THE LAVRENTIEV PHENOMENON FOR FREE DISCONTINUITY PROBLEMS

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Abstract. We study lower semicontinuity problems for a class of integral functionals depending on a bulk energy and a jump part energy. These functionals are naturally defined on functions of bounded variation, and can be extended by relaxation to a larger set of discontinuous functions. We give a representation formula for this extension showing the appearance of an additional "Lavrentiev" term, which values the energy necessary for the creation of a singularity with unbounded variation. In general the difference between the functional and its relaxation implies that the same minimization problem may have different solutions on BV functions and on larger spaces of discontinuous functions.

In the special case of autonomous functionals the Lavrentiev term can be expressed as an inf-convolution between the jump-part energy density and a rescaled bulk energy density.

1. Introduction

In many problems involving free discontinuities we come up with the study of minimum problems for functionals defined in some set of functions of bounded variation. In the simplest case of autonomous functionals in dimension one, these have the form

$$F(u) = \int_I f(u'(t))\,dt + \sum_{t \in S_u \cap I} g(u(t_+) - u(t_-)), \tag{1.1}$$

where I is some interval of the real line, u' is the approximate derivative of the function u, $u(x_+)$ and $u(x_-)$ are the right-hand side and left-hand side approximate values of u at the point x, and S_u is the set where these two values differ. For example, if we choose $f(\xi) = |\xi|^2$, and g the constant 1, the functional F is the 1- dimensional version of the Mumford & Shah functional of computer vision (see [18]). Functionals of the same type model also, in dimension 2 and 3, some static problems in fracture mechanics (see [2], [8]). Under some hypotheses of convexity on f, of subadditivity on g, and of compatibility between the two functions, the functional F is weakly lower semicontinuous in a suitable subspace of the space of functions of bounded variation, the space $SBV(I)$ introduced by E. De Giorgi & L. Ambrosio [13]. In this space we also get compactness, and hence existence for minimum problems, if f and g satisfy some growth conditions.

The expression in (1.1) may make sense for a larger class than the space $SBV(I)$. The problem arises then of finding whether this extension is sensible; *i.e.,* if the value given by F on some function u corresponds to some approximation of this function with functions (u_h) of bounded variation, for which the functional has a precise meaning. In the words of the theory of relaxation, this can be expressed by the question: does the functional F coincide with the relaxation of its restriction on $SBV(I)$? The answer is in general negative, and the functional must be "corrected" with an extra term. We show that the relaxed functional $\overline{F}$ can be expressed in many cases as $F(u) + L(u)$, where the functional L is not trivially 0, and it is given by an explicit formula. In the case of an autonomous functional as in (1.1) above, and of a function u that has a degenerate behaviour only at the point 0, the form of L is particularly simple:

$$L(u) = \liminf_{\varepsilon \to 0+} V(2\varepsilon, u(\varepsilon_+) - u((-\varepsilon)_-)), \tag{1.2}$$

where V is given by the inf-convolution

$$V(x, s) = (f^x \nabla g)(s), \tag{1.3}$$

and we define $f^x(t) = x\, f(\frac{t}{x})$.

We remark that in general the difference between F and $\overline{F}$ implies that the same minimization problem for the functional F may have different solutions on $SBV(I)$, and on larger spaces where $SBV(I)$ itself is dense. This phenomenon was first observed for integral functionals defined on Sobolev spaces by M. Lavrentiev [16], where the *rôle* of SBV is played by the class of Lipschitz functions (see also *e.g.* Manià [17] and Ball & Mizel [9]). The link between this kind of phenomena and the theory of relaxation has been recently investigated by G. Buttazzo & V.J. Mizel [11].

2. Preliminaries

2.1. The spaces SBV and GSBV

We shall consider some spaces of functions that have been introduced by E. De Giorgi & L. Ambrosio [13] in order to study variational problems which take into account free discontinuities. In the special case of dimension one, the definition of these spaces is particularly simple. These problems can be also included in the framework of the non-convex functionals defined on spaces of measures, studied by Bouchitté & Buttazzo [7].

Let I be an open interval of $\mathbb{R}$. We shall consider the space $SBV(I)$ of *special functions of bounded variation*, defined as the set of the functions

u of bounded variation whose measure first derivative, denoted by Du, is of the form

$$Du = u'\,dt + \sum_{k=1}^{\infty} a_k \delta_{t_k},$$

where $t_k \in I$, $a_k \in \mathbb{R}$, $\sum_{k=1}^{\infty} |a_k| < \infty$, and δ_t is the Dirac measure at t. The density u' of Du with respect of the Lebesgue measure is the approximate differential of u. Note that, if we define $u(t_+)$, $u(t_-)$, the right-hand and left-hand traces respectively of the function u at t, which exist for all $t \in I$, then $a_k = u(t_{k+}) - u(t_{k-})$. We define the set of the *jump points* of u as

$$S_u = \{t \in I \ : \ u(t_+) \neq u(t_-)\} = \{t \in I \ : \ \delta_t << Du\}.$$

Let I be an interval $]a, b[$. We shall consider boundary conditions on functions in SBV of the form $u(a) = \alpha$ and $u(b) = \beta$. It is well-known that these conditions are not well-posed for problems in $BV(I)$ and SBV (see Anzellotti & Giaquinta [6]). We have instead to *relax* these conditions, penalizing jumps also at $t = a, b$ by simply defining $u(a_-) = \alpha$, and $u(b_+) = \beta$, and extending the definition of S_u to the closed interval $[a, b]$. An equivalent approach is to extend the definition of u to the whole $\mathbb{R}$ by setting $u(t) = \alpha$ if $t \leq a$, and $u(t) = \beta$ if $t \geq b$.

