MAGNETOELASTIC ACTUATED MOTION OF FINE FERROMAGNETIC PARTICLES

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ABSTRACT. Wireless magnetic actuation offers precise control over microscopic devices, yet full planar manipulation of rigid, tethered magnetic particles remains challenging. We introduce a minimal variational model: a permanently magnetized planar ellipse anchored by two linear springs. First, we derive exact geometric conditions under which the springs can be configured so that the ellipse rotates freely without elastic penalty—producing a continuous family of zero-energy equilibria in which the ellipse's center traces a closed loop dictated solely by spring geometry. Next, we incorporate a uniform in-plane magnetic field and prove that the equilibrium magnetization aligns uniformly with the field. In the so-called full-controllability regime—when the spring rest length are long enough—rotating the external field directly prescribes the ellipse's orientation: the particle follows its zero-energy trajectory to maintain magnetic alignment, achieving a global energy minimum. For shorter springs, zero-energy configurations exist over a restricted orientation range; outside this range the ellipse is pinned at the origin. Our results yield exact criteria for planar control in this simplest magnetoelastic setting, offering clear guidelines for the design of microscale actuators and metamaterials.

1. INTRODUCTION

The remote actuation of soft devices without onboard power is crucial for advances in microrobotics, biomedical engineering, and adaptive materials. By embedding magnetic nanoparticles within elastic substrates, one can induce rapid deformations when exposed to external magnetic fields. As a result, magnetic actuation strategies have become widespread in applications ranging from microelectromechanical systems (MEMS) and microfluidics to targeted biomedical interventions [9, 10, 12, 13].

A central concept in these systems is *magnetoelastic coupling*, whereby mechanical deformations and magnetic responses influence one another, enabling both locomotion and sensing at small scales. In soft composites with ferromagnetic inclusions, externally applied fields can induce bending, buckling, and twisting. These magnetoelastic systems have been investigated primarily through continuum approximations and molecular dynamics (MD) simulations, predicting a variety of instabilities including buckling, dilation, and torsion [1, 2, 4, 11]. While these methods clarify many key mechanisms, their reliance on detailed simulations and approximate models often obscures the broader energy landscape and makes it difficult to extract simple, widely applicable design principles.

Here, we investigate a minimal magnetoelastic system: a uniformly magnetized rigid ellipse tethered by linear springs to a fixed frame, subject to a homogeneous in-plane magnetic field. Specifically, the ellipse (semi-major axis a) is attached to two linear springs of rest length L_0 and initial length L anchored at fixed points (cf. Figure 1). A uniform, in-plane external field h_a of fixed magnitude but variable direction ψ serves as the sole control input. As ψ varies, one seeks to guide the ellipse's center $c = (x, y) \in \mathbb{R}^2$ along a continuous closed trajectory while simultaneously adjusting its orientation θ. Remarkably, when $L_0 \geqslant L + 2a$, the system enters a *full-controllability regime*, allowing arbitrary planar positioning and rotation using only the single parameter ψ. Our analysis goes beyond small-deflection, linearized treatments by exploring the full nonlinear energy landscape associated with large deformations and rotations of the tethered ellipse.

Unlike more intricate metamaterials or multi-component microrobots, this simplified variational model reveals rich behavior through the interplay of elastic and magnetic energies alone. By focusing on this paradigmatic system, we aim to gain fundamental insights into the energy landscapes that govern magnetic particles embedded in elastic media.

Outline. The remainder of the paper is organized as follows. In Section 2 we introduce our minimal elastic model of a rigid, permanently magnetized ellipse tethered by two linear springs and derive an explicit expression for its elastic energy (12). Section 3 is devoted to the purely elastic problem: we characterize all zero-energy configurations and prove Theorem 1. In Section 4 we couple the elastic model to a uniform in-plane magnetic field, show that the magnetization remains spatially uniform at equilibrium, and—under the same rest-length hypothesis—demonstrate full planar controllability by rotating the external field (Theorem 2). We also analyze the non–full-controllability regime $L < L_0 < L + 2a$, identifying when the ellipse must "pin" at the origin versus when it can still follow a zero-energy path.

2. The elastic model

What we are going to derive works for general bounded domains in $\mathbb{R}^2$. However, for concreteness, we assume that Ω is an ellipse in $\mathbb{R}^2$ made of ferromagnetic material. We assume (cf. Figure 1) that in the reference configuration Ω is centered at the origin and that its major axis is aligned along the e_1 axis. We denote by $a > 0$ the length of its semi-major axis. The ellipse is elastically connected to two perpendicular walls at $w_1 := -(a + L)e_1$ and $w_2 := (a + L)e_1$ through linear springs, which are free to rotate about their pins. The springs are assumed to be at rest when their length is L_0. Hence, depending on the position of the walls, the initial length of the springs can be in an extension (if $L > L_0$) or compression (if $L < L_0$) state.

Assumption. *In this work, we assume that the initial (reference) length of the springs is in a compression state:*

$$L < L_0. \tag{1}$$

The regime $L \geqslant L_0$ can be investigated as well, but it is degenerate for our purposes as the critical points of $\mathcal{E}$ are isolated.

The state-space of Ω can be parameterized by the parameters $c \in \mathbb{R}^2$ and $\theta \in \mathbb{R}$, which identify the center of the new position of Ω and the angle θ between its semi-major axis (initially in e_1) and e_1. We denote by $\Omega_{c,\theta}$ the state in which Ω is centered at $c \in \mathbb{R}^2$ and rotated by an angle θ. Also, we denote by ℓ_1, ℓ_2 the increments of the springs connected,

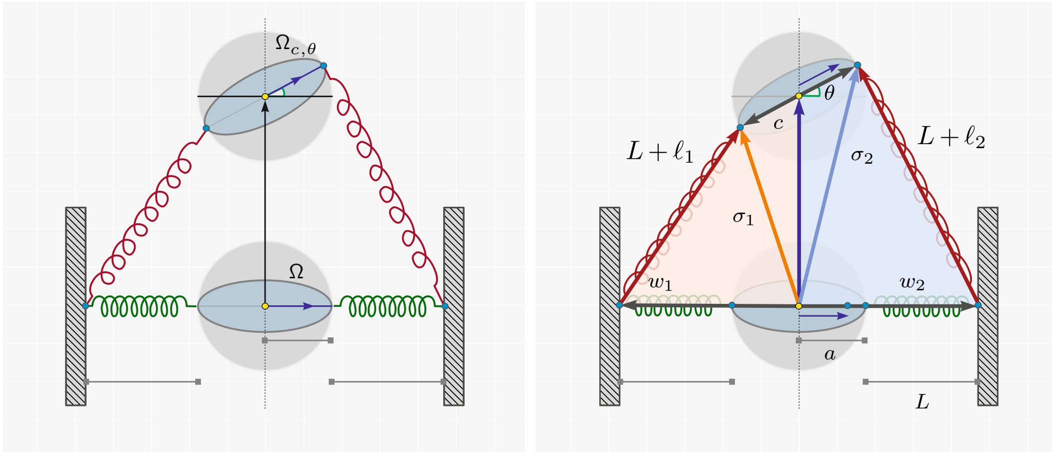

Figure 1. We assume that in the reference configuration Ω is centered at the origin and that its major axis is aligned along the e_1 axis. We denote by $a > 0$ the length of it semi-major axis. The ellipse is elastically connected to two perpendicular walls at $w_1 := -(a+L)e_1$ and $w_2 := (a+L)e_1$ through linear springs that are free to rotate about their pins. The springs are assumed to be at rest when their length is L_0. The state-space of Ω can be parameterized by the parameters $c \in \mathbb{R}^2$ and $\theta \in \mathbb{R}$ which identify the center of the new position of Ω and the angle θ between its semi-major axis (initially in e_1) and e_1.

respectively, to w_1 and w_2, when the magnetic ellipse occupies the region $\Omega_{c,\theta}$. The total elastic energy associated with the configuration $\Omega_{c,\theta}$ then reads as

$$\mathcal{E}(c,\theta) := \frac{1}{2}[(L-L_0)+\ell_1]^2 + \frac{1}{2}[(L-L_0)+\ell_2]^2. \tag{2}$$

