma 2.2. □

Proposition 2.4. *For $\varepsilon \in (0, \varepsilon_0]$, let u^ε be as in Proposition 2.3. Then, $u^\varepsilon \in W^{1,\infty}_{loc}(B) \cap W^{2,2}_{loc}(B)$.*

Proof. This follows by standard regularity theory (see for instance [22], [28]), since the functionals appearing in (16) have standard polynomial growth of order q.
□

Proposition 2.5. *With the same notation as above we have*

$$\lim_{\varepsilon \to 0} \|u^\varepsilon - U\|_{W^{1,p}(B_R)} = 0. \tag{23}$$

Proof. By using the uniform estimate (22) and the definition of U^ε we have that there exists a sequence $\{\varepsilon_k\}_{k \in \mathbb{N}}$, $0 < \varepsilon_k \le \varepsilon_0$, such that $\varepsilon_k \to 0$ and $u^{\varepsilon_k} - U^{\varepsilon_k}$ converges to some $\tilde{u} \in W^{1,p}_0(B_R)$ weakly in $W^{1,p}_0(B_R)$ and almost everywhere. By recalling that U^{ε_k} has been constructed by convolution, we also have that it converges strongly in $W^{1,p}(B_R)$ and almost everywhere to U. This permits to conclude that u^{ε_k} converges weakly and almost everywhere to $u := \tilde{u} + U$. Since U^ε is admissible for problem (21), we have that

$$\liminf_{k\to\infty} \mathcal{F}_{\varepsilon_k}[U^{\varepsilon_k}, B_R] \ge \liminf_{k\to\infty} \mathcal{F}_{\varepsilon_k}[u^{\varepsilon_k}, B_R] \ge \liminf_{k\to\infty} \mathcal{F}_0[u^{\varepsilon_k}, B_R] \ge \mathcal{F}_0[u, B_R] \tag{24}$$

by lower semicontinuity of $\mathcal{F}_0$. Next, we notice that by Lemma 2.2

$$\begin{aligned}\lim_{k\to\infty} |\mathcal{F}_{\varepsilon_k}[U^{\varepsilon_k}, B_R] - \mathcal{F}_0[U, B_R]| &\le \lim_{k\to\infty} |\mathcal{F}_0[U^{\varepsilon_k}, B_R] - \mathcal{F}_0[U, B_R]| \\ &\quad + \lim_{k\to\infty} \frac{\varepsilon_k}{1 + \|\nabla U^{\varepsilon_k}\|^q_{L^q(B_R)}} \int_{B_R} |\nabla U^{\varepsilon_k}|^q \, dx = 0\end{aligned} \tag{25}$$

and thus

$$\mathcal{F}_0[U, B_R] = \lim_{k\to\infty} \mathcal{F}_{\varepsilon_k}[U^{\varepsilon_k}, B_R] \ge \mathcal{F}_0[u, B_R]\,.$$

By the strict convexity of the functional, the minimizer must be unique and thus we get $u = U$, as desired. In order to prove (23) we can adapt the argument of [8, Lemma 2.6]. By (24), and taking into account (25) and the fact that $u = U$, we get

$$\lim_{k\to\infty}\sum_{i=1}^n \int_{B_R} \left(\left|u^{\varepsilon_k}_{x_i}\right|^p + a(x)\left|u^{\varepsilon_k}_{x_i}\right|^q\right) dx = \sum_{i=1}^n \int_{B_R} \left(|U_{x_i}|^p + a(x)\,|U_{x_i}|^q\right) dx. \tag{26}$$

For every $i = 1, \ldots, n$, we rely on the lower semicontinuity of the L^p norm to get

$$\liminf_{k\to\infty} \int_{B_R} \left|u^{\varepsilon_k}_{x_i}\right|^p dx \geq \int_{B_R} |U_{x_i}|^p\, dx,$$

as well as

$$\liminf_{k\to\infty}\sum_{i=1}^n \int_{B_R} a(x)\left|u^{\varepsilon_k}_{x_i}\right|^q dx \geq \sum_{i=1}^n \int_{B_R} a(x)\,|U_{x_i}|^q\, dx.$$

In connection with (26), this implies that

$$\lim_{k\to\infty} \int_{B_R} \left|u^{\varepsilon_k}_{x_i}\right|^p dx = \int_{B_R} |U_{x_i}|^p\, dx.$$

The convergence of the norms, combined with the weak convergence, permits to infer that $u^{\varepsilon_k}_{x_i}$ converges to U_{x_i} in $L^p(B_R)$ for every $i = 1, \ldots, n$. Finally, we observe that we can repeat this argument with any subsequence of the original family $\{u^\varepsilon\}_{0<\varepsilon\leq\varepsilon_0}$. Thus the above limit holds true for the whole family $\{u^\varepsilon\}_{0<\varepsilon\leq\varepsilon_0}$ instead of $\{u^{\varepsilon_k}\}_{k\in\mathbb{N}}$ and (23) follows.

□

3. Higher integrability

In this section, we establish Theorem 1.1. We assume throughout the section that $2 \leq p \leq q < +\infty$, the function $a(\cdot)$ satisfies (4) and $\frac{q}{p} < 1 + \frac{\alpha}{n}$.

Proof of Theorem 1.1. We adapt the proof of Theorem 5.1 in [14] to our setting.

We claim that if $B_R = B_R(x_0) \Subset \Omega$ then for every $\beta \in (n(q/p-1), \alpha)$ there exists $c = c(n, p, q, \beta) > 0$ such that

$$\left(\fint_{B_{R/2}} |\nabla U|_p^{\frac{np}{n-2\beta}}\, dx\right)^{\frac{n-2\beta}{n}} \leq c \fint_{B_R} |\nabla U|_p^p\, dx + \left(\|a\|^2_{L^\infty(B_R)} + R^{2\alpha}[a]^2_{0,\alpha}\right)^{b_1} \left(\fint_{B_R} |\nabla U|_p^p\right)^{b_2} \tag{27}$$

with

$$b_1 = \frac{\beta p}{\beta p - n(q-p)} \geq 1, \quad b_2 = \frac{\beta(2q-p) - n(q-p)}{\beta p - n(q-p)} \geq 1. \tag{28}$$

In particular, (27) implies (5) with an appropriate choice of β.

Under the transformations

$$\tilde{U}(x) = \frac{U(x_0 + Rx) - (U)_{B_R}}{R}, \quad \tilde{a}(x) = a(x_0 + Rx) \tag{29}$$

with $(U)_{B_R}$ the integral average of U on B_R, we have that $\tilde{U} \in W^{1,p}(B)$ is a minimizer of

$$\widetilde{\mathcal{F}}_0[\tilde{v}] = \int_B |\nabla\tilde{v}|_p^p + \tilde{a}(x)|\nabla\tilde{v}|_q^q\, dx,$$

where we have set $B = B_1$ the ball of center 0 and radius 1. We notice that inequality (27) becomes

$$\left\| |\nabla \tilde{U}|_p \right\|^p_{L^{np/(n-2\beta)}(B_{1/2})} \le c \left\| |\nabla \tilde{U}|_p \right\|^p_{L^p(B)} + c \left(\|\tilde{a}\|^2_{L^\infty(B)} + [\tilde{a}]^2_{0,\alpha} \right)^{b_1} \left\| |\nabla \tilde{U}|_p \right\|^{pb_2}_{L^p(B)} . \tag{30}$$

Next steps aim at proving inequality (30).

Step 1. We are in position to apply Lemma 2.2 and to consider a sequence $\{u_m\}_{m\in\mathbb{N}} \subset W^{1,\infty}(B)$ such that

$$u^m \to \tilde{U} \text{ in } W^{1,p}(B), \quad \widetilde{\mathcal{F}}_0[u^m] \to \widetilde{\mathcal{F}}_0[\tilde{U}]$$

where

$$\widetilde{\mathcal{F}}_0[v] := \int_B (|\nabla v|_p^p + \tilde{a}|\nabla v|_q^q)\, dx \qquad \text{for } v \in W^{1,p}(B).$$

For $m \in \mathbb{N}$, $x \in B$, and $z \in \mathbb{R}^n$ we now denote

$$\begin{cases} \sigma_m := \left(1 + m + \|\nabla u^m\|^{2(2q-p)}_{L^{2q-p}(B)}\right)^{-1} \\ F(x,z) = |z|_p^p + \tilde{a}(x)|z|_q^q \\ F_m(x,z) = F(x,z) + \sigma_m |z|^{2q-p}. \end{cases}$$

The functional

$$\mathfrak{F}_m[w] = \int_B F_m(x, \nabla w)\, dx \qquad \text{for } w \in W^{1,p}(B)$$

behaves like a standard functional of growth $2q-p$, and therefore admits a unique minimizer $v^m \in u^m + W^{1,2q-p}(B)$, which by standard techniques belongs to $W^{1,\infty}_{loc}(B)$ (see for instance [22]).

Step 2. We prove that for $\beta \in (0,\alpha)$ there exists a constant $c = c(n,p,q,\beta) > 0$ such that for every $m \in \mathbb{N}$

$$\||\nabla v^m|_p\|^p_{L^{np/(n-2\beta)}(B_{1/2})} \le c\||\nabla v^m|_p\|^p_{L^p(B)} + c\left(\|\tilde{a}\|_{L^\infty(B)} + [\tilde{a}]_{0,\alpha} + \sigma_m\right) \||\nabla v^m|_p\|^{2q-p}_{L^{2q-p}(B)}. \tag{31}$$

Consider the Euler-Lagrange equation for $\mathfrak{F}_m$:

$$\int_B \langle \nabla_z F_m(x, \nabla v_m), \nabla\varphi \rangle\, dx = 0 \qquad \text{for every } \varphi \in W^{1,2q-p}_0(B). \tag{32}$$

Let $\eta \in C_0^\infty(B)$ be such that $0 \le \eta \le 1$, $\eta \equiv 0$ outside $B_{3/4}$, $\eta \equiv 1$ on $B_{2/3}$ and $|\nabla\eta|^2 + |D^2\eta| \le M$, for some M. For $h \in \mathbb{R}^n$ sufficiently small, e.g., $|h| < 10^{-4}$, we test (32) with $\varphi = \tau_{-h}(\eta^2 \tau_h v^m)$, where $\tau_h v(x) := v(x+h) - v(x)$ for $x \in B$. We obtain

$$\begin{aligned} 0 &= \int_B \langle \nabla_z F_m(x, \nabla v^m), \nabla\left(\tau_{-h}(\eta^2 \tau_h v^m)\right)\rangle\, dx \\ &= \int_B \langle \nabla_z F_m(x, \nabla v^m), \tau_{-h}\nabla\left(\eta^2 \tau_h v^m\right)\rangle\, dx \\ &= \int_B \langle \nabla_z F_m(x, \nabla v^m), \nabla\left(\eta^2 \tau_h v^m\right)(x-h)\rangle\, dx \\ &\quad - \int_B \langle \nabla_z F_m(x, \nabla v^m), \nabla\left(\eta^2 \tau_h v^m\right)(x)\rangle\, dx \\ &= \int_B \langle \nabla_z F_m(x+h, \nabla v^m(x+h)), \nabla\left(\eta^2 \tau_h v^m\right)(x)\rangle\, dx \end{aligned}$$

$$
\begin{aligned}
&-\int_B \langle \nabla_z F_m(x, \nabla v^m(x)), \nabla\left(\eta^2 \tau_h v^m\right)(x)\rangle\, dx\\
&=\int_B \langle \tau_h\big(\nabla_z F_m(\cdot, \nabla v^m)\big), \nabla(\eta^2 \tau_h v^m)\rangle\, dx
\end{aligned}
$$

We write $\tau_h\big(\nabla_z F_m(\cdot, \nabla v^m)\big) = A_1 + A_2$ where, for every $z \in \mathbb{R}^n$ and $x \in B$,

$$
\begin{aligned}
\langle A_1(x), z\rangle &= \langle \tau_h\big[\nabla_z F_m(x+h, \nabla v^m(\cdot))\big](x), z\rangle\\
&= p\sum_{i=1}^n \left(|v^m_{x_i}(x+h)|^{p-2} v^m_{x_i}(x+h) - |v^m_{x_i}(x)|^{p-2} v^m_{x_i}(x)\right) z_i\\
&+ \tilde{a}(x+h) q \sum_{i=1}^n \left(|v^m_{x_i}(x+h)|^{q-2} v^m_{x_i}(x+h) - |v^m_{x_i}(x)|^{q-2} v^m_{x_i}(x)\right) z_i\\
&+ \sigma_m(2q-p)\sum_{i=1}^n \left(|\nabla v^m(x+h)|^{2q-p-2} v^m_{x_i}(x+h) - |\nabla v^m(x)|^{2q-p-2} v^m_{x_i}(x)\right) z_i
\end{aligned}
$$

and

$$
\begin{aligned}
\langle A_2(x), z\rangle &= \langle \tau_h\big[\nabla_z F_m(\cdot, \nabla v^m(x))\big](x), z\rangle\\
&= (\tilde{a}(x+h) - \tilde{a}(x)) q \sum_{i=1}^n |v^m_{x_i}(x)|^{q-2} v^m_{x_i}(x) z_i.
\end{aligned}
$$

