A Rockafellar Theorem for cyclically quasi-monotone maps: the regular non-vanishing case

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We study the connection between cyclic quasi-monotonicity and quasi-convexity, focusing on whether every cyclically quasi-monotone (possibly multivalued) map is included in the normal cone operator of a quasi-convex function, in analogy with Rockafellar's theorem for convex functions. We provide a positive answer for $\mathscr{C}^1$-regular, non-vanishing maps in any dimension, as well as for general multi-maps in dimension 1. We further discuss connections to revealed preference theory in economics and to L^∞ optimal transport. Finally, we present explicit constructions and examples, highlighting the main challenges that arise in the general case.

Keywords. quasi-monotonicity, quasi-convexity, optimal transport, L^∞ optimal transport, convex analysis, revealed preference, Rockafellar theorem.
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Contents

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1 Introduction

Our work deals with the relation between the notions of cyclic quasi-monotonicity and quasi-convexity. We shall start by discussion the latter, then move on to cyclic quasi-monotonicity.

A *quasi-convex function* $f : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ is a function whose sub-level sets $\{x : f(x) \leq \ell\}$ are convex for every $\ell \in \mathbb{R}$. Historically, quasi-convexity (or quasi-concavity) has been introduced in the first half of the 19th century through the interplay between mathematics and economics, specifically utility and decision theory. It seems (see [GM04]) that Von Neumann was the first one to make use of their defining properties, as a technical condition (without giving a definition) for his minimax theorem, in his monograph on game theory [von28]. The notion was properly introduced by de Finetti in his article on "convex stratifications" [de 49], where he studied their properties and their relation with totally ordered families of convex sets. The modern and rigorous study of sub-level sets of convex and quasi-convex functions was initiated by Fenchel in his 1953 monograph [Fen53, §7-8]. We refer to Guerragio's very interesting article [GM04] to know more about the origins of this notion.

Nowadays, quasi-convexity (or quasi-concavity) is widely used in optimization and game theory (as a common assumption for objective functionals or to apply Von Neumann's or Sion's minimax theorem), in economics (as a common assumption for utility functions in consumer theory), and calculus of variations (when supremal functionals are involved). We will elaborate later on the strong connections between the problem we address and both the theory of revealed preferences in economics and optimal transportation.

A *multi-map* or *multi-valued operator* $F : \mathbb{R}^d \multimap \mathbb{R}^d$ (which is nothing but a map from $\mathbb{R}^d$ to $2^{\mathbb{R}^d}$) is cyclically quasi-monotone if for every sequence of points $x_0, \ldots, x_N \in \mathbb{R}^d$ satisfying $x_N = x_0$, and every $p_0 \in F(x_0), \ldots, p_{N-1} \in F(x_{N-1})$ it holds:

$$\min_{0 \leq i < N} (x_{i+1} - x_i) \cdot p_i \leq 0.$$

The weaker notion of *quasi-monotonicity* corresponds to considering sequences only made of two points, i.e. imposing $N = 1$.

Cyclically monotone operators naturally arise from quasi-convex functions f through their *normal cone multi-map*, defined as

$$\forall x \in \mathbb{R}^d, \qquad \mathrm{N}f(x) := N_{\{y : f(y) \leq f(x)\}}(x)$$

where $N_C(x) := \partial\chi_C(x)$ is the normal cone[1] to the convex set C at x. Indeed, if f is quasi-convex then N_F is a cyclically quasi-monotone multi-map (as stated in Proposition 2.5). We can draw a parallel with the usual notions of convexity and cyclical monotonicity, the normal cone operator replacing the subdifferential operator of convex functions: indeed, it is well-known that ∂f is cyclically monotone whenever f is convex. In fact converse results exist for the classical notions of convexity and (cyclical) monotonity:

- by [Pol90; CJT94], for every function f, the Clarke-Rockafellar subdifferential ∂f is monotone (even cyclically) if and if f is convex ;
- by the celebrated theorem of Rockafellar (proved in [Roc66], see also [Roc15, Theorem 24.8]), an operator F is cyclically monotone if and only if $F \subseteq \partial f$ for some lower semicontinuous proper convex function $f : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$.

We are tempted to wonder if such converse statements exist for the corresponding "quasi" notions, and some suitable normal cone operator replacing ∂. Thus, recalling the questions originally studied by de Finetti [de 49] and Fenchel [Fen53, §7-8], we highlight three classes of problems related to (quasi-)convexity and (quasi-)monotonicity:

1. *The characterization of sublevel sets of convex functions.* Given a totally ordered family $\mathcal{C}$ of convex sets (indexed or not), is there a (quasi-)convex function f whose level sets belong to $\mathcal{C}$. If not, can we characterize those $\mathcal{C}$'s that are the sub-level sets of convex functions? The answer is true when the convex sets are closed, if we only ask f to be quasi-convex (an indexed version is shown in Corollary 2.13). The answer is no if one asks f to be convex, but some sufficient conditions were provided by de Finetti [de 49], Fenchel [Fen53, §7-8] and more recently by Rapcsák [Rap05a; Rap13] in particular. We refer to [Rap05b] for a survey on this problem, sometimes refered to as *Fenchel problem of level sets*.
2. *The subdifferential characterization of quasi-convexity.* Can we characterize, in the same way as for convexity, the quasi-convexity of a function f by the quasi-monotonicity of a suitable first-order operator Tf, like a subdifferential or normal cone operator? This has been answered positively in [ACL94] when T is the Clarke-Rockafellar subdifferential, and partial answers have been given for different notions of normal cone operators by several authors like Aussel, Crouzeix, Daniilidis and Penot to name a few [AD00; AD01; BC90; Pen98].
3. *The integration of cyclically quasi-monotone operators.* Is any cyclically quasi-monotone operator included in the graph of a first-order (of subdifferential or normal cone type) operator associated with a quasi-convex function? In other words, is there "quasi" counterpart to Rockafellar integration theorem known for cyclically monotone operators, for some replacement of the subdifferential operator? A partial answer has been given, under extra assumptions on the operator,

[1] Here χ_C stands for the convex indicator of C and ∂ to the classical Fenchel-Moreau subdifferential operator for convex functions.

with the lower subdifferential operator of Plastria by Bachir, Daniilidis and Penot in [BDP02]. Another partial answer has been given for the normal cone operator N, under some regularity and non-degeneracy assumptions by Crouzeix, Eberhard and Ralph in [ERC12], through an "indirect" statement that is not completely equivalent.

Our work deals with the third question, that can be set precisely as follows.

Problem 1.1. Given a cyclically quasi-monotone multi-map $F : \mathbb{R}^d \multimap \mathbb{R}^d$, can we build a quasi-convex function $f : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$, which satisfies

$$F \subseteq \mathrm{N}f?$$

Such a function f will be called a *potential* of F. We shall restrict ourselves to the case of finite and lower semi-continuous potentials, as justified in Remark 2.6.

Before stating our results and describing the structure of the article, let us stress the connection of this problem with consumer theory in economics, more precisely to revealed preference theory, and optimal transport type.

Relation with revealed preference theory

Revealed preference theory in economics, introduced by Samuelson in 1938 [Sam38] is a theory that postulates the rational behaviour of consumers. Given N bundles of goods $(x_1, \dots, x_N) \in \mathbb{R}^d$ the observation of prices $(p_1, \dots, p_N) \in \mathbb{R}^d$ at which the consumer bought those goods, we may infer a "preference relation" by the following: x_i is preferred to x, if $p_i \cdot x_i \geq p_i \cdot x$, i.e. the consumer could have chosen bundle x with the same (or a lower) budget $p_i \cdot x$ but decided to buy x_i instead.

In revealed preference theory, if certain axioms ensuring the consistency of the consumer's preferences are satisfied, it is possible to deduce that the observed choices result from the maximization of a utility function u; that is, the preference relation $\preceq$ corresponds (in our convention) to the minimization of a (quasi-)convex function $f = -u$. One says that f (or u) rationalizes the observed data. The axioms are called the *(weak/strong/generalized) axioms of revealed preference* (WARP, SARP, GARP respectively), and the main result in the area is the so-called *Afriat Theorem*: if finitely many data $(x_1, p_1), \dots, (x_N, p_N) \in \mathbb{R}^d$ satisfy (GARP), then there exists a continuous convex nonsatiated[2] function f rationalizing the data.

It turns out that $(x_1, p_1), \dots, (x_N, p_N)$ satisfied GARP if and only if the operator F defined by $F(x_i) = -p_i$ is cyclically quasi-monotone, and the rationalization condition that if x_i is preferred to x then $u(x_i) \geq u(x)$ rewrites as

$$\forall x \in \{f < f(x_i)\}, \quad F(x_i) \cdot (x - x_i) < 0,$$

which implies, by nonsatiation of f, to

$$\forall x \in \{f \leq f(x_i)\}, \quad F(x_i) \cdot (x - x_i) < 0,$$

[2] Meaning that f has no local minimum.

which exactly means that $F \subseteq \mathrm{N}f$. In particular, Afriat Theorem solves 1.1 in the case of an operator F with finite domain. We are actually trying to prove a slightly weaker version of Afriat Theorem but for infinite bundles. Of course, asking for concavity is too much in this case, but quasi-concavity is reasonable.

Relation with optimal transport

Classical optimal transport consists in the following minimization problem, given two compactly probability measures μ, ν over $\mathbb{R}^d$,

$$\inf \quad \left\{ \int_{\mathbb{R}^d \times \mathbb{R}^d} c(x, y)\, \mathrm{d}\gamma(x, y) : \gamma \in \Pi(\mu, \nu) \right\},$$

where γ is the set of probability measures on $\mathbb{R}^d \times \mathbb{R}^d$ having (μ, ν) as marginals, and $c : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ is a given cost function, typically $c(x, y) = |y - x|^2$ (see for instance [San15]). When c is continuous, it may be shown that γ is optimal if and only if $\operatorname{spt} \gamma$ is c-cyclically monotone, i.e. for every $(x_0, y_0), \ldots, (x_N, y_N)$ with $(x_N, y_N) = (x_0, y_0)$,

$$\sum_{i=0}^{N-1} c(x_i, y_i) \leq \sum_{i=0}^{N-1} c(x_i, y_{i+1}).$$

The first relation between optimal transport and our problem is through the revealed preference theory discussed above. Indeed, when $c(x, y) = \log(x \cdot y)$ and μ, ν as supported on $\mathbb{R}^d_+$, it turns out that c-cyclical monotonicity corresponds to yet another axiom or revelead preference, the *homogeneous axiom of revealed preference* (HARP), in which case finite data satisfying (HARP) can be rationalized by a positive homogeneous utility function, as shown in [KKN13]

The second link between our problem and optimal transport lies in L^∞ optimal transport, consisting in replacing the integral functional by a supremal cost:

$$\inf \quad \left\{ \gamma - \operatorname{ess\,sup}_{(x,y)} c(x, y) : \gamma \in \Pi(\mu, \nu) \right\}.$$

The problem has been thoroughly studied in [CDJ08] in the case of $c(x, y) = |y - x|$ (or any power of it, since the problem is unchanged). Notice that in the classical optimal transport problem, taking the quadratic cost $c(x, y) = |y - x|^2$ or $c(x, y) = -x \cdot y$ is equivalent, but this is no longer the case in the L^∞ framework. Besides, in L^∞ theory, c-cyclical monotonicity is replaced with $c\infty$-cyclical monotonicity:

$$\max_{0 \leq i < N} c(x_i, y_i) \leq \max_{0 \leq i < N} c(x_i, y_{i+1}).$$

It is known that $c\infty$-cyclical monotonicity of $\operatorname{spt} \gamma$ no longer characterizes general solutions of L^∞ optimal transport, but characterizes special solutions known as *restrictable solutions* (at least when c is the Euclidean distance). When $c(x, y) = -x \cdot y$, $c\infty$-cyclical monotonicity rewrites as

$$\min_{0 \leq i < N} x_i \cdot y_i \geq \min_{0 \leq i < N} x_i \cdot y_{i+1} = \min_{0 \leq i < N} (x_i \cdot y_i + x_i \cdot (y_{i+1} - y_i),$$

thus it implies

$$\min_{0\le i<N} x_i\cdot(y_{i+1}-y_i)\le 0,$$

which *exactly matches* the notion of cyclical quasi-monotonicity. Recovering a potential for cyclically quasi-monotone (multi-)maps appears then as an important step towards recovering some kind of replacement for Kantorovich potentials, in the context of L^∞ optimal transport with $c(x,y)=-x\cdot y$. This L^∞ optimal transport problem may have some application, since it corresponds to a limit Value-at-Risk functional VaR_α when $\alpha\to 0^+$ with correlation cost, i.e. to the minimization of $\mathrm{VaR}_0(X\cdot Y)$ among random variables (X,Y) with fixed marginals. This correlation cost with fixed marginals is considered in particular in [Rüs83].

