

CRACK PROPAGATION BY A REGULARIZATION OF THE PRINCIPLE OF LOCAL SYMMETRY

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ABSTRACT. For planar mixed mode crack propagation in brittle materials many similar criteria have been proposed. In this work the Principle of Local Symmetry together with Griffith Criterion will be the governing equations for the evolution. The Stress Intensity Factors, a crucial ingredient in the theory, will be employed in a 'non-local' (regularized) fashion. We prove existence of a Lipschitz path that satisfies the Principle of Local Symmetry (for the approximated stress intensity factors) and then existence of a BV -parametrization that satisfies Griffith Criterion (again for the approximated stress intensity factors).

1. Introduction. From the '60 several theoretical explanation have been proposed to predict the path along which a crack propagates when the body is under mixed mode loading; among them are widely used the Principle of Local Symmetry (PLS) [8], the Maximum Energy Release Rate [4] and the Maximum Circumferential (hoop) Stress [15]. All these theories are able to predict the crack angle at initiation (the kink) and the path with a good accuracy, at least within the order of the experimental errors. In general the crack paths obtained by the above criteria are very close and they may even coincide in some special cases. In this work we will take into account only the Principle of Local Symmetry.

Let us briefly review some classical results on PLS: [8], [5] and [1]. After the first, usually considered the original source for this criterion, the other two articles contain interesting analytical results. In both [5] and [1] it is provided an asymptotic expansion for the crack path in a small (right neighborhood) of the kink point. The classical technical machinery of analytic function theory is employed in all these three papers. In particular we would like to underlined the simplicity and applicability of the approach followed in [5] and the extreme accuracy of the analysis developed in [1].

In the (more rigorous) mathematical literature it is not easy to find a complete result on quasi-static crack propagation under mixed mode. Among the few we mention [7] and [2]. In the first the evolution is governed by another directional

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criterion, based on the vectorial J -integral, together with a rate-dependent regularization of Griffith's criterion. Despite these differences the paper is worth to mention since it provides, in a more rigorous setting, an asymptotic result like that of [5] and [1]; proofs are based again on a perturbation approach: first it is studied the case of a straight crack (under pure mode I) and then a linearization of its (non-linear) mixed mode perturbation. It is proved an existence result in $C^{1,\alpha}$ for $\alpha < 1/2$ by means of a fixed point theorem. In [2] the spirit is different. The subject of the paper is indeed the energy release rate and more specifically, the kink, i.e. the deflection angle at initiation; however, the paper is interesting also in the perspective of studying the evolution, in particular because it highlights some issues, related to the energy release rate, which seem to appear naturally when following a minimizing movement approach.

In our paper the goal is to provide an existence result for the crack path at least in a small, but finite, right neighborhood of the kink. To pursue this task we will neither linearize nor use a perturbation of the straight case; we will face instead the non-linear problem (with a regularization). Technically we will employ a fixed point theorem and functions of bounded variations (since the evolution is quasi-static). On the other hand, our approach requires a very rigorous analysis of the regularity of solutions, and in particular of the stress intensity factors; at the actual stage, the existence of the stress intensity factors has been proved completely only in the case of straight cracks [10] and it is reasonable to expect (see [12]) that the result holds true for crack path of class $C^{1,1}$. This regularity seems prohibitive for a direct existence result. For this reason we believe that it will be necessary to follow a much longer way: we consider first a regularized problem, for which we will prove existence and regularity of solutions, then we will use these regularized solutions to approximate the real solution. The first stage along this research direction is indeed the subject of this paper.

Let us introduce the equations that governs the evolution, i.e. the Principle of Local Symmetry and Griffith criterion. For the moment, we will not consider the above regularity issues. The evolution will be represented by a curve γ , so that $\gamma(t)$ is the position of the tip. Denote by $\Gamma_t = \gamma([0, t])$ the fracture set and by $K_i(t, \Gamma_t)$, for $i = I, II$, the stress intensity factors. Then, γ should solve

$$\tilde{K}_{II}(t, \Gamma_t) = 0 \quad (\text{Principle of Local Symmetry}),$$

$$\tilde{K}_I(t, \Gamma_t) \leq K_I^c \quad (\tilde{K}_I(t, \Gamma_t) - K_I^c) \dot{\gamma}(t) = 0 \quad (\text{Griffith criterion}),$$

for all $t \in [0, T]$ such that $\Gamma_t \supsetneq \Gamma_0$ (being Γ_0 the initial crack).

For the reasons explained above we will actually employ non-local stress intensity factors, of the form

$$\tilde{K}_i(\Gamma_t) = \int_{\Omega \setminus \Gamma_t} (u_t - \hat{u}_t) \cdot k_i(\vartheta_t) dx,$$

where u_t is the displacement in the equilibrium configuration (at time t), \hat{u}_t its average in a small ball, and $k_i(\theta_t)$ a suitable convolution kernel, supported in the ball $B_r(\gamma(t))$, and depending (in polar coordinates) on the argument ϑ_t of the vector $x - \gamma(t)$. The convolution kernels have been defined in order to provide a good approximation of the stress intensity factors and of their right limits by means of an approximated transfer matrix, proposed in [17] and employed in [5].

Formally we will solve first for the crack path and then for its parametrization. In particular we will find a path of class C^1 and a parametrization of class BV .

It is expected that the regularity of the path can be improved, ideally up to $C^{1,1}$. However, the regularity of the path is intimately connected to the regularity of the displacement field at equilibrium. Results à la Grisvard [10] are probably too strong to hold true for C^1 cracks. A feasible alternative could be higher integrability properties à la Meyers [13]; to our best knowledge results of this type are not known for domains with cracks and will be developed in the nearest future. Note that a higher regularity of the path would then lead to a higher regularity of (the absolutely continuous part of) the parametrization, however it would not be possible to avoid its discontinuities: mathematically they are indeed a typical feature of rate-independent system, physically they represent non-equilibrium regimes of propagation, which are known to occur in brittle fracture. The formal statement of the existence result is contained in § 4.

2. Setting and preliminaries. The reference un-cracked configuration is given by an open, bounded Lipschitz set Ω contained in \mathbb{R}^2 . For technical reasons and having in mind some standard experimental settings (such as the Single Edge Notch Tension) we assume that the initial crack $\Gamma_0 \subset \Omega$ is a closed line segment with one endpoint on the boundary $\partial\Omega$. For convenience we will fix a system of coordinates with the origin in the other endpoint (the crack tip) and \hat{e}_1 aligned with Γ_0 (see Figure 1). We will also assume that $\Omega \setminus \Gamma_0$ is connected and that it can be represented by the union of a finite number of Lipschitz subsets, so that Korn type inequalities hold true. Note that $\Omega \setminus \Gamma_0$ is no longer a Lipschitz set.

In the time interval $[0, T]$ our physical system is described (for each $t \in [0, T]$) by two kinematic variables: the crack set and the displacement, both depending on time. The first step is therefore to define the set of admissible cracks and the set of admissible displacements.

Considering the system of coordinates introduced above, the crack path will be represented by the graph of a Lipschitz function y belonging to the convex set

$$\mathcal{Y} = \{y \in C^{0,1}([0, S]) : y(0) = 0, \|y'\|_\infty \leq C_y\}; \quad (1)$$

the parameters $S, C_y > 0$ will be chosen later, respectively small and big enough. In this way, denoting $\Gamma_s = \Gamma_0 \cup \{(x, y(x)) : x \in [0, s]\}$ the domain $\Omega \setminus \Gamma_s$ is still connected and represented by a finite union of Lipschitz sets. For notational convenience, we introduce also the curve γ given by $\gamma(s) = (s, y(s))$.

For the displacement field u we set a Dirichlet boundary condition on a subset $\partial_D\Omega$ of $\partial\Omega \setminus \Gamma_0$ with $\mathcal{H}^1(\partial_D\Omega) > 0$. For technical reasons the boundary condition will be of proportional type, i.e. of the form $u = cg$, for $g \in H^{1/2}(\partial_D\Omega, \mathbb{R}^2)$ and $c \in W^{1,1}(0, T)$ with $c(0) = 0$. A proportional boundary condition, besides being realistic in many experiments, is theoretically very convenient combined with linearized elastic energy, indeed it allows to consider a single set of displacements and a time depending energy, instead of a time depending set of displacements. Therefore, given a path Γ_s the space of admissible displacements will be

$$\mathcal{U}(\Omega \setminus \Gamma_s) = \{u \in H^1(\Omega \setminus \Gamma_s, \mathbb{R}^2) : u = g \text{ on } \partial_D\Omega\}. \quad (2)$$

For the boundary condition $u = cg$ it will be sufficient to consider displacements fields of the form cu for $u \in \mathcal{U}(\Omega \setminus \Gamma_s)$. On the rest of the boundary $\partial(\Omega \setminus \Gamma_s) \setminus \partial_D\Omega$ we set an homogeneous Neumann condition. Note that $\partial(\Omega \setminus \Gamma_s) \setminus \partial_D\Omega$ contains

both $\partial\Omega \setminus \partial_D\Omega$ and the crack faces Γ_s^\pm . We employ linearized elasticity, so the energy is

$$E_s(u) = \int_{\Omega \setminus \Gamma_s} W(Du) dx,$$

with density $W(Du) = Du : \mathbf{C}[Du] = \mu|\varepsilon(u)|^2 + (\lambda/2)|\text{tr } \varepsilon(u)|^2$ ($\mu, \lambda > 0$ are the Lamè coefficients). In this setting, using for instance an analogous result contained in [3], it is not hard to see that a uniform Korn inequality holds true; more precisely, there exists a constant C_K such that for every $y \in \mathcal{Y}$ and $s \in [0, S]$

$$\int_{\Omega \setminus \Gamma_s} |Du - Dg|^2 dx \leq C_K \int_{\Omega \setminus \Gamma_s} W(Du - Dg) dx, \quad (3)$$

for every $u \in \mathcal{U}(\Omega \setminus \Gamma_s)$.