Thus, we can write

$$
\begin{aligned}
I_0 &:= \int_B \eta^2 \langle A_1, \tau_h \nabla v^m\rangle\, dx\\
&= -\int_B \eta^2\langle A_2, \tau_h\nabla v^m\rangle\, dx - 2\int_B \eta\langle A_2, \nabla\eta\rangle \tau_h v^m\, dx - 2\int_B \eta\langle A_1, \nabla\eta\rangle\tau_h v^m\, dx\\
&=: I_1 + I_2 + I_3.
\end{aligned}
\tag{33}
$$

We estimate I_0 by using (10) and (13):

$$
\begin{aligned}
I_0 &\gtrsim \int_B (\eta^2|\tau_h W_p(\nabla v^m)|^2 + \tilde{a}(x+h)\eta^2|\tau_h W_q(\nabla v^m)|^2 + \sigma_m\eta^2|\tau_h V_{2q-p}(\nabla v^m)|^2)\, dx\\
&\geq \int_B (\eta^2|\tau_h W_p(\nabla v^m)|^2 + \sigma_m\eta^2|\tau_h V_{2q-p}(\nabla v^m)|^2)\, dx.
\end{aligned}
\tag{34}
$$

We estimate I_1 using the Hölder continuity of $\tilde{a}$:

$$
\begin{aligned}
|I_1| &\lesssim [\tilde{a}]_{0,\alpha}|h|^\alpha \int_B \eta^2 \sum_{i=1}^n |v^m_{x_i}(x)|^{q-1}|\tau_h v^m_{x_i}(x)|\, dx\\
&\leq [\tilde{a}]_{0,\alpha}|h|^\alpha \int_B \eta^2 \sum_{i=1}^n (|v^m_{x_i}(x+h)| + |v^m_{x_i}(x)|)^{q-1}|\tau_h v^m_{x_i}(x)|\, dx.
\end{aligned}
$$

Writing $q-1=(q-\frac{p}{2})+(\frac{p}{2}-1)$, applying a weighted Young's inequality, inequality (11), and the fact that $\eta\equiv 0$ outside of $B_{3/4}$, for $\delta>0$ we obtain

$$\begin{aligned}|I_1| &\lesssim \delta\int_B \eta^2\sum_{i=1}^n(|v^m_{x_i}(x+h)|+|v^m_{x_i}(x)|)^{p-2}|\tau_h v^m_{x_i}(x)|^2\,dx\\ &+\frac{[\tilde a]^2_{0,\alpha}|h|^{2\alpha}}{\delta}\int_{B_{3/4}}\sum_{i=1}^n(|v^m_{x_i}(x+h)|+|v^m_{x_i}(x)|)^{2q-p}\,dx\\ &\lesssim \delta\int_B\eta^2|\tau_h W_p(\nabla v^m)|^2\,dx+\frac{[\tilde a]^2_{0,\alpha}|h|^{2\alpha}}{\delta}\int_B|\nabla v^m|^{2q-p}\,dx.\end{aligned}\tag{35}$$

Using the fact that $|V_{2q-p}(\nabla v^m)|^2=|\nabla v^m|^{2q-p}$ we have

$$|I_1|\lesssim\delta I_0+\frac{[\tilde a]^2_{0,\alpha}|h|^{2\alpha}}{\delta}\int_B|V_{2q-p}(\nabla v^m)|^2\,dx.$$

To estimate I_2 we recall that for a function $u\in W^{1,\gamma}(B_R)$, $\gamma\ge 1$, and $B_\rho\subset B_R$ a concentric ball, the following basic property of finite differences holds

$$\int_{B_\rho}|\tau_h u|^\gamma\,dx\le|h|^\gamma\int_{B_R}|\nabla u|^\gamma\,dx,$$

for $|h|\le R-\rho$. Hence, by Young and Hölder inequality we estimate I_2 as

$$\begin{aligned}|I_2| &\lesssim [\tilde a]_{0,\alpha}|h|^\alpha\int_{B_{3/4}}|\nabla v^m|^{q-1}|\tau_h v^m||\nabla\eta|\,dx\\ &\lesssim |h|^\alpha\left(\int_{B_{3/4}}|\nabla v^m|^{p-1}|\tau_h v^m|\,dx+[\tilde a]^2_{0,\alpha}\int_{B_{3/4}}|\nabla v^m|^{2q-p-1}|\tau_h v^m|\,dx\right)\\ &\lesssim |h|^\alpha\left(\int_B|\nabla v^m|^p\,dx\right)^{\frac{p-1}{p}}\left(\int_{B_{3/4}}|\tau_h v^m|^p\right)^{\frac1p}\\ &+[\tilde a]^2_{0,\alpha}|h|^\alpha\left(\int_B|\nabla v^m|^{2q-p}\right)^{\frac{2q-p-1}{2q-p}}\left(\int_{B_{3/4}}|\tau_h v^m|^{2q-p}\right)^{\frac{1}{2q-p}}\\ &\lesssim |h|^{2\alpha}\left(\int_B|\nabla v^m|^p+[\tilde a]^2_{0,\alpha}|\nabla v^m|^{2q-p}\,dx\right)\\ &\lesssim |h|^{2\alpha}\left(\int_B|\nabla v^m|^p_p+[\tilde a]^2_{0,\alpha}|\nabla v^m|^{2q-p}\,dx\right)\end{aligned}$$

where we used the fact that $|h|^{1+\alpha}\le|h|^{2\alpha}$.

We are now left with I_3. Since $p\ge 2$, we have

$$|I_3|\lesssim\|\nabla\eta\|_{L^\infty(B)}(I_{3,p}+\|\tilde a\|_{L^\infty(B)}I_{3,q}+\sigma_m J_{3,2q-p})\tag{36}$$

where

$$I_{3,\gamma}:=\int_B\eta|\tau_h v^m|\sum_{i=1}^n(|v^m_{x_i}(x+h)|+|v^m_{x_i}(x)|)^{\gamma-2}|\tau_h v^m_{x_i}|\,dx$$

for $\gamma = p, q$, and

$$J_{3,\gamma} := \int_B \eta(|\nabla v^m(x+h)| + |\nabla v^m(x)|)^{\gamma-2}|\tau_h \nabla v^m||\tau_h v^m|\,dx$$

for $\gamma = 2q-p$. Let us estimate these terms. Using Young's inequality as in (35) and recalling (11) we have

$$\begin{aligned}\|\tilde{a}\|_{L^\infty(B)} I_{3,q} &\lesssim \delta \int_{B_{3/4}} \eta^2 \sum_{i=1}^n (|v^m_{x_i}(x+h)| + |v^m_{x_i}(x)|)^{p-2}|\tau_h v^m_{x_i}|^2\,dx \\ &\qquad + \frac{\|\tilde{a}\|^2_{L^\infty(B)}}{\delta}\int_{B_{3/4}} |\tau_h v^m|^2 \sum_{i=1}^n (|v^m_{x_i}(x+h)| + |v^m_{x_i}(x)|)^{2q-p-2}\,dx \\ &\lesssim \delta \int_B \eta^2 |\tau_h W_p(\nabla v_m)|^2\,dx \\ &\qquad + \frac{\|\tilde{a}\|^2_{L^\infty(B)}}{\delta}\int_{B_{3/4}} |\tau_h v^m|^2 \sum_{i=1}^n (|v^m_{x_i}(x+h)| + |v^m_{x_i}(x)|)^{2q-p-2}\,dx.\end{aligned} \tag{37}$$

By Hölder inequality we further estimate

$$\begin{aligned}&\int_{B_{3/4}} |\tau_h v^m|^2 \sum_{i=1}^n (|v^m_{x_i}(x+h)| + |v^m_{x_i}(x)|)^{2q-p-2}\,dx \\ &\qquad\lesssim \int_{B_{3/4}} |\tau_h v^m|^2 (|\nabla v^m(x+h)| + |\nabla v^m(x)|)^{2q-p-2}\,dx \\ &\qquad\lesssim \|\nabla v^m\|^{2q-p-2}_{L^{2q-p}(B)} \|\tau_h v^m\|^2_{L^{2q-p}(B)} \le |h|^2 \|\nabla v^m\|^{2q-p}_{L^{2q-p}(B)}.\end{aligned} \tag{38}$$

We deduce from (37) and (38) that

$$\|\tilde{a}\|_{L^\infty(B)} I_{3,q} \lesssim \delta I_0 + \frac{|h|^2 \|\tilde{a}\|^2_{L^\infty(B)}}{\delta} \int_B |\nabla v^m|^{2q-p}\,dx. \tag{39}$$

Arguing in a similar way we conclude that

$$I_{3,p} + \sigma_m J_{3,2q-p} \lesssim \delta I_0 + \frac{|h|^2}{\delta}\int_B |\nabla v^m|_p^p\,dx + \frac{|h|^2 \sigma_m}{\delta}\int_B |\nabla v^m|^{2q-p}\,dx. \tag{40}$$

Combining the inequalities (33)–(40), using (9) and the relation $|h|^2 \le |h|^{2\alpha}$ we get

$$I_0 \lesssim \delta I_0 + \frac{|h|^{2\alpha}}{\delta}\int_B |\nabla v^m|_p^p\,dx + \frac{|h|^{2\alpha}}{\delta} T_m \int_B |\nabla v^m|^{2q-p}_{2q-p}\,dx,$$

where

$$T_m := \|\tilde{a}\|^2_{L^\infty(B)} + [\tilde{a}]^2_{0,\alpha} + \sigma_m\,.$$

Therefore, choosing $\delta > 0$ small enough, (34) yields

$$\int_B \eta^2 |\tau_h W_p(\nabla v^m)|^2\,dx \le c|h|^{2\alpha}\left(\int_B |\nabla v^m|_p^p\,dx + T_m \int_B |\nabla v^m|^{2q-p}_{2q-p}\,dx\right). \tag{41}$$

Inequality (41) implies that $W_p(\nabla v_m)$ belongs to the Nikol'skiĭ space $\mathcal{N}^{\alpha,2}(B_{2/3})$ and the validity of the embedding $\mathcal{N}^{\alpha,2} \hookrightarrow W^{\beta,2} \cap L^{2n/(n-2\beta)}$ for any $\beta \in (0,\alpha)$ (see [27]) implies

$$\|W_p(\nabla v^m)\|^2_{L^{\frac{2n}{n-2\beta}}(B_{1/2})} \le c\left(\||\nabla v^m|_p\|^p_{L^p(B)} + \|W_p(\nabla v^m)\|^2_{L^2(B)} + T_m\,\||\nabla v^m|_p\|^{2q-p}_{L^{2q-p}(B)}\right) \quad (42)$$

for some constant $c = c(n,p,q,\beta) > 0$. Using the equality $|W_p(z)|^2 = |z|_p^p$ and (9), (42) becomes

$$\||\nabla v^m|_p\|^p_{L^{\frac{np}{n-2\beta}}(B_{1/2})} \le c\left(\||\nabla v^m|_p\|^p_{L^p(B)} + T_m\,\||\nabla v^m|_p\|^{2q-p}_{L^{2q-p}(B)}\right), \quad (43)$$

which is precisely (31).

Step 3. We claim that, for a fixed $\beta \in (0,\alpha)$, there exists a constant $c_* = c_*(n,p,q,\beta) > 0$ such that for every $1/2 \le t < s \le 1$ the inequality

$$\||\nabla v^m|_p\|^p_{L^{\frac{np}{n-2\beta}}(B_t)} \le c_*\left(\frac{\||\nabla v^m|_p\|^p_{L^p(B_s)}}{(s-t)^{2\beta}} + \frac{T_m\,\||\nabla v^m|_p\|^{2q-p}_{L^{2q-p}(B_s)}}{(s-t)^{2\beta}}\right) \quad (44)$$

holds. To this aim, let us fix s,t and let $r := (s-t)/8$; we consider a covering of B_t with balls $\{B_{r/2}(y_i)\}_{i\in I}$ with $y_i \in B_t$ for every i, where $\#I \le c(n)r^{-n}$ for some dimensional constant $c(n) > 0$. We can assume that this family, as well as the family $\{B_r(y_i)\}_{i\in I}$, satisfy the finite intersection property: each ball of the family intersects at most $c'(n)$ other balls belonging to the family. Using again the rescaling argument (29), we get

$$\||\nabla v^m|_p\|^p_{L^{\frac{np}{n-2\beta}}(B_{r/2}(y_i))} \le c\left(\frac{\||\nabla v^m|_p\|^p_{L^p(B_r(y_i))}}{r^{2\beta}} + \frac{T_m\,\||\nabla v^m|_p\|^{2q-p}_{L^{2q-p}(B_r(y_i))}}{r^{2\beta}}\right).$$

for a positive constant $c = c(n,p,q,\beta)$. Summing over every ball of the family, using the finite intersection property and (9) we get (44).