In some problems it will be natural to consider a larger class of *general* functions: we will say that $u \in GSBV(I)$ if the truncations of u belong to the space $SBV(I)$; *i.e.,*

$$u_T = (-T) \vee (u \wedge T) \in SBV(I) \qquad \text{for every } T > 0.$$

If $u \in GSBV(I)$, we define S_u as

$$S_u = \bigcup_{T>0} S_{u_T}.$$

For the general properties and definitions of these spaces in higher dimensions we refer to the papers by Ambrosio [1], [2], [3].

We shall sometimes consider on $SBV(I)$ the weak topology of $BV(I)$, which is defined as the product topology of the strong topology of $\mathrm{L}^1(\Omega)$

for u, and of the weak* topology of measures for Du. Recall that every sequence in $BV(\Omega)$ with $\|u_h\|_{BV} \leq c$ admits a subsequence (u_{h_k}) such that $u_{h_k} \to u$ in $\mathrm{L}^1(\Omega)$, and $Du_{h_k} \rightharpoonup Du$ in the weak* topology of measures. For the properties of BV spaces we refer to Federer [14] and Ziemer [19].

2.2. Relaxation

Let X be a topological space and $F : X \to [-\infty, +\infty]$ be a functional on X. We define the *relaxation*, or (sequentially) lower-semicontinuous envelope, of F as the greatest sequentially lower semicontinuous functional $\overline{F}$ less than or equal F. For a general introduction to the theory of relaxation we refer the interested reader to the books by G. Buttazzo [10] and G. Dal Maso [12].

Let F be as above, and define the functional G by

$$G(u) = \inf\{\liminf_h F(u_h) \ : \ u_h \to u \text{ in } X\}.$$

It is easy to check that if G is sequentially lower semicontinuous then $G = \overline{F}$.

2.3. Notation

Let $g : \mathrm{I\!R} \to \mathrm{I\!R}$ be a function. If g is convex, we define

$$g^\infty(z) = \lim_{t \to +\infty} \frac{g(tz)}{t}.$$

We will say that g is *subadditive* if for every $s, t \in \mathrm{I\!R}$ we have

$$g(s+t) \leq g(s) + g(t).$$

If g is subadditive, we define

$$g^0(z) = \lim_{t \to 0+} \frac{g(tz)}{t}$$

(see Bouchitté & Buttazzo [7]).

The *inf-convolution* of two functions $f, g : \mathrm{I\!R} \to \mathrm{I\!R}$ is defined by

$$(f \nabla g)(t) = \inf\{f(s) + g(t-s) \ : \ s \in \mathrm{I\!R}\}.$$

The letter c will denote throughout the paper a strictly positive constant, whose value may vary from line to line, and which is independent from the parameters of the problems each time considered.

We end this Section with a lower semicontinuity result that will be needed in the sequel.

Theorem 2.1. *Let J be any open interval of $\mathbb{R}$. Let $f : J \times \mathbb{R} \times \mathbb{R} \to [0, +\infty[$ be a Carathéodory function, convex in the third variable, and such that there exists a convex function $\psi : \mathbb{R} \to \mathbb{R}$ with*

$$f(x, u, p) \geq \psi(|p|) \ \text{ for all } u, p \in \mathbb{R},\ x \in J, \qquad \lim_{t\to+\infty} \frac{\psi(t)}{t} = +\infty. \tag{2.1}$$

Let $g : J \times \mathbb{R} \times \mathbb{R} \to [0, +\infty[$ be a function, satisfying

(i) *$g(\cdot, u, v)$ is continuous for all $u, v \in \mathbb{R}$;*

(ii) *$g(t, \cdot, \cdot)$ is lower semicontinuous for every $t \in J$, and $g(t, u, v) \geq c > 0$ for $|u - v| \neq 0$ sufficiently small,*

(iii) *the subadditivity condition*

$$g(x, u, v) \leq g(x, u, w) + g(x, w, v) \ \text{ for all } u, v, w \in \mathbb{R}\ x \in J$$

(iv) *the growth hypothesis*

$$g(x, u, v) \geq c|u - v| \ \text{ for all } u, v \in \mathbb{R},\ x \in J,$$

for some constant $c > 0$.

Then the functional

$$F(u) = \int_J f(t, u(t), u'(t))\, dt + \sum_{t\in S_u\cap J} g(t, u(t_+), u(t_-))$$

is lower semicontinuous on $SBV(J)$ with respect to the topology of the convergence a.e.

Proof. We can follow word for word the proof of Braides & Coscia [8] Proposition 4.1 (see also Proposition 4.2 in [1]). ☐

Remark 2.2. Under the hypotheses above on f and the hypotheses (i)–(iii) on g, the functional F is lower semicontinuous on $SBV(J)$ with respect to the sequential weak topology of $BV(J)$ (or, equivalently, with respect to the $\mathrm{L}^1(J)$-topology along sequences with bounded BV norm). In fact, it suffices to consider the functionals

$$F_\varepsilon(u) = F(u) + \varepsilon \sum_{t \in S_u \cap J} |u(t_+) - u(t_-)|.$$

By Theorem 2.1 every F_ε is lower semicontinuous with respect to the $\mathrm{L}^1(J)$-topology. We now take a sequence $(u_h) \subset SBV(J)$ converging to $u \in SBV(J)$ with respect to the sequential weak topology of $BV(J)$. In particular $u_h \to u$ in $\mathrm{L}^1(J)$. We then have

$$\begin{aligned}\liminf_h F(u_h) \geq\ & \liminf_h F_\varepsilon(u_h) - \varepsilon \limsup_h \|Du_h\| \\ & \geq F_\varepsilon(u) - \varepsilon c \geq F(u) - 2\varepsilon c.\end{aligned}$$

By the arbitrariness of ε we have the lower semicontinuity of F.