To explicitly express the elastic energy $\mathcal{E}$ in terms of the state variables c, θ, we first observe (cf. Figure 1) that in terms of position vectors, the increments $L+\ell_i$ satisfy the relations

$$|L+\ell_1| = |\sigma_1 - w_1|, \tag{3}$$
$$|L+\ell_2| = |\sigma_2 - w_2|, \tag{4}$$

where σ_1, σ_2 are the position vectors of the extremities of the spring attached to $\Omega_{c,\theta}$ with respect to the origin. Simple vector algebra (cf. Figure 1) gives $\sigma_1 = c - aR_\theta e_1$ and $\sigma_2 = c + aR_\theta e_1$. Here, R_θ denotes the $2d$ rotation matrix

$$R_\theta := \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}. \tag{5}$$

Expanding the previous equations, we get

$$|L_0 + (L-L_0)+\ell_1| = |(a+L)e_1 + (c - aR_\theta e_1)|, \tag{6}$$
$$|L_0 + (L-L_0)+\ell_2| = |(a+L)e_1 - (c + aR_\theta e_1)|. \tag{7}$$

To further simplify the expressions (6) and (7), we rely on a classical physical assumption: the impenetrability of matter.

Assumption. *We assume the impenetrability of matter, i.e., that the compression cannot collapse the spring to more than a point. This amounts requiring that* $L+\ell_i \geqslant 0$ *for*

$i = 1, 2$. *Equivalently, we require that*

$$|\sigma_i - w_i| = L + \ell_i \quad (i = 1, 2). \tag{8}$$

Under assumption (8), the previous relations (6) and (7) can be rearranged under the form

$$(L - L_0) + \ell_1 = |(a + L)e_1 + (c - aR_\theta e_1)| - L_0, \tag{9}$$

$$(L - L_0) + \ell_2 = |(a + L)e_1 - (c + aR_\theta e_1)| - L_0. \tag{10}$$

Plugging the previous two relations into (2) we infer that in terms of the state variables c, θ, the total elastic energy reads as

$$\begin{aligned}\mathcal{E}\,(c,\theta) = \frac{1}{2}\left(|(a+L)e_1 + (c - aR_\theta e_1)| - L_0\right)^2 \\ + \frac{1}{2}\left(|(a+L)e_1 - (c + aR_\theta e_1)| - L_0\right)^2.\end{aligned} \tag{11}$$

To shorten notation, we set

$$\mathcal{E}\,(c,\theta) := \frac{1}{2}\left(|c - v_{a,L}(\theta)| - L_0\right)^2 + \frac{1}{2}\left(|c + v_{a,L}(\theta)| - L_0\right)^2, \tag{12}$$

with

$$v_{a,L}(\theta) = aR_\theta e_1 - (a+L)e_1 = a\begin{pmatrix}\cos\theta - (1 + L/a) \\ \sin\theta\end{pmatrix}. \tag{13}$$

Remark 2.1. Also, it is useful to keep in mind the geometric meaning of the vectors $c \pm v_{a,L}(\theta)$. They describe the segments occupied by the springs

$$c - v_{a,L}(\theta) = \sigma_1 - w_1, \tag{14}$$

$$c + v_{a,L}(\theta) = \sigma_2 - w_2. \tag{15}$$

For future reference, it is important to observe the estimates

$$|v_{a,L}(\theta)|^2 = a^2\sin^2\theta + (a(1-\cos\theta) + L)^2 \geqslant L^2 \tag{16}$$

and

$$|v_{a,L}(\theta)|^2 \leqslant a^2 + (a+L)^2 \leqslant 2(a+L)^2, \tag{17}$$

which hold uniformly with respect to $\theta \in \mathbb{R}$.

3. Minimizers of the elastic energy

In this section, we characterize the energy landscape described by the minimizers of the elastic energy (11). As we are going to show, the set of minimizers of $\mathcal{E}$ is degenerate in the sense that it is the image of a curve in $\mathbb{R}^2$.

Theorem 1. *Let*

$$\delta \;:=\; \frac{L_0^2 - L^2}{2\,a\,(a+L)},$$

and define the angular interval

$$I_\delta := \begin{cases}[-\arccos(1-\delta),\ \arccos(1-\delta)], & \textit{if } L \leqslant L_0 \leqslant L + 2a, \\ [-\pi, \pi], & \textit{if } L_0 \geqslant L + 2a.\end{cases} \tag{18}$$

Then the minimal value of the elastic energy $\mathcal{E}$ is zero. Moreover, this minimum is achieved if, and only if, (c, θ) belongs to the set $M \subseteq \mathbb{R}^2 \times [-\pi, \pi]$ generated by the graph of the closed curve

$$\gamma_\delta : \theta \in I_\delta \mapsto (c_1(\theta), c_2(\theta)) \in \mathbb{R}^2 \tag{19}$$

with

$$c_1(\theta) := \frac{c_2(\theta)\sin\theta}{(1 + L/a) - \cos\theta}, \tag{20}$$

$$c_2(\theta) := \left[\frac{(a(1-\cos\theta) + L)^2(L_0^2 - [L^2 + 2a(a+L)(1-\cos\theta)])}{L^2 + 2a(a+L)(1-\cos\theta)}\right]^{1/2}. \tag{21}$$

Precisely, there holds

$$M := \{(\gamma_\delta(\theta), \theta) : \theta \in I_\delta\} \cup \{(-\gamma_\delta(\theta), \theta) : \theta \in I_\delta\}. \tag{22}$$

In particular, if $L_0 \geqslant L + 2a$, we have full controllability in the angle, that is, for every $\theta \in [-\pi, \pi]$ there exist centers $c := \pm\gamma_\delta(\theta)$ where the energy is minimized. Therefore, if $L_0 \geqslant L + 2a$, then Ω can be stabilized in any rotation state θ.

Before proving Theorem 1, we make some observations.

Remark 3.1. Note that γ_δ is a closed curve. Indeed, $\gamma_\delta(\pm\cos^{-1}(1-\delta)) = (0, 0)$ if $L \leqslant L_0 \leqslant L + 2a$ and $\gamma_\delta(\pm\pi) = 0$ if $L_0 \geqslant L + 2a$. The maximum of $c_2(\theta)$ is always reached at $\theta = 0$ where $c_2^2(0) = L_0^2 - L^2$. Instead, the minimum value of $c_2(\theta)$ depends on the regime. It is $c_2(\theta) = 0$ (reached at $\theta = \pm\cos^{-1}(1-\delta)$) if $L \leqslant L_0 \leqslant L + 2a$, and $c_2^2(\theta) = L_0^2 - (2a + L)^2$ (reached at $\theta = \pm\pi$) if $L_0 \geqslant L + 2a$. In Figure 2 we sketch, for different values of $L_0 \geqslant L$, a qualitative plot of the family of trajectories that can be traced by the center c (i.e., by the curve γ_δ) without altering the minimal elastic energy. On the left part of the picture, we plot the regime $L \leqslant L_0 \leqslant L + 2a$, and on the right part of the picture, we plot the regime $L_0 \geqslant L + 2a$.