Step 4. We now want to prove that, for every $\beta \in \left(n\left(\frac{q}{p}-1\right),\alpha\right)$, there exists a constant $c = c(n,p,q,\beta) > 0$ such that

$$\||\nabla v^m|_p\|^p_{L^{\frac{np}{n-2\beta}}(B_{1/2})} \le c\,\||\nabla v^m|_p\|^p_{L^p(B)} + c\,T_m^{b_1}\,\||\nabla v^m|_p\|^{pb_2}_{L^p(B)} \quad (45)$$

with b_1, b_2 as in (28).

Assume $q > p$, since if $q = p$ the previous inequality becomes (43). Let us fix $\theta \in (0,1)$ such that

$$\frac{1}{2q-p} = \frac{1-\theta}{p} + \frac{\theta(n-2\beta)}{np} \implies \theta = \frac{n(q-p)}{\beta(2q-p)}.$$

By interpolation inequality, we obtain

$$\||\nabla v^m|_p\|_{L^{2q-p}(B_s)} \le \||\nabla v^m|_p\|^{1-\theta}_{L^p(B_s)}\,\||\nabla v^m|_p\|^{\theta}_{L^{np/(n-2\beta)}(B_s)}$$

Using Young's inequality with exponents $\frac{p}{\theta(2q-p)}$ and $\frac{p}{p-(2q-p)\theta}$ we infer that, for every $\delta \in (0,1)$,

$$\begin{aligned}\frac{T_m\,\||\nabla v^m|_p\|^{2q-p}_{L^{2q-p}(B_s)}}{(s-t)^{2\beta}} &\le \frac{T_m}{(s-t)^{2\beta}}\,\||\nabla v^m|_p\|^{(1-\theta)(2q-p)}_{L^p(B_s)}\,\||\nabla v^m|_p\|^{\theta(2q-p)}_{L^{\frac{np}{n-2\beta}}(B_s)} \\ &\le \delta\,\||\nabla v^m|_p\|^p_{L^{\frac{np}{n-2\beta}}(B_s)} + c_\delta\left(\frac{T_m}{(s-t)^{2\beta}}\,\||\nabla v^m|_p\|^{p\frac{b_2}{b_1}}_{L^p(B_s)}\right)^{b_1}.\end{aligned}$$

for some positive constant c_δ depending on δ, n, p, q, β. We substitute the above inequality in (44) with the choice $\delta = \frac{1}{2c_*}$ and obtain

$$\||\nabla v^m|_p\|^p_{L^{\frac{np}{n-2\beta}}(B_t)} \leq \frac{1}{2}\||\nabla v^m|_p\|^p_{L^{\frac{np}{n-2\beta}}(B_s)} + c\left(\frac{\||\nabla v^m|_p\|^p_{L^p(B)}}{(s-t)^{2\beta}} + \frac{T_m^{b_1}\||\nabla v^m|_p\|^{pb_2}_{L^p(B)}}{(s-t)^{2\beta b_1}}\right)$$

for a positive constant $c = c(n,p,q,\beta)$. Using Lemma 2.1 with $r = 1/2$ and $R = 1-\rho$ for some $\rho \in (0,1/2)$ and letting $\rho \to 0$ we conclude the proof of (45).

Step 5. We argue as in the proofs of Proposition 2.3 and Proposition 2.5. By the minimality of v^m we have

$$\begin{aligned}\int_B |\nabla v^m|_p^p\,dx \leq \int_B F_m(x,\nabla v^m)\,dx &\leq \int_B F_m(x,\nabla u^m)\,dx \\ &\leq \widetilde{\mathcal{F}}_0[u^m] + \sigma_m\|\nabla u^m\|^{2q-p}_{L^{2q-p}(B)}\,.\end{aligned}$$

By the definition of u^m and σ_m, the sequence v^m is bounded in $W^{1,p}(B)$ and, up to a subsequence, ∇v^m converges weakly to ∇w for some $w \in W^{1,p}(B)$. Since, by construction, $\sigma_m\|\nabla u_m\|^{2q-p}_{L^{2q-p}(B)} \to 0$, we have

$$\widetilde{\mathcal{F}}_0[w] \leq \liminf_{m\to\infty}\mathfrak{F}_m[v^m] \leq \limsup_{m\to\infty}\mathfrak{F}_m[v^m] \leq \widetilde{\mathcal{F}}_0[\tilde{U}]\,.$$

By the minimality of $\tilde{U}$ and the strict convexity of $\widetilde{\mathcal{F}}_0$ we obtain $w = \tilde{U}$. In particular

$$\widetilde{\mathcal{F}}_0[\tilde{U}] = \lim_{m\to\infty}\mathfrak{F}_m[v^m]$$

and this implies that

$$\lim_{m\to\infty}\int_B |\nabla v^m|_p^p\,dx = \int_B |\nabla\tilde{U}|_p^p\,dx\,.$$

Thus, we can pass to the limit in (45) to obtain (27), concluding the proof of (5).

$\square$

4. Local energy estimates

4.1. **Caccioppoli-type inequalities.** For $\varepsilon \in (0,\varepsilon_0]$ we consider from now on the minimizer u^ε of (21). The corresponding Euler-Lagrange equation reads as

$$\sum_{i=1}^n \int \left(p|u^\varepsilon_{x_i}|^{p-2} + q\sigma_\varepsilon|\nabla u^\varepsilon|^{q-2} + qa|u^\varepsilon_{x_i}|^{q-2}\right)u^\varepsilon_{x_i}\varphi_{x_i}\,dx = 0, \qquad \forall\varphi \in W^{1,q}_0(B_R), \tag{46}$$

where we denoted $\frac{\varepsilon}{1+\|\nabla U^\varepsilon\|^q_{L^q(B_R)}}$ by σ_ε. For every $j = 1,\cdots n$, we can use test functions φ_{x_j}, with $\varphi \in C^2$ compactly supported in B_R, and integrate by parts to obtain

$$\begin{aligned}&\sum_{i=1}^n \int \left[p(p-1)|u^\varepsilon_{x_i}|^{p-2} + q\sigma_\varepsilon|\nabla u^\varepsilon|^{q-2} + q(q-1)a|u^\varepsilon_{x_i}|^{q-2}\right]u^\varepsilon_{x_ix_j}\varphi_{x_i}\,dx \\ &+ q(q-2)\sigma_\varepsilon\int |\nabla u^\varepsilon|^{q-2}\langle\frac{\nabla u^\varepsilon\otimes\nabla u^\varepsilon}{|\nabla u^\varepsilon|^2}\nabla u^\varepsilon_{x_j},\nabla\varphi\rangle\,dx + q\sum_{i=1}^n\int a_{x_j}|u^\varepsilon_{x_i}|^{q-2}u^\varepsilon_{x_i}\varphi_{x_i}\,dx = 0.\end{aligned} \tag{47}$$

From now on, we denote by B the (generic) open ball B_R in $\mathbb{R}^n$. We further use the notation

$$\mathcal{M}_i^\varepsilon(\nabla u^\varepsilon) := p(p-1)|u^\varepsilon_{x_i}|^{p-2} + q\sigma_\varepsilon|\nabla u^\varepsilon|^{q-2} + q(q-1)a|u^\varepsilon_{x_i}|^{q-2} \tag{48}$$

for $i = 1, \dots, n$.

Proposition 4.1. *For $\varepsilon \in (0, \varepsilon_0)$, let u^ε be the minimizer of (21) and assume $a(\cdot)$ satisfies (6). Then, there exists a positive constant $C = C(p, q, n) > 0$ such that for every $\Lambda \in C^1(\mathbb{R})$ monotone non-decreasing and every Lipschitz function η compactly supported in B it holds*

$$\begin{aligned}\sum_{i=1}^n \int \mathcal{M}_i^\varepsilon(\nabla u^\varepsilon)\Lambda'(u^\varepsilon_{x_j})\left(u^\varepsilon_{x_ix_j}\right)^2 \eta^2\,dx &\le C\sum_{i=1}^n \int \mathcal{M}_i^\varepsilon(\nabla u^\varepsilon)\frac{\Lambda^2(u^\varepsilon_{x_j})}{\Lambda'(u^\varepsilon_{x_j})}|\nabla\eta|^2\,dx \\ &\quad + C\sum_{i=1}^n \int a_{x_j}^2|u^\varepsilon_{x_i}|^{2q-p}\Lambda'(u^\varepsilon_{x_j})\eta^2\,dx.\end{aligned} \tag{49}$$

Proof. Let us drop the dependence on ε in u^ε and $\mathcal{M}_i^\varepsilon$. We test equation (47) with test function $\varphi = \Lambda(u_{x_j})\eta^2$ with η as in the statement of the proposition:

$$\begin{aligned}&\sum_{i=1}^n \int \mathcal{M}_i(\nabla u)\Lambda'(u_{x_j})u^2_{x_ix_j}\eta^2\,dx + q(q-2)\sigma_\varepsilon\int|\nabla u|^{q-2}\frac{\langle\nabla u, \nabla u_{x_j}\rangle^2}{|\nabla u|^2}\Lambda'(u_{x_j})\,\eta^2\,dx \qquad (50)\\ &= -2\sum_{i=1}^n\int \mathcal{M}_i(\nabla u)\Lambda(u_{x_j})u_{x_ix_j}\eta\eta_{x_i}\,dx - 2q(q-2)\sigma_\varepsilon\int|\nabla u|^{q-2}\langle\frac{\nabla u\otimes\nabla u}{|\nabla u|^2}\nabla u_{x_j}, \nabla\eta\rangle\eta\Lambda(u_{x_j})\,dx \\ &- q\sum_{i=1}^n\int a_{x_j}|u_{x_i}|^{q-2}u_{x_i}\Lambda'(u_{x_j})u_{x_ix_j}\eta^2\,dx - 2q\sum_{i=1}^n\int a_{x_j}|u_{x_i}|^{q-2}u_{x_i}\Lambda(u_{x_j})\eta\eta_{x_i}\,dx.\end{aligned}$$

We first observe that the second term on left hand side is nonnegative. Using a weighted Young's inequality, we estimate the third term on the right-hand side of (50):

$$\begin{aligned}-\sum_{i=1}^n\int a_{x_j}|u_{x_i}|^{q-2}u_{x_i}u_{x_ix_j}\Lambda'(u_{x_j})\eta^2\,dx &\le \sum_{i=1}^n\int|a_{x_j}||u_{x_i}|^{q-1}|u_{x_ix_j}|\Lambda'(u_{x_j})\eta^2\,dx \\ &\le \delta\sum_{i=1}^n\int p(p-1)|u_{x_i}|^{p-2}u^2_{x_ix_j}\Lambda'(u_{x_j})\eta^2\,dx + C_\delta\sum_{i=1}^n\int a^2_{x_j}|u_{x_i}|^{2q-p}\Lambda'(u_{x_j})\eta^2\,dx \\ &\le \delta\sum_{i=1}^n\int\mathcal{M}_i(\nabla u)u^2_{x_ix_j}\Lambda'(u_{x_j})\eta^2\,dx + C_\delta\sum_{i=1}^n\int a^2_{x_j}|u_{x_i}|^{2q-p}\Lambda'(u_{x_j})\eta^2\,dx.\end{aligned}$$

We treat the first two terms on the right hand side: by Young's inequality, for every $\delta > 0$, we have

$$\begin{aligned}&-2\sum_{i=1}^n\int\mathcal{M}_i(\nabla u)\Lambda(u_{x_j})u_{x_ix_j}\eta\eta_{x_i}\,dx - 2q(q-2)\sigma_\varepsilon\int|\nabla u|^{q-2}\langle\frac{\nabla u\otimes\nabla u}{|\nabla u|^2}\nabla u_{x_j}, \nabla\eta\rangle\eta\Lambda(u_{x_j})\,dx \\ &\qquad\le C\sum_{i=1}^n\int\mathcal{M}_i(\nabla u)|\Lambda(u_{x_j})||u_{x_ix_j}||\eta||\nabla\eta|\,dx \\ &\qquad\le \delta\sum_{i=1}^n\int\mathcal{M}_i(\nabla u)\Lambda'(u_{x_j})u^2_{x_ix_j}\eta^2\,dx + C_\delta\sum_{i=1}^n\int\mathcal{M}_i(\nabla u)\frac{\Lambda^2(u_{x_j})}{\Lambda'(u_{x_j})}|\nabla\eta|^2\,dx.\end{aligned}$$

Similarly we get

$$
\begin{aligned}
&-2q\sum_{i=1}^{n}\int a_{x_j}|u_{x_i}|^{q-2}u_{x_i}\Lambda(u_{x_j})\eta\eta_{x_i}\,dx\\
&\lesssim \sum_{i=1}^{n}\int a_{x_j}^2|u_{x_i}|^{2q-p}\Lambda'(u_{x_j})\eta^2\,dx+\sum_{i=1}^{n}\int p(p-1)|u_{x_i}|^{p-2}\frac{\Lambda^2(u_{x_j})}{\Lambda'(u_{x_j})}\eta_{x_i}^2\,dx\\
&\lesssim \sum_{i=1}^{n}\int a_{x_j}^2|u_{x_i}|^{2q-p}\Lambda'(u_{x_j})\eta^2\,dx+\sum_{i=1}^{n}\int \mathcal{M}_i(\nabla u)\frac{\Lambda^2(u_{x_j})}{\Lambda'(u_{x_j})}\eta_{x_i}^2\,dx.
\end{aligned}
$$

Choosing δ small enough and combining the above inequalities yield the desired inequality. $\square$

As for the standard orthotropic functional a key role is played by the following sophisticated Caccioppoli-type inequality for the gradient, reminiscent of [9, Proposition 3.2].