Remark 2.3. In the case of $g(t,u,v) = \tilde{g}(u-v)$ hypothesis (ii) implies the strict positivity of $\tilde{g}$ near 0. This condition can be weakened requiring that $\tilde{g}^0(z) = +\infty$ for $z \neq 0$ (taking into account Lemma 4.1 by Ambrosio [1]). The condition of positivity of a general g near the points of the form (u,u) can be weakened in a similar way. Also note that in the autonomous case we can use the lower semicontinuity results for functionals defined on measures of Bouchitté & Buttazzo [7].

Note that in higher dimension the subadditivity condition on g becomes necessary, but it is not sufficient to obtain lower semicontinuity (cf. [5], [2]).

3. The Main Result

We attack now the problem of the computation of the lower-semicontinuous envelope of functionals on $GSBV(I)$ on a special yet meaningful model case, when it is possible to give a precise description of the relaxation. We shall fix an interval of $\mathbb{R}$, that we can take without loss of generality to be $I =]-1,1[$, and we shall focus our attention on the behaviour near a point of I where we may get a "degenerate" behaviour. Again, we can suppose, and we will, this point to be 0.

Our first result is a characterization of the behaviour of functionals with free discontinuities, when the "jump part" energy has a degenerate behaviour only at the point 0. We define the space of functions

$$\mathcal{A} = GSBV(I) \cap SBV_{\rm loc}(I \setminus \{0\}),$$

and the functionals

$$F(u) = \int_I f(t, u(t), u'(t))\, dt + \sum_{t \in S_u \cap I} g(t, u(t_+), u(t_-)) \qquad u \in SBV(I)$$

$$F_0(u) = \int_I f(t, u(t), u'(t))\, dt + \sum_{t \in S_u \cap I \setminus \{0\}} g(t, u(t_+), u(t_-)) \qquad u \in \mathcal{A}. \tag{3.1}$$

Remark that $F(u) = F_0(u)$ if $u \in \mathcal{A}$ and $0 \notin S_u$ (for example if $\lim\limits_{t \to 0} u(t) = +\infty$).

Theorem 3.1. *Let $f : I \times \mathbb{R} \times \mathbb{R} \to [0, +\infty[$ be a Carathéodory function, convex in the third variable, satisfying (2.1). Let $g : I \times \mathbb{R} \times \mathbb{R} \to [0, +\infty[$ be a function, satisfying for all $\eta > 0$*

- *(i) $g(\cdot, u, v)$ is continuous on $I \setminus \{0\}$ for all $u, v \in \mathbb{R}$;*
- *(ii) $g(t, \cdot, \cdot)$ is lower semicontinuous for every $t \in I$, and $g(t, u, v) \geq c(\eta) > 0$ for $|t| \geq \eta$, and for $|u - v| \neq 0$ sufficiently small,*
- *(iii) the subadditivity condition*

$$g(x, u, v) \leq g(x, u, w) + g(x, w, v) \quad \text{for all } u, v, w \in \mathbb{R}\ x \in I$$

- *(iv) the growth hypothesis*

$$g(x, u, v) \geq c(\eta)|u - v| \quad \text{for all } u, v \in \mathbb{R},\ |x| \geq \eta,$$

for some constant $c(\eta) > 0$.

Let F *and* F_0 *be defined as in (3.1), and define*

$$H(u) = \begin{cases} F(u) & \text{if } u \in SBV(I) \\ \\ +\infty & \text{if } u \in \mathcal{A} \setminus SBV(I). \end{cases} \tag{3.2}$$

Then the relaxation of H *with respect to the topology of a.e. convergence is given by*

$$\overline{H}(u) = F_0(u) + L(u) \tag{3.3}$$

for all $u \in \mathcal{A}$, *where the Lavrentiev term* L *is defined by*

$$L(u) = \liminf_{\varepsilon \to 0+} \overline{V}(\varepsilon, u(-\varepsilon_-), u(\varepsilon_+)), \tag{3.4}$$

with

$$\overline{V}(x, s, t) = \liminf_{(u,v) \to (s,t)} V(x, u, v), \tag{3.5}$$

and

$$\begin{aligned} V(x,s,t) = \inf\Big\{ &\int_{-x}^{x} f(y, u(y), u'(y))\, dy + \sum_{S_u \cap [-x,x]} g(y, u(y_+), u(y_-)) \\ &: \ u \in SBV(I),\ u(-x_-) = s, u(x_+) = \ t \Big\}. \end{aligned} \tag{3.6}$$

Our second result will take into account the case of autonomous integrands satisfying weaker growth condition. We shall obtain a result analogous to Theorem 3.1 with respect to a stronger convergence. We define a convergence on $\mathcal{A}$, by saying that a sequence $(u_h) \subset \mathcal{A}$ converges to u in $\mathcal{A}$ if $u_h \rightharpoonup u$ in $BV_{\mathrm{loc}}(I \setminus \{0\})$ (note that this convergence implies trivially a.e. convergence on I).

