Phase-field modelling of cohesive fracture. Part II: Reconstruction of the cohesive law

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This is the second paper of a three-part work the main aim of which is to provide a unified consistent framework for the phase-field modelling of cohesive fracture. Building on the theoretical foundations of the first paper, where Γ-convergence results have been derived, this second paper presents a systematic procedure for constructing phase-field models that reproduce prescribed cohesive laws. By either selecting the degradation function and determining the damage potential or vice versa, we enable the derivation of multiple phase-field models that exhibit the same cohesive fracture behavior but differ in their localized phase-field evolution. This methodology provides a flexible and rigorous strategy for tailoring phase-field models to specific cohesive responses, as shown by the several examples worked out. The mechanical responses associated with these examples, highlighting their features and validating the theoretical results, are investigated in the third paper from a more engineering-oriented and applied perspective.

Contents

1 Introduction

1.1 Background and Motivation

Accurately predicting fracture initiation and propagation is crucial for designing resilient mechanical systems and preventing structural failures. Griffith's brittle fracture theory [Gri21], despite its simplicity and widespread use, assumes that fracture energy dissipates entirely upon crack formation, neglecting any interaction between crack surfaces (Fig. 1-a). This leads to two major limitations: the inability to predict crack initiation in a pristine material [Mar10, TLB+18, KBFLP20] and unrealistic scale effects [Mar23].

To address these shortcomings, cohesive fracture models introduce nonzero forces between crack surfaces, the cohesive forces, making the surface energy density a function of the displacement jump. Originally proposed by Dugdale and Barenblatt [Dug60, Bar62], these models incorporate a critical stress and a characteristic length, offering a more realistic fracture representation (Fig. 1-b). Building on this foundation, the variational formulation of fracture, first developed for brittle materials in [FM98, AB95], has been extended to cohesive models in [BFM08].

The phase-field regularization of brittle fracture, [AT92, BFM00, Foc01], allows cracks to emerge naturally within the Γ-convergence framework, [Bra98], and enables the numerical simulation of complex fracture processes, making it nowadays a leading approach in fracture mechanics [BMMS14, MBK15]. For instance it captures crack nucleation, even without pre-existing singularities, and models complex crack patterns with straightforward numerical methods, often based on alternate minimization schemes.

Phase-field models for brittle fracture inherit a limitation from Griffith's theory: they do not allow independent control of critical strength, fracture toughness, and regularization length. As a consequence, the capability to model crack nucleation in both smooth and notched domains [BFM08, TLB+18, LPDFL24] is restricted.

Cohesive fracture models turned out to being able to overcome this limitation of Griffith's brittle fracture theory, [Mar23]. Building on the work of [PM10b, PM10a], [LCK11, LG11] developed a standalone gradient-damage (phase-field) model, not derived from a free-discontinuity fracture model, that is able to describe a cohesive fracture behavior. The model has been validated in a one-dimensional uniaxial tension scenario by a closed-form solution and in higher-dimensional problems with numerical simulations of large-scale domains. Despite its potential, this cohesive phase-field model was initially overlooked by many in the fracture mechanics community.

Later on, [CFI16] successfully developed a mathematically consistent variational phase-field model that regularizes the free-discontinuity cohesive fracture problem of [BFM08] (see also [CFI24, CFI25, Col25] for vectorial counterparts). The numerical implementation of this model, done by [FI17], encountered several challenges, partially overcome by adopting a backtracking algorithm and by further regularizing the degradation function, similarly to what has been done in [LCM23, LCM25]. Additionally, difficulties in adjusting the elastic degradation function to fit specific cohesive laws, along with the

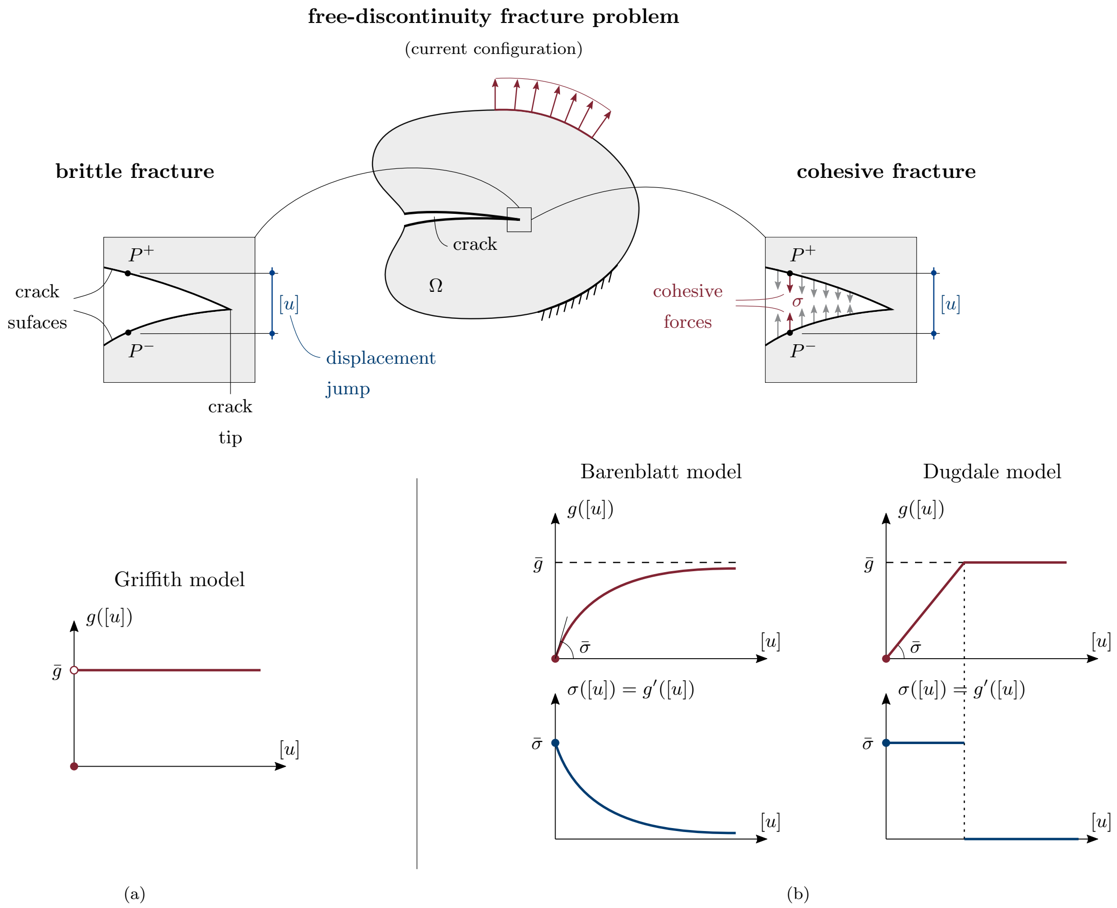

Figure 1: Qualitative trends of the surface energy density G_{lf} and associated cohesive stress $\bar{\sigma}$ with respect to the crack opening (displacement jump) δ for brittle and cohesive fracture models: (a) Griffith, (b) Dugdale and Barenblatt cohesive models. G_{lc} and $\bar{\sigma}_0$ denote the fracture toughness and the initial critical stress, respectively.

use of a fixed quadratic phase-field dissipation function, likely constrained the model's flexibility, limiting its wider acceptance in the fracture mechanics community.

More impactful progress was made by [Wu17, WN18], who, following [LCK11], proposed a next-generation phase-field cohesive fracture model within the familiar variational framework of gradient-damage models [PM10b, PM10a]. These models use polynomial crack geometric functions and rational energetic degradation functions, enabling the independent adjustment of critical stress, fracture toughness, and regularization length to match specific softening behaviors. A defining feature of the models proposed by [CFI16, Wu17, WN18] is the incorporation of the regularization length into the elastic degradation function, enhancing their ability to describe cohesive fracture failures. Examples of commonly used traction-separation laws in these models include linear, exponential, hyperbolic, and Cornelissen [CHR86] laws.

Despite the model's flexibility and accuracy, determining how to set the material functions to obtain a specific target softening traction-separation law was still unclear. A breakthrough came from [FFL21], who introduced an integral relation that links a single unknown function, defining both the degradation function and phase-field dissipation function, to the desired traction-separation law. In these phase-field models, both the regularization length and mesh size were found to have minimal impact on the overall mechanical response, provided the mesh adequately resolves the regularization length. One of the main goals of this Part II paper is exactly to validate analitically and to generalise the model-finding process to match a target cohesive fracture law.

Indeed, having established in the Part I paper [ACF25a] a Γ-convergence result in a one-dimensional setting for a broad class of phase-field energies for the description of cohesive fracture that encompasses the models proposed in [CFI16], [Wu17, WN18], [FFL21], and [LCM23, LCM25], in this second paper, we develop a general and flexible procedure for constructing phase-field models that reproduce prescribed cohesive traction-separation laws exhibiting either a linear or a superlinear behaviour for small jump amplitudes (for the analysis of the corresponding vector-valued model see [Col25]). This is achieved by either fixing the degradation function and determining the damage potential or vice versa.

It is worth pointing out that our approach rigorously justifies and extends the results for the linear case of [FFL21], enabling the derivation of multiple phase field models that correspond to the same target cohesive law but exhibit distinct localized phase field evolutions, as shown by the several examples worked out. In addition, the superlinear case has been shown to be particularly relevant for describing fracture processes in ductile materials by strain-gradient plasticity models [FCO14]. Our results, that provide a unified framework for the description of cohesive fracture, are validated in the Part III paper, [ACF25b], where the mechanical responses of different phase-field models are investigated in depth with a more engineering-oriented perspective. Moreover, the third paper also contains in its introduction a state-of-art timeline that illustrated the key milestones, their interconnections, and how this work fits within the broader variational framework of fracture mechanics.

1.2 A recap of the phase-field model and Γ-convergence

We briefly recall the phase-field model introduced and analyzed in [ACF25a] that encompasses at the same time several cohesive phase-field models present in the literature (see [CFI16, Wu17, WN18, LCM23, LCM25]). With this aim consider functions $l,\, Q\,, \omega : [0,1] \to [0,\infty)$, $f : [0,1) \to [0,\infty)$ and $\varphi : [0,\infty) \to [0,\infty)$ satisfying:

(Hp 1) $l,\, Q \in C^0([0,1],[0,\infty))$ with $l(1)=1$, $l^{-1}(\{0\}) = Q^{-1}(\{0\}) = \{0\}$, Q increasing in a right neighbourhood of the origin, and such that

$$[0,1) \ni t \mapsto f(t) := \left(\frac{l(t)}{Q(1-t)}\right)^{1/2}, \tag{1.1}$$

is increasing in a right neighbourhood of the origin;

(Hp 2) $\omega \in C^0([0,1],[0,\infty))$ such that $\omega^{-1}(\{0\}) = \{0\}$, and the following limit exists

$$\lim_{t\to 0^+} \left(\frac{\omega(t)}{Q(t)}\right)^{1/2} =: \overline{\varsigma} \in [0,\infty]\,; \tag{1.2}$$

(Hp 3) $\varphi \in C^0([0,\infty))$, φ is non-decreasing and $\varphi(\infty) := \lim_{t\to\infty} \varphi(t) \in (0,\infty)$;

(Hp 4) $\varphi^{-1}(0) = \{0\}$, and φ is (right-)differentiable in 0 with $\varphi'(0^+) \in (0,\infty)$.

Note that $[0,1] \ni t \mapsto f_\varepsilon(t)$ is increasing in a right neighbourhood of the origin, as well.

We adopt standard notation for Sobolev, BV, GBV spaces as in [AFP00]. We refer to the same book for results related to those function spaces. Then define the functionals $\mathcal{F}_\varepsilon : L^1(\Omega, \mathbb{R}^2) \times \mathcal{A}(\Omega) \to [0, \infty]$ by

$$\mathcal{F}_\varepsilon(u, v, A) := \int_A \left(\varphi(\varepsilon f^2(v))|u'|^2 + \frac{\omega(1-v)}{4\varepsilon} + \varepsilon |v'|^2 \right) \mathrm{d}x \tag{1.3}$$

if $(u, v) \in H^1(\Omega, \mathbb{R} \times [0, 1])$ and ∞ otherwise. In particular, in formula (1.3) above, the functions $[0, 1) \mapsto \varphi(\varepsilon f^2(t))$ are extended by continuity to $t = 1$ with value $\varphi(\infty)$ for every $\varepsilon > 0$. Now let $\mathscr{F}_{\overline{\varsigma}} : L^1(\Omega) \times \mathcal{A}(\Omega) \to [0, \infty]$ be given by

$$\mathscr{F}_{\overline{\varsigma}}(u, A) := \begin{cases} \displaystyle\int_A h^{**}_{\overline{\varsigma}}(|u'|) \mathrm{d}x + (\varphi'(0^+))^{1/2} \overline{\varsigma} |D^c u|(A) + \int_{J_u \cap A} g(|[u]|) \mathrm{d}\mathcal{H}^0 & \\ & \text{if } u \in GBV(\Omega) \\ \infty & \text{otherwise} \end{cases} \tag{1.4}$$

where for $\varsigma \in [0, \infty]$ we define $h_\varsigma : [0, \infty) \to [0, \infty)$ by

$$h_\varsigma(t) := \inf_{\tau \in (0, \infty)} \left\{ \varphi\left(\frac{1}{\tau}\right) t^2 + \frac{\varsigma^2}{4} \tau \right\}, \tag{1.5}$$

(in particular, $h_0(t) = 0$ and $h_\infty(t) = \varphi(\infty) t^2$ for every $t \in [0, \infty)$), $h^{**}_{\overline{\varsigma}}$ denotes the convex envelope of $h_{\overline{\varsigma}}$, and $g : [0, \infty) \to \mathbb{R}$ is defined by

$$g(s) := \inf_{(\gamma, \beta) \in \mathfrak{U}_s} \int_0^1 \Big(\omega(1 - \beta) \big(\varphi'(0^+) f^2(\beta) |\gamma'|^2 + |\beta'|^2 \big) \Big)^{1/2} \mathrm{d}x\,, \tag{1.6}$$

where $\mathfrak{U}_s$ is

$$\mathfrak{U}_s = \{\gamma, \beta \in H^1((0, 1)) : \gamma(0) = 0\,, \gamma(1) = s\,, 0 \le \beta \le 1\,, \beta(0) = \beta(1) = 1\}\,, \tag{1.7}$$

In [ACF25a, Theorem 1.1] we have established the following Γ-convergence result (cf. [DM93, Bra98] for theory and results on Γ-convergence). In what follows we adopt the convention $0 \cdot \infty = 0$.

Theorem 1.1. *Assume (Hp 1)-(Hp 4), and (1.2) hold with $\overline{\varsigma} \in (0, \infty]$. Let $\mathcal{F}_\varepsilon$ be the functional defined in (1.3). Then, for all $(u, v) \in L^1(\Omega, \mathbb{R}^2)$*

$$\Gamma(L^1)\text{-}\lim_{\varepsilon \to 0} \mathcal{F}_\varepsilon(u, v) = F_{\overline{\varsigma}}(u, v)\,,$$

where, $F_{\overline{\varsigma}} : L^1(\Omega, \mathbb{R}^2) \times \mathcal{A}(\Omega) \to [0, \infty]$ is defined by

$$F_{\overline{\varsigma}}(u, v) := \begin{cases} \mathscr{F}_{\overline{\varsigma}}(u) & \text{if } v = 1 \ \mathcal{L}^1\text{-a.e. on } \Omega \\ \infty & \text{otherwise.} \end{cases} \tag{1.8}$$

In particular, if $\mathscr{F}_\infty(u) < \infty$ above, then $|D^c u|(\Omega) = 0$ (cf. (1.4)), and thus we conclude that $u \in GSBV(\Omega)$ with $u' \in L^2(\Omega)$

We recall that changing the scaling in the degradation function $\varphi(\varepsilon f(t))$ to $\varphi(\gamma_\varepsilon f(t))$, with ${}^{\varepsilon}/_{\gamma_\varepsilon} \to 0$ as $\varepsilon \to 0^+$, then the corresponding functionals (1.3) Γ-converge to a brittle type functional (cf. [ACF25a, Theorem 1.2]).

1.3 Contents of this Paper

The great generality in which Theorem 1.1 is proven is instrumental to address the issue of constructing the phase-field model in order to obtain an assigned cohesive law g. In Section 3 we take advantage of Theorem 1.1 to solve the latter problem for a large class of concave functions with either a linear or a superlinear behaviour for small jump amplitudes, including several examples which are typically used in the engineering community for numerical simulations (cf. Section 3.3). The latter problem has already been discussed from a numerical and engineering perspective in [Wu17, WN18, FFL21, LCM25, Wu24].

Section 2 is devoted to provide an alternative characterization of the surface energy densities in the phase-field model. Our approach is different to the one introduced in [BCI21, BI24] (cf. the comments before Proposition 2.5 for a more detailed comparison). One of the main ideas we introduce is rewriting in a convenient way the minimization formula that defines the surface energy density $g(s)$ for every jump amplitude $s \geq 0$ (cf. (1.6) in Theorem 1.1). Indeed, the latter entails the solution of a vector-valued, one-dimensional variational problem with Dirichlet boundary conditions. Via a change of variable we are able to reduce it to a minimum problem for a family of scalar and one-dimensional functionals $G_{m,s}$, defined in (2.15), depending on the deformation gradient, parametrized by the minimum value m of the phase-field variable and finite on $W^{1,1}$ functions satisfying a suitable Dirichlet boundary condition depending on s (cf. Proposition 2.5). The new minimum problem is obtained by minimizing first each functional $G_{m,s}$ on L^1, and then minimizing the corresponding minimimum value $\mathfrak{G}_s(m)$ on the interval $(0,1)$ (cf. (2.17) and (2.25)), namely $g(s) = \inf_{m\in(0,1)} \mathfrak{G}_s(m)$ where $\mathfrak{G}_s(m) = \inf_{L^1} G_{m,s}$.