Proposition 4.2. *For $\varepsilon \in (0,\varepsilon_0]$, let u^ε be the minimizer of* (21) *and assume that $a(\cdot)$ satisfies* (6)*. Then, for every $\theta \in (0,2)$, $C_0 > 0$, every $\Phi \in C^1(\mathbb{R})$ monotone non-decreasing, $\Psi \in C^2(\mathbb{R}^+;\mathbb{R}^+)$ monotone non-decreasing, convex, with $\Psi(0) = 0$, and every Lipschitz function η compactly supported in B, there exists a constant $C = C(n,p,q) > 1$ such that*

$$
\begin{aligned}
&\sum_{i=1}^{n}\int \mathcal{M}_i^\varepsilon(\nabla u^\varepsilon)\Phi'(u^\varepsilon_{x_j})\Psi((u^\varepsilon_{x_k})^2)(u^\varepsilon_{x_ix_j})^2\eta^2dx\\
&\qquad\le C_0\sum_{i=1}^{n}\int \mathcal{M}_i^\varepsilon(\nabla u^\varepsilon)\Phi^2(u^\varepsilon_{x_j})\left(\Psi'((u^\varepsilon_{x_k})^2)\right)^\theta(u^\varepsilon_{x_ix_j})^2\eta^2dx\\
&\qquad\quad+C(1+C_0^{-1})\sum_{i=1}^{n}\int \mathcal{M}_i^\varepsilon(\nabla u^\varepsilon)\left(\Psi((u^\varepsilon_{x_k})^2)\right)^{2-\theta}|u^\varepsilon_{x_k}|^{2\theta}|\nabla\eta|^2dx\\
&\qquad\quad+C\sum_{i=1}^{n}\int \mathcal{M}_i^\varepsilon(\nabla u^\varepsilon)\frac{\Phi^2(u^\varepsilon_{x_j})}{\Phi'(u^\varepsilon_{x_j})}\Psi((u^\varepsilon_{x_k})^2)|\nabla\eta|^2dx\\
&\qquad\quad+C\sum_{i=1}^{n}\int |\nabla a|^2|u^\varepsilon_{x_i}|^{2q-p}\Phi'(u^\varepsilon_{x_j})\Psi((u^\varepsilon_{x_k})^2)\eta^2dx\\
&\qquad\quad+C\sum_{i=1}^{n}\int |\nabla a|^2|u^\varepsilon_{x_i}|^{2q-p}\Phi^2(u^\varepsilon_{x_j})\left(\Psi'((u^\varepsilon_{x_k})^2)\right)^\theta\eta^2dx\\
&\qquad\quad+C(1+C_0^{-1})\sum_{i=1}^{n}\int |\nabla a|^2|u^\varepsilon_{x_i}|^{2q-p}\Upsilon((u^\varepsilon_{x_k})^2)(u^\varepsilon_{x_k})^2\eta^2dx,
\end{aligned}
\tag{51}
$$

where we have set for $t \in [0,+\infty)$

$$
\Upsilon(t)=4\left(\Psi'(t)\right)^{2-\theta}+\left(\Psi(t)\right)^{1-\frac{\theta}{2}}\left(\Psi'(t)\right)^{-\frac{\theta}{2}}|\sqrt{t}|^{\theta-2}\left(2\Psi'(t)+\Psi''(t)t\right). \tag{52}
$$

Proof. Let us drop the index ε from u^ε and $\mathcal{M}_i^\varepsilon$. We test (47) with $\varphi = \Phi(u_{x_j})\Psi(u_{x_k}^2)\eta^2$:

$$\begin{aligned}
&\sum_{i=1}^{n}\int \mathcal{M}_i(\nabla u)\Phi'(u_{x_j})\Psi(u_{x_k}^2)u_{x_ix_j}^2\eta^2\,dx \\
&\quad + q(q-2)\sigma_\varepsilon\int |\nabla u|^{q-2}\frac{\langle \nabla u, \nabla u_{x_j}\rangle^2}{|\nabla u|^2}\Phi'(u_{x_j})\Psi(u_{x_k}^2)\eta^2 dx \\
&\quad = -2\sum_{i=1}^{n}\int \mathcal{M}_i(\nabla u)\Phi(u_{x_j})\Psi'(u_{x_k}^2)u_{x_k}u_{x_ix_j}u_{x_ix_k}\eta^2\,dx \\
&\quad - 2\sum_{i=1}^{n}\int \mathcal{M}_i(\nabla u)\Phi(u_{x_j})\Psi(u_{x_k}^2)u_{x_ix_j}\eta\eta_{x_i}\,dx \\
&\quad - 2q(q-2)\sigma_\varepsilon\int |\nabla u|^{q-2}\langle\frac{\nabla u\otimes\nabla u}{|\nabla u|^2}\nabla u_{x_j}, \nabla u_{x_k}\rangle\Phi(u_{x_j})\Psi'(u_{x_k}^2)u_{x_k}\eta^2 dx \\
&\quad - 2q(q-2)\sigma_\varepsilon\int |\nabla u|^{q-2}\langle\frac{\nabla u\otimes\nabla u}{|\nabla u|^2}\nabla u_{x_j}, \nabla \eta\rangle\Phi(u_{x_j})\Psi(u_{x_k}^2)\eta\,dx \\
&\qquad - q\sum_{i=1}^{n}\int a_{x_j}|u_{x_i}|^{q-2}u_{x_i}\Phi'(u_{x_j})\Psi(u_{x_k}^2)u_{x_ix_j}\eta^2\,dx \\
&\qquad - 2q\sum_{i=1}^{n}\int a_{x_j}|u_{x_i}|^{q-2}u_{x_i}\Phi(u_{x_j})\Psi'(u_{x_k}^2)u_{x_k}u_{x_ix_k}\eta^2\,dx \\
&\qquad - 2q\sum_{i=1}^{n}\int a_{x_j}|u_{x_i}|^{q-2}u_{x_i}\Phi(u_{x_j})\Psi(u_{x_k})\eta\eta_{x_i}\,dx \\
&\quad = (\mathrm{I}) + (\mathrm{II}) + (\mathrm{III}) + (\mathrm{IV}) + (\mathrm{V}) + (\mathrm{VI}) + (\mathrm{VII}).
\end{aligned} \tag{53}$$

Let us estimate these terms. Using Young's inequality we have

$$\begin{aligned}
(\mathrm{II}) + (\mathrm{IV}) \le& 2(q-1)\sum_{i=1}^{n}\int \mathcal{M}_i(\nabla u)|\Phi(u_{x_j})|\Psi(u_{x_k}^2)|u_{x_ix_j}||\eta||\nabla\eta|\,dx \\
&\le \delta\sum_{i=1}^{n}\int \mathcal{M}_i(\nabla u)\Phi'(u_{x_j})\Psi(u_{x_k}^2)u_{x_ix_j}^2\eta^2\,dx \\
&+ \frac{4(q-1)^2}{\delta}\sum_{i=1}^{n}\int \mathcal{M}_i(\nabla u)\frac{\Phi^2(u_{x_j})}{\Phi'(u_{x_j})}\Psi(u_{x_k}^2)|\nabla\eta|^2\,dx,
\end{aligned} \tag{54}$$

for some small $\delta \in (0, 1/10)$. Similarly we have

$$\begin{aligned}
(\mathrm{V}) \le& \ \delta\sum_{i=1}^{n}\int \mathcal{M}_i(\nabla u)\Phi'(u_{x_j})\Psi(u_{x_k}^2)u_{x_ix_j}^2\eta^2\,dx \\
&+ \frac{4q^2}{p(p-1)\delta}\sum_{i=1}^{n}\int a_{x_j}^2|u_{x_i}|^{2q-p}\Phi'(u_{x_j})\Psi(u_{x_k}^2)\eta^2\,dx,
\end{aligned} \tag{55}$$

$$
\begin{aligned}
(\mathrm{VII}) \leq\ & \sum_{i=1}^{n} \int \mathcal{M}_i(\nabla u) \frac{\Phi^2(u_{x_j})}{\Phi'(u_{x_j})} \Psi(u_{x_k}^2) \eta_{x_i}^2 \, dx \\
& + \frac{q^2}{p(p-1)} \sum_{i=1}^{n} \int a_{x_j}^2 |u_{x_i}|^{2q-p} \Phi'(u_{x_j}) \Psi(u_{x_k}^2) \eta^2 \, dx.
\end{aligned} \tag{56}
$$

To deal with the remaining terms, for any $\theta \in (0,2)$ we write $\Psi'(u_{x_k}^2) = \left(\Psi'(u_{x_k}^2)\right)^{\frac{\theta}{2}} \left(\Psi'(u_{x_k}^2)\right)^{1-\frac{\theta}{2}}$ and use Cauchy-Schwartz inequality to obtain

$$
\begin{aligned}
(\mathrm{I}) + (\mathrm{III}) \leq 2(q-1) & \left(\sum_{i=1}^{n} \int \mathcal{M}_i(\nabla u) \Phi^2(u_{x_j}) \left(\Psi'(u_{x_k}^2)\right)^{\theta} u_{x_i x_j}^2 \eta^2 \, dx \right)^{\frac{1}{2}} \\
& \times \left(\sum_{i=1}^{n} \int \mathcal{M}_i(\nabla u) \left(\Psi'(u_{x_k}^2)\right)^{2-\theta} u_{x_k}^2 u_{x_i x_k}^2 \eta^2 \, dx \right)^{\frac{1}{2}},
\end{aligned} \tag{57}
$$

$$
\begin{aligned}
(\mathrm{VI}) \leq \frac{2q}{p(p-1)} & \left(\sum_{i=1}^{n} \int a_{x_j}^2 |u_{x_i}|^{2q-p} \Phi^2(u_{x_j}) \left(\Psi'(u_{x_k}^2)\right)^{\theta} \eta^2 \, dx \right)^{\frac{1}{2}} \\
& \times \left(\sum_{i=1}^{n} \int \mathcal{M}_i(\nabla u) \left(\Psi'(u_{x_k}^2)\right)^{2-\theta} u_{x_k}^2 u_{x_i x_k}^2 \eta^2 \, dx \right)^{\frac{1}{2}}.
\end{aligned} \tag{58}
$$

We set $\Gamma(t) := \int_0^{t^2} \left(\Psi'(\tau)\right)^{1-\frac{\theta}{2}} \, d\tau$ and $\Lambda(t) := \Gamma(t)\Gamma'(t)$. We notice that

$$
\Gamma''(t) = 2(\Psi'(t^2))^{1-\frac{\theta}{2}} + 2t^2(2-\theta)(\Psi'(t^2))^{-\frac{\theta}{2}} \Psi''(t^2) \geq 0, \tag{59}
$$

$$
(\Gamma(t)\Gamma'(t))' = 4t^2(\Psi'(t^2))^{2-\theta} + \Gamma(t)\Gamma''(t) \geq 4t^2(\Psi'(t^2))^{2-\theta}, \tag{60}
$$

$$
\frac{(\Gamma(t)\Gamma'(t))^2}{(\Gamma(t)\Gamma'(t))'} = \frac{(\Gamma(t)\Gamma'(t))^2}{\Gamma(t)\Gamma''(t) + (\Gamma'(t))^2} \leq \frac{(\Gamma(t)\Gamma'(t))^2}{(\Gamma'(t))^2} = \Gamma^2(t). \tag{61}
$$

In particular, $\Lambda \in C^1(\mathbb{R})$ is non-decreasing. Applying Proposition 4.1 and (60)–(61) we obtain

$$
\begin{aligned}
\sum_{i=1}^{n} \int \mathcal{M}_i(\nabla u) \left(\Psi'(u_{x_k}^2)\right)^{2-\theta} u_{x_k}^2 u_{x_i x_k}^2 \eta^2 \, dx & \leq C \sum_{i=1}^{n} \int \mathcal{M}_i(\nabla u) \Gamma^2(u_{x_k}) |\nabla \eta|^2 \, dx \\
+ C \sum_{i=1}^{n} \int a_{x_k}^2 |u_{x_i}|^{2q-p} & \left(4 \left(\Psi'(u_{x_k}^2)\right)^{2-\theta} u_{x_k}^2 + \Gamma(u_{x_k})\Gamma''(u_{x_k}) \right) \eta^2 \, dx,
\end{aligned} \tag{62}
$$

where $C = C(p,q,n) > 0$. Using Jensen inequality, we further estimate

$$
\Gamma(t) = \int_0^{t^2} \left(\Psi'(\tau)\right)^{1-\frac{\theta}{2}} \, d\tau \leq t^2 \left(\frac{1}{t^2} \int_0^{t^2} \Psi'(\tau) \, d\tau \right)^{1-\frac{\theta}{2}} = |t|^{\theta} \Psi^{1-\frac{\theta}{2}}(t^2). \tag{63}
$$