Since we are interested in a quasi-static evolution, given Γ_s it is sufficient to consider the unique equilibrium configuration of the displacement, that is $\{u_s\} = \text{argmin}\{E_s(u) : u \in \mathcal{U}(\Omega \setminus \Gamma_s)\}$. Note that u_s depends on the path Γ_s and not only on s , as the notation would suggest. For the boundary condition $u = cg$ the equilibrium field is simply given by cu_s and its energy will be $c^2 E_s(u_s)$.

2.1. Continuity of displacements . Our proof of the existence of a crack path is based on Schauder Fixed Point Theorem. To this end it will be necessary to have at our disposal the continuity of displacements with respect to incremental and configurational variations of the crack path. The results are stated respectively in the next two Lemmas. Since we are not interested in quantitative estimates the proofs will be based only on continuity arguments; for both of them the framework will be that of Γ -convergence [6].

Lemma 2.1. *Given $y \in \mathcal{Y}$ the map $s \mapsto u_s$ is continuous from $[0, S]$ to $L^2(\Omega, \mathbb{R}^2)$.*

Proof. First of all we prove that the family $\{u_s\}$ of minimizers is (sequentially) pre-compact in $L^2(\Omega, \mathbb{R}^2)$. Remember that, given $y \in \mathcal{Y}$, the sets $\mathcal{U}(\Omega \setminus \Gamma_s)$ are increasing with respect to s . Thus, we can consider each u_s to be an element of the biggest set of admissible displacements, i.e. $\mathcal{U}(\Omega \setminus \Gamma_S)$. In the same spirit, we consider the functionals E_s to be defined in $\mathcal{U}(\Omega \setminus \Gamma_S)$, setting

$$E_s(u) = \begin{cases} \int_{\Omega \setminus \Gamma_s} W(Du) dx & \text{if } u \in \mathcal{U}(\Omega \setminus \Gamma_s) \\ +\infty & \text{otherwise.} \end{cases}$$

Note that $E_S(u_s) = E_s(u_s)$ and that $E_s(u_s) \leq E_0(u_0)$, hence the energies $E_S(u_s)$ are uniformly bounded. Then, thanks to Korn and Poincaré inequalities, the family $\{u_s\}$ is bounded in $H^1(\Omega \setminus \Gamma_S, \mathbb{R}^2)$. As a consequence, if $s_n \rightarrow s$ there exist a subsequence (not relabeled) and a limit \bar{u} such that $u_{s_n} \rightarrow \bar{u}$ in $L^2(\Omega \setminus \Gamma_S, \mathbb{R}^2)$. Clearly the convergence in $L^2(\Omega \setminus \Gamma_S, \mathbb{R}^2)$ implies the convergence in $L^2(\Omega, \mathbb{R}^2)$.

It remains to show that $\bar{u} = u_s$. To this end, we show that the sequence E_{s_n} converges to E_s (for $s_n \rightarrow s$) in the sense of Γ -convergence, with respect to the strong topology of $L^2(\Omega, \mathbb{R}^2)$. Then, by a standard result on Γ -convergence [6] it will follow that $\bar{u} = u_s$ (the unique minimizer of E_s).

The Γ -liminf inequality is a consequence of the lower semi-continuity of E_S ; indeed, if $v \in \mathcal{U}(\Omega \setminus \Gamma_s)$, $v_{s_n} \in \mathcal{U}(\Omega \setminus \Gamma_{s_n})$ and if $v_{s_n} \rightarrow v$ in $L^2(\Omega, \mathbb{R}^2)$ we can write

$$E_s(v) = E_S(v) \leq \liminf_{s_n \rightarrow s} E_S(v_{s_n}) = \liminf_{s_n \rightarrow s} E_{s_n}(v_{s_n}).$$

Next, we prove the Γ -limsup inequality. We will use a convenient density argument. Let \tilde{g} be a lifting of the boundary datum g with $\tilde{g} = 0$ in a neighborhood U of $\Gamma_S \setminus \Gamma_0$. Given $u \in \mathcal{U}(\Omega \setminus \Gamma_S)$ we introduce the auxiliary field $v = u - \tilde{g}$. Consider the sequence v_k obtained by truncation of v and defined for a.e. $x \in \Omega$ by

$$v_k(x) = \begin{cases} v_k(x) & \text{if } |v_k|_\infty(x) \leq k \\ kv_k(x)/|v_k|_\infty(x) & \text{otherwise.} \end{cases}$$

Then $v_k \rightarrow v$ in $H^1(\Omega \setminus \Gamma_S, \mathbb{R}^2)$ and $v_k = v = 0$ on $\partial_D \Omega$. Therefore, the sequence $u_k = v_k + \tilde{g}$ is contained in $\mathcal{U}(\Omega \setminus \Gamma_S)$ and converges strongly to u in $H^1(\Omega \setminus \Gamma_S, \mathbb{R}^2)$. As a consequence $E_s(u_k) \rightarrow E_s(u)$. Note that in the neighborhood U of $\Gamma_S \setminus \Gamma_0$ we have

$$\|u_k\|_\infty = \|v_k + \tilde{g}\|_\infty = \|v_k\| \leq k.$$

By a standard argument on Γ -convergence, it is then sufficient to find a recovery sequence for the fields u_k , i.e. a sequence $u_{s_n} \in \mathcal{U}(\Omega \setminus \Gamma_{s_n})$ such that $u_{s_n} \rightarrow u_k$ in $L^2(\Omega, \mathbb{R}^2)$ and $E_{s_n}(u_{s_n}) \rightarrow E_s(u_k)$. We will actually find a sequence such that $u_{s_n} \rightarrow u_k$ strongly in $H^1(\Omega \setminus \Gamma_S, \mathbb{R}^2)$. Unfortunately, the constant recovery sequence ($u_{s_n} \equiv u_k$) is not a good choice because in general u_k could be discontinuous on $\Gamma_S \setminus \Gamma_{s_n}$ and thus it may not belong to $\mathcal{U}(\Omega \setminus \Gamma_{s_n})$. This problem can be by-passed smoothing u_k on $\Gamma_S \setminus \Gamma_{s_n}$; to this end we will employ a sequence of capacity potentials for $\Gamma_S \triangle \Gamma_{s_n}$. Let $\psi_{s_n} \in C_0^\infty(U)$ with $\psi_{s_n} = 1$ in $\Gamma_S \triangle \Gamma_{s_n}$ and $\psi_{s_n} \rightarrow 0$ strongly in $H^1(\Omega)$. Consider the sequence $u_{s_n} = u_k(1 - \psi_{s_n})$. Then $u_{s_n} \in \mathcal{U}(\Omega \setminus \Gamma_{s_n})$, because $u_{s_n} = 0$ on $\Gamma_S \triangle \Gamma_{s_n}$. Moreover

$$\int_\Omega |u_{s_n} - u_k|^2 dx = \int_U |u_{s_n} - u_k|^2 dx = \int_U |u_k \psi_{s_n}|^2 dx \leq \|u_k\|_{L^\infty(U, \mathbb{R}^2)}^2 \|\psi_{s_n}\|_{L^2(U)}^2$$

and hence $u_{s_n} \rightarrow u_k$ in $L^2(\Omega, \mathbb{R}^2)$. Finally,

$$\begin{aligned} \int_\Omega |Du_{s_n} - Du_k|^2 dx &= \int_U |Du_{s_n} - Du_k|^2 dx = \|Du_k \psi_{s_n} - \nabla \psi_{s_n} \otimes u_k\|_{L^2(U, \mathbb{R}^{2 \times 2})}^2 \\ &\leq \|Du_k \psi_{s_n}\|_{L^2(U, \mathbb{R}^{2 \times 2})}^2 + \|u_k\|_{L^\infty(U, \mathbb{R}^2)}^2 \|\nabla \psi_{s_n}\|_{L^2(U, \mathbb{R}^2)}^2, \end{aligned}$$

the limit of first term vanishes by dominated convergence, the second by the definition of ψ_{s_n} . Hence, $u_{s_n} \rightarrow u_k$ strongly in $H^1(\Omega \setminus \Gamma_S, \mathbb{R}^2)$. \square

Lemma 2.2. *Let $y_n \in \mathcal{Y}$ such that $y_n \rightarrow y$ uniformly in $[0, S]$. (Denote by Γ_s^n the corresponding sequence of crack paths). Let $u_s^n \in \mathcal{U}(\Omega \setminus \Gamma_s^n)$ be the minimizer of the energy*

$$E_s^n(u) = \int_{\Omega \setminus \Gamma_s^n} W(Du) dx.$$

Then for every $s \in [0, S]$ the minimizers u_s^n converge to u_s in $L^2(\Omega, \mathbb{R}^2)$.

Proof. Let $\delta_n = \sup_{s \in [0, S]} |y_n(s) - y(s)|$. Denote by U_n the $(3\delta_n)$ -neighborhood of Γ_S , i.e.

$$U_n = \{(x, y) : d((x, y), \Gamma_S) < 3\delta_n\}.$$

Let $\eta_n \in C_0^\infty(\Omega)$ with $\eta_n = 1$ on Γ_S , $\eta_n = 0$ in $\Omega \setminus U_n$ and $\|\nabla \eta_n\|_\infty \leq 1/2\delta_n$. Define a change of variable with the map

$$\Psi_n(x, y) = (x, y) + \eta_n(x, y)(0, y_n(x) - y(x)).$$

Then $\Psi_n(\Gamma_s) = \Gamma_s^n$ for every $s \in [0, S]$. Let us check that Ψ_n is a bi-Lipschitz map of Ω in itself. We have

$$D\Psi_n(x, y) = I + \hat{e}_2 \otimes \hat{e}_1 (y_n'(x) - y'(x)) \eta_n(x, y) + \hat{e}_2 \otimes (y_n(x) - y(x)) \nabla \eta_n(x, y).$$

For convenience, let us write $D\Psi_n(x, y) = I + M_n(x, y)$. Then

$$\det D\Psi_n = 1 + \operatorname{tr} M_n + \det M_n = 1 + (y_n(x) - y(x)) \nabla \eta_n(x, y) \cdot \hat{e}_2 \geq 1 - \delta_n / 2\delta_n \geq 1/2.$$

Thus, Ψ_n is a bi-Lipschitz map of Ω in itself. Note that $\operatorname{supp}(M_n) \subset \operatorname{supp}(\eta_n) \subset U_n$ and thus $D\Psi_n \rightarrow I$ pointwise. Moreover $\|M_n\|_\infty \leq C$.