Remark 3.2. It can be useful to know under which conditions the configurations $\Omega_{c,n\pi}$ ($n \in \mathbb{N}$) are energy minimizing. For that, note that If $\cos\theta = 1$, i.e., if $\theta = n\pi$, $n \in \mathbb{Z}$ and n even, then from (21) we get $c_2^2 = L_0^2 - L^2$. Therefore, $\Omega_{c,0}$ is a minimizing configuration only when $L_0 \geqslant L$. The minimizing configurations are associated with the centers

$$c = \left(0, \pm\sqrt{L_0^2 - L^2}\right).$$

Similarly, if $\cos\theta = -1$, i.e., if $\theta = n\pi$, $n \in \mathbb{Z}$ and n odd, then from (21) we get $c_2^2 = L_0^2 - (2a + L)^2$. Therefore, $\Omega_{c,\pi}$ and $\Omega_{c,-\pi}$ are minimizing configurations only when $L_0 \geqslant 2a + L$. The minimizing configurations are associated with the centers

$$c = \left(0, \pm\sqrt{L_0^2 - (2a + L)^2}\right).$$

Remark 3.3. It is lightning to consider a concrete example. For $L = 2a$ and $L_0 = 4a$ we are in the limiting case $L_0 = L + 2a$. For every $\theta \in [-\pi, \pi]$ the equations (20) and (21)

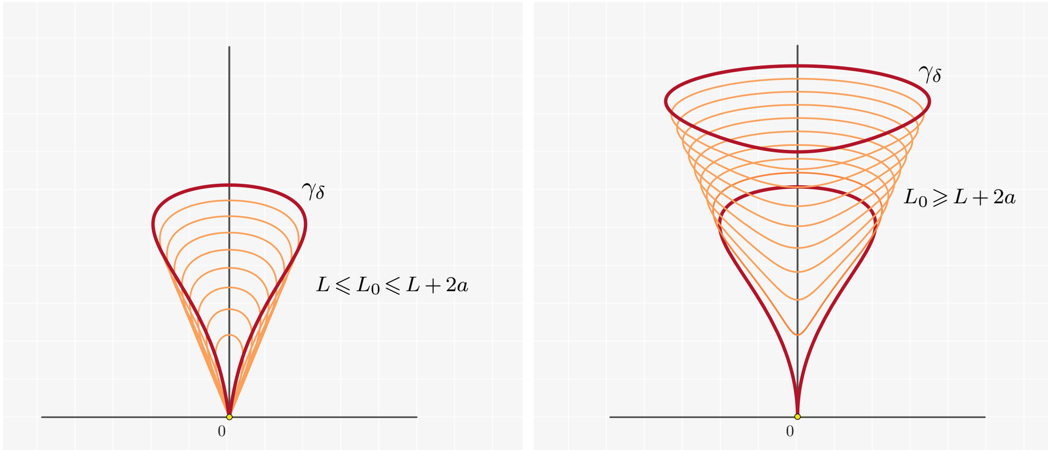

Figure 2. A qualitative plot of the family of trajectories that can be traced by the center c (without losing minimal elastic energy) for different values of $L_0 \geqslant L$. On the left part of the picture, we plot the regime $L \leqslant L_0 \leqslant L + 2a$, and on the right part of the picture, we plot the regime $L_0 \geqslant L + 2a$. Note that the curve γ_δ is a closed curve. Precisely, $\gamma_\delta(\pm\cos^{-1}(1-\delta)) = (0,0)$ if $L \leqslant L_0 \leqslant L + 2a$ and $\gamma_\delta(\pm\pi) = 0$ if $L_0 \geqslant L + 2a$. The maximum of $c_2(\theta)$ is reached at $\theta = 0$ where $c_2^2(0) = L_0^2 - L^2$. Instead, the minimum value of $c_2(\theta)$ depends on the regime. It is $c_2(\theta) = 0$ if $L \leqslant L_0 \leqslant L + 2a$, and $c_2^2(\theta) = L_0^2 - (2a+L)^2$ if $L_0 \geqslant L + 2a$.

reduces to

$$c_2^2(\theta) = 3a^2 \frac{(3-\cos\theta)^2(1+\cos\theta)}{(5-3\cos\theta)}, \tag{23}$$

$$c_1(\theta) = \frac{c_2 \sin\theta}{3-\cos\theta}. \tag{24}$$

Given the symmetries of the elastic system (cf. (28)), without loss of generality, we can focus on the positive brunch for $c_2(\theta)$. Plotting the curve $\gamma_\delta(\theta)$ for $\theta \in [-\pi, \pi]$, we get the admissible positions of the center c that minimize the energy. A plot in this limiting case $L_0 = L + 2a$, together with its physical meaning is given in Figure 3.

Proof. (of Theorem 1) We split the proof into three steps. The first step concerns the energy level associated with ground states, the second step is about symmetries of the minimizers.

Step 1. Minimizers have null energy. First, we show that if (c, θ) is a minimizer of $\mathcal{E}$, then $\mathcal{E}(c,\theta) = 0$. For that, we note that $v_{a,L}(0) = -Le_1$. Hence, for $\theta = 0$ and $c = c_2 e_2$ the energy reduces to

$$\mathcal{E}(c, 0) = \left(\sqrt{L^2 + c_2^2} - L_0\right)^2.$$

Taking $c_2^2 = L_0^2 - L^2 > 0$ (cf. (1)) we get

$$\mathcal{E}(c, 0) = \left(\sqrt{L^2 + c_2^2} - L_0\right)^2 = 0.$$

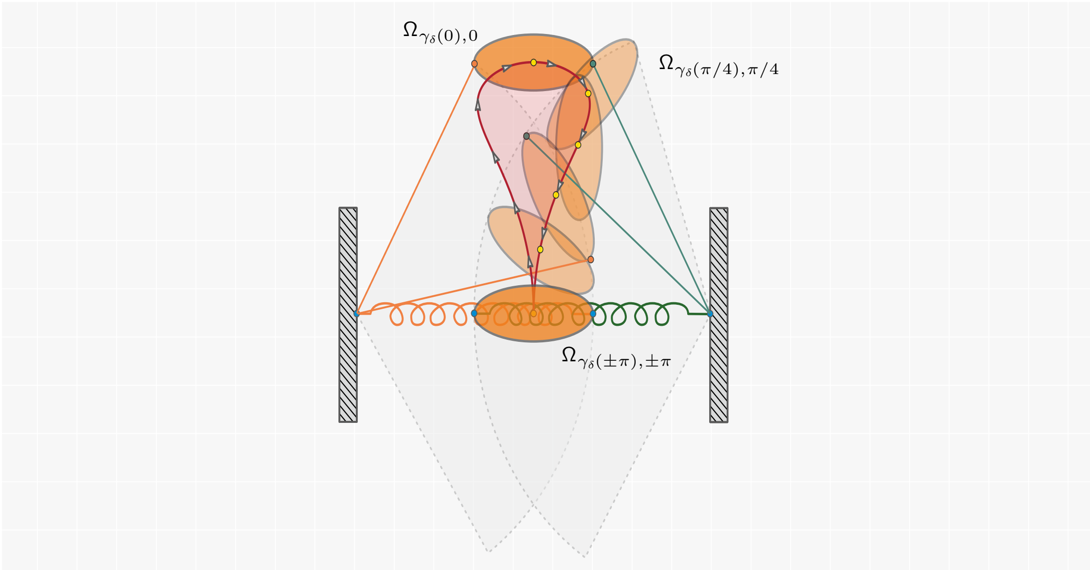

Figure 3. State-space of the system in the limiting case $L_0 = L + 2a$. The red curve represents $\gamma_\delta(\theta)$ which starts at $\theta = -\pi$ and ends at $\theta = \pi$. Note that for $\theta = \pm\pi$ the springs are attached to opposite vertices with respect to the reference configuration (this is energy favorable when $L_0 \geqslant L + 2a$).

Therefore, if (c, θ) is a minimizer of $\mathcal{E}$ then

$$(|c - v_{a,L}(\theta)| - L_0)^2 = (|c + v_{a,L}(\theta)| - L_0)^2 = 0. \tag{25}$$

Step 2. Symmetries of the minimizers. We now show that if (c, θ) is a minimizer of of $\mathcal{E}$, so are

$$(-c, \theta) = ((-c_1, -c_2), \theta), \tag{26}$$

$$(Z_{\pi/2}c, -\theta) = ((-c_1, c_2), -\theta), \tag{27}$$

$$(Z_\pi c, -\theta) = ((c_1, -c_2), -\theta). \tag{28}$$

First, by the invariance of the euclidean norm under rotations gives $\mathcal{E}(-c, \theta) = \mathcal{E}(c, \theta)$. Also, if we denote by Z_ϕ the reflection about a line through the origin that makes an angle ϕ with the x-axis, i.e.,

$$Z_\phi := \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}$$

then

$$Z_\phi v_{a,L}(\theta) = a \begin{pmatrix} \cos(\theta - 2\phi) - (1 + L/a)\cos 2\phi \\ -\sin(\theta - 2\phi) - (1 + L/a)\sin 2\phi \end{pmatrix}.$$

We deduce that $Z_\phi v_{a,L}(\theta) = -v_{a,L}(-\theta)$ when $\phi = +\pi/2$ (reflection about the y axis). Moreover, we have that $Z_\phi v_{a,L}(\theta) = v_{a,L}(-\theta)$ when $\phi = \pi$ (reflection about the x axis).

