A MATRIX-VALUED MEASURE ASSOCIATED TO THE DERIVATIVES OF A FUNCTION OF GENERALISED BOUNDED DEFORMATION

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Abstract. We associate to every function $u \in GBD(\Omega)$ a measure μ_u with values in the space of symmetric matrices, which generalises the distributional symmetric gradient Eu defined for functions of bounded deformation. We show that this measure μ_u admits a decomposition as the sum of three mutually singular matrix-valued measures μ_u^a, μ_u^c, and μ_u^j, the absolutely continuous part, the Cantor part, and the jump part, as in the case of $BD(\Omega)$ functions. We then characterise the space $GSBD(\Omega)$, originally defined only by slicing, as the space of functions $u \in GBD(\Omega)$ such that $\mu_u^c = 0$.

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1. Introduction

Given a bounded open set $\Omega \subset \mathbb{R}^d$, with $d \geq 1$, the spaces $GBD(\Omega)$ of *functions of generalised bounded deformation* and $GSBD(\Omega)$ of *special functions of generalised bounded deformation* were introduced in [20] to provide a functional framework for variational problems related to Griffith's energy in fracture mechanics (see [9,30]). The main feature of these spaces is that they avoid the unnatural L^∞ *a priori* bounds typically required for compactness in the space $SBD(\Omega)$ (see [8]).

Thanks to the very weak requirements appearing in the definitions of $GBD(\Omega)$ and $GSBD(\Omega)$, it was shown in [20] that compactness in these spaces is achieved under very mild assumptions (see also [4,14]). For the space $GSBD(\Omega)$ compactness results under even weaker conditions have later been obtained by Friedrich and Solombrino [33] in the planar case, and Chambolle and Crismale in the general case [12]. These results are similar to those available in the more restrictive setting of functions of bounded variation (see [5,29,31]). These advancements allow one to solve, in a weak sense, minimisation problems concerning Griffith's functional, thereby justifying the use of $GSBD(\Omega)$ for *brittle* models in fracture mechanics. Many more applications of the space $GSBD(\Omega)$ were also considered in the recent literature, see, for instance, [1–3,11,16–18,32,39].

The study of *cohesive* models for fracture mechanics in the *anti-plane* case carried out in [24,25] suggests that $GBD(\Omega)$ should be the appropriate space for the study of minimisation problems connected with these models, when the anti-plane hypothesis is dropped. This requires to extend to $GBD(\Omega)$ the structure theorems proved in [6] for the space $BD(\Omega)$ of *functions of bounded deformation* (see [6,37,40,41]), in analogy with what was done in [23] in the setting of functions of bounded variation.

The analysis of the *fine properties* of functions in $GBD(\Omega)$ carried out in [20] reveales that many of the key structural features of the space $BD(\Omega)$ of *functions of bounded deformation* (see [6,37,40,41]) naturally extend to $GBD(\Omega)$, albeit with suitable modifications to account for the weaker regularity of these functions. The fine properties of $BD(\Omega)$ were thoroughly examined in [6], where the authors show for a function $u\colon \Omega \to \mathbb{R}^d$ in $BD(\Omega)$ that the following conditions hold:

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(a) u admits an *approximate gradient* $\nabla u \in L^1(\Omega;\mathbb{R}^{d\times d})$, where $\mathbb{R}^{d\times d}$ is the space of $d\times d$ matrices with real entries, and $\mathcal{E}u := (\nabla u + \nabla u^T)/2$ defines an *approximate symmetric gradient* of u (see also [35]); moreover, if $\mathcal{H}^{d-1}$ is the $(d-1)$-dimensional Hausdorff measure, for every $\xi \in \mathbb{S}^{d-1} := \{\xi \in \mathbb{R}^d : |\xi| = 1\}$ and for $\mathcal{H}^{d-1}$-a.e $y \in \Pi^\xi := \{y \in \mathbb{R}^d : y\cdot\xi = 0\}$, the one-dimensional scalar function $t \mapsto u^\xi_y(t) := u(y+t\xi)\cdot\xi$ has bounded variation and

$$\mathcal{E}u(y+t\xi)\xi\cdot\xi = \nabla u^\xi_y(t) \quad \text{for } \mathcal{L}^1\text{-a.e. } t \in \{s\in\mathbb{R} : y+s\xi\in\Omega\}, \tag{1.1}$$

where $\mathcal{L}^1$ is the one-dimensional Lebesgue measure and ∇u^ξ_y denotes the absolutely continuous part of the distributional derivative Du^ξ_y of u^ξ_y;

(b) the jump set J_u (see Definition 2.3) is $(\mathcal{H}^{d-1}, d-1)$-rectifiable (see (2.1) and also [26]), with measure theoretical unit normal ν_u; in addition, for every $\xi \in \mathbb{S}^{d-1}$ and for $\mathcal{H}^{d-1}$-a.e. $y \in \Pi^\xi$, setting $[u] := u^+ - u^-$, where u^+ and u^- are the unilateral traces of u on J_u, and $(J^\xi_u)^\xi_y := \{t \in \mathbb{R} : x = y+t\xi \in J_u \text{ and } [u](x)\cdot\xi \neq 0\}$, we have

$$J_{u^\xi_y} = (J^\xi_u)^\xi_y \quad\text{and}\quad [u](y+t\xi)\cdot\xi = [u^\xi_y](t) \quad \text{for every } t \in (J^\xi_u)^\xi_y;$$

(c) the distributional symmetric gradient $Eu := (Du + Du^T)/2$, which by definition is a bounded Radon measure taking values in the space $\mathbb{R}^{d\times d}_{\rm sym}$ of $d\times d$ symmetric matrices, can be decomposed as the sum of three mutually singular measures:

$$Eu = (\mathcal{E}u)\mathcal{L}^d + E^c u + ([u]\odot\nu_u)\mathcal{H}^{d-1} \llcorner J_u, \tag{1.2}$$

where $\mathcal{L}^d$ is the Lebesgue measure in $\mathbb{R}^d$, $E^c u$ is called the *Cantor part* of Eu, $\odot$ is the symmetric tensor product, and $\mathcal{H}^{d-1}\llcorner J_u$ is the restriction of $\mathcal{H}^{d-1}$ to J_u; the Cantor part $E^c u$ is singular with respect to $\mathcal{L}^d$ and vanishes on all Borel sets that are σ-finite with respect to $\mathcal{H}^{d-1}$.

In $GBD(\Omega)$ property (c) would not make sense, since, in general, the symmetrised gradient Eu cannot be defined in the sense of distributions. In particular, it is not clear what is the analogue of $E^c u$ for a function $u \in GBD(\Omega)$. Understanding how to generalise this term is crucial for possible applications to cohesive fracture mechanics, as shown in the corresponding problems for the anti-plane case (see [24, 25]).

However, for $u \in GBD(\Omega)$ it is shown in [20] that property (b) still holds and also that u admits an *approximate symmetric gradient* $\mathcal{E}u$ that enjoys the slicing property (1.1). It is still an open question whether *every* $GBD(\Omega)$ function admits an approximate gradient.

In this paper we extend the analysis of the fine properties of functions in $GBD(\Omega)$ by introducing two *matrix-valued* measures μ_u and μ^c_u, which are closely related to the measures Eu and $E^c u$ when $u \in BD(\Omega)$. To describe this result, we fix some notation. Given $R > 0$, let $\tau_R\colon \mathbb{R}\to\mathbb{R}$ be the 1-Lipschitz function defined by

$$\tau_R(s) := \begin{cases} -\frac{R}{2} & \text{if } s \le -\frac{R}{2}, \\ s & \text{if } \frac{-R}{2} \le s \le \frac{R}{2}, \\ \frac{R}{2} & \text{if } s \ge \frac{R}{2}. \end{cases}$$

It follows from the definition of $GBD(\Omega)$ (see Definition 2.5 and Remark 2.8) that for every $\xi \in \mathbb{S}^{d-1}$ and $R > 0$ the distributional derivative of $\tau_R(u\cdot\xi)$ in direction ξ, denoted by $D_\xi(\tau_R(u\cdot\xi))$, is a scalar-valued bounded Radon measure on Ω.

The main result of our paper is that (see Corollary 8.6) for every $u \in GBD(\Omega)$ and $r > 0$, there exists a bounded Radon measure $\mu_{u,r}$ on Ω with values in $\mathbb{R}^{d\times d}_{\rm sym}$ such that for every $\xi \in \mathbb{S}^{d-1}$ we have

$$\mu_{u,r}(B)\xi\cdot\xi = \lim_{R\to+\infty} D_\xi(\tau_R(u\cdot\xi))(B) \tag{1.3}$$

for every Borel set $B \subset \Omega$ such that $B \cap J^r_u = \emptyset$, where $J^r_u := \{x \in J_u : |[u](x)| \ge r\}$.

If $u \in BD(\Omega)$ we can see that $\mu_{u,r}(B) = (Eu)(B)$ for every Borel set $B \subset \Omega$ with $B \cap J_u^r = \emptyset$ (see Remark 8.7). In the general case $u \in GBD(\Omega)$, the measure $\mu_{u,r}$ is not always the symmetrised distributional gradient Eu, and its connection with the distributional derivatives of u is given only by (1.3), which takes into account the directional derivatives of suitable truncations of the scalar components of u. However, the measure $\mu_{u,r}$ enjoys many of the formal properties of Eu and in particular an analogue of property (c) above holds for $\mu_{u,r}$.

More precisely, in Corollary 8.6, we prove that for every $u \in GBD(\Omega)$ the measure $\mu_{u,r}$ can be decomposed as the sum of three mutually singular measures:

$$\mu_{u,r} = \mu_u^a + \mu_u^c + \mu_{u,r}^j, \tag{1.4}$$

where for every Borel set $B \subset \Omega$

$$\mu_u^a(B) = \int_B \mathcal{E}u \,\mathrm{d}x, \qquad \mu_{u,r}^j(B) = \int_{(J_u \setminus J_u^r)\cap B} ([u] \odot \nu_u) \,\mathrm{d}\mathcal{H}^{d-1},$$

and μ_u^c is a singular measure (with respect to the d-dimensional Lebesgue measure) with values in $\mathbb{R}^{d\times d}_{\mathrm{sym}}$ that vanishes on all σ-finite Borel sets with respect to $\mathcal{H}^{d-1}$ (see Proposition 8.8). We remark that both μ_u^a and μ_u^c do not depend on r.

A slicing property of the measure μ_u^c, obtained in Proposition 8.5, allows us to characterise the space $GSBD(\Omega)$, introduced in [20, Definition 4.2] using slicing arguments, as the set of functions $u \in GBD(\Omega)$ such that $\mu_u^c = 0$, in analogy with what happens for $SBD(\Omega)$ (see [6, Definition 4.6]). Combining this result with the recent characterisation of the space $GBD(\Omega)$, proved by Chambolle and Crismale in [13], we obtain, in Theorem 8.10, an analogous characterisation for the space $GSBD(\Omega)$.

We now give a brief sketch of how we prove the existence of a measure $\mu_u = \mu_{u,1}$ which satisfies (1.3) with $r = 1$. Straightforward arguments show that the limit in the right-hand side of (1.3) exists for every $\xi \in \mathbb{R}^d \setminus \{0\}$ (see Proposition 4.3) and that this limit is equal to

$$\sigma_u^\xi(B) := |\xi| \int_{\Pi^\xi} Du_y^\xi((B \setminus J_u^1)_y^\xi) \,\mathrm{d}\mathcal{H}^{d-1}(y), \tag{1.5}$$

where for every $E \subset \mathbb{R}^d$ and for every $y \in \Pi^\xi$ we set $E_y^\xi := \{t \in \mathbb{R} : y + t\xi \in E\}$. By the definition of $GBD(\Omega)$ (see Definition 2.5) the expression above defines a bounded Radon measure for every $\xi \in \mathbb{R}^d$. To conclude, one needs to show that for every Borel set $B \subset \Omega$ there exists $\mu_u(B) \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$ such that

$$\mu_u(B)\xi \cdot \xi = \sigma_u^\xi(B) \tag{1.6}$$

for every $\xi \in \mathbb{R}^d \setminus \{0\}$. To prove this fact, we will show that for every Borel set $B \subset \Omega$ the function $\xi \mapsto \sigma^\xi(B)$ is 2-homogeneous, lower bounded, and satisfies the parallelogram identity, i.e.,

$$\sigma_u^{\xi+\eta}(B) + \sigma_u^{\xi-\eta}(B) = 2\sigma_u^\xi(B) + 2\sigma_u^\eta(B) \tag{1.7}$$

for every $\xi, \eta \in \mathbb{R}^d$. Indeed, these three conditions imply (see Proposition 2.2) the existence of a symmetric matrix $\mu_u(B)$ such that (1.6) holds. It is then easy to check that $B \mapsto \mu_u(B)$ is a bounded Radon measure. Since the 2-homogeneity and the lower boundedness are easily obtained, to conclude we only need to show that (1.7) holds.

This is done first in dimension $d = 2$ by means of a discretisation argument. We fix two linearly independent vectors $\xi, \eta \in \mathbb{R}^2$ and assume that B is a parallelogram with sides parallel to ξ and η. For every $\zeta \in \{\xi, \eta, \xi + \eta, \xi - \eta\}$ we approximate the integral $\sigma_u^\zeta(B)$ given by (1.5) by means of *Riemann sums* corresponding to a well-chosen grid of points y_j of Π^ζ and write each term $Du_{y_j}^\zeta((B \setminus J_u^1)_{y_j}^\zeta) = Du_{y_j}^\zeta(B_{y_j}^\zeta \setminus (J_u^1)_{y_j}^\zeta)$ as a sum over i of the numbers $Du_{y_j}^\zeta(I_i^j \setminus (J_u^1)_{y_j}^\zeta)$, where $I_i^j = [a_i^j, a_{i+1}^j)$ are well-chosen disjoint small intervals, whose union is the interval $B_{y_j}^\zeta$. The points y_j and a_i^j can be chosen by projecting onto the straight lines Π^ζ and $\{y_j + t\zeta : t \in \mathbb{R}\}$ the points $x_{i,j}$ of a two-dimensional grid, constructed using discrete linear combinations of ξ and η, and translated by a small vector ω to be chosen carefully. We observe that if $I_i^j \cap (J_u^1)_{y_j}^\zeta = \emptyset$, then

$$Du_{y_j}^\zeta(I_i^j \setminus (J_u^1)_{y_j}^\zeta) = u(y_j + a_{i+1}^j\zeta)\cdot\zeta - u(y_j + a_i^j\zeta)\cdot\zeta.$$

This leads to an approximation of $\sigma_u^\zeta(B)$ based on the difference of the values of u on neighbouring grid-points $x_{i,j}$. Hence, if $I_i^j \cap (J_u^1)_{y_j}^\zeta = \emptyset$, writing carefully this approximation for every $\zeta \in \{\xi, \eta, \xi+\eta, \xi-\eta\}$ we obtain a discrete version of the parallelogram identity (1.7). To conclude, we have to show that in this approximation we can neglect all terms that correspond to pairs i, j such that $I_i^j \cap (J_u^1)_{y_j}^\zeta \neq \emptyset$. This step constitutes the main difficulty of the proof and will be the content of Section 6. This result is obtained by regarding the sum of the contribution of these ill-behaved indices as sort of *Riemann sum* of arbitrarily small integrals.

Once the planar case $d = 2$ is settled, the general case $d > 2$ can be obtained by a Fubini-type argument, considering a sort of two dimensional slicing in the spirit of [6]. In our case, this is based on the properties of the restrictions of GBD functions to two dimensional slices proved in [20] (see Theorem 2.14 below).

The paper is structured as follows. In Section 2 we introduce the basic notions and the necessary tools we will use throughout the paper, while in Section 3 we present some technical results concerning approximation of Lebesgue integrals by means of Riemann sums. Then, we introduce in Section 4 the measures σ_u^ξ, which will be the main focus of the rest of the paper, and prove several properties of these measures. Section 5 is devoted to the proof of the quadraticity of the function $\xi \mapsto \sigma_u^\xi(B)$ in the planar case $d = 2$. In Section 6 we complete this proof by means of some technical arguments. This result is then extended to every dimension in Section 7. In Section 8, we prove the decomposition (1.4) and deduce from it several consequences. The Appendix is devoted to proving the measurability of several auxiliary functions appearing in the arguments of Section 6.

2. Notation and preliminary results

In this section we fix the notation and lay down the basic tools used in this paper. Ω is a bounded open set of $\mathbb{R}^d$ with $d \geq 1$. The scalar product in $\mathbb{R}^d$ is denoted by $\cdot$, while the Euclidean norm of $\mathbb{R}^d$ is denoted by $|\ |$. For every $\rho > 0$ and $x \in \mathbb{R}^d$ the open ball of centre x and radius ρ is denoted by $B_\rho(x)$. The unit sphere of $\mathbb{R}^d$ is denoted by $\mathbb{S}^{d-1} := \{\xi \in \mathbb{R}^d : |\xi| = 1\}$. The vector space $\mathbb{R}^{d\times d}$ is identified with the space of $d \times d$ matrices. Given $A \in \mathbb{R}^{d\times d}$, its ij-th component is denoted by A_{ij}. For $A \in \mathbb{R}^{d\times d}$ and $\xi \in \mathbb{R}^d$, $A\xi \in \mathbb{R}^d$ is defined via the standard rules of matrix multiplication. The symbol $\mathbb{R}^{d\times d}_{\rm sym}$ denotes the space of all $d\times d$ symmetric matrices, that is, the space of those matrices $A \in \mathbb{R}^{d\times d}$ such that $A = A^T$, where A^T is the transpose of A. We recall that all matrices $A \in \mathbb{R}^{d\times d}_{\rm sym}$ satisfy the polarisation identity, i.e.,

$$A\xi\cdot\eta = \frac{1}{4}\big(A(\xi+\eta)\cdot(\xi+\eta) - A(\xi-\eta)\cdot(\xi-\eta)\big)$$

for every $\xi, \eta \in \mathbb{R}^d$.

We recall the definition of quadratic function.

Definition 2.1. A function $f\colon \mathbb{R}^d \to \mathbb{R}$ is quadratic if there exists a matrix $A \in \mathbb{R}^{d\times d}_{\rm sym}$ such that $f(\xi) = A\xi\cdot\xi$ for every $\xi \in \mathbb{R}^d$.

We recall the following characterisation of quadratic functions.

Proposition 2.2. *A function $f\colon \mathbb{R}^d \to \mathbb{R}$ is quadratic if and only if the following conditions are satisfied:*

(a) 2-homogeneity: $f(t\xi) = t^2 f(\xi)$ for every $\xi \in \mathbb{R}^d$ and every $t \in \mathbb{R}$;
(b) parallelogram identity: for every $\xi, \eta \in \mathbb{R}^d$ we have

$$f(\xi+\eta) + f(\xi-\eta) = 2f(\xi) + 2f(\eta);$$

(c) lower bound: there exists a constant $c > 0$ *such that*

$$f(\xi) \geq -c|\xi|^2 \quad \textit{for every } \xi \in \mathbb{R}^d.$$

Proof. Assume that (a)-(c) hold. By applying [19, Proposition 11.9] to the function $g(\xi) = f(\xi) + c|\xi|^2$ we obtain a matrix $B \in \mathbb{R}^{d\times d}_{\rm sym}$ such that $g(\xi) = B\xi \cdot \xi$ for every $\xi \in \mathbb{R}^d$. Setting $A := B - cI$, where I is the identity matrix, we obtain $f(\xi) = A\xi \cdot \xi$ for every $\xi \in \mathbb{R}^d$.

The converse implication is trivial. □

Given two vectors $\xi, \eta \in \mathbb{R}^d$, the symmetric tensor product $\xi \odot \eta \in \mathbb{R}^{d\times d}_{\rm sym}$ is defined by $(\xi \odot \eta)_{ij} := \frac{1}{2}(\xi_i\eta_j + \xi_j\eta_i)$. Given two distinct points $x_1, x_2 \in \mathbb{R}^d$ we set

$$[x_1, x_2] := \{tx_1 + (1-t)x_2 : t \in [0,1]\}.$$

The notation naturally extends to $[x_1, x_2)$, $(x_1, x_2]$, and (x_1, x_2) replacing $[0,1]$ by $[0,1)$, $(0,1]$, and $(0,1)$, respectively. Given a set $A \subset \Omega$, we say that A is relatively compact in Ω and write $A \subset\subset \Omega$ if there exists a compact set $K \subset \Omega$ such that $A \subset K$. Given $k \in \mathbb{N}$ and $E \subset \mathbb{R}^k$ the characteristic function of E is the function $\chi_E \colon \mathbb{R}^k \to \{0,1\}$ such that $\chi_E(x) = 1$ if $x \in E$ and 0 otherwise.

Given a finite dimensional real normed vector space X, $\mathcal{M}_b(\Omega; X)$ is the space of all bounded Radon measures with values in X; the indication of X is omitted if $X = \mathbb{R}$. The symbol $\mathcal{M}_b^+(\Omega)$ denotes the space of all positive bounded Radon measures. Given $\mu \in \mathcal{M}_b(\Omega; X)$ and $\lambda \in \mathcal{M}_b^+(\Omega)$, $\mathrm{d}\mu/\mathrm{d}\lambda$ is the Radon-Nikodým derivative of μ with respect to λ. Given a measure $\mu \in \mathcal{M}_b(\Omega; X)$, the total variation measure of μ with respect to the norm $|\ |$ on X, is the Borel measure defined for Borel set $B \subset \Omega$ by

$$|\mu|(B) := \sup \sum_{i\in I} |\mu(B_i)|,$$

where the *supremum* is taken over all finite sets $I \subset \mathbb{N}$ and all Borel partitions $(B_i)_{i\in I}$ of B. Given a measure $\lambda \in \mathcal{M}_b^+(\Omega)$ and a Borel measurable function $f \colon \Omega \to X$, the symbol $f\lambda$ denotes the X-valued measure defined for every Borel set $B \subset \Omega$ by

$$f\mu(B) := \int_B f \, \mathrm{d}\lambda.$$

The k-dimensional Lebesgue measure and the k-dimensional Hausdorff measure are denoted by $\mathcal{L}^k$ and $\mathcal{H}^k$, respectively. Given a measure μ on Ω and a μ-measurable set $B \subset \Omega$, we introduce the measure $\mu \llcorner B$ defined by $(\mu \llcorner B)(E) := \mu(B \cap E)$ for every Borel set $E \subset \Omega$.

We say that $E \subset \mathbb{R}^d$ is $(\mathcal{H}^{d-1}, d-1)$-*rectifiable* if there exist a collection of compact sets $(K_i)_{i\in\mathbb{N}}$, a collection of $(d-1)$-dimensional C^1 manifolds $M_i \subset \mathbb{R}^d$, with $K_i \subset M_i$ and $\mathcal{H}^{d-1}(M_i) < +\infty$ for every $i \in \mathbb{N}$, and a set N_0 with $\mathcal{H}^{d-1}(N_0) = 0$ such that

$$E = N_0 \cup \Big(\bigcup_{i\in\mathbb{N}} K_i \Big). \tag{2.1}$$

We refer the reader to [7, Chapter 2] and to [28, Chapter 3] for the properties of these sets.