Using the triangle and the weighted Young inequality we infer that for every $x_1, x_2, x_3, x_4 \geq 0$ it holds

$$
(x_1^{1/2} + x_2^{1/2})(x_3 + x_4)^{1/2} \leq (c_1 + c_2)x_1 + (c_3 + c_4)x_2 + \frac{(c_1 + c_3)x_3}{4c_1c_3} + \frac{(c_2 + c_4)x_4}{4c_2c_4}
$$

for arbitrary positive constants c_1, c_2, c_3, c_4. Combining (57)–(63) and properly choosing the constants $\{c_i\}_{i=1}^4$ above, we have, for some $C = C(n,p,q,\delta) \geq 1$,

$$
\begin{aligned}
(\mathrm{I}) + (\mathrm{III}) + (\mathrm{VI}) \leq{} & (1-2\delta) C_0 \sum_{i=1}^n \int \mathcal{M}_i(\nabla u) \Phi^2(u_{x_j}) \left(\Psi'(u_{x_k}^2)\right)^\theta u_{x_i x_j}^2 \eta^2 \, dx \\
& + C \sum_{i=1}^n \int |\nabla a|^2 |u_{x_i}|^{2q-p} \Phi^2(u_{x_j}) \left(\Psi'(u_{x_k}^2)\right)^\theta \eta^2 \, dx \\
& + C(1 + C_0^{-1}) \sum_{i=1}^n \int \mathcal{M}_i(\nabla u) \left(\Psi(u_{x_k}^2)\right)^{2-\theta} |u_{x_k}|^{2\theta} |\nabla \eta|^2 dx \\
& + C(1 + C_0^{-1}) \sum_{i=1}^n \int |\nabla a|^2 |u_{x_i}|^{2q-p} \Upsilon(u_{x_k}^2) u_{x_k}^2 \eta^2 \, dx
\end{aligned}
\tag{64}
$$

where Υ is the function defined in (52). Combining (64) with (53)–(56) and observing that the second term in the left hand-side of (53) is nonnegative, we conclude for (51). $\square$

As in [9, Proposition 4.1], applying Proposition 4.2 to suitable powers, we obtain the following result.

Corollary 4.3. *For $\varepsilon \in (0, \varepsilon_0]$, let u^ε be the minimizer of (21) and assume that $a(\cdot)$ satisfies (6). Then, there exists a positive constant $C = C(n,p,q)$ such that for every integer $m \geq 1$, every $s \in [1, m]$, and every Lipschitz function η compactly supported in B it holds*

$$
\begin{aligned}
& \sum_{i=1}^n \int \mathcal{M}_i^\varepsilon(\nabla u^\varepsilon) |u_{x_j}^\varepsilon|^{2s-2} |u_{x_k}^\varepsilon|^{2m} (u_{x_i x_j}^\varepsilon)^2 \eta^2 dx \\
& \quad \leq \sum_{i=1}^n \int \mathcal{M}_i^\varepsilon(\nabla u^\varepsilon) |u_{x_j}^\varepsilon|^{4s-2} |u_{x_k}^\varepsilon|^{2m-2s} (u_{x_i x_j}^\varepsilon)^2 \eta^2 dx \\
& \qquad + C \sum_{i=1}^n \int \mathcal{M}_i^\varepsilon(\nabla u^\varepsilon) |u_{x_j}^\varepsilon|^{2s+2m} |\nabla \eta|^2 dx \\
& \qquad + Cm \sum_{i=1}^n \int |\nabla a|^2 |u_{x_i}^\varepsilon|^{2q-p} |u_{x_j}^\varepsilon|^{2m+2s-2} \eta^2 dx \\
& \qquad + C(m+1) \sum_{i=1}^n \int \mathcal{M}_i^\varepsilon(\nabla u^\varepsilon) |u_{x_k}^\varepsilon|^{2m+2s} |\nabla \eta|^2 dx \\
& \qquad + C(m^3+1) \sum_{i=1}^n \int |\nabla a|^2 |u_{x_i}^\varepsilon|^{2q-p} |u_{x_k}^\varepsilon|^{2m+2s-2} \eta^2 dx.
\end{aligned}
\tag{65}
$$

Proof. As usual, we drop the index ε from u^ε and $\mathcal{M}_i^\varepsilon$. We apply Proposition 4.2 with $\Phi(t) = \frac{|t|^{2s-2}t}{2s-1}$, $\Psi(t) = t^m$ and

$$
\theta = \begin{cases} \frac{m-s}{m-1} & \text{if } m > 1 \\ 1 & \text{if } m = 1\,. \end{cases}
$$

Hence, (51) becomes

$$\begin{aligned}
&\sum_{i=1}^{n}\int \mathcal{M}_i(\nabla u)|u_{x_j}|^{2s-2}|u_{x_k}|^{2m}u^2_{x_ix_j}\eta^2dx\\
&\quad\le \frac{C_0m^\theta}{(2s-1)^2}\sum_{i=1}^{n}\int \mathcal{M}_i(\nabla u)|u_{x_j}|^{4s-2}|u_{x_k}|^{2m-2s}u^2_{x_ix_j}\eta^2dx\\
&\quad\quad + C(1+C_0^{-1})\sum_{i=1}^{n}\int \mathcal{M}_i(\nabla u)|u_{x_k}|^{2m+2s}|\nabla\eta|^2dx\\
&\quad\quad + \frac{C}{(2s-1)^2}\sum_{i=1}^{n}\int \mathcal{M}_i(\nabla u)|u_{x_j}|^{2s}|u_{x_k}|^{2m}|\nabla\eta|^2dx\\
&\quad\quad + C\sum_{i=1}^{n}\int |\nabla a|^2|u_{x_i}|^{2q-p}|u_{x_j}|^{2s-2}|u_{x_k}|^{2m}\eta^2dx\\
&\quad\quad + \frac{Cm^\theta}{(2s-1)^2}\sum_{i=1}^{n}\int |\nabla a|^2|u_{x_i}|^{2q-p}|u_{x_j}|^{4s-2}|u_{x_k}|^{2m-2s}\eta^2dx\\
&\quad\quad + 7C(1+C_0^{-1})m^2\sum_{i=1}^{n}\int |\nabla a|^2|u_{x_i}|^{2q-p}|u_{x_k}|^{2m+2s-2}\eta^2dx.
\end{aligned}$$

Choosing $C_0 = m^{-1}$ gives, after simplifying some inequalities

$$\begin{aligned}
&\sum_{i=1}^{n}\int \mathcal{M}_i(\nabla u)|u_{x_j}|^{2s-2}|u_{x_k}|^{2m}u^2_{x_ix_j}\eta^2dx\\
&\quad\le \sum_{i=1}^{n}\int \mathcal{M}_i(\nabla u)|u_{x_j}|^{4s-2}|u_{x_k}|^{2m-2s}u^2_{x_ix_j}\eta^2dx\\
&\quad\quad + C(m+1)\sum_{i=1}^{n}\int \mathcal{M}_i(\nabla u)|u_{x_k}|^{2m+2s}|\nabla\eta|^2dx\\
&\quad\quad + C\sum_{i=1}^{n}\int \mathcal{M}_i(\nabla u)|u_{x_j}|^{2s}|u_{x_k}|^{2m}|\nabla\eta|^2dx\\
&\quad\quad + C\sum_{i=1}^{n}\int |\nabla a|^2|u_{x_i}|^{2q-p}|u_{x_j}|^{2s-2}|u_{x_k}|^{2m}\eta^2dx\\
&\quad\quad + Cm^\theta\sum_{i=1}^{n}\int |\nabla a|^2|u_{x_i}|^{2q-p}|u_{x_j}|^{4s-2}|u_{x_k}|^{2m-2s}\eta^2dx\\
&\quad\quad + C(m^3+1)\sum_{i=1}^{n}\int |\nabla a|^2|u_{x_i}|^{2q-p}|u_{x_k}|^{2m+2s-2}\eta^2dx\,.
\end{aligned}$$

Using Young inequalities with the right weights gives (65). □

Proposition 4.4. *For $\varepsilon \in (0, \varepsilon_0]$, let u^ε be the minimizer of (21) and assume that $a(\cdot)$ satisfies (6). Let ℓ_0 be a positive integer and set $\sigma := 2^{\ell_0} - 1$. Then there exists a positive constant $C = C(n,p,q)$ such that for every Lipschitz function η compactly supported in B*

$$\begin{aligned}\int \left(\left| \nabla \left(|u^\varepsilon_{x_k}|^{\sigma+\frac{p}{2}} \right) \right|^2 + a \left| \nabla \left(|u^\varepsilon_{x_k}|^{\sigma+\frac{q}{2}} \right) \right|^2 \right) \eta^2 dx \\ \leq C\sigma^3 \sum_{i,j=1}^n \int \mathcal{M}_i^\varepsilon(\nabla u^\varepsilon) |u^\varepsilon_{x_j}|^{2\sigma+2} |\nabla \eta|^2 dx + C\sigma^4 \sum_{i,j=1}^n \int |\nabla a|^2 |u^\varepsilon_{x_i}|^{2q-p} |u^\varepsilon_{x_j}|^{2\sigma} \eta^2 dx \\ + C\sigma^4 \sum_{i=1}^n \int \mathcal{M}_i^\varepsilon(\nabla u^\varepsilon) |u^\varepsilon_{x_k}|^{2\sigma+2} |\nabla \eta|^2 dx + C\sigma^6 \sum_{i=1}^n \int |\nabla a|^2 |u^\varepsilon_{x_i}|^{2q-p} |u^\varepsilon_{x_k}|^{2\sigma} \eta^2 dx.\end{aligned} \tag{66}$$

Proof. We drop the index ε. For $\ell = 0, 1, \ldots, \ell_0$ we take

$$s_\ell = 2^\ell, \quad m_\ell = 2^{\ell_0} - 2^\ell = \sigma + 1 - 2^\ell.$$

These families satisfy

$$\begin{aligned}&s_0 = 1,\ m_0 = \sigma,\ s_{\ell_0} = \sigma + 1,\ m_{\ell_0} = 0\,, \\ &1 \leq s_\ell \leq m_\ell \quad \forall \ell = 0, \ldots, \ell_0 - 1\,, \\ &s_\ell + m_\ell = \sigma + 1 \quad \forall \ell = 0, \ldots, \ell_0\,, \\ &4s_\ell - 2 = 2s_{\ell+1} - 2, \quad 2m_\ell - 2s_\ell = 2m_{\ell+1} \quad \forall \ell = 0, \ldots, \ell_0 - 1.\end{aligned}$$

Substituting in (65) gives

$$\begin{aligned}\sum_{i=1}^n \int \mathcal{M}_i(\nabla u) |u_{x_j}|^{2s_\ell - 2} |u_{x_k}|^{2m_\ell} u^2_{x_i x_j} \eta^2 dx \\ \leq \sum_{i=1}^n \int \mathcal{M}_i(\nabla u) |u_{x_j}|^{2s_{\ell+1} - 2} |u_{x_k}|^{2m_{\ell+1}} u^2_{x_i x_j} \eta^2 dx \\ + C \sum_{i=1}^n \int \mathcal{M}_i(\nabla u) |u_{x_j}|^{2\sigma+2} |\nabla \eta|^2 dx + C m_\ell \sum_{i=1}^n \int |\nabla a|^2 |u_{x_i}|^{2q-p} |u_{x_j}|^{2\sigma} \eta^2 dx \\ + C(m_\ell + 1) \sum_{i=1}^n \int \mathcal{M}_i(\nabla u) |u_{x_k}|^{2\sigma+2} |\nabla \eta|^2 dx \\ + C(m_\ell^3 + 1) \sum_{i=1}^n \int |\nabla a|^2 |u_{x_i}|^{2q-p} |u_{x_k}|^{2\sigma} \eta^2 dx\,.\end{aligned}$$