Step 3. Characterization of the minimizers. By (25) we know that the minimal elastic energy is reached when the relations $|c - v_{a,L}(\theta)| = |c + v_{a,L}(\theta)| = L_0$ are satisfied. A direct computation shows that the equation $|c - v_{a,L}(\theta)|^2 = |c + v_{a,L}(\theta)|^2$ is satisfied when

$$c_1 := \frac{c_2 \sin\theta}{(1+L/a) - \cos\theta}. \tag{29}$$

Note that the denominator is always strictly positive. Expanding the equation

$$|c + v_{a,L}(\theta)|^2 = L_0^2 \tag{30}$$

and taking into account the expression of c_1 in (29), we get that

$$c_2^2 = \frac{(a(1-\cos\theta)+L)^2(L_0^2 - [L^2 + 2a(a+L)(1-\cos\theta)])}{L^2 + 2a(a+L)(1-\cos\theta)}. \tag{31}$$

The existence of a solution is constrained to the condition that the right-hand side of (31) is nonnegative, i.e., provided that $L_0^2 \geqslant L^2 + 2a(a+L)(1-\cos\theta)$. This is equivalent to

$$\cos\theta \geqslant \frac{L^2 + 2a(a+L) - L_0^2}{2a(a+L)} = 1 - \frac{L_0^2 - L^2}{2a(a+L)}. \tag{32}$$

Observe that the previous inequality in θ is well-posed due to the regime of compression we are investigating where $L_0 \geqslant L$. After that, any $\theta \in \mathbb{R}$ is a solution of (32) if

$$1 - \frac{L_0^2 - L^2}{2a(a+L)} \leqslant -1 \tag{33}$$

i.e., when $L_0 \geqslant L + 2a$. Instead, if $0 < L_0 < L + 2a$, then we have only a subset of angles that solves (32) and this is given by the interval where

$$|\theta| \leqslant \cos^{-1}(1-\delta), \quad \delta := \frac{L_0^2 - L^2}{2a(a+L)} \tag{34}$$

Overall, it follows that for any given θ there exists *at most* a solution $(c_1(\theta), c_2(\theta))$, and this is given by (20) and (21). Moreover, from the symmetries of the elastic system (cf. (28)), we deduce that if $L_0 \geqslant L + 2a$, then for every θ there exists a unique $(c_1, c_2) \in \mathbb{R}^2, c_2 \geqslant 0$, such that $\Omega_{c,\theta}$ is at minimal energy with angle θ. In particular, we have full controllability in the angle θ when $L_0 \geqslant L + 2a$. Instead, if $L < L_0 < L + 2a$, then the minimizers of the elastic energy are the image of the curve

$$\gamma_\delta(\theta) = (c_1(\theta), c_2(\theta)) \quad \text{with} \quad \theta \in I_\delta := [-\cos^{-1}(1-\delta), \cos^{-1}(1-\delta)].$$

In particular, for any $|\theta| \leqslant \cos^{-1}(1-\delta)$ there exists a unique $c \in \mathbb{R}^2, c_2 \geqslant 0$, such that $\Omega_{c,\theta}$ is at minimal energy with angle θ. Combining these observations with the symmetry properties of the elastic system (cf. (28)), we get the characterization of the energy landscape given in (22). □

4. Minimizers of the magnetoelastic system

Our investigation targets the *small-scale* regime, in which device dimensions become comparable to the magnetic exchange length. In this limit, the variational theory of micromagnetics offers a natural description: the magnetization is represented by a unit-length vector field that minimizes a total energy functional comprised of exchange, crystalline anisotropy, Zeeman, and demagnetizing terms [3,5,8]. Crucially, this variational setting is well-suited to include elastic effects by introducing a coupled energy that depends jointly on the strain field and the magnetization. In a two-dimensional formulation, the magnetic energy of an elastic ellipse $\Omega_{c,\theta}$ in the deformed state characterized by parameters (c,θ) is expressed as follows [6, 7]:

$$\mathcal{F}_{\Omega_{c,\theta}}(m;\theta) := \frac{1}{|\Omega_{c,\theta}|}\Big[\int_{\Omega_{c,\theta}} a_{\text{ex}}^2 |\nabla m|^2 + \kappa^2 \int_{\Omega_{c,\theta}} (m\cdot e_3)^2 - \int_{\Omega_{c,\theta}} h_a \cdot m + \frac{\kappa_{\text{an}}^2}{2}\int_{\Omega_{c,\theta}} |m\times e_\theta|^2\Big] \tag{35}$$

for $m \in H^1(\Omega_{c,\theta},\mathbb{S}^2)$. If Ω is the reference ellipse centered at the origin as in Figure 1, then $\Omega_{c,\theta}$ is the image under Ω of the map

$$\mu_{c,\theta} : x \in \Omega \mapsto (R_\theta x + c) \in \Omega_{c,\theta}.$$

Here R_θ is the three-dimensional rotation matrix about the out-of-plane e_3-axis. To simplify the exposition, we assume that in the reference configuration the direction of the anisotropy axis corresponds to e_1. Therefore $e_\theta := R_\theta e_1$. Note that, with a small abuse of the notation, we denote by the same symbols the basis vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$. The context clarifies what is meant.

The total magnetoelastic energy associated with the system in the configuration $\Omega_{c,\theta}$ reads as

$$\mathcal{G}(m;c,\theta) = \mathcal{F}_{\Omega_{c,\theta}}(m;\theta) + \mathcal{E}(c,\theta).$$

We are interested in the minimization problem

$$\min_{(c,\theta)\in\mathbb{R}^2\times[-\pi,\pi]} \left(\min_{m\in H^1(\Omega_{c,\theta},\mathbb{S}^2)} [\mathcal{F}_{\Omega_{c,\theta}}(m;\theta)] + \mathcal{E}(c,\theta)\right). \tag{36}$$

Our main result reads as follows.

Theorem 2. *Let $h_a \neq 0$ and $\psi \in [-\pi,\pi]$ the angle that the applied field h_a makes with e_1. If $(m;c,\theta)$ in $H^1(\Omega_{c,\theta},\mathbb{S}^2)\times\mathbb{R}^2\times[-\pi,\pi]$ is a minimizer of the energy functional $\mathcal{G}$ then $m \in \mathbb{S}^1$, i.e., m is constant in $\Omega_{c,\theta}$.*

Moreover, if

$$\psi \in [-\cos^{-1}(1-\delta), \cos^{-1}(1-\delta)], \quad \delta := \frac{L_0^2 - L^2}{2a(a+L)}, \tag{37}$$

then the magnetoelastic minimizers are given by

$$m = h_a/|h_a|, \qquad e_\theta = \pm h_a/|h_a|, \qquad c = \gamma_\delta(\theta),$$

with $\theta = \psi + k\pi$, $k\in\mathbb{Z}$, and $c = \gamma_\delta(\theta)$ the curve characterized in Theorem 1. *The minimal value of the energy is then $\mathcal{G}(m;c,\theta) = -|h_a|$. In particular, in the full controllability regime $L_0 \geqslant L + 2a$, the characterization holds for every $\psi\in[-\pi,\pi]$.*