Approximate limits. Let E be a Lebesgue measurable subset of $\mathbb{R}^d$ and let $x \in \mathbb{R}^d$ be such that

$$\limsup_{\rho\to 0} \frac{\mathcal{L}^d(E \cap B_\rho(x))}{\rho^d} > 0. \tag{2.2}$$

We say that an $\mathcal{L}^d$-measurable function $u \colon E \to \mathbb{R}^m$ has approximate limit $\widetilde{u}(x) \in \mathbb{R}^m$ at x, in symbols

$$\operatorname*{ap\lim}_{\substack{y\to x\\ y\in E}} u(y) = \widetilde{u}(x), \tag{2.3}$$

if for every $\varepsilon > 0$ we have

$$\lim_{\rho\to 0} \frac{\mathcal{L}^d(\{y \in E : |u(y) - \widetilde{u}(x)| > \varepsilon\} \cap B_\rho(x))}{\rho^d} = 0.$$

Throughout the paper the symbol $\widetilde{u}(x)$ is always used to denote the approximate limit of u considered in (2.3). By (2.2) the vector $\widetilde{u}(x)$ is uniquely defined. The set S_u is defined as the complement of the points where the approximate limit exists. It is well-known that for an $\mathcal{L}^d$-measurable function $u\colon \mathbb{R}^d \to \mathbb{R}^m$ it holds $\mathcal{L}^d(S_u) = 0$.

Jump set. We now give the definition of jump set of a measurable function.

Definition 2.3. Let U be an open set of $\mathbb{R}^d$ and $u\colon U \to \mathbb{R}^m$ an $\mathcal{L}^d$-measurable function. The *jump set* J_u is the set of all points $x \in U$ such that there exists $(u^+(x), u^-(x), \nu_u(x)) \in \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{S}^{d-1}$, with $u^+(x) \neq u^-(x)$, such that

$$\operatorname*{ap\,lim}_{\substack{y\to x\\ y\in H^\pm(x)\cap U}} u(y) = u^\pm(x),$$

where $H^\pm(x) := \{y \in \mathbb{R}^d : \pm(y - x)\cdot \nu_u(x) > 0\}$. The triple $(u^+(x), u^-(x), \nu_u(x))$ is uniquely defined up to swapping $u^+(x)$ and $u^-(x)$ and changing the sign of $\nu_u(x)$. Given $x \in J_u$, we set $[u](x) := u^+(x) - u^-(x)$. For $r > 0$ we also introduce J_u^r as the set defined by

$$J_u^r := \{x \in J_u : |[u](x)| \geq r\}. \tag{2.4}$$

Slicing. For every $\xi \in \mathbb{R}^d \setminus \{0\}$, $y \in \mathbb{R}^d$, and $A \subset \Omega$ we define

$$A_y^\xi := \{t \in \mathbb{R} : y + t\xi \in A\}.$$

Given a function $u\colon A \to \mathbb{R}^d$, we define the slice in direction ξ of the ξ-component of u as the function $u_y^\xi\colon \mathbb{R} \to \mathbb{R}^d$ defined by

$$u_y^\xi(t) := \begin{cases} u(y + t\xi)\cdot\xi & \text{if } t \in A_y^\xi,\\ 0 & \text{otherwise.}\end{cases}$$

Π^ξ denotes the hyperplane orthogonal to ξ and passing through 0, that is,

$$\Pi^\xi := \{y \in \mathbb{R}^d : y\cdot\xi = 0\}.$$

The projection map onto Π^ξ is denoted by $\pi^\xi\colon \mathbb{R}^d \to \Pi^\xi$. We shall use the following estimate.

Lemma 2.4. *Let $\xi \in \mathbb{R}^d \setminus \{0\}$ and let $B \subset \mathbb{R}^d$ be a Borel set. Then the function $y \mapsto \mathcal{H}^0(B_y^\xi)$ is $\mathcal{H}^{d-1}$-measurable on Π^ξ and*

$$\int_{\Pi^\xi} \mathcal{H}^0(B_y^\xi)\, \mathrm{d}\mathcal{H}^{d-1}(y) \leq \mathcal{H}^{d-1}(B). \tag{2.5}$$

Proof. For every $k \in \mathbb{N}$ and $i \in \mathbb{Z}$ we set $B_{k,i} := \{x \in B : i/2^k \leq x\cdot\xi < (i+1)/2^k\}$ and we consider the function $f_k\colon \Pi^\xi \to [0, +\infty]$ defined by

$$f_k(y) := \sum_{i\in\mathbb{Z}} \chi_{\pi^\xi(B_{k,i})}(y), \tag{2.6}$$

where $\chi_{\pi^\xi(B_{k,i})}$ is the characteristic function of the projection of $B_{k,i}$ onto Π^ξ. We observe that $f_k \leq f_{k+1}$ for every $k \in \mathbb{N}$.

It follows from the definition of B_y^ξ that

$$\mathcal{H}^0(B_y^\xi) = \lim_{k\to+\infty} f_k(y) \quad \text{for every } y \in \Pi^\xi. \tag{2.7}$$

By the Projection Theorem (see, e.g., [15, Proposition 8.4.4]) for every $k \in \mathbb{N}$ and $i \in \mathbb{Z}$ the set $\pi^\xi(B_{k,i})$ is $\mathcal{H}^{d-1}$-measurable on Π^ξ. By (2.6) and (2.7) this implies that the same is true for the function $y \mapsto \mathcal{H}^0(B^\xi_y)$. By the Monotone Convergence Theorem we obtain from (2.7) that

$$\int_{\Pi^\xi} \mathcal{H}^0(B^\xi_y)\,\mathrm{d}\mathcal{H}^{d-1}(y) = \lim_{k\to+\infty} \int_{\Pi^\xi} f_k(y)\,\mathrm{d}\mathcal{H}^{d-1}(y). \tag{2.8}$$

By (2.6) we have

$$\int_{\Pi^\xi} f_k(y)\,\mathrm{d}\mathcal{H}^{d-1}(y) = \sum_{i\in\mathbb{Z}} \mathcal{H}^{d-1}(\pi^\xi(B_{k,i})) \le \sum_{i\in\mathbb{Z}} \mathcal{H}^{d-1}(B_{k,i}) = \mathcal{H}^{d-1}(B),$$

which, together with (2.8), gives (2.5). □

Functions of generalised bounded deformation. We recall the definition of the space of *functions of generalised bounded deformation*, introduced in [20, Definition 4.1]. This definition uses the collection $\mathcal{T}$ of regular truncation functions defined by

$$\mathcal{T} := \{\tau \in C^1(\mathbb{R}) : -1/2 \le \tau \le 1/2 \ \text{ and } \ 0 \le \tau' \le 1\}.$$

Definition 2.5. The space $GBD(\Omega)$ is the space of all $\mathcal{L}^d$-measurable functions $u\colon \Omega \to \mathbb{R}^d$ for which there exists a measure $\lambda \in \mathcal{M}^+_b(\Omega)$ such that the following equivalent conditions are satisfied for every $\xi \in \mathbb{S}^{d-1}$:

(a) for every $\tau \in \mathcal{T}$ we have $D_\xi(\tau(u\cdot\xi)) \in \mathcal{M}_b(\Omega)$ and

$$|D_\xi(\tau(u\cdot\xi))|(B) \le \lambda(B) \quad \text{for every Borel set } B \subset \Omega; \tag{2.9}$$

(b) for $\mathcal{H}^{d-1}$-a.e. $y \in \Pi^\xi$ we have $u^\xi_y \in BV_{\mathrm{loc}}(\Omega^\xi_y)$ and

$$\int_{\Pi^\xi} \big(|Du^\xi_y|(B^\xi_y \setminus J^1_{u^\xi_y}) + \mathcal{H}^0(B^\xi_y \cap J^1_{u^\xi_y})\big)\mathrm{d}\mathcal{H}^{d-1}(y) \le \lambda(B) \ \text{ for every Borel set } B \subset \Omega. \tag{2.10}$$

Remark 2.6. If $u \in GBD(\Omega)$ it follows from (b) of Definition 2.5 that for every $\xi \in \mathbb{S}^{d-1}$ there exists a Borel set $N_\xi \subset \Pi^\xi$, with $\mathcal{H}^{d-1}(N_\xi) = 0$, such that $u^\xi_y \in BV_{\mathrm{loc}}(\Omega^\xi_y)$ and $|Du^\xi_y|(\Omega^\xi_y) < +\infty$ for every $y \in \Pi^\xi \setminus N_\xi$. In particular, if Ω^ξ_y is an interval and $y \in \Pi^\xi \setminus N_\xi$, then $u^\xi_y \in BV(\Omega^\xi_y)$.

Remark 2.7. The previous remark implies that, if $d = 1$, then $GBD(\Omega) := \{u \in BV_{\mathrm{loc}}(\Omega) : |Du|(\Omega) < +\infty\}$.

Remark 2.8. Condition (a) of Definition 2.5 can be strengthened by requiring that (2.9) holds also for every $\tau \in \mathcal{T}_{\mathrm{Lip}}$, where

$$\mathcal{T}_{\mathrm{Lip}} := \{\tau \in \mathrm{Lip}(\mathbb{R}) : -1/2 \le \tau \le 1/2,\ 0 \le \tau' \le 1,\ \tau' \text{ has compact support}\}.$$

Indeed, every function $\tau \in \mathcal{T}_{\mathrm{Lip}}$ can be approximated uniformly on $\mathbb{R}$ by a sequence $(\tau_n)_n \subset \mathcal{T}$, so that $\tau_n(u\cdot\xi) \to \tau(u\cdot\xi)$ in $L^1(\Omega)$. Since for every $U \subset \Omega$ open, the function $v \mapsto |D_\xi v|(U)$ is lower semicontinuous with respect to the $L^1(\Omega)$ convergence, by (2.9) we obtain that $D_\xi(\tau(u\cdot\xi)) \in \mathcal{M}_b(\Omega)$ and that

$$|D_\xi(\tau(u\cdot\xi))|(U) \le \liminf_{n\to+\infty} |D_\xi(\tau_n(u\cdot\xi))|(U) \le \lambda(U)$$

for every $U \subset \Omega$ open. Inequality (2.9) for a general Borel set $B \subset \Omega$ follows by approximation with open sets.

Remark 2.9. Inequality (2.10) can be extended to non-unitary vectors. Elementary arguments show that for every $t \neq 0$, $\xi \in \mathbb{R}^d \setminus \{0\}$, $y \in \Pi^\xi$, and $A \subset \mathbb{R}^d$

$$A^\xi_y = tA^{t\xi}_y. \tag{2.11}$$

Since $u^{t\xi}_y(s) = tu^\xi_y(st)$ for every $s \in \mathbb{R}$, we have

$$Du^{t\xi}_y(S) = |t|Du^\xi_y(tS) \quad \text{and} \quad |Du^{t\xi}_y|(S) = |t||Du^\xi_y|(tS),$$

$$tJ_{u^{t\xi}_y} = J_{u^\xi_y} \quad \text{and} \quad tJ^{|t|}_{u^{t\xi}_y} = J^1_{u^\xi_y} \tag{2.12}$$

for every Borel set $S \subset \Omega^{t\xi}_y$. In particular, if $B \subset \Omega$ is a Borel set, taking $S = B^{t\xi}_y \setminus J^{|t|}_{u^{t\xi}_y}$ and using (2.11) and (2.12) we get

$$|Du^{t\xi}_y|(B^{t\xi}_y \setminus J^{|t|}_{u^{t\xi}_y}) = |t||Du^{\xi}_y|(B^{\xi}_y \setminus J^{1}_{u^{\xi}_y}), \tag{2.13}$$

while, taking $S = B^{t\xi}_y$ and using (2.11)

$$Du^{t\xi}_y(B^{t\xi}_y) = |t|Du^{\xi}_y(B^{\xi}_y), \tag{2.14}$$

which will be used later in the proof of Proposition 4.4. Applying (2.12) and (2.13) with $t = |\xi|$ and ξ replaced by $\xi/|\xi|$, from (2.10) we obtain

$$\int_{\Pi^\xi} |\xi||Du^{\xi}_y|(B^{\xi}_y \setminus J^{|\xi|}_{u^{\xi}_y}) \mathrm{d}\mathcal{H}^{d-1}(y) \leq |\xi|^2 \lambda(B), \tag{2.15}$$
$$\int_{\Pi^\xi} \mathcal{H}^0(B^{\xi}_y \cap J^{|\xi|}_{u^{\xi}_y}) \mathrm{d}\mathcal{H}^{d-1}(y) \leq \lambda(B),$$

for every $\xi \in \mathbb{R}^d \setminus \{0\}$ and every Borel set $B \subset \Omega$.

Remark 2.10. It follows from [4, Remark 3.6] that, if $u \in GBD(\Omega)$, then $\mathcal{H}^{d-1}(J^1_u) < +\infty$. More precisely, it is shown that, if $\lambda \in \mathcal{M}^+_b(\Omega)$ satisfies (2.10), then for every Borel set $B \subset \Omega$ we have the inequality

$$\mathcal{H}^{d-1}(J^1_u \cap B) \leq 4d\lambda(B). \tag{2.16}$$

Since by Theorem 2.12 below for $\mathcal{H}^{d-1}$-a.e. $y \in \Pi^\xi$ we have

$$J^{|\xi|}_{u^{\xi}_y} \subset (J^1_u)^{\xi}_y,$$

it follows from (2.5), (2.15), and (2.16) that

$$\int_{\Pi^\xi} |\xi||Du^{\xi}_y|((B \setminus J^1_u)^{\xi}_y) \mathrm{d}\mathcal{H}^{d-1}(y) \leq |\xi|^2 \lambda(B), \tag{2.17}$$
$$\int_{\Pi^\xi} \mathcal{H}^0((B \cap J^1_u)^{\xi}_y) \mathrm{d}\mathcal{H}^{d-1}(y) \leq 4d\lambda(B).$$

Given $0 < r \leq 1$, we can consider the function $v = u/r$, which by [20, Remark 4.6] satisfies (2.10) with λ/r. Applying (2.16) to this function, we obtain

$$\mathcal{H}^{d-1}(J^r_u \cap B) \leq \frac{4d}{r}\lambda(B) \tag{2.18}$$

for every Borel set $B \subset \Omega$. In particular, this implies that $[u] \in L^1_{\mathrm{weak}}(J_u, \mathcal{H}^{d-1})$. Moreover, since $J_u = \bigcup_{0<r\leq 1} J^r_u$, we have that J_u is σ-finite with respect to $\mathcal{H}^{d-1}$.

Definition 2.11. For every $u \in GBD(\Omega)$ and $\xi \in \mathbb{R}^d \setminus \{0\}$ we introduce the bounded Radon measure λ^ξ_u defined for every $\xi \in \mathbb{R}^d \setminus \{0\}$ and every Borel set $B \subset \Omega$ by

$$\lambda^\xi_u(B) := \int_{\Pi^\xi} \big(|\xi||Du^{\xi}_y|(B^{\xi}_y \setminus J^{|\xi|}_{u^{\xi}_y}) + \mathcal{H}^0(B^{\xi}_y \cap J^{|\xi|}_{u^{\xi}_y})\big) \mathrm{d}\mathcal{H}^{d-1}(y). \tag{2.19}$$

Given $u \in GBD(\Omega)$, let λ_u be the smallest measure λ for which (a) and (b) of Definition 2.5 hold true. It can be shown (see [20, Proposition 4.17]) that

$$\lambda_u(B) = \sup \sum_{i=1}^k \lambda^{\xi_i}_u(B_i) \quad \text{for every Borel set } B \subset \Omega, \tag{2.20}$$

where the supremum is taken over all $k \in \mathbb{N}$, all families $(\xi_i)^k_{i=1}$ of vectors of $\mathbb{S}^{d-1}$, and all Borel partitions $(B_i)^k_{i=1}$ of B.

The following theorem collects some of the fine properties of $GBD(\Omega)$ functions, proved in [20, Proposition 6.1 and Theorems 6.2, 8.1, 9.1].

Theorem 2.12. *Let $u \in GBD(\Omega)$. Then the following properties hold:*

(a) *existence of the approximate symmetric gradient: there exists $\mathcal{E}u \in L^1(\Omega;\mathbb{R}^{d\times d}_{\rm sym})$ such that for $\mathcal{L}^d$-a.e. $x \in \Omega$ we have*

$$\operatorname*{aplim}_{y\to x} \frac{(u(y)-u(x)-\mathcal{E}u(x)(y-x))\cdot(y-x)}{|x-y|^2} = 0;$$

moreover, for every $\xi \in \mathbb{R}^d \setminus \{0\}$ and for $\mathcal{H}^{d-1}$-a.e. $y \in \Pi^\xi$ we have

$$\mathcal{E}u(y+t\xi)\xi\cdot\xi = \nabla u^\xi_y(t)$$

for $\mathcal{L}^1$-a.e. $t \in \Omega^\xi_y$, where ∇u^ξ_y denotes the density of the absolutely continuous part of Du^ξ_y with respect to $\mathcal{L}^1$;

(b) *J_u and its slices: the jump set J_u is $(\mathcal{H}^{d-1}, d-1)$-rectifiable; for every $\xi \in \mathbb{R}^d \setminus \{0\}$ and for $\mathcal{H}^{d-1}$-a.e. $y \in \Pi^\xi$ we have*

$$J_{u^\xi_y} = \{x \in J_u : [u](x)\cdot\xi \neq 0\}^\xi_y \subset (J_u)^\xi_y, \tag{2.21}$$

$$(u^\xi_y)^\pm(t) = u^\pm(y+t\xi)\cdot\xi \quad \textit{for every } t \in (J_u)^\xi_y, \tag{2.22}$$

$$J^{|\xi|}_{u^\xi_y} \subset (J^1_u)^\xi_y, \tag{2.23}$$

where the normals at J_u and $J_{u^\xi_y}$ are oriented in such a way that $\nu_u\cdot\xi \geq 0$ and $\nu_{u^\xi_y} = 1$; moreover, setting

$$\Theta_u := \Big\{x \in \Omega : \limsup_{\rho\to 0} \frac{\lambda_u(B_\rho(x))}{\rho^{d-1}} > 0\Big\}, \tag{2.24}$$

we have that Θ_u is $(\mathcal{H}^{d-1}, d-1)$-rectifiable, $J_u \subset \Theta_u$, and that $\mathcal{H}^{d-1}(\Theta_u \setminus J_u) = 0$.

It is easy to see that for a function $u \in GBD(\Omega) \setminus BD(\Omega)$ its jump $[u]$ may be not integrable on J_u with respect to $\mathcal{H}^{d-1}$ (see, for instance, [20, Example 12.3]). More precisely, one can show (see, for instance, [6, Proposition 3.2] or [38, Remark 2.17]) that, if $u \in GBD(\Omega) \cap L^1(\Omega;\mathbb{R}^d)$ and $[u] \in L^1(J_u,\mathcal{H}^{d-1})$, then u belongs to $BD(\Omega)$. Nonetheless, we now show that $[u]$ is integrable on $J_u \setminus J^1_u$ with respect to $\mathcal{H}^{d-1}$. This is the content of the following proposition.

Proposition 2.13. *Let $u \in GBD(\Omega)$. Then*

$$\int_{J_u\setminus J^1_u} |([u]\odot\nu_u)\xi\cdot\xi| \,\mathrm{d}\mathcal{H}^{d-1} < +\infty \tag{2.25}$$

for every $\xi \in \mathbb{R}^d$. This is equivalent to $[u] \in L^1(J_u \setminus J^1_u, \mathcal{H}^{d-1})$.

Proof. By homogeneity it is enough to show (2.25) for a fixed $\xi \in \mathbb{S}^{d-1}$. By definition of $\odot$ we have $([u]\odot\nu_u)\xi\cdot\xi = ([u]\cdot\xi)(\nu\cdot\xi)$. Therefore, by the Area Formula [36, 12.4], we can write

$$\begin{aligned}\int_{J_u\setminus J^1_u} |([u]\odot\nu_u)\xi\cdot\xi| \,\mathrm{d}\mathcal{H}^{d-1} &= \int_{J_u\setminus J^1_u} |([u]\cdot\xi)(\nu_u\cdot\xi)| \,\mathrm{d}\mathcal{H}^{d-1}\\ &= \int_{\Pi^\xi}\Big(\int_{(J_u\setminus J^1_u)^\xi_y} |[u](y+t\xi)\cdot\xi| \,\mathrm{d}\mathcal{H}^0(t)\Big)\mathrm{d}\mathcal{H}^{d-1}(y)\\ &\leq \int_{\Pi^\xi}\Big(\int_{J_{u^\xi_y}\setminus J^1_{u^\xi_y}} |[u^\xi_y](t)| \,\mathrm{d}\mathcal{H}^0(t)\Big)\mathrm{d}\mathcal{H}^{d-1}(y),\end{aligned} \tag{2.26}$$

where in the inequality we have used (2.21)-(2.23). By (b) of Definition 2.5, for $\mathcal{H}^{d-1}$-a.e. $y \in \Pi^\xi$ we have $u^\xi_y \in BV_{\rm loc}(\Omega^\xi_y)$, so that

$$\begin{aligned}\int_{\Pi^\xi}\Big(\int_{J_{u^\xi_y}\setminus J^1_{u^\xi_y}} |[u^\xi_y](t)| \,\mathrm{d}\mathcal{H}^0(t)\Big)\mathrm{d}\mathcal{H}^{d-1}(y) &= \int_{\Pi^\xi} |Du^\xi_y|(J_{u^\xi_y}\setminus J^1_{u^\xi_y})\mathrm{d}\mathcal{H}^{d-1}(y)\\ &\leq \int_{\Pi^\xi} |Du^\xi_y|(\Omega^\xi_y\setminus J^1_{u^\xi_y})\mathrm{d}\mathcal{H}^{d-1}(y) \leq \lambda(\Omega) < +\infty,\end{aligned}$$

which, together with (2.26), concludes the proof of (2.25).

From the polarisation identity it follows that for every $\xi,\eta\in\mathbb{R}^d\setminus\{0\}$ one has

$$|([u]\odot\nu_u)\xi\cdot\eta|\le\frac14\big(|([u]\odot\nu_u)(\xi+\eta)\cdot(\xi+\eta)|+|([u]\odot\nu_u)(\xi-\eta)\cdot(\xi-\eta)|\big)$$

on J_u. Hence, by (2.25) we deduce that $[u]\odot\nu_u\in L^1(J_u\setminus J_u^1,\mathcal{H}^{d-1})$. Since for every $a,b\in\mathbb{R}^d$ we have $|a||b|\le\sqrt{2}|a\odot b|$, the proof is concluded. □

The space $GBD(\Omega)$ behaves nicely with respect to restriction to affine subspaces of the domain Ω. This fact is made rigorous by the following result.

Theorem 2.14. *Assume $d\ge 2$. Let $u\in GBD(\Omega)$, let V be a vector subspace of $\mathbb{R}^d$ of dimension k, with $1\le k\le d-1$, let $V^\perp$ be its orthogonal subspace, and let $\pi_V\colon\mathbb{R}^d\to V$ be the orthogonal projection onto V. For every $z\in V^\perp$ and $E\subset\Omega$ we set $E^V_z:=\{x\in V: z+x\in E\}=V\cap(E-z)$ and consider the function $u^V_z\colon\Omega^V_z\to V$ defined by $u^V_z(x):=\pi_V(u(z+x))$. Then the following properties hold:*

(a) for $\mathcal{H}^{d-k}$-a.e. $z\in V^\perp$ we have $u^V_z\in GBD(\Omega^V_z)$;

(b) for $\mathcal{H}^{d-k}$-a.e. $z\in V^\perp$ we have $J_{u^V_z}\subset(J_u)^V_z\cup N_z$ for a Borel set $N_z\subset V$ with $\mathcal{H}^{k-1}(N_z)=0$.