Theorem 3.2. *Let* $f, g : \mathbb{R} \to [0, +\infty[$ *be functions satisfying*

(a) f *is convex;*

(b) g *is lower semicontinuous and subadditive;*

(c) $f^\infty(z) = g^0(z) = +\infty$ *if* $z \neq 0$.

Define F *on* $SBV(I)$ *by setting*

$$F(u) = \int_I f(u'(t))\,dt + \sum_{t\in S_u\cap I} g(u(t_+) - u(t_-)),$$

and H *as in (3.2). Then the relaxation of* H *in the topology of* $\mathcal{A}$ *is given by*

$$\overline{H}(u) = F_0(u) + L(u)$$

for all $u \in \mathcal{A}$*, where*

$$F_0(u) = \int_I f(u'(t))\,dt + \sum_{t\in S_u\cap I\setminus\{0\}} g(u(t_+) - u(t_-)),$$

and the Lavrentiev term L *is defined by*

$$L(u) = \liminf_{\varepsilon\to 0+} V_1(2\varepsilon, u(\varepsilon_+) - u(-\varepsilon_-)),$$

with V *given by the inf-convolution*

$$V_1(x,s) = ((x\,f(\frac{\cdot}{x}))\nabla g)(s) \tag{3.7}$$

Remark 3.3. 1) If the functions f and g in Theorem 3.1 do not depend on the first variable it may be useful to rewrite the formula for V in (3.6), using a change of variables, as a minimum problem in the whole $SBV(I)$:

$$V(x,s,t) = \inf\Big\{x\int_I f(u(y), \frac{u'(y)}{x})\,dt + \sum_{S_u\cap[-1,1]} g(u(y_+), u(y_-))\ :$$
$$:\ u\in SBV(I),\ u(-1_-) = s, u(1_+) = t\Big\};$$

in particular if f is positively γ-homogeneous with respect to the last variable we have

$$V(x,s,t) = \inf\Big\{x^{1-\gamma}\int_I f(u(y), u'(y))\,dt + \sum_{S_u\cap[-1,1]} g(u(y_+), u(y_-)) \tag{3.8}$$
$$:\ u\in SBV(I),\ u(-1_-) = s, u(1_+) = t\Big\}.$$

2) Theorem 3.2 can be extended to functions satisfying conditions (i)–(iii) of Theorem 2.1, provided that we define $L(u)$ as in (3.4)–(3.6).

Proof of Theorem 3.1. Define for every $u \in \mathcal{A}$

$$G(u) = \inf\{\liminf_h H(u_h) \ : \ u_h \to u \text{ a.e.}\}.$$

We now check that $F_0 + L = G$ on $\mathcal{A}$.

We shall prove first that $G \leq F_0 + L$; *i.e.,* that for every $u \in \mathcal{A}$ there exists a sequence $(u_h) \in SBV(I)$ with $u_h \to u$ a.e., such that

$$F_0(u) + L(u) \geq \lim_{h \to +\infty} H(u_h). \tag{3.9}$$

By the definition of $L(u)$ for every $h \in \mathbb{N}$ there exists $\varepsilon_h \in]0, 1/h[$ such that

$$\overline{V}(\varepsilon_h, u(-\varepsilon_{h-}), u(\varepsilon_{h+})) \leq L(u) + \frac{1}{2h}.$$

Again, by (3.5) and (3.6) there exist $v_h \in SBV(I)$ such that

$$|v_h(-\varepsilon_{h-}) - u_h(-\varepsilon_{h-})| + |v_h(\varepsilon_{h+}) - u_h(\varepsilon_{h+})| \leq \frac{1}{h},$$

and

$$\int_{-\varepsilon_h}^{\varepsilon_h} f(t, v_h(t), v_h'(t))\, dt + \sum_{t \in S_{v_h} \cap [-\varepsilon_h, \varepsilon_h]} g(t, v_h(t_+), v_h(t_-)) \leq L(u) + \frac{1}{h}.$$

If we define u_h by setting

$$u_h(t) = \begin{cases} u(t) + z_h^1 & \text{if } t \in]-1, -\varepsilon_h] \\ v_h(t) & \text{if } t \in]-\varepsilon_h, \varepsilon_h[\\ u(t) + z_h^2 & \text{if } t \in [\varepsilon_h, 1[\end{cases}$$

where

$$z_h^1 = v_h(-\varepsilon_{h-}) - u_h(-\varepsilon_{h-}), \qquad z_h^2 = v_h(\varepsilon_{h+}) - u_h(\varepsilon_{h+}),$$

we obtain

$$\liminf_h H(u_h) =$$

$$\liminf_h \Big(\int_{]-1,-\varepsilon_h]} f(t, u(t) + z^1_h, u'(t))\, dt + \sum_{S_u \cap]-1,-\varepsilon_h]} g(t, u(t_+), u(t_-))$$

$$+ \int_{[\varepsilon_h,1[} f(t, u(t) + z^2_h, u'(t))\, dt + \sum_{S_u \cap [\varepsilon_h,1[} g(t, u(t_+), u(t_-))$$

$$+ \int_{[-\varepsilon_h,\varepsilon_h]} f(t, v_h(t), v'_h(t))\, dt + \sum_{t\in S_{v_h} \cap [-\varepsilon_h,\varepsilon_h]} g(t, v_h(t_+), v_h(t_-)) \Big)$$

$$\geq F_0(u) + L(u).$$

Note that $u_h \to u$ a.e.