Another key step in our approach is the variational analysis of the energies $G_{m,s}$ introduced in (2.15). In particular, for each m we determine the relaxed functional with respect to the L^1 topology of $G_{m,s}$, which turns out to be finite on BV, and to have a unique minimizer that it is either $W^{1,1}$ or $SBV \setminus W^{1,1}$ with the singular part of the distributional derivative concentrated in the point $t = m$ in the second instance (cf. Theorem 2.7 and Corollary 2.8). The analysis of the function $\mathfrak{G}_s$ exploits an auxiliary key function Φ, defined in (2.30), determining its monotonicity properties. More precisely, $\mathfrak{G}_s$ turns out to be increasing on intervals for which the minimizer $w_{m,s}$ of the relaxation of $G_{m,s}$ is $SBV \setminus W^{1,1}$ and decreasing otherwise (for the precise statement see Lemma 2.13). In addition, the regularity of the minimizer $w_{m,s}$ depends on whether m belongs or not to the s sublevel set of Φ. Given this, we are able to show that $g(s)$ is reduced either to $\inf\{\mathfrak{G}_s(m) : \Phi(m) = s\}$ or to its behaviour for $m = 0$ and $m = 1$, for every s in the range of Φ (for the precise statement see Proposition 2.14). In particular, in case Φ is strictly decreasing then g turns out to be concave and of class C^1, with derivative expressed exactly in terms of $\Phi^{-1}(s)$ (cf. Theorems 2.15). In turn, this information is used in Theorems 3.1 and 3.2 for the linear case, and in Theorems 3.4 and 3.5 in the superlinear one, to define a phase-field model choosing either the damage potential and fixing the degradation function in the former or vice versa in the latter, with corresponding Γ-limit that has surface energy density exactly equal to the assigned function g. With this aim we are led to solve the Abel integral equation corresponding to the equality $g(s) = \mathfrak{G}_s(\Phi^{-1}(s))$ (cf. (3.4)). A similar use of Abel integral equation is also proposed in [FFL21, Section 3.2] for the heuristic calibration of the damage potential in the phase-field model for numerical convergence purposes. Instead, in the linear case if Φ is non-decreasing the function g always corresponds to the Dugdale cohesive law (cf. Theorems 2.18).

2 Analysis of the surface energy densities arising in the phase-field approximation

In this section we study in depth the surface energy densities g obtained via the phase-field approximation in Theorem 1.1. For solving the reconstruction problem it is key to provide an equivalent and more maneageable characterization with respect to its definition in formula (1.6).

With this aim, throughout the whole section l, Q, ω and φ are assumed to satisfy (Hp 1)-(Hp 4) even if not explicitly mentioned.

It is convenient to recall several qualitative properties of g that follows directly from (1.6) established in [ACF25a, Section 2.3]. With this aim we introduce the function $\Psi : [0,1] \to [0,\infty)$ given by

$$\Psi(t) := \int_0^t \omega^{1/2}(1-\tau)d\tau\,. \tag{2.1}$$

We start with the case in which g is linear for small jump amplitudes.

Proposition 2.1. *Under the assumptions of Theorem 1.1 with $\overline{\varsigma} \in (0,\infty)$, the function g defined in (1.6) enjoys the following properties:*

(i) $g(0)=0$, *g is non-decreasing, and subadditive;*

(ii) $g(s) \le (\varphi'(0^+))^{1/2}\,\overline{\varsigma} s \wedge 2\Psi(1)$, *$g$ is Lipschitz continuous with Lipschitz constant equal to* $(\varphi'(0^+))^{1/2}\,\overline{\varsigma}$*;*

(iii) the ensuing limit exists and

$$\lim_{s\to\infty} g(s) = 2\Psi(1); \tag{2.2}$$

(iv) the ensuing limit exists and

$$\lim_{s\to 0} \frac{g(s)}{s} = (\varphi'(0^+))^{1/2}\,\varsigma. \tag{2.3}$$

We then consider the superlinear setting.

Proposition 2.2. *Under the assumptions of Theorem 1.1 with $\overline{\varsigma} = \infty$, the function g defined in (1.6) enjoys the following properties:*

(i) $g(0)=0$, *g is non-decreasing, and subadditive;*

(ii) $g \in C^0([0,\infty))$, $0 \le g(s) \le \widehat{g}(s)$ *for every* $s \ge 0$, *where*

$$\widehat{g}(s) := \inf_{x\in(0,1]} \left\{ 2(\Psi(1)-\Psi(1-x)) + \left(\varphi'(0^+)l(1-x)\frac{\omega(x)}{Q(x)}\right)^{1/2} s \right\}, \tag{2.4}$$

$\widehat{g} \in C^0([0,\infty))$ *is concave, non-decreasing,* $\widehat{g}(s) \le 2\Psi(1)$, *and*

$$\lim_{s\to 0^+} \frac{\widehat{g}(s)}{s} = \infty; \tag{2.5}$$

(iii)

$$\lim_{s\to\infty} g(s) = 2\Psi(1);$$

(iv)

$$2^{-1/2} \le \liminf_{s\to 0^+} \frac{g(s)}{\widehat{g}(s)} \le \limsup_{s\to 0^+} \frac{g(s)}{\widehat{g}(s)} \le 1\,. \tag{2.6}$$

For the sake of notational simplicity we also assume that the function φ in the definition of the degradation function obeys $\varphi'(0^+)=1$ (clearly, this restriction can be easily dropped). Finally, we also assume that Q, l and ω satify

(Hp 5) $\vartheta:[0,1)\to[0,\infty)$ is strictly increasing, where

$$\vartheta(t):=\frac{\omega(1-t)}{Q(1-t)}l(t)\,.$$

Note that $\vartheta(t)=\omega(1-t)f(t)$ (cf. (1.1)). Necessarily, $\vartheta\in C^0([0,1))$, with $\vartheta(0)=0$ and $\vartheta(1^-)=\bar{\varsigma}^2\in(0,\infty]$ (cf. (1.2) and recall that $l(1)=1$). For some results we will need to strengthen (Hp 5) (cf. assumption (Hp 5′) below).

With this notation at hand, formula (1.6) rewrites as

$$g(s)=\inf_{(u,v)\in\mathfrak{U}_s}\mathcal{G}(u,v) \tag{2.7}$$

for every $s\in[0,\infty)$, where $\mathfrak{U}_s$ is defined in (1.7) (see also $\mathfrak{U}_{s,1}$ below), and $\mathcal{G}:W^{1,1}((0,1);\mathbb{R}^2)\to[0,\infty)$ is defined by

$$\mathcal{G}(u,v):=\int_0^1\Big(\vartheta(v)|u'|^2+\omega(1-v)|v'|^2\Big)^{1/2}\mathrm{d}x\,. \tag{2.8}$$

We remark that the infimum defining g can be equivalently taken on $\mathfrak{U}_{s,1}$, where

$$\mathfrak{U}_{s,1}:=\{u\in W^{1,1}((0,1)),v\in H^1((0,1)):u(0)=0\,,u(1)=s\,,\,0\le v\le 1\,,v(0)=v(1)=1\}\,, \tag{2.9}$$

as the functional $\mathcal{G}$ is continuous in the $W^{1,1}$ topology.

2.1 An equivalent characterization of g

In this section we introduce a different minimization problem defining g which will be more suitable for our purposes. With this aim, in order to reduce first the space of competitors for v to those satisfying suitable monotonicity properties we first prove a technical lemma.

Lemma 2.3. *Let $v\in W^{1,p}((0,1))\cap C^0([0,1])$ with $p\in[1,\infty]$, and for every $x\in[0,1]$ set*

$$T_0v(x):=\inf_{t\in[0,x]}v(t),\quad \textit{and}\quad T_1v(x):=\inf_{t\in[x,1]}v(t)\,.$$

Then,

(i) T_0v, $T_1v\le v$ on $[0,1]$, T_0v is non-increasing with $T_0v(0)=v(0)$, T_1v is non-decreasing with $T_1v(1)=v(1)$;

(ii) T_0v, $T_1v\in W^{1,p}((0,1))\cap C^0([0,1])$, with $|(T_0v)'|$, $|(T_1v)'|\le|v'|$ $\mathcal{L}^1$ a.e on $(0,1)$;

(iii) $(T_iv)'=0$ $\mathcal{L}^1$ a.e on $\{T_iv\neq v\}$, $i\in\{0,1\}$.

Proof. We give the proof of the statements for T_0v, that for T_1v being analogous.

The monotonicity of T_0v follows straightforwardly from the very definition, as well as the fact that $T_0v\le v$.

Let $x_1, x_2 \in (0,1)$, with $x_1 < x_2$, if $T_0 v(x_2) < T_0 v(x_1)$ there exist $y_1 \in [0, x_1]$ and $y_2 \in (x_1, x_2]$ such that

$$v(x_1) \geq v(y_1) = T_0 v(x_1) > T_0 v(x_2) = v(y_2)\,.$$

In particular, it follows that

$$|T_0 v(x_1) - T_0 v(x_2)| \leq |v(x_1) - v(y_2)| \leq \int_{x_1}^{x_2} |v'(t)| \mathrm{d}t\,.$$

Therefore, $T_0 v$ is absolutely continuous and the statements in item (ii) holds.

Finally, note that if $T_0 v(x) = v(y)$ for some $y \in [0, x]$, then $T_0 v(t) = v(y)$ for every $t \in [y, x]$ by item (i). Thus, if $I = (x_1, x_2)$ is a connected component of the open set $\{T_0 v < v\}$ (by continuity of both v and $T_0 v$), then $T_0 v(x) = v(x_1)$ for every $x \in I$, and finally $T_0 v(x_2) = v(x_1)$ by continuity of $T_0 v$. $\square$

Proposition 2.4. *Assume (Hp 1)-(Hp 5). For every* $s \in [0, \infty)$

$$g(s) = \inf_{(u,v) \in \mathfrak{U}_{s,1}} \mathcal{G}(u, v). \tag{2.10}$$

Moreover, for every $s \in (0, \infty)$*,* $\eta > 0$*, and* $(u, v) \in \mathfrak{U}_{s,1}$ *there exists* $\overline{v} \in H^1((0,1))$ *such that* $\overline{v}(0) = \overline{v}(1) = 1$*,* $\overline{v}$ *is strictly positive, strictly decreasing on* $[0, x_0]$*, strictly increasing on* $[x_0, 1]$*, for some* $x_0 \in (0,1)$*, and with Lipschitz inverse when restricted both to* $[0, x_0]$ *and to* $[x_0, 1]$*, and such that*

$$\mathcal{G}(u, \overline{v}) \leq \mathcal{G}(u, v) + \eta\,. \tag{2.11}$$

Proof. Fix $(u, v) \in \mathfrak{U}_{s,1}$ such that $\mathcal{G}(u, v) < \infty$. Let (z_k) be an increasing sequence of positive functions in $L^\infty((0,1))$ such that $z_k(x) \to |u'(x)|$ for $\mathcal{L}^1$-a.e. $x \in (0,1)$. In particular setting

$$w_k := \frac{s}{\int_0^1 z_k \mathrm{d}x} z_k$$

we have that $(u_k, v) \in \mathfrak{U}_{s,1}$ with $u_k(t) := \int_0^t w_k \mathrm{d}x \in W^{1,\infty}((0,1))$ and

$$\lim_{k \to \infty} \mathcal{G}(u_k, v) \leq \mathcal{G}(u, v)\,,$$

so that (2.10) easily follows.

Let v be as in the statement above, without loss of generality we assume that $v \in C^0([0,1])$. Let $v_k := v \vee \frac{1}{k}$ then $v_k(x) \to v(x)$, $\omega(1 - v_k(x))|v_k'(x)|^2 = 0$ $\mathcal{L}^1$-a.e. on $\{v \leq \frac{1}{k}\}$, and $v_k = v$ otherwise, so that $\omega(1 - v_k(x))|v_k'(x)|^2 \leq \omega(1 - v(x))|v'(x)|^2$ $\mathcal{L}^1$-a.e. $(0,1)$. In particular, $\mathcal{G}(u, v_k) \to \mathcal{G}(u, v)$ for $k \to \infty$, as for every $k \geq 2$

$$\begin{aligned}
\mathcal{G}(u, v_k) &\leq \int_0^1 \Big(\vartheta(v_k)|u'|^2 + \omega(1-v)|v'|^2\Big)^{1/2} \mathrm{d}x \\
&= \int_{\{v \leq \frac{1}{2}\}} \Big(\vartheta(v_k)|u'|^2 + \omega(1-v)|v'|^2\Big)^{1/2} \mathrm{d}x + \int_{\{v > \frac{1}{2}\}} \Big(\vartheta(v)|u'|^2 + \omega(1-v)|v'|^2\Big)^{1/2} \mathrm{d}x.
\end{aligned}$$

Hence, without loss of generality we can suppose v strictly positive. Let then $x_0 \in (0,1)$ be a global minimum point of v, and define $\hat{v} : [0,1] \to [0,1]$ by

$$\hat{v}(x) := T_0 v(x)\chi_{[0,x_0]}(x) + T_1 v(x)\chi_{[x_0,1]}(x) .$$

Lemma 2.3 yields that $\hat{v} \in H^1((0,1))$, $\hat{v}(0) = \hat{v}(1) = 1$, $\hat{v}$ is non-increasing on $[0,x_0]$ and non-decreasing on $[x_0,1]$. Moreover, by item (iii) there

$$\mathcal{G}(u,\hat{v}) \le \mathcal{G}(u,v)$$

as ϑ is non-decreasing and $\hat{v}' = 0$ $\mathcal{L}^1$-a.e. on $\{\hat{v} \ne v\}$. Next, consider the piecewise affine function

$$\zeta(x) := \begin{cases} \frac{x}{x_0} & \text{if } 0 \le x \le x_0 \\ -\frac{x-x_0}{1-x_0} + 1 & \text{if } x_0 \le x \le 1, \end{cases}$$

observe that $\hat{v} - \delta\zeta$ is strictly positive for δ small enough, and converges to $\hat{v}$ monotonically as $\delta \to 0$, so that $\mathcal{G}(u, \hat{v} - \delta\zeta) \to \mathcal{G}(u,\hat{v})$ as $\delta \to 0$. Therefore, (2.11) is satisfied. Moreover, the restrictions of $\hat{v} - \delta\zeta$ to $[0,x_0]$ and $[x_0,1]$ are invertible with, respectively, derivative less than or equal to $-\delta$ and bigger than or equal to δ, and thus they are Lipschitz continuos. □

We are now ready to give a new characterization of g as a minimum problem obtained by first minimizing over the variable u (belonging to a suitable admissible set) a family of functionals labeled through the minimum value of v, and then minimizing on the latter. This is alternative to the approach in [BCI21, BI24] where g is expressed as a minimum problem of a single functional depending on v loosely speaking using u to change variables. With this aim it is convenient to denote by $\zeta_{m,s}$ the unique affine function with $\zeta_{m,s}(2m-1) = 0$ and $\zeta_{m,s}(1) = s$.

Proposition 2.5. *Assume (Hp 1)-(Hp 5). For every $m \in (0,1)$ let $G_m : W^{1,1}((2m-1,1)) \to [0,\infty]$ be given by*

$$G_m(w) := \int_{2m-1}^{1} \Big(\vartheta_m(t)|w'|^2 + \omega_m(1-t)\Big)^{1/2} \mathrm{d}t \tag{2.12}$$

where ϑ_m and ω_m are, respectively, the even extensions to $[2m-1,m]$ with respect to $t = m$ of ϑ and $\omega(1-\cdot)$ restricted to $[m,1]$. Then, for every $s \in [0,\infty)$

$$g(s) = \inf_{m\in(0,1)} \inf\{G_m(w) \ : \ w \in \zeta_{m,s} + W^{1,1}_0((2m-1,1))\} . \tag{2.13}$$

Proof. Let $s \in (0,\infty)$ and $m \in (0,1)$. In view of Proposition 2.4 we may take $v \in H^1((0,1)) \cap C^0([0,1])$ such that $0 \le v \le 1$, $v(0) = v(1) = 1$, $v(x_0) = \min_{[0,1]} v = m$, for some $x_0 \in (0,1)$, and $v|_{[0,x_0]}$ and $v|_{[x_0,1]}$ invertible with Lipschitz continuous inverses. For every $u \in \zeta_{1/2,s} + W^{1,1}_0((0,1))$, define $w \in \zeta_{m,s} + W^{1,1}_0((2m-1,1))$ by

$$w(t) = \begin{cases} u \circ (v|_{[0,x_0]})^{-1}(2m-t) & t \in [2m-1,m] \\ u \circ (v|_{[x_0,1]})^{-1}(t) & t \in [m,1]. \end{cases} \tag{2.14}$$

By changing variable on the intervals $[0,x_0]$ and $[x_0,1]$ by means of the one-dimensional area formula [AT04, Theorem 3.4.6] we obtain the equality $\mathcal{G}(u,v) = G_m(w)$. Vice versa, being v absolutely continuous, given $w \in \zeta_{m,s} + W_0^{1,1}((2m-1,1))$ one can define $u \in \zeta_{1/2,s} + W_0^{1,1}((0,1))$ via (2.14) in a way that $\mathcal{G}(u,v) = G_m(w)$. In particular, by equality (2.10) in Proposition 2.4 we can conclude that (2.13) holds. □

2.2 Analysis of the equivalent characterization

In this section we study the new characterization of g provided in Proposition 2.5. With this aim, we recall first a relaxation result (cf. [GMS79, Section 2], [BB90]). We recall that we have adopted the convention $0 \cdot \infty = 0$.

Proposition 2.6. *Assume (Hp 1)-(Hp 5). Let $m \in (0,1)$, $s \in (0,\infty)$, and $G_{m,s} : L^1((2m-1,1)) \to [0,\infty]$ be the functional given by*

$$G_{m,s}(w) := \begin{cases} G_m(w) & \text{if } w \in \zeta_{m,s} + W_0^{1,1}((2m-1,1)), \\ \infty & \text{otherwise,} \end{cases} \tag{2.15}$$

where G_m is defined in (2.12). Then the L^1-lower semicontinuous envelope of $G_{m,s}$ is given by

$$\begin{aligned} sc^- G_{m,s}(w) = \int_{2m-1}^{1} \Big(\vartheta_m(t)|w'|^2 + \omega_m(1-t)\Big)^{1/2} \mathrm{d}t + \int_{2m-1}^{1} \vartheta_m^{1/2}(t) \mathrm{d}|D^s w| \\ + \vartheta^{1/2}(1^-)|w((2m-1)^+)| + \vartheta^{1/2}(1^-)|w(1^-) - s|, \end{aligned} \tag{2.16}$$

for every $w \in BV((2m-1,1))$, and $+\infty$ otherwise on $L^1((2m-1,1))$. In particular, $w((2m-1)^+) = w(1^-) - s = 0$ if $\vartheta(1^-) = \infty$.

Moreover, the following holds

$$g(s) = \inf_{m\in(0,1)} \inf_{w \in L^1((2m-1,1))} sc^- G_{m,s}(w). \tag{2.17}$$

Proof. If $\vartheta(1^-) < \infty$ the claim is a direct consequence of [GMS79, Section 2].