To prove property (b) we need the following result concerning the relation between the jump points of a function $u\in GBD(\Omega)$ and the jump points of its restriction to a hyperplane that does not intersect the set S_u of approximate discontinuity points.

Proposition 2.15. *Assume $d\ge2$. Let $u\in GBD(\Omega)$, $x_0\in\Omega$, and $\xi\in\mathbb{S}^{d-1}$. Assume that*

$$\mathcal{H}^{d-1}(S_u\cap(x_0+\Pi^\xi))=0. \tag{2.27}$$

Let $v\colon(\Omega-x_0)\cap\Pi^\xi\to\Pi^\xi$ be the function defined by $v(y):=\pi^\xi(\widetilde{u}(x_0+y))$ for $\mathcal{H}^{d-1}$-a.e. $y\in(\Omega-x_0)\cap\Pi^\xi$. Assume that there exist a direction $\nu\in\mathbb{S}^{d-1}\cap\Pi^\xi$ and two vectors $b^\pm\in\Pi^\xi$, with $b^+\ne b^-$, such that for every $\varepsilon>0$ we have

$$\limsup_{\rho\to0^+}\frac{\mathcal{H}^{d-1}(\{y\in B_\rho(0)\cap\Pi^\xi:\pm y\cdot\nu>0,\,|v(y)-b^\pm|>\varepsilon\})}{\rho^{d-1}}=0. \tag{2.28}$$

Then $x_0\in\Theta_u$.

Proof. See [20, Theorem 7.1]. □

Proof of Theorem 2.14. The proof of (a) can be found in [20, Theorem 4.19].

We divide the proof of (b) into two steps.

Step 1. Assume that $k=d-1$ and let $\xi\in\mathbb{S}^{d-1}$ be such that $V=\Pi^\xi$. We claim that for $\mathcal{L}^1$-a.e. $s\in\mathbb{R}$ there exists a Borel set $N_s\subset V$, with $\mathcal{H}^{d-2}(N_s)=0$, such that

$$J_{u^V_{s\xi}}\subset(J_u)^V_{s\xi}+N_s.$$

To prove this property, we observe that by the Fubini Theorem the equality $\mathcal{L}^d(S_u)=0$ implies that for $\mathcal{L}^1$-a.e. $s\in\mathbb{R}$ and for every $y_0\in\Pi^\xi=V$ condition (2.27) holds with $x_0=s\xi+y_0$, while the equality $\widetilde{u}=u$ $\mathcal{L}^d$-a.e. in Ω implies that for $\mathcal{L}^1$-a.e. $s\in\mathbb{R}$ we have

$$\widetilde{u}(s\xi+y)=u(s\xi+y)\quad\text{for }\mathcal{H}^{d-1}\text{-a.e. }y\in\Pi^\xi. \tag{2.29}$$

Let us fix $s\in\mathbb{R}$ with these properties. Given $y_0\in J_{u^V_{s\xi}}$, we consider the function $v(y):=\pi^\xi(\widetilde{u}(s\xi+y_0+y))$ for $y\in\Pi^\xi$ and observe that by (2.29) we have

$$v(y)=u^V_{s\xi}(y_0+y)\qquad\text{for }\mathcal{H}^{d-1}\text{-a.e. }y\in\Pi^\xi.$$

Since $y_0\in J_{u^V_{s\xi}}$, v satisfies (2.28). We can then apply Proposition 2.15 to obtain that $s\xi+y_0\in\Theta_u$, which gives $y_0\in(\Theta_u)^V_{s\xi}$. Setting $N_s:=(\Theta_u\setminus J_u)^V_{s\xi}$, we have $y_0\in(J_u)^V_{s\xi}\cup N_s$, hence $J_{u^V_{s\xi}}\subset(J_u)^V_{s\xi}\cup N_s$. Since by Theorem 2.12 we have $\mathcal{H}^{d-1}(\Theta_u\setminus J_u)=0$, it follows from [28, Theorem 2.10.25] that $\mathcal{H}^{d-2}(N_s)=0$ for $\mathcal{L}^1$-a.e. $s\in\mathbb{R}$. This concludes the proof of the claim. Of course, if $d=2$ this Step also completes the proof of (b).

Step 2. Assume $d>2$. We now prove (b) by induction on the codimension of V. Step 1 gives (b) when the dimension of V is $d-1$. Given $1\le k\le d-2$, we assume now that (b) holds for every subspace of dimension $k+1$ and we want to prove that it holds in dimension k. Let us fix a subspace V of dimension k and $\xi\in V^\perp\cap\mathbb{S}^{d-1}$. We consider the vector space $\widetilde{V}$ generated by V and ξ. By (a) for $\mathcal{H}^{d-k-1}$-a.e. $z\in\widetilde{V}^\perp$ we have that $u_z^{\widetilde{V}}\in GBD(\Omega_z^{\widetilde{V}})$. Using the inductive hypothesis, we deduce that for $\mathcal{H}^{d-k-1}$-a.e. $z\in\widetilde{V}^\perp$ there exists a Borel set $\widetilde{N}_z\subset\widetilde{V}$, with $\mathcal{H}^k(\widetilde{N}_z)$=0, such that

$$J_{u_z^{\widetilde{V}}}\subset(J_u)_z^{\widetilde{V}}\cup\widetilde{N}_z. \tag{2.30}$$

Let us fix $z\in\widetilde{V}^\perp$ satisfying both these properties. We observe that for every $s\in\mathbb{R}$ and $E\subset\Omega$ we have

$$E^V_{z+s\xi}=(E_z^{\widetilde{V}})^V_{s\xi}\quad\text{and}\quad u^V_{z+s\xi}=(u_z^{\widetilde{V}})^V_{s\xi}\quad\text{on}\in\Omega^V_{z+s\xi}. \tag{2.31}$$

Applying Step 1 with $\mathbb{R}^d$ replaced by $\widetilde{V}$ and with u replaced by $u_z^{\widetilde{V}}$, we have that for $\mathcal{L}^1$-a.e. $s\in\mathbb{R}$ there exists a Borel set $N_s\subset V$, with $\mathcal{H}^{k-1}(N_s)=0$, such that

$$J_{(u_z^{\widetilde{V}})^V_{s\xi}}\subset(J_{u_z^{\widetilde{V}}})^V_{s\xi}\cup N_s.$$

By (2.30) this implies that

$$J_{(u_z^{\widetilde{V}})^V_{s\xi}}\subset((J_u)_z^{\widetilde{V}})^V_{s\xi}\cup(\widetilde{N}_z)^V_{s\xi}\cup N_s.$$

By (2.31) this gives

$$J_{u^V_{z+s\xi}}\subset(J_u)^V_{z+s\xi}\cup(\widetilde{N}_z)^V_{s\xi}\cup N_s. \tag{2.32}$$

By [28, Theorem 2.10.25] we deduce that for $\mathcal{L}^1$-a.e. $s\in\mathbb{R}$ we have $\mathcal{H}^{k-1}((\widetilde{N}_z)^V_{s\xi})=0$. Setting $N_{z+s\xi}:=N_s\cup(\widetilde{N}_z)^V_{s\xi}$, it follows that $\mathcal{H}^{k-1}(N_{z+s\xi})=0$ for $\mathcal{L}^1$-a.e. $s\in\mathbb{R}$. Finally, from (2.32) we deduce that

$$J_{u^V_{z+s\xi}}\subset(J_u)^V_{z+s\xi}\cup N_{z+s\xi}. \tag{2.33}$$

Since $V^\perp$ is the space generated by $\widetilde{V}^\perp$ and ξ and (2.33) holds for $\mathcal{L}^1$-a.e. $s\in\mathbb{R}$ and for $\mathcal{H}^{d-k-1}$-a.e. $z\in\widetilde{V}^\perp$, property (b) for V follows from the Fubini Theorem. This concludes the proof of the inductive step and hence of the theorem. □

3. Approximations by Riemann sums

In the proof of the main result of this paper, we will approximate various integrals with well-chosen Riemann sums, using a suitable version of a result that goes back to Hahn (see [34]). Similar results are proved also in [10, Lemma A.1]. For technical reasons, we use a construction described in [27, Page 63] and further developed in [21, Lemma 4.12]. Since this result is crucial for our arguments, we give here the precise statement and a detailed proof.

Lemma 3.1. *Let $I=[a,b]\subset\mathbb{R}$ be a bounded closed interval. For every $z\in I$, $k\in\mathbb{N}$, and $i\in\mathbb{Z}$ let $t_i^k:=z+i/k$. Let $(X,\|\cdot\|)$ be a Banach space and let $f\colon\mathbb{R}\to X$ be a Bochner integrable function such that $f=0$ on $\mathbb{R}\setminus I$. Then there exist an infinite subset $K\subset\mathbb{N}$ and an $\mathcal{L}^1$-negligible set $N\subset I$ such that*

$$\lim_{\substack{k\to+\infty\\ k\in K}}\sum_{i\in\mathbb{Z}}\int_{t_i^k}^{t_{i+1}^k}\|f(t_i^k)-f(t)\|\,\mathrm{d}t=0, \tag{3.1}$$

for every $z \in I \setminus N$. Moreover, given $h \in \mathbb{N}$ and setting $\mathcal{I}^k_h := \{i \in \mathbb{Z} : [t^k_{i-h}, t^k_{i+h}] \subset I\}$ and $\mathcal{F}^k_h := \{i \in \mathbb{Z} : t^k_i \in I\} \setminus \mathcal{I}^k_h$, for every $\varepsilon > 0$ there exists a Borel set $I_\varepsilon \subset I$, with $\mathcal{L}^1(I \setminus I_\varepsilon) \le \varepsilon$, such that

$$\lim_{\substack{k\to+\infty\\ k\in K}} \sum_{i\in\mathbb{Z}} \int_{t^k_i}^{t^k_{i+1}} \|f(t^k_i) - f(t)\| \,\mathrm{d}t = 0 \qquad \textit{uniformly for } z \in I_\varepsilon, \tag{3.2}$$

$$\lim_{\substack{k\to+\infty\\ k\in K}} \frac{1}{k} \sum_{i\in\mathcal{I}^k_h} f(t^k_i) = \int_I f(t)\,\mathrm{d}t \qquad \textit{uniformly for } z \in I_\varepsilon, \tag{3.3}$$

$$\lim_{\substack{k\to+\infty\\ k\in K}} \frac{1}{k} \sum_{i\in\mathcal{F}^k_h} \|f(t^k_i)\| = 0 \qquad \textit{uniformly for } z \in I_\varepsilon. \tag{3.4}$$

Proof. We follow closely the proof of [21, Lemma 4.12]. Given $k \in \mathbb{N}$, we consider the set

$$\mathcal{J}^k := \{i \in \mathbb{Z} : k(a-b) - 1 \le i \le k(b-a)\}. \tag{3.5}$$

Note that for every $z \in I$ we have

$$[t^k_i, t^k_{i+1}] \cap I \neq \emptyset \quad \Longrightarrow \quad i \in \mathcal{J}^k.$$

Integrating with respect to z the sum on the left-hand side of (3.1) and using Fubini's theorem we obtain

$$\begin{aligned}
\int_I \Big(\sum_{i\in\mathbb{Z}} \int_{t^k_i}^{t^k_{i+1}} \|f(t^k_i) - f(t)\| \,\mathrm{d}t\Big)\mathrm{d}z &= \int_I \Big(\sum_{i\in\mathcal{J}^k} \int_{t^k_i}^{t^k_{i+1}} \|f(t^k_i) - f(t)\| \,\mathrm{d}t\Big)\mathrm{d}z \\
&= \int_I \Big(\sum_{i\in\mathcal{J}^k} \int_0^{\frac{1}{k}} \|f(z + \tfrac{i}{k}) - f(z + \tfrac{i}{k} + s)\| \,\mathrm{d}s\Big)\mathrm{d}z \\
&\le \sum_{i\in\mathcal{J}^k} \int_0^{\frac{1}{k}} \Big(\int_{-\infty}^{+\infty} \|f(z + \tfrac{i}{k}) - f(z + \tfrac{i}{k} + s)\| \,\mathrm{d}z\Big)\mathrm{d}s \\
&= \sum_{i\in\mathcal{J}^k} \int_0^{\frac{1}{k}} \Big(\int_{-\infty}^{+\infty} \|f(z) - f(z+s)\| \,\mathrm{d}z\Big)\mathrm{d}s.
\end{aligned} \tag{3.6}$$

By the L^1-continuity of translations for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\int_{-\infty}^{+\infty} \|f(z) - f(z+s)\| \,\mathrm{d}z < \varepsilon$$

whenever $0 \le s < \delta$. Hence,

$$\int_0^{\frac{1}{k}} \Big(\int_{-\infty}^{+\infty} \|f(z) - f(z+s)\| \,\mathrm{d}z\Big)\mathrm{d}s < \frac{\varepsilon}{k}$$

for $k > 1/\delta$. Observing that the number of elements of $\mathcal{J}^k$ satisfies $\mathcal{H}^0(\mathcal{J}^k) \le 2k(b-a)+2$, from the previous inequality and (3.6) we deduce that

$$\lim_{k\to+\infty} \int_I \Big(\sum_{i\in\mathbb{Z}} \int_{t^k_i}^{t^k_{i+1}} \|f(t^k_i) - f(t)\| \,\mathrm{d}t\Big)\mathrm{d}z = 0.$$

Hence, there exists an infinite set $K \subset \mathbb{N}$ and an $\mathcal{L}^1$-negligible set $N \subset I$ such that (3.1) holds.

To prove the second part of the statement, we first observe that by Egorov's theorem there exists a Borel set $I_\varepsilon \subset I$, with $\mathcal{L}^1(I \setminus I_\varepsilon) \le \varepsilon$, such that (3.2) holds.

To prove (3.3) and (3.4), we first observe that, since the number of elements of

$$\widehat{\mathcal{F}}^k_h := \{i \in \mathbb{Z} : [t^k_i, t^k_{i+1}] \cap I \neq \emptyset\} \setminus \mathcal{I}^k_h$$

is less than $2h+2$, by the absolute continuity of the integral we have

$$\lim_{k\to+\infty}\sum_{i\in\widehat{\mathcal{F}}^k_h}\int_{t^k_i}^{t^k_{i+1}}\|f(t)\|\,\mathrm{d}t=0\quad\text{uniformly for } z\in I_\varepsilon. \tag{3.7}$$

Since

$$\frac{1}{k}\sum_{i\in\mathcal{F}^k_h}\|f(t^k_i)\|\le\sum_{i\in\mathbb{Z}}\int_{t^k_i}^{t^k_{i+1}}\|f(t^k_i)-f(t)\|\,\mathrm{d}t+\sum_{i\in\mathcal{F}^k_h}\int_{t^k_i}^{t^k_{i+1}}\|f(t)\|\,\mathrm{d}t,$$

(3.4) follows from (3.2), (3.7), and from the inclusion $\mathcal{F}^k_h\subset\widehat{\mathcal{F}}^k_h$. Similarly, the inequality

$$\Big\|\frac{1}{k}\sum_{i\in\mathcal{I}^k_h}f(t^k_i)-\int_I f(t)\,\mathrm{d}t\Big\|\le\sum_{i\in\mathcal{I}^k_h}\int_{t^k_i}^{t^k_{i+1}}\|f(t^k_i)-f(t)\|\,\mathrm{d}t+\sum_{i\in\widehat{\mathcal{F}}^k_h}\int_{t^k_i}^{t^k_{i+1}}\|f(t)\|\,\mathrm{d}t,$$

together with (3.2) and (3.7), gives (3.3). □

In the next technical result we will consider the Riemann sums associated with functions converging to 0 in L^1 and depending on an additional parameter. This result will be crucial in Section 6.

Lemma 3.2. *Let $I:=[a,b]$ and $J:=[c,d]$ be bounded closed intervals. For every $z_1\in I$, $k\in\mathbb{N}$, and $i\in\mathbb{Z}$ let $t^k_i:=z_1+\frac{i}{k}$. Let $(X,\|\cdot\|)$ be a Banach space and let $f_k\colon\mathbb{R}\times\mathbb{R}\to X$ be a sequence of Bochner integrable functions, with $f_k=0$ on $(\mathbb{R}\times\mathbb{R})\setminus(I\times J)$, such that*

$$\lim_{k\to+\infty}f_k(t,z_2)=0\quad\text{for } \mathcal{L}^1\text{-a.e. } t\in I \text{ and } \mathcal{L}^1\text{ -a.e. } z_2\in J. \tag{3.8}$$

Assume also that there exists an integrable function $g\colon\mathbb{R}\to[0,+\infty)$ such that

$$\|f_k(t,z_2)\|\le g(t)\quad\text{for } \mathcal{L}^1\text{-a.e. } t\in I,\ \mathcal{L}^1\text{-a.e. } z_2\in J, \text{ and every } k\in\mathbb{N}. \tag{3.9}$$

Then there exists an $\mathcal{L}^2$-negligible set $N\subset I\times J$ and a infinite set $K\subset\mathbb{N}$ such that for every $(z_1,z_2)\in(I\times J)\setminus N$ we have

$$\lim_{\substack{k\to+\infty\\ k\in K}}\frac{1}{k}\sum_{i\in\mathbb{Z}}\|f_k(t^k_i,z_2)\|=0. \tag{3.10}$$

Moreover, for every $\varepsilon>0$ there exists a Borel set $A_\varepsilon\subset I\times J$, with $\mathcal{L}^2((I\times J)\setminus A_\varepsilon)\le\varepsilon$, such that

$$\lim_{\substack{k\to+\infty\\ k\in K}}\frac{1}{k}\sum_{i\in\mathbb{Z}}\|f_k(t^k_i,z_2)\|=0\ \text{uniformly for } (z_1,z_2)\in A_\varepsilon. \tag{3.11}$$

Proof. For $k\in\mathbb{N}$ let $\mathcal{J}^k$ be given by (3.5). Integrating the sum on the left-hand side of (3.10) with respect to $z=(z_1,z_2)\in I\times J$

$$\int_J\Big(\int_I\Big(\frac{1}{k}\sum_{i\in\mathbb{Z}}\|f_k(t^k_i,z_2)\|\Big)\mathrm{d}z_1\Big)\mathrm{d}z_2=\int_J\Big(\int_I\Big(\frac{1}{k}\sum_{i\in\mathbb{Z}}\|f_k(z_1+\tfrac{i}{k},z_2)\|\Big)\mathrm{d}z_1\Big)\mathrm{d}z_2$$

$$\le\frac{1}{k}\sum_{i\in\mathcal{J}^k}\int_{\mathbb{R}}\Big(\int_{\mathbb{R}}\|f_k(z_1+\tfrac{i}{k},z_2)\|\,\mathrm{d}z_1\Big)\mathrm{d}z_2=\frac{\mathcal{H}^0(\mathcal{J}^k)}{k}\int_{\mathbb{R}}\Big(\int_{\mathbb{R}}\|f_k(z_1,z_2)\|\,\mathrm{d}z_1\Big)\mathrm{d}z_2, \tag{3.12}$$

where in the inequality we have used the hypothesis on the supports of f_k. By (3.8) for $\mathcal{L}^2$-a.e. every $z\in I\times J$ the sequence $\|f_k(z_1,z_2)\|$ converges to 0 as $k\to+\infty$. Thanks to (3.9), by the Dominated Convergence Theorem we have $\lim_k\|f_k\|_{L^1(\mathbb{R}^2;X)}=0$. This fact, together with (3.12) and the boundedness of $\mathcal{H}^0(\mathcal{J}^k)/k$, implies that

$$\lim_{k\to+\infty}\int_{I\times J}\Big(\frac{1}{k}\sum_{i\in\mathbb{Z}}\|f_k(t^k_i,z_2)\|\Big)\mathrm{d}z=0,$$

which gives (3.10). By Egorov's Theorem we obtain also (3.11), concluding the proof. □

Remark 3.3. Suppose that for every $n \in \mathbb{N}$ we have a function (f^n) that satisfies the hypotheses of Lemma 3.1. Arguing as in [21, Remark 4.13], one can find sets K and I_ε as in the statement of Lemma 3.2 for which (3.2)-(3.4) and (3.11) hold with $f = f^n$ for every $n \in \mathbb{N}$.

Similarly, if for every $n \in \mathbb{N}$ we have a sequence of functions $(f^n_k)_k$ that satisfies the hypotheses of Lemma 3.2, the same argument shows that one can find sets K and U_ε as in the statement of Lemma 3.2 for which (3.10) and (3.11) hold with $f_k = f^n_k$ for every $n \in \mathbb{N}$.

4. An auxiliary family of measures

Given $u \in GBD(\Omega)$, we associate to it a family of measures, closely related to the measures λ^ξ_u introduced in (2.19). For every $\xi \in \mathbb{R}^d \setminus \{0\}$ and every Borel set $B \subset \Omega$ we set

$$\sigma^\xi_u(B) := |\xi| \int_{\Pi^\xi} Du^\xi_y((B \setminus J^1_u)^\xi_y) \, \mathrm{d}\mathcal{H}^{d-1}(y). \tag{4.1}$$

For $\xi = 0$, we set $\sigma^\xi_u(B) = 0$ for every Borel set $B \subset \Omega$.

Remark 4.1. Suppose that $u \in BD(\Omega)$, the space of *functions of bounded deformation*, and let Eu the distributional symmetric gradient of u, which by definition belongs to the space $\mathcal{M}_b(\Omega; \mathbb{R}^{d\times d}_{\mathrm{sym}})$. From the Structure Theorem for $BD(\Omega)$ functions [6, Theorem 4.5] it follows that

$$\sigma^\xi_u(B) = Eu(B \setminus J^1_u)\xi \cdot \xi$$

for every Borel set $B \subset \Omega$ and $\xi \in \mathbb{R}^d$. Since σ^ξ_u is a measure, this equality can be extended to all functions $u \in BD_{\mathrm{loc}}(\Omega)$ such that $Eu \in \mathcal{M}_b(\Omega; \mathbb{R}^{d\times d}_{\mathrm{sym}})$. This implies that in this case the function $\xi \mapsto \sigma^\xi_u(B)$ is quadratic in the sense of Definition 2.1. In particular, if $d = 1$ this happens for every $u \in GBD(\Omega)$ thanks to Remark 2.7.

Remark 4.2. Let λ^ξ_u be the measure introduced in (2.19). One can see that for every $\xi \in \mathbb{S}^{d-1}$, we have the equality $|\sigma^\xi_u| = \lambda^\xi_u \llcorner (\Omega \setminus J^1_u)$ as Borel measures on Ω. In light of [7, Theorem 3.103], this is an easy consequence of (4.1) and of the fact that, by Theorem 2.12, for $\mathcal{H}^{d-1}$-a.e. $y \in \Pi^\xi$ we have the inclusion $J^1_{u^\xi_y} \subset (J^1_u)^\xi_y$. This fact, together with (2.20) and the 2-homogeneity of the function $\xi \mapsto \sigma^\xi_u(\Omega)$ proved in Proposition 4.4 below, implies that

$$|\sigma^\xi_u|(\Omega) \le |\xi|^2 \lambda_u(\Omega \setminus J^1_u). \tag{4.2}$$

For $R > 0$ let $\tau_R \colon \mathbb{R} \to \mathbb{R}$ be the 1-Lipschitz functions defined by

$$\tau_R(s) := \begin{cases} -\frac{R}{2} & \text{if } s \le -\frac{R}{2}, \\ s & \text{if } \frac{-R}{2} \le s \le \frac{R}{2}, \\ \frac{R}{2} & \text{if } s \ge \frac{R}{2}. \end{cases} \tag{4.3}$$

Thanks to Remark 2.8, we have that $D_\xi(\tau_R(u \cdot \xi)) \in \mathcal{M}_b(\Omega; \mathbb{R})$ for every $u \in GBD(\Omega)$, $\xi \in \mathbb{S}^{d-1}$, and $R > 0$. The following result shows that for Borel sets that do not intersect J^1_u we can obtain the value of $\sigma^\xi_u(B)$ by considering the limit of $D_\xi(\tau_R(u\cdot\xi))(B)$ as $R \to +\infty$.