Instead, if (37) does not hold, i.e., if $\psi \notin [-\cos^{-1}(1-\delta), \cos^{-1}(1-\delta)]$ and, therefore, necessarily $L < L_0 < L + 2a$, the following assertions hold. If $(m;c,\theta)$ is a minimizer of $\mathcal{G}$ the following dichotomy holds:

i. *Either $\cos\theta \geqslant 1-\delta$; in which case $\mathcal{E}(c,\theta) = 0$ with the optimal center c determined by the curve γ_δ defined by (20) and (21),* *ii. or else,* $\cos\theta < 1-\delta$*; in which case* $c = 0$*.*

Therefore, if $\cos\theta < 1-\delta$ *then any minimizer* $(m; c, \theta)$ *of* $\mathcal{G}$ *is centered at the origin, i.e., of the form* $(m; 0, \theta)$*, with* (m, θ) *minimizer in* $\mathbb{S}^1 \times [-\pi, \pi]$ *of the energy*

$$\mathcal{F}(m;\theta) := -h_a \cdot m + \frac{\kappa_{\rm an}^2}{2} |m \times e_\theta|^2 + (|v_{a,L}(\theta)| - L_0)^2 . \quad (38)$$

Equivalently, if we express the quantities in polar coordinates through the angles (ϕ, ψ) *such that* $m := R_\phi e_1$ *and* $h_a := |h_a|\, R_\psi e_1$*, then* (38) *reads as*

$$\mathcal{F}(m;\theta) = \frac{\kappa_{\rm an}^2}{2} \sin^2(\theta - \phi) - |h_a| \cos(\psi - \phi) + (|v_{a,L}(\theta)| - L_0)^2 . \quad (39)$$

In Figure 4 we represent the magnetoelastic energy minimizers associated with a full rotation of the applied field h_a (encoded in the angle ψ) in the interval $[0, 2\pi]$. In order to catch the nonfull controllability regime $L < L_0 < L+2a$, we set $L = 3a$ and $L_0 = L+a$. Also, we set $|h_a| = 1$ and $\kappa_{\rm an} = 2$. This choice entails that the minimal magnetization m is not aligned with the applied field h_a when the associated angle ψ is such that $\cos\psi < 1-\delta$ (see the configuration at $c = 0$). Note that, without any loss of generality, in Figure 4 we can restrict the visualization to the case $c_2 \geqslant 0$. Indeed, we already pointed out that if (c, θ) is a minimizer of the elastic energy $\mathcal{E}$, so are (cf. (28)) $(-c, \theta)$, $(Z_{\pi/2}c, -\theta)$, $(Z_\pi c, -\theta)$. This is because of

$$\mathcal{E}(c, \theta) = \mathcal{E}(-c, \theta) = \mathcal{E}(Z_{\pi/2}c, -\theta) = \mathcal{E}(Z_\pi c, -\theta) .$$

The previous relations imply similar symmetry relations on the magnetoelastic energy $\mathcal{F}$. Precisely, if we set

$$\mathcal{F}_\psi(m; c, \theta) = \frac{\kappa_{\rm an}^2}{2} \sin^2(\theta - \phi) - |h_a| \cos(\psi - \phi) + \mathcal{E}(c, \theta)$$

so that the notation for $\mathcal{F}$ also the dependence from the external field direction ψ, then we have

$$\mathcal{F}_\psi(\phi; c, \theta) = \mathcal{F}_\psi(\phi; -c, \theta) = \mathcal{F}_{-\psi}(-\phi, Z_{\pi/2}c, -\theta) = \mathcal{F}_{-\psi}(-\phi, Z_\pi c, -\theta) .$$

Therefore, the visualization of the energy landscape for $c_2 \geqslant 0$ does not affect any generality.

Before the proof of Theorem 2 we make some observations and prove complementary results.

Remark 4.1. The analysis of the minimizers in the elastic regime is independent of the shape of Ω. Indeed, it depends only on the segment of length $2a$ passing through Ω to which the springs are connected.

Remark 4.2. Intuitively, the energy in (39) reveals that the higher is $|h_a|$, the more the magnetization m (in terms of ϕ) tends to be aligned with the field h_a. Also, the higher is $\kappa_{\rm an}^2$, the more the axis of Ω (expressed in terms of θ) tends to follow the orientation of m. To make these statements quantitative, we prove the following result.

Proposition 1. *If* $(m; c, \theta)$ *is a minimum point for* $\mathcal{F}$ *in* (39) *then*

$$|\sin(\phi - \psi)| \leqslant \frac{1}{2\,|h_a|} \quad (40)$$

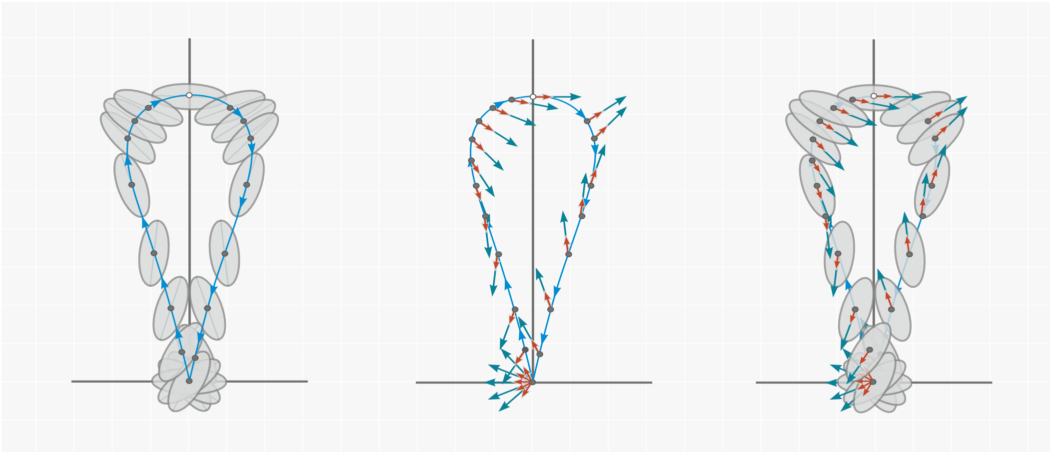

Figure 4. Numerical computation of the magnetoelastic energy landscape associated with a full rotation of the applied field h_a (encoded in the angle ψ) in the interval $[0, 2\pi]$. Here, we set $L = 3a$ and $L_0 = L + a$ so that we are in the nonfull controllability regime $L < L_0 < L + 2a$. Also, we set $|h_a| = 1$ and $\kappa_{\text{an}} = 2$. This choice entails that the minimal magnetization m (in red in the Figure) is not aligned with the applied field h_a (in cyan in the Figure) when the associated angle ψ is outside of the interval $[-\cos^{-1}(1-\delta), \cos^{-1}(1-\delta)]$ (see the configuration pinned at $c = 0$). Left, position and orientation of the ellipse as ψ varies in the interval $[0, 2\pi]$. When $\psi = 0$, the center of the ellipse is on the y axis and is represented by a white dot. As ψ varies in $[0, 2\pi]$ the ellipse rotates and moves along the curve γ_δ (represented in light blue in the picture. Center, for clarity, we represent the minimal magnetization m (in dark red) associated with the applied field h_a. Right, we overlap the two plots so as to have a complete picture of the induced dynamics.

and

$$|\sin(2(\theta - \phi))| \leqslant \frac{1}{\kappa_{\text{an}}^2}. \tag{41}$$

Remark 4.3. Relation (41) gives a quantitative justification to the observation that the bigger is κ_{an}^2, the more is $(\theta - \phi)$ close to the set $\{-\pi/2, 0, \pi/2, \pi\}$. Also, (40) tells us that the bigger is $|h_a|$ the more is $(\phi - \psi)$ close to the set $\{0, \pi\}$. In terms of limiting relations, this tells us that

$$\sphericalangle\,(m \cdot e_\theta) \xrightarrow{\kappa_{\text{an}} \to \infty} \{-\pi/2, 0, \pi/2, \pi\}, \quad \sphericalangle\,(m \cdot h_a) \xrightarrow{|h_a| \to \infty} \{0, \pi\},$$

where e_θ coincides with the axis of the ellipse to the extremities of which the springs are attached. On the other hand, by minimality, we know that if $(m; c, \theta)$ minimizes the energy, then necessarily $\sphericalangle\,(m \cdot h_a) \leqslant \pi/2$ because if $\sphericalangle\,(m \cdot h_a) > \pi/2$, one can decrease the energy by reversing the magnetization from m to $-m$. This is because the only part of the energy that depends on m is the density $-h_a \cdot m + \frac{\kappa_{\text{an}}^2}{2} |m \times e_\theta|^2$ and, therefore, switching from m to $-m$ does not alter the anisotropy energy while reducing the Zeeman energy. Overall, we get that

$$\sphericalangle\,(m \cdot e_\theta) \xrightarrow{\kappa_{\text{an}} \to \infty} \{-\pi/2, 0, \pi/2, \pi\}, \quad \sphericalangle\,(m \cdot h_a) \xrightarrow{|h_a| \to \infty} \{0\}.$$