Instead, if $\vartheta(1^-) = \infty$, we first prove a lower bound for the relaxed functional by approximation. Indeed, fix $s \in (0,\infty)$, $m \in (0,1)$ and denote by $\mathscr{G}_{m,s}$ the functional on the right hand side of (2.16). Let $\vartheta^{(j)}(t) := \vartheta(t) \wedge j$, $t \in [0,1]$, and apply the equality in (2.16) to the corresponding functional in (2.15) obtained by substituting ϑ_m with $\vartheta_m^{(j)}$. By monotone convergence it is then easy to conclude that $sc^- G_{m,s}(w) \geq \mathscr{G}_{m,s}(w)$ for every $w \in BV((2m-1,1))$ with $w(2m-1) = 0$ and $w(1) = s$, and that $sc^- G_{m,s}(w) = \infty$ otherwise.

For the reverse inequality, first choose $w \in BV((2m-1,1))$ with $w \equiv 0$ on $(2m-1, 2m-1+\lambda)$ and $w \equiv s$ on $(1-\lambda, 1)$, for some λ sufficiently small. Thanks to the result in the case $\vartheta(1^-) < \infty$, which actually holds for every interval $(a,b) \subset (0,1)$, there exists e sequence $w_j \in H^1((2m-1+\lambda, 1-\lambda))$ such that $w_j \to w$ in $L^1((2m-1+\lambda, 1-\lambda))$ as $j \to \infty$ and

$$\begin{aligned} \lim_{j\to\infty} \int_{2m-1+\lambda}^{1-\lambda} \Big(\vartheta_m(t)|w_j'|^2 + \omega_m(1-t)\Big)^{1/2} \mathrm{d}t \\ = \int_{2m-1+\lambda}^{1-\lambda} \Big(\vartheta_m(t)|w'|^2 + \omega_m(1-t)\Big)^{1/2} \mathrm{d}t + \int_{2m-1+\lambda}^{1-\lambda} \vartheta_m^{1/2}(t)\mathrm{d}|D^s w|\,. \end{aligned} \tag{2.18}$$

Clearly, extending w_j to 0 on $(2m-1, 2m-1+\lambda)$, and to s on $(1-\lambda, 1)$ implies the $L^1((2m-1,1))$

convergence of the extensions, still denoted by w_j with a slight abuse of notation, to w. Moreover, using (2.18) a simple calculation shows that

$$sc^{-}G_{m,s}(w) \le \lim_{j\to\infty} G_{m,s}(w_j) = \mathscr{G}_{m,s}(w)\,,$$

and thus the identity in (2.16) follows for w as above. To establish the latter in general, let $w \in BV((2m-1,1))$ with $w(2m-1)=0$, $w(1)=s$ and $\mathscr{G}_{m,s}(w)<\infty$, otherwise the proof is completed by the lower bound. For every λ sufficiently small let $\varphi_\lambda : [2m-1,1] \to [2m-1+\lambda, 1-\lambda]$ given by $\varphi_\lambda(t) := m_\lambda(t-m)+m$, with $m_\lambda := \frac{1-m-\delta}{1-m}$. Then define the function $w_\lambda \in BV((2m-1,1))$ by

$$w_\lambda(\tau) := \begin{cases} 0 & \tau\in[2m-1, 2m-1+\lambda] \\ w(\varphi_\lambda^{-1}(\tau)) & \tau\in[2m-1+\lambda, 1-\lambda] \\ s & \tau\in[1-\lambda,1] \end{cases}. \tag{2.19}$$

Note that $w_\lambda \to w$ in $L^1((2m-1,1))$ as $\lambda\to 0$, and moreover that $Dw_\lambda = (\varphi_\lambda)_\#(Dw)$ (the push-forward measure through φ_λ). Thus, we compute as follows

$$\begin{aligned}
&sc^{-}G_{m,s}(w_\lambda) \\
&\quad = \int_{2m-1+\lambda}^{1-\lambda} \Big(\vartheta_m(\tau)|w_\lambda'|^2 + \omega_m(1-\tau)\Big)^{1/2} \mathrm{d}\tau + \int_{2m-1-\lambda}^{1-\lambda} \vartheta_m^{1/2}(\tau)\mathrm{d}|D^s w_\lambda| \\
&\qquad + \int_{2m-1}^{2m-1+\lambda} \omega_m^{1/2}(1-\tau)\mathrm{d}\tau + \int_{1-\lambda}^{1} \omega_m^{1/2}(1-t)\mathrm{d}\tau \\
&\quad = \int_{2m-1}^{1} \Big(\vartheta_m(\varphi_\lambda(t))|w'|^2 + \omega_m(1-\varphi_\lambda(t))\, m_\lambda^2\Big)^{1/2} \mathrm{d}t \\
&\qquad + \int_{2m-1}^{1} \vartheta_m^{1/2}(\varphi_\lambda(t))\mathrm{d}|D^s w| + 2\int_{1-\lambda}^{1} \omega^{1/2}(1-t)\mathrm{d}t \\
&\quad \le \int_{2m-1}^{1} \Big(\vartheta_m(t)|w'|^2 + \omega_m(1-\varphi_\lambda(t))\Big)^{1/2} \mathrm{d}t \\
&\qquad + \int_{2m-1}^{1} \vartheta_m^{1/2}(t)\mathrm{d}|D^s w| + 2\int_{1-\lambda}^{1} \omega^{1/2}(1-t)\mathrm{d}t\,,
\end{aligned}$$

where in the last inequality we have used that $m_\lambda < 1$, and that $\vartheta_m(\varphi_\lambda(t)) \le \vartheta_m(t)$ on $[2m-1,1]$ by taking into account that ϑ_m is decreasing on $[2m-1,m]$ together with the inequality $\varphi_\lambda(t)\ge t$ on the same interval, and increasing on $[m,1]$ and the inequality $\varphi_\lambda(t)\le t$ on the same interval. The conclusion the follows at once by Dominated Convergence Theorem and then $L^1((2m-1,1))$ lower semicontinuity of $sc^{-}G_{m,s}$. □

Next, we investigate several properties of the minimizers of $G_{m,s}$ at $m\in(0,1)$ and $s\in(0,\infty)$ fixed.

Theorem 2.7. *Assume (Hp 1)-(Hp 5). For every $m\in(0,1)$ and $s\in(0,\infty)$, let $G_{m,s}$ be defined in (2.15). Then there exists a function $w_{m,s}$ such that*

$$sc^{-}G_{m,s}(w_{m,s}) = \inf_{w\in L^1((2m-1,1))} sc^{-}G_{m,s}(w). \tag{2.20}$$

Moreover, $w_{m,s}$ is the unique minimizer of $sc^{-}G_{m,s}$ on $L^1((2m-1,1))$, $w_{m,s}\in SBV((2m-1,1))$, is *strictly increasing and odd symmetric with respect to $t = m$, $w_{m,s}(2m-1) = 0$, $w_{m,s}(1) = s$, and for (a unique) $\lambda_{m,s} \in (0, \vartheta^{1/2}(m)]$*

$$w'_{m,s}(t) = \left(\frac{\lambda^2_{m,s}\omega_m(1-t)}{\vartheta_m(t)(\vartheta_m(t) - \lambda^2_{m,s})} \right)^{1/2} \tag{2.21}$$

for $\mathcal{L}^1$-a.e $t \in (2m-1, 1)$, and either

(1) $w_{m,s} \in W^{1,1}((2m-1,1))$, with $w_{m,s}(m) = \frac{s}{2}$;

or

(2) $w_{m,s} \in SBV((2m-1,1))$, with $D^j w_{m,s}$ concentrated on $t = m$, and $\lambda_{m,s} = \vartheta^{1/2}(m)$.

Proof. **Step 1. Existence of a minimizer.** We note that $\inf_{[2m-1,1]} \vartheta_m = \vartheta_m(m) > 0$ thanks to the condition $m > 0$. Therefore, $sc^- G_{m,s}$ is a coercive functional on $BV((2m-1,1))$ (cf. (2.16)). The existence of a minimizer $w_{m,s}$ of $sc^- G_{m,s}$ follows from the Direct Method of the Calculus of Variations and the BV compactness theorem.

Step 2. Regularity of minimizers. We prove that any minimizer $w_{m,s}$ attains the boundary data and the singular part of its distributional derivative is concentrated in $t = m$. With this aim the crucial observation is that the function ϑ_m assumes its unique minimum value in $t = m$. Then, consider

$$\hat{w}(t) := \int_{2m-1}^{t} w'_{m,s}(\tau)d\tau + D^s w_{m,s}((2m-1,1))\chi_{(m,1)}(t)\,,$$

and define the test function

$$w(t) := \begin{cases} \hat{w}(t) & t \in (2m-1, m] \\ \hat{w}(t) + (s - \hat{w}(1)) & t \in (m, 1) \end{cases}.$$

Clearly, $w \in BV((2m-1,1))$, with

$$Dw = w'_{m,s}\mathcal{L}^1 \llcorner (2m-1,1) + (D^s w_{m,s}((2m-1,1)) + s - \hat{w}(1))\delta_{\{t=m\}},$$

$w(2m-1) = 0$, and $w(1) = s$. Since $\hat{w}(1) = Dw_{m,s}((2m-1,1)) = w_{m,s}(1) - w_{m,s}(2m-1)$ we have that $sc^- G_{m,s}(w) < sc^- G_{m,s}(w_{m,s})$ if either the boundary data are not attained by $w_{m,s}$ or $D^s w_{m,s}$ is not concentrated on $t = m$. Indeed, computing the energy for w shows that the contributions for the energy of $w_{m,s}$ corresponding either to the boundary terms or to the singular part of the distributional derivative, which are both multiplied by $\vartheta_m^{1/2}(t)$ for some $t \neq m$, have been either moved to or concentrated in $t = m$, which is the unique minimum point of the same function $\vartheta_m^{1/2}(t)$.

Step 3. Euler-Lagrange equation and structural properties. By Step 2 any minimizer $w_{m,s}$ is $SBV((2m-1,1))$, $w_{m,s}(2m-1) = 0$, $w_{m,s}(1) = s$, and $D^s w_{m,s}$ is at most concentrated on $t = m$. Note also that by truncation necessarily $w_{m,s} \in [0, s]$ $\mathcal{L}^1$-a.e. on $(2m-1,1)$.

Assume first that $w_{m,s} \in W^{1,1}((2m-1,1))$, then by using smooth outer variations, i.e. $w_{m,s} + \varepsilon\phi$ with $\phi \in C^1_c((2m-1,1))$, we obtain the following Euler-Lagrange equation

$$\frac{\vartheta_m(t) w'_{m,s}(t)}{\left(\vartheta_m(t)|w'_{m,s}(t)|^2 + \omega_m(1-t)\right)^{1/2}} = \lambda_{m,s} \tag{2.22}$$

for $\mathcal{L}^1$-a.e. $t \in (2m-1,1)$ and for some $\lambda_{m,s} \in \mathbb{R}$. Actually, $\lambda_{m,s} \in (0,\infty)$ as $w_{m,s} \in [0,s]$ $\mathcal{L}^1$-a.e. on $(2m-1,1)$ and attains the boundary data, in turn implying that $w'_{m,s} > 0$ $\mathcal{L}^1$-a.e. on $(2m-1,1)$, and thus $\lambda_{m,s} \le \vartheta^{1/2}(m)$ (cf. (2.22)). Furthermore, (2.22) shows that $w_{m,s}$ is strictly increasing and odd symmetric with respect to m. Solving $w'_{m,s}$ in (2.22) and integrating on $(2m-1,1)$ gives

$$s = 2\int_m^1 \Big(\frac{\lambda_{m,s}^2 \omega(1-t)}{\vartheta(t)(\vartheta(t)-\lambda_{m,s}^2)}\Big)^{1/2} \mathrm{d}t.$$

Therefore, we conclude the uniqueness of $\lambda_{m,s}$ that satisfies (2.21), and in turn that $w_{m,s}$ is the only possible minimizer of $sc^-G_{m,s}$ among $W^{1,1}$ functions, using the strict monotonicity of

$$\lambda \mapsto \int_m^1 \Big(\frac{\lambda^2 \omega(1-t)}{\vartheta(t)(\vartheta(t)-\lambda^2)}\Big)^{1/2} dt\,. \tag{2.23}$$

Instead, if $w_{m,s} \in SBV \setminus W^{1,1}((2m-1,1))$, we consider again smooth outer variations separately on $(2m-1,m)$ and $(m,1)$ to infer the Euler-Lagrange equation in (2.22) with constants $\lambda^-_{m,s}$ and $\lambda^+_{m,s}$ on each interval, respectively. Since, $w_{m,s}$ takes value in $[0,s]$ $\mathcal{L}^1$-a.e. on $(2m-1,1)$, necessarily $\lambda^{\pm}_{m,s} \ge 0$.

Next, we use variations on the whole interval that move the value of the jump point in $t=m$, i.e. $w_{m,s} + \varepsilon\phi$ with $\phi \in C^1((2m-1,m)\cup(m,1)) \cap SBV((2m-1,1))$, and $\phi(2m-1)=\phi(1)=0$. Note that for ε sufficiently small $[w_{m,s}+\varepsilon\phi](m)$ has the same sign of $[w_{m,s}](m)$, therefore we get

$$\lambda^-_{m,s}\phi(m^-) - \lambda^+_{m,s}\phi(m^+) \pm \vartheta^{1/2}(m)(\phi(m^+)-\phi(m^-)) = 0,$$

where the sign $+$ corresponds to the case $[w_{m,s}](m) > 0$ and $-$ to the other case. In particular, it follows that $\lambda^+_{m,s} = \lambda^-_{m,s} = \pm\vartheta^{1/2}(m)$, and since they are both non negative we conclude (2.21). Hence, $w_{m,s}$ is odd symmetric with respect to $t=m$. Moreover, $w_{m,s}(m^-) < \frac{s}{2}$ since otherwise having assumed $w_{m,s} \in SBV \setminus W^{1,1}((2m-1,1))$ then $w_{m,s}(m^-) > \frac{s}{2}$ and the function $\hat{w} = w_{m,s}\chi_{(2m-1,m)} + (w_{m,s} \vee w_{m,s}(m^-))\chi_{[m,1)}$ would be $W^{1,1}((2m-1,1))$ with $sc^-G_{m,s}(\hat{w}) < sc^-G_{m,s}(w_{m,s})$. Therefore, $w_{m,s}$ is strictly increasing and it is the only possible minimizer of $sc^-G_{m,s}$ belonging to $SBV \setminus W^{1,1}((2m-1,1))$ in view of (2.21). □

By taking into account the odd symmetry of the minimizer $w_{m,s}$ with respect to $t=m$ we rewrite the minimal value of the energy $sc^-G_{m,s}$ in an alternative and more efficient form. With this aim it is convenient to introduce the function $\mathscr{G} : \{(m,\lambda) \in (0,1)^2 : m \in (0,1),\, \lambda \in (0,\vartheta^{1/2}(m)]\} \to [0,\infty]$ defined by

$$\begin{aligned}\mathscr{G}(m,\lambda) &:= 2\int_m^1 \Big(\frac{\vartheta(t)-\lambda^2}{\vartheta(t)}\omega(1-t)\Big)^{1/2}\mathrm{d}t + 2\lambda\int_m^1 \Big(\frac{\lambda^2\omega(1-t)}{\vartheta(t)(\vartheta(t)-\lambda^2)}\Big)^{1/2}\mathrm{d}t \\ &=: A(m,\lambda) + \lambda B(m,\lambda)\,.\end{aligned} \tag{2.24}$$

Note that $\mathscr{G}$ is infinite if and only if B is infinite, and the only points in which this can happen are of the type $(m,\vartheta^{1/2}(m))$. In addition we set $\mathscr{G}(0,0) := 2\Psi(1)$.

Corollary 2.8. *Assume (Hp 1)-(Hp 5). For every $m \in (0,1)$, and $s \in (0,\infty)$, if $w_{m,s}$ denotes the* *minimizer of $sc^-G_{m,s}$ in Theorem 2.7, we have*

$$\begin{aligned}\mathfrak{G}_s(m) &:= sc^-G_{m,s}(w_{m,s})\\ &= 2\int_m^1 \Big(\vartheta(t)|w'_{m,s}|^2+\omega(1-t)\Big)^{1/2}\mathrm{d}t + 2\vartheta^{1/2}(m)(w_{m,s}(m^+)-\frac{s}{2})\,,\end{aligned}\tag{2.25}$$

and either

$$w_{m,s}\in W^{1,1}((2m-1,1)) \Longleftrightarrow s = B(m,\lambda_{m,s})\,,\tag{2.26}$$

where $\lambda_{m,s}\in(0,\vartheta^{1/2}(m)]$ is as in Theorem 2.7, and

$$\mathfrak{G}_s(m) = 2\int_m^1 \Big(\frac{\vartheta(t)\omega(1-t)}{\vartheta(t)-\lambda^2_{m,s}}\Big)^{1/2}\mathrm{d}t = \mathscr{G}(m,\lambda_{m,s}) = A(m,\lambda_{m,s})+\lambda_{m,s}s\,,\tag{2.27}$$

or

$$w_{m,s}\in SBV\setminus W^{1,1}((2m-1,1)) \Longleftrightarrow s > B(m,\vartheta^{1/2}(m))\,,\tag{2.28}$$

and

$$w_{m,s}(m^+) = s-\frac{1}{2}B(m,\vartheta^{1/2}(m))\,,$$

and

$$\begin{aligned}\mathfrak{G}_s(m) &= 2\int_m^1 \Big(\frac{\vartheta(t)\omega(1-t)}{\vartheta(t)-\vartheta(m)}\Big)^{1/2}\mathrm{d}t + \vartheta^{1/2}(m)(s-B(m,\vartheta^{1/2}(m)))\\ &= \mathscr{G}(m,\vartheta^{1/2}(m))+\vartheta^{1/2}(m)(s-B(m,\vartheta^{1/2}(m))) = A(m,\vartheta^{1/2}(m))+\vartheta^{1/2}(m)s\,.\end{aligned}\tag{2.29}$$

Remark 2.9. *We stress that the value $\mathscr{G}(m,\vartheta^{1/2}(m))$ in (2.29) is necessarily finite due to (2.28), analogously due to (2.27) if $\lambda_{m,s}=\vartheta^{1/2}(m)$.*

Proof of Corollary 2.8. The proof of (2.25) follows from a simple calculation using the explicit form of $sc^-G_{m,s}$ in (2.16), the odd symmetry of $w_{m,s}$ and the fact that it is increasing. Moreover, the same properties of $w_{m,s}$ yield that

$$B(m,\lambda_{m,s}) = 2\int_m^1 \Big(\frac{\lambda^2_{m,s}\omega(1-t)}{\vartheta(t)(\vartheta(t)-\lambda^2_{m,s})}\Big)^{1/2}\mathrm{d}t = \int_{2m-1}^1 w'_{m,s}\mathrm{d}t\,,$$

implying (2.26) and (2.28).