We prove now the inequality $G \geq F_0 + L$; *i.e.*, for every $u \in \mathcal{A}$, and for every sequence $(u_h) \in SBV(I)$ with $u_h \to u$ a.e., we have

$$F_0(u) + L(u) \leq \liminf_{h\to+\infty} H(u_h).$$

We fix $\eta \in]0, 1[$ such that $u_h(-\eta_-) \to u(-\eta_-)$, and $u_h(\eta_+) \to u(\eta_+)$. For every $h \in \mathbb{N}$ we have

$$H(u_h) = \int_{I\setminus[-\eta,\eta]} f(t, u_h(t), u'_h(t))dt + \sum_{t\in S_{u_h}\setminus[-\eta,\eta]} g(t, u_h(t_+), u_h(t_-))$$

$$+ \int_{[-\eta,\eta]} f(t, u_h(t), {u_h}'_h(t))dt + \sum_{t\in S_{u_h}\cap[-\eta,\eta]} g(t, u_h(t_+), u_h(t_-))$$

$$\geq \int_{I\setminus[-\eta,\eta]} f(t, u_h(t), {u_h}'_h(t))dt + \sum_{t\in S_{u_h}\setminus[-\eta,\eta]} g(t, u_h(t_+), u_h(t_-))$$

$$+V(\eta, u_h(-\eta_-), u_h(\eta_+))$$

(we have used the definition of $V(x, s, t)$ with the function u_h). Since $\overline{V} \leq V$, and recalling that functionals of the type of F are weakly lower semicontinuous with respect to the a.e. convergence on $SBV(I \setminus [-\eta, \eta])$, we obtain

$$\begin{aligned}
&\liminf_h H(u_h) \\
&\quad \geq \liminf_h \Big(\int_{I \setminus [-\eta,\eta]} f(t, u_h(t), {u_h}'(t)) dt + \sum_{t \in S_{u_h} \setminus [-\eta,\eta]} g(t, u_h(t_+), u_h(t_-)) \Big) \\
&\quad\quad + \liminf_h \overline{V}(\eta, u_h(\eta_-), u_h(-\eta_+)) \\
&\quad \geq \int_{I \setminus [-\eta,\eta]} f(t, u(t), u'(t)) dt + \sum_{t \in S_u \setminus [-\eta,\eta]} g(t, u(t_+), u(t_-)) \\
&\quad\quad + \overline{V}(\eta, u(\eta_-), u(-\eta_+)).
\end{aligned}$$

We have made use of the lower semicontinuity of $\overline{V}$ with respect to the last two variables, and of the convergence

$$u_h(\eta_+) \to u(\eta_+) \qquad u_h(-\eta_-) \to u(-\eta_-). \tag{3.10}$$

At this point we can pass to the lim inf as $\eta \to 0+$, recalling that the set of $\eta \in]0, 1[$ for which (3.10) does not hold is negligible, and obtain then

$$\begin{aligned}
&\liminf_h H(u_h) \\
&\quad \geq \liminf_{\eta \to 0+} \Big(\int_{I \setminus [-\eta,\eta]} f(t, u(t), u'(t))\, dt + \sum_{t \in S_u \setminus [-\eta,\eta]} g(t, u(t_+) - u(t_-)) \Big) \\
&\quad + \liminf_{\eta \to 0+} \overline{V}(\eta, u(\eta_-), u(-\eta_+)) \\
&\quad = \int_I f(t, u(t), u'(t))\, dt + \sum_{t \in S_u \cap I \setminus \{0\}} g(t, u(t_+), u(t_-)) + L(u) \\
&\quad = F_0(u) + L(u).
\end{aligned}$$

The proof of Theorem 3.1 will be completed by showing that $F_0 + L$ is sequentially lower semicontinuous on $\mathcal{A}$; *i.e.,* that if $u_h \to u$ a.e., then we have

$$F_0(u) + L(u) \leq \liminf_h (F_0(u_h) + L(u_h)).$$

Without loss of generality we can suppose the existence of the limit

$$\lim_h (F_0(u_h) + L(u_h)) < +\infty.$$

Choose a decreasing sequence (t_h) with $t_h \to 0$, and $v_h \in SBV(I)$ such that

$$|v_h(t_{h+}) - u_h(t_{h+})| + |v_h(-t_{h-}) - u_h(-t_{h-})| \leq \frac{1}{h},$$

and

$$\int_{[-t_h,t_h]} f(t, v_h(t), v_h'(t))\,dt \leq \overline{V}(t_h, u_h(t_{h-}), u_h(-t_{h+})) + \frac{1}{h}.$$