At this point we can perform a change of variable in the energies, writing

$$E_s^n(u) = \int_{\Omega \setminus \Gamma_s^n} W(Du) dx = \int_{\Omega \setminus \Gamma_s} W_n(D(u \circ \Psi_n)) dx$$

where

$$W_n(F) = D\Psi_n^{-1} F : \mathbf{C}[FD\Psi_n^{-1}] \det D\Psi_n.$$

Introduce the energy

$$\bar{E}_s^n(z) = \int_{\Omega \setminus \Gamma_s} W_n(Dz) dx$$

defined in $\mathcal{U}(\Omega \setminus \Gamma_s)$. With the above change of variable the minimizer u_s^n of E_s^n is mapped to $\bar{u}_s^n = u_s^n \circ \Psi_n$, minimizer of \bar{E}_s^n . We check that the energies \bar{E}_s^n Γ -converge to E_s and that the sequence \bar{u}_s^n is pre-compact in the weak topology of $H^1(\Omega \setminus \Gamma_s, \mathbb{R}^2)$.

Note that $D\Psi_n^{-1} \rightarrow I$ pointwise with $\|D\Psi_n^{-1}\|_\infty \leq C$ and that $\det D\Psi_n \rightarrow 1$ pointwise with $\|\det D\Psi_n\|_\infty \leq C$. If $Dz_n \rightarrow Dz$ in $L^2(\Omega, \mathbb{R}^{2 \times 2})$ then, by dominated convergence, $Dz_n D\Psi_n^{-1} \det^{1/2} D\Psi_n \rightharpoonup Dz$ in $L^2(\Omega, \mathbb{R}^{2 \times 2})$. As a consequence, by lower semi-continuity we have the Γ -liminf inequality

$$\begin{aligned} \int_{\Omega \setminus \Gamma_s} Dz : \mathbf{C}[Dz] dx &\leq \liminf_{n \rightarrow +\infty} \int_{\Omega \setminus \Gamma_s} Dz_n D\Psi_n^{-1} : \mathbf{C}[Dz_n D\Psi_n^{-1}] \det D\Psi_n dx \\ &= \liminf_{n \rightarrow +\infty} \int_{\Omega \setminus \Gamma_s} W_n(Dz_n) dx. \end{aligned}$$

For the Γ -limsup inequality it is sufficient to employ the constant recovery sequence.

Now we prove that the sequence \bar{u}_s^n is bounded in $H^1(\Omega \setminus \Gamma_s, \mathbb{R}^2)$. Note that $\bar{E}_s^n(\bar{u}_s^n) = E_s^n(u_s^n) \leq E_0(u_0)$, therefore (thanks to Korn and Poincaré inequalities) \bar{u}_s^n is bounded in $H^1(\Omega \setminus \Gamma_s, \mathbb{R}^2)$. Hence the sequence \bar{u}_s^n is weakly pre-compact.

From the standard theory of Γ -convergence we get that the minimizers \bar{u}_s^n converge to the minimizer u_s strongly in $L^2(\Omega, \mathbb{R}^2)$. To conclude the proof it is then sufficient to apply the inverse change of variable. \square

3. Stress intensity factors. If Γ_s is locally (in a neighborhood of the crack tip) a line segment then from [10] there exists a unique couple of real values K_I, K_{II} (the stress intensity factors) such that, in a small neighborhood B of the tip, the displacement u_s can be represented in the form

$$u_s = K_I \hat{u}_I + K_{II} \hat{u}_{II} + \bar{u}_s, \quad (4)$$

where $\bar{u}_s \in H^2(B \setminus \Gamma_s)$ and

$$\hat{u}_I = (2\mu)^{-1} (\rho/2\pi)^{1/2} (\cos(\theta/2)(k - \cos \theta), \sin(\theta/2)(k - \cos \theta)) \quad (5)$$

$$\hat{u}_{II} = (2\mu)^{-1} (\rho/2\pi)^{1/2} (\sin(\theta/2)(k + 2 + \cos \theta), -\cos(\theta/2)(k - 2 + \cos \theta)), \quad (6)$$

for $k = (3 - 4\nu)$, and $\nu = \lambda/2(\lambda + \mu)$ the Poisson ratio. Here ρ and θ denote the polar coordinates in the usual local system of coordinates with center at the tip $\gamma(s)$ and polar axis along $\gamma'(s)$ (see Figure 2).

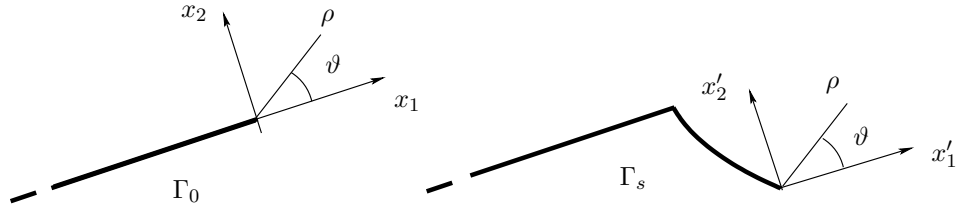


FIGURE 1. Absolute (left) and local (right) systems of Cartesian and polar coordinates employed in this work. The local system translates with the crack.

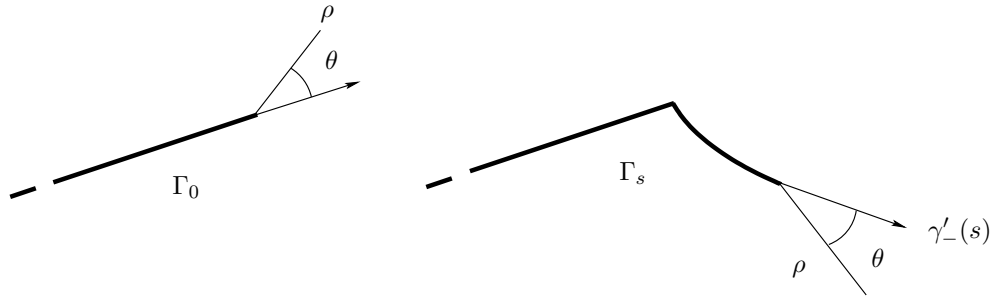


FIGURE 2. The usual local systems of polar coordinates translating and rotating with the crack.

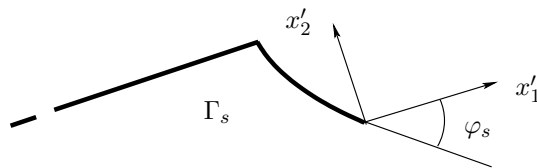


FIGURE 3. The angle φ_s such that $\tilde{K}_i(\Gamma_s) = \tilde{K}_i(\Gamma_s, \varphi_s)$.

Remark 1. Employing the system of coordinates in Figure 1 instead of that in Figure 2 is not substantial, however the former simplifies several proofs since it depends only on the position $\gamma(s)$ and not on the tangent $\gamma'_-(s)$.

3.1. Approximated stress intensity factors. In general, for a path of class $C^{0,1}$ it is not known whether a representation like (4) holds. On the base of [12] it is reasonable to expect that it holds for crack paths of class $C^{1,1}$ but this class is too restrictive to provide an existence result, at least at the present stage. Thus, a notion of stress intensity factors for a larger class of crack paths is needed. Our philosophy,

to provide such a notion, is to use a suitable volume integral representation, among which there are clearly several possible choices. The one given hereafter embeds an approximation of the transfer matrix [17] and provides a good approximation in the case of straight, curved and kinked cracks. Other integral representations, such as the interaction integrals [9] and the vectorial J -integral [7] could be feasible alternatives. Finally, it is interesting to remark that in the anti-plane setting an integral representation like ours provides the exact value of the stress intensity factor for a straight crack [11].

Let us consider the system of local polar coordinates of Figure 1. For $x \in \Omega \setminus \Gamma_s$ let ϑ denote the argument of $x - \gamma(s)$. Given $r > 0$ let $c_r = (2\pi)^{-1}r^{-5/2}$ and for $i = 1, 2$ let

$$\begin{aligned} k_1(\vartheta) &= c_r (a_1 \cos(\vartheta/2) + a_3 \cos(3\vartheta/2), a_2 \sin(\vartheta/2) + a_4 \sin(3\vartheta/2)) \\ k_2(\vartheta) &= c_r (b_1 \sin(\vartheta/2) + b_3 \sin(3\vartheta/2), b_2 \cos(\vartheta/2) + b_4 \cos(3\vartheta/2)). \end{aligned} \quad (7)$$

The choice of the coefficients a_i and b_i will be done in the sequel (see Appendix A.1). For $\varphi \in (-\pi, \pi)$ let us introduce the functions

$$\tilde{K}_i(\Gamma_s, \varphi) = \int_{B_r \setminus \Gamma_s} (u_s - \hat{u}_s) \cdot k_i(\vartheta - \varphi) dx, \quad (8)$$

where B_r denotes the ball $B_r(\gamma(s))$ and \hat{u}_s is the average of u in the ball $B_{r/2}(\gamma(s))$. We remark that in the notation \tilde{K}_i there is no explicit dependence on the radius r . Note also that $\tilde{K}_i(\Gamma_s, \varphi)$ does not depend on the tangent vector $\gamma'_-(s)$.