Proof. (of Proposition 1) Let $(m;c,\theta)$ be a minimizer of $\mathcal{F}$ in (39). From the stationary condition $\partial_m \mathcal{F}(m;c,\theta)=0$ we get that the following Euler–Lagrange holds

$$(m\cdot e_\theta)\, e_\theta + h_a = \lambda m, \qquad \lambda := (m\cdot e_\theta)^2 + m\cdot h_a, \tag{42}$$

with λ the Lagrange multiplies coming from the nonconvex constraint $m\in\mathbb{S}^1$. By cross multiplying both sides of the Euler–Lagrange equation by e_θ we get $(h_a\times e_\theta)=\lambda\,(m\times e_\theta)$, from which

$$|h_a|^2 - (e_\theta\cdot h_a)^2 = \left((m\cdot e_\theta)^2 + (m\cdot h_a)\right)^2\left(1-(m\cdot e_\theta)^2\right). \tag{43}$$

Dot multiplying both sides of of the Euler–Lagrange equation by e_θ we get $(m\cdot e_\theta)+(e_\theta\cdot h_a)=\lambda\,(m\cdot e_\theta)$, from which it follows that

$$(e_\theta\cdot h_a) = \left((m\cdot e_\theta)^2 + (m\cdot h_a) - 1\right)(m\cdot e_\theta). \tag{44}$$

Combining (43) and (44) through the in common term $(e_\theta\cdot h_a)$ we get the relation

$$|h_a|^2 - \left((m\cdot e_\theta)^2 + (m\cdot h_a) - 1\right)^2 (m\cdot e_\theta)^2 = \left((m\cdot e_\theta)^2 + (m\cdot h_a)\right)^2\left(1-(m\cdot e_\theta)^2\right),$$

which after some algebra reduces to the equation

$$(m\cdot e_\theta)^4 - (m\cdot e_\theta)^2 + |h_a\times m|^2 = 0. \tag{45}$$

Therefore the solutions of (45) satisfy the relation

$$(m\cdot e_\theta)^2 = \frac{1\pm\sqrt{1-4\,|h_a\times m|^2}}{2}. \tag{46}$$

Since we know about the existence of solutions, this implies that

$$\frac{1}{4}\geqslant |h_a\times m|^2. \tag{47}$$

In terms of angles, the previous relation reads as $|\sin(\phi-\psi)|\leqslant 1/2\,|h_a|$, which is exactly (40). After that, if $(m;c,\theta)$ is a minimizer, then also the stationary condition $\partial_\phi\mathcal{F}(m;c,\theta)=0$ holds. In polar coordinates, the stationary condition $\partial_\phi\mathcal{F}(m;c,\theta)=0$ reads as

$$\frac{\kappa_{\rm an}^2}{2}\sin(2(\theta-\phi)) = |h_a|\sin(\phi-\psi). \tag{48}$$

Combining (40) and (48) we get (41). □

Proof. (*of* Theorem 2) We divide the proof into three steps.

Step 1. Magnetoelastic minimizers have $m\in\mathbb{S}^1$. Our first step shows that if $(m;c,\theta)$ is a minimizer in $H^1(\Omega_{c,\theta},\mathbb{S}^2)\times\mathbb{R}^2\times[-\pi,\pi]$ of the energy functional $\mathcal{G}$ then $m\in\mathbb{S}^1$, i.e., m is constant in $\Omega_{c,\theta}$. Moreover, the minimization problem (36) is equivalent to

$$\min_{(c,\theta)\in\mathbb{R}^2\times[-\pi,\pi]}\left(\min_{m\in\mathbb{S}^1}\left[-h_a\cdot m + \frac{\kappa_{\rm an}^2}{2}|m\times e_\theta|^2\right] + \mathcal{E}(c,\theta)\right). \tag{49}$$

Indeed, we have

$$\begin{aligned}\mathcal{G}(m;c,\theta) &\geqslant \min_{(c,\theta)\in\mathbb{R}^2\times[-\pi,\pi]}\left(\min_{m\in\mathbb{S}^2}\left[\frac{\kappa^2}{2}(m\cdot e_3)^2 - h_a\cdot m + \frac{\kappa_{\text{an}}^2}{2}|m\times e_\theta|^2\right] + \mathcal{E}(c,\theta)\right)\\ &\geqslant \min_{(c,\theta)\in\mathbb{R}^2\times[-\pi,\pi]}\left(\min_{m\in\mathbb{S}^2}\left[-h_a\cdot m + \frac{\kappa_{\text{an}}^2}{2}|m\times e_\theta|^2\right] + \mathcal{E}(c,\theta)\right)\\ &= \min_{(c,\theta)\in\mathbb{R}^2\times[-\pi,\pi]}\left(\min_{m\in\mathbb{S}^1}\left[-h_a\cdot m + \frac{\kappa_{\text{an}}^2}{2}|m\times e_\theta|^2\right] + \mathcal{E}(c,\theta)\right). \qquad (50)\end{aligned}$$

The last equality (50), which reduces a minimization problem on $\mathbb{S}^2$ to a minimization problem on $\mathbb{S}^1$, is justified by the following argument. First, we observe that by using $m = \pm e_\theta$ as competitors, one can deduce the bound

$$\min_{m\in\mathbb{S}^2}\left[-h_a\cdot m + \frac{\kappa_{\text{an}}^2}{2}|m\times e_\theta|^2\right] \leqslant -|h_a\cdot e_\theta| \leqslant 0. \qquad (51)$$

Thus, the minimization problem in (51) is solved by configuration with nonpositive energy. After that, we can exclude that $m = \pm e_3$ are minimizers of (51) given that the resulting energy would be strictly positive because of

$$-h_a\cdot m + \frac{\kappa_{\text{an}}^2}{2}|m\times e_\theta|^2 = \frac{\kappa_{\text{an}}^2}{2}.$$

Also, note that any minimizer of (51) has to be such that $h_a\cdot m = h_a\cdot m^\perp \geqslant 0$; otherwise, again, the energy would be strictly positive. That said, if $m\cdot e_3 \neq 0$, $m \neq \pm e_3$, and $h_a\cdot m^\perp \geqslant 0$ then

$$\begin{aligned}-h_a\cdot m + \frac{\kappa_{\text{an}}^2}{2}|m\times e_\theta|^2 &= \frac{\kappa_{\text{an}}^2}{2} - \left|h_a\cdot m^\perp\right| - \frac{\kappa_{\text{an}}^2}{2}\left(m^\perp\cdot e_\theta\right)^2\\ &\geqslant \frac{\kappa_{\text{an}}^2}{2} - \left|h_a\cdot\frac{m^\perp}{|m^\perp|}\right| - \frac{\kappa_{\text{an}}^2}{2}\left(\frac{m^\perp}{|m^\perp|}\cdot e_\theta\right)^2\\ &= -h_a\cdot\frac{m^\perp}{|m^\perp|} + \frac{\kappa_{\text{an}}^2}{2}\left|\frac{m^\perp}{|m^\perp|}\times e_\theta\right|^2,\end{aligned}$$

from which it follows that $m = \frac{m^\perp}{|m^\perp|}$ has a lower energy density. Overall, we get that $\mathbb{S}^2$-minimizers of $\frac{\kappa_{\text{an}}^2}{2}|m\times e_\theta|^2 - h_a\cdot m$ are actually elements of $\mathbb{S}^1$.