Main results, strategy and structure of the paper

Our main contribution in this article is the resolution of Problem 1.1 when $F:\mathbb{R}^d\to\mathbb{R}^d$ is a map (single-valued) which is of class $\mathscr{C}^1$ and never vanishes, as stated in Theorem 4.5. The proof consists in the following strategy: build from F a preference relation that has convex and closed sub-level sets, then a lower semi-continuous quasi-convex functions with these exact levels. In all generality, the relation we define is a pre-order and not a preference relation, since it is not total. Nevertheless, we also provide a solution under no extra assumption in dimension $d=1$, as well as a conditional result, stating that the problem is solved when the relation is a preference relation. Finally, we try to illustrate the difficulties ahead for the general case with a series of examples.

The article is organized as follows: in Section 2 we show the identification between classes of potentials, preference relations and set families, thus recovering a potential from a preference relation, and we define our candidate preference relation for a given cyclically quasi-monotone maps ; Section 3 is devoted to the 1-dimensional case ; Section 4 to the d-dimensional case with the $\mathscr{C}^1$ regularity and non-vanishing conditions ; the article ends with Section 5 providing many examples.

2 From cyclically quasi-monotone maps to quasi-convex potentials

A *multi-valued map* (or *multi-map*) $F:\mathbb{R}^d\multimap\mathbb{R}^d$ is a map from $\mathbb{R}^d$ to $2^{\mathbb{R}^d}$. Its *domain* consists of all points x such that $F(x)\neq\emptyset$. In our work we shall always consider maps with *full domain*, i.e. whose domain is $\mathbb{R}^d$. Moreover, we shall, from time to time, assimilate F with its graph $\{(x,y)\in\mathbb{R}^d\times\mathbb{R}^d : y\in F(x)\}$, thus writing $(x,y)\in F$ instead of $y\in F(x)$.

Definition 2.1 (Paths, cycles, ascending paths and cycles)**.** A *discrete path* on $\mathbb{R}^d$ is a finite sequence of points $x_0,\dots,x_N\in\mathbb{R}^d$. Such a path is called a *cycle* if $x_0=x_N$, and it is said *F-ascending*[3] if for every $0\le i<N$ there exists $p_i\in F(x_i)$ such that $(x_{i+1}-x_i)\cdot p_i>0$.

[3] Or just ascending when there is no confusion.

2.1 Cyclically quasi-monotone maps and quasi-convex potentials

Let us reformulate the definition of cyclic quasi-monotonicity, given in the introduction, with the notion of F-ascending cycle.

Definition 2.2 (Cyclically quasi-monotone maps)**.** A multi-valued map $F : \mathbb{R}^d \multimap \mathbb{R}^d$ with full domain is *cyclically quasi-monotone* if there is no F-ascending cycle. In other words, for every cycle $x_0, \dots, x_N \in \mathbb{R}^d$ (where $x_N = x_0$), and every $p_0 \in F(x_0), \dots, p_{N-1} \in F(x_{N-1})$ it holds:

$$\min_{0 \le i < N} (x_{i+1} - x_i) \cdot p_i \le 0.$$

Remark 2.3. If $F : \mathbb{R}^d \multimap \mathbb{R}^d$ is cyclically quasi-monotone, then so are:

- its closure, $\operatorname{cl} F$, defined as its minimal closed extension (as a graph) ;
- its convex cone envelope, $\operatorname{cone} F$, defined as its minimal extension whose image at every point is a convex cone, or equivalently

$$(\operatorname{cone} F)(x) = \Big\{ \sum_{0 \le i < N} \lambda_i p_i : \forall i, \lambda_i \ge 0 \text{ and } p_i \in F(x) \Big\} \quad (\forall x);$$

- its closed convex cone envelope, $\overline{\operatorname{cone}}\, F$, defined as the minimal extension which is closed and whose image at every point is a (necessarily closed) convex cone;
- $\mathbf{1}_M(x) F(x)$ where $\mathbf{1}_M$ is the characteristic function (value in $\{0, 1\}$) of a subset $M \subset \mathbb{R}^d$.

As said in the introduction, it turns out that cyclically quasi-monotone maps are closely related *quasi-convex* functions, whose definition we now recall.

Definition 2.4 (Quasi-convex function and normal cone multi-map)**.** A function $f : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ is *quasi-convex* if its sub-level sets $\{x : f(x) \le \ell\}$ are convex for every $\ell \in \mathbb{R}$. The *normal cone multi-map* $\mathrm{N} f : \mathbb{R}^d \multimap \mathbb{R}^d$ of such a function f is defined by

$$\forall x \in \mathbb{R}^d, \qquad \mathrm{N} f(x) = N_{C_f(x)}(x)$$

where $N_C(x) \coloneqq \partial \chi_C(x)$ is the normal cone to the convex set C at x, and $C_f(x) \coloneqq \{w \in \mathbb{R}^d : f(w) \le f(x)\}$. Equivalently, for every $x \in \mathbb{R}^d$,

$$\mathrm{N} f(x) = \{p \in \mathbb{R}^d : \forall w, f(w) \le f(x) \implies p \cdot (w - x) \le 0\}. \tag{2.1}$$

In the following proposition, we show that quasi-convex functions induce cyclically quasi-monotone operators through their normal cone operator.

Proposition 2.5. *Let $F : \mathbb{R}^d \multimap \mathbb{R}^d$ be a multi-map included in the normal cone operator of some quasi-convex function $f : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$, namely*

$$\forall x \in \mathbb{R}^d, \quad F(x) \subset N_f(x).$$

Then F is cyclically quasi-monotone.

Proof. Let us consider $(x_0, p_0), \dots (x_N, p_N)$ an $\mathrm{N}f$-ascending path. Let $0 \le i < N$. Since $p_i \cdot (x_{i+1} - x_i) > 0$ and $p_i \in \mathrm{N}f(x_i)$, by (2.1) we have:

$$f(x_{i+1}) > f(x_i).$$

As a consequence $f(x_N) > f(x_0)$ thus $x_0 \neq x_N$ and there is no $\mathrm{N}f$-ascending cycle. $\square$

Our goal is to tackle Problem 1.1, i.e. to show that the normal cone multi-maps associated with quasi-convex functions are, in some sense, the *only* cyclically quasi-monotone multi-maps, at least under some assumptions. As explained in the introduction, let us explain that can restrict ourselves to lower semi-continuous potentials that only take finite values.

Remark 2.6 (Finiteness and lower semi-continuity of potentials). First of all, by composing f with a strictly increasing map from $\mathbb{R} \cup \{+\infty\}$ to $\mathbb{R}$ (for example $\arctan(t)$), we can assume f to be valued in $\mathbb{R}$.

Moreover, notice that in Definition 2.4, $C_f(x)$ may be replaced with its closure because

$$p \in \partial\chi_{C_f(x)}(x) \iff p \in \partial\chi_{\overline{C_f(x)}}(x),$$

thus one is tempted to look for potentials f which are lower semi-continuous, ensuring that $C_f(x)$ is closed for every $x \in \mathbb{R}^d$. Indeed, this will be possible, since by Corollary 2.13 for any given quasi-convex function f we can find a lower semi-continuous function $\tilde{f}$ such that for every $x \in \mathbb{R}^d$, $C_{\tilde{f}}(x) = \overline{C_f(x)}$, which implies that $\mathrm{N}f(x) = \mathrm{N}\tilde{f}(x)$.

2.2 Potentials, preference relations and set families

Before tackling Problem 1.1, i.e. the question of existence of a (finite and lower semi-continuous) quasi-convex potential f, what about uniqueness? Notice that if there is a solution f, then $\phi \circ f$ is also a potential of F whenever $\phi : \mathbb{R} \to \mathbb{R}$ is a strictly increasing function, thus there is no uniqueness. What matters is actually the binary relation $\preceq_f$ induced by f, defined by

$$x \preceq_f y \iff f(x) \le f(x),$$

which remains unchanged after composition with ϕ. It is a quasi-convex and lower semi-continuous *preference relation*, a notion which we shall define just after recalling some useful vocabulary on binary relations.

Definition 2.7 (Vocabulary on binary relations). A binary relation $\mathcal{R}$ on a set X is

- *reflexive* if $x \mathrel{\mathcal{R}} x$ is true for every $x \in X$,
- *irreflexive* if $x \mathrel{\mathcal{R}} x$ is false for every $x \in X$,
- *asymmetric* if $x \mathrel{\mathcal{R}} y$ and $x \mathrel{\mathcal{R}} y$ cannot be simultaneously true, for every $x, y \in X$,
- *transitive* if $x \mathrel{\mathcal{R}} y$ and $y \mathrel{\mathcal{R}} z$ implies $x \mathrel{\mathcal{R}} z$ for every $x, y, z \in X$,
- *total* (or *connected*) if either $x \mathrel{\mathcal{R}} y$ or $y \mathrel{\mathcal{R}} x$ is true, for every $x, y \in X$.

Definition 2.8 (Pre-order and preference relation)**.** A binary relation $\preceq$ is a *pre-order* if it is reflexive and transitive, and it is a *preference relation* if it is a total pre-order. The pre-order $\preceq$ is said

- *lower semi-continuous* if for every x, $\{w : w \preceq x\}$ is closed,
- *quasi-convex* if for every x, $\{w : w \preceq x\}$ is convex.

All the properties of the preference relation $\preceq$ can be equivalently expressed in terms of the family (indexed by $\mathbb{R}^d$) of subsets

$$\mathcal{C}_{\preceq} := (C(x))_{x \in \mathbb{R}^d} \quad \text{where} \quad C(x) = \{w : w \preceq x\}$$

for every $x \in \mathbb{R}^d$. In order to build a quasi-convex potential for F (up to strictly increasing composition), we may therefore try to build three different types of objects: (quotiented) potentials, preference relations or set families, as defined below.

Definition 2.9 (Spaces of quotiented potentials, preference relations and set families)**.** We denote:

F_d the set of quasi-convex lsc functions $f : \mathbb{R}^d \to \mathbb{R}$ quotiented by the relation $\sim$ defined by $f \sim g \iff f = \phi \circ g$ for some strictly increasing function $\phi : g(\mathbb{R}^d) \to \mathbb{R}$;

P_d the set of lower semi-continuous quasi-convex preference relations over $\mathbb{R}^d$;

C_d the set of families $\mathcal{C} = (C(x))_{x \in \mathbb{R}^d}$ of totally ordered[4] closed convex subsets of $\mathbb{R}^d$ which satisfy $x \in C(x)$ and $(w \in C(x) \implies C(w) \subseteq C(x))$ for every $x, w \in \mathbb{R}^d$.

We shall prove that these three sets are in bijection, through the canonical maps:

$$\begin{array}{rccc} \Phi : & \mathsf{F}_d & \longrightarrow & \mathsf{P}_d \\ & f/\sim & \longmapsto & \preceq_f \end{array} \quad \text{and} \quad \begin{array}{rccl} \Psi : & \mathsf{P}_d & \longrightarrow & \mathsf{C}_d \\ & \preceq & \longmapsto & \mathcal{C}_{\preceq} = (\{w : w \preceq x\})_{x \in \mathbb{R}^d} \end{array}, \tag{2.2}$$

the map Φ being well-defined because two equivalent functions produce the same preference relation, as already noticed.

The only difficulty in showing that they are bijective is to inverse Φ, more precisely to build a lower semi-continuous quasi-convex function f from a quasi-convex lower semi-continuous relation $\preceq$ that satisfies $\preceq = \preceq_f$. To obtain a lower semi-continuous function f, we shall use the following lemma, providing a "sublevel-stable" lower semi-continuous envelope (sublsc envelope for short).