Finally, formulas (2.27) and (2.29) follows by direct computations using (2.21) and (2.25). □

2.3 A key auxiliary function

Under suitable assumptions on s, ϑ and ω we will show that there is a unique value m_s such that $g(s) = sc^-G_{m_s,s}(w_{m_s,s})$. With this aim we define the auxiliary function $\Phi:(0,1)\to[0,\infty]$ as follows

$$\Phi(x) := B(x,\vartheta^{1/2}(x))\,.\tag{2.30}$$

The importance of Φ is related to the fact that (2.26) rewrites equivalently as $\Phi(m)\geq s$, and (2.28) as $\Phi(m)<s$. In the next proposition we determine several properties of Φ, as well as that its definition is well posed provided that the following assumption on ϑ which strengthens (Hp 5) holds:

(Hp 5′) $\vartheta \in C^1((0,1))$ with $\vartheta'(t) > 0$ for every $t \in (0,1)$.

Recall that Ψ is the function introduced in (2.1).

Proposition 2.10. *Assume (Hp 1)-(Hp 4), and (Hp 5′). Then*

(i) $\Phi \in C^0((0,1))$*;*

(ii) if the following limit exists

$$\lim_{x\to 0^+} (\vartheta^{1/2})'(x) =: (\vartheta^{1/2})'(0^+) \in [0,\infty]\,, \tag{2.31}$$

then Φ *has right limit in* $x = 0$ *given by*

$$\lim_{x\to 0^+} \Phi(x) = \frac{\pi \omega^{1/2}(1)}{2(\vartheta^{1/2})'(0^+)}\,; \tag{2.32}$$

(iii) if $\vartheta(1^-) = \overline{\varsigma}^2 \in (0,\infty)$*, and there is* $p > 2$ *such that for some* $z \in (0,\overline{\varsigma}^2)$

$$(0,\overline{\varsigma}^2) \ni t \mapsto F(t) := \frac{(\Psi\circ\vartheta^{-1})'(\overline{\varsigma}^2 - t)}{(\overline{\varsigma}^2 - t)^{1/2}} \in L^p((0,z))\,, \tag{2.33}$$

then

$$\lim_{x\to 1^-} \Phi(x) = 0\,. \tag{2.34}$$

If, in addition, (2.33) holds for every $z \in (0,1)$ *then* $\Phi \in C^{1/2-1/p}_{\mathrm{loc}}([0,1))$*;*

(iv) if $[0,1] \ni x \mapsto (x(1-x))^{1/2}F(xt)$ *is strictly decreasing (non-increasing, strictly increasing, non-decreasing) for* $\mathcal{L}^1$*-a.e.* $t \in (0,1)$*, then* Φ *is strictly increasing (non-decreasing, strictly decreasing, non-increasing);*

(v) if $(\vartheta\circ\Psi^{-1})^{1/2}$ *is convex, then* Φ *is strictly decreasing.*

Proof. Let $\gamma \in (0, 1/2)$, then there is $\delta \in (0,\gamma/2)$ such that $\vartheta(t) - \vartheta(y) \geq \frac{\vartheta'(y)}{2}(t-y)$ for every $y \in [\gamma, 1-\gamma]$ and t such that $|t-y| \leq \delta$. Therefore, recalling that $\vartheta'(t) > 0$ for every $t \in (0,1)$, Φ is well defined for every $x \in (0,1)$.

In addition, if $x_j \to x$ we have for every $\varepsilon > 0$ sufficiently small

$$\Phi(x_j) = 2\int_{x_j}^{x_j+\varepsilon} \Big(\frac{\vartheta(x_j)\omega(1-t)}{\vartheta(t)(\vartheta(t)-\vartheta(x_j))}\Big)^{1/2}\mathrm{d}t + 2\int_{x_j+\varepsilon}^{1} \Big(\frac{\vartheta(x_j)\omega(1-t)}{\vartheta(t)(\vartheta(t)-\vartheta(x_j))}\Big)^{1/2}\mathrm{d}t =: I_j + K_j\,.$$

It is clear from the continuity assumption on ϑ that

$$\lim_j K_j = 2\int_{x+\varepsilon}^{1} \Big(\frac{\vartheta(x)\omega(1-t)}{\vartheta(t)(\vartheta(t)-\vartheta(x))}\Big)^{1/2}\mathrm{d}t\,.$$

Moreover, we estimate I_j as follows

$$0 \leq I_j \leq C \sup_{(x-\varepsilon,x+2\varepsilon)} (\vartheta\vartheta')^{-1/2} \int_{x_j}^{x_j+\varepsilon} (t-x_j)^{-1/2}\mathrm{d}t = 2C \sup_{(x-\varepsilon,x+2\varepsilon)} (\vartheta\vartheta')^{-1/2}\varepsilon^{1/2}\,,$$

and the conclusion follows.

Next, we prove that the right limit of Φ in 0 exists and the equality in (2.32) holds. With this aim, note that for every $\delta>0$ being ϑ and ω continuous on $[0,1]$ and $\vartheta(0)=0$, we have

$$\Phi(x)=2(\omega^{1/2}(1)+o(1))\int_x^\delta\Big(\frac{\vartheta(x)}{\vartheta(t)(\vartheta(t)-\vartheta(x))}\Big)^{1/2}\mathrm{d}t+o(1)$$

as $x\to 0^+$. Changing variable with $\tau=(\vartheta(t)/\vartheta(x))^{1/2}$ and applying the Mean Value Theorem gives some $\bar t\in[x,\delta]$ such that

$$\begin{aligned}\Phi(x)&=4(\omega^{1/2}(1)+o(1))\frac{\vartheta^{1/2}(\bar t)}{\vartheta'(\bar t)}\int_1^{(\frac{\vartheta(\delta)}{\vartheta(x)})^{1/2}}\frac{1}{\tau(\tau^2-1)^{1/2}}\mathrm{d}\tau+o(1)\\&=\frac{2(\omega^{1/2}(1)+o(1))}{(\vartheta^{1/2})'(\bar t)}\left(\arctan\left(\Big(\frac{\vartheta(\delta)}{\vartheta(x)}\Big)^{1/2}+\Big(\frac{\vartheta(\delta)}{\vartheta(x)}-1\Big)^{1/2}\right)-\frac{\pi}{4}\right)+o(1)\end{aligned}$$

as $x\to 0^+$, where the last equality follows from an elementary integration argument, and the fact that $\vartheta^{1/2}\in C^1((0,1))$ (cf. the assumption on ϑ). Eventually (2.32) follows at once by letting first $x\to 0^+$ and then $\delta\to 0^+$, using (2.31).

Next, we prove (2.34). Recalling that $\vartheta(1^-)=\bar\varsigma^2\in(0,\infty)$, changing variables with $t=\vartheta^{-1}(\bar\varsigma^2-\tau)$ in the definition of Φ (cf. (2.24) and (2.30)) and letting $\lambda=\bar\varsigma^2-\vartheta(x)$, we deduce the following Abel integral equation involving the function F defined in the statement

$$\frac{\Phi(\vartheta^{-1}(\bar\varsigma^2-\lambda))}{2(\bar\varsigma^2-\lambda)^{1/2}}=\int_0^\lambda\frac{(\Psi\circ\vartheta^{-1})'(\bar\varsigma^2-\tau)}{(\bar\varsigma^2-\tau)^{1/2}(\bar\varsigma^2-\vartheta(x)-\tau)^{1/2}}\mathrm{d}\tau=\int_0^\lambda\frac{F(\tau)}{(\lambda-\tau)^{1/2}}\mathrm{d}\tau\,, \tag{2.35}$$

(see [GV91] for references on such a topic). By assumption $F\in L^p((0,z))$, $p>2$, for some $z\in(0,\bar\varsigma^2)$, so that if $\frac1q+\frac1p=1$, $q\in(1,2)$, and if $\lambda\in(0,z)$ we get

$$|\Phi(\vartheta^{-1}(\bar\varsigma^2-\lambda))|\le 2(\bar\varsigma^2-\lambda)^{1/2}\|F\|_{L^p((0,z))}\Big(\frac{2}{2-q}\lambda^{\frac{2-q}{2}}\Big)^{1/q},$$

and thus the limit of $\Phi(\vartheta^{-1}(\bar\varsigma^2-\lambda))$ as $\lambda\to 0^+$ is infinitesimal, corresponding to the left limit of Φ in 1. In addition, if $F\in L^p((0,z))$, $p>2$, for every $z\in(0,\bar\varsigma^2)$ the left hand side of (2.35) is $C^{1/2-1/p}_{\mathrm{loc}}([0,\bar\varsigma^2))$ by [GV91, Theorem 4.1.4].

Next, we prove the monotonicity assertions. We start with the claim in item (iv), focusing on the strictly increasing case, the other cases being analogous. With this aim we use the change of variable $\tau=tz$ in (2.35) to get

$$\Phi(\vartheta^{-1}(1-z))=2(z(1-z))^{1/2}\int_0^1\Big(\frac{t}{1-t}\Big)^{1/2}F(tz)\mathrm{d}t,$$

and the conclusion follows being $[0,1]\ni z\mapsto(z(1-z))^{1/2}F(tz)$ strictly decreasing for $\mathcal{L}^1$-a.e. $t\in(0,1)$, and ϑ strictly increasing.

To establish the claim in item (v) we first use the change of variable $\tau=\frac{\vartheta(t)}{\vartheta(x)}$ in the definition of Φ

(cf. (2.30)) to deduce that

$$\Phi(x) = 2\int_1^{\frac{1}{\vartheta(x)}} \Big(\frac{\vartheta(x)}{\tau(\tau-1)}\Big)^{1/2} \frac{\Psi'(\vartheta^{-1}(\vartheta(x)\tau))}{\vartheta'(\vartheta^{-1}(\vartheta(x)\tau))}\mathrm{d}\tau = \int_1^{\frac{1}{\vartheta(x)}} \frac{\zeta'\big((\vartheta(x)\tau)^{1/2}\big)}{\tau(\tau-1)^{1/2}}\mathrm{d}\tau, \tag{2.36}$$

where ζ denotes the inverse of $(\vartheta\circ\Psi^{-1})^{1/2}$, namely $\zeta(t) = \Psi(\vartheta^{-1}(t^2))$. We conclude using the fact that ϑ is strictly increasing, and $(\vartheta\circ\Psi^{-1})^{1/2}$ is convex and increasing. □

Remark 2.11. *More generally, using the one-dimensional area formula (see [AT04, Theorem 3.4.6]), the definition of Φ is well posed and the conclusions in item (iii) hold assuming $\vartheta^{-1}\in W^{1,1}((0,1))$.*

We note that formula (2.32) corresponds to [BI24, (3.11) in Proposition 3.7] in case $\omega(t) = t^2$ (there it is equivalently stated in terms of the differentiability in $t=0$ of the function f in (1.1)).

The monotonicity of Φ is a key tool in the analysis below of the variational problem defining g as it implies that, for every $s\geq 0$, the energies $\mathfrak{G}_s$ are minimized either in the interior of $[0,1]$ in the strictly decreasing case, or at the boundary in the increasing case. In this respect, the convexity assumption on $(\vartheta\circ\Psi^{-1})^{1/2}$ is equivalent to the assumption used in [BCI21, BI24] (with $Q(t) = \omega(t) = t^2$) which in our notation reads as the convexity of $(\vartheta\circ(2(\Psi(1)-\Psi))^{-1})^{1/2}$. We remark that the concavity of $(\vartheta\circ(2(\Psi(1)-\Psi))(t^{1/2})$ has been used in [LCM25] (with $Q(t) = \omega(t) = t^2$) to infer the concavity of the corresponding energy. We have not been able to show that such a condition implies neither that Φ is increasing nor that the energy $\mathfrak{G}_s$ is concave. Despite this, using item (iv) simple calculations show that the explicit example given in [LCM25] (namely, $\vartheta(t) = 1-(1-t)^4$ and $\omega(t) = t^2$) yields that Φ is increasing.

Remark 2.12. *The fact that Φ is strictly decreasing is also satisfied, for instance, in the following particular cases for every non-decreasing damage potential ω*

(i) $\vartheta(t) := t^\alpha$, *with* $\alpha\geq 2$;

(ii) $\vartheta(t) := \frac{e^t-1}{e-1}$;

where Ψ is defined in (2.1). For instance, it suffices to take $Q=\omega$ and $l(t) = \vartheta(t) = t^\alpha$ in case (i), analogously in case (ii). To prove the claim we use the change of variable $\frac{t}{x} = \tau$, for every $x\in(0,1)$, to get

$$\Phi(x) = 2\int_1^{\frac{1}{x}} x\Big(\frac{\vartheta(x)\omega(1-\tau x)}{\vartheta(\tau x)(\vartheta(\tau x)-\vartheta(x))}\Big)^{1/2}\mathrm{d}\tau\,.$$

In case $\vartheta(t) = t^\alpha$, we obtain for every $x\in(0,1)$

$$\Phi(x) = 2\int_1^{\frac{1}{x}} x^{1-\frac{\alpha}{2}}\Big(\frac{\omega(1-\tau x)}{\tau^\alpha(\tau^\alpha-1)}\Big)^{1/2}\mathrm{d}\tau\,,$$

which is strictly decreasing, being $\alpha\geq 2$ and ω non-decreasing by assumption. Similarly, one can argue for the exponential. Note also that the result in item (iii) of Proposition 2.10 applies to $\vartheta(t) = t^\alpha$ for $\alpha\geq 2$.

2.4 Dependence of g on the parameters of the phase-field model

In this section we show the explicit dependence of g on the damage potential and the degradation function. With this aim, we prove a technical result which will be instrumental in what follows.

Lemma 2.13. *Assume (Hp 1)-(Hp 5). Let $s \in (0,\infty)$ and $0 < m_0 < m_1 < 1$, consider $\lambda_{m,s} \in (0, \vartheta^{1/2}(m)]$ in Theorem 2.7, and $\mathfrak{G}_s$ in (2.25). Then*

(i) if $\lambda_{m,s} \in (0, \vartheta^{1/2}(m))$ for every $m \in (m_0, m_1)$, then $\mathfrak{G}_s \in C^1((m_0,m_1))$ and is strictly decreasing. Moreover, $(m_0,m_1) \mapsto \lambda_{m,s} \in C^1((m_0,m_1))$ and is strictly increasing;

(ii) if $\lambda_{m,s} = \vartheta^{1/2}(m)$ for every $m \in (m_0,m_1)$ and $\vartheta \in C^1((m_0,m_1))$ with $\vartheta'(t) > 0$ for every $t \in (0,1)$, then $\mathfrak{G}_s \in C^1((m_0,m_1))$ and increasing.

More precisely, let $w_{m,s}$ be the function defined in Theorem 2.7 then

(a) if $w_{m,s} \in SBV \setminus W^{1,1}((2m-1,1))$ for every $m \in (m_0,m_1)$ then $\mathfrak{G}_s$ is strictly increasing on (m_0,m_1).

(b) if $w_{m,s} \in W^{1,1}((2m-1,1))$ for some $m \in (m_0,m_1)$ then $\mathfrak{G}'_s(m) = 0$.

Proof. We start with the first case noting that in particular (2.26) necessarily holds, and thus $w_{m,s} \in W^{1,1}((2m-1,1))$ in view of Corollary 2.8. Moreover, (2.27) yields that

$$\mathfrak{G}_s(m) = 2\int_m^1 \left(\frac{\vartheta(t)\omega(1-t)}{\vartheta(t)-\lambda_{m,s}^2}\right)^{1/2} \mathrm{d}t\,. \tag{2.37}$$

Next, we claim that $(m_0,m_1) \mapsto \lambda_{m,s}$ is continuously differentiable and strictly increasing thanks to the implicit function theorem applied to the function B defined in (2.24) defined on the open set

$$T := \Big\{(m,\lambda)\ :\ m \in (m_0,m_1),\ \lambda \in (0,\vartheta^{1/2}(m))\Big\}\,.$$

Indeed, item (i) in Theorem 2.7 ensures that for every $m \in (m_0,m_1)$ the unique value of λ such that $B(m,\lambda) = s$ is $\lambda_{m,s} \in (0,\vartheta^{1/2}(m))$, and

$$\begin{aligned}\frac{\partial B}{\partial \lambda}(m,\lambda) &= 2\int_m^1 \left(\frac{\omega(1-t)}{\vartheta(t)(\vartheta(t)-\lambda^2)}\right)^{1/2} \mathrm{d}t + 2\lambda^2 \int_m^1 \left(\frac{\omega(1-t)}{\vartheta(t)(\vartheta(t)-\lambda^2)^3}\right)^{1/2} \mathrm{d}t \\ &= 2\int_m^1 \left(\frac{\vartheta(t)\omega(1-t)}{(\vartheta(t)-\lambda^2)^3}\right)^{1/2} \mathrm{d}t > 0\,,\end{aligned} \tag{2.38}$$

Thus, by the implicit function theorem

$$\begin{aligned}\lambda'_{m,s} &= -\frac{\partial B}{\partial m}(m,\lambda_{m,s})\left(\frac{\partial B}{\partial \lambda}(m,\lambda_{m,s})\right)^{-1} \\ &= 2\left(\frac{\lambda_{m,s}^2\omega(1-m)}{\vartheta(m)(\vartheta(m)-\lambda_{m,s}^2)}\right)^{1/2}\left(\frac{\partial B}{\partial \lambda}(m,\lambda_{m,s})\right)^{-1} > 0\,.\end{aligned}$$

Therefore, $(m_0,m_1) \mapsto \lambda_{m,s}$ is strictly increasing. In turn, this implies that $\mathfrak{G}_s$ in (2.37) is differentiable on (m_0,m_1). Indeed, using (2.27), namely $\mathfrak{G}_s(m) = A(m,\lambda_{m,s}) + \lambda_{m,s}s$, (2.26) and (2.38) we get

$$\mathfrak{G}'_s(m) = \left(\frac{\partial A}{\partial \lambda}(m,\lambda_{m,s}) + s\right)\lambda'_{m,s} + \frac{\partial A}{\partial m}(m,\lambda_{m,s}) = \frac{\partial A}{\partial m}(m,\lambda_{m,s}) < 0 \tag{2.39}$$

as from the very definition of A in (2.24) easy calculations lead to

$$\frac{\partial A}{\partial \lambda}(m,\lambda) = -B(m,\lambda)\,, \qquad \frac{\partial A}{\partial m}(m,\lambda) = -2\left(\frac{\vartheta(m)-\lambda^2}{\vartheta(m)}\omega(1-m)\right)^{1/2}.$$

Next, we deal with the second case. In particular note that $w_{m,s}$ can be either $W^{1,1}$ or $SBV \setminus W^{1,1}$, and in both cases $\lambda_{m,s} = \vartheta^{1/2}(m)$. We claim that $\mathfrak{G}_s \in C^1((m_0,m_1))$ with

$$\mathfrak{G}_s'(m) = \frac{\vartheta'(m)}{2\vartheta^{1/2}(m)}(s-\Phi(m)) \tag{2.40}$$

for every $m \in (m_0,m_1)$. Given this for granted, the right hand side in (2.40) is either null if $w_{m,s} \in W^{1,1}$ thanks to $s = B(m,\vartheta^{1/2}(m)) = \Phi(m)$ (cf. (2.26)) and (2.30)), or is strictly positive if $w_{m,s} \in SBV \setminus W^{1,1}$ $s > B(m,\vartheta^{1/2}(m)) = \Phi(m)$ (cf. (2.28)), and thus we conclude the monotonicity of $\mathfrak{G}_s$.