Step 2. Solution of the minimization problem in the full controllability setting. From the previous step and Theorem 1, we get that if $L_0 \geqslant L + 2a$ then, as a function of $m\in\mathbb{S}^1$, the energy density $(\kappa_{\text{an}}^2/2)|m\times e_\theta|^2 - h_a\cdot m$ is minimized when θ is such that $e_\theta = \pm h_a/|h_a|$, and $m = h_a/|h_a|$ because in this case, we have

$$\frac{\kappa_{\text{an}}^2}{2}|m\times e_\theta|^2 - h_a\cdot m = -|h_a|.$$

After that, we recall that in the full controllability regime, the elastic energy has a minimizer $\Omega_{c,\theta}$ for every $\theta\in[\pi,\pi]$. It is sufficient to take $c = \gamma_\delta(\theta)$. This implies that the minimal magnetoelastic energy is reached in the state $(m;c,\theta)$ with

$$m = \frac{h_a}{|h_a|}, \quad R_\theta e_1 = \pm\frac{h_a}{|h_a|}, \quad c = \gamma_\delta(\theta). \qquad (52)$$

Indeed, for every $m \in \mathbb{S}^1$, the following energy lower bound holds

$$\mathcal{G}\left(m; c, \theta\right) \geqslant -\left|h_a\right|,$$

and with the choices in (52) we reach the equality $\mathcal{G}\left(m; c, \theta\right) = -\left|h_a\right|$. This is the reason why we refer to the case $L_0 \geqslant L + 2a$ as the full controllability regime.

Instead, if $L < L_0 < L + 2a$, then, depending on the direction of the applied field h_a, we have to distinguish between two possible scenarios. First, recall from (18) that the elastic energy vanishes whenever

$$\theta \in [-\cos^{-1}(1-\delta), \cos^{-1}(1-\delta)], \quad \delta := \frac{L_0^2 - L^2}{2a(a+L)}.$$

Given $h_a \neq 0$, we denote by $\psi \in [-\pi, \pi]$ the angle that the applied field h_a makes with e_1. If $h_a/\left|h_a\right| = R_\psi e_1$ for some $\psi \in [-\cos^{-1}(1-\delta), \cos^{-1}(1-\delta)]$, then the magnetoelastic minimizer is given by $(m; c, \theta)$ with $m = h_a/\left|h_a\right|$, θ such that $e_\theta = R_\theta e_1 = \pm h_a/\left|h_a\right|$ (i.e., $\theta = \psi + k\pi$, $k \in \mathbb{Z}$), and $c = \gamma_\delta(\theta)$. Indeed, arguing as before, in this configuration we have that $m \times e_\theta = 0$ and, therefore,

$$\mathcal{G}\left(m; c, \theta\right) = \mathcal{F}_{\Omega_{c,\theta}}\left(m; \theta\right) = -\left|h_a\right|.$$

It remains to understand the minimal configuration when $h_a/\left|h_a\right| = R_\psi e_1$ and $|\psi| > \cos^{-1}(1-\delta)$. For that, we need to investigate the whole energy functional, which thanks to Step 1 can be written as (cf. (49))

$$\mathcal{F}\left(m; c, \theta\right) := -h_a \cdot m + \frac{\kappa_{\text{an}}^2}{2}\left|m \times e_\theta\right|^2 + \mathcal{E}\left(c, \theta\right), \tag{53}$$

with

$$\mathcal{E}\left(c, \theta\right) = \frac{1}{2}\left(|c - v_{a,L}(\theta)| - L_0\right)^2 + \frac{1}{2}\left(|c + v_{a,L}(\theta)| - L_0\right)^2. \tag{54}$$

Note that the minimization problem for $\mathcal{F}$ is finite-dimensional. However, given the nonconvex constraint $m \in \mathbb{S}^1$, the critical points are solutions of a high order equation in the powers of $m \cdot e_\theta$ and $m \cdot h_a$. In order to proceed, it is convenient to express the quantities in polar coordinates through the angles (ϕ, ψ) whose meaning is given by

$$m := R_\phi e_1, \quad h_a := \left|h_a\right| R_\psi e_1. \tag{55}$$

In this way, we have

$$h_a \cdot m = \left|h_a\right| R_\psi e_1 \cdot R_\phi e_1 = \left|h_a\right| \cos(\psi - \phi). \tag{56}$$

Similarly, we have

$$\left|m \times e_\theta\right|^2 = \left|m \times R_\theta e_1\right|^2 = 1 - \left(e_1 \cdot R_\theta^\top R_\phi e_1\right)^2 = \sin^2(\theta - \phi).$$

Therefore the total magnetoelastic energy (38) reads as

$$\mathcal{F}\left(m; c, \theta\right) = \frac{\kappa_{\text{an}}^2}{2} \sin^2(\theta - \phi) - \left|h_a\right| \cos(\psi - \phi) + \mathcal{E}\left(c, \theta\right) \tag{57}$$

The energy functional $\mathcal{F}$ has to be minimized in the configuration space $(c, \theta, \phi) \in \mathbb{R}_+^2 \times [-\pi, \pi]^3$.

Step 3. Proof of (38). According to Proposition 1, depending on the strength of the physical parameters κ^2 and $|h_a|$, the minimal angle θ can differ from the direction of the applied field (encoded in the angle ψ) and, in general, only numerical simulations can predict the behavior of the minimal magneto-elastic states. However, we can still predict the position of the center c depending on the minimal angle θ. For that, we observe that if (c,θ) is a critical point of the elastic energy $\mathcal{E}$, then $\partial_c\mathcal{E}\,(c,\theta)=0$. In expanded form, the stationary condition $\partial_c\mathcal{E}\,(c,\theta)=0$ reads as

$$\begin{aligned}\partial_c\mathcal{E}\,(c,\theta) = (|c+v_{a,L}(\theta)|-L_0)\,\frac{c+v_{a,L}(\theta)}{|c+v_{a,L}(\theta)|}\\ +(|c-v_{a,L}(\theta)|-L_0)\,\frac{c-v_{a,L}(\theta)}{|c-v_{a,L}(\theta)|}=0.\end{aligned} \tag{58}$$

In writing the previous relation, we assumed that $|c+v_{a,L}(\theta)|\neq 0$ and $|c-v_{a,L}(\theta)|\neq 0$. But actually this is always the case for minimizers. To see this, we observe that if $|c+v_{a,L}(\theta)|=0$ or $|c-v_{a,L}(\theta)|=0$, then one of the two springs is at the maximal compression. Suppose that $|c-v_{a,L}(\theta)|=0$, then $c=v_{a,L}(\theta)$ and the value of the associated elastic energy is

$$\mathcal{E}\,(v_{a,L}(\theta),\theta)=\frac{1}{2}L_0^2+\frac{1}{2}\left(2\,|v_{a,L}(\theta)|-L_0\right)^2.$$

On the other hand, $c=0$ has lower energy regardless of the angle θ. Indeed, we have

$$\begin{aligned}\mathcal{E}\,(v_{a,L}(\theta),\theta)-\mathcal{E}(0,\theta) &= \frac{1}{2}L_0^2+\frac{1}{2}\left(2\,|v_{a,L}(\theta)|-L_0\right)^2-\left(|v_{a,L}(\theta)|-L_0\right)^2\\ &=\frac{1}{2}L_0^2+\frac{1}{2}\left(4\,|v_{a,L}(\theta)|^2+L_0^2-4L_0\,|v_{a,L}(\theta)|\right)\\ &\qquad -\left(|v_{a,L}(\theta)|^2+L_0^2-2L_0\,|v_{a,L}(\theta)|\right)\\ &=\frac{1}{2}L_0^2+2\,|v_{a,L}(\theta)|^2+\frac{1}{2}L_0^2-2L_0\,|v_{a,L}(\theta)|\\ &\qquad -|v_{a,L}(\theta)|^2-L_0^2+2L_0\,|v_{a,L}(\theta)|\\ &=|v_{a,L}(\theta)|^2,\end{aligned}$$

and we know from (16) that $|v_{a,L}(\theta)|^2\geqslant L^2$. This means that the choice $c=v_{a,L}(\theta)$ never gives an energy minimizing configuration. The same argument applies to the $|c+v_{a,L}(\theta)|=0$. Overall, we have shown that if (c,θ) is a minimum point of the elastic energy $\mathcal{E}$ then $|c\pm v_{a,L}(\theta)|\neq 0$, i.e.,

$$c\neq \pm v_{a,L}(\theta). \tag{59}$$

As a side remark, note that this observation holds regardless of the current regime $L<L_0<L+2a$.