Lemma 2.10 (Sublevel-stable lower semi-continuous envelope)**.** *Let $g : \mathbb{R}^d \to \mathbb{R}$. If $\{g \le g(x)\}$ is closed for every $x \in \mathbb{R}^d$, then the function $\hat{g} : \mathbb{R}^d \to \mathbb{R}$ defined by*

$$\hat{g}(x) = \inf \left\{ \liminf_{n \to +\infty} g(x'_n) : x'_n \to x' \text{ and } g(x') = g(x) \right\} \quad (\forall x \in \mathbb{R}^d) \tag{2.3}$$

is lower semi-continuous and satisfies

$$\{g \le g(x)\} = \{\hat{g} \le \hat{g}(x)\} \quad (\forall x \in \mathbb{R}^d). \tag{2.4}$$

[4] For the inclusion.

Proof. It is immediate to see that

$$\hat{g} \leq g \tag{2.5}$$

by taking $x' = x$ and $x'_n = x$ for every n in the definition of $\hat{g}(x)$. Denoting by $\bar{g}$ the lower semi-continuous envelope of g, (2.3) may be rewritten as

$$\hat{g}(x) = \inf\{\bar{g}(x') : g(x) = g(x')\} \tag{2.6}$$

and thus

$$\hat{g}(x) = \hat{g}(x') \quad \text{whenever} \quad g(x) = g(x'). \tag{2.7}$$

We first prove (2.4). We shall successively prove the following, for every $x, y \in \mathbb{R}^d$:

$$g(x) < g(y) \implies g(x) \leq \hat{g}(y) \tag{2.8}$$

$$g(x) \leq g(y) \implies \hat{g}(x) \leq \hat{g}(y) \tag{2.9}$$

$$\hat{g}(x) \leq \hat{g}(y) \implies g(x) \leq g(y). \tag{2.10}$$

Obviously (2.9) and (2.10) yield (2.4), while (2.5), (2.7) and (2.8) imply (2.9). To alleviate notation, we shall denote $C(x) = \{g \leq g(x)\}$ for every $x \in \mathbb{R}^d$.

Let us show (2.8). Consider $x, y \in \mathbb{R}^d$ such that $g(x) < g(y)$. Take a point y' competitor in the definition of $\hat{g}(y)$, so such that $g(y') = g(y)$ and consider a sequence (y'_n) converging to y'. We know that $y' \notin C(x)$ otherwise we would have $g(y) = g(y') \leq g(x)$: a contradiction of $g(x) < g(y)$. Since $C(x)$ is closed, $y'_n \notin C(x)$ for n large enough, thus $g(y'_n) > g(x)$, and taking the inferior limit yields:

$$\liminf_{n \to +\infty} g(y'_n) \geq g(x).$$

This holds for any $y'_n \to y'$ such that $g(y') = g(y)$, hence we get $\hat{g}(y) \geq g(x)$, and (2.8) is proved.

Let us show (2.10) by contradiction, assuming that there exists x, y such that $\hat{g}(x) \leq \hat{g}(y)$ but $g(x) > g(y)$. By (2.8) we have

$$\hat{g}(y) \leq g(y) \leq \hat{g}(x) \implies \hat{g}(x) = \hat{g}(y) = g(y) < g(x). \tag{2.11}$$

We distinguish two cases.

Case 1: there exists z such that $g(y) < g(z) < g(x)$. In that case $g(z) \leq \hat{g}(x)$ by (2.8), hence

$$\hat{g}(y) = g(y) < g(z) \leq \hat{g}(x),$$

a contradiction with $\hat{g}(y) = \hat{g}(x)$ in (2.11).

Case 2: $\forall z, g(y) \leq g(z) \leq g(x) \implies g(z) = g(y)$ **or** $g(z) = g(x)$**.** Consider $x'_n \to x'$ such that $g(x') = g(x)$ as candidate in the definition of $\hat{g}(x)$. We may assume without loss of generality that $g(x'_n) \leq g(x')$ (otherwise we replace x'_n with x' for all indices n such that $g(x'_n) > g(x')$). Since $g(y) < g(x) = g(x')$, necessarily $x' \notin C(y)$ thus $x'_n \notin C(y)$ for n large enough because $C(y)$ is closed, which yields

$$g(x) = g(x') \geq g(x'_n) > g(y).$$

In the present case, it implies that $g(x'_n)$ is equal either to $g(y)$ or to $g(x)$, thus necessarily to $g(x)$: $g(x'_n) = g(x)$ for every n large enough. This is true for every candidate sequence (x'_n) in the definition of $\hat{g}(x)$ thus $\hat{g}(x) = g(x)$: a contradiction with (2.11).

In both cases we have found a contradiction, hence (2.10) holds true, which concludes the proof of (2.4).

Finally, we show that $\hat{g}$ is lower semi-continuous. Let $(x_k)_{k\in\mathbb{N}}$ be a sequence converging to some x, and up to taking a subsequence, assume that $\liminf_{k\to+\infty} \hat{g}(x_k) = \lim_{k\to+\infty} \hat{g}(x_k)$. We want to show that $\hat{g}(x) \leq \lim_{k\to+\infty} \hat{g}(x_k)$. Notice that, thanks to (2.4), $\hat{g}(x) \leq \hat{g}(x_k)$ whenever $g(x) \leq g(x_k)$, thus we may without loss of generality remove all the corresponding indices k from the sequence $(x_k)_{k\in\mathbb{N}}$ (assuming there remains infinitely many), so that for every k,

$$\hat{g}(x_k) < \hat{g}(x), \quad \text{or equivalently,} \quad g(x_k) < g(x). \tag{2.12}$$

Moreover, one can extract a subsequence such that the real sequence $(\hat{g}(x_k))_{k\in\mathbb{N}}$ is monotone, and $(g(x_k))_{k\in\mathbb{N}}$ has the same monotonicity. Let us justify that $(\hat{g}(x_k))_{k\in\mathbb{N}}$ must be nondecreasing and eventually increasing. If not, it would eventually be nonincreasing. Thus for large k, $g(x_k) \leq g(x_N)$ for some fixed N, i.e. $x_k \in C(x_N)$, and since $C(x_N)$ is closed, $x = \lim_{k\to+\infty} x_k \in C(x_N)$, which means that $g(x) \leq g(x_N)$: a contradiction with (2.12). Thus, up to subsequence $(\hat{g}(x_k))_{k\in\mathbb{N}}$ is increasing. For every k, since $g(x_k) < g(x_{k+1})$ we know by (2.8) that $g(x_k) \leq \hat{g}(x_{k+1})$, thus

$$\liminf_{k\to+\infty} \hat{g}(x_k) = \liminf_{k\to+\infty} \hat{g}(x_{k+1}) \geq \liminf_{k\to+\infty} g(x_k) \geq \liminf_{k\to+\infty} \bar{g}(x_k) \geq \bar{g}(x) \geq \hat{g}(x),$$

where we have used (2.6) in the last inequality. □

Proposition 2.11 ($\mathsf{F}_d \simeq \mathsf{P}_d \simeq \mathsf{C}_d$)**.** *The functions $\Phi : \mathsf{F}_d \to \mathsf{P}_d$ and $\Psi : \mathsf{P}_d \to \mathsf{C}_d$ defined in (2.2) are bijective.*

Moreover, for every $\mathcal{C} = (C(x))_{x\in\mathbb{R}^d} \in \mathsf{C}_d$,

$$\Psi^{-1}(\mathcal{C}) \text{ is the relation } \preceq \text{ defined by } (w \preceq x \iff w \in C(x)), \tag{2.13}$$

and

$$(\Psi \circ \Phi)^{-1}(\mathcal{C}) = \hat{g}/\!\sim, \tag{2.14}$$

where $\hat{g}$ is, for instance, the sublsc envelope (defined in (2.3) of Lemma 2.10) of the real-valued quasi-convex function g given by

$$g(x) = \dim C(x) + \frac{1}{(2\pi)^{\dim C(x)/2}} \int_{C(x)} e^{-|w|^2/2} \, \mathrm{d}\mathscr{H}^{\dim C(x)}(w), \tag{2.15}$$

for every $x \in \mathbb{R}^d$.

Remark 2.12. We shall see that the function g defined in (2.15) satisfies $C(x) = \{g \le g(x)\}$ for every $x \in \mathbb{R}^d$, which implies that g is quasi-convex, but it is not necessarily lsc. Indeed, although all sublevel sets of the form $\{g \le g(x)\}$ are closed, it may happen that for some levels $\ell \notin g(\mathbb{R}^d)$, $\{g \le \ell\}$ is not closed. Replacing g with its lsc envelope $\bar{g}$ would not work since in general $\{g \le g(x)\} \ne \{\bar{g} \le \bar{g}(x)\}$. This is why we introduced the sublevel-stable lsc envelope $\hat{g}$ in Lemma 2.10.

Proof. First of all, it is straightforward to check that Ψ is bijective with inverse given by (2.13), thus it suffices to show that $\Theta := \Psi \circ \Phi$ is bijective with inverse given by (2.14).

Let $\mathcal{C} = (C(x))_{x\in\mathbb{R}^d} \in \mathsf{C}_d$ and $\preceq := \Psi^{-1}(\mathcal{C}) \in \mathsf{P}_d$ the associated preference relation. Let us show that the function g defined in (2.15) satisfies

$$C(x) = \{g \le g(x)\} := C_g(x) \quad (\forall x \in \mathbb{R}^d). \tag{2.16}$$

This will follow from the fact that g is strictly increasing w.r.t. the family $\mathcal{C}$, in the sense that $g(x') \le g(x)$ whenever $C(x') \subseteq C(x)$ (this part is obvious from the expression of g) and that $g(x') = g(x) \implies C(x') = C(x)$.

Let us prove the latter implication. Assume that $g(x') = g(x)$, and since $\mathcal{C}$ is totally ordered we may assume for example that $C(x') \subseteq C(x)$. If $\dim C(x') < \dim C(x)$ then

$$\begin{aligned}
g(x') &= \dim C(x') + \frac{1}{(2\pi)^{\dim C(x')/2}} \int_{C(x')} e^{-|w|^2/2} \,\mathrm{d}\mathscr{H}^{\dim C(x')}(w) \\
&\le \dim C(x') + 1 \\
&< \dim C(x) + \frac{1}{(2\pi)^{\dim C(x)/2}} \int_{C(x)} e^{-|w|^2/2} \,\mathrm{d}\mathscr{H}^{\dim C(x)}(w) = g(x),
\end{aligned}$$

which is a contradiction. Thus $\dim C(x') = \dim C(x) = k$, and $C(x), C(x')$ are subsets of a k-dimensional affine subspace V_k. If $C(x') \subsetneq C(x)$, since both $C(x), C(x')$ are closed convex sets and closed convex sets are the closure of their relative interior, $\overline{\mathrm{ri}\, C(x)} \setminus C(x') \ne \emptyset$, which implies that $\mathrm{ri}(C(x) \setminus C(x') \ne \emptyset$, thus we can find a small ball $B_{V_k}(y, r)$ in V_k with $r > 0$ such that $B_{V_k}(y, r) \subseteq C(x) \setminus C(x')$. As a consequence

$$\begin{aligned}
g(x) &= k + \int_{C(x)} e^{-|w|^2/2} \,\mathrm{d}\mathscr{H}^k(w) \\
&\ge k + \int_{C(x')} e^{-|w|^2/2} \,\mathrm{d}\mathscr{H}^k(w) + \int_{B_{V_k}(y,r)} e^{-|w|^2/2} \,\mathrm{d}\mathscr{H}^k(w) > g(x'),
\end{aligned}$$

which contradicts $g(x) = g(x')$, thus $C(x') = C(x)$. We have thus proven that g is strictly increasing w.r.t. the family $\mathcal{C}$.

Let us conclude the proof of (2.16). If $x' \in C(x)$ then by definition of C_d we have $C(x') \subseteq C(x)$, thus $g(x') \le g(x)$. Conversely, if $x' \notin C(x)$ then $C(x') \not\subseteq C(x)$ and necessarily (by definition of C_d again) $C(x) \subseteq C(x')$ thus $g(x) \le g(x')$. Since $C(x') \ne C(x)$ we cannot have $g(x) = g(x')$, thus $g(x) < g(x')$ and $x' \notin \{g \le g(x)\}$. We conclude that $C(x) = \{g \le g(x)\}$. This shows in particular that g is quasi-convex.