Let us see how the functions $\tilde{K}_i(\Gamma_s, \varphi)$ provide an approximation of the stress intensity factors and of their right limits. Since y is Lipschitz continuous, for a.e. $s \in [0, S]$ there exists the left tangent vector $\gamma'_-(s) = (1, y'_-(s))$. This vector defines the usual local system of polar coordinates (see Figure 2) with center at the tip $\gamma(s)$ and polar axis along $\gamma'_-(s)$. Let $\varphi_s = \arctan(y'_-(s))$. Then, for a.e. $s \in [0, S]$ the approximated stress intensity factors will be given by (see Figure 3)

$$\tilde{K}_i(\Gamma_s) = \tilde{K}_i(\Gamma_s, \varphi_s) = \int_{B_r \setminus \Gamma_s} (u_s - \hat{u}_s) \cdot k_i(\vartheta - \varphi_s) dx.$$

For Γ_0 we have $\varphi_0 = 0$, thus will also write

$$\tilde{K}_i(\Gamma_0) = \int_{B_r \setminus \Gamma_0} (u_s - \hat{u}_0) \cdot k_i(\vartheta) dx.$$

Now, let us see how (8) provides approximated right stress intensity factors. For $s < S$, given $\varphi \in (-\pi, \pi)$ consider a crack 'extension' in direction φ , i.e. a function $\tilde{y} \in \mathcal{Y} \cap C^1([s, S])$ with $\tilde{y}(\tau) = y(\tau)$ for $\tau \leq s$ and $\tilde{y}'_+(s) = \tan \varphi$.

We denote by $\tilde{\Gamma}_s$ the associated path. Then we define¹

$$\tilde{K}_i^*(\Gamma_s, \varphi) = \lim_{\tau \rightarrow s^+} \tilde{K}_i(\tilde{\Gamma}_\tau).$$

The next Lemma provides a representation of the right stress intensity factors $\tilde{K}_i^*(\Gamma_s, \varphi)$ in terms of $\tilde{K}_i(\Gamma_s, \varphi)$.

Lemma 3.1. *With the above notation*

$$\tilde{K}_i^*(\Gamma_s, \varphi) = \lim_{\tau \rightarrow s^+} \tilde{K}_i(\tilde{\Gamma}_\tau) = \tilde{K}_i(\Gamma_s, \varphi). \quad (9)$$

¹In the literature on Fracture Mechanics the right limits of the stress intensity factors are usually denoted by K^* .

Proof. Let $\varphi_\tau = \arctan(y'(\tau))$ for $\tau > s$ and let $\varphi_s = \arctan(y'_+(s))$. Denote by ϑ_τ the argument of $x - \gamma(\tau)$ in $\Omega \setminus \Gamma_\tau$. We have to show that

$$\int_{B_r \setminus \Gamma_\tau} (u_\tau - \hat{u}_\tau) \cdot k_i(\vartheta_\tau - \varphi_\tau) dx \rightarrow \int_{B_r \setminus \Gamma_s} (u_s - \hat{u}_s) \cdot k_i(\vartheta_s - \varphi_s) dx. \quad (10)$$

By Lemma 2.1 we already know that $u_\tau \rightarrow u_s$ strongly in $L^2(\Omega, \mathbb{R}^2)$. Denote by χ_τ the characteristic function of ball $B_r(\gamma(\tau))$. Then $\chi_\tau \rightarrow \chi_s$ strongly $L^2(\Omega)$. It follows that $\hat{u}_\tau \rightarrow \hat{u}_s$. As $\tilde{y} \in C^1([s, S])$ we have $\varphi_\tau \rightarrow \varphi_s$ and $\vartheta_\tau \rightarrow \vartheta_s$ a.e. in Ω . As the kernel k_i is continuous with respect to its argument, we get $k_i(\vartheta_\tau - \varphi_\tau) \rightarrow k_i(\vartheta_s - \varphi_s)$ a.e. in Ω . Since k_i is uniformly bounded in Ω by dominated convergence we get that $k_i(\vartheta_\tau - \varphi_\tau) \rightarrow k_i(\vartheta_s - \varphi_s)$ in $L^2(\Omega, \mathbb{R}^2)$. As a consequence we get (10). \square

Lemma 3.2. *Given $y \in \mathcal{Y}$ the functions \tilde{K}_i and $\partial_\varphi \tilde{K}_i$ are continuous with respect to the variables s and φ .*

Proof. We will denote by ϑ_s the argument of $x - \gamma(s)$ in $\Omega \setminus \Gamma_s$. Remember that

$$\tilde{K}_i(\Gamma_s, \varphi) = \int_{\Omega \setminus \Gamma_s} (u_s - \hat{u}_s) \cdot k_i(\vartheta_s - \varphi) dx.$$

By Lemma 2.1 we know that u_s is continuous in $L^2(\Omega, \mathbb{R}^2)$ from which it follows that \hat{u}_s is continuous. Moreover, the kernels k_i are Lipschitz functions, thus

$$\begin{aligned} \int_{\Omega \setminus \Gamma_s} |k_i(\vartheta_s - \varphi_a) - k_i(\vartheta_t - \varphi_b)|^2 dx &\leq C \int_{\Omega \setminus \Gamma_s} |\vartheta_s - \vartheta_t|^2 + |\varphi_a - \varphi_b|^2 dx \\ &\leq C \|\vartheta_s - \vartheta_t\|_{L^2}^2 + C |\Omega| |\varphi_a - \varphi_b|^2. \end{aligned}$$

Note that for $s \rightarrow t$ we have $\vartheta_s \rightarrow \vartheta_t$ a.e. in Ω , thus by dominated convergence $k_i(\vartheta_s - \varphi)$ is continuous in $L^2(\Omega, \mathbb{R}^2)$ with respect to both s and φ . It follows that $\tilde{K}_i(\Gamma_s, \varphi)$ is continuous.

Moreover, for $s \in [0, S]$

$$\partial_\varphi \tilde{K}_i(\Gamma_s, \varphi) = - \int_{\Omega \setminus \Gamma_s} (u_s - \hat{u}_s) \cdot k'_i(\vartheta_s - \varphi) dx,$$

where k'_i is the derivative of k_i with respect to its argument, say ϑ in (7). Arguing as above we get that also $\partial_\varphi \tilde{K}_i$ is continuous with respect to s and φ . \square

Using the above argument it is easy to prove the following Corollary.

Corollary 1. *Given $y \in \mathcal{Y}$ the functions $\tilde{K}_i(\Gamma_s, \cdot)$ converges uniformly to $\tilde{K}_i(\Gamma_t, \cdot)$ for $s \rightarrow t$. A similar property holds true for $\partial_\varphi \tilde{K}_i(\Gamma_s, \cdot)$.*

In the sequel we will need also the continuity of $\tilde{K}_i(\Gamma_s, \varphi)$ with respect to variations of the crack path. This is proved in the next Lemma.

Lemma 3.3. *Let $y_n \in \mathcal{Y}$ such that $y_n \rightarrow y$ uniformly in $[0, S]$. Then, for every $s \in [0, S]$ the functions $\tilde{K}_i(\Gamma_s^n, \cdot)$ converge uniformly to $\tilde{K}_i(\Gamma_s, \cdot)$.*

Proof. Denote by ϑ_s^n the argument of $x - \gamma_n(s)$ in $\Omega \setminus \Gamma_s^n$. Given $\varphi \in (-\pi, \pi)$, we can write

$$\begin{aligned} \tilde{K}_i(\Gamma_s^n, \varphi) - \tilde{K}_i(\Gamma_s, \varphi) &= \int_{\Omega} (u_s^n - \hat{u}_s^n) \cdot k_i(\vartheta_s^n - \varphi) - (u_s - \hat{u}_s) \cdot k_i(\vartheta_s - \varphi) dx \\ &= \int_{\Omega} (u_s^n - \hat{u}_s^n) \cdot k_i(\vartheta_s^n - \varphi) - (u_s - \hat{u}_s) \cdot k_i(\vartheta_s^n - \varphi) dx + \\ &\quad + \int_{\Omega} (u_s - \hat{u}_s) \cdot k_i(\vartheta_s^n - \varphi) - (u_s - \hat{u}_s) \cdot k_i(\vartheta_s - \varphi) dx. \end{aligned}$$

Re-arranging the terms in the first integral we get

$$\begin{aligned} |\tilde{K}_i(\Gamma_s^n, \varphi) - \tilde{K}_i(\Gamma_s, \varphi)| &\leq \|u_s^n - u_s\|_{L^2} \|k_i(\vartheta_s^n - \varphi)\|_{L^2} + |\hat{u}_s^n - \hat{u}_s| \|k_i(\vartheta_s^n - \varphi)\|_{L^1} + \\ &\quad + \|u_s - \hat{u}_s\|_{L^2} \|k_i(\vartheta_s^n - \varphi) - k_i(\vartheta_s - \varphi)\|_{L^2}. \end{aligned}$$

By Lemma 2.2 we know that $u_s^n \rightarrow u_s$ in $L^2(\Omega, \mathbb{R}^2)$ and hence $\hat{u}_s^n \rightarrow \hat{u}_s$. The kernel k_i is bounded and supported in a ball, hence $k_i(\vartheta_s^n - \varphi)$ is bounded (uniformly with respect to φ) both in $L^1(\Omega, \mathbb{R}^2)$ and $L^2(\Omega, \mathbb{R}^2)$. Moreover, the kernel k_i is Lipschitz continuous with respect to its argument, thus $|k_i(\vartheta_s^n - \varphi) - k_i(\vartheta_s - \varphi)| \leq C|\vartheta_s^n - \vartheta_s|$ and hence

$$\|k_i(\vartheta_s^n - \varphi) - k_i(\vartheta_s - \varphi)\|_{L^2} \leq C\|\vartheta_s^n - \vartheta_s\|_{L^2}.$$

As $\vartheta_s^n \rightarrow \vartheta_s$ a.e. in Ω by dominated convergence we get that $\vartheta_s^n \rightarrow \vartheta_s$ in $L^2(\Omega)$ (uniformly with respect to φ). \square

3.2. Error estimate . This section deals with the relationship between the real stress intensity factors K_i and their approximations \tilde{K}_i . To this end, it is clearly necessary to assume that the stress intensity factors exist, i.e. that a representation like $u = K_I \hat{u}_I + K_{II} \hat{u}_{II} + \bar{u}$ holds for $\bar{u} \in H^2(B_R \setminus \Gamma)$. In our specific setting only u_0 satisfies this (strict) requirement. Thus, to be rigorous, the estimate of this section will apply only to u_0 .