If we define $w_h \in SBV(I)$ by setting

$$w_h(t) = \begin{cases} u_h(t) + v_h(-t_{h-}) - u_h(-t_{h-}) & \text{if } t \leq -t_h \\ v_h(t) & \text{if } |t| < t_h \\ u_h(t) + v_h(t_{h+}) - u_h(t_{h+}) & \text{if } t \geq t_h, \end{cases}$$

then we have $w_h \to u$ a.e., and

$$F_0(u) + L(u) \leq \liminf_h F(w_h) = \lim_h (F_0(u_h) + L(u_h)).$$

The proof of Theorem 3.1 is thus complete. □

Proof of Theorem 3.2. The representation of the relaxation as $\overline{H} = F_0 + L$ on $\mathcal{A}$, where L is given by (3.4)–(3.6), follows immediately from the proof of Theorem 3.1, and from Remarks 2.2 and 2.3. We have only to show that we can define V as in formula (3.7) (remark that this formula gives a lower-semicontinuous function of s, and hence $\overline{V} = V$). We can modify slightly (3.6), and define

$$V_1(x,s) = \inf\Big\{\int_{-x/2}^{x/2} f(u'(y))\,dt + \sum_{S_u\cap[-x/2,x/2]} g(u(y_+)-u(y_-)) \ : \ : u \in SBV(I),\ u(-\frac{x}{2}_-) = 0, u(\frac{x}{2}_+) = s\Big\}. \tag{3.11}$$

We have then, by Theorem 3.1,

$$L(u) = \liminf_{x\to 0+} \overline{V}_1(2x, u(x_+) - u(x_-))$$

($\overline{V}_1(x,\cdot)$ being the lower-semicontinuous envelope of $V_1(x,\cdot)$).

We now check the value of $V_1(x,s)$. Given any $u \in SBV(I)$, by the convexity of f we can suppose that $u'(y) = a$, a constant, a.e., and hence we get the sum

$$xf(a) + \sum_{S_u\cap[-\frac{x}{2},\frac{x}{2}]} g(u(y_+)-u(y_-)). \tag{3.12}$$

By the boundary conditions we must have

$$\sum_{S_u\cap[-\frac{x}{2},\frac{x}{2}]} (u(y_+)-u(y_-)) = s - ax.$$

With fixed $a \in \mathbb{R}$, by the subadditivity of g, a minimum for the functional in (3.12) is given by the function $u(t) = a\,(t + \frac{x}{2})$, which has at most one jump, for $t = \frac{x}{2}$, of size $s - ax$ (remember that by the boundary conditions we must take $u(x_+) = s$). Hence we get

$$\begin{aligned} V_1(x,s) =&\ \inf\{xf(a) + g(s-ax) : a \in \mathbb{R}\} \\ =&\ \inf\{xf(\frac{y}{x}) + g(s-y) : y \in \mathbb{R}\} = \Big(\Big(x\,f(\frac{\cdot}{x})\Big)\nabla g\Big)(s); \end{aligned} \tag{3.13}$$

that is, the desired expression for V_1. □

4. Examples

In this section we shall illustrate the results of Section 3 with some examples. We shall limit our analysis mainly to the case of autonomous integrands.

Example 4.1. (Lavrentiev Phenomenon) Let $f : \mathbb{R} \to [0,+\infty]$ be convex and lower semicontinuous, and let $g : \mathbb{R} \to [0,+\infty[$ be subadditive and lower semicontinuous. Fix $s \in \mathbb{R}$ and consider the minimum problem

$$\min\Big\{\int_{]-1/2,1/2[} f(u'(t))\,dt + \sum_{t\in S_u\cap[-1/2,1/2]} g(u(t_+)-u(t_-)) \ :$$
$$: \ u \in SBV(I),\ u(\frac{1}{2}_+) = -u(-\frac{1}{2}_-) = \frac{s}{2}\Big\}.$$

By the convexity of f and the subadditivity of g we get, as in the proof of Theorem 3.2, that a minimizer for this problem is given, for example, by

$$u(t) = \begin{cases} \frac{1}{2}(s-a)+at & \text{if } t>0 \\ \frac{1}{2}(a-s)+at & \text{if } t<0, \end{cases}$$

where a verifies

$$f(a)+g(s-a) = (f\nabla g)(s),$$

and hence the minimum value is exactly $(f\nabla g)(s)$.

Suppose now that there exists $\delta > 0$ such that

$$\inf_{x>\delta} g(x) = \inf_{x<-\delta} g(x) = 0.$$

For every $k \in \mathbb{N}$ consider two sequences $(t^k_h) \subset]\delta,+\infty[$ and $(s^k_h) \subset]-\infty,-\delta[$ such that

$$\sum_h g(t^k_h) < \frac{1}{k} \quad \text{and} \qquad \sum_h g(s^k_h) < \frac{1}{k}.$$

Let $a_k \in \mathbb{R}$ such that

$$f(a_k) < \inf_{\mathbb{R}} f + \frac{1}{k}.$$

We define $u_k \in GSBV(I)$ by setting

$$u_k(t) = \begin{cases} -\frac{1}{2}s & \text{if } t = -\frac{1}{2} \\[2ex] -\frac{1}{2}(s - a_k) + a_k t + \sum_{j=1}^{h} t_j^k & \text{if } -\frac{1}{h+1} < t \le -\frac{1}{h+2},\ h \in \mathbb{N} \\[2ex] \frac{1}{2}(s - a_k) + a_k t - \sum_{j=1}^{h} s_j^k & \text{if } \frac{1}{h+2} \le t < \frac{1}{h+1},\ h \in \mathbb{N} \\[2ex] \frac{1}{2}s & \text{if } t = \frac{1}{2}. \end{cases}$$