By Lemma 2.10, $\hat{g}$ is lower semicontinuous and still satisfies

$$C(x) = \{\hat{g} \leq \hat{g}(x)\} \quad (\forall x \in \mathbb{R}^d), \tag{2.17}$$

thus $\hat{g}$ is quasi-convex, $\hat{g}/\!\sim\, \in \mathsf{F}_d$ and $\Theta(\hat{g}/\!\sim) = \mathcal{C}$. In particular Θ is surjective. To conclude that Θ is bijective with inverse given by (2.14), it suffices to show that Θ is injective, thus to show that if two quasi-convex lsc functions g_1, g_2 have the same (spatially-indexed) sublevel families, i.e.

$$C_1(x) := \{g_1 \leq g_1(x)\} = \{g_2 \leq g_2(x)\} =: C_2(x) \quad (\forall x \in \mathbb{R}^d),$$

then $g_1 \sim g_2$. Notice that they must also have the same (spatially-indexed) level families:

$$\{g = g_1(x)\} = C_1(x) \setminus \bigcup_{y \notin C_1(x)} C_1(y) = C_2(x) \setminus \bigcup_{y \notin C_2(x)} C_2(y) = \{g = g_2(x)\}.$$

As a consequence, it is possible to define a function $\phi : g_1(\mathbb{R}^d) \to g_2(\mathbb{R}^d)$ by setting $\ell(g_1(x)) = g_2(x)$ for every $x \in \mathbb{R}^d$. Notice that if $g_1(x') < g_1(x)$ then $C_2(x') = C_1(x') \subsetneq C_1(x) = C_2(x)$ which implies that $g_2(x') < g_2(x)$, so that ϕ is increasing and $g_1 \sim g_2$. $\square$

We get the following immediate corollary.

Corollary 2.13. *(1) If $\mathcal{C} = (C(x))_{x\in\mathbb{R}^d}$ is a family of convex closed sets belonging to C_d, i.e. which satisfy $x \in C(x)$ and ($w \in C(x) \implies C(w) \subseteq C(x)$) for every $x, w \in \mathbb{R}^d$, then there exists a quasi-convex lsc function f such that*

$$C_f(x) := \{f \leq f(x)\} = C(x) \quad (\forall x \in \mathbb{R}^d).$$

(2) If g is a quasi convex function, there exists a quasi-convex lsc function f such that

$$C_f(x) := \{f \leq f(x)\} = \overline{\{g \leq g(x)\}} \quad (\forall x \in \mathbb{R}^d).$$

Proof. The first item is merely a rephrasing of the surjectivity of the map $\Psi \circ \Phi : \mathsf{P}_d \to \mathsf{C}_d$ established in Proposition 2.11, and the second item is a consequence of the first one, applied to the family $\mathcal{C} := (\overline{\{g \leq g(x)\}})_{x\in\mathbb{R}^d} \in \mathsf{C}_d$. $\square$

2.3 From cyclically quasi-monotone maps to preference relations and set families

Given a multi-valued map $F : \mathbb{R}^d \multimap \mathbb{R}^d$ (with full domain) which is cyclically quasi-monotone, we start by defining a binary relation $\prec_F$ as follows:

$$x \prec_F y \iff \text{there is an } F\text{-ascending path from } x \text{ to } y \tag{2.18}$$

i.e.

$$x \prec_F y \iff \text{there exists } (x_0, p_0), \dots, (x_{N-1}, p_{N-1}) \in F, x_N \in \mathbb{R}^d \text{ s.t. } x_0 = x, x_N = y \text{ and } \forall i < N, (x_{i+1} - x_i) \cdot p_i > 0.$$

From this we define the binary relation $\preceq_F$ that will be our candidate preference relation, as

$$x \preceq_F y \iff (\forall w \in \mathbb{R}^d, w \prec_F x \implies w \prec_F y) \tag{2.19}$$

or equivalently

$$x \preceq_F y \iff \{w : w \prec_F x\} \subseteq \{w : w \prec_F y\}. \tag{2.20}$$

Recalling the vocabulary on binary relations given in Definition 2.7, let us state the first properties satisfied by $\prec_F$ and $\preceq_F$.

Proposition 2.14. *Let $F : \mathbb{R}^d \multimap \mathbb{R}^d$ be a cyclically quasi-monotone multi-map.*

(i) The relation $\prec_F$ defined in (2.18) is a strict pre-order*, i.e. an irreflexive and transitive relation. As such, it is also asymmetric.*

(ii) The relation $\preceq_F$ defined in (2.19) is a pre-order, i.e. a reflexive and transitive relation.

(iii) The strict relation is stronger than the large one: for every $x, y \in \mathbb{R}^d$,

$$x \prec_F y \implies x \preceq_F y. \tag{2.21}$$

(iv) The following mixed transitivity relation holds: for every $x, y, z \in \mathbb{R}^d$,

$$x \prec_F y \text{ and } y \preceq_F z \implies x \prec_F z. \tag{2.22}$$

Proof. The mixed transitivity property (iv) holds by definition of $\preceq_F$ as given in (2.19).

The transitivity of the strict relation $\prec_F$ is an immediate consequence of the fact that concatenating an F-ascending path from x to y with an F-ascending path from y to z yields an F-ascending path from x to z. Besides, there exists $x \in \mathbb{R}^d$ such that $x \prec_F x$ if and only if there exists an F-ascending cycle, thus the cyclical quasi-monotonicity of F is equivalent to the irreflexivity of $\prec_F$, and (i) holds true.

As for (ii), it is a consequence of the characterization of $x \preceq_F y$, given in (2.20), by the inclusion $\{\cdot \prec_F x\} \subseteq \{\cdot \prec_F y\}$. Thus, the reflexivity and transitivity of $\preceq_F$ follows from the reflexivity and transitivity of the inclusion relation $\subseteq$.

Finally, take $x \prec_F y$ and $y \preceq_F z$. The latter means that $\{\cdot \prec_F y\} \subseteq \{\cdot \prec_F z\}$, thus $x \prec_F z$, and (iii) is proved. □

Remark 2.15 (On the definition of the pre-order). There is a very natural way to define a strict pre-order from a pre-order $\preceq$, by setting $x \prec y$ if $x \preceq y$ but $y \npreceq x$. Notice that setting $x \prec_{\neq} y$ if $x \preceq y$ but $x \neq y$ would not work for a pre-order which is not an order, since $\prec_{\neq}$ could fail to be transitive. It seems $\prec$ is the only natural choice.

However, doing the converse is more complicated, since one can think of many natural ways to define a pre-order $\preceq$ from a strict pre-order $\prec$, since there are many possibilities to add equivalent points, i.e. points such that $x \preceq y$ and $y \preceq x$. One possibility is to take $x \preceq' y \iff y \nprec x$. When $\prec$ is such that the incomparability relation $\sim$, defined by $x \sim y$ iff neither $x \prec y$ nor $y \prec x$, is transitive – in which case $\prec$ is said to be a *strict partial order* – then $\preceq'$ is a preference relation. Moreover the map $\prec\, \mapsto\, \preceq'$ is a bijection between strict partial orders and preference relations, and is the inverse of the map $\preceq\, \mapsto\, \prec$ defined in the paragraph just above. Nonetheless, when $\prec$ is not a strict partial order, then $\preceq'$ may fail to be transitive, although it is reflexive and total. We decided to trade totality for transitivity: defining, as we did, $x \preceq y$ if $\{\cdot \prec x\} \subseteq \{\cdot \prec y\}$, yields a reflexive and transitive relation, i.e. a pre-order, that may lack totality to be a preference relation. We found it more convenient to study a relation that is transitive and look for totality rather that to study a relation that is total but lacks transitivity. Finally, let us notice that both relations $\preceq$ and $\preceq'$ are equal when $\prec$ is a strict partial order.

We continue with an easy but useful characterization of $\preceq_F$.

Lemma 2.16 (One-point characterization of $\preceq_F$)**.** *Given a cyclically quasi-monotone multi-map $F : \mathbb{R}^d \multimap \mathbb{R}^d$. For every $x, y \in \mathbb{R}^d$, the pre-order $\preceq_F$ defined in (2.19) satisfies:*

$$x \preceq_F y \iff (\forall (w, p_w) \in F, \quad (x - w) \cdot p_w > 0 \implies w \prec_F y). \tag{2.23}$$

Proof. The direct inclusion is straightforward, because if $(w, p_w) \in F$ is such that $(x - w) \cdot p_w > 0$ then (x, w) is an F-ascending path, so that $w \prec_F x$, and the mixed transitivity relation (2.22) gives $w \prec_F y$.

For the converse implication, we assume the right-hand side of (2.23) is satisfied and we consider $x, y \in \mathbb{R}^d$. For every $w \prec_F x$, we take any F-ascending path $w = x_0, \dots x_N = x$, together with $p_0 \in F(x_0), \dots, p_{N-1} \in F(x_{N-1})$ such that $(x_{i+1} - x_i) \cdot p_i > 0$ for every $i < N$. By assumption, $(x - x_{N-1}) \cdot p_{N-1} > 0$ implies that $x_{N-1} \prec_F x$. Thus we have $w = x_0 \prec_F x_1, x_1 \prec_F x_2, \dots, x_{N-1} \prec_F x$, and by transitivity of $\prec_F$ (Proposition 2.14 (i)) we get $w \prec_F y$. We have shown $w \prec_F x \implies w \prec_F y$, which exactly means that $x \preceq_F y$. □

We are now ready to show that $\preceq_F$ is quasi-convex lsc and relate the normal cones of the convex sets belonging to the family

$$\mathcal{C}^F \coloneqq (C^F(x))_{x \in \mathbb{R}^d} \coloneqq \Phi(\preceq_F) \tag{2.24}$$

to the original multi-map F. In other words $y \in C^F(x)$ if $y \preceq_F x$, which means that for all $z \prec_F y$, it holds $z \prec_F x$. In turn, $z \prec_F y$ means that there exist an F-ascending path from z to y.

Proposition 2.17. *Let $F : \mathbb{R}^d \multimap \mathbb{R}^d$ be a cyclically quasi-monotone multi-map and $\preceq_F$ the pre-order defined in (2.19). The following holds:*

(i) the pre-order $\preceq_F$ is quasi-convex and lsc ;

(ii) for every $x \in \mathbb{R}^d$, $F(x) \subseteq N_{C^F(x)}(x)$, $C^F(x)$ being defined in (2.24).

Proof. Let us take $x \in \mathbb{R}^d$. By Lemma 2.16, a point x' satisfies $x' \preceq_F x$ if and only if

$$\forall w \not\preceq_F x, \forall p_w \in F(w), \quad (x' - w) \cdot p_w \leq 0,$$

which means that

$$C^F(x) = \bigcap_{(w,p_w)\in F : w \not\preceq_F x} \{(\cdot - w) \cdot p_w \leq 0, \} \tag{2.25}$$

thus $C^F(x)$ is closed and convex as an intersection of closed half-spaces. We have proved the first item.

For the second item, consider again a point $x \in \mathbb{R}^d$ and take $p_x \in F(x)$. For every $y \in C^F(x)$, $y \preceq_F x$ thus by Proposition 2.14 $x \not\prec_F y$, therefore $(y - x) \cdot p_x$ is necessarily smaller or equal than 0. This is true for every $y \in C^F(x)$ thus $p_x \in N_{C^F(x))}(x)$ by definition of the normal cone. $\square$

Whenever $\preceq_F$ is total Problem 1.1 can be solved, as stated in the following.

Proposition 2.18 (Conditional resolution of Problem 1.1)**.** *Let $F : \mathbb{R}^d \multimap \mathbb{R}^d$ be a cyclically quasi-monotone multi-map. If the pre-order $\preceq_F$ defined in (2.19) is total, then it is a quasi-convex and lsc preference relation, and there exists a lsc quasi-convex function $f : \mathbb{R}^d \to \mathbb{R}$ such that*

$$\forall x \in \mathbb{R}^d, \quad F(x) \subseteq \mathrm{N}f(x).$$

Proof. By Proposition 2.14, if $\preceq_F$ is total then it is a preference relation, and by Proposition 2.17 (i) it is quasi-convex and lsc, thus $\preceq_F \in \mathsf{P}_d$. Since the family $\mathcal{C}^F$ induced by $\preceq_F$ belongs to C_d (see Proposition 2.11), we may apply Corollary 2.13 to find a quasi-convex lsc potential $f : \mathbb{R}^d \to \mathbb{R}$ such that $C_f(x) = C^F(x)$ for every $x \in \mathbb{R}^d$. By Proposition 2.17 (ii), it yields:

$$\forall x \in \mathbb{R}^d, \quad F(x) \subseteq N_{C^F(x)}(x) = N_{C_f(x)}(x) = \mathrm{N}f(x).$$

$\square$

3 The one-dimensional case

In the 1-d case we can give a positive solution of Problem 1.1. Let $F : \mathbb{R} \multimap \mathbb{R} \cup \{+\infty\}$ be a cyclically quasi-monotone map (with full domain). In this section $\preceq$ will always refer to $\preceq_F$ and C_* to the following convex set:

$$C_* := \bigcap_{x\in\mathbb{R}} C^F(x) = \{w \in \mathbb{R} \ : \ w \preceq x \ (\forall x)\}.$$

Notice that if $x \in C_*$ then $C^F(x) = C_*$, and moreover if $x \in \mathring{C}_*$ then $F(x) = \{0\}$ by (ii) of Proposition 2.17. The set C_* may be empty or not, and we will distinguish the two cases later on.