Having proved that (58) is well-defined at minima, we observe that (58) implies, in particular, that at minima *equipartition of the energy* holds:

$$\frac{1}{2}\left(|c-v_{a,L}(\theta)|-L_0\right)^2=\frac{1}{2}\left(|c+v_{a,L}(\theta)|-L_0\right)^2. \tag{60}$$

In other words, at equilibrium, the variation in absolute value of the length of the two springs (computed with respect to the rest length L_0) is the same for the two springs.

The previous relation allows us to focus on the case $|c \pm v_{a,L}(\theta)| - L_0 \neq 0$. Indeed, if θ is such that $|c \pm v_{a,L}(\theta)| = L_0$, then we already know from Theorem 1 that minimizers have zero elastic energy, the angle θ is such that

$$\cos\theta \geqslant 1 - \delta \quad \text{with} \quad \delta := \frac{L_0^2 - L^2}{2a(a+L)}, \tag{61}$$

and the position of c is given by (20) and (21).

From (58) and (60) we get that if $|c \pm v_{a,L}(\theta)| - L_0 \neq 0$, then

$$\frac{c + v_{a,L}(\theta)}{|c + v_{a,L}(\theta)|} = \pm \frac{c - v_{a,L}(\theta)}{|c - v_{a,L}(\theta)|}. \tag{62}$$

In particular, the wedge product of $c + v_{a,L}(\theta)$ with $c - v_{a,L}(\theta)$ vanishes, i.e.,

$$(c + v_{a,L}(\theta)) \times (c - v_{a,L}(\theta)) = 0. \tag{63}$$

The previous expression simplifies to

$$c \times v_{a,L}(\theta) = 0. \tag{64}$$

The previous relation (64) implies that if (c, θ) is a stationary point for $\mathcal{E}$ (satisfying (59)) then $c = \lambda v_{a,L}(\theta)$ for some $\lambda \in \mathbb{R}$ with $\lambda \neq \pm 1$.

However, a direct computation shows that either $\lambda = 0$ or $\lambda = \frac{L_0}{|v_{a,L}(\theta)|}$. Indeed, for $c = \lambda v_{a,L}(\theta)$ we have

$$\begin{aligned} \partial_c \mathcal{E}(c,\theta) = (|1+\lambda| \cdot |v_{a,L}(\theta)| - L_0) \cdot \operatorname{sgn}(1+\lambda) \frac{v_{a,L}(\theta)}{|v_{a,L}(\theta)|} \\ - (|1-\lambda| \cdot |v_{a,L}(\theta)| - L_0) \operatorname{sgn}(1-\lambda) \frac{v_{a,L}(\theta)}{|v_{a,L}(\theta)|}. \end{aligned} \tag{65}$$

Therefore, if $|\lambda| < 1$ then

$$\begin{aligned} \partial_c \mathcal{E}(c,\theta) &= ((1+\lambda) \cdot |v_{a,L}(\theta)| - L_0) \frac{v_{a,L}(\theta)}{|v_{a,L}(\theta)|} \\ &\quad - ((1-\lambda) \cdot |v_{a,L}(\theta)| - L_0) \frac{v_{a,L}(\theta)}{|v_{a,L}(\theta)|} \\ &= 2\lambda \, |v_{a,L}(\theta)| \end{aligned} \tag{66}$$

and the right-hand side vanishes if, and only if, $\lambda = 0$. On the other hand, if $\lambda > 1$ then

$$\begin{aligned} \partial_c \mathcal{E}(c,\theta) &= ((1+\lambda) \cdot |v_{a,L}(\theta)| - L_0) \frac{v_{a,L}(\theta)}{|v_{a,L}(\theta)|} \\ &\quad + ((\lambda-1) \cdot |v_{a,L}(\theta)| - L_0) \frac{v_{a,L}(\theta)}{|v_{a,L}(\theta)|} \\ &= 2 (\lambda \, |v_{a,L}(\theta)| - L_0) \frac{v_{a,L}(\theta)}{|v_{a,L}(\theta)|} \end{aligned} \tag{67}$$

and the right-hand side vanishes if, and only if, $\lambda = \frac{L_0}{|v_{a,L}(\theta)|}$. Similarly, if $\lambda < -1$ then

$$\partial_c \mathcal{E}(c,\theta) = 2 (\lambda \cdot |v_{a,L}(\theta)| + L_0) \frac{v_{a,L}(\theta)}{|v_{a,L}(\theta)|}$$

and the right-hand side vanishes if, and only if, $\lambda = -L_0/|v_{a,L}(\theta)|$.

Summarizing, in the regime $L < L_0 < L + 2a$ we have that if $(m; c, \theta)$ is a stationary point of the magnetoelastic energy, then the following alternatives are possible for the position of the center c (in what follows, as usual, $\delta := (L_0^2 - L^2)/(2a(a+L))$):

i. There holds $\cos\theta \geqslant 1 - \delta$; in which case $\mathcal{E}(c, \theta) = 0$ with the optimal center c determined by the curve γ_δ defined by (20) and (21).

ii. There holds $\cos\theta < 1 - \delta$ and $\lambda = 0$; in which case $c = 0$.

iii. There holds $\cos\theta < 1 - \delta$ and $\lambda = \pm L_0/|v_{a,L}(\theta)|$ with $L_0 > |v_{a,L}(\theta)|$ and; in which case $c = \pm L_0 v_{a,L}(\theta)/|v_{a,L}(\theta)|$ (and $|c| = L_0$).

From the previous possibilities it appears clear that a full characterization of the minimal center c associated with θ is achieved as soon as we compare the energy associated with possibilities *ii* and *iii*. For that, recalling that $|\lambda| > 1$, we have

$$\begin{aligned}
\mathcal{E}(\lambda v_{a,L}(\theta), \theta) &= (|\lambda - 1|\, |v_{a,L}(\theta)| - L_0)^2 \\
&= \left(\left| \pm \frac{L_0}{|v_{a,L}(\theta)|} - 1 \right| |v_{a,L}(\theta)| - L_0 \right)^2 \\
&= (|\pm L_0 - |v_{a,L}(\theta)|| - L_0)^2 \\
&= |v_{a,L}(\theta)|^2 .
\end{aligned}$$

Also, we have

$$\mathcal{E}(0, \theta) = \qquad (|v_{a,L}(\theta)| - L_0)^2 = |v_{a,L}(\theta)|^2 - 2L_0\, |v_{a,L}(\theta)| + L_0^2.$$

Hence, when $\cos\theta < 1 - \delta$, the center $c = 0$ is energetically favored whenever $|v_{a,L}(\theta)| > L_0/2$. From the expression of $|v_{a,L}(\theta)|$ we get that this happens if, and only if,

$$a^2 \sin^2\theta + (a(1 - \cos\theta) + L)^2 > L_0^2/4.$$

Simplifying the previous expression we see that this happens when

$$\cos(\theta) < \frac{2a^2 + 2aL + L^2 - L_0^2/4}{2a(a+L)} = \frac{2a(a+L) + L^2 - L_0^2/4}{2a(a+L)} = 1 - \frac{L_0^2/4 - L^2}{2a(a+L)}.$$

But this is always the case because by assumption $\cos\theta < 1 - \delta$ and, on the other hand,

$$1 - \delta < 1 - \frac{L_0^2/4 - L^2}{2a(a+L)}.$$

This concludes the proof. □

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