Proposition 4.3. *Let $u \in GBD(\Omega)$. Then for every $\xi \in \mathbb{S}^{d-1}$ we have*

$$\sigma^\xi_u(B) = \lim_{R\to+\infty} D_\xi(\tau_R(u \cdot \xi))(B), \tag{4.4}$$

$$\lambda^\xi_u(B) = \lim_{R\to+\infty} |D_\xi(\tau_R(u \cdot \xi))|(B), \tag{4.5}$$

for every Borel set $B \subset \Omega$ with $\mathcal{H}^{d-1}(B \cap J^1_u) = 0$.

Proof. Let us fix $\xi \in \mathbb{S}^{d-1}$ and a Borel set $B \subset \Omega$ as in the statement. Since $\mathcal{H}^{d-1}(B \cap J_u^1) = 0$, it follows that $\mathcal{H}^{d-1}(\pi^\xi(B \cap J_u^1)) = 0$ and hence $B_y^\xi \cap (J_u^1)_y^\xi = \emptyset$ for $\mathcal{H}^{d-1}$-a.e. $y \in \Pi^\xi$. Recalling that by Theorem 2.12 for $\mathcal{H}^{d-1}$-a.e. $y \in \Pi^\xi$ the inclusion in (2.23) holds, we also have that

$$B_y^\xi \cap J^1_{u_y^\xi} = \emptyset \text{ for } \mathcal{H}^{d-1}\text{-a.e. } y \in \Pi^\xi. \tag{4.6}$$

Thus, by (2.19) and (4.1) we have

$$\sigma_u^\xi(B) = \int_{\Pi^\xi} Du_y^\xi((B \setminus J_u^1)_y^\xi)\, \mathrm{d}\mathcal{H}^{d-1}(y) = \int_{\Pi^\xi} Du_y^\xi(B_y^\xi)\, \mathrm{d}\mathcal{H}^{d-1}(y), \tag{4.7}$$

$$\lambda_u^\xi(B) = \int_{\Pi^\xi} |Du_y^\xi|(B_y^\xi \setminus J^1_{u_y^\xi})\, \mathrm{d}\mathcal{H}^{d-1}(y) = \int_{\Pi^\xi} |Du_y^\xi|(B_y^\xi)\, \mathrm{d}\mathcal{H}^{d-1}(y). \tag{4.8}$$

In particular, since $\lambda_u^\xi \in \mathcal{M}_b^+(\Omega)$, we have

$$\int_{\Pi^\xi} |Du_y^\xi|(B_y^\xi)\, \mathrm{d}\mathcal{H}^{d-1}(y) < +\infty. \tag{4.9}$$

We then remark that from (b) of Definition 2.5 and the chain rule for BV-functions (see [7, Theorem 3.99]) it follows that for $\mathcal{H}^{d-1}$-a.e. $y \in \Pi^\xi$ we have $\tau_R(u_y^\xi) \in BV_{\mathrm{loc}}(\Omega_y^\xi)$ and

$$D(\tau_R(u_y^\xi)) = \tau_R'(v_y^\xi)\nabla u_y^\xi \mathcal{L}^1 + \tau_R'(v_y^\xi) D^c u_y^\xi + [\tau_R(u_y^\xi)]\mathcal{H}^0 \llcorner J_{\tau_R(u_y^\xi)}, \tag{4.10}$$

where for every $t \in \Omega_y^\xi \setminus J_{u_y^\xi}$ the function v_y^ξ is defined by

$$v_y^\xi(t) := \lim_{\varepsilon \to 0^+} \frac{1}{2\varepsilon} \int_{-\varepsilon}^{\varepsilon} u_y^\xi(t+s)\, \mathrm{d}s,$$

and we set $\tau_R'(\pm\frac{R}{2}) = 0$. Since the measures on the right-hand side of (4.10) are mutually singular, we have also

$$|D(\tau_R(u_y^\xi))| = |\tau_R'(v_y^\xi)\nabla u_y^\xi|\mathcal{L}^1 + |\tau_R'(v_y^\xi)||D^c u_y^\xi| + |[\tau_R(u_y^\xi)]|\mathcal{H}^0 \llcorner J_{\tau_R(u_y^\xi)}. \tag{4.11}$$

Moreover, we observe that for every $y \in \Pi^\xi$ such that $u_y^\xi \in BV_{\mathrm{loc}}(\Omega_y^\xi)$ we have

$$|D^c u_y^\xi|(\Omega_y^\xi \setminus J_{u_y^\xi}) = |D^c u_y^\xi|(\Omega_y^\xi),$$

$$\lim_{R \to +\infty} \tau_R'(v_y^\xi)(t) = 1 \quad \text{for } \mathcal{L}^1\text{-a.e. } t \in \Omega_y^\xi, \tag{4.12}$$

$$\lim_{R \to +\infty} \tau_R'(v_y^\xi)(t) = 1 \quad \text{for } |D^c u_y^\xi|\text{-a.e. } t \in \Omega_y^\xi, \tag{4.13}$$

$$J_{u_y^\xi} = \bigcup_{R>0} J_{\tau_R(u_y^\xi)} \quad \text{and} \quad \lim_{R \to +\infty} [\tau_R(u_y^\xi)](t) = [u_y^\xi](t) \quad \text{for every } t \in J_{u_y^\xi}. \tag{4.14}$$

Additionally, from (4.3) and (4.11) it follows that

$$|D(\tau_R(u_y^\xi))|(B_y^\xi) \leq \int_{B_y^\xi} |\nabla u_y^\xi|\, \mathrm{d}t + |D^c u_y^\xi|(B_y^\xi) + \int_{B_y^\xi \cap J_{u_y^\xi}} |[u_y^\xi]|\, \mathrm{d}\mathcal{H}^0 = |Du_y^\xi|(B_y^\xi). \tag{4.15}$$

Recalling that by Remark 2.6 for $\mathcal{H}^{d-1}$-a.e. $y \in \Pi^\xi$ we have $|Du_y^\xi|(B_y^\xi) < +\infty$, from (4.10)-(4.14) and the Dominated Convergence Theorem we deduce for $\mathcal{H}^{d-1}$-a.e $y \in \Pi^\xi$ that

$$\lim_{R \to +\infty} D(\tau_R(u_y^\xi))(B_y^\xi) = Du_y^\xi(B_y^\xi), \tag{4.16}$$

$$\lim_{R \to +\infty} |D(\tau_R(u_y^\xi))|(B_y^\xi) = |Du_y^\xi|(B_y^\xi). \tag{4.17}$$

Thanks to the general theory of slicing (see [7, Theorem 3.107]) and [20, Proposition 3.1]) for every $R > 0$ we have that

$$D_\xi(\tau_R(u\cdot\xi))(B) = \int_{\Pi^\xi} D(\tau_R(u^\xi_y))(B^\xi_y)\,\mathrm{d}\mathcal{H}^{d-1}(y), \tag{4.18}$$

$$|D_\xi(\tau_R(u\cdot\xi))|(B) = \int_{\Pi^\xi} |D(\tau_R(u^\xi_y))|(B^\xi_y)\,\mathrm{d}\mathcal{H}^{d-1}(y), \tag{4.19}$$

so that by (4.10) and (4.11) we have

$$\begin{aligned} D_\xi(\tau_R(u\cdot\xi))(B) = \int_{\Pi^\xi}\Big(\int_{B^\xi_y}\tau_R'(v^\xi_y)\nabla u^\xi_y\,\mathrm{d}t \\ + \int_{B^\xi_y}\tau_R'(v^\xi_y)\frac{\mathrm{d}D^c u^\xi_y}{\mathrm{d}|D^c u^\xi_y|}\,\mathrm{d}|D^c u^\xi_y| + \int_{B^\xi_y\cap J_{\tau_R(u^\xi_y)}}[\tau_R(u^\xi_y)]\,\mathrm{d}\mathcal{H}^0\Big)\mathrm{d}\mathcal{H}^{d-1}(y),\\ |D_\xi(\tau_R(u\cdot\xi))|(B) = \int_{\Pi^\xi}\Big(\int_{B^\xi_y}|\tau_R'(v^\xi_y)\nabla u^\xi_y|\,\mathrm{d}t \\ + \int_{B^\xi_y}\tau_R'(v^\xi_y)\,\mathrm{d}|D^c u^\xi_y| + \int_{B^\xi_y\cap J_{\tau_R(u^\xi_y)}}|[\tau_R(u^\xi_y)]|\,\mathrm{d}\mathcal{H}^0\Big)\mathrm{d}\mathcal{H}^{d-1}(y), \end{aligned}$$

Finally, recalling (2.10), (4.6), and (4.15), we can apply the Dominated Convergence Theorem and from (4.7), (4.8), and (4.16)-(4.19) we obtain (4.4) and (4.5). □

The main goal of Sections 5-7 will be proving that for every $u \in GBD(\Omega)$ and every Borel set $B \subset \Omega$ the function $\xi \mapsto \sigma^\xi_u(B)$ is quadratic. Toward this end, in the rest of this section we investigate some of the properties of the measure defined by (4.1). We first show that the function $\xi \mapsto \sigma^\xi_u(B)$ is 2-homogeneous.

Proposition 4.4. *Let $u \in GBD(\Omega)$ and let $B \subset \Omega$ be a Borel set. Then the function $\xi \mapsto \sigma^\xi_u(B)$ is 2-homogeneous.*

Proof. Let us fix $t \neq 0$ and $\xi \in \mathbb{R}^d \setminus \{0\}$. By (2.14), with B replaced by $B \setminus J^1_u$, we have

$$Du^{t\xi}_y((B\setminus J^1_u)^{t\xi}_y) = |t| Du^\xi_y((B\setminus J^1_u)^\xi_y).$$

Hence,

$$\sigma^{t\xi}_u(B) = |t\xi|\int_{\Pi^\xi} Du^{t\xi}_y((B\setminus J^1_u)^{t\xi}_y)\,\mathrm{d}\mathcal{H}^1(y) = t^2|\xi|\int_{\Pi^\xi} Du^\xi_y\big((B\setminus J^1_u)^\xi_y\big)\,\mathrm{d}\mathcal{H}^1(y) = t^2\sigma^\xi_u(B).$$

This shows that $\xi \mapsto \sigma^\xi_u(B)$ is 2-homogeneous, concluding the proof. □

In the next proposition we give an explicit formula for $\sigma^\xi_u(B)$ when the Borel set B is contained in J_u. This shows in particular that in this case the function $\xi \mapsto \sigma^\xi_u(B)$ is quadratic in the sense of Definition 2.1.

Proposition 4.5. *Let $u \in GBD(\Omega)$ and let $B \subset J_u$ be a Borel set. Then for every $\xi \in \mathbb{R}^d \setminus \{0\}$ we have*

$$\sigma^\xi_u(B) = \int_{B\setminus J^1_u}(([u]\odot\nu_u)\xi\cdot\xi)\,\mathrm{d}\mathcal{H}^{d-1}(y). \tag{4.20}$$

Proof. Let us fix $\xi \in \mathbb{R}^d \setminus \{0\}$. Since a change of sign of ν_u implies a change of sign of $[u]$, we may assume without loss of generality that $\nu_u \cdot \xi \geq 0$. Thanks to Proposition 2.13 the integral in the right-hand side of (4.20) is well-defined. Hence, by (2.22) and the Area Formula [36, 12.4] we infer

$$\begin{aligned}\sigma^\xi_u(B) &= |\xi|\int_{\Pi^\xi} Du^\xi_y((B\setminus J^1_u)^\xi_y)\,\mathrm{d}\mathcal{H}^{d-1}(y) = |\xi|\int_{\Pi^\xi}\Big(\int_{(B\setminus J^1_u)^\xi_y}[u^\xi_y](t)\,\mathrm{d}\mathcal{H}^0(t)\Big)\mathrm{d}\mathcal{H}^{d-1}(y)\\ &= \int_{B\setminus J^1_u}([u]\cdot\xi)(\nu_u\cdot\xi)\,\mathrm{d}\mathcal{H}^{d-1}(y) = \int_{B\setminus J^1_u}(([u]\odot\nu_u)\xi\cdot\xi)\,\mathrm{d}\mathcal{H}^{d-1}(y),\end{aligned}$$

which concludes the proof of (4.20). □

Given two $\mathcal{H}^{d-1}$-measurable sets $A, B \subset \mathbb{R}^d$, we write $A \simeq B$ when $\mathcal{H}^{d-1}(A \triangle B) = 0$, where $\triangle$ denotes the symmetric difference. We now present a decomposition result for $GBD(\Omega)$ functions, which states that any function $u \in GBD(\Omega)$ can be written as the sum of two functions v and w, with $v \in SBV(\Omega;\mathbb{R}^d)$, $w \in GBD(\Omega)$, $J_v \simeq J_u \setminus J_u^1$, and $J_w \simeq J_u^1$. We refer to [7] for the definition and properties of the space SBV of *special functions of bounded variation.*

Proposition 4.6. *Let $u \in GBD(\Omega)$. Then there exists $v \in SBV(\Omega;\mathbb{R}^d)$ such that*

$$J_v \simeq J_u \setminus J_u^1, \tag{4.21}$$

$$[v] = [u] \quad \text{and} \quad \nu_v = \nu_u \quad \mathcal{H}^{d-1}\text{-a.e. on } J_u \setminus J_u^1. \tag{4.22}$$

In particular, setting $w := u - v$, we have $w \in GBD(\Omega)$, $u = w + v$,

$$J_w \simeq J_w^1 \simeq J_u^1, \tag{4.23}$$

$$[w] = [u] \quad \mathcal{H}^{d-1}\text{-a.e. } J_u^1. \tag{4.24}$$

Proof. Since $J_u \setminus J_u^1$ is $(\mathcal{H}^{d-1}, d-1)$-rectifiable and $[u]$ is integrable on $J_u \setminus J_u^1$ by Proposition 2.13, the proof of the statements concerning v can be obtained arguing as in [22, Theorems 3.1 and 4.1].

The inclusion $w \in GBD(\Omega)$ is due to the vector space properties of $GBD(\Omega)$. Equalities (4.23) and (4.24) follow from (4.21) and (4.22). □

From this result, we derive the following useful consequence.

Lemma 4.7. *Let $u \in GBD(\Omega)$ and $v \in SBV(\Omega;\mathbb{R}^d)$ be as in Proposition 4.6 and let $w := u - v$. Assume that for a Borel set $B \subset \Omega$ the function $\xi \mapsto \sigma_w^\xi(B)$ is quadratic. Then the function $\xi \mapsto \sigma_u^\xi(B)$ is quadratic as well.*

Proof. As $v \in SBV(\Omega;\mathbb{R}^d)$, from Remark 4.1 it follows immediately that $\xi \mapsto \sigma_v^\xi(B)$ is quadratic, so that the function $\xi \mapsto \sigma_w^\xi(B) + \sigma_v^\xi(B)$ is also quadratic. We claim that for every $\xi \in \mathbb{R}^d \setminus \{0\}$ we have

$$\sigma_u^\xi(B) = \sigma_w^\xi(B) + \sigma_v^\xi(B). \tag{4.25}$$

To prove this, let us fix $\xi \in \mathbb{R}^d \setminus \{0\}$. By definition we have $u = w + v$. Thus, for $\mathcal{H}^{d-1}$-a.e. $y \in \Pi^\xi$ it holds

$$Du_y^\xi = Dw_y^\xi + Dv_y^\xi \tag{4.26}$$

as Borel measures on Ω_y^ξ. Moreover, by construction we have $J_v^1 \simeq \emptyset$ and $J_w \simeq J_w^1 \simeq J_u^1$. This implies that

$$Dw_y^\xi((B \setminus J_w^1)_y^\xi)) = Dw_y^\xi((B \setminus J_u^1)_y^\xi) \quad \text{for } \mathcal{H}^{d-1}\text{-a.e. } y \in \Pi^\xi.$$

Since $v \in SBV(\Omega;\mathbb{R}^d)$, from (4.21) we deduce that $|Dv|(J_u^1) = 0$. By slicing we obtain that

$$|Dv_y^\xi|((J_u^1)_y^\xi) = 0 \quad \text{for } \mathcal{H}^{d-1}\text{-a.e. } y \in \Pi^\xi.$$

for $\mathcal{H}^{d-1}$-a.e. $y \in \Pi^\xi$. Recalling that $J_v^1 \simeq \emptyset$ by (4.21) and (4.22), this implies that

$$Dv_y^\xi((B \setminus J_v^1)_y^\xi) = Dv_y^\xi(B_y^\xi) = Dv_y^\xi((B \setminus J_u^1)_y^\xi).$$

These remarks, together with (4.26), imply that for $\mathcal{H}^{d-1}$-a.e. $y \in \Pi^\xi$ it holds

$$Dw_y^\xi((B \setminus J_w^1)_y^\xi) + Dv_y^\xi((B \setminus J_v^1)_y^\xi) = D(w_y^\xi + v_y^\xi)((B \setminus J_u^1)_y^\xi) = Du_y^\xi((B \setminus J_u^1)_y^\xi).$$

Integrating this equality, by (4.1) we obtain (4.25), concluding the proof. □

5. The case of dimension $d = 2$

In this and the next section we assume that $\Omega \subset \mathbb{R}^2$. Our aim is to prove the following result.

Theorem 5.1. *Let $u \in GBD(\Omega)$ and let $B \subset \Omega$ be a Borel set. Then the function $\xi \mapsto \sigma_u^\xi(B)$ is quadratic.*

Proof. Thanks to Proposition 4.6 and Lemma 4.7, it is not restrictive to assume that $J_u \simeq J_u^1$. Moreover, it is enough to prove the result when B is an open set, which will be denoted by U.

To prove that $\xi \mapsto \sigma_u^\xi(U)$ is quadratic we use Proposition 2.2. Since by Proposition 4.4 the function $\xi \mapsto \sigma_u^\xi(U)$ is 2-homogeneous and by Remark 4.2 it satisfies the lower bound (c) of Proposition 4.4, we are left with proving the *parallelogram identity*

$$\sigma_u^{\xi+\eta}(U) + \sigma_u^{\xi-\eta}(U) = 2\sigma_u^\xi(U) + 2\sigma_u^\eta(U), \tag{5.1}$$

for every $\xi, \eta \in \mathbb{R}^2 \setminus \{0\}$.

To this aim, we fix $\xi, \eta \in \mathbb{R}^2 \setminus \{0\}$. Note that, if ξ and η are not linearly independent, then the parallelogram identity follows from 2-homogeneity, so we may assume ξ and η to be linearly independent. We also note that it is not restrictive to assume that U is a parallelogram of the form

$$U = \{s\xi + t\eta : s \in (0, \alpha) \text{ and } t \in (0, \beta)\}, \tag{5.2}$$

for suitable constants $\alpha, \beta > 0$ and with $U \subset\subset \Omega$. Indeed, every open set U contained in Ω can be approximated by a sequence $(U_k)_k$ of disjoint unions of such parallelograms for which $\sigma_u^\zeta(U) = \lim_k \sigma_u^\zeta(U_k)$ for every $\zeta \in \{\xi, \eta, \xi + \eta, \xi - \eta\}$.

The vectors ξ and η, as well as the parallelogram U, are kept fixed throughout the rest of the proof.

To prove the parallelogram identity (5.1), we will use Lemma 3.1 to approximate, by means of Riemann sums, each integral appearing in the definition of the terms occurring in (5.1). This will allow us to prove that the obtained approximations satisfy, up to an arbitrarily small error, the parallelogram identity.

In order to construct these approximations, we need to introduce some notation first. Given a point $\omega \in \mathbb{R}^2$, for every $k \in \mathbb{N}$ and for every $i, j \in \mathbb{Z}$ we set (see Figure 1)

$$x_{i,j}^k := \omega + \tfrac{i}{k}\xi + \tfrac{j}{k}\eta. \tag{5.3}$$

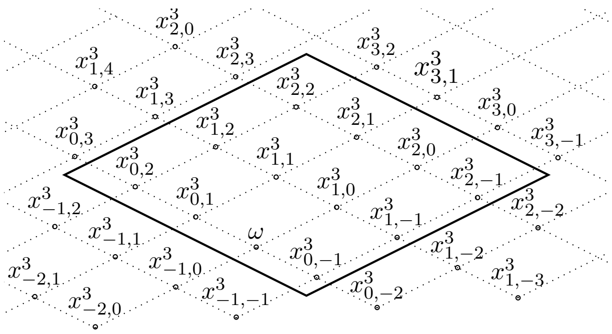

Figure 1. The parallelogram U and the grid of points $x_{i,j}^k$ associated to $\omega \in U$ and $k = 3$

Since the points $x_{i,j}^k$ will be instrumental to the discretisation of the summands in (5.1), which are integrals over the straight lines Π^ζ for $\zeta \in \{\xi, \eta, \xi + \eta, \xi - \eta\}$, we consider also the projections of the points $x^k_{i,j}$ onto these lines. For every $k \in \mathbb{N}$, $i, j \in \mathbb{Z}$, and $\zeta \in \{\xi, \eta, \xi+\eta, \xi-\eta\}$ we set

$$y^{k,\zeta}_{i,j} := \pi^\zeta(x^k_{i,j}) = \pi^\zeta(\omega) + \tfrac{i}{k}\pi^\zeta(\xi) + \tfrac{j}{k}\pi^\zeta(\eta) \in \Pi^\zeta. \tag{5.4}$$

We observe that $y^{k,\xi}_{i,j}$ depends only on j and $y^{k,\eta}_{i,j}$ depends only on i, while $y^{k,\xi+\eta}_{i,j}$ depends only on $i-j$ and $y^{k,\xi-\eta}_{i,j}$ depends only on $i+j$. When we want to underline the dependence of these families on a single index, we set

$$\begin{gathered} y^{k,\xi}_j = y^{k,\xi}_{0,j}, \qquad y^{k,\eta}_i = y^{k,\eta}_{i,0}, \\ y^{k,\xi+\eta}_j = y^{k,\xi+\eta}_{j,0} = y^{k,\xi+\eta}_{0,-j}, \quad y^{k,\xi-\eta}_j = y^{k,\xi-\eta}_{j,0} = y^{k,\xi-\eta}_{0,j}, \end{gathered} \tag{5.5}$$

see Figure 2.