Iterating from $\ell=0$ to $\ell=\ell_0-1$ gives

$$\begin{aligned}\sum_{i=1}^n\int &\mathcal{M}_i(\nabla u)|u_{x_k}|^{2\sigma}u^2_{x_ix_j}\eta^2dx\\
&\le\sum_{i=1}^n\int\mathcal{M}_i(\nabla u)|u_{x_j}|^{2\sigma}u^2_{x_ix_j}\eta^2dx+C\ell_0\sum_{i=1}^n\int\mathcal{M}_i(\nabla u)|u_{x_j}|^{2\sigma+2}|\nabla\eta|^2dxr\\
&\quad+C\ell_0\sigma\sum_{i=1}^n\int|\nabla a|^2|u_{x_i}|^{2q-p}|u_{x_j}|^{2\sigma}\eta^2dx+C\ell_0\sigma\sum_{i=1}^n\int\mathcal{M}_i(\nabla u)|u_{x_k}|^{2\sigma+2}|\nabla\eta|^2dx\\
&\quad+C\ell_0\sigma^3\sum_{i=1}^n\int|\nabla a|^2|u_{x_i}|^{2q-p}|u_{x_k}|^{2\sigma}\eta^2dx.\end{aligned}\tag{67}$$

We now use (49) with $\Lambda(t)=\frac{|t|^{2\sigma}t}{2\sigma+1}$ to estimate

$$\begin{aligned}\sum_{i=1}^n\int &\mathcal{M}_i(\nabla u)|u_{x_j}|^{2\sigma}u^2_{x_ix_j}\eta^2dx\\
&\le\frac{C}{(2\sigma+1)^2}\sum_{i=1}^n\int\mathcal{M}_i(\nabla u)|u_{x_j}|^{2\sigma+2}|\nabla\eta|^2dx+C\sum_{i=1}^n\int|\nabla a|^2|u_{x_i}|^{2q-p}|u_{x_j}|^{2\sigma}\eta^2dx\\
&\le C\sum_{i=1}^n\int\mathcal{M}_i(\nabla u)|u_{x_j}|^{2\sigma+2}|\nabla\eta|^2dx+C\sum_{i=1}^n\int|\nabla a|^2|u_{x_i}|^{2q-p}|u_{x_j}|^{2\sigma}\eta^2dx\,.\end{aligned}$$

Combining the above inequality with (67) we get

$$\begin{aligned}\sum_{i=1}^n\int &\mathcal{M}_i(\nabla u)|u_{x_k}|^{2\sigma}u^2_{x_ix_k}\eta^2dx\\
&\le C\ell_0\sum_{i=1}^n\int\mathcal{M}_i(\nabla u)|u_{x_j}|^{2\sigma+2}|\nabla\eta|^2\,dx+C\ell_0\sigma\sum_{i=1}^n\int|\nabla a|^2|u_{x_i}|^{2q-p}|u_{x_j}|^{2\sigma}\eta^2dx\\
&+C\ell_0\sigma\sum_{i=1}^n\int\mathcal{M}_i(\nabla u)|u_{x_k}|^{2\sigma+2}|\nabla\eta|^2dx+C\ell_0\sigma^3\sum_{i=1}^n\int|\nabla a|^2|u_{x_i}|^{2q-p}|u_{x_k}|^{2\sigma}\eta^2dx.\end{aligned}\tag{68}$$

Notice that for $\gamma=p,q$ we have

$$\sum_{i=1}^n|u_{x_i}|^{\gamma-2}|u_{x_k}|^{2\sigma}u^2_{x_ix_j}\ge|u_{x_k}|^{\gamma-2}|u_{x_k}|^{2\sigma}u^2_{x_kx_j}=\left(\sigma+\frac{\gamma}{2}\right)^{-2}\left|\left(|u_{x_k}|^{\sigma+\frac{\gamma}{2}}\right)_{x_j}\right|^2.$$

By excluding the nonnegative term with $|\nabla u|^{q-2}$ from $\mathcal{M}_i$ in the left hand side of (68), since $\ell_0\le\sigma$ we obtain

$$\begin{aligned}\int&\left(\left|\left(|u_{x_k}|^{\sigma+\frac{p}{2}}\right)_{x_j}\right|^2+a\left|\left(|u_{x_k}|^{\sigma+\frac{q}{2}}\right)_{x_j}\right|^2\right)\eta^2dx\\
&\le C\sigma^3\sum_{i=1}^n\int\mathcal{M}_i(\nabla u)|u_{x_j}|^{2\sigma+2}|\nabla\eta|^2dx+C\sigma^4\sum_{i=1}^n\int|\nabla a|^2|u_{x_i}|^{2q-p}|u_{x_j}|^{2\sigma}\eta^2dx\end{aligned}$$

$$+ C\sigma^4 \sum_{i=1}^n \int \mathcal{M}_i(\nabla u)|u_{x_k}|^{2\sigma+2}|\nabla\eta|^2 dx + C\sigma^6 \sum_{i=1}^n \int |\nabla a|^2 |u_{x_i}|^{2q-p}|u_{x_k}|^{2\sigma}\eta^2 dx.$$

Summing up over j gives (66), and the proof is thus concluded. □

5. Proof of Theorem 1.2

5.1. **The key estimate.** The core of the proof is the following Lipschitz estimate. The proof is very similar to that of [9, Theorem 5.1], though some important technical modifications have to be taken into account.

Proposition 5.1. *For $\varepsilon \in (0, \varepsilon_0]$, let u^ε be the minimizer of* (21) *and assume that $a(\cdot)$ satisfies* (6). *Then, there exist positive constants $C, \mu > 0$ and $\Theta \geq 1$ (depending on n, p, q, a) such that for every $0 < r < R < 1$ and every concentric balls $B_r \subset B_R$ it holds*

$$\|\nabla u^\varepsilon\|_{L^\infty(B_r)} \leq \frac{C}{(R-r)^\mu} \left(\int_{B_R} |\nabla u^\varepsilon|^p + 1 \right)^{\frac{\Theta}{p}}. \tag{69}$$

Proof. We prove the proposition in the case $n \geq 3$, so that 2^* is finite. An analogous argument works in dimension $n = 2$ replacing 2^* with a sufficiently large exponent.

As usual, we drop the superscript ε for simplicity. For any Lipschitz continuous function η compactly supported in $B = B_R$, we add to both sides of (66) (skipping the nonnegative second term in the left hand side) the term $\int |\nabla\eta|^2 |u_{x_k}|^{p+2\sigma}\, dx$, obtaining

$$\begin{aligned}
&\int \left| \nabla \left(|u_{x_k}|^{\sigma + \frac{p-2}{2}} u_{x_k} \eta \right) \right|^2 dx \\
&\leq \; C\sigma^3 \sum_{i,j=1}^n \int \mathcal{M}_i(\nabla u)|u_{x_j}|^{2\sigma+2}|\nabla\eta|^2 dx + C\sigma^4 \sum_{i,j=1}^n \int |\nabla a|^2 |u_{x_i}|^{2q-p}|u_{x_j}|^{2\sigma}\eta^2 dx \\
&\quad + C\sigma^4 \sum_{i=1}^n \int \mathcal{M}_i(\nabla u)|u_{x_k}|^{2\sigma+2}|\nabla\eta|^2 dx + C\sigma^6 \sum_{i=1}^n \int |\nabla a|^2 |u_{x_i}|^{2q-p}|u_{x_k}|^{2\sigma}\eta^2 dx \\
&\quad + C \int |\nabla\eta|^2 |u_{x_k}|^{p+2\sigma}\, dx.
\end{aligned}$$

We apply Sobolev inequality on the left hand side:

$$\begin{aligned}
&\left(\int |u_{x_k}|^{\frac{2^*}{2}(p+2\sigma)} \eta^{2^*} dx \right)^{2/2^*} \\
&\leq \; C\sigma^3 \sum_{i,j=1}^n \int \mathcal{M}_i(\nabla u)|u_{x_j}|^{2\sigma+2}|\nabla\eta|^2 dx + C\sigma^4 \sum_{i,j=1}^n \int |\nabla a|^2 |u_{x_i}|^{2q-p}|u_{x_j}|^{2\sigma}\eta^2 dx \\
&\quad + C\sigma^4 \sum_{i=1}^n \int \mathcal{M}_i(\nabla u)|u_{x_k}|^{2\sigma+2}|\nabla\eta|^2 dx + C\sigma^6 \sum_{i=1}^n \int |\nabla a|^2 |u_{x_i}|^{2q-p}|u_{x_k}|^{2\sigma}\eta^2 dx \\
&\quad + C \int |\nabla\eta|^2 |u_{x_k}|^{p+2\sigma} dx.
\end{aligned}$$

Let us now sum over $k=1,\dots,n$. Minkowski inequality implies that

$$\sum_{k=1}^{n}\left(\int |u_{x_k}|^{\frac{2^*}{2}(p+2\sigma)}\eta^{2^*}\,dx\right)^{2/2^*}=\sum_{k=1}^{n}\left\||u_{x_k}|^{p+2\sigma}\eta^2\right\|_{L^{2^*/2}}\geq\left\|\sum_{k=1}^{n}|u_{x_k}|^{p+2\sigma}\eta^2\right\|_{L^{2^*/2}}$$

so that (recall that $\sigma>1$):

$$\begin{aligned}\left(\int\left(\sum_{k=1}^{n}|u_{x_k}|^{(p+2\sigma)}\right)^{\frac{2^*}{2}}\eta^{2^*}\,dx\right)^{\frac{2}{2^*}}&\\ \leq C\sigma^4\sum_{i,k=1}^{n}\int\mathcal{M}_i(\nabla u)|u_{x_k}|^{2\sigma+2}|\nabla\eta|^2dx&+C\sigma^6\sum_{i,k=1}^{n}\int|\nabla a|^2|u_{x_i}|^{2q-p}|u_{x_k}|^{2\sigma}\eta^2dx\\ +C\sum_{k=1}^{n}\int|\nabla\eta|^2|u_{x_k}|^{p+2\sigma}dx.&\end{aligned}$$

We define

$$\mathcal{U}(x):=\max_{k=1,\dots,n}|u_{x_k}(x)|$$

which satisfies, for every $\alpha>0$,

$$\mathcal{U}^\alpha\leq\sum_{k=1}^{n}|u_{x_k}|^\alpha\leq n\,\mathcal{U}^\alpha\qquad\text{and}\qquad|\nabla u|^\alpha\leq n^{\alpha/2}\,\mathcal{U}^\alpha\,.$$

Using this function and recalling the definition of $\mathcal{M}_i(\nabla u)$ (cf. (48)) we have

$$\begin{aligned}\left(\int\mathcal{U}^{\frac{2^*}{2}(p+2\sigma)}\,\eta^{2^*}\,dx\right)^{\frac{2}{2^*}}\leq\ &C\sigma^4\int\mathcal{U}^{p+2\sigma}\,|\nabla\eta|^2dx\\ &+C\sigma^4\int(a(x)+1)\,\mathcal{U}^{q+2\sigma}\,|\nabla\eta|^2dx+C\sigma^6\int|\nabla a|^2\,\mathcal{U}^{2q-p+2\sigma}\,\eta^2dx.\end{aligned}\tag{70}$$

Let $0<r<\rho<R<1$ and let $B_s\subset B_t$ be two concentric balls with $r\leq s<t\leq\rho$. Let us choose $\eta=\eta_{s,t}$ such that $\operatorname{supp}(\eta)=B_t$, $0\leq\eta\leq1$, $\eta=1$ on B_s, and $\|\nabla\eta\|_{L^\infty}\leq\frac{C}{t-s}$. By Hölder inequality, we continue in (70) with

$$\begin{aligned}\left(\int_{B_s}\mathcal{U}^{\frac{2^*}{2}(p+2\sigma)}\,dx\right)^{\frac{2}{2^*}}\leq\frac{C\sigma^4}{(t-s)^2}\int_{B_t}\mathcal{U}^{p+2\sigma}\,dx&+\frac{C\sigma^4(\|a\|_{L^\infty}+1)}{(t-s)^2}\int_{B_t}\mathcal{U}^{q+2\sigma}\,dx\\ &+C\sigma^6\|\nabla a\|^2_{L^{\frac{n}{1-\alpha}}}\left(\int_{B_t}\mathcal{U}^{(2q-p+2\sigma)h}\,\eta^2\,dx\right)^{\frac{1}{h}},\end{aligned}$$

where we have set $h:=\frac{n}{n-2(1-\alpha)}$. We further simplify the above expression with

$$\left(\int_{B_s}\mathcal{U}^{\frac{2^*}{2}(p+2\sigma)}\,dx\right)^{\frac{2}{2^*}}\leq\frac{C\sigma^6}{(t-s)^2}\left(1+\|a\|_{L^\infty}+\|\nabla a\|^2_{L^{\frac{n}{1-\alpha}}}\right)\left(\int_{B_t}\mathcal{U}^{(2q-p+2\sigma)h}\,dx+1\right)^{\frac{1}{h}}.\tag{71}$$

For every $j\in\mathbb{N}$, let us consider the following quantities

$$\sigma_j:=2^{j+1}-1\,,\tag{72}$$

$$\gamma_j := (2q-p+2\sigma_j)h = \frac{n(2q-p+2^{j+2}-2)}{n-2+2\alpha}\,, \tag{73}$$

$$\hat{\gamma}_j := \frac{2^*}{2}(p+2\sigma_j) = \frac{n(p+2^{j+2}-2)}{n-2}\,. \tag{74}$$