Notice that $u_k \in SBV_{\mathrm{loc}}(]-1/2, 1/2[\setminus\{0\})$, and that $\lim\limits_{t\to 0} u_k(t) = +\infty$; hence we have $0 \notin S_{u_k}$. We obtain

$$\int_{]-1/2,1/2[} f(u_k'(t))\,dt + \sum_{t\in S_{u_k}\cap[-1/2,1/2]} g(u_k(t_+) - u_k(t_-))$$

$$= f(a_k) + \sum_h g(t_h^k) + \sum_h g(s_h^k) \le \inf_{\mathbb{R}} f + \frac{3}{k}.$$

This shows that

$$\inf\Big\{ \int_{]-1/2,1/2[} f(u'(t))\,dt + \sum_{t\in S_u\cap[-1/2,1/2]} g(u(t_+) - u(t_-)) \ :$$
$$u \in GSBV(]-1/2, 1/2[),\ u(\frac{1}{2}_+) = -u(-\frac{1}{2}_-) = \frac{s}{2}\Big\} = \inf_{\mathbb{R}} f.$$

Here we tacitly assume the functional defined for all functions $u \in GSBV$ such that $\{t \in S_u \ : \ |u(t_+)| + |u(t_-)| = +\infty\} = \emptyset$, in order to have a meaningful expression for the summation. The same reasoning is valid for all $u \in GSBV$, without any restriction, if we define the values $g(-\infty)$ and $g(+\infty)$. Note that the inf is actually a minimum if and only if f reaches its minimum and there exist two points $t_0 > 0$, and $s_0 < 0$ such that $g(t_0) = g(s_0) = 0$ (in this case we take $a_k \equiv a$, where $f(a) = \min f$, $s_h^k \equiv s_0$, and $t_h^k \equiv t_0$).

If $\inf\limits_{\mathbb{R}} f \ne (f\nabla g)(s)$ the minimum values for the same boundary value problem are different in SBV and in $GSBV$.

Note that the condition $\inf\limits_{x>\delta} g(x) = 0$ is equivalent to $\liminf\limits_{t\to+\infty} g(t) = 0$. In fact, take $(t_h) \subset]\delta, +\infty[$ such that $\lim_h g(t_h) = 0$. If $\lim_h t_h = +\infty$

then $\liminf_{t\to+\infty} g(t) = 0$; otherwise, we can suppose $t_h \to T \geq \delta$. By the lower semicontinuity of g we have then $g(T) = 0$, and by the subadditivity, and the positivity, of g we get $g(kT) = 0$ for all $k = 1, 2, \ldots$ This again implies that $\liminf_{t\to+\infty} g(t) = 0$. In the same way we can rewrite the condition $\inf_{x<-\delta} g(x) = 0$. $\square$

Remark that in general the expression for F

$$F(u) = \int_I f(u'(t))\,dt + \sum_{x\in S_u\cap I} g(u(x_+) - u(x_-)), \tag{4.1}$$

does *not* give a meaningful energy on $GSBV(I)$ or on the space $\mathcal{A}$ as defined in Section 2, even for functions u such that $0 \notin S_u$, as shown in the next example.

Example 4.2. Consider $f(p) = |p|^2$, and the function g_1 given by

$$g_1(s) = \begin{cases} 1 & \text{if } s \notin \mathbb{Z} \\ \\ 0 & \text{if } s \in \mathbb{Z}. \end{cases} \tag{4.2}$$

We have $f^\infty(z) = g_1^0(z) = +\infty$ for $z \neq 0$; moreover g_1 is clearly subadditive and lower semicontinuous.

Define, for every function $u \in GSBV(I) \cap \mathrm{L}^1(I)$ such that $\{t \in S_u : |u(t_+)| + |u(t_-)| = +\infty\} = \emptyset$,

$$F_1(u) = \int_I |u'(t)|^2\,dt + \sum_{t\in S_u\cap I} g_1(u(t_+) - u(t_-)). \tag{4.3}$$

Then, the relaxation of F_1 in the strong topology of $\mathrm{L}^1(I)$ is the functional identically equal to 0. To prove this fact, it suffices to check that it is 0 on all functions of the form

$$u(t) = \begin{cases} a & \text{if } t < 0 \\ b & \text{if } t \geq 0, \end{cases}$$

and show that this implies that $\overline{F}_1$ be 0 on all piecewise constant functions in $SBV(I)$, that are dense in $\mathrm{L}^1(I)$.

We can take the function v, defined on $\mathbb{R}$ by

$$v(t) = \begin{cases} a & \text{if } t < -1 \\ a+k & \text{if } -\frac{1}{k^2} \le t < -\frac{1}{(k+1)^2},\ k \in \mathbb{N} \\ b+k & \text{if } \frac{1}{(k+1)^2} < t \le \frac{1}{k^2},\ k \in \mathbb{N} \\ b & \text{if } t > 1, \end{cases} \tag{4.4}$$

and the sequence $(v_h) \subset GSBV(I) \cap \mathrm{L}^1(I)$ of piecewise constant functions, defined by $v_h(t) = v(ht)$. This sequence converges in $\mathrm{L}^1(I)$ to the function u. Moreover, if $t \in S_{v_h}$ we have $v_h(t_+) - v_h(t_-) = 1$ if $t < 0$, and $v_h(t_+) - v_h(t_-) = -1$ if $t > 0$, and hence $F_1(v_h) = 0$. This shows that $\overline{F}_1(u) = 0$. Since the sequence v_h modifies u only on an arbitrarily small neighbourhood of 0, a similar procedure shows that F_1 is 0 on all piecewise constant $u \in SBV(I)$.