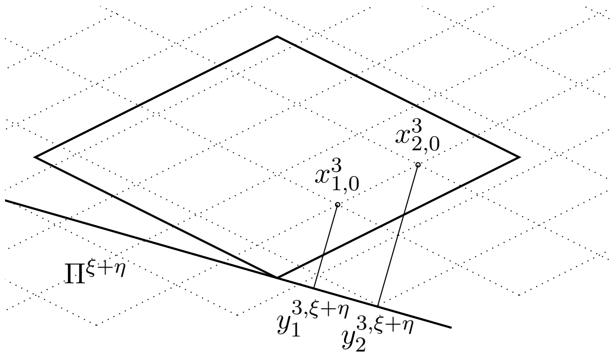

FIGURE 2. Projections of the points $x^k_{i,j}$ onto the straight line $\Pi^{\xi+\eta}$

It is clear from these definitions that for every $\zeta \in \{\xi, \eta, \xi+\eta, \xi-\eta\}$ and $i, j \in \mathbb{Z}$ there exists a unique real number $t^{k,\zeta}_{i,j}$ such that

$$x^k_{i,j} = y^{k,\zeta}_{i,j} + t^{k,\zeta}_{i,j}\zeta. \tag{5.6}$$

Let $C_{\xi,\eta} := \big(|\xi|^2|\eta|^2 - (\xi\cdot\eta)^2\big)^{1/2} > 0$. We observe that for $i, j \in \mathbb{Z}$ we have

$$\begin{aligned} k|y^{k,\xi}_{i,j+1} - y^{k,\xi}_{i,j}| &= |\pi^\xi(\eta)| = \frac{1}{|\xi|}C_{\xi,\eta}, \\ k|y^{k,\eta}_{i+1,j} - y^{k,\eta}_{i,j}| &= |\pi^\eta(\xi)| = \frac{1}{|\eta|}C_{\xi,\eta} \\ k|y^{k,\xi+\eta}_{i,j} - y^{k,\xi+\eta}_{i,j+1}| = k|y^{k,\xi+\eta}_{i,j} - y^{k,\xi+\eta}_{i+1,j}| &= |\pi^{\xi+\eta}(\xi)| = |\pi^{\xi+\eta}(\eta)| = \frac{1}{|\xi+\eta|}C_{\xi,\eta}, \\ k|y^{k,\xi-\eta}_{i,j} - y^{k,\xi-\eta}_{i,j+1}| = k|y^{k,\xi-\eta}_{i,j} - y^{k,\xi-\eta}_{i-1,j}| &= |\pi^{\xi-\eta}(\xi)| = |\pi^{\xi-\eta}(\eta)| = \frac{1}{|\xi-\eta|}C_{\xi,\eta}. \end{aligned} \tag{5.7}$$

For technical reasons, which will appear in Lemmas 6.2 and A.2, it is convenient to replace the set $J_u = J^1_u$ by a set $J \subset U$ that can be written as countable union of compact sets. Since $\mathcal{H}^1(J^1_u) < +\infty$ by Remark 2.10, there exist a countable family of compact sets $K_n \subset J^1_u \cap U$ and two Borel sets $N_1 \subset N \subset U$ with $\mathcal{H}^1(N_1) = \mathcal{H}^1(N) = 0$ such that

$$J^1_u \cap U = \Big(\bigcup_{n\in\mathbb{N}} K_n\Big) \cup N_1 \quad \text{and} \quad J_u \cap U = \Big(\bigcup_{n\in\mathbb{N}} K_n\Big) \cup N. \tag{5.8}$$

We set

$$J := \bigcup_{n\in\mathbb{N}} K_n \tag{5.9}$$

and observe that for every $\zeta \in \{\xi, \eta, \xi+\eta, \xi-\eta\}$ and for $\mathcal{H}^1$-a.e. $y \in \Pi^\zeta$ we have the equality $(J^1_u \cap U)^\zeta_y = (J_u \cap U)^\zeta_y = J^\zeta_y$. In particular, by (4.1) we have that

$$\sigma^\zeta_u(B) := |\zeta| \int_{\Pi^\zeta} Du^\zeta_y((B \setminus J)^\zeta_y)\, \mathrm{d}\mathcal{H}^1(y) \tag{5.10}$$

for every $\zeta \in \{\xi, \eta, \xi+\eta, \xi-\eta\}$ and every Borel set $B \subset U$.

The following two lemmas will be used in the choice of ω to obtain an approximation of $\sigma_u^\zeta(U)$ by means of suitable Riemann sums.

Lemma 5.2. *Let $\zeta \in \{\xi, \eta, \xi+\eta, \xi-\eta\}$. Then for $\mathcal{L}^2$-a.e. $\omega \in U$ the following conditions are simultaneously satisfied:*

(a) properties of the slices: for every $k \in \mathbb{N}$ and $i, j \in \mathbb{Z}$ we have

$$u^\zeta_{y^{k,\zeta}_{i,j}} \in BV(U^\zeta_{y^{k,\zeta}_{i,j}}) \qquad \textit{and} \qquad (J^1_u \cap U)^\zeta_{y^{k,\zeta}_{i,j}} = J^\zeta_{y^{k,\zeta}_{i,j}};$$

(b) the points $x^k_{i,j}$ are directional Lebesgue points: for every $k \in \mathbb{N}$ and $i, j \in \mathbb{Z}$, with $x^k_{i,j} \in U$, we have

$$\lim_{\varepsilon \to 0^+} \frac{1}{2\varepsilon} \int_{-\varepsilon}^{\varepsilon} |u(x^k_{i,j} + s\zeta)\cdot\zeta - u(x^k_{i,j})\cdot\zeta| \,\mathrm{d}s = 0. \tag{5.11}$$

Proof. Taking into account that $U \subset\subset \Omega$ and recalling the definition of $GBD(\Omega)$ and Remark 2.6, from (5.8) and (5.9) it follows that there exists a Borel set $N_\zeta \subset \Pi^\zeta$, with $\mathcal{H}^1(N_\zeta) = 0$, such that for every $y \in \Pi^\zeta \setminus N_\zeta$ we have $u^\zeta_y \in BV(U^\zeta_y)$ and $(J^1_u \cap U)^\zeta_y = J^\zeta_y$. Let $N^\infty_\zeta := \bigcup_{(i,j)\in\mathbb{Z}^2} \big(N_\zeta - \frac{i}{k}\pi^\zeta(\xi) - \frac{j}{k}\pi^\zeta(\eta)\big)$. It is immediate to check that $\mathcal{H}^1(N^\infty_\zeta) = 0$. By (5.4) we have

$$y^{k,\zeta}_{i,j} = \pi^\zeta(\omega) + \tfrac{i}{k}\pi^\zeta(\xi) + \tfrac{j}{k}\pi^\zeta(\eta),$$

so that if $\pi^\zeta(\omega) \notin N^\infty_\zeta$, we have $u^\zeta_{y^{k,\zeta}_{i,j}} \in BV(U^\zeta_{y^{k,\zeta}_{i,j}})$ and that $(J^1_u \cap U)^\zeta_{y^{k,\zeta}_{i,j}} = J^\zeta_{y^{k,\zeta}_{i,j}}$. This proves that for $\mathcal{L}^2$-a.e $\omega \in U$ condition (a).

Let us prove (b). Let B be the $\mathcal{L}^2$-measurable set defined by

$$B := \Big\{x \in U : \limsup_{\varepsilon\to 0^+} \frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon} |u(x+s\zeta)\cdot\zeta - u(x)\cdot\zeta| \,\mathrm{d}s > 0\Big\}.$$

For every $y \in \Pi^\zeta \setminus N_\zeta$ we have $u^\zeta_y \in BV(U^\zeta_y)$ and the slices B^ζ_y satisfy

$$B^\zeta_y = \Big\{t \in U^\zeta_y : \limsup_{\varepsilon\to 0^+} \frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon} |u^\zeta_y(s+t) - u^\zeta_y(t)| \,\mathrm{d}s > 0\Big\}.$$

Therefore, by the Lebesgue Differentiation Theorem $\mathcal{L}^1(B^\zeta_y) = 0$ for every $y \in \Pi^\zeta \setminus N_\zeta$ and by the Fubini Theorem this implies that $\mathcal{L}^2(B) = 0$. We observe that

$$\lim_{\varepsilon\to 0^+} \frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon} |u^\zeta_y(t+s) - u^\zeta_y(t)| \,\mathrm{d}s = 0 \tag{5.12}$$

for every $t \in \Pi^\zeta_y \setminus B^\zeta_y$.

Recalling that by (5.6) we have that

$$x^k_{i,j} = y^{k,\zeta}_{i,j} + t^{k,\zeta}_{i,j}\zeta, \tag{5.13}$$

and that $y^{k,\zeta}_{i,j} \notin N_\zeta$ by the first step, from (5.12) we deduce that (5.11) holds whenever

$$t^{k,\zeta}_{i,j} \notin N^\zeta_{y^{k,\zeta}_{i,j}}. \tag{5.14}$$

Thus, to prove (b) it is enough to show that, for given i, j, and ζ, condition (5.14) holds for $\mathcal{L}^2$-a.e. $\omega \in U$. Observing that $y^{k,\zeta}_{i,j}\cdot\zeta = 0$ and recalling (5.3), if we multiply (5.13) by $\zeta/|\zeta|^2$ we obtain that

$$t^{k,\zeta}_{i,j} = \tfrac{x^k_{i,j}\cdot\zeta}{|\zeta|^2} = \tfrac{\omega\cdot\zeta}{|\zeta|^2} + \tfrac{i}{k}\tfrac{\xi\cdot\zeta}{|\zeta|^2} + \tfrac{j}{k}\tfrac{\eta\cdot\zeta}{|\zeta|^2}.$$

Hence, (5.14) holds whenever

$$\tfrac{\omega\cdot\zeta}{|\zeta|^2} \notin N^\zeta_{y^{k,\zeta}_{i,j}} - \tfrac{i}{k}\tfrac{\xi\cdot\zeta}{|\zeta|^2} - \tfrac{j}{k}\tfrac{\eta\cdot\zeta}{|\zeta|^2}. \tag{5.15}$$

Recalling that by (5.4) $y^{k,\zeta}_{i,j}$ has the form $\pi^\zeta(\omega)+z$ for some $z\in\Pi^\zeta$, depending on k, ζ, i, and j, we deduce that (5.15) holds for $\mathcal{L}^2$-a.e. $\omega\in U$. This proves that (5.14) holds for $\mathcal{L}^2$-a.e. $\omega\in U$, concluding the proof. □

Proof of Theorem 5.1 (continuation). Given $i,j\in\mathbb{Z}$ and $\zeta\in\{\xi,\eta,\xi+\eta,\xi-\eta\}$ we set

$$I^{k,\zeta}_{i,j}:=[t^{k,\zeta}_{i,j},t^{k,\zeta}_{i,j}+\tfrac{1}{k}), \tag{5.16}$$

where $t^{k,\zeta}_{i,j}$ are defined in (5.6). We note that

$$[x^k_{i,j},x^k_{i,j}+\tfrac{1}{k}\zeta)=\{y^{k,\zeta}_{i,j}+t\zeta : t\in I^{k,\zeta}_{i,j}\}. \tag{5.17}$$

For $h\in\mathbb{N}$ we set

$$\mathcal{J}^k_h:=\{(i,j)\in\mathbb{Z}^2 : x^k_{i,j}\pm\tfrac{h}{k}\zeta\in U \text{ for every } \zeta\in\{\xi,\eta,\xi+\eta,\xi-\eta\}\}. \tag{5.18}$$

Since every $\omega\in\mathbb{R}^2$ can be written in a unique way as $\omega=s\xi+t\eta$ with $s,t\in\mathbb{R}$, by (5.2) and (5.3) we have

$$\mathcal{J}^k_h:=\{(i,j)\in\mathbb{Z}^2 : 0<s+\tfrac{i\pm h}{k}<\alpha \text{ and } 0<t+\tfrac{j\pm h}{k}<\beta\}. \tag{5.19}$$

In the following lemma, given a sequence $(\omega_k)_k$ of elements of U, we consider the points $x^k_{i,j}$ and $y^{k,\zeta}_{i,j}$ defined by (5.3) and (5.4) with $\omega=\omega_k$. We recall that $C_{\xi,\eta}>0$ is the constant which appears in (5.7).

Lemma 5.3. *There exists an infinite set $K\subset\mathbb{N}$ and, for every $\varepsilon>0$, a Borel set $U_\varepsilon\subset U$, with $\mathcal{L}^2(U\setminus U_\varepsilon)\le\varepsilon$, such that for every sequence $(\omega_k)_{k\in\mathbb{N}}$ in U_ε and for every $\zeta\in\{\xi,\eta,\xi+\eta,\xi-\eta\}$ conditions (a) and (b) of Lemma 5.2 are satisfied and*

$$\lim_{\substack{k\to+\infty\\ k\in K}}\frac{C_{\xi,\eta}}{k}\sum_{(i,j)\in\mathcal{J}^k} Du^\zeta_{y^{k,\zeta}_{i,j}}(I^{k,\zeta}_{i,j}\setminus J^\zeta_{y^{k,\zeta}_{i,j}})=\sigma^\zeta_u(U) \tag{5.20}$$

for every sequence $(\mathcal{J}^k)$ in $\mathbb{Z}^2$ for which there exists $h\in\mathbb{N}$ such that $\mathcal{J}^k_h\subset\mathcal{J}^k\subset\mathcal{J}^k_1$ for every $k\in\mathbb{N}$.

Proof. Thanks to Lemma 5.2, there exists a Borel set $U_0\subset U$, with $\mathcal{L}^2(U_0)=\mathcal{L}^2(U)$, such that (a) and (b) hold for every $\omega\in U_0$, so that we only need to show that there exist an infinite set $K\subset\mathbb{N}$ and for every $\varepsilon>0$, a Borel set $U_\varepsilon\subset U_0$, with $\mathcal{L}^2(U\setminus U_\varepsilon)\le\varepsilon$, such that (5.20) holds. The proof for every $\zeta\in\{\xi,\eta,\xi+\eta,\xi-\eta\}$ will be carried out in four steps, first for $\zeta=\xi$, then for $\zeta=\eta$, next for $\zeta=\xi+\eta$, and finally for $\zeta=\xi-\eta$. Starting from the second step, $\mathbb{N}$ is replaced by the set K of the previous step and we may assume that U_ε is contained in the corresponding set of the previous step, so that the sets K and U_ε obtained at the end satisfy (5.20) for every $\zeta\in\{\xi,\eta,\xi+\eta,\xi-\eta\}$.

We begin by proving the result for $\zeta=\xi$. We observe that every $\omega\in\mathbb{R}^2$ can be written in a unique way as

$$\omega=z_1\eta+z_2\xi,$$

with $z_1,z_2\in\mathbb{R}$. For every $k\in\mathbb{N}$ and $i,j\in\mathbb{Z}$ by (5.4) we have

$$y^{k,\xi}_{i,j}=\big(z_1+\tfrac{j}{k}\big)\pi^\xi(\eta). \tag{5.21}$$

We set $I:=(0,\beta)=\{t\in\mathbb{R} : t\pi^\xi(\eta)\in\pi^\xi(U)\}$. For every $h,k\in\mathbb{N}$ and $z_1\in I$ we define

$$\mathcal{I}^k_h(z_1):=\big\{j\in\mathbb{Z} : z_1+\tfrac{j\pm h}{k}\in I\big\}=\big\{j\in\mathbb{Z} : y^{k,\xi}_{0,j}\pm\tfrac{h}{k}\pi^\xi(\eta)\in\pi^\xi(U)\big\},$$
$$\mathcal{F}^k_h(z_1):=\big\{j\in\mathbb{Z} : z_1+\tfrac{j}{k}\in I\big\}\setminus\mathcal{I}^k_h(z_1).$$

Let $N_\xi\subset\Pi^\xi$ be the $\mathcal{H}^1$-negligible Borel set introduced at the beginning of the proof of Lemma 5.2 for $\zeta=\xi$ and consider the Borel set $M_\xi:=\{t\in\mathbb{R} : t\pi^\xi(\eta)\in N_\xi\}$. Applying Lemma 3.1 to the function defined for $t\in\mathbb{R}$ by

$$f(t):=\begin{cases} Du^\xi_{t\pi^\xi(\eta)}((U\setminus J)^\xi_{t\pi^\xi(\eta)}) & \text{if } t\in\mathbb{R}\setminus M_\xi,\\ 0 & \text{if } t\in M_\xi,\end{cases}$$

which vanishes out of I, and recalling (5.7), (5.10), and (5.21), we obtain an infinite set $H \subset \mathbb{N}$ and, for every $\varepsilon > 0$, a Borel set $I_\varepsilon \subset I$, with $\mathcal{L}^1(I \setminus I_\varepsilon) \le \varepsilon$ and $I_\varepsilon \cap M_\xi = \emptyset$, such that for every $h \in \mathbb{N}$ we have

$$\lim_{\substack{k\to+\infty\\ k\in H}} \frac{C_{\xi,\eta}}{k} \sum_{j\in\mathcal{I}_h^k(z_1)} Du^\xi_{y^{k,\xi}_{0,j}}((U\setminus J)^\xi_{y^{k,\xi}_{0,j}}) = |\xi||\pi^\xi(\eta)| \int_I Du^\xi_{s\pi^\xi(\eta)}((U\setminus J)^\xi_{s\pi^\xi(\eta)})\,\mathrm{d}s$$

$$= |\xi| \int_{\Pi^\xi} Du^\xi_y((U\setminus J)^\xi_y)\,\mathrm{d}\mathcal{H}^1(y) = \sigma^\xi_u(U) \qquad \text{uniformly for } z_1 \in I_\varepsilon, \tag{5.22}$$

$$\lim_{\substack{k\to+\infty\\ k\in H}} \frac{1}{k} \sum_{j\in\mathcal{F}_h^k(z_1)} Du^\xi_{y^{k,\xi}_{0,j}}((U\setminus J)^\xi_{y^{k,\xi}_{0,j}}) = 0 \qquad \text{uniformly for } z_1 \in I_\varepsilon. \tag{5.23}$$

We set

$$V_\varepsilon := \{\omega \in U : \omega = z_1\eta + z_2\xi,\ z_1 \in I_\varepsilon,\ z_2 \in \mathbb{R}\} \tag{5.24}$$

and observe that $\mathcal{L}^2(U \setminus V_\varepsilon) \le c\varepsilon$ for a constant $c > 0$ depending only on ξ, η, α, and β. For every $h, k \in \mathbb{N}$ we set

$$W^k_h := \{x \in U :\ \text{either } x + \tfrac{h}{k}\xi \ \text{ or } \ x - \tfrac{h}{k}\xi \quad \text{does not belong to } U\}, \tag{5.25}$$

$$f^k_h(t, z_2) := \begin{cases} |Du^\xi_{t\pi^\xi(\eta)}|((W^k_h \setminus J)^\xi_{t\pi^\xi(\eta)}) & \text{if } t \in I \setminus M_\xi \text{ and } z_2 \in (0, \alpha+\beta), \\ 0 & \text{otherwise,} \end{cases} \tag{5.26}$$

$$g(t) := \begin{cases} |Du^\xi_{t\pi^\xi(\eta)}|((U \setminus J)^\xi_{t\pi^\xi(\eta)}) & \text{for } t \in I \setminus M_\xi, \\ 0 & \text{otherwise.} \end{cases} \tag{5.27}$$

We observe that $0 \le f^k_h(t, z_2) \le g(t)$ for $\mathcal{L}^1$-a.e. $t \in I$ and $\mathcal{L}^1$-a.e. $z_2 \in \mathbb{R}$. Let λ be a measure as in Definition 2.5. Since $\lambda(W^k_h)$ converges to 0 as $k \to +\infty$, from (2.15) and (2.23) we deduce that the sequence (f^k_h) converges to 0 in $L^1(I \times \mathbb{R})$ as $k \to +\infty$. Thanks to Lemma 3.2 and Remark 3.3, applied with $\mathbb{N}$ replaced by H, we can find an infinite set $K \subset H \subset \mathbb{N}$ and a Borel set $U_\varepsilon \subset V_\varepsilon \subset U$, with $\mathcal{L}^2(U \setminus U_\varepsilon) \le c\varepsilon$ for a constant $c > 0$ depending only on ξ, η, α, and β, such that for every $h \in \mathbb{N}$

$$\lim_{\substack{k\to+\infty\\ k\in K}} \frac{1}{k} \sum_{j\in\mathcal{I}_h^k(z_1)} |Du^\xi_{y^{k,\xi}_{0,j}}|((W^k_h\setminus J)^\xi_{y^{k,\xi}_{0,j}}) = \lim_{\substack{k\to+\infty\\ k\in K}} \frac{1}{k} \sum_{j\in\mathcal{I}_h^k(z_1)} f^k_h(z_1 + \tfrac{j}{k}, z_2) = 0 \tag{5.28}$$

uniformly for $\omega = z_1\eta + z_2\xi \in U_\varepsilon$, where the first equality follows from (5.21).

For every $j \in \mathbb{Z}$ let $\mathcal{J}^k(j) := \{i \in \mathbb{Z} : (i,j) \in \mathcal{J}^k\}$. Since $\mathcal{J}^k_h \subset \mathcal{J}^k \subset \mathcal{J}^k_1$, by (5.16) and (5.25) for every $j \in \mathcal{I}^k_h(z_1)$ it holds

$$(U\setminus J)^\xi_{y^{k,\xi}_{0,j}} = (W^k_h \setminus J)^\xi_{y^{k,\xi}_{0,j}} \cup \bigcup_{i\in\mathcal{J}^k(j)} (I^{k,\xi}_{i,j} \setminus J^\xi_{y^{k,\xi}_{0,j}}).$$

Hence,

$$\Big| Du^\xi_{y^{k,\xi}_{0,j}}\big((U\setminus J)^\xi_{y^{k,\xi}_{0,j}}\big) - \sum_{i\in\mathcal{J}^k(j)} Du^\xi_{y^{k,\xi}_{0,j}}\big(I^{k,\xi}_{i,j} \setminus J^\xi_{y^{k,\xi}_{0,j}}\big)\Big| \le |Du^\xi_{y^{k,\xi}_{0,j}}|\big((W^k_h\setminus J)^\xi_{y^{k,\xi}_{0,j}}\big).$$

Recalling that $y^{k,\xi}_{i,j} = y^{k,\xi}_{0,j}$, the previous inequality gives

$$\Big| \sum_{j\in\mathcal{I}_h^k(z_1)} Du^\xi_{y^{k,\xi}_{0,j}}\big((U\setminus J)^\xi_{y^{k,\xi}_{0,j}}\big) - \sum_{(i,j)\in\mathcal{J}^k} Du^\xi_{y^{k,\xi}_{0,j}}\big(I^{k,\xi}_{i,j} \setminus J^\xi_{y^{k,\xi}_{0,j}}\big)\Big|$$

$$\le |Du^\xi_{y^{k,\xi}_{0,j}}|\big((W^k_h\setminus J)^\xi_{y^{k,\xi}_{0,j}}\big) + \sum_{j\in\mathcal{F}_h^k(z_1)} \big|Du^\xi_{y^{k,\xi}_{0,j}}\big((U\setminus J)^\xi_{y^{k,\xi}_{0,j}}\big)\big|.$$

Combining (5.22), (5.23), and (5.28), we obtain (5.20) for $\zeta = \xi$. The proof for the case $\zeta = \eta$ can be obtained by arguing as above, exchanging the roles of ξ and η.