We thus rewrite (71):

$$\left(\int_{B_s} \mathcal{U}^{\hat{\gamma}_j}\,dx\right)^{\frac{2}{2^*}} \le \frac{C2^{6j}}{(t-s)^2}\left(1+\|a\|_{L^\infty}+\|\nabla a\|^2_{L^{\frac{n}{1-\alpha}}}\right)\left(\int_{B_t} \mathcal{U}^{\gamma_j}\,dx+1\right)^{\frac{1}{h}}. \tag{75}$$

We now use the interpolation inequality in Lebesgue spaces to deduce an inequality involving L^{γ_j} and $L^{\gamma_{j-1}}$ norms. First, we notice that

$$\hat{\gamma}_j > \gamma_j \iff h(2q-p-2+2^{j+2}) < \frac{2^*}{2}(p-2+2^{j+2}) \iff 2^{j+2} > \frac{n-2}{\alpha}(q-p)-(p-2)\,.$$

Since

$$\frac{q}{p} < 1+\frac{\alpha}{n} < 1+\frac{\alpha}{n-2},$$

we immediately deduce that

$$\frac{n-2}{\alpha}(q-p)-(p-2) < 2 \tag{76}$$

and thus $\gamma_{j-1} < \gamma_j < \hat{\gamma}_j$ for all $j \ge 0$. We can therefore consider the interpolation inequality

$$\int_{B_s} \mathcal{U}^{\gamma_j}\,dx \le \left(\int_{B_s} \mathcal{U}^{\gamma_{j-1}}\,dx\right)^{\frac{\tau_j\gamma_j}{\gamma_{j-1}}}\left(\int_{B_s} \mathcal{U}^{\hat{\gamma}_j}\,dx\right)^{\frac{(1-\tau_j)\gamma_j}{\hat{\gamma}_j}} \tag{77}$$

where the value of $\tau_j \in (0,1)$ is

$$\tau_j := \frac{\dfrac{\hat{\gamma}_j}{\gamma_j}-1}{\dfrac{\hat{\gamma}_j}{\gamma_j}\cdot\dfrac{\gamma_j}{\gamma_{j-1}}-1}. \tag{78}$$

The sequence $\{\tau_j\}_{j\in\mathbb{N}}$ satisfies $\lim\limits_{j\to+\infty}\tau_j = \tau_\infty \in (0,1)$.

Let us further consider the positive quantity

$$M_j := (1-\tau_j)\frac{2^*}{2h}\cdot\frac{\gamma_j}{\hat{\gamma}_j} = (1-\tau_j)\frac{2q-p+2\sigma_j}{p+2\sigma_j}.$$

We can write

$$M_j = \frac{\frac{\hat{\gamma}_j}{\gamma_{j-1}}-\frac{\hat{\gamma}_j}{\gamma_j}}{\frac{\hat{\gamma}_j}{\gamma_{j-1}}-1}\cdot\frac{\gamma_j}{\hat{\gamma}_j}\cdot\frac{2^*}{2h} = \frac{\gamma_j-\gamma_{j-1}}{\hat{\gamma}_j-\gamma_{j-1}}\cdot\frac{2^*}{2h} = \frac{2^{j+1}}{\frac{2^*}{2h}(p-2+2^{j+2})-(2q-p-2+2^{j+1})}\cdot\frac{2^*}{2h}$$

so that $M_j < 1$ if

$$\frac{2^*}{2h}(p-2+2^{j+2})-(2q-p-2+2^{j+1}) > 2^{j+1}\cdot\frac{2^*}{2h}$$

and therefore

$$M_j < 1 \iff 2^{j+1} > \frac{n-2}{\alpha}(q-p)-(p-2).$$

Since, for every $j \geq 0$, the right hand side is smaller than 2, as we already observed in (76), and since $M_j \to 1 - \tau_\infty \in (0,1)$, we have that there exist two constants C_1, C_2 only depending on p, q, α, and n, such that

$$0 < C_1 \leq M_j \leq C_2 < 1 \quad \forall j \geq 0. \tag{79}$$

Applying (75) to (77), using Young inequality and (79) we infer that

$$\begin{aligned}
\int_{B_s} \mathcal{U}^{\gamma_j}\,dx &\leq \left(\frac{C2^{6j}}{(t-s)^2}\right)^{\frac{2^*(1-\tau_j)\gamma_j}{2\hat{\gamma}_j}} \left(\int_{B_s} \mathcal{U}^{\gamma_{j-1}}\,dx\right)^{\frac{\tau_j\gamma_j}{\gamma_{j-1}}} \left(\int_{B_t} \mathcal{U}^{\gamma_j}\,dx + 1\right)^{\frac{2^*}{2h}\cdot\frac{(1-\tau_j)\gamma_j}{\hat{\gamma}_j}} \\
&= \left(\left(\frac{C2^{6j}}{(t-s)^2}\right)^{\frac{2^*(1-\tau_j)\gamma_j}{2\tau_j\hat{\gamma}_j}} \left(\int_{B_s} \mathcal{U}^{\gamma_{j-1}}\,dx\right)^{\frac{\gamma_j}{\gamma_{j-1}}}\right)^{\tau_j} \left(\int_{B_t} \mathcal{U}^{\gamma_j}\,dx + 1\right)^{M_j} \\
&\leq C_2\left(\int_{B_t} \mathcal{U}^{\gamma_j}\,dx + 1\right) + (1 - C_1)\left(\frac{C2^{6j}c_a}{(t-s)^2}\right)^{\beta} \left(\int_{B_s} \mathcal{U}^{\gamma_{j-1}}\,dx\right)^{\frac{\tau_j\gamma_j}{\gamma_{j-1}}\cdot\left(\frac{1}{M_j}\right)'}
\end{aligned} \tag{80}$$

where $C = C(n,p,q,a)$ and $\beta \in (1,+\infty)$ is a suitable exponent dependent on p, q, α, n.

Defining

$$\kappa_j := \tau_j \left(\frac{1}{M_j}\right)' - 1$$

we rewrite (80) as

$$\int_{B_s} \mathcal{U}^{\gamma_j}\,dx \leq C_2 \int_{B_t} \mathcal{U}^{\gamma_j}\,dx + (1 - C_1)\left(\frac{C2^{6j}}{(t-s)^2}\right)^{\beta} \left(\int_{B_s} \mathcal{U}^{\gamma_{j-1}}\,dx\right)^{\frac{\gamma_j}{\gamma_{j-1}}(1+\kappa_j)} + C_2.$$

We now apply Lemma 2.1 with

$$\begin{aligned}
&Z(y) = \int_{B_y} \mathcal{U}^{\gamma_j}\,dx, \quad \mathcal{A} = (1 - C_1)\big(C2^{6j}\big)^{\beta} \left(\int_{B_\rho} \mathcal{U}^{\gamma_{j-1}}\,dx\right)^{\frac{\gamma_j}{\gamma_{j-1}}(1+\kappa_j)}, \\
&\mathcal{B} = 0\,, \quad \mathcal{C} = C_2\,, \quad \alpha_0 = 2\beta\,, \quad \theta = C_2\,.
\end{aligned}$$

Hence, we obtain that, for all $j \geq 0$,

$$\int_{B_r} \mathcal{U}^{\gamma_j}\,dx \leq C\left(\left(\frac{2^{6j}}{(\rho-r)^2}\right)^{\beta} \left(\int_{B_\rho} \mathcal{U}^{\gamma_{j-1}}\,dx\right)^{\frac{\gamma_j}{\gamma_{j-1}}(1+\kappa_j)} + 1\right) \tag{81}$$

with $C = C(n,p,q,a) > 1$.

We now want to iterate the previous estimate on a sequence of shrinking balls. We fix two radii $0 < r < \rho \leq 1$, then we consider the sequence

$$R_j = r + \frac{\rho - r}{2^j}$$

and we apply (81) with $R_{j+1} < R_j$ instead of $r < \rho$. For later use, we define

$$\lambda_j := \frac{2h}{2^*}\frac{\hat{\gamma}_j}{\gamma_j} = \frac{p + 2\sigma_j}{2q - p + 2\sigma_j} \quad \in (0,1] \tag{82}$$

and notice that we may rewrite

$$\begin{aligned}\kappa_j &= -1+\tau_j\cdot\left(\frac{2h}{2^*}\frac{1}{1-\tau_j}\frac{\hat\gamma_j}{\gamma_j}\right)' = -1+\tau_j\cdot\left(\frac{\lambda_j}{1-\tau_j}\right)' \\ &= -1+\frac{\tau_j}{1-\frac{1-\tau_j}{\lambda_j}} = \frac{(1-\lambda_j)(1-\tau_j)}{\lambda_j+\tau_j-1}\,.\end{aligned} \tag{83}$$

In particular, κ_j is nonnegative for $j\geq 0$, as $\gamma_{j-1}<\gamma_j<\hat\gamma_j$ and $\frac{2h}{2^*}>1$. To simplify our notation, we denote

$$Y_j := \int_{B_{R_j}} \mathcal{U}^{\gamma_{j-1}}\,dx$$

so that (81) writes as

$$Y_{j+1}\leq C\left(2^{8\beta j}(\rho-r)^{-2\beta}\,Y_j^{\frac{\gamma_j}{\gamma_{j-1}}(1+\kappa_j)}+1\right)\leq\left(C2^{8\beta}(\rho-r)^{-2\beta}\right)^j(Y_j+1)^{\frac{\gamma_j}{\gamma_{j-1}}(1+\kappa_j)}$$

since $C>1$ and $\rho\leq 1$. Iterating this inequality gives

$$\begin{aligned}Y_{j+1}&\leq\left(C2^{8\beta}(\rho-r)^{-2\beta}\right)^j(Y_j+1)^{\frac{\gamma_j}{\gamma_{j-1}}(1+\kappa_j)}\\ &\leq\left(C2^{8\beta}(\rho-r)^{-2\beta}\right)^j\left(\left(C2^{8\beta}(\rho-r)^{-2\beta}\right)^{j-1}(Y_{j-1}+1)^{\frac{\gamma_{j-1}}{\gamma_{j-2}}(1+\kappa_{j-1})}+1\right)^{\frac{\gamma_j}{\gamma_{j-1}}(1+\kappa_j)}\\ &\leq\left(C2^{8\beta}(\rho-r)^{-2\beta}\right)^j\left(2\left(C2^{8\beta}(R-r)^{-2\beta}\right)^{j-1}(Y_{j-1}+1)^{\frac{\gamma_{j-1}}{\gamma_{j-2}}(1+\kappa_{j-1})}\right)^{\frac{\gamma_j}{\gamma_{j-1}}(1+\kappa_j)}.\end{aligned}$$

Therefore, we infer that

$$Y_{N+1}\leq\left(C2^{8\beta+1}(\rho-r)^{-2\beta}\right)^{\sum_{j=1}^N\left(j\frac{\gamma_N}{\gamma_j}\prod_{k=j}^N(1+\kappa_k)\right)}(Y_1+1)^{\frac{\gamma_N}{\gamma_0}\prod_{j=1}^N(1+\kappa_j)}\,. \tag{84}$$

Denoting the constant $C2^{8\beta+1}$ again with C and setting

$$\Theta_N := \prod_{j=1}^N(1+\kappa_j)\,,$$

we take the power $1/\gamma_N$ on both sides of (84)

$$\begin{aligned}Y_{N+1}^{\frac{1}{\gamma_N}}&\leq\left(C(\rho-r)^{-2\beta}\right)^{\sum_{j=1}^N\left(\frac{j}{\gamma_j}\prod_{k=j}^N(1+\kappa_k)\right)}(Y_j+1)^{\frac{1}{\gamma_0}\prod_{j=1}^N(1+\kappa_j)}\\ &\leq\left(C(\rho-r)^{-2\beta}\right)^{\Theta_N\sum_{j=1}^N\left(\frac{j}{\gamma_j}\right)}(Y_1+1)^{\frac{\Theta_N}{\gamma_0}}\,.\end{aligned} \tag{85}$$

Let us now estimate κ_j as $j\to\infty$. Recalling the definitions (73), (74), (78), (82), and (83), we have $\tau_j\to\tau_\infty\in(0,1)$ and

$$1-\lambda_j=\frac{2q-2p}{2q-p+2\sigma_j}\sim\frac{q-p}{2^{j+1}}$$

so that

$$\kappa_j\sim\frac{1-\tau_\infty}{\tau_\infty}\cdot\frac{q-p}{2^{j+1}}\qquad\text{as } j\to\infty.$$

Since

$$\log\left(\prod_{j=1}^{N}(1+\kappa_j)\right)=\sum_{j=1}^{N}\log(1+\kappa_j)\leq\sum_{j=1}^{N}\kappa_j$$

we have that

$$\lim_{N\to\infty}\Theta_N=\Theta_\infty<+\infty\,.$$

Using the fact that $\gamma_j\sim 2^{j+1}$ we infer that $\sum_{j=1}^{\infty}\frac{j}{\gamma_j}$ also converges. Therefore, we infer from (85) that

$$\|\,\mathcal{U}\,\|_{L^\infty(B_r)}=\lim_{N\to\infty}\left(\int_{B_r}\mathcal{U}^{\gamma_N}\,dx\right)^{\frac{1}{\gamma_N}}=\lim_{N\to\infty}Y_{N+1}^{\frac{1}{\gamma_N}}\leq\frac{C}{(\rho-r)^{\beta'}}\left(\int_{B_\rho}\mathcal{U}^{\gamma_0}+1\right)^{\frac{\Theta_\infty}{\gamma_0}}.$$

By definition of $\mathcal{U}$ and of γ_0 we deduce that

$$\|\nabla u\|_{L^\infty(B_r)}\leq\frac{C}{(\rho-r)^{\beta'}}\left(\int_{B_\rho}|\nabla u|^{h(2q-p+2)}+1\right)^{\frac{\Theta_\infty}{h(2q-p+2)}}\tag{86}$$

for some $C,\beta'>0$ depending on n,p,q,a.