First of all, write $u_0 = K_I(\Gamma_0) \hat{u}_I + K_{II}(\Gamma_0) \hat{u}_{II} + \bar{u}_0$. Next, we introduce the matrix $\mathbf{M}(\varphi)$ with elements $M_{ij}(\varphi)$ (for $i = I, II$ and $j = 1, 2$) given by

$$M_{ij}(\varphi) = \int_{B_r \setminus \Gamma_0} \hat{u}_j \cdot k_i(\vartheta - \varphi) dx.$$

Then we define

$$\bar{K}_i(\Gamma_0, \varphi) = \int_{B_r \setminus \Gamma_0} \bar{u}_0 \cdot k_i(\vartheta - \varphi) dx, \quad \dot{K}_i(\Gamma_0, \varphi) = \int_{B_r \setminus \Gamma_0} \hat{u}_0 \cdot k_i(\vartheta - \varphi) dx.$$

Using the (column) vectors $K, \tilde{K}, \bar{K}, \dot{K}$ with components K_I, K_{II} etc. we can write

$$\tilde{K}(\Gamma_0, \varphi) = \mathbf{M}(\varphi)K(\Gamma_0) + (\bar{K}(\Gamma_0, \varphi) - \dot{K}(\Gamma_0, \varphi)). \quad (11)$$

Since $\bar{K}(\Gamma_0, \varphi) - \dot{K}(\Gamma_0, \varphi) = o(1)$ for $r \searrow 0$ (see next subsection) the matrix \mathbf{M} plays the role of the transfer matrix \mathbf{T} . In particular we can choose the coefficients a_i and b_i (see Appendix A) in such a way that $\mathbf{M}(\varphi)$ coincides with an approximation $\tilde{\mathbf{T}}(\varphi)$ proposed in [17] and given by

$$\tilde{\mathbf{T}}(\vartheta) = \frac{1}{2} \cos(\vartheta/2) \begin{pmatrix} \cos \vartheta + 1 & -3 \sin \vartheta \\ \sin \vartheta & 3 \cos \vartheta - 1 \end{pmatrix}. \quad (12)$$

Note that for $\varphi = 0$ we have $\tilde{\mathbf{T}}(0) = \mathbf{M}(0) = \mathbf{I}$. Thus, we can write

$$\tilde{K}(\Gamma_0) = K(\Gamma_0) + (\bar{K}(\Gamma_0) - \dot{K}(\Gamma_0)). \quad (13)$$

Our next goal is to provide an estimate of the error term $\bar{K}(\Gamma_0) - \mathring{K}(\Gamma_0)$ as the support of the convolution kernels k_i vanishes.

Lemma 3.4. *For every $\varepsilon > 0$ there exists C_ε such that (for r sufficiently small)*

$$|K(\Gamma_0) - \tilde{K}(\Gamma_0)| \leq C_\varepsilon r^{1/2-\varepsilon}. \quad (14)$$

Proof. Let $B_R^\pm = \{x \in B_R : \text{sign}(x \cdot \hat{e}_2) = \pm 1\}$. By Sobolev embedding $\bar{u} \in C^{0,\alpha}(B_R^\pm)$. It follows that $|\bar{u}(x) - \bar{u}(0)| \leq C_\alpha |x|^\alpha$ for $0 < \alpha < 1$. Moreover, by the square root singularity of the functions \hat{u}_i we get that $u \in C^{0,1/2}(B_R^\pm)$. Note also that $\bar{u}(0) = u(0)$.

Next, write the error term in (13) as

$$\begin{aligned} \bar{K}_i(\Gamma_0) - \mathring{K}_i(\Gamma_0) &= \int_{B_r \setminus \Gamma_0} (\bar{u}_0 - \hat{u}_0) \cdot k_i(\vartheta) dx \\ &= \int_{B_r \setminus \Gamma_0} (\bar{u}_0 - \bar{u}(0)) \cdot k_i(\vartheta) dx + \int_{B_r \setminus \Gamma_0} (\bar{u}(0) - \hat{u}_0) \cdot k_i(\vartheta) dx \end{aligned} \quad (15)$$

Using the Hölder regularity of \bar{u} and the fact that $|k_i| \leq Cr^{-5/2}$ we can write that for every $0 < \alpha < 1$ there exists C_α such that

$$\left| \int_{B_r \setminus \Gamma_0} (\bar{u}_0 - \bar{u}(0)) \cdot k_i(\vartheta) dx \right| \leq C_\alpha r^{\alpha-1/2},$$

therefore for every $\varepsilon > 0$ there exists C_ε such that

$$\left| \int_{B_r \setminus \Gamma_0} (\bar{u}_0 - \bar{u}(0)) \cdot k_i(\vartheta) dx \right| \leq C_\varepsilon r^{1/2-\varepsilon}.$$

For the second term in (15) it is sufficient to estimate $\bar{u}(0) - \hat{u}_0 = u(0) - \hat{u}_0$. So,

$$\begin{aligned} |\hat{u}_0 - u(0)| &\leq \left| \int_{B_{r,2}} u(x) - u(0) dx \right| \leq \int_{B_{r,2}} |u(x) - u(0)| dx \\ &\leq C \int_{B_{r,2}} |x|^{1/2} dx \leq C' r. \end{aligned}$$

Since $|k_i| \leq Cr^{-5/2}$, we get

$$\left| \int_{B_r \setminus \Gamma_0} (\hat{u}_0 - \bar{u}(0)) \cdot k_i(\vartheta) dx \right| \leq C'' r^{1/2}.$$

In conclusion we can write that for every $\varepsilon > 0$ there exists C_ε such that (14) holds. \square

Arguing as in the proof of the previous Lemma we can prove also the following Corollary which provides an estimate for (11).

Corollary 2. *For every $\varepsilon > 0$ there exists C_ε such that (for r sufficiently small)*

$$|\bar{K}(\Gamma_0, \varphi) - \mathring{K}(\Gamma_0, \varphi)| \leq C_\varepsilon r^{1/2-\varepsilon}, \quad (16)$$

$$|\partial_\varphi \bar{K}(\Gamma_0, \varphi) - \partial_\varphi \mathring{K}(\Gamma_0, \varphi)| \leq C_\varepsilon r^{1/2-\varepsilon}, \quad (17)$$

for every $\varphi \in (-\pi, \pi)$.

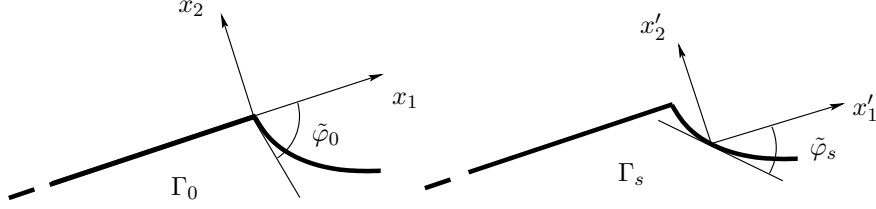


FIGURE 4. Absolute (left) and local (right) systems of Cartesian and polar coordinates employed in this work. The local system translates with the crack.

4. Existence . As discussed before, in our model the evolution of the crack is governed by the Principle of Local Symmetry together with Griffith's criterion. Taking advantage of the fact that the boundary conditions are proportional and that the constitutive law is linear, we can solve first for the path and then for its parametrization. First, we study the kink angle obtained by applying the Principle of Local Symmetry to the approximated stress intensity factors.

4.1. Prediction of the kink angle.

Lemma 4.1. *Assume that $K_i(\Gamma_0) > 0$ for $i = I, II$. Then, if r is sufficiently small, there exists a unique $\tilde{\varphi}_0 \in (-\arccos(1/3), 0)$ such that $\tilde{K}_{II}(\Gamma_0, \tilde{\varphi}_0) = 0$. Moreover $\partial_\varphi \tilde{K}_{II}(\Gamma_0, \tilde{\varphi}_0) > 0$.*

Proof. In this proof we will write explicitly the dependence on the radius $r > 0$ in the stress intensity factors \tilde{K}_i . Remember that $\tilde{K}(\Gamma_0, \varphi, r)$ is given by

$$\tilde{K}(\Gamma_0, \varphi, r) = \tilde{\mathbf{T}}(\varphi)K(\Gamma_0) + [\tilde{K}(\Gamma_0, \varphi, r) - \mathring{K}(\Gamma_0, \varphi, r)] = \tilde{\mathbf{T}}(\varphi)K(\Gamma_0) + E(\Gamma_0, \varphi, r).$$

Hence the equation $\tilde{K}_{II}(\Gamma_0, \varphi, r) = 0$ reads

$$\tilde{K}_{II}(\Gamma_0, \varphi, r) = C_{21}(\varphi)K_I(\Gamma_0) + C_{22}(\varphi)K_{II}(\Gamma_0) + E_2(\Gamma_0, \varphi, r) = 0,$$

where C_{ij} are the elements of the matrix $\tilde{\mathbf{T}}$ given by (12), i.e.

$$C_{21}(\varphi) = \cos(\varphi/2)(\sin \varphi)/2, \quad C_{22}(\varphi) = \cos(\varphi/2)(3 \cos \varphi - 1)/2.$$

For convenience let us introduce the auxiliary function K_a given by

$$K_a(\varphi) = C_{21}(\varphi)K_I(\Gamma_0) + C_{22}(\varphi)K_{II}(\Gamma_0),$$

so that $K_{II}(\Gamma_0, \varphi, r) = K_a(\varphi) + E_2(\Gamma_0, \varphi, r)$.