In the same way we obtain that the relaxation of F_1 with respect to a.e. convergence is equal to 0 on the whole $GSBV(I)$. Notice that F_1 is lower semicontinuous on $SBV(I)$ with respect to the weak convergence of BV. □

Example 4.3 Consider the functional F_1 defined on SBV by (4.3). Then, by formula (3.13) we get the following expression for the "cost" function V_1 defined in (3.11):

$$\begin{aligned} V_1(x,s) = {} & \inf\{x\,a^2 + g_1(s-ax) : a \in \mathbb{R}\} \\ = {} & \min\Big\{\frac{(|s| - [\,|s|\,])^2}{x}, \frac{(|s| - [\,|s|\,] - 1)^2}{x}, 1\Big\} \end{aligned}$$

(we denote by $[t]$ the integral part of $t \in \mathbb{R}$). In the case of the function v defined in (4.4) we have $v(x_+) - v(-x_-) = b - a$ for all $x \in I$, and hence

$$V_1(2x, v(x_+) - v(-x_-)) = V_1(2x, b-a) =$$

$$= \min\Big\{\frac{(|b-a| - [\,|b-a|\,])^2}{2x}, \frac{(|b-a| - [\,|b-a|\,] - 1)^2}{2x}, 1\Big\}.$$

We obtain then

$$
\begin{aligned}
L(v) &= \liminf_{x\to 0+} V_1(2x, v(x_+)-v(-x_-)) \\
&= \begin{cases} 0 & \text{if } b-a\in \mathbb{Z} \\ 1 & \text{if } b-a\notin \mathbb{Z} \end{cases} = g_1(b-a).
\end{aligned}
\tag{4.5}
$$

Note that in the previous example a recovery sequence for the function u, along which we can reach the value of the the relaxation can be obtained simply by considering the "truncations" of u, and choosing among them those with minor "cost" for V_1; for example we can take $u_h = u \wedge (a+h)$. In general though, we cannot restrict our analysis to these functions, as explained in the next example.

Example 4.4. Consider $f(p)=|p|^\alpha$, with $\alpha>1$, and g_1 as in (4.2), and the functional

$$
F_\alpha(u)=\int_I |u'(t)|^\alpha\,dt+\sum_{t\in S_u\cap I} g_1(u(t_+)-u(t_-)). \tag{4.6}
$$

Fixed $\beta>0$, consider the function

$$
v_\beta(t)=\begin{cases} [-\frac{1}{t}]+|t|^\beta & \text{if } t<0 \\ [\frac{1}{t}] & \text{if } t>0. \end{cases} \tag{4.7}
$$

For every $x\in]0,1[$, we have

$$
v_\beta(x_+)-v_\beta(-x_-)=-|t|^{-\beta},
$$

and hence, since in this case $V(x,u,v)=V(x,u+w,v+w)$ for all $x,u,v,w\in \mathbb{R}$, we get

$$
\begin{aligned}
V(x,v_\beta(x_-),v_\beta(-x_+)) &= V(x,0,|x|^\beta) \\
&= \min\{1, 2^{1-\alpha}\, x^{1-\alpha+\alpha\beta}\}.
\end{aligned}
$$

We then obtain

$$L(v_\beta) = \liminf_{x \to 0+} V(x, v_\beta(x_-), v_\beta(-x_+)) = \min\Big\{1, 2^{1-\alpha} \lim_{k \to +\infty} k^{\alpha-1-\alpha\beta}\Big\}$$
$$= \begin{cases} 1 & \text{if } \beta < 1 - \frac{1}{\alpha} \\ 2^{1-\alpha} & \text{if } \beta = 1 - \frac{1}{\alpha} \\ 0 & \text{if } \beta > 1 - \frac{1}{\alpha}. \end{cases}$$

Note that if $\beta \geq 1 - \frac{1}{\alpha}$ the truncations of v_β do not give a good approximation for $\overline{F}_\alpha(v_\beta)$, since we have

$$\liminf_{T \to +\infty} F_\alpha(v_\beta \wedge T) = 1.$$

Remark 4.5. The functions f^x defined by $f^x(t) = x\, f\big(\frac{t}{x}\big)$ approach the function f^∞ as $x \to 0+$. This may suggest, since $f^\infty(t) = +\infty$ when $t \neq 0$, that $L(u)$ could be expressed directly in terms of g (note that $f^\infty \nabla g = g$). The previous example shows that this is not the case, since if $\beta = 1 - \frac{1}{\alpha}$ we have $L(u) \neq g_1(t)$ for all $t \in \mathrm{I\!R}$.

Example 4.6. Let $f = f(t, p) = |t||p|^2$, and $g(t) \equiv 1$. By Theorem 3.1 we get that $\overline{H} = F_0$; *i.e.,* $L(u) = 0$ for all u. To prove this, it suffices to show that $V(x, s, t) = 0$ for all $x, s, t \in \mathrm{I\!R}$, and this, in its turn, will follow from

$$\inf\{\int_{-x}^{x} f(t, u'(t))\, dt \;:\; u \in \mathrm{H}^1_0(]{-x}, x[), u(-x) = -a, u(x) = a\} = 0$$

To check that this infimum is 0 it suffices to choose for example the sequence

$$u_h(t) = \frac{t}{|t|}\left(\frac{1}{h}\log\Big(\frac{|t|}{x}\Big) + a\right)^+$$

(whose choice follows from the examination of the Euler-Lagrange equation).

In this case the functional F is not lower semicontinuous in SBV, and its relaxation is given by F_0.

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