In the case $\zeta = \xi + \eta$ we argue as follows. First, we write every $\omega \in \mathbb{R}^2$ as

$$\omega = z_1 \eta + z_2(\xi + \eta),$$

with $z_1, z_2 \in \mathbb{R}$, so that by (5.4)

$$y_{i,j}^{k,\xi+\eta} = (z_1 + \tfrac{j}{k})\pi^{\xi+\eta}(\eta) + \tfrac{i}{k}\pi^{\xi+\eta}(\xi) = (z_1 + \tfrac{j-i}{k})\pi^{\xi+\eta}(\eta).$$

Setting $m := j - i$, we have

$$y_{i,j}^{k,\xi+\eta} = (z_1 + \tfrac{m}{k})\pi^{\xi+\eta}(\eta). \tag{5.29}$$

We now set $I := \{t \in \mathbb{R} : t\pi^{\xi+\eta}(\eta) \in \pi^{\xi+\eta}(U)\}$ and for every $k \in \mathbb{N}$ and $z_1 \in I$ we define

$$\mathcal{I}^k(z_1) := \{m \in \mathbb{Z} : z_1 + \tfrac{m}{k} \in I\}. \tag{5.30}$$

Let $N_{\xi+\eta}$ be the $\mathcal{H}^1$-negligible Borel introduced at the beginning of the proof of Lemma 5.2 for $\zeta = \xi + \eta$ and consider the Borel set $M_{\xi+\eta} := \{t \in \mathbb{R} : t\pi^{\xi+\eta}(\eta) \in N_{\xi+\eta}\}$. We can apply Lemma 3.1 to the function defined for $t \in \mathbb{R}$ by

$$h(t) := \begin{cases} Du^{\xi+\eta}_{t\pi^{\xi+\eta}(\eta)}((U \setminus J)^{\xi+\eta}_{t\pi^{\xi+\eta}(\eta)}) & \text{if } t \in \mathbb{R} \setminus M_{\xi+\eta}, \\ 0 & \text{if } t \in M_{\xi+\eta}. \end{cases}$$

and arguing as in the previous part of the proof we obtain an infinite set H contained in the set K obtained in the previous steps and, for every $\varepsilon > 0$, a Borel set $I_\varepsilon \subset I$, with $\mathcal{L}^1(I \setminus I_\varepsilon) \le \varepsilon$ and $I_\varepsilon \cap M_{\xi+\eta} = \emptyset$, such that

$$\lim_{\substack{k\to+\infty\\ k\in H}} \frac{C_{\xi,\eta}}{k} \sum_{m \in \mathcal{I}^k(z_1)} Du^{\xi+\eta}_{y^{k,\xi+\eta}_{0,m}}((U \setminus J)^{\xi}_{y^{k,\xi}_{0,m}}) = \sigma^{\xi+\eta}_u(U) \qquad \text{uniformly for } z_1 \in I_\varepsilon. \tag{5.31}$$

For every $h, k \in \mathbb{N}$ now define

$$W_h^k := \{x \in U : \text{there exists } \zeta \in \{\pm\xi, \pm\eta, \pm(\xi+\eta), \pm(\xi-\eta)\} \text{ such that } x + \tfrac{h}{k}\zeta \notin U\}$$

and we observe that (5.19) and the inclusions $\mathcal{J}_h^k \subset \mathcal{J}^k \subset \mathcal{J}_1^k$ imply that

$$\{(i,j) \in \mathbb{Z}^2 : x_{i,j}^k \in U \setminus W_h^k\} \subset \mathcal{J}^k,$$

$$\mathcal{J}^k \subset \{(i,j) \in \mathbb{Z}^2 : [x_{i,j}^k, x_{i,j}^k \pm \tfrac{1}{k}\zeta] \subset U \text{ for every } \zeta \in \{\xi, \eta, \xi+\eta, \xi-\eta\}\}.$$

It follows from (5.16) and (5.17) that

$$(U \setminus J)^{\xi+\eta}_{y^{k,\xi+\eta}_{0,m}} = (W_h^k \setminus J)^{\xi+\eta}_{y^{k,\xi+\eta}_{0,m}} \cup \bigcup_{\substack{(i,j)\in\mathcal{J}^k\\ j-i=m}} (I_{i,j}^{k,\xi+\eta} \setminus J^{\xi+\eta}_{y^{k,\xi+\eta}_{0,m}}). \tag{5.32}$$

For every $k \in \mathbb{N}$ we now define V_ε , W_h^k, f_h^k, and g as in (5.24)-(5.27), with ξ replaced by $\xi + \eta$. Arguing as in the first part of the proof, we obtain that (f_h^k) converges to 0 in $L^1(I \times \mathbb{R})$ as $k \to +\infty$. Hence, recalling (5.19), we may apply Lemma 3.2, with $\mathbb{N}$ replaced by H, to obtain an infinite set $K \subset H \subset \mathbb{N}$ and a Borel set $U_\varepsilon \subset V_\varepsilon \subset U$, with $\mathcal{L}^2(U \setminus U_\varepsilon) \le c\varepsilon$ for a constant $c > 0$ depending only on ξ, η, α, and β, such that

$$\lim_{\substack{k\to+\infty\\ k\in K}} \frac{1}{k} \sum_{m\in\mathcal{I}^k(z_1)} |Du^{\xi+\eta}_{y^{k,\xi+\eta}_{0,m}}|((W_h^k \setminus J)^{\xi+\eta}_{y^{k,\xi+\eta}_{0,m}}) = \lim_{\substack{k\to+\infty\\ k\in K}} \frac{1}{k} \sum_{m\in\mathcal{I}^k(z_1)} f_h^k(z_1 + \tfrac{m}{k}, z_2) = 0 \tag{5.33}$$

uniformly for $\omega = z_1\eta + z_2(\xi+\eta) \in U_\varepsilon$. Recalling the equality $y_{i,j}^{k,\xi+\eta} = y_{0,m}^{k,\xi+\eta}$ for $j - i = m$, from (5.32) it follows that

$$|Du^{\xi+\eta}_{y^{k,\xi+\eta}_{0,m}}((U \setminus J)^{\xi+\eta}_{y^{k,\xi+\eta}_{0,m}}) - \sum_{\substack{(i,j)\in\mathcal{J}^k\\ j-i=m}} Du^{\xi+\eta}_{y^{k,\xi+\eta}_{0,m}}(I_{i,j}^{k,\xi+\eta} \setminus J^{\xi+\eta}_{y^{k,\xi+\eta}_{0,m}})| \le |Du^{\xi+\eta}_{y^{k,\xi+\eta}_{0,m}}|((W_h^k \setminus J)^{\xi+\eta}_{y^{k,\xi+\eta}_{0,m}}).$$

Since by (5.29) and (5.30) we have

$$\sum_{m\in\mathcal{I}^k(z_1)} \sum_{\substack{(i,j)\in\mathcal{J}^k\\ j-i=m}} = \sum_{(i,j)\in\mathcal{J}^k},$$

combining (5.31) and (5.33) with the previous inequality, we obtain (5.20) for $\zeta = \xi + \eta$. The proof for $\zeta = \xi - \eta$ is similar. □

Proof of Theorem 5.1 (continuation). By Lemma 5.3 we may choose a sequence $(\omega_k)_k \subset U$ and an infinite set $K \subset \mathbb{N}$ such that for every $\zeta \in \{\xi, \eta, \xi+\eta, \xi-\eta\}$ conditions (a) and (b) of Lemma 5.2 hold and

$$\lim_{\substack{k\to+\infty\\ k\in K}} \frac{C_{\xi,\eta}}{k} \sum_{(i,j)\in\mathcal{J}^k} Du^{\zeta}_{y^{k,\zeta}_{i,j}}(I^{k,\zeta}_{i,j} \setminus J^{\zeta}_{y^{k,\zeta}_{i,j}}) = \sigma^{\zeta}_u(U),$$

for every $\mathcal{J}^k$ such that $\mathcal{J}^k_3 \subset \mathcal{J}^k \subset \mathcal{J}^k_1$ (see (5.18)), where the projections $y^{k,\zeta}_{i,j}$ are defined taking $\omega = \omega_k$ in (5.4).

To present the technique we will employ in the sequel, let us assume for a moment that for every $(i,j) \in \mathcal{J}^k_1$ the segments $[x^k_{i,j}, x^k_{i+1,j}]$, $[x^k_{i,j}, x^k_{i,j+1}]$, $[x^k_{i,j}, x^k_{i+1,j+1}]$, $[x^k_{i+1,j}, x^k_{i+1,j+1}]$, $[x^k_{i,j+1}, x^k_{i+1,j+1}]$, and $[x^k_{i+1,j}, x^k_{i,j+1}]$ do not intersect the set J, which implies $I^{k,\zeta}_{i,j} \setminus J^{\zeta}_{y^{k,\zeta}_{i,j}} = I^{k,\zeta}_{i,j}$, hence

$$\lim_{\substack{k\to+\infty\\ k\in K}} \frac{C_{\xi,\eta}}{k} \sum_{(i,j)\in\mathcal{J}^k} Du^{\zeta}_{y^{k,\zeta}_{i,j}}(I^{k,\zeta}_{i,j}) = \sigma^{\zeta}_u(U), \tag{5.34}$$

whenever $\mathcal{J}^k_3 \subset \mathcal{J}^k \subset \mathcal{J}^k_1$. By (a) and (b) of Lemma 5.2 and (5.17), for every $k \in \mathbb{N}$ and $(i,j) \in \mathcal{J}^k_1$ we have that $u^{\zeta}_{y^{k,\zeta}_{i,j}} \in BV(U^{\zeta}_{y^{k,\zeta}_{i,j}})$ and

$$Du^{\zeta}_{y^{k,\zeta}_{i,j}}(I^{k,\zeta}_{i,j}) = (u(x^{k,\zeta}_{i,j} + \tfrac{1}{k}\zeta) - u(x^{k,\zeta}_{i,j}))\cdot\zeta \quad \text{for every } \zeta \in \{\xi, \eta, \xi+\eta, \xi-\eta\}. \tag{5.35}$$

Let $e_1 := (1,0)$ and $e_2 := (0,1)$ and set

$$\mathcal{J}^k_2 + e_1 := \{(i+1,j) : (i,j) \in \mathcal{J}^k_2\} \quad \text{and} \quad \mathcal{J}^k_2 + e_2 := \{(i,j+1) : (i,j) \in \mathcal{J}^k_2\},$$

and observe that by (5.19) we have $\mathcal{J}^k_3 \subset \mathcal{J}^k_2 + e_1 \subset \mathcal{J}^k_1$ and $\mathcal{J}^k_3 \subset \mathcal{J}^k_2 + e_2 \subset \mathcal{J}^k_1$. From (5.3) and (5.35) it follows that

$$\begin{aligned}
\sum_{(i,j)\in\mathcal{J}^k_2} Du^{\xi}_{y^{k,\xi}_{i,j}}(I^{k,\xi}_{i,j}) &= \sum_{(i,j)\in\mathcal{J}^k_2} \big(u(x^k_{i+1,j}) - u(x^k_{i,j})\big)\cdot\xi,\\
\sum_{(i,j)\in\mathcal{J}^k_2+e_2} Du^{\xi}_{y^{k,\xi}_{i,j}}(I^{k,\xi}_{i,j}) &= \sum_{(i,j)\in\mathcal{J}^k_2+e_2} \big(u(x^k_{i+1,j}) - u(x^k_{i,j})\big)\cdot\xi,\\
\sum_{(i,j)\in\mathcal{J}^k_2} Du^{\eta}_{y^{k,\eta}_{i,j}}(I^{k,\eta}_{i,j}) &= \sum_{(i,j)\in\mathcal{J}^k_2} \big(u(x^k_{i,j+1}) - u(x^k_{i,j})\big)\cdot\eta,\\
\sum_{(i,j)\in\mathcal{J}^k_2+e_1} Du^{\eta}_{y^{k,\eta}_{i,j}}(I^{k,\eta}_{i,j}) &= \sum_{(i,j)\in\mathcal{J}^k_2+e_1} \big(u(x^k_{i,j+1}) - u(x^k_{i,j})\big)\cdot\eta,\\
\sum_{(i,j)\in\mathcal{J}^k_2} Du^{\xi+\eta}_{y^{k,\xi+\eta}_{i,j}}(I^{k,\xi+\eta}_{i,j}) &= \sum_{(i,j)\in\mathcal{J}^k_2} \big(u(x^k_{i+1,j+1}) - u(x^k_{i,j})\big)\cdot(\xi+\eta),\\
\sum_{(i,j)\in\mathcal{J}^k_2} Du^{\xi-\eta}_{y^{k,\xi-\eta}_{i,j}}(I^{k,\xi-\eta}_{i,j}) &= \sum_{(i,j)\in\mathcal{J}^k_2} \big(u(x^k_{i+1,j}) - u(x^k_{i,j+1})\big)\cdot(\xi-\eta).
\end{aligned}$$

Thus,

$$\begin{aligned}
&\sum_{(i,j)\in\mathcal{J}^k_2} Du^{\xi+\eta}_{y^{k,\xi+\eta}_{i,j}}(I^{k,\xi+\eta}_{i,j}) + \sum_{(i,j)\in\mathcal{J}^k_2} Du^{\xi-\eta}_{y^{k,\xi-\eta}_{i,j}}(I^{k,\xi-\eta}_{i,j})\\
&= \sum_{(i,j)\in\mathcal{J}^k_2} \big(u(x^k_{i+1,j+1}) - u(x^k_{i,j})\big)\cdot(\xi+\eta) + \sum_{(i,j)\in\mathcal{J}^k_2} \big(u(x^k_{i+1,j}) - u(x^k_{i,j+1})\big)\cdot(\xi-\eta)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{(i,j)\in\mathcal{J}_2^k} (u(x^k_{i+1,j+1}) - u(x^k_{i,j+1}))\cdot\xi + \sum_{(i,j)\in\mathcal{J}_2^k} (u(x^k_{i+1,j}) - u(x^k_{i,j}))\cdot\xi \\
&\quad + \sum_{(i,j)\in\mathcal{J}_2^k} (u(x^k_{i+1,j+1}) - u(x^k_{i+1,j}))\cdot\eta + \sum_{(i,j)\in\mathcal{J}_2^k} (u(x^k_{i,j+1}) - u(x^k_{i,j}))\cdot\eta \\
&= \sum_{(i,j)\in\mathcal{J}_2^k} Du^\xi_{y^{k,\xi}_{i,j}}(I^{k,\xi}_{i,j}) + \sum_{(i,j)\in\mathcal{J}_2^k+e_2} Du^\xi_{y^{k,\xi}_{i,j}}(I^{k,\xi}_{i,j}) + \sum_{(i,j)\in\mathcal{J}_2^k+e_1} Du^\eta_{y^{k,\eta}_{i,j}}(I^{k,\eta}_{i,j}) + \sum_{(i,j)\in\mathcal{J}_2^k} Du^\eta_{y^{k,\eta}_{i,j}}(I^{k,\eta}_{i,j}).
\end{aligned}
\tag{5.36}
$$

Thanks to (5.34) we obtain

$$
\sigma_u^{\xi+\eta}(U) + \sigma_u^{\xi-\eta}(U) = 2\sigma_u^\xi(U) + 2\sigma_u^\eta(U),
$$

which implies that $\xi \mapsto \sigma_u^\xi(U)$ is quadratic.

Unfortunately, the hypothesis that for every $k\in\mathbb{N}$ and $(i,j)\in\mathcal{J}_1^k$ the every one of the six segments $[x^k_{i,j}, x^k_{i+1,j}]$, $[x^k_{i,j}, x^k_{i,j+1}]$, $[x^k_{i,j}, x^k_{i+1,j+1}]$, $[x^k_{i+1,j}, x^k_{i+1,j+1}]$, $[x^k_{i,j+1}, x^k_{i+1,j+1}]$, and $[x^k_{i+1,j}, x^k_{i,j+1}]$ do not intersect the set J is almost never satisfied. Therefore, for every $k\in\mathbb{N}$ we introduce the set $\mathcal{G}^k\subset\mathbb{Z}^2$ of good indices, defined as

$$
\begin{aligned}
\mathcal{G}^k := \big\{(i,j)\in \mathcal{J}_2^k :\ &\text{none of the segments } [x^k_{i,j}, x^k_{i+1,j}],\ [x^k_{i,j}, x^k_{i,j+1}]\,,\ [x^k_{i,j}, x^k_{i+1,j+1}], \\
&[x^k_{i+1,j}, x^k_{i+1,j+1}],\ [x^k_{i,j+1}, x^k_{i+1,j+1}],\ [x^k_{i+1,j}, x^k_{i,j+1}] \text{ intersects } J\big\},
\end{aligned}
\tag{5.37}
$$

Note that by (5.17) we have

$$
\begin{aligned}
I^{k,\zeta}_{i,j}\cap J^\zeta_{y^{k,\zeta}_{i,j}} &= \emptyset \qquad \text{for every } (i,j)\in\mathcal{G}^k, \\
I^{k,\xi}_{i,j}\cap J^\xi_{y^{k,\xi}_{i,j}} &= \emptyset \qquad \text{for every } (i,j)\in\mathcal{G}^k+e_2, \\
I^{k,\eta}_{i,j}\cap J^\eta_{y^{k,\eta}_{i,j}} &= \emptyset \qquad \text{for every } (i,j)\in\mathcal{G}^k+e_1,
\end{aligned}
\tag{5.38}
$$

where

$$
\mathcal{G}^k + e_1 := \{(i+1,j)\in\mathbb{Z}^2 : (i,j)\in\mathcal{G}^k\} \qquad\text{and}\qquad \mathcal{G}^k+e_2 := \{(i,j+1) : (i,j)\in\mathcal{G}^k\}.
$$

To prove the result in the general case, in the next section (see Theorem 6.1) we shall show that the sequence $(\omega_k)_k\subset U$ and the infinite set $K\subset\mathbb{N}$ can be chosen in such a way that conditions (a) and (b) of Lemma 5.2 hold and, in addition, for every $\zeta\in\{\xi,\eta,\xi+\eta,\xi-\eta\}$,

$$
\begin{aligned}
\lim_{\substack{k\to+\infty\\ k\in K}} \frac{C_{\xi,\eta}}{k} \sum_{(i,j)\in\mathcal{G}^k} Du^\zeta_{y^{k,\zeta}_{i,j}}(I^{k,\zeta}_{i,j}) &= \sigma_u^\zeta(U), \\
\lim_{\substack{k\to+\infty\\ k\in K}} \frac{C_{\xi,\eta}}{k} \sum_{(i,j)\in\mathcal{G}^k+e_2} Du^\xi_{y^{k,\xi}_{i,j}}(I^{k,\xi}_{i,j}) &= \sigma_u^\xi(U), \\
\lim_{\substack{k\to+\infty\\ k\in K}} \frac{C_{\xi,\eta}}{k} \sum_{(i,j)\in\mathcal{G}^k+e_1} Du^\eta_{y^{k,\eta}_{i,j}}(I^{k,\eta}_{i,j}) &= \sigma_u^\eta(U).
\end{aligned}
\tag{5.39}
$$

Assuming that these equalities hold, we now conclude the proof in the general case. Observing that (5.35) still holds for $(i,j)\in\mathcal{G}^k$, and also for $(i,j)\in\mathcal{G}^k+e_1$ when $\zeta=\eta$ and for $(i,j)\in\mathcal{G}^k+e_2$ when $\zeta=\xi$, repeating the arguments that led to (5.36) we obtain

$$
\begin{aligned}
&\sum_{(i,j)\in\mathcal{G}^k} Du^{\xi+\eta}_{y^{k,\xi+\eta}_{i,j}}(I^{k,\xi+\eta}_{i,j}) + \sum_{(i,j)\in\mathcal{G}^k} Du^{\xi-\eta}_{y^{k,\xi-\eta}_{i,j}}(I^{k,\xi-\eta}_{i,j}) \\
&= \sum_{(i,j)\in\mathcal{G}^k} Du^\xi_{y^{k,\xi}_{i,j}}(I^{k,\xi}_{i,j}) + \sum_{(i,j)\in\mathcal{G}^k+e_2} Du^\xi_{y^{k,\xi}_{i,j}}(I^{k,\xi}_{i,j}) \\
&\quad + \sum_{(i,j)\in\mathcal{G}^k+e_1} Du^\eta_{y^{k,\eta}_{i,j}}(I^{k,\eta}_{i,j}) + \sum_{(i,j)\in\mathcal{G}^k} Du^\eta_{y^{k,\eta}_{i,j}}(I^{k,\eta}_{i,j}).
\end{aligned}
$$

Multiplying the previous equality by $C_{\xi,\eta}/k$ and using (5.39) we obtain (5.1). This concludes the proof. □

6. Conclusion of the proof in dimension $d = 2$

In this section we prove a technical result, which concludes the proof of Theorem 5.1. Throughout this section u, ξ, η, and U are as in Section 5 and we use the notation introduced in the proof of Theorem 5.1. In particular, we recall that $\mathcal{G}^k$ is defined by (5.37). Before stating the main result of this section we introduce the set of bad indices $\mathcal{B}^k \subset \mathbb{Z}^2$, defined as

$$\begin{aligned}\mathcal{B}^k := \big\{(i,j) \in \mathcal{J}_2^k :\ &\text{one of the segments } [x^k_{i,j}, x^k_{i+1,j}],\ [x^k_{i,j}, x^k_{i,j+1}]\,, [x^k_{i,j}, x^k_{i+1,j+1}],\\ &[x^k_{i+1,j}, x^k_{i+1,j+1}],\ [x^k_{i,j+1}, x^k_{i+1,j+1}],\ [x^k_{i+1,j}, x^k_{i,j+1}] \text{ intersects } J\big\},\end{aligned} \tag{6.1}$$

Theorem 6.1. *There exist an infinite set $K \subset \mathbb{N}$ and a sequence $(\omega_k)_{k\in\mathbb{N}} \subset U$ such that for every $\zeta \in \{\xi, \eta, \xi+\eta, \xi-\eta\}$ conditions (a), (b) of Lemma 5.2 and (5.20) of Lemma 5.3 hold and the following equalities are satisfied:*

$$\lim_{\substack{k\to+\infty\\ k\in K}} \frac{C_{\xi,\eta}}{k} \sum_{(i,j)\in\mathcal{G}^k} Du^\zeta_{y^{k,\zeta}_{i,j}}(I^{k,\zeta}_{i,j}) = \sigma^\zeta_u(U), \tag{6.2}$$

$$\lim_{\substack{k\to+\infty\\ k\in K}} \frac{C_{\xi,\eta}}{k} \sum_{(i,j)\in\mathcal{G}^k_2+e_2} Du^\xi_{y^{k,\xi}_{i,j}}(I^{k,\xi}_{i,j}) = \sigma^\xi_u(U), \tag{6.3}$$

$$\lim_{\substack{k\to+\infty\\ k\in K}} \frac{C_{\xi,\eta}}{k} \sum_{(i,j)\in\mathcal{G}^k_2+e_1} Du^\eta_{y^{k,\eta}_{i,j}}(I^{k,\eta}_{i,j}) = \sigma^\eta_u(U), \tag{6.4}$$

$$\lim_{\substack{k\to+\infty\\ k\in K}} \frac{1}{k} \sum_{(i,j)\in\mathcal{B}^k} |Du^\zeta_{y^{k,\zeta}_{i,j}}|(I^{k,\zeta}_{i,j} \setminus J^\zeta_{y^{k,\zeta}_{i,j}}) = 0, \tag{6.5}$$

$$\lim_{\substack{k\to+\infty\\ k\in K}} \frac{1}{k} \sum_{(i,j)\in\mathcal{B}^k+e_2} |Du^\xi_{y^{k,\xi}_{i,j}}|(I^{k,\xi}_{i,j} \setminus J^\xi_{y^{k,\xi}_{i,j}}) = 0, \tag{6.6}$$

$$\lim_{\substack{k\to+\infty\\ k\in K}} \frac{1}{k} \sum_{(i,j)\in\mathcal{B}^k+e_1} |Du^\eta_{y^{k,\eta}_{i,j}}|(I^{k,\eta}_{i,j} \setminus J^\eta_{y^{k,\eta}_{i,j}}) = 0, \tag{6.7}$$

where the points $y^{k,\zeta}_{i,j}$ introduced in (5.5) are defined by taking $\omega = \omega_k$.