To prove (2.40) define the map $L : T' \to \mathbb{R}$ by

$$L(m,\rho) = 2\int_m^1 \left(\frac{(\vartheta(t)-\vartheta(\rho))\omega(1-t)}{\vartheta(t)}\right)^{1/2} \mathrm{d}t + s\vartheta^{1/2}(m),$$

where $T' := \{(m,\rho) : m \in (m_0,m_1),\ \rho \in (0,m)\}$. Note that $L \in C^0(\overline{T'})$, and $\mathfrak{G}_s(m) = L(m,m)$ on (m_0,m_1). Thanks to the assumption $\vartheta \in C^1((m_0,m_1))$, then $L \in C^1(T')$ with

$$\frac{\partial L}{\partial m}(m,\rho) = -2\left(\frac{(\vartheta(m)-\vartheta(\rho))\omega(1-m)}{\vartheta(m)}\right)^{1/2} + \frac{s\vartheta'(m)}{2\vartheta^{1/2}(m)}\,, \tag{2.41}$$

and

$$\frac{\partial L}{\partial \rho}(m,\rho) = -\vartheta'(\rho)\int_m^1 \left(\frac{\omega(1-t)}{\vartheta(t)(\vartheta(t)-\vartheta(\rho))}\right)^{1/2} \mathrm{d}t = -\vartheta'(\rho)\frac{B(m,\vartheta^{1/2}(\rho))}{2\vartheta^{1/2}(\rho)}. \tag{2.42}$$

Being ϑ^{-1} locally Lipschitz on (m_0,m_1), we can extend ∇L to $\overline{T}$ continuously. Therefore, an elementary argument shows that $\mathfrak{G}_s \in C^1((m_0,m_1))$ and

$$\mathfrak{G}_s'(m) = \frac{\partial L}{\partial m}(m,m) + \frac{\partial L}{\partial \rho}(m,m) = \frac{\vartheta'(m)}{2\vartheta^{1/2}(m)}(s-\Phi(m))\,,$$

where we have used (2.41), (2.42), and (2.30) to deduce (2.40). □

We are now ready to infer some interesting consequences of the previous statement for the minimum problem defining g. With this aim, we introduce

$$s_{\mathrm{frac}} := \sup\{s \in [0,\infty) \mid g(s) < 2\Psi(1)\} \in (0,\infty], \tag{2.43}$$

where Ψ is defined in (2.1). The value s_{frac} is well defined in view of either item (ii) in Proposition 2.1 in case $\vartheta(1^-) < \infty$ or item (iii) in Proposition 2.2 otherwise. Moreover, $g(s) < 2\Psi(1)$ for every $s \in [0,s_{\mathrm{frac}})$ by taking into account that g is non-decreasing.

Proposition 2.14. *Assume (Hp 1)-(Hp 4), and (Hp 5′). Then*

(i) $g(s) = \overline{\varsigma}s$ *for every* $s \in [0,\inf\Phi]$*. In particular,* $\inf\Phi = 0$ *if* $\vartheta(1^-) = \infty$*;*

(ii) $g(s) = (\inf_{\Phi^{-1}(s)} \mathfrak{G}_s) \wedge (\liminf_{m\to 0^+} \mathfrak{G}_s) \wedge (\liminf_{m\to 1^-} \mathfrak{G}_s)$ *for every* $s \in (\inf\Phi,\sup\Phi)$*;*

(iii) $g(s) = 2\Psi(1)$ *for every* $s \in [\sup \Phi, \infty)$.

In particular, $s_{\mathrm{frac}} \in [\inf \Phi, \sup \Phi]$.

Proof. **Step 1.** Assume $\inf \Phi > 0$ otherwise the statement is trivial. Let $s \in [0, \inf \Phi)$, then $w_{m,s} \in W^{1,1}((2m-1, 1))$ for every $m \in (0,1)$. Therefore, the energy $\mathfrak{G}_s$ is strictly decreasing on $(0,1)$ by Lemma 2.13, so that we may use formula (2.25) and the monotonicity of ϑ to bound the energy from below as follows:

$$\mathfrak{G}_s(m) \geq 2\vartheta^{1/2}(m) \int_m^1 w'_{m,s} \mathrm{d}t = \vartheta^{1/2}(m) s\,.$$

Therefore, $\vartheta(1^-) < \infty$ and

$$\lim_{m \to 1^-} \mathfrak{G}_s(m) \geq \bar{\varsigma} s\,. \tag{2.44}$$

Hence, we deduce that $g(s) \geq \bar{\varsigma} s$ for every $s \in (s_{\mathrm{frac}}, \inf \Phi)$. On the other hand, $g(s) \leq \bar{\varsigma} s \wedge (2\Psi(1))$ by item (ii) in Proposition 2.1. Hence, $s_{\mathrm{frac}} \geq \inf \Phi$ and $g(s) = \bar{\varsigma} s$ for $s \in (s_{\mathrm{frac}}, \inf \Phi)$. By continuity of g we have that the last equality extends to $s = \inf \Phi$.

Step 2. Let $s \in (\inf \Phi, \sup \Phi)$, then consider the open sub-level set $\{m \in (0,1) : \Phi(m) < s\}$ and the open super-level set $\{m \in (0,1) : \Phi(m) > s\}$. Let $(m_0, m_1) \subset \{m \in (0,1) : \Phi(m) < s\}$ be a connected component, then $w_{m,s} \in SBV \setminus W^{1,1}((2m-1, 1))$ for every $m \in (m_0, m_1)$. By case (ii)-(a) in Lemma 2.13 the energy is increasing on (m_0, m_1) so that

$$\inf_{(m_0, m_1)} \mathfrak{G}_s = \lim_{m \to m_0^+} \mathfrak{G}_s(m).$$

In particular, Lemma 2.13 itself implies the continuity of $\mathfrak{G}_s$ on (m_0, m_1), and if $m_0 > 0$ we claim that $\mathfrak{G}_s$ is right continuous in m_0. Indeed, using (2.29) we obtain

$$\lim_{m \to m_0^+} \mathfrak{G}_s(m) = \lim_{m \to m_0^+} A(m, \vartheta^{1/2}(m)) + \vartheta^{1/2}(m_0) s = \mathfrak{G}_s(m_0)\,, \tag{2.45}$$

where in the last equality we have used that $A \in C^0(\{(m,\lambda) \in (0,1)^2 : m \in (0,1),\, \lambda \in (0, \vartheta^{1/2}(m)]\})$. Therefore, it follows that $\mathfrak{G}_s$ is continuous on $[m_0, 1)$. Note that in this case $\Phi(m_0) = s$.

Now let $(m_0, m_1) \subset \{m \in (0,1) : \Phi(m) > s\}$ be a connected component, then $w_{m,s} \in W^{1,1}((2m-1, 1))$ for every $m \in (m_0, m_1)$. By case (i) in Lemma 2.13 the energy is strictly decreasing on (m_0, m_1) so that

$$\inf_{(m_0, m_1)} \mathfrak{G}_s = \lim_{m \to m_1^-} \mathfrak{G}_s(m)\,,$$

with $\mathfrak{G}_s$ continuous on (m_0, m_1). We claim that if $m_1 < 1$ then $\mathfrak{G}_s$ is left-continuous in m_1 as well. Indeed, we note that $(m_0, m_1) \mapsto \lambda_{m,s}$ is continuously differentiable and strictly increasing (cf. Lemma 2.13). Then necessarily $\lim\limits_{m \to m_1^-} \lambda_{m,s} = \vartheta^{1/2}(m_1)$ by (2.26) as $\Phi(m_1) = s$ by continuity of Φ in $(0,1)$. To conclude we argue as follows:

$$\begin{aligned} \mathfrak{G}_s(m) &= 2 \int_m^1 \left(\frac{\vartheta(t) \omega(1-t)}{\vartheta(t) - \lambda_{m,s}^2} \right)^{1/2} \mathrm{d}t \\ &= 2 \int_m^{m_1} \left(\frac{\vartheta(t) \omega(1-t)}{\vartheta(t) - \lambda_{m,s}^2} \right)^{1/2} \mathrm{d}t + 2 \int_{m_1}^1 \left(\frac{\vartheta(t) \omega(1-t)}{\vartheta(t) - \lambda_{m,s}^2} \right)^{1/2} \mathrm{d}t\,. \end{aligned}$$

The first summand above is infinitesimal by the local Lipschitz continuity of ϑ^{-1}. Indeed, if K^2 is the maximum value of $\frac{\vartheta(t)\omega(1-t)}{(\vartheta^{-1})'(t)}$ on $[\frac{1}{2}\vartheta^{1/2}(m_1), \vartheta^{1/2}(m_1)]$ then for m sufficiently close to m_1 we get

$$\int_m^{m_1} \Big(\frac{\vartheta(t)\omega(1-t)}{\vartheta(t)-\lambda_{m,s}^2}\Big)^{1/2} \mathrm{d}t \le K \int_m^{m_1} \frac{1}{\big(t-\vartheta^{-1}(\lambda_{m,s}^2)\big)^{1/2}} \mathrm{d}t \le 2K(m_1-\vartheta^{-1}(\lambda_{m,s}^2))^{1/2},$$

and the conclusion follows by the continuity as $\lim_{m\to m_1^-} \lambda_{m,s} = \vartheta^{1/2}(m_1)$. Therefore, we have

$$\begin{aligned}\lim_{m\to m_1^-} \mathfrak{G}_s(m) &= \lim_{m\to m_1^-} 2\int_{m_1}^1 \Big(\frac{\vartheta(t)\omega(1-t)}{\vartheta(t)-\lambda_{m,s}^2}\Big)^{1/2} \mathrm{d}t \\ &= 2\int_{m_1}^1 \Big(\frac{\vartheta(t)\omega(1-t)}{\vartheta(t)-\vartheta(m_1)}\Big)^{1/2} \mathrm{d}t = \mathfrak{G}_s(m_1),\end{aligned}$$

by the monotone convergence theorem.

Therefore, we have shown that $g(s) \ge (\inf_{\Phi^{-1}(s)} \mathfrak{G}_s) \wedge (\liminf_{m\to 0^+} \mathfrak{G}_s) \wedge (\liminf_{m\to 1^-} \mathfrak{G}_s)$. The opposite inequality is trivial as $g(s) = \inf_{(0,1)} \mathfrak{G}_s$.

Step 3. Next, consider any $s > \sup \Phi$, then by (2.28) necessarily $w_{m,s} \in SBV \setminus W^{1,1}((2m-1,1))$ for every $m \in (0,1)$. Therefore, $\mathfrak{G}_s$ is strictly increasing on $(0,1)$ by case 2 in Lemma 2.13, and thus

$$g(s) = \inf_{(0,1)} \mathfrak{G}_s = \lim_{m\to 0^+} \mathfrak{G}_s(m) = \lim_{m\to 0^+} A(m, \vartheta^{1/2}(m))$$

where in the last inequality we have used the explicit formula (2.29) for $\mathfrak{G}_s$ and the fact that the second summand in there is infinitesimal. On the other hand, by using Fatou's lemma we conclude the continuity of $[0,1) \ni m \mapsto A(m, \vartheta^{1/2}(m))$ in $m = 0$ and the thesis. Indeed, by Fatou's lemma we have

$$\begin{aligned}g(s) &= \lim_{m\to 0^+} A(m, \vartheta^{1/2}(m)) \\ &= 2\lim_{m\to 0^+} \int_0^1 \Big(\frac{(\vartheta(t)-\vartheta(m))\vee 0}{\vartheta(t)}\,\omega(1-t)\Big)^{1/2} \mathrm{d}t \ge 2\Psi(1)\,.\end{aligned} \tag{2.46}$$

The conclusion for $s > \sup \Phi$ follows from the estimate $g(s) \le \overline{\varsigma} s \wedge (2\Psi(1))$ if $\vartheta(1^-) < \infty$ (cf. item (ii) in Proposition 2.1), and the estimate $g(s) \le g_0(s) \le 2\Psi(1)$ if $\vartheta(1^-) = \infty$ (cf. item (ii) in Proposition 2.2). By continuity of g we obtain the equality $g(\sup \Phi) = 2\Psi(1)$. Therefore, $s_{\mathrm{frac}} \le \sup \Phi$. □

Next, we assume Φ to be strictly decreasing, as discussed for instance in Remark 2.12, to identify a unique minimizing value (defined in terms of Φ itself) for the problem defining $g(s)$ introduced in (2.17).

Theorem 2.15. *Assume (Hp 1)-(Hp 4), and (Hp 5′), and that $\Phi : (0,1) \to (0,\infty)$ in (2.30) is strictly decreasing. Then, the following hold*

(i) $s_{\mathrm{frac}} = \sup \Phi = \Phi(0^+)$*;*

(ii) $g(s) = \overline{\varsigma} s$ *for every* $s \in [0, \inf \Phi]$*. Moreover, for every* $s \in (\inf \Phi, s_{\mathrm{frac}})$ *let* $m_s := \Phi^{-1}(s)$*, then* $w_{m_s,s} \in W^{1,1}((2m_s-1,1))$*,* $\lambda_{m_s,s} = \vartheta^{1/2}(m_s)$ *and*

$$g(s) = \mathfrak{G}_s(m_s)\,, \tag{2.47}$$

and m_s *is the unique value* m *such that* $g(s) = \mathfrak{G}_s(m)$*;*

(iii) $g \in C^1([0,\infty))$ *if* $\vartheta(1^-) < \infty$*, and* $g \in C^1((0,\infty)) \cap C^0([0,\infty))$ *if* $\vartheta(1^-) = \infty$*, with*

$$g'(s) = \vartheta^{1/2}(m_s)\,, \tag{2.48}$$

for $s \in [\inf \Phi, s_{\rm frac}]$*,* $g'(s) = \overline{\varsigma}$ *for* $s \in [0, \inf \Phi]$*, and* $g'(s) = 0$ *for* $s \geq s_{\rm frac}$*. In particular,* g *is concave (actually strictly concave on* $[\inf \Phi, s_{\rm frac}]$*). Moreover,* $\overline{\varsigma} s_{\rm frac} \geq 2\Psi(1)$ *if* $\vartheta(1^-) < \infty$*.*

Proof. Being Φ strictly decreasing $\inf \Phi = \Phi(1^-)$ and $\sup \Phi = \Phi(0^+)$, then Proposition 2.14 yields that $s_{\rm frac} \in [\Phi(1^-), \Phi(0^+)]$, and $\Phi((0,1)) = (\Phi(1^-), \Phi(0^+))$ by continuity of Φ (cf. Proposition 2.10).

Step 1: minimality of the relaxed energy at $m = m_s$**.** Fix $s \in (\Phi(1^-), \Phi(0^+))$, and let $m_s := \Phi^{-1}(s) \in (0,1)$. Then $w_s := w_{m_s,s} \in W^{1,1}((2m_s - 1, 1))$ by (2.26), with $\lambda_{m_s,s} = \vartheta^{1/2}(m_s)$, $w_s(2m_s - 1) = 0$, $w_s(1) = s$, and moreover in view of (2.21)

$$w_s(t) = \frac{s}{2} + \int_{m_s}^t \Big(\frac{\vartheta(m_s)\omega(1-\tau)}{\vartheta(\tau)(\vartheta(\tau) - \vartheta(m_s))} \Big)^{1/2} \mathrm{d}\tau\,.$$

In addition, the following claims hold due to the strict monotonicity of Φ, using that $B(m,\cdot)$ is strictly increasing for all m, and $B(\cdot,\lambda)$ is strictly decreasing for all λ

(i) $w_{m,s} \in W^{1,1}((2m-1,1))$ and $\lambda_{m,s} \in (0, \vartheta^{1/2}(m))$ for every $m \in (0, m_s)$,

(ii) $w_{m,s} \in SBV \setminus W^{1,1}((2m-1,1))$ for every $m \in (m_s, 1)$.

To conclude (2.47) we apply case (ii) in Proposition 2.14 noting that $\mathfrak{G}_s(m_s) < (\liminf_{m\to 0^+} \mathfrak{G}_s) \wedge (\liminf_{m\to 1^-} \mathfrak{G}_s)$ (the latter inferior limits are actually limits being Φ monotone) thanks to Lemma 2.13 and items (i) and (ii) above.

The equality $g(s) = \overline{\varsigma} s$ for $s \in [0, \Phi(1^-)]$ follows directly from case (i) in Proposition 2.14.