Let us start considering only the term K_a , i.e. solving

$$K_a(\varphi) = C_{21}(\varphi)K_I(\Gamma_0) + C_{22}(\varphi)K_{II}(\Gamma_0) = 0.$$

For $t = K_{II}(\Gamma_0)/K_I(\Gamma_0) > 0$ and $\varphi \in (-\arccos(1/3), 0)$ the above equation is equivalent to

$$(\sin \varphi)/(3 \cos \varphi - 1) = -t.$$

The function $(\sin \varphi)/(3 \cos \varphi - 1)$ is strictly increasing and continuous for φ in the interval $(-\arccos(1/3), 0)$, it is unbounded from below and null in zero, thus, being

$-t < 0$, there exists a unique $\varphi'_0 \in (-\arccos(1/3), 0)$ such that $(\sin \varphi'_0)/(3 \cos \varphi'_0 - 1) = -t$, i.e. such that $K_a(\varphi'_0) = 0$.

For the general case, the equation to solve is

$$\tilde{K}_{\text{II}}(\Gamma_0, \varphi, r) = C_{21}(\varphi)K_{\text{I}}(\Gamma_0) + C_{22}(\varphi)K_{\text{II}}(\Gamma_0) + E_2(\Gamma_0, \varphi, r) = K_a(\varphi) + E_2(\Gamma_0, \varphi, r) = 0.$$

We will use a perturbation argument. First let us compute the partial derivative $\partial_\varphi \tilde{K}_{\text{II}} = \partial_\varphi K_a + \partial_\varphi E_2$. Let us start with

$$\partial_\varphi K_a = -\sin(\varphi/2)[K_{\text{I}} \sin \varphi + K_{\text{II}}(3 \cos \varphi - 1)]/4 + \cos(\varphi/2)[K_{\text{I}} \cos \varphi - 3K_{\text{II}} \sin \varphi]/2.$$

Since $K_a(\varphi'_0) = [K_{\text{I}} \sin \varphi'_0 + K_{\text{II}}(3 \cos \varphi'_0 - 1)] = 0$ by continuity $\sin(\varphi/2)[K_{\text{I}} \sin \varphi + K_{\text{II}}(3 \cos \varphi - 1)]/4$ can be made arbitrarily small for $|\varphi - \varphi'_0| \leq \delta$ and $\delta > 0$ sufficiently small. For the second term in the derivative remember that $K_i > 0$ and that $\varphi \in (-\arccos(1/3), 0)$, thus

$$\cos(\varphi/2)[K_{\text{I}} \cos \varphi - 3K_{\text{II}} \sin \varphi]/2 \geq K_{\text{I}} \cos(\varphi/2)(\cos \varphi)/2 \geq K_{\text{I}} \sqrt{2/3}/6 > 0.$$

It follows that, $\partial_\varphi K_a(\varphi) \geq C > 0$ for $|\varphi - \varphi'_0| < \delta$. Finally, write

$$E_2(\Gamma_0, \varphi, r) = \int_{B_r \setminus \Gamma_0} (\bar{u}_0 - \dot{u}_0) \cdot k_2(\vartheta - \varphi) dx,$$

$$\partial_\varphi E_2(\Gamma_0, \varphi, r) = - \int_{B_r \setminus \Gamma_0} (\bar{u}_0 - \dot{u}_0) \cdot k'_2(\vartheta - \varphi) dx,$$

where k'_2 denote the derivative of k_2 with respect to its argument. By Corollary 2 we get that $E_2(\Gamma_0, \varphi, r) \rightarrow 0$ and $\partial_\varphi E_2(\Gamma_0, \varphi, r) \rightarrow 0$ for $r \rightarrow 0$ uniformly with respect to φ in the interval $(-\arccos(1/3), 0)$.

At this point, we have all the ingredients to conclude the proof. Being $K_a(\varphi'_0) = 0$ and $\partial_\varphi K_a(\varphi) \geq C > 0$ for $|\varphi - \varphi'_0| \leq \delta$ it follows that $K_a(\varphi'_0 - \delta) < 0 < K_a(\varphi'_0 + \delta)$. Moreover $E_2(\Gamma_0, \varphi, r) \rightarrow 0$ uniformly, thus for r small enough

$$\tilde{K}_{\text{II}}(\Gamma_0, \varphi'_0 - \delta, r) = K_a(\varphi'_0 - \delta) + E_2(\Gamma_0, \varphi'_0 - \delta, r) < C_1 < 0$$

$$K_{\text{II}}(\Gamma_0, \varphi'_0 + \delta, r) = K_a(\varphi'_0 + \delta) + E_2(\Gamma_0, \varphi'_0 + \delta, r) > C_2 > 0.$$

Moreover, for r and δ sufficiently small we have $\partial_\varphi K_a(\varphi) \geq C > 0$ and $\partial_\varphi E_2(\Gamma_0, \varphi, r)$ uniformly small, thus

$$\partial_\varphi \tilde{K}_{\text{II}}(\Gamma_0, \varphi, r) = \partial_\varphi K_a(\varphi) + \partial_\varphi E_2(\Gamma_0, \varphi, r) \geq C' > 0$$

whenever $|\varphi - \varphi'_0| \leq \delta$. It follows that there exists a unique $\tilde{\varphi}_0 \in (\varphi'_0 - \delta, \varphi'_0 + \delta)$ with $\tilde{K}_{\text{II}}(\Gamma_0, \tilde{\varphi}_0, r) = 0$. Finally, note that

$$K_a(-\arccos(1/3)) = C_{21}(-\arccos(1/3))K_{\text{I}}(\Gamma_0) < 0, \quad K_a(0) = K_{\text{II}}(\Gamma_0) > 0$$

and remember that $K_a(\varphi'_0 - \delta) < 0 < K_a(\varphi'_0 + \delta)$. Since there exists a unique solution to $K_a(\varphi) = 0$ and since K_a is continuous we get that $\min |K_a(\varphi)| \geq C'' > 0$ for $|\varphi - \varphi'_0| \geq \delta$. As $E_2(\Gamma_0, \varphi, r) \rightarrow 0$ uniformly for $r \rightarrow 0$, we get $\min |\tilde{K}_{\text{II}}(\Gamma_0, \varphi, r)| \geq C''' > 0$ for $|\varphi - \varphi'_0| \geq \delta$. Therefore, the solution to $\tilde{K}_{\text{II}}(\Gamma_0, \varphi, r) = 0$ is unique in the interval $(-\arccos(1/3), 0)$. \square

Remark 2. Clearly, the kink angle $\tilde{\varphi}_0$ given by the previous Lemma depends on the radius $r > 0$. However, if $r \searrow 0$ then $\tilde{\varphi}_0$ converges to the angle φ'_0 obtained in the first part of the proof, i.e. neglecting the error E_2 . We remark that the value φ'_0 is usually considered a good theoretical prediction of the kink angle in accordance with the experimental data (see for instance [16]).

4.2. Functional differential equation for the crack path. Before proceeding, we remind that Γ_0 is straight, therefore there exists the stress intensity factors $K_i(\Gamma_0)$. We will assume that both $K_i(\Gamma_0)$ are positive and denote by $\tilde{\varphi}_0$ the angle given by Lemma 4.1. The crack path will be represented, in the absolute system of coordinates, see Figure 1, by the graph of functions in the set (1).

Given $y \in \mathcal{Y}$ and $s \in [0, S]$ let $V(\Gamma_s) = \text{tg } \tilde{\varphi}_s$ where $\tilde{\varphi}_s$ solves $\tilde{K}_{II}(\Gamma_s, \tilde{\varphi}_s) = 0$ (we will prove existence and uniqueness of $\tilde{\varphi}_s$ in Lemma 4.2). The crack path is found by solving the first order functional differential equation

$$\begin{cases} y'(s) = V(\Gamma_s) & \text{for a.e. } s \in (0, S) \\ y(0) = 0. \end{cases}$$

Indeed, if $y'(s) = V(\Gamma_s) = \text{tg } \tilde{\varphi}_s$ then (by definition) $\tilde{K}_{II}(\Gamma_s, \tilde{\varphi}_s) = \tilde{K}_{II}(\Gamma_s) = 0$.

First, we will prove that V is well defined and that it depends continuously on s .

Lemma 4.2. *Assume that $K_i(\Gamma_0) > 0$ for $i = I, II$. Let r and S be sufficiently small. Then, for every $y \in \mathcal{Y}$ and $s \in [0, S]$ there exists a unique $\tilde{\varphi}_s \in (-\arccos(1/3), 0)$ such that $\tilde{K}_{II}(\Gamma_s, \tilde{\varphi}_s) = 0$ and $\partial_\varphi \tilde{K}_{II}(\Gamma_s, \tilde{\varphi}_s) > 0$. Moreover, $\tilde{\varphi}_s$ is continuous in $[0, S]$.*

Proof. By Lemma 3.2 we know that \tilde{K}_i is continuous with respect to the variables s and φ .

Remember that by Lemma 4.1 there exists a unique $\tilde{\varphi}_0 \in (-\arccos(1/3), 0)$ such that $\tilde{K}_{II}(\Gamma_0, \tilde{\varphi}_0) = 0$. Moreover there exists $\delta > 0$ such that $\partial_\varphi \tilde{K}_{II}(\Gamma_0, \varphi) \geq C' > 0$ for $|\varphi - \tilde{\varphi}_0| \leq \delta$ and $|\tilde{K}_{II}(\Gamma_0, \varphi)| \geq C > 0$ for $|\varphi - \tilde{\varphi}_0| > \delta$. Hence, by Corollary 1 for s sufficiently small we have $\partial_\varphi \tilde{K}_{II}(\Gamma_s, \varphi) \geq C'/2 > 0$ for $|\varphi - \tilde{\varphi}_0| \leq \delta$ and $|\tilde{K}_{II}(\Gamma_s, \varphi)| \geq C/2 > 0$ for $|\varphi - \tilde{\varphi}_0| > \delta$. Thus by continuity of $\tilde{K}_{II}(\Gamma_s, \cdot)$ there exists a unique $\tilde{\varphi}_s \in (-\arccos(1/3), 0)$ with $\tilde{K}_{II}(\Gamma_s, \tilde{\varphi}_s) = 0$.