The crucial part in the proof of this result is proving (6.5)-(6.7), as (6.2)-(6.4) can then be obtained from (5.20) by difference, using (5.38). We only prove (6.5), as the proof of (6.6) and (6.7) are similar. This proof is extremely technical. The arguments we are going to use require some additional notation.

Given $\zeta \in \{\xi, \eta, \xi-\eta, \xi+\eta\}$, we set

$$\bar\zeta := \xi \ \text{ if } \ \zeta \in \{\eta, \xi+\eta\} \qquad \text{and} \qquad \bar\zeta := \eta \ \text{ if } \ \zeta \in \{\xi, \xi-\eta\}. \tag{6.8}$$

We observe that ζ and $\bar\zeta$ are linearly independent. For $y \in \mathbb{R}^2$, $k \in \mathbb{N}$, $j \in \mathbb{Z}$, and $\zeta \in \{\xi, \eta, \xi+\eta, \xi-\eta\}$ let $t^{k,\zeta}_j(y)$ be the real number characterised by

$$y + t^{k,\zeta}_j(y)\zeta \in \{\omega + \tfrac{j}{k}\zeta + s\bar\zeta : s \in \mathbb{R}\}. \tag{6.9}$$

We also set

$$x^{k,\zeta}_j(y) := y + t^{k,\zeta}_j(y)\zeta. \tag{6.10}$$

In other words, $x^{k,\zeta}_j(y)$ is the intersection of the straight lines $\{y + t\zeta : t \in \mathbb{R}\}$ and $\{\omega + \frac{j}{k}\zeta + s\bar\zeta : s \in \mathbb{R}\}$. Note that the family of straight lines $(\{\omega + \frac{j}{k}\zeta + s\bar\zeta : s \in \mathbb{R}\})_{j\in\mathbb{Z}}$ coincides with the family of the straight lines parallel to $\bar\zeta$ passing through one of the points $x^k_{i,j}$ for $i, j \in \mathbb{Z}$.

Note that for every $j \in \mathbb{Z}$ and $t \in \mathbb{R}$ we have

$$t_{j+1}^{k,\zeta}(y) = t_j^{k,\zeta}(y) + \tfrac{1}{k} \quad \text{and} \quad t_j^{k,\zeta}(y + t\zeta) = t_j^{k,\zeta}(y) - t,$$

which give

$$x_{j+1}^{k,\zeta}(y) = x_j^{k,\zeta}(y) + \tfrac{1}{k}\zeta \quad \text{and} \quad x_j^{k,\zeta}(y + t\zeta) = x_j^{k,\zeta}(y). \tag{6.11}$$

Moreover,

$$[x_j^{k,\zeta}(y), x_{j+1}^{k,\zeta}(y)) = \{y + t\zeta : t \in [t_j^{k,\zeta}(y), t_{j+1}^{k,\zeta}(y))\}. \tag{6.12}$$

Therefore for every $y \in \mathbb{R}^2$ each straight line $\{y + t\zeta : t \in \mathbb{R}\}$ can be written as disjoint union of segments in the following way

$$\{y + t\zeta : t \in \mathbb{R}\} = \bigcup_{j \in \mathbb{Z}} [x_j^{k,\zeta}(y), x_{j+1}^{k,\zeta}(y)). \tag{6.13}$$

We need to introduce some sets which are useful to establish (6.5) and whose definition requires some additional notation. Let us fix $\zeta \in \{\xi, \eta, \xi + \eta, \xi - \eta\}$ and $k \in \mathbb{N}$. In view of (6.13), for every $x \in \mathbb{R}^2$ there exists a unique $j \in \mathbb{Z}$ such that

$$x \in [x_j^{k,\zeta}(x), x_{j+1}^{k,\zeta}(x)). \tag{6.14}$$

We define the map $z^{k,\zeta} \colon \mathbb{R}^2 \to \mathbb{R}^2$ (see Figure 3) as

$$z^{k,\zeta}(x) = x_j^{k,\zeta}(x), \tag{6.15}$$

where $j \in \mathbb{Z}$ is the unique index such that (6.14) holds.

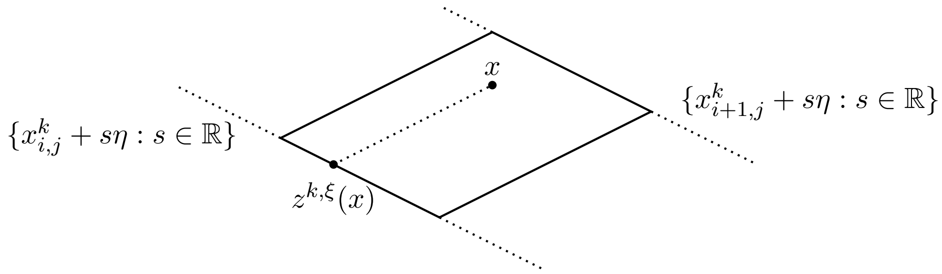

FIGURE 3. The map $x \mapsto z^{k,\xi}(x)$.

By (6.11) and (6.12) for every $y \in \mathbb{R}^2$

$$z^{k,\zeta}(y + t\zeta) = x_j^{k,\zeta}(y) \quad \text{for every } t \in [t_j^{k,\zeta}(y), t_{j+1}^{k,\zeta}(y)). \tag{6.16}$$

Geometrically (see Figure 3), $z^{k,\zeta}(x)$ is given by $x - t\zeta$ where $t \geq 0$ is the smallest number such that $x - t\zeta$ belongs to one of the straight lines parallel to $\bar{\zeta}$ passing through one of the points $x_{i,j}^k$. By this geometric characterisation, $z^{k,\zeta}$ is a Borel function.

We consider the union S of the sides and the diagonals of the parallelogram of vertices at 0, $\xi, \xi + \eta, \eta$, i.e.,

$$S := [0, \xi] \cup [0, \eta] \cup [0, \xi + \eta] \cup [\eta, \xi] \cup [\eta, \xi + \eta] \cup [\xi, \xi + \eta]. \tag{6.17}$$

For $k \in \mathbb{N}$ and $\zeta \in \{\xi, \eta, \xi + \eta, \xi - \eta\}$ we introduce the set

$$E^{k,\zeta} := \{x \in \mathbb{R}^2 : (z^{k,\zeta}(x) + \tfrac{1}{k}S) \cap J \neq \emptyset\}. \tag{6.18}$$

For $y \in \mathbb{R}^2$ we define

$$\begin{gathered} \mathcal{I}^{k,\zeta}(y) := \{i \in \mathbb{Z} : (x_i^{k,\zeta}(y) + \tfrac{1}{k}S) \cap J \neq \emptyset\}, \quad \mathcal{N}^{k,\zeta}(y) := \mathcal{H}^0(\mathcal{I}^{k,\zeta}(y)), \\ E^{k,\zeta}(y) := \bigcup_{i \in \mathcal{I}^{k,\zeta}(y)} [t_i^{k,\zeta}(y), t_{i+1}^{k,\zeta}(y)). \end{gathered} \tag{6.19}$$

By (6.16) we have the equality

$$E^{k,\zeta}(y) = (E^{k,\zeta})_y^\zeta \quad \text{for every } y \in \mathbb{R}^2 \tag{6.20}$$

and by (6.11) we have

$$\mathcal{I}^{k,\zeta}(y+t\zeta)=\mathcal{I}^{k,\zeta}(y) \quad\text{and}\quad \mathcal{N}^{k,\zeta}(y+t\zeta)=\mathcal{N}^{k,\zeta}(y) \quad \text{for every } t\in\mathbb{R}. \tag{6.21}$$

Let $x\in\mathbb{R}^2$ and let $j\in\mathbb{Z}$ be the unique index such that (6.14) holds. By (6.11) for every $i\in\mathbb{Z}$ we have

$$x_i^{k,\zeta}(x)=x_i^{k,\zeta}(x+\tfrac{i-j}{k}\zeta)=z^{k,\zeta}(x+\tfrac{i-j}{k}\zeta),$$

where the last equality follows from (6.15), since $x+\frac{i-j}{k}\zeta\in[x_i^{k,\zeta}(x+\frac{i-j}{k}\zeta),x_{i+1}^{k,\zeta}(x+\frac{i-j}{k}\zeta))$ by (6.11). Recalling the definition of $E^{k,\zeta}$ in (6.18), the equalities above imply that

$$\mathcal{I}^{k,\zeta}(x)=\{i\in\mathbb{Z}:(z^{k,\zeta}(x+\tfrac{i-j}{k}\zeta)+\tfrac{1}{k}S)\cap J\neq\emptyset\}=\{i\in\mathbb{Z}:x+\tfrac{i-j}{k}\zeta\in E^{k,\zeta}\},$$

which gives

$$\mathcal{N}^{k,\zeta}(x)=\mathcal{H}^0(\{i\in\mathbb{Z}:x+\tfrac{i}{k}\zeta\in E^{k,\zeta}\}). \tag{6.22}$$

For $\zeta\in\{\xi,\eta,\xi+\eta,\xi-\eta\}$, $k,m\in\mathbb{N}$, and $y\in\mathbb{R}^2$ we set

$$\hat{E}_m^{k,\zeta}:=\{x\in E^{k,\zeta}:\mathcal{N}^{k,\zeta}(x)\le m\},\quad \check{E}_m^{k,\zeta}:=\{x\in E^{k,\zeta}:\mathcal{N}^{k,\zeta}(x)> m\}, \tag{6.23}$$

$$\hat{E}_m^{k,\zeta}(y):=\begin{cases}E^{k,\zeta}(y) & \text{if } \mathcal{N}^{k,\zeta}(y)\le m,\\ \emptyset & \text{otherwise,}\end{cases}\quad \check{E}_m^{k,\zeta}(y):=\begin{cases}E^{k,\zeta}(y) & \text{if } \mathcal{N}^{k,\zeta}(y)> m,\\ \emptyset & \text{otherwise.}\end{cases} \tag{6.24}$$

By (6.20) and (6.21) we have

$$\hat{E}_m^{k,\zeta}(y)=(\hat{E}_m^{k,\zeta})_y^\zeta \quad\text{and}\quad \check{E}_m^{k,\zeta}(y)=(\check{E}_m^{k,\zeta})_y^\zeta. \tag{6.25}$$

All these sets are Borel measurable as the following lemma shows.

Lemma 6.2. *The sets $E^{k,\zeta}$, $\hat{E}_m^{k,\zeta}$, and $\check{E}_m^{k,\zeta}$ are Borel measurable. Moreover, the function $\mathcal{N}^{k,\zeta}$ is Borel measurable on $\mathbb{R}^2$.*

Proof. For every set $B\subset\mathbb{R}^2$ we define

$$E_B^{k,\zeta}:=\{x\in\mathbb{R}^2:(z^{k,\zeta}(x)+\tfrac{1}{k}S)\cap B\neq\emptyset\},$$
$$F_B:=\{z\in\mathbb{R}^2:(z+\tfrac{1}{k}S)\cap B\neq\emptyset\}.$$

We begin by proving that for a compact set $K\subset\mathbb{R}^2$ the set $E_K^{k,\zeta}$ is Borel measurable. To this aim we note that the set F_K is closed and that $E_K^{k,\zeta}=\{x\in\mathbb{R}^2:z^{k,\zeta}(x)\in F_K\}$. Recalling that $z^{k,\zeta}$ is Borel measurable, we conclude that $E_K^{k,\zeta}$ is Borel measurable. By (5.9) we have $J=\bigcup_{n\in\mathbb{N}}K_n$, where K_n are compact sets. This gives that $E^{k,\zeta}=\bigcup_{n\in\mathbb{N}}E_{K_n}^{k,\zeta}$. Since the sets $E_{K_n}^{k,\zeta}$ are Borel measurable, so is $E^{k,\zeta}$.

To prove that $\hat{E}_m^{k,\zeta}$ Borel measurable, we observe that by (6.22) and (6.23) a point x belongs to $\hat{E}_m^{k,\zeta}$ if and only if the number of indices $i\in\mathbb{Z}$ such that $x\in E^{k,\zeta}-\frac{i}{k}\zeta$ is less than or equal to m. This implies that, setting $E_i^{k,\zeta}:=E^{k,\zeta}-\frac{i}{k}\zeta$, we have

$$\hat{E}_m^{k,\zeta}=\big\{x\in\mathbb{R}^2:\textstyle\sum_i\chi_{E_i^{k,\zeta}}(x)\le m\big\},$$

where $\chi_{E_i^{k,\zeta}}$ is the characteristic function of $E_i^{k,\zeta}$. Since the sets $E_i^{k,\zeta}$ are Borel measurable, we deduce that $\hat{E}_m^{k,\zeta}$ is Borel measurable. The Borel measurability of $\check{E}_m^{k,\zeta}$ follows from the equality $\check{E}_m^{k,\zeta}=E^{k,\zeta}\setminus\hat{E}_m^{k,\zeta}$.

To prove that the function $\mathcal{N}^{k,\zeta}$ is Borel measurable, it is enough to observe that by (6.22) we have $\mathcal{N}^{k,\zeta}=\sum_{i\in\mathbb{Z}}\chi_{E_i^{k,\zeta}}$. □

Remark 6.3. All the sets defined in (6.18) and (6.23) depend non-trivially on $\omega\in\mathbb{R}^2$, since by (5.3) every choice of ω determines different points $x_{i,j}^k$ and thus, by (6.10), different sets in (6.18) and (6.23) as well. Not to overburden the notation, we do not indicate the dependence of such objects on ω. The measurability issues with respect to ω will be dealt with in the Appendix.

To use the sets $\hat{E}^{k,\zeta}_m$ and $\check{E}^{k,\zeta}_m$ in our estimates we need the properties proved in the following two lemmas, whose proofs are postponed. We observe that, since ξ and η are linearly independent, given $\zeta \in \{\xi, \eta, \xi+\eta, \xi-\eta\}$, every point $\omega \in \mathbb{R}^2$ can be written in a unique way as

$$\omega = z_1\bar{\zeta} + z_2\zeta, \tag{6.26}$$

for suitable $z_1, z_2 \in \mathbb{R}$. We set

$$I^\zeta := \begin{cases} (0,\alpha) & \text{if } \zeta = \xi \text{ or } \zeta = \xi - \eta, \\ (0,\beta) & \text{if } \zeta = \eta \text{ or } \zeta = \xi + \eta, \end{cases} \tag{6.27}$$

and we observe that by (5.2)

$$\pi^{\bar{\zeta}}(U) := \{z_2 \pi^{\bar{\zeta}}(\zeta) : z_2 \in I^\zeta\}. \tag{6.28}$$

Lemma 6.4. *There exist a constant $C > 0$ such that for every $\zeta \in \{\xi, \eta, \xi+\eta, \xi-\eta\}$ and for every $\varepsilon > 0$ there exist an infinite set $K^\zeta_\varepsilon \subset \mathbb{N}$ and a Borel set $I^\zeta_\varepsilon \subset I^\zeta$, with $\mathcal{L}^1(I^\zeta \setminus I^\zeta_\varepsilon) \le \varepsilon$, such that*

$$\mathcal{H}^1\big(\pi^\zeta(\check{E}^{k,\zeta}_m)\big) \le C/m \tag{6.29}$$

for every $m \in \mathbb{N}$, $\omega \in U^\zeta_\varepsilon := U \cap \{z_1\bar{\zeta} + z_2\zeta : z_1 \in \mathbb{R},\, z_2 \in I^\zeta_\varepsilon\}$, and $k \in K^\zeta_\varepsilon$.

Lemma 6.5. *Let $m \in \mathbb{N}$ and $\zeta \in \{\xi, \eta, \xi+\eta, \xi-\eta\}$. Then we have*

$$\lim_{k\to+\infty} |Du^\zeta_y|(\hat{E}^{k,\zeta}_m(y) \cap U^\zeta_y \setminus J^\zeta_y) = 0 \quad \text{for } \mathcal{H}^1\text{-a.e. } y \in \Pi^\zeta, \tag{6.30}$$

$$\lim_{k\to+\infty} \int_{\Pi^\zeta} |Du^\zeta_y|(\hat{E}^{k,\zeta}_m(y) \cap U^\zeta_y \setminus J^\zeta_y)\, \mathrm{d}\mathcal{H}^1(y) = 0. \tag{6.31}$$

To prove Lemma 6.4 we need the following elementary result.

Lemma 6.6. *Let $F \subset \mathbb{R}$ be a finite set and let $a > 0$ and $b < c$. Then*

$$\int_{\mathbb{R}} \mathcal{H}^0([at+b, at+c] \cap F)\, \mathrm{d}t = \frac{c-b}{a}\mathcal{H}^0(F).$$

Proof. By the Fubini Theorem we have that

$$\int_{\mathbb{R}} \mathcal{H}^0([at+b, at+c] \cap F)\, \mathrm{d}t = \int_{\mathbb{R}} \Big(\int_F \chi_{[at+b,at+c]}(s)\, \mathrm{d}\mathcal{H}^0(s) \Big) \mathrm{d}t$$
$$= \int_F \Big(\int_{\mathbb{R}} \chi_{[\frac{s-c}{a}, \frac{s-b}{a}]}(t)\, \mathrm{d}t \Big) \mathrm{d}\mathcal{H}^0(s) = \frac{c-b}{a}\mathcal{H}^0(F).$$

This concludes the proof. □

Proof of Lemma 6.4. We observe that $\pi^\zeta(\check{E}^{k,\zeta}_m) = \{y \in \Pi^\zeta : \mathcal{N}^{k,\zeta}(y) > m\}$ by (6.21) and (6.23). Since $\mathcal{N}^{k,\zeta}$ is Borel measurable by Lemma 6.2, $\pi^\zeta(\check{E}^{k,\zeta}_m)$ is a Borel set. By Čebyšëv's inequality we obtain

$$\mathcal{H}^1\big(\pi^\zeta(\check{E}^{k,\zeta}_m)\big) = \mathcal{H}^1\big(\{y \in \Pi^\zeta : \mathcal{N}^{k,\zeta}(y) > m\}\big) \le \frac{1}{m} \int_{\Pi^\zeta} \mathcal{N}^{k,\zeta}(y)\, \mathrm{d}\mathcal{H}^1(y) \tag{6.32}$$

for every $\zeta \in \{\xi, \eta, \xi+\eta, \xi-\eta\}$, $k, m \in \mathbb{N}$, $\omega \in U$. To conclude the proof it is enough to show that there exist a constant $C > 0$ and, for every $\varepsilon > 0$ and $\zeta \in \{\xi, \eta, \xi+\eta, \xi-\eta\}$, an infinite set $K^\zeta_\varepsilon \subset \mathbb{N}$ and a Borel set $I^\zeta_\varepsilon \subset I^\zeta$, with $\mathcal{L}^1(I^\zeta \setminus I^\zeta_\varepsilon) \le \varepsilon$, such that

$$\int_{\Pi^\zeta} \mathcal{N}^{k,\zeta}(y)\, \mathrm{d}\mathcal{H}^1(y) \le C \tag{6.33}$$

for every $\omega \in U^\zeta_\varepsilon := U \cap \{z_1\bar{\zeta} + z_2\zeta : z_1 \in \mathbb{R},\, z_2 \in I^\zeta_\varepsilon\}$ and $k \in K^\zeta_\varepsilon$.

We observe that by (6.17) we can write $S = S_1 \cup \dots \cup S_6$, where

$$S_1 := [0, \bar{\zeta}], \qquad S_2 := \begin{cases} S_1 + \xi & \text{if } \quad \zeta \in \{\xi, \xi-\eta\}, \\ S_1 + \eta & \text{if } \quad \zeta \in \{\eta, \xi+\eta\}, \end{cases} \tag{6.34}$$

while S_3,..., S_6 are the other four segments of S, which are transversal to $\bar\zeta$. By (6.19) this implies that

$$\mathcal{I}^{k,\zeta}(y) = \bigcup_{h=1}^{6} \{j \in \mathbb{Z} : (x_j^{k,\zeta}(y) + S_h) \cap J \neq \emptyset\}$$

for every $y \in \Pi^\zeta$, hence

$$\mathcal{N}^{k,\zeta}(y) \leq \sum_{h=1}^{6} \mathcal{H}^0(\{j \in \mathbb{Z} : (x_j^{k,\zeta}(y) + S_h) \cap J \neq \emptyset\}).$$

Thus,

$$\int_{\Pi^\zeta} \mathcal{N}^{k,\zeta}(y)\, \mathrm{d}\mathcal{H}^1(y) \leq \sum_{h=1}^{6} \int_{\Pi^\zeta} \mathcal{H}^0(\{j \in \mathbb{Z} : (x_j^{k,\zeta}(y) + \tfrac{1}{k} S_h) \cap J \neq \emptyset\})\, \mathrm{d}\mathcal{H}^1(y). \tag{6.35}$$

Let us fix $\zeta \in \{\xi, \eta, \xi+\eta, \xi-\eta\}$ and $\varepsilon > 0$. We claim that there exist a constant $c_1 > 0$, independent of ω, k, and ε, an infinite set $K_\varepsilon^\zeta \subset \mathbb{N}$, independent of ω, and a Borel set $I_\varepsilon^\zeta \subset I^\zeta$, with $\mathcal{L}^1(I^\zeta \setminus I_\varepsilon^\zeta) \leq \varepsilon$, such that

$$\int_{\Pi^\zeta} \mathcal{H}^0(\{j \in \mathbb{Z} : (x_j^{k,\zeta}(y) + \tfrac{1}{k} S_h) \cap J \neq \emptyset\})\, \mathrm{d}\mathcal{H}^1(y) \leq c_1, \tag{6.36}$$

for every $\omega \in U_\varepsilon^\zeta$, $k \in K_\varepsilon^\zeta$, and $h \in \{1, ..., 6\}$. To prove this claim, we first observe that for every $y \in \Pi^\zeta$ we have

$$\mathcal{H}^0(\{j \in \mathbb{Z} : (x_j^{k,\zeta}(y) + \tfrac{1}{k} S_h) \cap J \neq \emptyset\}) \leq \sum_{j \in \mathbb{Z}} \mathcal{H}^0((x_j^{k,\zeta}(y) + \tfrac{1}{k} S_h) \cap J). \tag{6.37}$$

We consider first the case $h = 1$. We prove that there exists a constant $c > 0$, independent of ω, k, ε, and j, such that

$$\int_{\Pi^\zeta} \mathcal{H}^0((x_j^{k,\zeta}(y) + \tfrac{1}{k} S_1) \cap J)\, \mathrm{d}\mathcal{H}^1(y) \leq \frac{c}{k} \mathcal{H}^0(J^{\bar\zeta}_{y_j^{k,\bar\zeta}}) \tag{6.38}$$

for every $k \in \mathbb{N}$ and $j \in \mathbb{Z}$. If $\mathcal{H}^0(J^{\bar\zeta}_{y_j^{k,\bar\zeta}}) = +\infty$ there is nothing to prove. Let us fix k and j such that $\mathcal{H}^0(J^{\bar\zeta}_{y_j^{k,\bar\zeta}}) < +\infty$.