Step 2. Differentiability of g**.** To establish the differentiability of g and prove (2.48), fix $s_1, s_2 \in (\Phi(1^-), \Phi(0^+))$. Thanks to (2.47), using the competitor $w = \frac{s_1}{s_2} w_{s_2} \in W^{1,1}((2m_2 - 1, 1))$, where $m_2 := m_{s_2}$ and $w_{s_2} = w_{s_2,m_2}$, we obtain that

$$\begin{aligned}
g(s_1) &= \inf_{m\in(0,1)} \inf_{L^1((2m-1,1))} sc^- G_{m,s_1} \leq sc^- G_{m_2,s_1}(w) \\
&= 2\int_{m_2}^1 \Big(\vartheta(t)|w'|^2 + \omega(1-t) \Big)^{1/2} \mathrm{d}t = g(s_2) \\
&+ 2\left(\frac{s_1^2}{s_2^2} - 1\right) \int_{m_2}^1 \frac{\vartheta(t)|w'_{s_2}|^2}{\big((\frac{s_1}{s_2})^2 \vartheta(t)|w'_{s_2}|^2 + \omega(1-t)\big)^{1/2} + \big(\vartheta(t)|w'_{s_2}|^2 + \omega(1-t)\big)^{1/2}} \mathrm{d}t.
\end{aligned} \tag{2.49}$$

In particular, if $s_1 < s_2$ we get

$$\begin{aligned}
&\frac{g(s_1) - g(s_2)}{s_1 - s_2} \\
&\geq 2\frac{s_2 + s_1}{s_2^2} \int_{m_2}^1 \frac{\vartheta(t)|w'_{s_2}|^2}{\big((\frac{s_1}{s_2})^2 \vartheta(t)|w'_{s_2}|^2 + \omega(1-t)\big)^{1/2} + \big(\vartheta(t)|w'_{s_2}|^2 + \omega(1-t)\big)^{1/2}} \mathrm{d}t,
\end{aligned} \tag{2.50}$$

therefore using (2.21) (see also (2.22)), we infer

$$\begin{aligned}\liminf_{s_1\to s_2^-}\frac{g(s_1)-g(s_2)}{s_1-s_2} &\geq \frac{2}{s_2}\int_{m_2}^1 \frac{\vartheta(t)|w'_{s_2}|^2}{\big(\vartheta(t)|w'_{s_2}|^2+\omega(1-t)\big)^{1/2}}\mathrm{d}t\\ &= \frac{2}{s_2}\int_{m_2}^1 \vartheta^{1/2}(m_2)w'_2\mathrm{d}t = \vartheta^{1/2}(m_2).\end{aligned}$$

Using (2.49) with $s_1 > s_2$ we obtain

$$\begin{aligned}&\limsup_{s_2\to s_1^-}\frac{g(s_2)-g(s_1)}{s_2-s_1} \leq \limsup_{s_2\to s_1^-}\frac{s_1+s_2}{s_2^2}\int_{m_2}^1 \frac{\vartheta(t)|w'_{s_2}|^2}{\big(\vartheta(t)|w'_{s_2}|^2+\omega(1-t)\big)^{1/2}}\mathrm{d}t\\ &= \limsup_{s_2\to s_1^-}\frac{s_2+s_1}{s_2^2}\int_{m_2}^1 \vartheta^{1/2}(m_2)w'_2\mathrm{d}t = \limsup_{s_2\to s_1^-}\frac{s_2+s_1}{2s_2}\vartheta^{1/2}(m_2) = \vartheta^{1/2}(m_1),\end{aligned} \tag{2.51}$$

where in the last equality we have used the continuity of Φ^{-1}. In particular, we have shown that at each point $s\in(\Phi(1^-),\Phi(0^+))$ the left derivative of g exists and equals $\vartheta^{1/2}(m_s)$.

The conclusion for the right derivative can be deduced analogously by passing to the inferior limit as $s_2\to s_1^+$ in (2.50), and to the superior limit as $s_1\to s_2^-$ in (2.51).

To conclude, we need only to prove the differentiability of g in $\Phi(1^-)$ if $\vartheta(1^-)<\infty$, and in $\Phi(0^+)$ in general assuming the latter finite. Indeed, $g'(s)=\overline{\varsigma}$ for $s\in[0,\Phi(1^-))$ if $\vartheta(1^-)<\infty$, and $g'(s)=0$ if $s>\Phi(0^+)$. Thus, the formula for g' in (2.48) and the continuity of ϑ respectively in $\Phi(1^-)$ if $\vartheta(1^-)<\infty$ and in $\Phi(0^+)$ in general, imply that

$$\lim_{s\to\Phi(1^-)^+} g'(s)=\vartheta^{1/2}(1)=\overline{\varsigma}, \qquad \lim_{s\to\Phi(0^+)^-} g'(s)=\vartheta^{1/2}(0)=0\,.$$

Furthermore, we deduce that $s_{\mathrm{frac}}=\Phi(0^+)$ thanks to (2.48) and as by definition $g(s)=2\Psi(1)$ for every $s\geq s_{\mathrm{frac}}$.

Moreover, $(\Phi(1^-),\Phi(0^+))\ni s\mapsto m_s$ is non-increasing as Φ is so, in turn implying the concavity of g by (2.48). Eventually, if $\vartheta(1^-)<\infty$, necessarily $\overline{\varsigma}s_{\mathrm{frac}}\geq 2\Psi(1)$ as $g'(0)=\overline{\varsigma}$ and g is concave. □

In some cases under a more restrictive set of assumptions an expression for g similar to that in (2.47), and the differentiability of g, have been first established in [BI24, Proposition 3.3] and [BI24, Proposition 3.6], respectively.

Remark 2.16. *We point out that for $s\in(\inf\Phi,\sup\Phi)$ the energy $\mathfrak{G}_s$ in Theorem 2.15 turns out to be $C^1((0,1))$ with $\mathfrak{G}'_s(m_s)=0$ thanks to item (ii) case (b) in Lemma 2.13.*

Remark 2.17. *If Φ is non-increasing on $(0,1)$, one cannot expect g to be of class C^1. Indeed, within the hypothesis of Proposition 2.14, assume that $\Phi(m)=s_0$ if $m\in[m_0,m_1]$, for some $0<m_0<m_1<1$, and strictly decreasing otherwise. Then, from the condition $\Phi(m)=s_0$ for every $m\in[m_0,m_1]$ it follows that the corresponding minimizers $w_{m,s}\in W^{1,1}$. Defining $\overline{m}:=\inf\{m\in[m_0,m_1]:\lambda_{m,s}=\vartheta^{1/2}(m)\}$, and setting $\overline{m}=m_1$ if the latter set is empty, we have that $g(s_0)=\mathfrak{G}_{s_0}(m)$ for every $m\in[\overline{m},m_1]$. Indeed, item (ii) case (b) in Lemma 2.13 yields that $\mathfrak{G}'_s(m)=0$ on the interval $(\overline{m},m_1)$, if $\overline{m}<m_1$.*

Otherwise, on $(0,1)\setminus[m_0,m_1]$ we may argue as in Theorem 2.15 to obtain formula (2.47). Moreover (2.48) hold for every $s\in(0,s_0)\cup(s_0,1)$, where $s_{\mathrm{frac}}=\Phi(0^+)$. Therefore, $g\in C^1((0,s_0)\cup(s_0,1))$, and g

is not differentiable in s_0 as

$$\lim_{s\to s_0^-} g'(s) = \vartheta^{1/2}(m_1) \neq \lim_{s\to s_0^+} g'(s) = \vartheta^{1/2}(m_0)\,.$$

Next, we deal with the case Φ increasing and $\vartheta(1^-) < \infty$, we show that the cohesive law g always corresponds to that in the Dugdale cohesive model.

Theorem 2.18. *Assume (Hp 1)-(Hp 4), and (Hp 5′), $\vartheta(1^-) < \infty$, and that $\Phi : (0,1) \to (0,\infty)$ in (2.30) is non-decreasing. Then, $g(s) = \overline{\varsigma} s \wedge (2\Psi(1))$ for every $s \geq 0$.*

Proof. The proof is similar to that of Theorem 2.15. Recall that by continuity $\Phi((0,1)) = (\Phi(0^+), \Phi(1^-))$. Next, we use Proposition 2.14 to infer that $g(s) = \overline{\varsigma} s$ for all $s \in [0, \Phi(0^+)]$ by case (i) there, and $g(s) = 2\Psi(1)$ for all $s \geq \Phi(1^-)$ by case (iii) there. Thus, we are left with considering the case $\Phi(0^+) < \Phi(1^-)$ and $s \in (\Phi(0^+), \Phi(1^-))$. Let $m \in [m_0, m_1] = \Phi^{-1}(s) \in (0,1)$ and note that $w_{m,s} \in W^{1,1}((2m-1,1))$ by (2.26), with $\lambda_{m,s} = \vartheta^{1/2}(m)$, and moreover in view of (2.21)

$$w_{m,s}(t) = \frac{s}{2} + \int_m^t \Big(\frac{\vartheta(m)\omega(1-\tau)}{\vartheta(\tau)(\vartheta(\tau)-\vartheta(m))}\Big)^{1/2} \mathrm{d}\tau\,.$$

In addition, the following claims hold due to the monotonicity of Φ

(i) $w_{m,s} \in SBV \setminus W^{1,1}((2m-1,1))$ for every $m \in (0, m_0)$

(ii) $w_{m,s} \in W^{1,1}((2m-1,1))$ and $\lambda_{m,s} \in (0, \vartheta^{1/2}(m))$ for every $m \in (m_1, 1)$.

To conclude $g(s) = \overline{\varsigma} s \wedge (2\Psi(1))$ we apply case (ii) in Proposition 2.14 noting that $\mathfrak{G}_s(m_s) > (\liminf_{m\to 0^+} \mathfrak{G}_s) \wedge (\liminf_{m\to 1^-} \mathfrak{G}_s)$ (the latter inferior limits are actually limits by monotonicity) thanks to Lemma 2.13 and items (i) and (ii) above. Therefore, we have that

$$g(s) = \inf_{(0,1)} \mathfrak{G}_s(m) = \big(\lim_{m\to 0^+} \mathfrak{G}_s(m)\big) \wedge \big(\lim_{m\to 1^-} \mathfrak{G}_s(m)\big)\,.$$

The first limit is easily seen to be bigger than $2\Psi(1)$ arguing as to deduce (2.46) in Proposition 2.14). In addition, the second limit above is bigger than $\overline{\varsigma} s$ arguing as to deduce (2.44) in Proposition 2.14). Therefore, $g(s) \geq \overline{\varsigma} s \wedge (2\Psi(1))$ for every $s \in (\Phi(0^+), \Phi(1^-))$, the reverse inequality is contained in item (ii) in Proposition 2.1. □

3 Reconstructing the cohesive law g

In this section we show how to assign a given cohesive law g_0, either with a linear or a superlinear behaviour for small amplitudes of the jump, by appropriately choosing the parameters of the phase-field model in (1.3). More precisely, either fixing the damage potential and choosing appropriately the degradation function or vice versa, we consider the corresponding functionals $\mathcal{F}_\varepsilon$ and show that g_0 equals the surface energy density of their Γ-limit according to Theorem 1.1.

A scaling argument shows that the assumption $\varphi'(0^+) = 1$ in Theorems 3.1, 3.2, 3.4, and 3.5 below can be easily dispensed with. The functions in the conclusions of those statements would then be depending on the value $\varphi'(0^+) \neq 1$.

3.1 The linear case

We start with reconstructing cohesive laws that behave linearly for small jump amplitudes. More precisely, let

(Hp 6) $g_0 \in C^1([0,\infty))$, $g_0^{-1}(0) = \{0\}$, g_0 bounded and non-decreasing;

(Hp 7) g_0 concave, $g_0'(0) = \overline{\varsigma} \in (0,\infty)$, g_0' is strictly decreasing on $[0,(s_{\text{frac}})_0)$ where

$$(s_{\text{frac}})_0 := \sup\{s \in [0,\infty) : g_0(s) < \lim_{s\to\infty} g_0(s)\} \in (0,\infty]\,.$$

In addition, it is convenient to define the auxiliary function $R : [0,1] \to [0, g_0(\infty)]$ by

$$R(t) := \begin{cases} g_0\big((g_0')^{-1}((\overline{\varsigma}^2 - t)^{1/2})\big) & \text{if } t \in [0,\overline{\varsigma}^2) \\ g_0\big((s_{\text{frac}})_0\big) & \text{if } t = \overline{\varsigma}^2. \end{cases} \tag{3.1}$$

As a consequence of (Hp 6), (Hp 7) and the definition of $(s_{\text{frac}})_0$, R is strictly increasing. Moreover, $g_0'(s) \to 0$ for $s \to (s_{\text{frac}})_0^-$ as $g_0 \in C^1([0,\infty))$ is concave and strictly decreasing, so that $g_0(\infty) < \infty$. Therefore, R is continuous. We further assume that

(Hp 8) R is convex on $[0,\overline{\varsigma}^2]$ and $W^{1,p}((0,\overline{\varsigma}^2))$ for some $p > 2$.

Having introduced R, for every $\tau \in [0,\overline{\varsigma}^2]$ consider

$$\phi(\tau) := \frac{1}{\pi}\int_0^\tau \frac{R'(t)}{(\tau - t)^{1/2}}\mathrm{d}t\,. \tag{3.2}$$

By Abel's inversion theorem [GV91, Theorem 1.A.1] (see also formula (1.B.1ii)), ϕ is the unique $L^1((0,1))$ function satisfying the following Abel integral equation

$$R(t) = \int_0^t \frac{\phi(\tau)}{(t-\tau)^{1/2}}\mathrm{d}\tau\,,$$

for every $t \in [0,\overline{\varsigma}^2]$. This is a consequence of the smoothness of R and of $R(0) = 0$. Moreover, the regularity, the strict monotonicity and the convexity of R in (Hp 8) yield that $\phi \in C^0([0,\overline{\varsigma}^2])$, $\phi^{-1}(0) = \{0\}$, and ϕ is strictly increasing (cf. [GV91, Theorem 4.1.4]). More generally, by changing variables in the very definition of ϕ above (i.e., $t = \tau y$), the strict monotonicity of ϕ follows provided $R'(t)t^{1/2}$ is non-decreasing and not constant on $(0,\overline{\varsigma}^2)$.

We are now ready to reconstruct g_0. In the ensuing two results we will always choose $\omega = \overline{\varsigma}^2 Q$, so that $\vartheta = \overline{\varsigma}^2 l$. We start fixing ϑ with $\vartheta'(t) > 0$ for every $t \in (0,1)$, and find the appropriate ω. In particular, $\vartheta(t) = \overline{\varsigma}^2 t^2$ satisfies all the conditions below.

Theorem 3.1. *Assume that g_0 satisfy (Hp 6)-(Hp 7), R satisfy (Hp 8), and ϑ_0 is such that $\vartheta_0(0) = 0$, $\vartheta_0(1^-) = \overline{\varsigma}^2$, $\vartheta_0^{1/2} \in C^1([0,1])$ with $\vartheta_0'(t) > 0$ for every $t \in (0,1)$, and concave in a left neighborhood of $t = 1$ (in particular, ϑ_0 satisfies (Hp 5′)).*

Then there exists $\omega_0 \in C^0([0,1])$ such that $l(t) := \overline{\varsigma}^{-2}\vartheta_0(t)$, $\overline{\varsigma}^2 Q(t) = \omega(t) := \omega_0(t)$ on $[0,1]$ satisfy (Hp 1)-(Hp 2), and for any φ satisfying (Hp 3)-(Hp 4) with $\varphi'(0^+) = 1$ the corresponding functionals $\mathcal{F}_\varepsilon$ in (1.3) Γ-converge to $F_{\overline{\varsigma}}$ in (1.8) with $g = g_0$.

Proof. Let us first assume $\overline{\varsigma} = 1$. Let ϕ be defined in (3.2), and then set for every $t \in [0,1]$

$$\omega_0(1-t) := \big((\vartheta_0^{1/2}(t))'\phi(1-\vartheta_0(t))\big)^2 . \tag{3.3}$$

Clearly, $\omega_0 \in C^0([0,1])$ and with the choices $l := \vartheta_0$ and $Q = \omega := \omega_0$, (Hp 1) and (Hp 2) are satisfied as ϕ is strictly increasing and $\vartheta_0^{1/2}$ is concave in a left neighborhood of $t = 1$. Moreover, (1.2) holds with $\overline{\varsigma} = 1$. In particular, choosing any φ that satisfies (Hp 3) and (Hp 4) with $\varphi'(0^+) = 1$, by Theorem 1.1 $(\mathcal{F}_\varepsilon)_\varepsilon$ Γ-converges to F_1, where the surface energy density g is given by

$$g(s) := \inf_{(u,v)\in\mathfrak{U}_s} \int_0^1 \Big(\vartheta_0(v)|u'|^2 + \omega_0(1-v)|v'|^2\Big)^{1/2} \mathrm{d}x.$$

Note that $\Phi(1^-) = 0$ by (2.34) in Proposition 2.10. Indeed, using (3.3) we have

$$F(t) = \frac{(\Psi_0 \circ \vartheta_0^{-1})'(1-t)}{(1-t)^{1/2}} = \frac{\phi(t)}{2(1-t)} \in C^0([0,z])$$

for every $z \in (0,1)$ being $\phi \in C^0([0,1])$. In addition, Φ is strictly decreasing as (3.3) implies that for every $t \in [0,1]$

$$\big(\Psi_0(\vartheta_0^{-1}(t^2))\big)' = 2t\big(\Psi_0 \circ \vartheta_0^{-1}\big)'(t^2) = \phi(1-t^2)\,,$$

the claim thus follows from item (v) in Proposition 2.10, and the fact that ϕ is strictly increasing. Therefore, we may apply Theorem 2.15 to deduce for every $s \in (0, s_{\mathrm{frac}})$

$$\begin{aligned} g(s) &= 2\int_{\vartheta_0^{-1}((g'(s))^2)}^1 \Big(\frac{\vartheta_0(t)\omega_0(1-t)}{\vartheta_0(t)-(g'(s))^2}\Big)^{1/2} \mathrm{d}t \\ &= 2\int_0^{1-(g'(s))^2} \Big((1-r)\frac{\omega_0(1-\vartheta_0^{-1}(1-r))}{1-(g'(s))^2-r}\Big)^{1/2} \frac{1}{\vartheta_0'(\vartheta_0^{-1}(1-r))}\mathrm{d}r\,, \end{aligned} \tag{3.4}$$

having used the change of variables $r = 1-\vartheta_0(t)$. Therefore, on setting $\lambda = 1-(g'(s))^2$ and using the very definition of ω_0 we conclude for every $\lambda \in [0,1)$

$$g((g')^{-1})\big((1-\lambda)^{1/2}\big) = \int_0^\lambda \frac{\phi(r)}{(\lambda-r)^{1/2}}\mathrm{d}r = R(\lambda) = g_0\big((g_0')^{-1}((1-\lambda)^{1/2}\big)\,,$$

where we have used that g' is invertible in view of (2.48). Inverting $g \circ (g')^{-1}$ on $[0, g(s_{\mathrm{frac}}))$ and $g_0 \circ (g_0')^{-1}$ on $[0, g_0((s_{\mathrm{frac}})_0))$ we conclude that $(g^{-1})' = (g_0^{-1})'$ on $[0, g_0((s_{\mathrm{frac}})_0) \wedge g(s_{\mathrm{frac}}))$. From this, using the very definitions of $(s_{\mathrm{frac}})_0$ and s_{frac}, together with $g_0'((s_{\mathrm{frac}})_0) = g'(s_{\mathrm{frac}}) = 0$, $g_0' > 0$ on $(0,(s_{\mathrm{frac}})_0)$, and $g' > 0$ on $(0, s_{\mathrm{frac}})$, we deduce that $(s_{\mathrm{frac}})_0 = s_{\mathrm{frac}}$. Being $g^{-1}(0) = g_0^{-1}(0) = 0$, we conclude that $g(s) = g_0(s)$ for every $s \in [0,\infty)$.