We start with a lemma expressing that in $\mathbb{R}$, in the definition of $\prec$ we need not consider paths made of two points.

Lemma 3.1. *For every $x, y \in \mathbb{R}$, $x \prec y$ if and only if there exists $p \in F(x)$ such that $p \cdot (y - x) > 0$.*

Proof. Assume $x \prec y$, so that there exists a path $x = x_0, \dots, x_N = y$ and reals $p_i \in F(x_i)$ such that $p_i \cdot (x_{i+1} - x_i) > 0$ for every $0 \le i < N$. Assume without loss of generality that $p_0 > 0$. If $p_i > 0$ for every i then $x_0 < x_N$ thus $p_0 \cdot (x_N - x_0) > 0$, while if that is not the case taking the minimal $j > i$ such that $p_j < 0$ yields $\min\{p_0 \cdot (x_j - x_0), p_j \cdot (x_0 - x_j)\} > 0$ which violates the cyclical quasi-monotonicity of F. Thus we are in the first case and $p_0 \cdot (y - x) > 0$. $\square$

Now let us make use of C_*.

Lemma 3.2. *Assume $C_* \neq \emptyset$ and let $\gamma \in C_*$, then for all x with[5] $\gamma < x$,*

$$F(x) \subseteq \mathbb{R}_+,$$

while for $x < \gamma$,

$$F(x) \subseteq \mathbb{R}_-.$$

Proof. Let $x > \alpha$, if $p \in F(x)$ and $p < 0$ then $p \cdot (\alpha - x) > 0$ which implies $x \prec \alpha$ so $\alpha \notin C_*$. The proof of the second part is analogue. $\square$

There are two possible cases, considered in the two following propositions.

Proposition 3.3. *Assume that C_* is a non-empty and compact interval $[\alpha, \beta]$. For all $\alpha < x < x'$ we have $x \prec x'$ if $F(x) \neq \{0\}$, and $x \preceq x'$ otherwise. A symmetric statement holds for $x' < x < \beta$. Besides, the function $f(x) = \left|x - \frac{\alpha+\beta}{2}\right|^q$ for any $q > 0$ is a potential for F, i.e. a solution of Problem 1.1.*

Proof. By Lemma 3.2, if $F(x) \neq \{0\}$ then for every $p \in F(x) \setminus \{0\}$, $p > 0$ so that $p \cdot (x' - x) > 0$ and then $x \prec x'$. Besides, for every y satisfying $y \prec x$, Lemma 3.1 implies that $p_y \cdot (x - y) > 0$ for some $p_y \in F(y)$. If $p_y < 0$ we would have $p_y \cdot (\alpha - y) > p_y \cdot (x - y) > 0$ thus $y \prec \alpha$ which is impossible. Therefore $p_y > 0$ and $p_y \cdot (x' - y) > p_y \cdot (x - y) > 0$, which implies that $y \prec x'$. We have shown $\{\cdot \prec x\} \subseteq \{\cdot \prec x\}$ i.e. that $x \preceq x$.

Now, consider the function f as in the statement. We see that $\mathrm{N}f(x) = \mathbb{R}_+$ for $x > \frac{\alpha+\beta}{2}$, $\mathrm{N}f(x) = \mathbb{R}_-$ for $x < \frac{\alpha+\beta}{2}$ and $\mathrm{N}f(x) = \mathbb{R}$ for $x = \frac{\alpha+\beta}{2}$. Using Lemma 3.2, we get that $F(x) = \{0\}$ on (α, β), $F(x) \subseteq \mathbb{R}_+$ for $x > \frac{\alpha+\beta}{2}$ and $F(x) \subseteq \mathbb{R}_-$ for $x < \frac{\alpha+\beta}{2}$, which yields $F(x) \subseteq \mathrm{N}f(x)$ for every $x \in \mathbb{R}$. $\square$

Remark 3.4. We observe that $\beta = \inf\{x \ : \ F(x) \cap (0, +\infty) \neq \emptyset\}$.

Proposition 3.5. *If C_* is empty or is a closed half-line, then the order $\preceq$ is total, i.e. for all x, $\overline{x} \in \mathbb{R}$ either $x \preceq \overline{x}$ or vice-versa. In this case, one can consider $f(x) = x$ of $f(x) = -x$ depending on the sign of F*

[5] $\le$ and $<$ designate respectively the usual large and strict order on $\mathbb{R}$.

Proof. If C_* is empty, then either $F(x) \geq 0$ for every x or $F(x) \leq 0$ for every x and in at least one point $F(x) \neq 0$. Is $F(x) \equiv 0$ then $C_* = \mathbb{R}$ and $f \equiv 0$. Then we may assume that for some $\tilde{x} \in \mathbb{R}$ we have, for example, $F(\tilde{x}) > 0$. The quasi-cyclical monotonicity, then, implies that $F(\overline{x}) \geq 0$ for all $\tilde{x} \leq \overline{x}$. We now prove that if there exists a $y < \tilde{x}$ such that $F(y) < 0$ then $C_* \neq \emptyset$. In fact, let $\alpha = \sup\{x \;:\; F(x) < 0\}$. By what we just said $\alpha < \tilde{x}$. Let $\beta = \inf\{x \;:\; F(x) > 0\}$ and observe that $\alpha \leq \beta$. The, possibly degenerate, interval $[\alpha, \beta] \subset C_*$.

□

Proposition 3.3 and Proposition 3.5 provide the full solution of Problem 1.1 in 1-d, as stated in the following theorem.

Theorem 3.6. *If $F : \mathbb{R} \multimap \mathbb{R} \cup \{+\infty\}$ be a cyclically quasi-monotone multi-map then there exists a lower semicontinuous quasi-convex function $f : \mathbb{R}^d \to \mathbb{R}$ such that $F \subseteq \mathrm{N}f$.*

4 The regular and non-vanishing case

In this section we will give positive answer to Problem 1.1 in the case of a single-valued operator $F : \mathbb{R}^d \to \mathbb{R}^d$ which is cyclically quasi-monotone and satisfies the following non-vanishing condition:

(NV) $S(x) \neq 0$ for every $x \in \mathbb{R}^d$.

We start with a general lemma providing information on the interior, boundary, and boundedness of the convex sets belonging to the family $\mathcal{C}^F$ generated by F (see (2.24)).

Lemma 4.1. *Let F be a continuous and cyclically quasi-monotone field satisfying (NV). For every $x \in \mathbb{R}^d$, we have:*

(i) $\emptyset \neq \{z : z \prec x\} \subseteq \operatorname{int} C^F(x)$ and $C^F(x) \neq \mathbb{R}^d$,

(ii) for every $y \in \partial C^F(x)$, $N_{C^F(x)}(y) = \operatorname{span}_+ F(y)$, and $C^F(x)$ is of class C^1,

(iii) $C^F(x)$ is unbounded.

Proof. Let us consider $z_t \coloneqq x - tF(x)$ where $t \in (0, 1)$. For t small enough, by continuity of F we have
$$(x - z_t) \cdot F(z_t) = tF(x) \cdot F(z_t) > 0,$$
thus $z_t \prec x$ and $\{z : z \prec x\} \neq \emptyset$. Now take any $z \prec x$: there is an ascending path $z = x_0, x_1, \ldots, x_n = x$, i.e. such that $F(x_i) \cdot (x_{i+1} - x_i) > 0$ for every $i \in \{0, \ldots, n\}$. By continuity of F, there exists a small $\varepsilon > 0$ such that for every $z' \in B_\varepsilon(z)$,
$$(x_1 - z') \cdot S(z') > 0,$$
thus the path $z', x_1, \ldots, x_n = x$ is an ascending path and $z' \prec x$. Finally, since for $t > 0$, $x \prec (x + tF(x))$, $x + tF(x) \notin C^F(x)$ thus $C^F(x) \neq \mathbb{R}^d$. We have proved (i).

To prove (ii), notice first that, by the discussion above, $x \in \partial C^F(x)$ which is, then, not empty. Then take a point $y \in \partial C^F(x)$ and consider a sequence $y_n \notin C(x)$ converging to y as $n \to +\infty$. Necessarily,

$$\forall z \in C^F(x), \quad (z - y_n) \cdot F(y_n) \leq 0$$

otherwise we would have $y_n \prec z$ and since $z \preceq x$ then $y_n \prec x$ which contradicts $y_n \notin C^F(x)$. We take the limit $n \to +\infty$ so as to get

$$(z - y) \cdot F(y) \leq 0. \tag{4.1}$$

Therefore $F(y) \in N_{C^F(x)}(y)$. Let us prove by contradiction that $N_{C^F(x)}(y)$ is a half line at every $y \in C^F(x)$, which will thus show the result. Let $y \in \partial C^F(x)$ be such that there exists two unit normal vectors $\nu, \nu' \in N_{C^F(x)}(y)$ which generate extreme rays. Since C_x has non empty interior, the boundary of $C^F(x)$ is locally the graph of a convex (or concave) function $g : \mathbb{R}^{d-1} \to \mathbb{R}$. By the regularity of convex functions, there exist two sequences of points $(y_n), (y_n') \subseteq \partial C^F(x)$ where there exists unique unit normal vectors ν_n, ν_n' and such that $\nu_n \to \nu$, $\nu_n' \to \nu'$. In that case, by what we just proved and thanks to (NV), for every $n \in \mathbb{N}$ we have

$$\nu_n = \frac{F(y_n)}{|F(y_n)|} \quad \text{and} \quad \nu_n' = \frac{F(y_n')}{|F(y_n')|},$$

and by continuity of F,

$$\frac{F(y)}{|F(y)|} = \lim_{n\to+\infty} \frac{F(y_n)}{|F(y_n)|} = \nu \neq \nu' = \lim_{n\to+\infty} \frac{F(y_n')}{|F(y_n')|} = \frac{F(y)}{|F(y)|},$$

which is a contradiction. This shows that $F/|F|$ is a the unit normal field at the boundary of $C^F(x)$ and since F is if class C^0 then, again writing that $\partial C^F(x)$ is locally the graph of a convex function, we conclude that $C^F(x)$ is of class C^1, concluding the proof of (ii).

Let us show that $C^F(x)$ is unbounded. We argue by contradiction, assuming that it is bounded. In that case the set $C^F(x) = \{z : z \preceq_F x\}$ equipped with $\preceq_F$ is inductive: if a subset V is totally ordered, the set $U := \bigcap_{z\in U} C^F(z)$ is nonempty as the intersection of totally ordered nonempty compact sets, and any $z \in U$ is a minorant of V. Zorn's lemma then guarantees the existence of a minimal element z^* in $C^F(x)$. By (i) the set $\{z : z \prec z^*\} \subseteq E$ is nonempty: a contradiction to the minimality of z^*. □

We proceed by showing that that the convex sets in the family $\mathcal{C}^F$ always have a non-empty pairwise intersection.

Lemma 4.2. *Assume that F is a cyclically quasi-monotone and continuous field that satisfies (NV). For every $x, x' \in \mathbb{R}^d$, $\operatorname{int}(C^F(x)) \cap \operatorname{int}(C^F(x')) \neq \emptyset$.*

Proof. We start by showing that $C^F(x) \cap C^F(x') \neq \emptyset$ for every $x, x' \in \mathbb{R}^d$. We argue by contradiction, assuming that for some $x \neq x'$, $C^F(x) \cap C^F(x') = \emptyset$. We distinguish two cases.