To conclude, let us check that $\tilde{\varphi}_s$ is continuous with respect to s . Given $\varepsilon > 0$ (sufficiently small) we know that

$$\tilde{K}_{II}(\Gamma_s, \tilde{\varphi}_s - \varepsilon) < \tilde{K}_{II}(\Gamma_s, \tilde{\varphi}_s) < \tilde{K}_{II}(\Gamma_s, \tilde{\varphi}_s + \varepsilon),$$

because $\partial_\varphi \tilde{K}_{II}(\Gamma_s, \varphi) \geq C' > 0$ for $|\varphi - \tilde{\varphi}_s| < \delta$. Remember that $\tilde{K}_{II}(\Gamma_s, \tilde{\varphi}_s) = 0$, thus, for $|t - s|$ small enough, by uniform convergence of $\tilde{K}_{II}(\Gamma_s, \cdot)$ we have

$$\tilde{K}_{II}(\Gamma_t, \tilde{\varphi}_s - \varepsilon) < 0 < \tilde{K}_{II}(\Gamma_t, \tilde{\varphi}_s + \varepsilon).$$

It follows that the unique angle $\tilde{\varphi}_t$ such that $\tilde{K}_{II}(\Gamma_t, \tilde{\varphi}_t) = 0$ must belong to the interval $(\tilde{\varphi}_s - \varepsilon, \tilde{\varphi}_s + \varepsilon)$. \square

4.3. Fixed point problem for the crack path. Remember that the crack path is the graph of a function in the set

$$\mathcal{Y} = \{y \in C^{0,1}([0, S]) : y(0) = 0, \|y'\|_\infty \leq C_{\mathcal{Y}}\}.$$

By Lemma 4.2, (if S is small) the "deflection angle" $\tilde{\varphi}_s$ belongs to $(-\arccos(1/3), 0)$. Remember that $\tilde{\varphi}_s$ is the "deflection angle" in a system of coordinates translating with the crack. For this reason, we will choose $C_{\mathcal{Y}} \geq \tan(\arccos(1/3))$. Let us consider on \mathcal{Y} the functional \mathcal{F} defined by

$$[\mathcal{F}(y)](s) = \int_0^s V(\Gamma_z) dz.$$

Note that, if S is small enough then \mathcal{F} takes values in \mathcal{Y} since $[\mathcal{F}(y)](0) = 0$ and since for a.e. s we have $|[\mathcal{F}(y)]'(s)| = |V(\Gamma_s)| = |\tan \tilde{\varphi}_s| \leq |\tan(\arccos(1/3))| \leq C_{\mathcal{Y}}$. Let us state our existence result for the crack path.

Theorem 4.3. *For $S > 0$ sufficiently small there exists a fixed point for \mathcal{F} in $\mathcal{Y} \cap C^1([0, S])$. In other terms, there exists a crack path such that $\tilde{K}_{\text{II}}(\Gamma_0, \tilde{\varphi}_0) = 0$ and $\tilde{K}_{\text{II}}(\Gamma_s) = 0$ for every $s \in (0, S]$.*

Proof. We will prove existence by means of Schauder Theorem. To this end we consider the set \mathcal{Y} to be endowed with the topology of $C^0([0, S])$. The set \mathcal{Y} is convex, closed and compact, by Ascoli-Arzelà Theorem. We have already seen that \mathcal{F} takes values in \mathcal{Y} . It remains to see that it is (sequentially) continuous. Let y_n be a sequence in \mathcal{Y} such that $y_n \rightarrow y$ uniformly. Given $s \in [0, S]$, let $\tilde{\varphi}_s$ such that $\tilde{K}_{\text{II}}(\Gamma_s, \tilde{\varphi}_s) = 0$. Remember that $\partial_{\varphi} \tilde{K}_{\text{II}}(\Gamma_s, \tilde{\varphi}_s) > 0$. Then for ε small enough we can write

$$\tilde{K}_{\text{II}}(\Gamma_s, \tilde{\varphi}_s - \varepsilon) < 0 < \tilde{K}_{\text{II}}(\Gamma_s, \tilde{\varphi}_s + \varepsilon).$$

By the uniform convergence of $\tilde{K}_{\text{II}}(\Gamma_s^n, \cdot)$ (Lemma 3.3) for n sufficiently large we have

$$\tilde{K}_{\text{II}}(\Gamma_s^n, \tilde{\varphi}_s - \varepsilon) < 0 < \tilde{K}_{\text{II}}(\Gamma_s^n, \tilde{\varphi}_s + \varepsilon).$$

As a consequence, the angle $\tilde{\varphi}_s^n$ such that $\tilde{K}_{\text{II}}(\Gamma_s^n, \tilde{\varphi}_s^n) = 0$ must satisfy $\tilde{\varphi}_s - \varepsilon < \tilde{\varphi}_s^n < \tilde{\varphi}_s + \varepsilon$. Thus, $\tilde{\varphi}_s^n \rightarrow \tilde{\varphi}_s$ pointwise in $[0, S]$ and hence $V(\Gamma_s^n) = \tan \tilde{\varphi}_s^n \rightarrow V(\Gamma_s) = \tan \tilde{\varphi}_s$ pointwise as well. As $\tilde{\varphi}_s^n \in (-\arccos(1/3), 0)$, $V(\Gamma_s^n)$ is uniformly bounded in $[0, S]$; it follows that $V(\Gamma_s^n) \rightarrow V(\Gamma_s)$ in $L^1(0, S)$. As a consequence $\mathcal{F}(y_n) \rightarrow \mathcal{F}(y)$ in $W^{1,1}(0, S)$ and thus, by Sobolev embedding, in $C^0([0, S])$.

At this point it is sufficient to apply Schauder Theorem to gain the existence of a fixed point in \mathcal{Y} . Finally, by Lemma 4.2 the function $\tilde{\varphi}_s$ is continuous in $[0, S]$, thus $V(\Gamma_s)$ is continuous. In particular, the fixed point will be of class $C^1([0, S])$. \square

To conclude this section, let us consider how the approximated stress intensity factors depend on time. By linearity, denote $u_{t,s}(x) = c(t)u_s(x)$ the equilibrium solution with boundary condition $c(t)g$ and crack path Γ_s . Since \tilde{K}_i is linear with respect to the displacement, it turns out that the approximated stress intensity factors for $u_{t,s}$ are

$$\tilde{K}_i(t, \Gamma_s, \varphi) = \int_{B_r \setminus \Gamma_s} (u_{t,s} - \hat{u}_{t,s}) \cdot k_i(\vartheta - \varphi) dx = c(t) \tilde{K}_i(\Gamma_s, \varphi).$$

As a consequence, we can represent the evolution as the composition of the curve $\gamma(s) = (s, y(s))$ (given by Theorem 4.3) with a parametrization $s(t)$. Indeed, denoting $\Gamma_t = \Gamma_{s(t)}$ then

$$\tilde{K}_{\text{II}}(t, \Gamma_t) = c(t) \tilde{K}_{\text{II}}(\Gamma_{s(t)}) = 0$$

for every $t \in [0, T]$ such that $s(t) > 0$ (i.e. for $\Gamma_t \supsetneq \Gamma_0$) while

$$\tilde{K}_{\text{II}}(t, \Gamma_t, \tilde{\varphi}_0) = c(t) \tilde{K}_{\text{II}}(\Gamma_0, \tilde{\varphi}_0) = 0$$

for $s(t) = 0$ (i.e. for $\Gamma_t = \Gamma_0$). At this point it is convenient to introduce

$$\tilde{K}_{\text{II}}^*(t, \Gamma_s) = \begin{cases} \tilde{K}_{\text{II}}(t, \Gamma_0, \tilde{\varphi}_0) & \text{for } \Gamma_s = \Gamma_0 \\ \tilde{K}_{\text{II}}(t, \Gamma_s) & \text{otherwise.} \end{cases}$$

In this way it is necessary to distinguish between Γ_0 and $\Gamma_{s(t)}$ for $s(t) > 0$; we will write, more simply, that $\tilde{K}_{\text{II}}^*(t, \Gamma_t) = 0$ for every $t \in [0, T]$.

4.4. Existence of a parametrization. Once the path Γ_s is given (by Theorem 4.3) the approximated Principle of Local Symmetry $\tilde{K}_{\Pi}^*(t, \Gamma_t) = 0$ holds true for every parametrization $s(t)$. We are therefore free to choose the parametrization in such a way that the approximated Griffith's criterion is satisfied. As we did above we define

$$\tilde{K}_I^*(t, \Gamma_s) = \begin{cases} \tilde{K}_I(t, \Gamma_0, \tilde{\varphi}_0) & \text{for } \Gamma_s = \Gamma_0 \\ \tilde{K}_I(t, \Gamma_s) & \text{otherwise.} \end{cases}$$

The precise statement, which characterizes the quasi-static evolution, is contained in the next Theorem.

Theorem 4.4. *Given a fixed point $y \in \mathcal{Y} \cap C^1([0, S])$ there exists a non-decreasing, left-continuous parametrization $s(t)$ such that for $\Gamma_t = \Gamma_{s(t)}$ the following Kuhn-Tucker conditions are satisfied:*

$$\tilde{K}_I^*(t, \Gamma_t) \leq K_I^c \quad \text{for } t \in [0, T]$$

$$(\tilde{K}_I^*(t, \Gamma_t) - K_I^c) ds(t) = 0 \quad (\text{in the sense of measures}) \text{ in } [0, T].$$

Moreover for $t \in J(s)$ we have

$$\tilde{K}_I^*(t, l) \geq K_I^c \quad \text{for } l \in [s^-(t), s^+(t)],$$

so that discontinuities represents the non-equilibrium regimes of the evolution. Finally, if $c(t)$ is strictly increasing the left-continuous parametrization s is unique.