If $\overline{\varsigma} \neq 1$, apply the previous argument to $\widetilde{g}_0 := \overline{\varsigma}^{-1} g_0$ and $\widetilde{l}_0(t) = t^2$, to find $\widetilde{\omega}_0$ such that $\widetilde{g}_0$ is the surface energy density of the Γ-limit of the corresponding functionals $\mathcal{F}_\varepsilon$ with $\widetilde{Q} = \widetilde{\omega} := \widetilde{\omega}_0$ and $\widetilde{l} := \widetilde{l}_0$. The conclusion for g_0 then follows by taking $l(t) := t^2$, $\omega := \overline{\varsigma}^2\widetilde{\omega}_0$, and $Q := \widetilde{\omega}_0$. □

Next, given g_0, we fix ω (and $Q = \overline{\varsigma}^{-2}\omega$), and determine ϑ, equivalently l.

Theorem 3.2. *Let g_0 satisfy (Hp 6)-(Hp 7), R satisfy (Hp 8), and $\omega_0 \in C^0([0,1],[0,\infty))$, $\omega_0^{-1}(\{0\}) = \{0\}$, ω_0 increasing in a right neighbourhood of 0 and such that $2\Psi_0(1) = g_0(\infty)$, where Ψ_0 is defined as* *in* (2.1) *integrating* ω_0.

Then, there exists $l_0 \in C^0([0,1]) \cap C^1((0,1))$ *with* $l_0' > 0$ *on* $(0,1)$*, such that* $l := l_0$*,* $\bar{\varsigma}^2 Q = \omega := \omega_0$ *satisfy (Hp 1), (Hp 2),* $\vartheta := \bar{\varsigma}^2 l$ *satisfies (Hp 5′), and for any* φ *satisfying (Hp 3)-(Hp 4) with* $\varphi'(0^+) = 1$*, the functionals* $\mathcal{F}_\varepsilon$ *in* (1.3) Γ*-converge to* $F_{\bar{\varsigma}}$ *in* (1.8) *with* $g = g_0$*.*

Proof. Let us first assume $\bar{\varsigma} = 1$. With ϕ defined in (3.2), let $l_0 : [0,1] \to [0,1]$ be for every $t \in [0,1]$

$$l_0^{-1}(1-t) := \Psi_0^{-1}\left(\frac{g_0(\infty)}{2} - \frac{1}{2}\int_0^t \frac{\phi(\tau)}{(1-\tau)^{1/2}}\mathrm{d}\tau\right).$$

Then $l_0 \in C^0([0,1])$, l_0 is strictly increasing, $l_0(0) = 0$ as $R(1) = g_0(\infty)$, $l_0(1) = 1$ as $2\Psi_0(1) = g_0(\infty)$, $l_0 \in C^1((0,1))$ with $l_0' > 0$ in $(0,1)$, $l_0^{-1} \in W^{1,1}((0,1))$ and for every $t \in (0,1)$

$$2\big((1-t)\omega_0(1 - l_0^{-1}(1-t))\big)^{1/2}(l_0^{-1})'(1-t) = \phi(t). \tag{3.5}$$

We observe that setting $l = \vartheta := l_0$ and $Q = \omega := \omega_0$, (Hp 1) and (Hp 2) are satisfied. In particular choosing any φ that satisfies (Hp 3) and (Hp 4) with $\varphi'(0^+) = 1$, Theorem 1.1 implies that $\mathcal{F}_\varepsilon$ in (1.3) Γ-converge to F_1 where g is given by

$$g(s) = \inf_{(u,v)\in\mathfrak{U}_s} \int_0^1 \Big(l_0(v)|u'|^2 + \omega_0(1-v)|v'|^2\Big)^{1/2}\mathrm{d}x. \tag{3.6}$$

Moreover $(\vartheta \circ \Psi_0^{-1})^{1/2}$ is convex, because it is non-decreasing and its inverse $\Psi_0(\vartheta^{-1}(t^2))$ is concave. Indeed, (3.5) yields that the derivative of the latter function is given by

$$2t\omega_0^{1/2}(1 - l_0^{-1}(t^2))(l_0^{-1})'(t^2) = \phi(1-t^2)$$

for every $t \in (0,1)$. Hence, the corresponding function Φ is strictly decreasing, and $\Phi(1^-) = 0$ by (2.34) in Proposition 2.10. Moreover, the condition in (2.33) is satisfied as $F(t) = \frac{\phi(t)}{2(1-t)} \in C^0([0,1))$, so that $\Phi(1^-) = 0$. Hence, Theorem 2.15 yields that for every $s \in (0, s_{\mathrm{frac}})$

$$g(s) = 2\int_{l_0^{-1}((g'(s))^2)}^1 \left(\frac{l_0(t)\omega_0(1-t)}{l_0(t) - (g'(s))^2}\right)^{1/2}\mathrm{d}t\,.$$

Setting $\lambda = 1 - (g'(s))^2$, and changing variable with $\tau = 1 - l_0(t)$ we find

$$g\big((g')^{-1}((1-\lambda)^{1/2})\big) = \int_0^\lambda \frac{\phi(t)}{(\lambda-t)^{1/2}}\mathrm{d}t = R(\lambda) = g_0\big((g_0')^{-1}((1-\lambda)^{1/2})\big)\,,$$

for every $\lambda \in [0,1)$. Arguing as in Theorem 3.1 the conclusion $g = g_0$ follows at once.

If $\bar{\varsigma} \neq 1$, apply the previous argument to $\widetilde{g}_0 := \bar{\varsigma}^{-1}g_0$ and $\widetilde{\omega}_0 := \bar{\varsigma}^{-2}\omega_0$ to find $\widetilde{l}_0$ such that $\widetilde{g}_0$ is the surface energy density of the Γ-limit of the corresponding functionals $\mathcal{F}_\varepsilon$ with $\widetilde{Q} = \widetilde{\omega} := \widetilde{\omega}_0$ and $\widetilde{l} := \widetilde{l}_0$. The conclusion for g_0 then follows by taking $l := \widetilde{l}_0$, $\omega := \omega_0$, and $Q := \bar{\varsigma}^{-2}\omega_0$. ☐

3.2 The superlinear case

In this section we show how to reconstruct suitable cohesive laws with superlinear behaviour for small jump amplitudes. Let

(Hp 6′) $g_0 \in C^1((0,\infty)) \cap C^0([0,\infty))$, $g_0^{-1}(0) = \{0\}$, g_0 is bounded and non-decreasing;

(Hp 7′) $g'(0^+) = \infty$, g_0' is convex on $(0, (s_{\mathrm{frac}})_0]$ where

$$(s_{\mathrm{frac}})_0 := \sup\{s \in [0,\infty) : g_0(s) < \lim_{s\to\infty} g_0(s)\} < \infty\,,$$

and $g_0 \in C^2((0, (s_{\mathrm{frac}})_0))$ with $g_0''((s_{\mathrm{frac}})_0^-) < 0$.

In particular, g_0 is concave on $[0,\infty)$, and g_0' is strictly decreasing on $(0, (s_{\mathrm{frac}})_0]$. Next, define the auxiliary function $R : [0,\infty) \to (0, g_0(\infty)]$ by

$$R(t) := g_0((g_0')^{-1}(t^{1/2})) \tag{3.7}$$

Observe that $R(0^+) = g_0(\infty)$, $R(\infty) = 0$, R is strictly decreasing and convex, $R \in C^1([0,\infty))$ with

$$R'(t) = \begin{cases} (2g_0''((g_0')^{-1}(t^{1/2})))^{-1} & \text{if } t \in (0,\infty) \\ (2g_0''((s_{\mathrm{frac}})_0^-))^{-1} & \text{if } t = 0. \end{cases}$$

For every $\tau \in [0,\infty)$ then define

$$\phi(\tau) := -\frac{1}{\pi}\int_\tau^\infty \frac{R'(t)}{(t-\tau)^{1/2}}\mathrm{d}t\,. \tag{3.8}$$

In the following proposition we establish some basic properties of the function ϕ as a solution of an Abel equation with datum R. Despite being a natural outcome of the theory developed in [GV91], we have not been able to find an explicit reference there. Hence, we give a proof below.

Proposition 3.3. *Assume (Hp 6′) and (Hp 7′), and let ϕ be defined by (3.8). Then, $\phi \in C^0([0,\infty))$ is strictly positive, non-increasing and satisfies the Abel equation*

$$R(t) = \int_t^\infty \frac{\phi(\tau)}{(\tau - t)^{1/2}}\mathrm{d}\tau \tag{3.9}$$

for every $t \in [0,\infty)$.

Proof. First we observe that ϕ is well defined and strictly positive being $R'(t) < 0$ for every $t \in [0,\infty)$. Moreover, by the change of variable $\tau = (g_0')^{-1}(t^{1/2})$

$$\phi(0) = -\int_0^\infty \frac{R'(t)}{t^{1/2}}\mathrm{d}t = -\int_0^\infty \frac{1}{2g_0''((g_0')^{-1}(t^{1/2}))t^{1/2}}\mathrm{d}t = (s_{\mathrm{frac}})_0 < \infty\,.$$

For every $\tau_1, \tau_2 \in [0,\infty)$ with $\tau_1 < \tau_2$ we have that

$$\begin{aligned}\phi(\tau_1) &= -\frac{1}{\pi}\int_{\tau_1}^\infty \frac{R'(t)}{(t-\tau_1)^{1/2}}\mathrm{d}t = -\frac{1}{\pi}\int_{\tau_2}^\infty \frac{R'(t+\tau_1-\tau_2)}{(t-\tau_2)^{1/2}}\mathrm{d}t \\ &\geq -\frac{1}{\pi}\int_{\tau_2}^\infty \frac{R'(t)}{(t-\tau_2)^{1/2}}\mathrm{d}t = \phi(\tau_2),\end{aligned}$$

being R' non-decreasing. Therefore, $\phi(\tau) \in (0,\infty)$ for every $\tau \in [0,\infty)$.

To prove the continuity of ϕ fix $t_0 \in [0,\infty)$ and $(t_j)_j$ such that $t_j \to t_0$ as $j \to \infty$. For every $\varepsilon \in (0,1)$ we have that

$$\phi(t_j) = \int_{t_j}^{t_j+\varepsilon} \frac{R'(t)}{(t-t_j)^{1/2}}\mathrm{d}t + \int_{t_j+\varepsilon}^\infty \frac{R'(t)}{(t-t_j)^{1/2}}\mathrm{d}t.$$

By Lebesgue Dominated Convergence Theorem we have that

$$\lim_{j\to\infty}\int_{t_j+\varepsilon}^{\infty}\frac{R'(t)}{(t-t_j)^{1/2}}\mathrm{d}t=\int_{t_0+\varepsilon}^{\infty}\frac{R'(t)}{(t-t_0)^{1/2}}\mathrm{d}t,$$

while

$$\left|\int_{t_j}^{t_j+\varepsilon}\frac{R'(t)}{(t-t_j)^{1/2}}\mathrm{d}t\right|\leq 2|R'(t_j)|\varepsilon^{1/2}.$$

Thus

$$\limsup_{j\to\infty}|\phi(t_j)-\phi(t_0)|\leq 4|R'(t_0)|\varepsilon^{1/2}.$$

Taking $\varepsilon\to 0$ we infer the continuity of ϕ.

To obtain (3.9), we use Fubini's Theorem and the elementary identity

$$\int_{t_1}^{t_2}\frac{1}{(\tau-t_1)^{1/2}(t_2-\tau)^{1/2}}\mathrm{d}\tau=\pi$$

for every t_1, $t_2\in\mathbb{R}$ with $t_1<t_2$. Indeed

$$\int_t^{\infty}\frac{\phi(\tau)}{(\tau-t)^{1/2}}\mathrm{d}\tau=-\frac{1}{\pi}\int_t^{\infty}\frac{1}{(\tau-t)^{1/2}}\int_\tau^{\infty}\frac{R'(r)}{(r-\tau)^{1/2}}\mathrm{d}r=$$
$$-\frac{1}{\pi}\int_t^{\infty}R'(r)\int_t^{r}\frac{1}{(\tau-t)^{1/2}(r-\tau)^{1/2}}\mathrm{d}\tau\mathrm{d}r=R(t)$$

because $R(\tau)\to 0$ as $\tau\to\infty$. □

Next, we fix ω and determine both l and Q, and thus ϑ.

Theorem 3.4. *Let g_0 satisfy (Hp 6')-(Hp 7'), and let $\omega_0\in C^0([0,1],[0,\infty))$, $\omega_0^{-1}(\{0\})=\{0\}$, $2\Psi_0(1)=g_0(\infty)$, Ψ_0 defined as in (2.1) integrating ω_0.*

Then, there exist l_0 and $Q_0\in C^0([0,1])$ such that $l:=l_0$, $Q:=Q_0$, $\omega:=\omega_0$ satisfy (Hp 1), (Hp 2) with $\bar\varsigma=\infty$, $\vartheta:=\bar\varsigma^2 l$ satisfies (Hp 5'), and for any φ satisfying (Hp 3)-(Hp 4) with $\varphi'(0^+)=1$ the functionals $\mathcal{F}_\varepsilon$ in (1.3) Γ-converge to F_∞ in (1.8) with $g=g_0$.

Proof. Let ϕ be defined in (3.8), then let $\vartheta_0^{-1}:[0,\infty)\to[0,1]$ be

$$\vartheta_0^{-1}(t):=\Psi_0^{-1}\left(\frac{g_0(\infty)}{2}-\frac{1}{2}\int_t^{\infty}\frac{\phi(\tau)}{\tau^{1/2}}\mathrm{d}\tau\right)$$

Then $\vartheta_0\in C^0([0,1))\cap C^1((0,1))$ by Proposition 3.3. Moreover, $\vartheta_0(0)=0$ as $R(0)=g_0(\infty)$, $\vartheta_0(t)\to\infty$ as $t\to 1$ as $2\Psi_0(1)=g_0(\infty)$, $\vartheta_0'(t)>0$ for every $t\in(0,1)$ and

$$2(\vartheta_0^{-1})'(t)\big(t\omega_0(1-\vartheta_0^{-1}(t))\big)^{1/2}=\phi(t)$$

for every $t\in(0,1)$. We observe that setting $l(t):=\vartheta_0(t)\wedge 1$, $Q(0):=0$, $Q(t):=\frac{\omega_0(t)l(1-t)}{\vartheta_0(1-t)}$ if $t\in(0,1]$ and $\omega:=\omega_0$, (Hp 1) and (Hp 2) are satisfied. In particular, for every φ as in the statement, Theorem 1.1 implies that $\mathcal{F}_\varepsilon$ in (1.3) Γ-converge to F_∞ where g is given by

$$g(s)=\inf_{(u,v)\in\mathfrak{U}_s}\int_0^1\Big(\vartheta_0(v)|u'|^2+\omega_0(1-v)|v'|^2\Big)^{1/2}\mathrm{d}x. \tag{3.10}$$

Moreover $(\vartheta_0 \circ \Psi^{-1})^{1/2}$ is convex, because it is non-decreasing and its inverse $\Psi(\vartheta_0^{-1}(t^2))$ is concave. Indeed, the derivative of the latter function is

$$(\omega_0(1 - \vartheta_0^{-1}(t^2)))^{1/2} (\vartheta_0^{-1})'(t^2)\, 2t = \phi(t^2).$$

Therefore, by Theorem 2.15 we infer that

$$g(s) = 2 \int_{\vartheta_0^{-1}((g'(s))^2)}^{1} \Big(\frac{\vartheta_0(t)\omega_0(1-t)}{\vartheta_0(t) - (g'(s))^2} \Big)^{1/2} \mathrm{d}t$$

and changing variable with $\vartheta_0(t) = \tau$ and setting $\lambda = (g'(s))^2$, we find

$$\begin{aligned} g\big((g')^{-1}(\lambda^{1/2})\big) &= 2 \int_{\lambda}^{\infty} \Big(\frac{\tau\, \omega_0(1 - \vartheta_0^{-1}(\tau))}{\tau - \lambda} \Big)^{1/2} (\vartheta_0^{-1})'(\tau) \mathrm{d}\tau \\ &= \int_{\lambda}^{\infty} \frac{\phi(\tau)}{(\tau - \lambda)^{1/2}} \mathrm{d}\tau = R(\lambda) = g_0\big((g_0')^{-1}(\lambda^{1/2})\big)\,, \end{aligned}$$

for every $\lambda \in [0, \infty)$. Arguing as in Theorem 3.1 the conclusion $g = g_0$ follows at once. □

Finally, with fixed ϑ we determine l, Q and ω.

Theorem 3.5. *Assume that g_0 satisfy (Hp 6′)-(Hp 7′), and ϑ_0 is such that $\vartheta_0(0) = 0$, $\vartheta_0(1^-) = \infty$, $\vartheta_0^{1/2} \in C^1([0, 1))$ with $\vartheta_0'(t) > 0$ for every $t \in (0, 1)$, concave in a left neighborhood of $t = 1$, and such that $(\vartheta_0^{1/2}(t))'\phi(\vartheta_0(t))$ is decreasing and infinitesimal as $t \to 1$, where ϕ is defined in (3.8) in particular ϑ_0 satisfies (Hp 5′).*

Then there exist ω_0 and $Q_0 \in C^0([0, 1])$ such that $l(t) := \vartheta_0(t) \wedge 1$, $Q := Q_0$, $\omega := \omega_0$ satisfy (Hp 1)-(Hp 2) with $\bar\varsigma = \infty$; and for any φ satisfying (Hp 3)-(Hp 4) with $\varphi'(0^+) = 1$ the functionals $\mathcal{F}_\varepsilon$ in (1.3) Γ-converge to F_∞ in (1.8) with $g = g_0$.

Proof. Let ϕ be defined in (3.8), and then set for every $t \in [0, 1)$

$$\omega_0(1 - t) := \big((\vartheta_0^{1/2}(t))' \phi(\vartheta_0(t)) \big)^2 , \tag{3.11}$$

and $\omega_0(0) := 0$. Note that $\omega_0 \in C^0([0, 1])$ by the assumptions on ϑ_0. With the choices $l(t) := \vartheta_0(t) \wedge 1$, $Q(0) := 0$ and $Q(1 - t) := \frac{\omega_0(1-t)l(t)}{\vartheta_0(t)}$ for $t \in [0, 1)$, (Hp 1) and (Hp 2) are satisfied as ϕ is strictly increasing and $\vartheta_0^{1/2}$ is concave in a left neighborhood of $t = 1$. Moreover, (1.2) holds with $\bar\varsigma = \infty$. In particular, choosing any φ that satisfies (Hp 3) and (Hp 4) with $\varphi'(0^+) = 1$, by Theorem 1.1 $(\mathcal{F}_\varepsilon)_\varepsilon$ Γ-converges to F_∞, where the surface energy density g is given by

$$g(s) := \inf_{(u,v) \in \mathfrak{U}_s} \int_0^1 \Big(\vartheta_0(v)|u'|^2 + \omega_0(1 - v)|v'|^2 \Big)^{1/2} \mathrm{d}x.$$

The rest of the proof follows exactly as in Theorem 3.4. □

3.3 Examples

In this section, we apply Theorems 2.18, 3.1, 3.2 and 3.4 to find closed form solutions for phase-field models whose Γ-limit have as surface energy density assigned cohesive laws g frequently used in applications, Fig. 2. From a numerical point of view the problem has already been discussed extensively in the papers [Wu17, WN18, FFL21, LCM25, Wu24]. We note that the reconstruction problem for the Dugdale cohesive law has been addressed in [CFI16, Section 7.1] and [LCM25] using a truncated degradation function of the form $\varphi(t) = 1 \wedge t$, as discussed in [ACF25a, formula (1.2)]. In contrast, we obtain here Dugdale cohesive law by using the smooth degradation function $\varphi(t) = \frac{t}{1+t}$, as shown in [ACF25a, formula (1.3)]. Consequently, in a one-dimensional setup, the mechanical global response corresponding to the Dugdale cohesive law with the truncated degradation function is obtained only in the limit as $\varepsilon \to 0$, since the critical stress also depends on the regularizing length ε [LCM25, Fig. 2]. By employing the smooth degradation function, as done in this work, the regularizing length ε only influences the phase-field profile, ensuring that the mechanical global response is immediately and correctly described. Further details about this point can be found in the application-oriented part of this work, [ACF25b].