Case 1: $d(C^F(x), C^F(x'))$ is attained. Let $y \in C^F(x), y' \in C^F(x')$ such that

$$d(C^F(x), C^F(x')) = |y - y'| > 0.$$

We know that $y' = p_{C^F(x')}(y)$ and $y = p_{C^F(x)}(y')$, and since by (ii) of Lemma 4.1 F is a normal field to $C^F(x)$ and $C^F(x')$, we have:

$$y' - y \in N_{C^F(x)}(y) = \operatorname{span}_+(F(y)) \quad \text{and} \quad y - y' \in N_{C^F(x')}(y') = \operatorname{span}_+(F(y')).$$

As a consequence $(y' - y) \cdot F(y) > 0$ and $(y - y') \cdot S(y') > 0$: a contradiction the quasi-monotonicity of F.

Case 2: $d(C^F(x), C^F(x'))$ is not attained. We shall use Ekeland's variational principle. Let us define $\Phi = \Phi_1 + \Phi_2$ where $\Phi_1, \Phi_2 : \mathbb{R}^d \times \mathbb{R}^d \to [0, +\infty]$ are given by

$$\Phi_1(Y) = |y - y'|, \qquad \Phi_2(Y) = \chi_{C^F(x) \times C^F(x')}(y, y'),$$

for every $Y = (y, y') \in \mathbb{R}^d \times \mathbb{R}^d$. It is a proper and lower semi-continuous convex functions on $\mathbb{R}^d \times \mathbb{R}^d$ and it is bounded from below (by 0). For every $\varepsilon > 0$ we may apply Ekeland's variational principle to obtain a point $Y_* = (y_*, y'_*)$ which satisfies in particular:

$$\forall Y \neq Y_*, \quad \Phi(Y) + \varepsilon|Y - Y_*| > \Phi(Y_*).$$

In particular, Y_* is a minimizer of the map $\Phi + \Phi_3$ where $\Phi_3 : Y \mapsto \varepsilon|Y - Y_*|$. The subgradients can be easily computed:

$$\begin{aligned} \partial\Phi_1(y, y') &= \left\{ \left(\frac{y - y'}{|y - y'|}, -\frac{y - y'}{|y - y'|} \right) \right\} \qquad &(\forall y \neq y'), \\ \partial\chi_{C^F(x) \times C^F(x')}(y, y') &= \partial\chi_{C^F(x)}(y) \times \partial\chi_{C^F(x')}(y') \qquad &(\forall y, y'), \\ \partial\Phi_3(Y_*) &= \varepsilon\mathbb{S}^{2d-1}. \end{aligned}$$

Besides,

$$\bigcap_{i \in \{1,2,3\}} \operatorname{int}(\operatorname{dom}\Phi_i) = \operatorname{int}(\operatorname{dom}\Phi_2) = \operatorname{int}(C^F(x)) \times \operatorname{int}(C^F(x')) \neq \emptyset,$$

thus the sub-gradient of $\Phi_1 + \Phi_2 + \Phi_3$ is the sum of the sub-gradients of the three functions. Since $C^F(x) \cap C^F(x') = \emptyset$, $y_* \neq y'_*$, and and the first-order optimality condition yields $0 \in \partial\Phi(Y_*)$, i.e. the the existence of $(\nu, \nu') \in \partial\chi_{C^F(x)}(y_*) \times \partial\chi_{C^F(x')}(y'_*)$ and $(u, u') \in \mathbb{S}^{2d-1}$ such that

$$0 = \begin{pmatrix} \dfrac{y_* - y'_*}{|y_* - y'_*|} + \varepsilon u + \nu \\ \dfrac{y'_* - y_*}{|y'_* - y_*|} + \varepsilon u' + \nu' \end{pmatrix}.$$

As a consequence for $\varepsilon < 1/2$,

$$\nu \cdot \frac{y'_* - y_*}{|y'_* - y_*|} = 1 - \varepsilon u \cdot \nu \geq \frac{1}{2} > 0$$

and similarly

$$\nu' \cdot \frac{y_* - y'_*}{|y_* - y'_*|} > 0.$$

Since $\nu \in \operatorname{span}_+ F(y_*)$ and $\nu' \in \operatorname{span}_+ F(y'_*)$, we obtain $y_* \prec y'_* \prec y_*$: a contradiction to the quasi-monotonicity of F. We have thus shown that $C^F(x) \cap C^F(x') \neq \emptyset$ for every $x, x' \in \mathbb{R}^d$.

Now let us prove that they always intersect in their interior. Assume by contradiction that $\operatorname{int}(C^F(x)) \cap \operatorname{int}(C^F(x')) = \emptyset$ for some $x \neq x'$. Necessarily we have $\operatorname{int}(C^F(x)) \cap \partial C^F(x') = \operatorname{int}(C^F(x')) \cap \partial C^F(x) = \emptyset$: otherwise if for example $\operatorname{int}(C^F(x)) \cap \partial C^F(x') \neq \emptyset$, since $C^F(x') = \overline{\operatorname{int}(C^F(x'))}$ we would have $\operatorname{int}(C^F(x)) \cap \operatorname{int}(C^F(x')) \neq \emptyset$. Since $C^F(x) \cap C^F(x') \neq \emptyset$ we must have $\partial C^F(x) \cap \partial C^F(x') \neq \emptyset$. Take a point $y \in \partial C^F(x) \cap \partial C^F(x')$: since the convex sets intersect at y but their interior have empty intersection, they are tangent at y, and because they are of class $\mathscr{C}^1$, by Lemma 4.1, they have opposite normal at y. By the same lemma the normal to both convex sets is positively spanned by $F(y)$, hence a contradiction. $\square$

In order to show that the convex sets of the family $\mathcal{C}^F$ are totally ordered, we need a final perturbation lemma.

Lemma 4.3. *Let $F : \mathbb{R}^d \to \mathbb{R}^d$ be a cyclically quasi-monotone map satisfying (NV) and assume that F is locally Lipschitz continuous. Let $p \in \mathbb{R}^d$, K a compact subset of $\partial C(p)$ and $T > 0$. For every $\varepsilon > 0$, there exists $\delta_0 = \delta_0(\varepsilon, F, K, T) > 0$ satisfying the following property: for every $0 < \delta \leq \delta_0$ and every tuple of points $(x_0, \dots, x_n) \subseteq K$ such that*

$$n\delta \leq 2T \qquad \textit{and} \qquad |x_{i+1} - x_i| = \delta \qquad (\forall i \in \{0, \dots, n-1\}),$$

there exists a perturbed family $(\tilde{x}_0, \dots, \tilde{x}_n) \subseteq \mathbb{R}^d$ such that

(i) $\tilde{x}_0 = x_0$,

(ii) $|\tilde{x}_i - x_i| \leq \varepsilon$ *for every* $i \in \{0, \dots, n\}$,

(iii) $(\tilde{x}_{i+1} - \tilde{x}_i) \cdot F(\tilde{x}_i) > 0$ *for every* $i \in \{0, \dots, n-1\}$.

Proof. Let $\varepsilon > 0$. First of all, let us notice that thanks to (NV), up to replacing F with $S/|S|$, we may assume that $|S(x)| = 1$ for every $x \in \mathbb{R}^d$.

We can write, locally, as in Lemma 4.1 above, $\partial C(p)$ as the graph of a function which is $C^{1,1}$, since in this lemma F is locally Lipschitz. Then we obtain that for $\delta > 0$ small enough and some constant A, it holds:

$$\forall x, y \in K, \quad |y - x| \leq \delta \implies d(y, \operatorname{Tan}_x C^F(p)) \leq A|y - x|^2.$$

Notice that by Lemma 4.1, $\operatorname{Tan}_x C^F(p) = x + F(x)^\perp$ thus this is equivalent to

$$\forall x, y \in K, \quad |y - x| \leq \delta \implies |(y - x) \cdot F(x)| \leq A|y - x|^2. \tag{4.2}$$

Let us take a tuple of points $(x_0, \dots, x_n) \subseteq K$ as in the statement:

$$|x_{i+1} - x_i| = \delta \qquad (\forall i \in \{0, \dots, n-1\}).$$

We define the perturbed tuple $(\tilde{x}_i)_{0 \leq i \leq n}$ as follows:

$$\begin{cases} \tilde{x}_0 = x_0 \\ \tilde{x}_i = x_i + \varepsilon_i F(x_{i-1}) \qquad (0 < i \leq n) \end{cases},$$

for a suitable tuple $(\varepsilon_1, \dots, \varepsilon_n)$ of values in $(0, 1)$ to be chosen later.

Let us estimate from below the scalar product $F(\tilde{x}_i) \cdot (x_{i+1} - x_i)$ for $i = 0$ and $1 < i < n$:

$$\begin{aligned} (\tilde{x}_1 - \tilde{x}_0) \cdot F(\tilde{x}_0) &= (x_1 + \varepsilon_1 F(x_0) - x_0) \cdot F(x_0) \\ &= (x_1 - x_0) \cdot F(x_0) + \varepsilon_1 \geq -A\delta^2 + \varepsilon_1, \end{aligned} \tag{4.3}$$

and

$$\begin{aligned} &(\tilde{x}_{i+1} - \tilde{x}_i) \cdot F(\tilde{x}_i) \\ &= (x_{i+1} - x_i) \cdot F(\tilde{x}_i) + (\varepsilon_{i+1} F(x_i) - \varepsilon_i F(x_{i-1})) \cdot F(\tilde{x}_i) \\ &= (x_{i+1} - x_i) \cdot F(\tilde{x}_i) + (\varepsilon_{i+1} - \varepsilon_i) F(x_i) + \varepsilon_i (F(x_i) - F(x_{i-1})) \cdot F(\tilde{x}_i) \\ &= (x_{i+1} - x_i) \cdot F(x_i) + (x_{i+1} - x_i) \cdot (F(\tilde{x}_i) - F(x_i)) \\ &\quad + (\varepsilon_{i+1} - \varepsilon_i) F(x_i) \cdot F(\tilde{x}_i) + \varepsilon_i (F(x_i) - F(x_{i-1})) \cdot F(\tilde{x}_i). \end{aligned} \tag{4.4}$$

We may lower bound the last four terms of (4.4) using (4.2) (for the first term), the fact that F is Lipschitz on $K + B(0, 1)$ with constant L, so as to obtain:

$$(\tilde{x}_{i+1} - \tilde{x}_i) \cdot F(\tilde{x}_i) \geq -A\delta^2 - 2L\delta\varepsilon_i + (\varepsilon_{i+1} - \varepsilon_i)(1 - L\varepsilon_i). \tag{4.5}$$

To get a positive scalar product in (4.3) and (4.4) we define $(\varepsilon_i)_{1 \leq i \leq n}$ as follows:

$$\begin{cases} \varepsilon_1 = \dfrac{2A}{L}\delta \\ \varepsilon_{i+1} = \left(1 + \dfrac{3L\delta}{1 - ML\delta}\right) \varepsilon_i \qquad (0 < i \leq n) \end{cases}.$$

The constant $M > 0$ will be chosen (large) later on, and δ_0 will also be chosen small enough so as to ensure $ML\delta \leq 1/2$ for every $\delta \leq \delta_0$. Obviously for every i,

$$\begin{aligned} \varepsilon_i \leq \varepsilon_n = \left(1 + \frac{3L\delta}{1 - ML\delta}\right)^n \varepsilon_1 &\leq \exp\left(n \log\left(1 + \frac{3L\delta}{1 - ML\delta}\right)\right) \frac{2A}{L}\delta \\ &\leq \frac{2A}{L} \exp\left(\frac{3LT}{1 - ML\delta_0}\right) \delta, \end{aligned} \tag{4.6}$$

where we have used the concavity of the logarithm and the bound $n\delta \leq 2T$. Now we choose $M > 0$ large enough so that

$$\frac{2A}{L} \exp(6LT) \leq M$$

then $\delta_0 > 0$ small enough so that (4.2) holds for every $0 < \delta \leq \delta_0$ and so that

$$ML\delta_0 \leq 1/2 \qquad \text{and} \qquad \delta_0 M \leq \varepsilon. \tag{4.7}$$

With these choices of M and δ_0 and resuming from (4.6) we have for every i:

$$\varepsilon_i \leq \frac{2A}{L} \exp\left(\frac{3LT}{1 - ML\delta_0}\right)\delta \leq \frac{2A}{L} \exp(6LT)\,\delta \leq M\delta. \tag{4.8}$$