Let us now improve inequality (86). To simplify the notation, we write $K=\frac{2^*}{2}$. Since

$$p<h(2q-p+2)<K(p+2)\,,$$

using interpolation we can write for every $t\leq R$

$$\left(\int_{B_t}|\nabla u|^{h(2q-p+2)}\,dx\right)^{\frac{1}{h(2q-p+2)}}\leq\left(\int_{B_t}|\nabla u|^p\,dx\right)^{\frac{\zeta}{p}}\left(\int_{B_t}|\nabla u|^{K(p+2)}\,dx\right)^{\frac{1-\zeta}{K(p+2)}}\tag{87}$$

with the choice

$$\frac{1}{h(2q-p+2)}=\frac{\zeta}{p}+\frac{1-\zeta}{K(p+2)}\,.$$

We take $\rho=\frac{r+R}{2}$ and $\rho\leq s<t\leq R$ and we rewrite (71) with $\sigma=\sigma_0=1$ (i.e. $j=0$). Hence we have

$$\left(\int_{B_s}|\nabla u|^{K(p+2)}\,dx\right)^{\frac{1}{K}}\leq\frac{C}{(t-s)^2}\left(\int_{B_t}|\nabla u|^{h(2q-p+2)}\,dx+1\right)^{\frac{1}{h}}.\tag{88}$$

Plugging (88) in (87) gives

$$\begin{aligned}&\left(\int_{B_s}|\nabla u|^{h(2q-p+2)}\,dx\right)^{\frac{1}{h(2q-p+2)}}\\&\quad\leq\frac{C}{(t-s)^{\upsilon}}\left(\int_{B_t}|\nabla u|^p\,dx\right)^{\frac{\zeta}{p}}\left(\left(\int_{B_t}|\nabla u|^{h(2q-p+2)}\,dx\right)^{\frac{1-\zeta}{h(p+2)}}+1\right),\end{aligned}\tag{89}$$

where we have set $\upsilon:=\frac{2(1-\zeta)}{p+2}$. We claim that

$$M:=(1-\zeta)\frac{2q-p+2}{p+2}<1.\tag{90}$$

In order to prove (90), let us denote

$$A:=2q-p+2,\quad B:=p,\quad D:=p+2,$$

so that

$$\zeta = \frac{\dfrac{1}{hA} - \dfrac{1}{KD}}{\dfrac{1}{B} - \dfrac{1}{KD}}$$

and

$$M = (1-\zeta)\frac{A}{D} = \frac{\frac{1}{B} - \frac{1}{hA}}{\dfrac{1}{B} - \dfrac{1}{KD}} \cdot \frac{A}{D} = \frac{\dfrac{A}{B} - \dfrac{1}{h}}{\dfrac{D}{B} - \dfrac{1}{K}} = \frac{K}{h} \cdot \frac{hA - B}{KD - B}.$$

Hence, the condition $M < 1$ is equivalent to

$$K(hA - B) < h(KD - B) \iff K(A - D) < \left(\frac{K}{h} - 1\right) B.$$

Writing the above equivalence explicitly, we get

$$\frac{n}{n-2}(2q - 2p) < \frac{2\alpha}{n-2} p \iff n(q-p) < \alpha p \iff \frac{q}{p} < 1 + \frac{\alpha}{n},$$

which is true by assumption, so that (90) holds. Using this and applying Young inequality to (89) we have:

$$\begin{aligned} \int_{B_s} |\nabla u|^{h(2q-p+2)}\, dx \leq{}& (1-\zeta)\frac{2q-p+2}{p+2}\left(\int_{B_t} |\nabla u|^{h(2q-p+2)}\, dx\right) \\ &+ \frac{C}{(t-s)^{\upsilon\omega}}\left(\int_{B_R} |\nabla u|^p\, dx\right)^{\frac{\zeta h(2q-p+2)\omega}{p}} \\ &+ \frac{C}{(t-s)^{\upsilon}}\left(\int_{B_R} |\nabla u|^p\, dx\right)^{\frac{\zeta h(2q-p+2)}{p}} \end{aligned} \tag{91}$$

with $\omega = \left(\frac{p+2}{(1-\zeta)(2q-p+2)}\right)' = \frac{1}{1-M}$. Lemma 2.1, (91), and (86) finally yield

$$\|\nabla u\|_{L^\infty(B_r)} \leq \frac{C}{(R-r)^\mu}\left(\int_{B_R} |\nabla u|^p + 1\right)^{\frac{\Theta}{p}}$$

for some $C, \mu > 0$, and $\Theta \geq 1$ depending on n, p, q, a. This concludes the proof of (69). □

5.2. **Main proof.** We now proceed to prove the main theorem.

Proof of Theorem 1.2. Proposition 5.1 gives

$$\|\nabla u^\varepsilon\|_{L^\infty(B_{R/3})} \leq C\left(\int_{B_R} |\nabla u^\varepsilon|^p + 1\right)^{\Theta/p} \tag{92}$$

for every ball B_R with radius $R \leq 1$, where C depends on n, p, q, a, R. Thanks to Proposition 2.5 $u^\varepsilon \to U$ strongly in $W^{1,p}(B_R)$ and thus, passing to the limit in (92), we get

$$\|\nabla u\|_{L^\infty(B_{R/3})} \leq C\left(\int_{B_R} |\nabla u|^p + 1\right)^{\Theta/p}.$$

This concludes the proof of the theorem. □

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References

[1] P. Baroni, M. Colombo, and G. Mingione, *Regularity for general functionals with double phase*, Calc. Var. Partial Differential Equations, 57 (2018), pp. Paper No. 62, 48.

[2] L. Beck and G. Mingione, *Lipschitz bounds and nonuniform ellipticity*, Comm. Pure Appl. Math., 73 (2020), pp. 944–1034.

[3] M. Borowski, I. Chlebicka, and B. Miasojedow, *Absence of Lavrentiev's gap for anisotropic functionals*, Nonlinear Anal., 246 (2024), p. 113584.

[4] P. Bousquet, *Another look to the orthotropic functional in the plane*, in Bruno Pini Mathematical Analysis Seminar 2020, vol. 11(2) of Bruno Pini Math. Anal. Semin., Univ. Bologna, Alma Mater Stud., Bologna, 2021, pp. 1–29.

[5] P. Bousquet and L. Brasco, *C^1 regularity of orthotropic p-harmonic functions in the plane*, Anal. PDE, 11 (2018), pp. 813–854.

[6] ——, *Lipschitz regularity for orthotropic functionals with nonstandard growth conditions*, Rev. Mat. Iberoam., 36 (2020), pp. 1989–2032.

[7] P. Bousquet, L. Brasco, and V. Julin, *Lipschitz regularity for local minimizers of some widely degenerate problems*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5), 16 (2016), pp. 1235–1274.

[8] P. Bousquet, L. Brasco, and C. Leone, *Singular orthotropic functionals with nonstandard growth conditions*, Rev. Mat. Iberoam., 40 (2024), pp. 753–802.

[9] P. Bousquet, L. Brasco, C. Leone, and A. Verde, *On the Lipschitz character of orthotropic p-harmonic functions*, Calc. Var. Partial Differential Equations, 57 (2018), pp. Paper No. 88, 33.

[10] L. Brasco and G. Carlier, *On certain anisotropic elliptic equations arising in congested optimal transport: local gradient bounds*, Adv. Calc. Var., 7 (2014), pp. 379–407.

[11] L. Brasco, G. Carlier, and F. Santambrogio, *Congested traffic dynamics, weak flows and very degenerate elliptic equations*, J. Math. Pures Appl. (9), 93 (2010), pp. 652–671.

[12] L. Brasco, C. Leone, G. Pisante, and A. Verde, *Sobolev and Lipschitz regularity for local minimizers of widely degenerate anisotropic functionals*, Nonlinear Anal., 153 (2016), pp. 169–199.

[13] A. Cavaliere, A. D'Ottavio, F. Leonetti, and M. Longobardi, *Differentiability for minimizers of anisotropic integrals*, Comment. Math. Univ. Carolin., 39 (1998), pp. 685—696.

[14] M. Colombo and G. Mingione, *Regularity for double phase variational problems*, Archive for Rational Mechanics and Analysis, 215 (2015), pp. 443–496.

[15] C. De Filippis and G. Mingione, *Lipschitz bounds and nonautonomous integrals*, Arch. Ration. Mech. Anal., 242 (2021), pp. 973–1057.

[16] F. Demengel, *Lipschitz interior regularity for the viscosity and weak solutions of the pseudo p-Laplacian equation*, Adv. Differential Equations, 21 (2016), pp. 373–400.

[17] M. Eleuteri, P. Marcellini, and E. Mascolo, *Lipschitz estimates for systems with ellipticity conditions at infinity*, Ann. Mat. Pura Appl. (4), 195 (2016), pp. 1575–1603.

[18] ——, *Regularity for scalar integrals without structure conditions*, Adv. Calc. Var., 13 (2020), pp. 279–300.

[19] L. Esposito, F. Leonetti, and G. Mingione, *Sharp regularity for functionals with (p, q) growth*, J. Differential Equations, 204 (2004), pp. 5–55.

[20] E. Giusti, *Direct methods in the calculus of variations*, Springer, 2003.
[21] P. Hästö and J. Ok, *Maximal regularity for local minimizers of non-autonomous functionals*, J. Eur. Math. Soc. (JEMS), 24 (2022), pp. 1285–1334.
[22] T. Kuusi and G. Mingione, *New perturbation methods for nonlinear parabolic problems*, J. Math. Pures Appl., 98 (2012), pp. 390–427.
[23] P. Lindqvist and D. Ricciotti, *Regularity for an anisotropic equation in the plane*, Nonlinear Anal., 177 (2018), pp. 628–636.
[24] P. Marcellini, *Regularity of minimizers of integrals of the calculus of variations with nonstandard growth conditions*, Arch. Rational Mech. Anal., 105 (1989), pp. 267–284.
[25] ——, *Regularity and existence of solutions of elliptic equations with p, q-growth conditions*, J. Differential Equations, 90 (1991), pp. 1–30.
[26] G. Mingione and V. t. Rădulescu, *Recent developments in problems with nonstandard growth and nonuniform ellipticity*, J. Math. Anal. Appl., 501 (2021), pp. Paper No. 125197, 41.
[27] S. Nikol'skiĭ, *Approximation of functions of several variables and imbedding theorems*, Springer-Verlag, 1975.
[28] A. Passarelli di Napoli, *Regularity results for non-autonomous variational integrals with discontinuous coefficients*, Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur., 26 (2015), pp. 475–496.
[29] D. Ricciotti, *Regularity of the derivatives of p-orthotropic functions in the plane for $1 < p < 2$*, Ann. Acad. Sci. Fenn. Math., 44 (2019), pp. 1093–1099.
[30] S. Russo, *Higher differentiability of minimizers for non-autonomous orthotropic functionals*, Nonlinear Anal. Real World Appl., 87 (2026), p. Paper No. 104450.
[31] N. N. Ural'tseva and A. B. Urdaletova, *Boundedness of gradients of generalized solutions of degenerate nonuniformly elliptic quasilinear equations*, Vestnik Leningrad. Univ. Mat. Mekh. Astronom., (1983), pp. 50–56.
[32] V. V. Zhikov, *Problems of convergence, duality, and averaging for a class of functionals of the calculus of variations*, Dokl. Akad. Nauk SSSR, 267 (1982), pp. 524–528.
[33] ——, *Averaging of functionals of the calculus of variations and elasticity theory*, Izv. Akad. Nauk SSSR Ser. Mat., 50 (1986), pp. 675–710, 877.
[34] ——, *On Lavrentiev's phenomenon*, Russian J. Math. Phys., 3 (1995), pp. 249–269.
[35] ——, *On some variational problems*, Russian J. Math. Phys., 5 (1997), pp. 105–116.