Using the same argument of [14, § 6], the proof of the previous Theorem follows by the fact that $\tilde{K}_I(t, \Gamma_s) = c(t)\tilde{K}_I(\Gamma_s)$ is continuous with respect to s and t .

Appendix A. Approximation of the right stress intensity factors.

A.1. Choice of the kernel. First of all, we will see how to choose the coefficients a_i and b_i , appearing in (7). Our choice is inspired by [17]. Writing u_0 in polar coordinates (8) takes the form

$$\tilde{K}_i(\Gamma_0, \varphi) = \int_0^r \int_{-\pi}^{\pi} (u_0(\rho, \vartheta) - \hat{u}_0) \cdot k_i(\vartheta - \varphi) \rho d\vartheta d\rho. \quad (18)$$

Next, we introduce the matrix $\mathbf{M}(\varphi)$ with elements $M_{ij}(\varphi)$ given by

$$M_{ij}(\varphi) = \int_0^r \int_{-\pi}^{\pi} \hat{u}_j(\rho, \vartheta) \cdot k_i(\vartheta - \varphi) \rho d\vartheta d\rho,$$

where \hat{u}_j are the singular functions appearing in (4). A lengthy evaluation of trigonometric integrals yields

$$\begin{aligned} M_{11}(\varphi) &= \cos(\varphi/2) [(2k-1)(a_1+a_2)] + \cos(3\varphi/2) [a_4-a_3], \\ M_{12}(\varphi) &= \sin(\varphi/2) [a_1(2k+3) + a_2(2k-3)] + \sin(3\varphi/2) [a_3+a_4], \\ M_{21}(\varphi) &= \sin(\varphi/2) [(2k-1)(b_2-b_1)] + \sin(3\varphi/2) [b_3+b_4], \\ M_{22}(\varphi) &= \cos(\varphi/2) [b_1(2k+3) - b_2(2k-3)] + \cos(3\varphi/2) [b_3-b_4]. \end{aligned}$$

The identity $\tilde{\mathbf{T}}(\varphi) = \mathbf{M}(\varphi)$ holds if and only if the coefficients a_i and b_i satisfy

$$\left(\begin{array}{cc|cc} 2k-1 & 2k-1 & 0 & 0 \\ 2k+3 & 2k-3 & 0 & 0 \\ \hline 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 1 \end{array} \right) \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} = \frac{1}{4} \begin{pmatrix} 3 \\ -3 \\ 1 \\ -3 \end{pmatrix},$$

$$\left(\begin{array}{cc|cc} 1-2k & 2k-1 & 0 & 0 \\ 2k+3 & 3-2k & 0 & 0 \\ \hline 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{array} \right) \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix} = \frac{1}{4} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 3 \end{pmatrix}.$$

It is easy to check that both the linear systems admits a unique solution if and only if $2k - 1 \neq 0$. Remembering that $k = 3 - 4\nu$ it turns out that there exists a unique solution if and only if $\nu \neq 0.6$, which is true since Poisson ratio ν is always less than 0.5. The explicit solutions are

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} = \frac{1}{4} \begin{pmatrix} (2-2k)/(2k-1) \\ (2k+1)/(2k-1) \\ -1 \\ -2 \end{pmatrix}, \quad \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix} = \frac{1}{4} \begin{pmatrix} (2k-2)/(6k-3) \\ (2k+1)/(6k-3) \\ 2 \\ -1 \end{pmatrix}.$$

In conclusion (thanks to the fact that $u_0 = K_I \hat{u}_I + K_{II} \hat{u}_{II} + \bar{u}_0$) we get

$$\tilde{K}(\Gamma_0, \varphi) = \tilde{T}(\varphi)K(\Gamma_0) + [\bar{K}(\Gamma_0, \varphi) - \dot{K}(\Gamma_0, \varphi)]. \quad (19)$$

In particular, for $\varphi = 0$ we have $\tilde{K}(\Gamma_0, 0) = \tilde{K}(\Gamma_0)$ etc. and $\tilde{T}(0) = \mathbf{I}$, therefore

$$\tilde{K}(\Gamma_0) = K(\Gamma_0) + [\bar{K}(\Gamma_0) - \dot{K}(\Gamma_0)].$$

A.2. Convergence. In our analysis and in several other applications the radius of the ball B_r , appearing in the evaluation of the stress intensity factors, is kept constant. As a consequence, just after initiation (for $s \ll 1$) the crack set $\Gamma_s \cap B_r$ is not "regular" and it contains a corner point (the kink); in this case it is likely that the non-local quantities \tilde{K}_i are not a good approximation of the stress intensity factors. More precisely, the error estimate of the previous section holds true if the representation (4) of u_s is valid in a ball B_R with $R > r$. But, if B_r contains a kink then (in a small neighborhood of the kink itself) other two couples of singular functions appears; their singularity with respect to ρ (the distance from the kink) is of the type ρ^{π/φ_0^\pm} , where φ_0^\pm are the inner angles at the kink. However, our choice of the convolution kernels k_i has been crafted in such a way that \tilde{K}_i provide an accurate value even close to the kink angle (this is why the coefficients a_i and b_i have been tailored to the transfer matrix). As we will see, this property is crucial in order to have a realistic prediction of the kink angle.

Let us try to give a more general picture. Given $\varphi \in (-\pi/2, \pi/2)$ let $\tilde{\gamma}(s) = s(\cos \varphi, \sin \varphi)$. In this way the (true) stress intensity factors do exist for every s and the convergence property of section 3.2 holds. Let $K(\tilde{\Gamma}_s)$ be the vector of the (true) stress intensity factors; the corresponding vector of the approximated stress intensity factors will be $\tilde{K}(\tilde{\Gamma}_s, r)$ (note here the explicit dependence on the convolution radius r). Let their right limits be

$$K(\Gamma_0, \varphi) = \lim_{s \rightarrow 0^+} K(\Gamma_s), \quad \tilde{K}(\Gamma_0, \varphi, r) = \lim_{s \rightarrow 0^+} \tilde{K}(\Gamma_s, r).$$

The relationship between these four quantities are drawn in this diagram:

$$\begin{array}{ccc} \tilde{K}(\Gamma_s, r) & \xrightarrow{r \rightarrow 0} & K(\Gamma_s) \\ s \rightarrow 0 \downarrow & & s \rightarrow 0 \downarrow \\ \tilde{K}(\Gamma_0, \varphi, r) & \not\rightarrow & K(\Gamma_0, \varphi). \end{array}$$

The first line, for $s > 0$, comes from the convergence property of section 3.2. The vertical lines are just the above definition. The last line is the issue: according again section 3.2

$$\tilde{K}(\Gamma_0, \varphi, r) \xrightarrow{r \rightarrow 0} \tilde{\mathbf{T}}(\varphi) K(\Gamma_0) \neq \mathbf{T}(\varphi) K(\Gamma_0) = K(\Gamma_0, \varphi),$$

where \mathbf{T} and $\tilde{\mathbf{T}}$ are respectively the transfer matrix and its approximation (12). Mathematically speaking, if we consider the family of functions \tilde{K}_r , parametrized by r and defined in $[0, S]$, we have that \tilde{K}_r convergence pointwise to K only for $s > 0$. In general, as it is in this case, pointwise convergence in an open set does not imply pointwise convergence on its boundary.

Now, let us study the error in the evaluation of the stress intensity factors for $s \searrow 0$. We will provide a 'low cost' estimate just to understand the issue and the idea behind the integral approximation formula (8). By (19) the right limits of the approximated stress intensity factors reads

$$\tilde{K}(\Gamma_0, \varphi) = \tilde{\mathbf{T}}(\varphi) K(\Gamma_0) + [\tilde{K}(\Gamma_0, \varphi) - \dot{K}(\Gamma_0, \varphi)].$$

Denoting by $K(\Gamma_0, \varphi)$ the vector of the (true) right stress intensity factors and using the transfer matrix \mathbf{T} to represent it, we can write

$$\tilde{K}(\Gamma_0, \varphi) - K(\Gamma_0, \varphi) = [\tilde{\mathbf{T}}(\varphi) - \mathbf{T}(\varphi)] K(\Gamma_0) + [\tilde{K}(\Gamma_0, \varphi) - \dot{K}(\Gamma_0, \varphi)].$$

Hence, the error is bounded by

$$|\tilde{K}(\Gamma_0, \varphi) - K(\Gamma_0, \varphi)| \leq \max_{\varphi} \|\tilde{\mathbf{T}} - \mathbf{T}\| |K(\Gamma_0)| + C_{\varepsilon} r^{1/2-\varepsilon}.$$

As \mathbf{T} is not known explicitly it is hard to find a good estimate for $\|\tilde{\mathbf{T}} - \mathbf{T}\|$; for simplicity, here we will rely on [5] where it has been estimated that $\|\tilde{\mathbf{T}} - \mathbf{T}\|_{\infty} \leq 0.05$ for φ in the range $(-\arccos(1/3), 0)$. Therefore, if r is sufficiently small we can say that the relative error

$$\frac{|\tilde{K}(\Gamma_0, \varphi) - K(\Gamma_0, \varphi)|}{|K(\Gamma_0)|}$$

is of the order of 5%. Note that, to improve this estimate it is not enough to take r smaller, because $\tilde{K}(\Gamma_0, \varphi, r) \not\rightarrow K(\Gamma_0, \varphi)$ as $r \rightarrow 0$ (see the graph above). It is instead necessary to improve also the approximation of the transfer matrix \mathbf{T} and accordingly the approximation of the stress intensity factors. Otherwise, in order to find $K(\Gamma_0, \varphi)$ it is necessary to pass to the limit first with respect to r and then with respect to s (see again the graph above).

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