Now from the induction relation satisfied by ε_i we have:

$$\varepsilon_{i+1} = \left(1 + \frac{3L\delta}{1 - ML\delta}\right)\varepsilon_i \geq \left(1 + \frac{3L\delta}{1 - L\varepsilon_i}\right)\varepsilon_i,$$

which is equivalent to the relation

$$(\varepsilon_{i+1} - \varepsilon_i)(1 - L\varepsilon_i) \geq 3L\delta\varepsilon_i$$

and putting this into (4.5) we obtain:

$$\begin{aligned}
(\tilde{x}_{i+1} - \tilde{x}_i) \cdot F(\tilde{x}_i) &\geq -A\delta^2 - 2L\delta\varepsilon_i + +3L\delta\varepsilon_i \\
&\geq -A\delta^2 + L\delta\varepsilon_1 \\
&= -A\delta^2 + 2A\delta^2 \\
&= A\delta^2 > 0.
\end{aligned}$$

We have thus shown (iii). Besides, by (4.8) and (4.7) we have $\varepsilon_i \leq M\delta \leq \varepsilon$ for every $\delta \leq \delta_0$, and (ii) holds true. □

Proposition 4.4. *Let $F : \mathbb{R}^d \to \mathbb{R}^d$ be a locally Lipschitz cyclically quasi-monotone vector field satisfying (NV). The family of convex sets $\mathcal{C}^F$ is totally ordered.*

Proof. Let $x \neq x'$ be two points of $\mathbb{R}^d$. By contradiction, we assume that they are not totally ordered, i.e. $C^F(x) \nsubseteq C^F(x')$ and $C^F(x') \nsubseteq C^F(x)$. By Lemma 4.2, $\mathrm{int}(C^F(x)) \cap \mathrm{int}(C^F(x')) \neq \emptyset$. Let us justify that

$$\partial C^F(x) \cap \mathrm{int}(C^F(x')).$$

Take a point $y \in \mathrm{int}(C^F(x)) \cap \mathrm{int}(C^F(x'))$, and $z \in C^F(x') \setminus C^F(x)$. Define $y_t = (1-t)y + tz$ for $t \in [0,1]$ and $t^* = \sup\{t : y_t \in C^F(x)\}$. Since $C^F(x)$ is closed, $y_{t^*} \in \partial C^F(x)$, and since $y \in \mathrm{int}(C^F(x)), z \notin C^F(x)$ we have $t^* \in (0,1)$. Thus y_{t^*} belongs to the open segment (y,z) with $y \in \mathrm{int}(C^F(x')), z \in C^F(x')$, which implies that $y_{t^*} \in \mathrm{int}(C^F(x'))$. We have thus found a point $x^* := y_{t^*} \in \partial C^F(x) \cap \mathrm{int}(C^F(x'))$.

The boundary $\partial C^F(x)$ is a path-connected manifold of class $\mathscr{C}^1$ which contains x (thanks to Lemma 4.1) and x^*. We may consider a $\mathscr{C}^1$ curve $\gamma : [0,T] \to \partial C^F(x)$ which is parameterized by arc length such that $\gamma(0) = x$ and $\gamma(T) = x^*$. Take $\varepsilon > 0$ small so that $B(x^*, \varepsilon) \subseteq C^F(x')$. We shall choose $\delta > 0$ small enough such that in particular $\delta \leq \min\{L, \varepsilon\}/2$. We start by defining $t_0 = 0, x_0 = \gamma(t_0) = x$, then $t_1 > t_0$ and $x_1 = \gamma(t_1)$ such that $|x_1 - x_0| = \delta$ and continue inductively so as to define $t_0 < \ldots, t_{n-1}$ and $x_0 = \gamma(t_0), \ldots, \gamma(t_{n-1})$ up to a step $n-1$ such that $T - t_{n-1} \leq \delta$. Set finally $t_n = L$ and $x_n = \gamma(t_n)$, so that $|x_n - x_{n-1}| \leq \delta$ because γ is 1-Lipschitz. It is not difficult to show that as $\delta \to 0$

$$\sum_{i=0}^{n-1} |x_{i+1} - x_i| \sim_{\delta \to 0} T$$

but obviously $\sum_{i=0}^{n-1} |x_{i+1} - x_i| \sim n\delta$ thus for δ small enough it holds $n\delta \leq 2T$. Consider δ small enough so that $\delta \leq \delta_0(\varepsilon, \gamma([0,L]), T)$ as given in Lemma 4.3. Applying this lemma yields the existence of a sequence of points $\tilde{x}_0, \ldots, \tilde{x}_n$ satisfying the three items (ii), (ii) and (iii) of Lemma 4.3. In particular $(\tilde{x}_0, \ldots, \tilde{x}_n)$ is a strictly increasing path from $\tilde{x}_0 = x_0 = x$ to $\tilde{x}_n$. Thus $x \prec \tilde{x}_n$. But $\tilde{x}_n \in B(x^*, \varepsilon) \subseteq C(x')$ thus $x_n \preceq x'$ and $x \preceq x'$: a contradiction with the fact that $C^F(x) \nsubseteq C^F(x')$. □

From Proposition 2.18 and Proposition 4.4, recalling that $\mathcal{C}^F$ being totally ordered means that $\preceq_F$ is total, we have solved Problem 1.1 in the case of regular and non-vanishing cyclically quasi-monotone maps, as stated in the following theorem.

Theorem 4.5 (Existence of potentials – regular and non-vanishing case)**.** *Let $F : \mathbb{R}^d \to \mathbb{R}^d$ be a cyclically quasi-monotone map which is of class $\mathscr{C}^1$ and satisfies (NV). There exists a lower semi-continuous quasi-convex function $f : \mathbb{R}^d \to \mathbb{R}$ such that*

$$\forall x \in \mathbb{R}^d, \quad F(x) \in \mathrm{N}f(x).$$

5 Examples

In this section we present several examples of cyclically quasi-monotone fields that we selected to highlight the, sometimes odd, behaviour of the convex sets (and the candidate quasi-convex functions) obtained from our construction, as well as some difficulties that arise in building a quasi-convex function generating such fields. For each example we compute the convex sets induced by the preference relation $\preceq_F$ and we provide one or several quasi-convex functions that generate the field. In most of the examples we will only give a figure with the set C^F so let's briefly summarize how to obtain them. We use essentially two ways. First we recall $\mathcal{C}^F = \{C^F(x)\}_{x \in \mathbb{R}^d}$ and $C^F(x) = \{z \ : \ z \preceq_F x\}$. To obtain the sets, we first compute the set $\{z \ : \ z \prec_F x\}$ building explicitly some F-ascending paths (this could, sometime, be very difficult. In the examples below it is not the case). Notice that, in order to have $z \prec_F x$, it must be at least $F(z) \neq \{0\}$. The $\prec_F$ condition is described in (2.18). The passage from $\prec_F$ to $\preceq_F$ is then described in (2.19), (2.20) but, in particular in *Lemma* 2.16.

Example 5.1 (Hedgehog). We start with an easy example where proposition 4.4 does not apply, but $\mathcal{C}^F$ is totally ordered and so proposition 2.18 still give us a potential. On $\mathbb{R}^d$, $d \geq 1$, we define

$$F(x) := \begin{cases} \operatorname{span}_+(x) & \text{if } x \neq 0, \\ \mathbb{R}^d & \text{if } x = 0. \end{cases}$$

F is cyclically quasi-monotone since it is the normal cone multi-map associated with the euclidean norm. We stress that F does not satisfy the assumptions of proposition 4.4 since it is not single-valued. We could take a single-valued selection of F, but we cannot ask it to be both continuous and non-vanishing, since continuity would imply $F(0) = 0$.

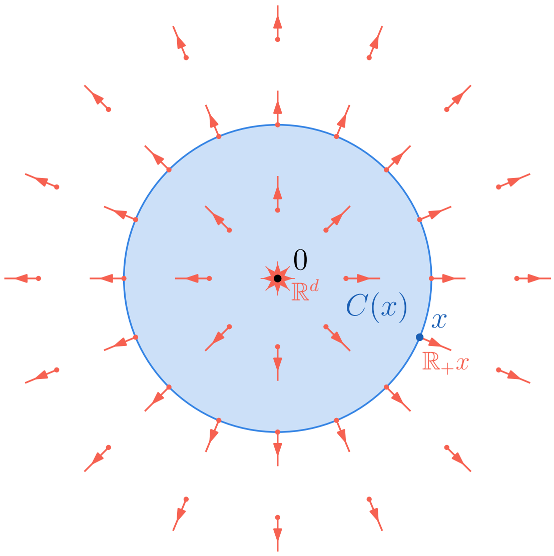

Figure 1: Hedgehog field F of Example 5.1

In dimension $d = 1$, the family $\mathcal{C}^F$ is not totally ordered since $C^F(x) = [0, x]$ when $x \geq 0$, $C^F(x) = [x, 0]$ when $x \neq 0$, but Section 3 shows how to build one (actually several) quasi-convex functions whose normal cone multi-map contains F.

In dimension $d \geq 2$, we can also compute the family of convex sets $\mathcal{C}^F$ given by our construction: for any given x, identify all points $y \prec x$ (the open ball centered at 0 and radius $|x|$), then $C^F(x)$ is the intersection of all half spaces $\{z : (z - y) \cdot p \leq 0\}$ for $y \not\prec x$ and $p \in F(y)$. One may check, in this case, that $C^F(x) = \bar{B}(0, |x|)$ for every $x \in \mathbb{R}^d$, thus the family $\mathcal{C}^F$ is totally ordered and Proposition 2.18 provides the existence of many quasi-convex functions whose normal cone multi-map contains F.

Example 5.2 (Plateau and jump). On $\mathbb{R}^d$, $d \geq 1$, we set for every $x \in \mathbb{R}^d$:

$$f(x) := |x| \wedge ((|x| - 2)_+ + 1) = \begin{cases} |x| & \text{if } |x| \leq 1 \\ 1 & \text{if } 1 < |x| \leq 2 \\ |x| - 1 & \text{if } |x| > 2. \end{cases}$$

then

$$F(x) \coloneqq \mathrm{N} f.$$

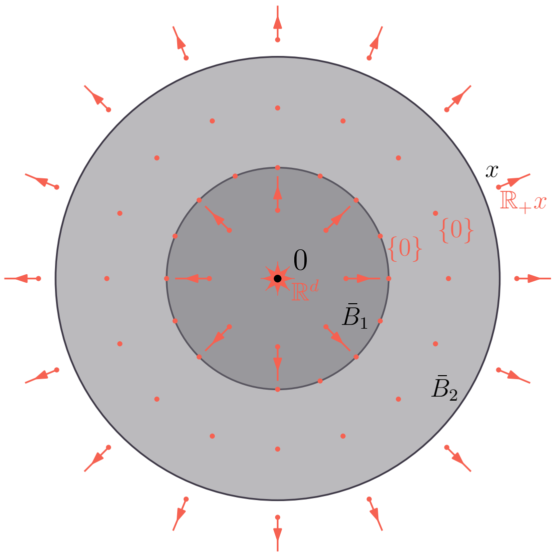

Figure 2: A field with a plateau, F of Example 5.2

By construction f is a continuous quasi-convex function (its sub-level sets being centered balls), that has a plateau at the value 1: $f \equiv 1$ on $\bar{B}_2 \setminus B_1$. Thus F is a cyclically quasi-monotone field such that $F(x) = \{0\}$ on $B_2 \setminus B_1$. More precisely, the sub-level sets C_f are given by

$$C_f(x) = \begin{cases} \bar{B}_{|x|} & \text{if } |x| < 1 \text{ or } 2 \leq |x| \\ \bar{B}_2 & \text{if } 1 \leq |x| < 2, \end{cases}$$

and the field F by

$$F(x) = \begin{cases} \mathbb{R}^d & \text{if } x = 0 \\ \operatorname{span}_+ x & \text{if } 0 < |x| < 1 \text{ or } 2 \leq |x| \\ 0 & \text{if } 1 \leq |x| < 2. \end{cases}$$

Computing the family of convex sets $\mathcal{C}^F(x)$ is not difficult. For any given x, identify all points $y \prec x$, then $C_2(x)$ is the intersection of all half spaces $\{z : (z-y) \cdot p \leq 0\}$ for $y \not\prec x$ and $p \in F(y)$. One may check in such a case that we recover the same family of convex sets:

$$C^F(x) = C_f(x) \qquad (\forall x \in \mathbb{R}^d). \tag{5.1}$$

However, (5.1) is not true anymore for the closure $\bar{F}$ of F, defined as the closure of the graph of F in $\mathbb{R}^d \times \mathbb{R}^d$. Indeed, $F(x) = \bar{F}(x)$ when $|x| \neq 1$, while for $|x| = 1$ the cone is now nontrivial: $F(x) = \operatorname{span}_+ x$. Denoting by $C'(x)$ the convex sets in the family $\mathcal{C}^{\bar{F}}$, we may check that $C(x) = C'(x)$ when $|x| \neq 1$, while $C'(x) = \bar{B}_1$ if $|x| = 1$. For the family

$(C'(x))_{x\in\mathbb{R}^d}$ to be the sub-level sets family of a function $\tilde{f}$ it must be discontinuous on $S(0,1)$. Take for instance:

$$\tilde{f}(x) := \begin{cases} |x| & \text{if } |x| \le 1 \\ 2 & \text{if } 1 < |x| \le 2 \\ |x| & \text{if } |x| > 2. \end{cases}$$

This function is lower semi-continuous but not continuous. Moreover

$$\bar{F}(x) = \mathrm{N}\tilde{f}(x) \quad \text{and} \quad C'(x) = C_{\tilde{f}}(x) \qquad (\forall x \in \mathbb{R}^d). \tag{5.2}$$

This example shows in particular that a field F and its closure $\bar{F}$ may produce, from our construction, different families of totally ordered convex sets. But for both F and $\bar{F}$ there are quasi-convex functions f and $\tilde{f}$ such that $F = \mathrm{N}f, \bar{F} = \mathrm{N}_{\tilde{f}}, \mathcal{C}^F = \mathcal{C}_f, \mathcal{C}^{\bar{F}} = \mathcal{C}_{\tilde{f}}$, although they may not be continuous (as $\tilde{f}$). Notice finally that if we only require the fields to be *included* in the normal field of a quasi-convex function, another solution (for both F and $\bar{F}$) is given by the norm $|\cdot|$, which is continuous.