We will use the notation of section 3. Recall that we have assumed (Hp 1)-(Hp 4), and moreover (Hp 5′). To be consistent with section 3, we suppose that the function φ satisfies $\varphi'(0^+) = 1$. For the sake of simplicity, if $\vartheta(1^-) < \infty$ we shall choose $Q = \omega$ so that $g'(0^+) = 1$. This is clearly not a limitation using the rescaling argument employed in the proofs of Theorems 3.1 and 3.2.

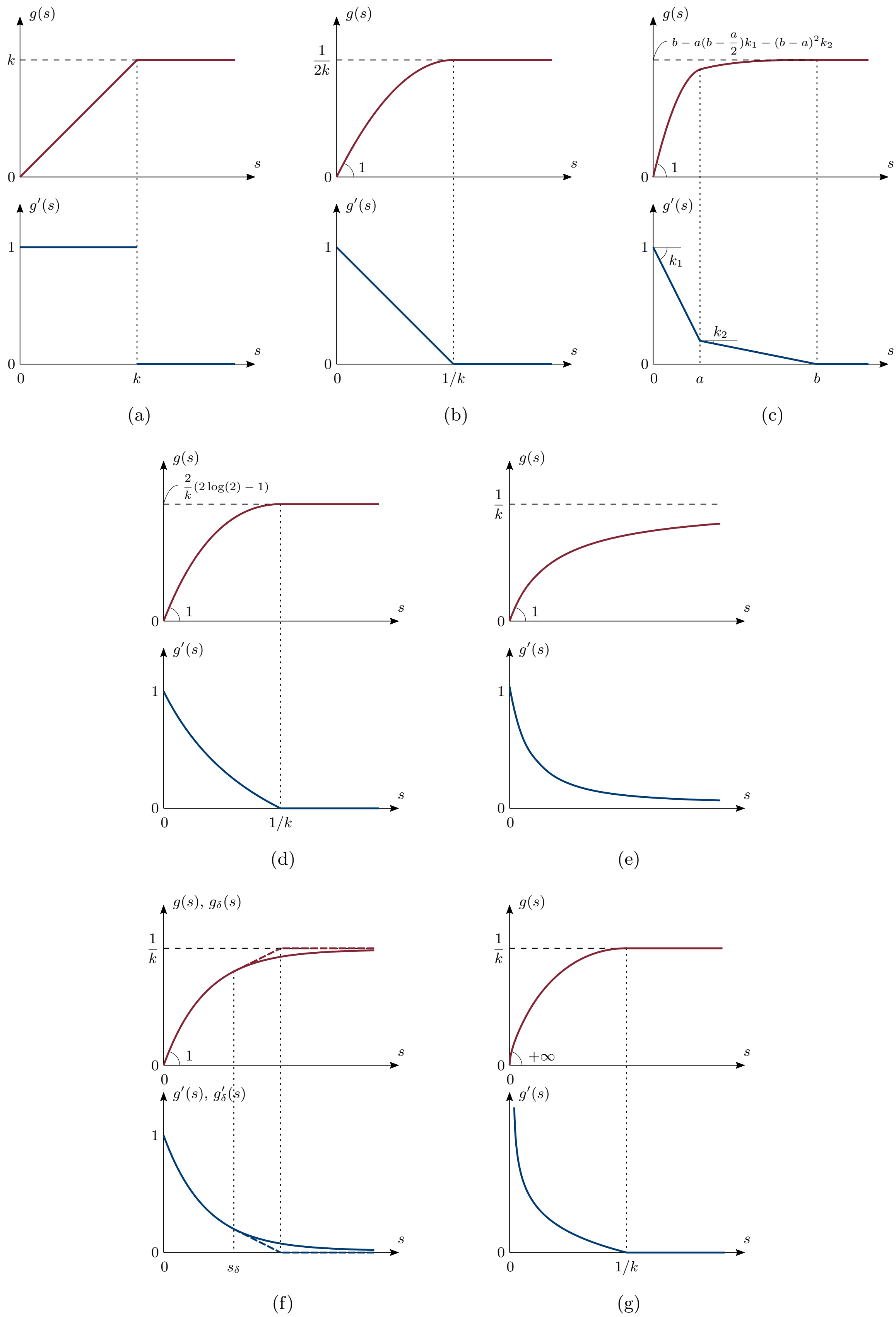

Figure 2: Cohesive laws considered in the examples of section 3.3: (a) Dugdale, (b) Linear, (c) Bilinear, (d) hyperbolic, (e) quadratic hyperbolic, (f) exponential and (g) logarithmic cohesive softening laws.

3.3.1 Dugdale's model

Consider the Dugdale's model given by the cohesive law

$$g(s) = \begin{cases} s & \text{if } s \in [0,k] \\ k & \text{if } s \in (k,\infty) \end{cases} \tag{3.12}$$

where $k \in (0,\infty)$. We observe that if l and ω are such that $l \in C^1((0,1))$ with $l'(t) > 0$ for every $t \in (0,1)$ and

$$\Psi(l^{-1}(\tau^2)) = \frac{k}{\pi}(\sin^{-1}(\tau) - \tau(1-\tau^2)^{1/2})$$

for every $\tau \in [0,1]$, then an easy computation gives $\Phi(x) = 2kl^{1/2}(x)$ for every $x \in (0,1)$, where Ψ and Φ are respectively defined in (2.1) and (2.30). Therefore, if $\mathcal{F}_\varepsilon$ are the functionals defined in (1.3) with $Q = \omega$, then Theorem 2.18 implies that $\Gamma\text{-}\lim_{\varepsilon\to 0} \mathcal{F}_\varepsilon$ is a functional as in (1.8) with g as surface energy density. In particular we obtain the following

- if $\omega(1-t) = k^2(1-t)^2$ then $l^{-1}(t) = 1 - \left[1 - \frac{2}{\pi}\left(\sin^{-1}(t^{\frac{1}{2}}) - (t-t^2)^{\frac{1}{2}}\right)\right]^{\frac{1}{2}}$
- if $\omega(1-t) = \frac{9k^2}{16}(1-t)$ then $l^{-1}(t) = 1 - \left[1 - \frac{2}{\pi}\left(\sin^{-1}(t^{\frac{1}{2}}) - (t-t^2)^{\frac{1}{2}}\right)\right]^{\frac{2}{3}}$

3.3.2 Linear softening

Let $k \in (0,\infty)$, and consider softening laws of the form

$$g'(s) = \begin{cases} 1-ks & \text{if } s \in [0,\frac{1}{k}] \\ 0 & \text{if } s \in (\frac{1}{k},\infty) \end{cases}$$

One can easily verify that the corresponding g satisfies (Hp 6)-(Hp 7), $R(t) = \frac{t}{2k}$ fulfills (Hp 8), and that $\phi(t) = \frac{t^{1/2}}{k\pi}$ by (3.2). In particular, we deduce that

- if $l(t) = t^2$ then $\omega(1-t) = \frac{1-t^2}{k^2\pi^2}$,
- if $\omega(t) = \frac{t^2}{4k^2}$ then $l^{-1}(t) = 1 - \left(1 - \frac{2}{\pi}((t-t^2)^{1/2} + \cos^{-1}((1-t)^{1/2}))\right)^{1/2}$,
- if $\omega(t) = \frac{9t}{64k^2}$ then $l^{-1}(t) = 1 - \left(1 - \frac{2}{\pi}((t-t^2)^{1/2} + \cos^{-1}((1-t)^{1/2})\right)^{2/3}$.

We remark that in case $l(t) = t^2$ we obtain exactly the same damage potential ω as in [FFL21, Section 4.1]

3.3.3 Bilinear softening

In this case we fix $a, b, k_1, k_2 \in (0,\infty)$ such that $a < b$, $k_2 < k_1$ and

$$g'(s) = \begin{cases} 1-k_1 s & \text{if } s \in [0,a] \\ (1-k_1 a + k_2 a) - k_2 s & \text{if } s \in (a,b] \\ 0 & \text{if } s \in (b,\infty) \end{cases}$$

is continuous. Therefore, the corresponding g satisfies (Hp 6)-(Hp 7) and simple calculations yield $R \in W^{1,\infty}((0,1))$ with

$$R'(t) = \begin{cases} \frac{1}{2k_1} & \text{if } t \in [0, 1-(1-k_1a)^2] \\ \frac{1}{2k_2} & \text{if } t \in (1-(1-k_1a)^2, 1]. \end{cases}$$

In particular, R fulfills (Hp 8), and if $l(t) = t^2$ then

$$\omega(1-t) = \begin{cases} \left(\frac{(1-t^2)^{1/2}}{k_1\pi} + (\frac{1}{k_2\pi} - \frac{1}{k_1\pi})((1-k_1a)^2 - t^2)^{1/2} \right)^2 & \text{if } t \in [0, 1-k_1a] \\ \frac{1-t^2}{k_1^2\pi^2} & \text{if } t \in (1-k_1a, 1]. \end{cases}$$

3.3.4 Hyperbolic softening

Let $k \in (0, \infty)$ and consider hyperbolic softening laws of the form

$$g'(s) = \begin{cases} \frac{2}{1+ks} - 1 & \text{if } s \in [0, \frac{1}{k}] \\ 0 & \text{if } s \in (\frac{1}{k}, \infty). \end{cases}$$

Then

$$g(s) = \begin{cases} \frac{2}{k}\log(1+ks) - s & \text{if } s \in [0, \frac{1}{k}] \\ \frac{1}{k}(2\log(2) - 1) & \text{if } s \in (\frac{1}{k}, \infty) \end{cases}$$

satisfies (Hp 6)-(Hp 7) and the corresponding function R fulfills (Hp 8) with

$$R'(t) = \frac{1}{k(1+(1-t)^{\frac{1}{2}})^2}$$

for every $t \in [0,1]$. In particular, by an easy computation and Theorem 3.1, we obtain, with $l(t) = t^2$, that

$$\omega(1-t) = \begin{cases} \frac{(2t^2\log(t)+1-t^2)^2}{k^2\pi^2(1-t^2)^3} & t \in [0,1) \\ 0 & t = 1 \end{cases}.$$

3.3.5 Quadratic hyperbolic softening

Let $k \in (0, \infty)$ and consider quadratic hyperbolic softening laws of the form $g'(s) = \frac{1}{(1+ks)^2}$. Then g satisfies (Hp 6)-(Hp 7) with

$$R'(t) = \frac{4k}{(1-t)^{\frac{1}{4}}} \tag{3.13}$$

for every $t \in [0,1)$. In particular $R' \in L^p((0,1))$ for every $p \in [1,4)$ and thus R fulfills (Hp 8). From a simple computation and Theorem 3.1 we can infer that if $l(t) = t^2$ then $\omega(1-t) = \phi(1-t^2)^2$ where

$$\phi(t) = \frac{16k}{3\pi(1-t)^{\frac{1}{2}}} \left[(1-t)^{\frac{3}{4}} {}_2F_1(1/2, 3/4; 7/4; 1) - {}_2F_1(1/2, 3/4; 7/4; 1/(1-t)) \right] \tag{3.14}$$

and ${}_2F_1(\cdot\,,\cdot\,;\cdot\,;\cdot)$ is the Hypergeometric function.

3.3.6 Exponential softening

Let $k \in (0,\infty)$ and consider exponential softening laws of the form $g'(s) = e^{-ks}$, namely $g(s) = \frac{1}{k}(1 - e^{-ks})$ for $s \geq 0$. It is easy to verify that g fulfills (Hp 6)-(Hp 7) with $R(t) = \frac{1}{k}(1-(1-t)^{1/2})$. Note that $R' \notin L^p((0,1))$ for any $p > 2$, and thus R does not satisfy (Hp 8) so that we can apply neither Theorem 3.1 nor 3.2. Assumption (Hp 8) ensures the continuity of the function ϕ on $[0,1]$, which indeed fails in this particular case as $\phi(t) = \frac{1}{k\pi}\cosh^{-1}\left(\frac{1}{(1-t)^{1/2}}\right)$. Therefore, using the constructions in Theorem 3.1 and 3.2 we would get that either l or ω is not in $C^0([0,1])$. On the other hand, the continuity of both l and ω is a crucial assumption in Theorem 1.1 (cf. (Hp 1)-(Hp 2)).

To overcome this problem, we slightly modify g as follows: let $\delta > 0$ and $s_\delta \to \infty$ as $\delta \to 0^+$, we linearize g' in (s_δ, ∞) to obtain a new cohesive law g_δ given by

$$g'_\delta(s) := \begin{cases} e^{-ks} & \text{if } s \in [0, s_\delta] \\ (e^{-ks_\delta} - ke^{-ks_\delta}(s - s_\delta)) \vee 0 & \text{if } s \in (s_\delta, \infty). \end{cases}$$

For every $\delta \in (0,1)$, g_δ satisfies (Hp 6)-(Hp 7) with

$$R_\delta(t) = \begin{cases} \frac{1}{k}(1-(1-t)^{1/2}) & \text{if } t \in [0, \hat{s}_\delta] \\ \frac{1}{k}(1-(1-\hat{s}_\delta)^{1/2}) & \text{if } t \in (\hat{s}_\delta, 1], \end{cases}$$

for some $\hat{s}_\delta$ such that $\hat{s}_\delta \to 1$ as $\delta \to 0^+$. Moreover, R_δ fulfills (Hp 8) and $g_\delta \to g$ uniformly and monotonically on $[0,\infty)$. In particular, one may choose $\hat{s}_\delta = 1-\delta$ to obtain

$$\phi_\delta(t) = \begin{cases} \frac{1}{k\pi}\cosh^{-1}\left(\frac{1}{(1-t)^{1/2}}\right) & \text{if } t \in [0, 1-\delta] \\ \frac{1}{k\pi}\left(\cosh^{-1}\left(\frac{1}{(1-t)^{1/2}}\right) - \cosh^{-1}\left((\frac{\delta}{1-t})^{1/2}\right) + \left(\frac{\delta-(1-t)}{\delta}\right)^{1/2}\right) & \\ & \text{if } t \in (1-\delta, 1]\,, \end{cases}$$

and hence get from Theorem 3.1 for $l(t) = t^2$

$$\omega_\delta(1-t) = \begin{cases} \frac{1}{k^2\pi^2}\left(\cosh^{-1}(\frac{1}{t}) - \cosh^{-1}(\frac{\delta^{1/2}}{t}) + \left(\frac{\delta - t^2}{\delta}\right)^{1/2}\right)^2 & \text{if } t \in [t, \delta^{1/2}] \\ \frac{1}{k^2\pi^2}(\cosh^{-1})^2(\frac{1}{t}) & \text{if } t \in (\delta^{1/2}, 1]. \end{cases}$$

Let then $\mathcal{F}_{\varepsilon,\delta}$ be defined as in (1.3) with $l(t) = t^2$ and $Q = \omega = \omega_\delta$. Theorem 1.1 implies that $\Gamma\text{-}\lim_{\varepsilon\to 0} \mathcal{F}_{\varepsilon,\delta}$ is a functional as in (1.8) with g_δ as surface energy density, and diffuse energy densities independent from δ. Noting that $g_{\delta_2} \geq g_{\delta_1}$ for $0 < \delta_1 < \delta_2 < 1$, [DM93, Proposition 5.4] yields that $\Gamma\text{-}\lim_{\delta\to 0}\left(\Gamma\text{-}\lim_{\varepsilon\to 0}\mathcal{F}_{\varepsilon,\delta}\right)$ is a functional as in (1.8) with surface energy density $g(s) = \frac{1}{k}(1-e^{-ks})$, $s \geq 0$.

In conclusion, we observe that ω_δ coincides with the damage potential found in [FFL21, Section 4.2] on $[\delta^{1/2}, 1]$ for every $\delta \in (0,1)$, using the identity $\cosh^{-1}(1/t) = \tanh^{-1}((1-t^2)^{1/2})$.

3.3.7 Logarithmic softening

In this section we provide an application of Theorem 3.4 in order to approximate cohesive laws with logarithmic behaviour for small jump amplitudes. Let $k \in (0, \infty)$ and consider

$$g'(s) = \begin{cases} -\log(ks) & \text{if } s \in (0, 1/k] \\ 0 & \text{if } s \in (1/k, \infty), \end{cases}$$

then

$$g(s) = \begin{cases} s(1 - \log(ks)) & \text{if } s \in [0, \frac{1}{k}] \\ \frac{1}{k} & \text{if } s \in (\frac{1}{k}, \infty) \end{cases}$$

satisfies (Hp 6′)-(Hp 7′) and the corresponding function R is given by

$$R'(t) = -\frac{1}{2k} e^{-t^{1/2}}$$

for every $t \in [0, \infty)$. In particular, by (3.8), we have that

$$\phi(t) = \frac{1}{2k\pi} \int_t^\infty \frac{e^{-x^{1/2}}}{(x-t)^{1/2}} \mathrm{d}x$$

and from Theorem 3.4, if $\omega(t) = \frac{9}{16k} t$, then we obtain

$$f^2(t) := \frac{l(t)}{Q(1-t)} = \frac{16k\vartheta(t)}{9(1-t)}$$

where

$$\vartheta^{-1}(t) = 1 - \left[k \int_t^\infty \frac{\phi(x)}{x^{1/2}} \mathrm{d}x \right]^{\frac{2}{3}}.$$

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Data availability statement

Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

Declarations

The authors have no relevant financial or non-financial interests to disclose.

The authors have no conflicts of interest to declare that are relevant to the content of this article.

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