Example 5.3 (Stadium field). In $\mathbb{R}^2$ let Σ be the segment $0\times[-1,1]$ and let $d_\Sigma : \mathbb{R}^2 \to \mathbb{R}^+$ be the distance from Σ (i.e. $d_\Sigma(x) = dist(x,\Sigma)$).

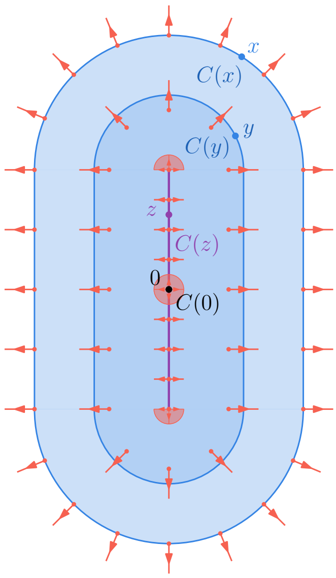

Figure 3: Stadium field of Example 5.3

Consider the multi-valued vector field

$$F(x) := \begin{cases} \mathbb{R}^2 & \text{if } x = 0 \\ \mathbb{R}\times\{0\} & \text{if } x \text{ belongs to the relative interior of } \Sigma \\ \mathbb{R}\times[0,+\infty) & \text{if } x \text{ is the upper extremum of } \Sigma \\ \mathbb{R}\times(-\infty,0] & \text{if } x \text{ is the lower extremum of } \Sigma \\ \nabla d_\Sigma(x) & \text{otherwise.} \end{cases}$$

In this case, the sets $C^F(x)$ are not essential to find a lsc and quasi-convex potential f, however it is easy to see that they are like in the figure ($C^F(0) = 0$ since $0 \prec x$ for every $x \neq 0$ and it is slightly difficult to see it in the figure) and, besides the points of Σ, coincide with the level sets of d_Σ. Computing the sets $C^F(x)$ requires, in this case, F-ascending paths of length at most 2. It is also immediate to see that $F(x) \subset \partial\chi_{C^F(x)}$. As for the potential f, we may consider (see (2.15))

$$f(x) := \begin{cases} 0 & \text{if } x = 0 \\ 3 = (dim(\Sigma) + length(\Sigma)) & \text{if } x \in \Sigma \setminus 0 \\ 2 + \mathcal{L}^2(C(x)) & \text{otherwise.} \end{cases}$$

Example 5.4 ($\frac{3}{4}$ Hedgehog). Here we consider a variant of the example 5.1. The vector field $F : \mathbb{R}^2 \multimap \mathbb{R}^2$

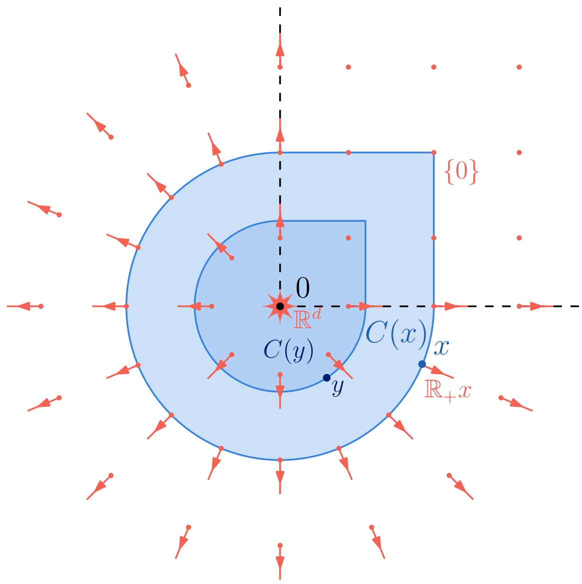

Figure 4: $\frac{3}{4}$ hedgehog field of Example 5.4

$$F(x) := \begin{cases} \text{span}_+(x) & \text{if } x \neq 0 \text{ and } \min\{x_1, x_2\} \leq 0; \\ \mathbb{R}^d & \text{if } x = 0, \\ 0 & \text{otherwise.} \end{cases}$$

In this case, still using F-ascending paths of length at most 2, one can see that the sets $C^F(x)$ are the drop-shaped set in the pictures (with the exception of $C^F(0) = 0$ as in example 5.1). The family $\mathcal{C}^F$ is totally ordered so that proposition 2.18 gives a quasi-convex and lsc potential f. In this case it is slightly more difficult to produce an explicit f and this start to show the necessity of an existence theorem.

Example 5.5 (Single circle). In this example the family $\mathcal{C}^F$ will not be totally ordered. Let $d = 2$ and

$$F(x) = \begin{cases} x & \text{if } |x| = 1; \\ 0 & \text{otherwise.} \end{cases}$$

$F(x) \in \mathcal{N}_{f(x)}$ for several quasi-convex functions f, for example $f(x) = |x|$, and then S is cyclically quasi-monotone. Another example of potential is

$$f_1(x) := \begin{cases} 0 & \text{if } |x| \leq 0; \\ 1 & \text{otherwise.} \end{cases}$$

Notice that in the case of f_1 it holds $\operatorname{span}_+ F(x) = \mathcal{N}_{f_1(x)}$

We can describe the sets $C(x)$ explicitly. If $|x| \leq 1$ then $C^F(x) = \overline{B(0,1)}$ the unitary, closed ball. If $|x| > 1$ then $C^F(x)$ is the sharp-drop like set in the figure. In this case neither Proposition 4.4 nor Proposition 2.18 apply but we explicitly showed two potentials for the vector field.

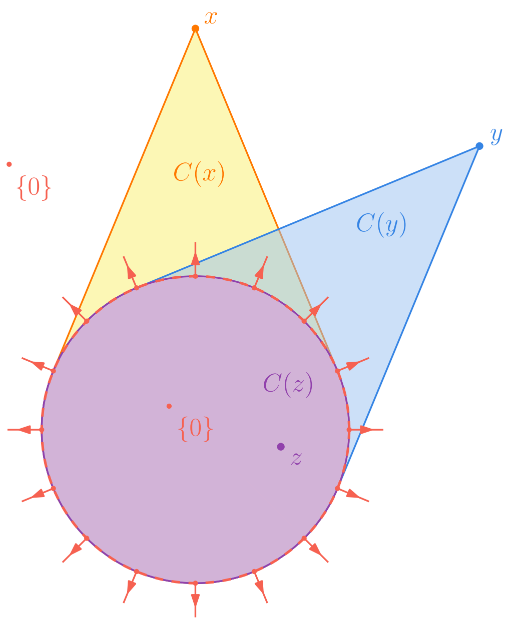

Figure 5: Single circle field of Example 5.5

Example 5.6 ($\frac{1}{2}$ hedgehog). On $\mathbb{R}^d$, $d \geq 1$, we define

$$F(x) := \begin{cases} \operatorname{span}_+(x) & \text{if } x \neq 0 \text{ and } x_1 \leq 0, \\ \chi_{\{x_1 \leq 0\}} \mathbb{R}^d & \text{if } x = 0, \\ 0 & \text{otherwise.} \end{cases}$$

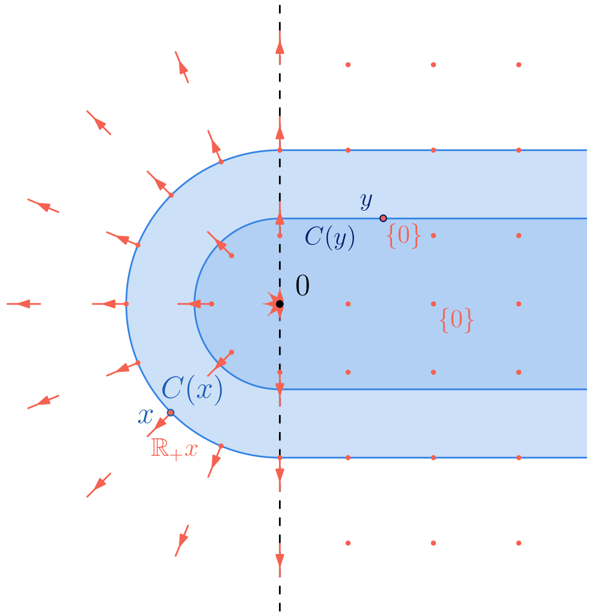

Figure 6: $\frac{1}{2}$ hedgehog field of Example 5.6

In this case, the family $\mathcal{C}^F$ is totally ordered and is depicted in the figure above. Denoting by Σ the half-line $\{x \in \mathbb{R}^d \ : \ x_1 \geq 0\}$ An explicit potential is given by

$$f(x) = d_\Sigma(x).$$

Example 5.7 ($\frac{1}{2}$ norm - $\frac{1}{2}$ constant)*.* We modify the previous example choosing

$$F(x) = (1, 0, \dots, 0), \text{ if } x_1 > 0.$$

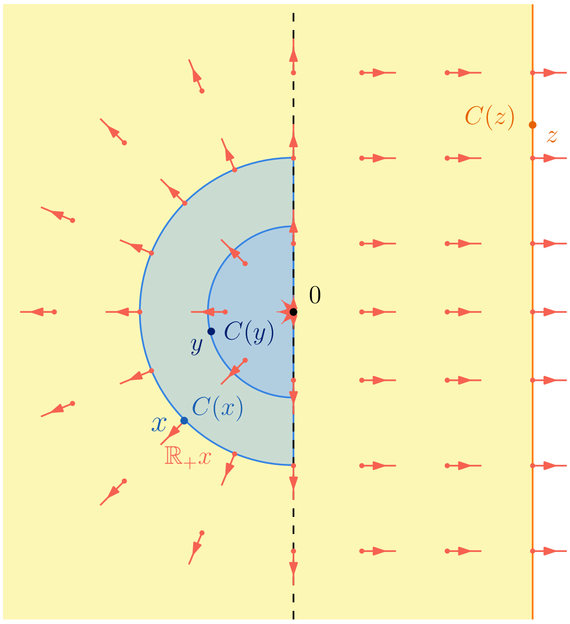

Figure 7: $\frac{1}{2}$ hedgehog - $\frac{1}{2}$ constant field of Example 5.7

Also in this case the family $\mathcal{C}^F$ is totally ordered and it is described as follows (see also the figure). If $\overline{x}_1 \leq 0$ then

$$C^F(\overline{x}) = \overline{B(0, \|\overline{x}\|)} \cap \{x \ : \ x_1 \leq 0\}$$

i.e. a closed ball intersected with "the negative half-space". If $\overline{x}_1 > 0$ then

$$C^F(\overline{x}) = \{x \ : \ x_1 \leq \overline{x}_1\},$$

i.e. half-space.

In this case it is more difficult to give an explicit quasi-convex potential f but we can rely con Proposition 2.18.

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