We parametrise Π^ζ by $y = s\pi^\zeta(\bar\zeta)$ with $s \in \mathbb{R}$. and observe that (6.38) is equivalent to

$$\int_{\mathbb{R}} \mathcal{H}^0((x_j^{k,\zeta}(s\pi^\zeta(\bar\zeta)) + \tfrac{1}{k} S_1) \cap J)\, \mathrm{d}s \leq \frac{c}{k} \mathcal{H}^0(J^{\bar\zeta}_{y_j^{k,\bar\zeta}}) \tag{6.39}$$

for a possibly different constant c, independent of ω, k , ε, and j. By (5.4), (5.5), and (6.8) we have that $\pi^{\bar\zeta}(\omega + \frac{j}{k}\zeta) = y_j^{k,\bar\zeta}$. Therefore by the comments after (6.10)

$$\{x_j^{k,\zeta}(y)\} = \{x_j^{k,\zeta}(s\pi^\zeta(\bar\zeta))\} = \{s\pi^\zeta(\bar\zeta) + t\zeta : t \in \mathbb{R}\} \cap \{y_j^{k,\bar\zeta} + t\bar\zeta : t \in \mathbb{R}\}. \tag{6.40}$$

This implies that for every $s \in \mathbb{R}$ there exists a unique $\tau_j^{k,\bar\zeta}(s) \in \mathbb{R}$ such that $y_j^{k,\bar\zeta} + \tau_j^{k,\bar\zeta}(s)\bar\zeta = x_j^{k,\zeta}(s\pi^\zeta(\bar\zeta))$, so that by (6.34) we have

$$\mathcal{H}^0((x_j^{k,\zeta}(s\pi^\zeta(\bar\zeta)) + \tfrac{1}{k} S_1) \cap J) = \mathcal{H}^0([\tau_j^{k,\bar\zeta}(s), \tau_j^{k,\bar\zeta}(s) + \tfrac{1}{k}] \cap J^{\bar\zeta}_{y_j^{k,\bar\zeta}}).$$

Elementary geometric arguments show that there exist a constant $c > 0$, independent of ω, k, ε, and j, and a constant $d \in \mathbb{R}$, depending on ω, k, and j, such that

$$\tau_j^{k,\bar\zeta}(s) = \frac{s}{c} + d \qquad \text{for every } s \in \mathbb{R}.$$

Since by assumption $\mathcal{H}^0(J^{\bar\zeta}_{y_j^{k,\bar\zeta}}) < +\infty$, we may apply Lemma 6.6 to obtain that

$$\int_{\mathbb{R}} \mathcal{H}^0([\tau_j^{k,\bar\zeta}(s), \tau_j^{k,\bar\zeta}(s) + \tfrac{1}{k}] \cap J^{\bar\zeta}_{y_j^{k,\bar\zeta}})\,\mathrm{d}s \le \frac{c}{k}\mathcal{H}^0(J^{\bar\zeta}_{y_j^{k,\bar\zeta}}). \tag{6.41}$$

As the constant c depends only on ξ and η, this proves (6.39), which gives (6.38). Arguing in a similar way, we prove that

$$\int_{\Pi^\zeta} \mathcal{H}^0((x_j^{k,\zeta}(y) + \tfrac{1}{k}S_2) \cap J)\,\mathrm{d}\mathcal{H}^1(y) \le \frac{c}{k}\mathcal{H}^0(J^{\bar\zeta}_{y_{j\pm1}^{k,\bar\zeta}}), \tag{6.42}$$

where the sign in ±1 depends on the specific value of ζ, according to (6.34).

By (5.4), (5.5), (6.8), and (6.26) we have the equality $y_j^{k,\bar\zeta} = (z_2 + \frac{j}{k})\pi^{\bar\zeta}(\zeta)$ for every $j \in \mathbb{Z}$ and for a suitable $z_2 \in \mathbb{R}$. Let $f\colon \mathbb{R} \to [0, +\infty]$ be defined by

$$f(s) := \mathcal{H}^0(J^{\bar\zeta}_{s\pi^{\bar\zeta}(\zeta)}) \quad \text{for every } s \in \mathbb{R}.$$

Observe that, since $J \subset U$, by (6.28) the function f vanishes outside of I^ζ. Since by (5.8) and (5.9) we have $\mathcal{H}^1(J) = \mathcal{H}^1(J_u^1 \cap U)$ and $\mathcal{H}^1(J_u^1 \cap U) < +\infty$ by Remark 2.10, the function f is integrable by Lemma 2.4. Thus, we may apply Lemma 3.1 and we obtain an infinite set $K_\varepsilon^\zeta \subset \mathbb{N}$ and a Borel set $I_\varepsilon^\zeta \subset I^\zeta$, with $\mathcal{L}^1(I^\zeta \setminus I_\varepsilon^\zeta) \le \varepsilon$, such that

$$\lim_{\substack{k\to+\infty\\ k\in K_\varepsilon^\zeta}} \frac{1}{k}\sum_{j\in\mathbb{Z}} \mathcal{H}^0(J^{\bar\zeta}_{y_j^{k,\bar\zeta}}) = \int_{I^\zeta} \mathcal{H}^0(J^{\bar\zeta}_{s\pi^{\bar\zeta}(\zeta)})\,\mathrm{d}s = \frac{1}{|\pi^{\bar\zeta}(\zeta)|}\int_{\Pi^{\bar\zeta}} \mathcal{H}^0(J^{\bar\zeta}_y)\,\mathrm{d}\mathcal{H}^1(y) \le \frac{1}{|\pi^{\bar\zeta}(\zeta)|}\mathcal{H}^1(J)$$

uniformly for $z_2 \in I_\varepsilon^\zeta$. Hence, up to removing a finite number of elements from K_ε^ζ, we may assume that

$$\frac{1}{k}\sum_{j\in\mathbb{Z}} \mathcal{H}^0(J^{\bar\zeta}_{y_j^{k,\bar\zeta}}) \le \frac{1}{|\pi^{\bar\zeta}(\zeta)|}\mathcal{H}^1(J) + 1 \tag{6.43}$$

for every $k \in K_\varepsilon^\zeta$ and $z_2 \in I_\varepsilon^\zeta$. Together with (6.37), (6.38), and (6.42), this implies (6.36) for $h = 1$ and $h = 2$.

Let us now fix $h \in \{3, \dots, 6\}$. For every $j \in \mathbb{Z}$, let L_j be the strip defined by $L_j := \{x \in \mathbb{R}^2 : \pi^{\bar\zeta}(x) \in [y_{j-1}^{k,\bar\zeta}, y_{j+1}^{k,\bar\zeta}]\}$, that is to say, the region of the plane between the straight lines $\{y_{j-1}^{k,\bar\zeta} + s\bar\zeta : s \in \mathbb{R}\}$ and $\{y_{j+1}^{k,\bar\zeta} + s\bar\zeta : s \in \mathbb{R}\}$. Since $\pi^{\bar\zeta}(S_h)$ is equal to $[0, \pi^\xi(\eta)]$ or $[\pi^\xi(\eta), 0]$ if $\bar\zeta = \xi$, while $\pi^{\bar\zeta}(S_h) = [0, \pi^\eta(\xi)]$ if $\bar\zeta = \eta$, from (5.4) and (5.5) we see that in every case we have the inclusion

$$\bigcup_{s\in\mathbb{R}} \{y_j^{k,\bar\zeta} + s\bar\zeta + \tfrac{1}{k}S_h\} \subset L_j \quad \text{for every } j \in \mathbb{Z}.$$

Let $\tilde\zeta \in \mathbb{R}^2 \setminus \{0\}$ a vector in the direction of the segment S_h. Recalling that $x_j^{k,\zeta}(y) \in \{y_j^{k,\bar\zeta} + s\bar\zeta : s \in \mathbb{R}\}$ for every $y \in \Pi^\zeta$ by (6.40), we deduce from the previous inclusion that $(x_j^{k,\zeta}(y) + \frac{1}{k}S_h) \cap J \subset J \cap L_j$, hence

$$\mathcal{H}^0((x_j^{k,\zeta}(y) + \tfrac{1}{k}S_h) \cap J) \le \mathcal{H}^0((J \cap L_j)^{\tilde\zeta}_{\pi^{\tilde\zeta}(x_j^{k,\zeta}(y))}) \tag{6.44}$$

for every $j \in \mathbb{Z}$. As ζ and $\bar\zeta$ are linearly independent and the same holds for $\tilde\zeta$ and $\bar\zeta$, the map $y \mapsto \pi^{\tilde\zeta}(x_j^{k,\zeta}(y))$ from Π^ζ into $\Pi^{\tilde\zeta}$ is affine and invertible. Moreover, its linear part is independent of ω, k, and j, since it depends only on ζ, $\bar\zeta$, and $\tilde\zeta$. By Lemma 2.4 we then have

$$\int_{\Pi^{\tilde\zeta}} \mathcal{H}^0((J \cap L_j)^{\tilde\zeta}_z)\,\mathrm{d}\mathcal{H}^1(z) \le \mathcal{H}^1(J \cap L_j).$$

Using the map $y \mapsto \pi^{\tilde{\zeta}}(x_j^{k,\zeta}(y))$ as change of variables, from the previous inequality and (6.44) it follows that

$$\int_{\Pi^\zeta} \mathcal{H}^0((x_j^{k,\zeta}(y) + \tfrac{1}{k} S_h) \cap J)\, \mathrm{d}\mathcal{H}^1(y) \leq \gamma \mathcal{H}^1(J \cap L_j)$$

for every $j \in \mathbb{Z}$, $k \in \mathbb{N}$, and $\omega \in U$, with $\gamma > 0$, a constant depending only on ζ, $\bar{\zeta}$, and $\tilde{\zeta}$. Observing that every point $x \in \mathbb{R}^2$ belongs at most to three strips of the form L_j, from the previous inequality it follows that

$$\sum_{j\in\mathbb{Z}} \int_{\Pi^\zeta} \mathcal{H}^0((x_j^{k,\zeta}(y) + \tfrac{1}{k} S_h) \cap J)\, \mathrm{d}\mathcal{H}^1(y) \leq 3\gamma \mathcal{H}^1(J).$$

This inequality, together with (6.37), yields

$$\int_{\Pi^\zeta} \mathcal{H}^0(\{j \in \mathbb{Z} : (x_j^{k,\zeta}(y) + \tfrac{1}{k} S_h) \cap J \neq \emptyset\})\, \mathrm{d}\mathcal{H}^1(y) \leq 3\gamma \mathcal{H}^1(J).$$

Since $\mathcal{H}^1(J) < +\infty$, this proves (6.36) for $h \in \{3, \dots, 6\}$. Therefore, (6.36) holds for every $h \in \{1, \dots, 6\}$. Thanks to (6.35), from (6.36) we obtain (6.33), which by (6.32) concludes the proof. □

Before proving Lemma 6.5, we state a result about one-dimensional measures, which shows that non-atomic measures satisfy a suitable uniform absolute continuity property.

Lemma 6.7. *Let $I = [a, b] \subset \mathbb{R}$ be a bounded closed interval and let $\mu \in \mathcal{M}_b^+(I)$ be a measure such that $\mu(\{t\}) = 0$ for every $t \in I$. Then for every $m \in \mathbb{N}$ and $\varepsilon > 0$ there exists $\delta(\varepsilon, m) > 0$ such that for every $\delta \in (0, \delta(\varepsilon, m))$ we have*

$$\mu\Big(\bigcup_{\ell=1}^{m} (t_\ell - \delta, t_\ell + \delta) \cap I\Big) \leq \varepsilon \quad \text{for every } (t_1, ..., t_m) \in I^m.$$

Proof. We argue by contradiction. Suppose that there exist $m \in \mathbb{N}$ and $\varepsilon > 0$ such that for a sequence $\delta_k > 0$ converging to 0 we have

$$\mu\Big(\bigcup_{\ell=1}^{m} (t_\ell^k - \delta_k, t_\ell^k + \delta_k) \cap I\Big) > \varepsilon \quad \text{for some } (t_1^k, ..., t_m^k) \in I^m.$$

Given $\delta > 0$, this implies that

$$\mu\Big(\bigcup_{\ell=1}^{m} (t_\ell^k - \delta, t_\ell^k + \delta) \cap I\Big) > \varepsilon \qquad \text{for all } k \in \mathbb{N} \text{ sufficiently large.}$$

Since I is compact, there exists a subsequence, not relabelled, and a point $(t_1, ..., t_m) \in I^m$ such that $(t_1^k, ..., t_m^k)$ converges to $(t_1, ..., t_m)$ as $k \to +\infty$. Since $(t_\ell^k - \delta, t_\ell^k + \delta) \subset (t_\ell - 2\delta, t_\ell + 2\delta)$ for k large we deduce that

$$\mu\Big(\bigcup_{\ell=1}^{m} (t_\ell - 2\delta, t_\ell + 2\delta) \cap I\Big) > \varepsilon.$$

Since $\delta > 0$ is arbitrary, we obtain

$$\mu\Big(\bigcup_{\ell=1}^{m} \{t_\ell\}\Big) \geq \varepsilon,$$

in contradiction with our hypotheses. This concludes the proof. □

Proof of Lemma 6.5. We begin by noting that for $\mathcal{H}^1$-a.e. $y \in \Pi^\zeta$ we have $u_y^\zeta \in BV(U_y^\zeta)$ by Remark 2.6 and, recalling (5.8) and (5.9), by Theorem 2.12 we have also $J_{u_y^\zeta} \cap U_y^\zeta \subset J_y^\zeta = (J_u^1)_y^\zeta$. Let us fix $y \in \Pi^\zeta$ such that these two conditions hold. By standard properties of BV-functions in dimension one we have $Du_y^\xi(\{t\}) = 0$ for every $t \in (U \setminus J)_y^\zeta$. We recall that by definition $\hat{E}^{k,\zeta}_m(y)$ is the union of at most m intervals of length $\frac{1}{k}$ (see (6.19)). Hence, Lemma 6.7, applied to the measure $\mu := |Du^\xi_y| \llcorner (U^\zeta_y \setminus J^\zeta_y)$ defined on the closure of the interval U^ζ_y, implies that (6.30) holds.

We observe that the integral in (6.31) is well-defined, since by Lemma 6.2 the set $\hat{E}^{k,\zeta}_m$ is Borel measurable and $\hat{E}^{k,\zeta}_m(y)$ is its slice by (6.25). Since $|Du^\zeta_y|(\hat{E}^{k,\zeta}_m(y) \cap U^\zeta_y \setminus J^\zeta_y) \le |Du^\zeta_y|(U^\zeta_y \setminus J^\zeta_y)$ for $\mathcal{H}^1$-a.e. $y \in \Pi^\zeta$ and the function $y \mapsto |Du^\zeta_y|(U^\zeta_y \setminus J^\zeta_y)$ is integrable on Π^ζ with respect to $\mathcal{H}^1$ by (2.17), the Dominated Convergence Theorem implies (6.31), concluding the proof. □

For technical reasons, instead of a single $\omega \in U$ we have to consider a sequence $(\omega_k)_k \subset U$; accordingly, the points $y^{k,\zeta}_j$, introduced in (5.5), and the sets $\hat{E}^{k,\zeta}_m(y)$ and $\check{E}^{k,\zeta}_m(y)$ are defined by taking $\omega = \omega_k$.

Lemma 6.8. *For every $\varepsilon > 0$ there exist $m_\varepsilon \in \mathbb{N}$ and an infinite set $K_\varepsilon \subset \mathbb{N}$ with the following property: for every $k \in K_\varepsilon$ there exists a Borel set $U^k_\varepsilon \subset U$, with $\mathcal{L}^2(U \setminus U^k_\varepsilon) \le \varepsilon$, such that for every $\zeta \in \{\xi, \eta, \xi+\eta, \xi-\eta\}$ and for every sequence $(\omega_k)_{k\in K_\varepsilon}$, with $\omega_k \in U^k_\varepsilon$ for every $k \in K_\varepsilon$, the following conditions are simultaneously satisfied:*

$$\frac{1}{k}\sum_{j\in\mathbb{Z}} |Du^\zeta_{y^{k,\zeta}_j}|(\check{E}^{k,\zeta}_{m_\varepsilon}(y^{k,\zeta}_j) \cap U^\zeta_{y^{k,\zeta}_j} \setminus J^\zeta_{y^{k,\zeta}_j}) \le \varepsilon \quad \textit{for every } k \in K_\varepsilon, \tag{6.45}$$

$$\lim_{\substack{k\to+\infty \\ k\in K_\varepsilon}} \frac{1}{k}\sum_{j\in\mathbb{Z}} |Du^\zeta_{y^{k,\zeta}_j}|(\hat{E}^{k,\zeta}_{m_\varepsilon}(y^{k,\zeta}_j) \cap U^\zeta_{y^{k,\zeta}_j} \setminus J^\zeta_{y^{k,\zeta}_j}) = 0. \tag{6.46}$$

Proof. Let us fix $\zeta \in \{\xi, \eta, \xi+\eta, \xi-\eta\}$ and $\varepsilon > 0$. Arguing as in Lemma 5.3 we see that it is enough to prove that there exist $m_\varepsilon \in \mathbb{N}$ and an infinite set $K_\varepsilon \subset \mathbb{N}$ with the following property: for every $k \in K_\varepsilon$ there exists a Borel set $U^k_\varepsilon \subset U$, with $\mathcal{L}^2(U \setminus U^k_\varepsilon) \le \varepsilon$, such that for every sequence $(\omega_k)_{k\in K_\varepsilon}$, with $\omega_k \in U^k_\varepsilon$ for every $k \in K_\varepsilon$, conditions (6.45) and (6.46) hold for this particular ζ.

We observe that every $\omega \in \mathbb{R}^2$ can be written in a unique way as $\omega = z_1\bar{\zeta} + z_2\zeta$, where $z_1, z_2 \in \mathbb{R}$ and $\bar{\zeta}$ is defined by (6.8). By (6.9) the numbers $t^{k,\zeta}_i(y)$ depend on ω only through z_2, hence the same holds for $x^{k,\zeta}_j(y)$, $\hat{E}^{k,\zeta}_m(y)$, and $\check{E}^{k,\zeta}_m(y)$ (see (6.10) and (6.24)). We also remark that there exists a constant $c_1 > 0$ such that if $\omega \in U$ then $|z_1| \le c_1$. Moreover, there exists a constant $c_2 > 0$ such that, if $B \subset \mathbb{R}^2$ is a Borel set and $A = \{\omega \in U : \omega = z_1\bar{\zeta} + z_2\zeta \text{ with } (z_1, z_2) \in B\}$, then

$$\mathcal{L}^2(A) \le c_2 \mathcal{L}^2(B). \tag{6.47}$$

Let $C > 0$ be the constant of Lemma 6.4. Thanks to Lemma 6.4, we can find an infinite set $H^\zeta_\varepsilon \subset \mathbb{N}$ and a Borel set $I^\zeta_\varepsilon \subset I^\zeta$, with

$$\mathcal{L}^1(I^\zeta \setminus I^\zeta_\varepsilon) \le \varepsilon/(4c_1c_2), \tag{6.48}$$

such that (6.29) holds for every $m \in \mathbb{N}$, $k \in H^\zeta_\varepsilon$, and $\omega \in U^k_\varepsilon := U \cap \{z_1\bar{\zeta} + z_2\zeta : z_1, \in \mathbb{R},\ z_2 \in I^\zeta_\varepsilon\}$. Moreover, let $N_\zeta \subset \Pi^\zeta$ be the $\mathcal{H}^1$-negligible Borel set introduced at the beginning of the proof of Lemma 5.2.

For every $m \in \mathbb{N}$, $k \in H^\zeta_\varepsilon$, $y \in \Pi^\zeta$, and $z_2 \in \mathbb{R}$ we define

$$h^{k,\zeta}_{m,\varepsilon}(y, z_2) := \begin{cases} |Du^\zeta_y|(\check{E}^{k,\zeta}_m(y) \cap U^\zeta_y \setminus J^\zeta_y) & \text{if } y \in \Pi^\zeta \setminus N_\zeta \text{ and } z_2 \in I^\zeta_\varepsilon, \\ 0 & \text{if } y \in N_\zeta \text{ or } z_2 \notin I^\zeta_\varepsilon, \end{cases} \tag{6.49}$$

where $\check{E}^{k,\zeta}_m(y)$ is defined using ω of the form $z_1\bar{\zeta} + z_2\zeta$ with an arbitrary $z_1 \in \mathbb{R}$. We observe that the function $h^{k,\zeta}_{m,\varepsilon}$ is Borel measurable on $\Pi^\zeta \times \mathbb{R}$ thanks to Lemma A.4. We also define

$$g^\zeta(y) := \begin{cases} |Du^\zeta_y|((U \setminus J)^\zeta_y) & \text{if } y \in \Pi^\zeta \setminus N_\zeta, \\ 0 & \text{if } y \in N_\zeta. \end{cases} \tag{6.50}$$

Note that for every $y \in \Pi^\zeta$ we have $\check{E}^{k,\zeta}_m(y) \cap U^\zeta_y \setminus J^\zeta_y \subset (U \setminus J)^\zeta_y$, hence $h^{k,\zeta}_{m,\varepsilon}(y, z_2) \le g^\zeta(y)$ for every $y \in \Pi^\zeta$ and $z_2 \in \mathbb{R}$.

We set

$$M^\zeta := \mathcal{H}^1(\pi^\zeta(U)) + 1.$$

By (2.17) we have $g^\zeta \in L^1(\Pi^\zeta, \mathcal{H}^1)$, so that by the absolute continuity of the integral there exists $\delta_\varepsilon > 0$ such that

$$\int_B g^\zeta(y)\, \mathrm{d}\mathcal{H}^1(y) < \frac{\varepsilon^2 |\pi^\zeta(\bar\zeta)|^2}{4 \max\{\alpha, \beta\} M^\zeta c_2} \tag{6.51}$$

for every $\mathcal{H}^1$-measurable set $B \subset \Pi^\zeta$ with $\mathcal{H}^1(B) < \delta_\varepsilon$.

Since by (6.25) and (6.49) the inequality $h^{k,\zeta}_{m,\varepsilon}(y, z_2) > 0$ implies $y \in \pi^\zeta(\check{E}^{k,\zeta}_m)$ and $z_2 \in I^\zeta_\varepsilon$, we have

$$\begin{aligned} \int_{\mathbb{R}} \Big(\int_{\mathbb{R}} h^{k,\zeta}_{m,\varepsilon}(z_1 \pi^\zeta(\bar\zeta), z_2)\, \mathrm{d}z_1 \Big) \mathrm{d}z_2 &= |\pi^\zeta(\bar\zeta)|^{-1} \int_{I^\zeta_\varepsilon} \Big(\int_{\pi^\zeta(\check{E}^{k,\zeta}_m)} h^{k,\zeta}_{m,\varepsilon}(y, z_2)\, \mathrm{d}\mathcal{H}^1(y) \Big) \mathrm{d}z_2 \\ &\le |\pi^\zeta(\bar\zeta)|^{-1} \int_{I^\zeta_\varepsilon} \Big(\int_{\pi^\zeta(\check{E}^{k,\zeta}_m)} g^\zeta(y)\, \mathrm{d}\mathcal{H}^1(y) \Big) \mathrm{d}z_2. \end{aligned} \tag{6.52}$$

Let us fix $m_\varepsilon \in \mathbb{N}$ such that $m_\varepsilon > C\delta_\varepsilon^{-1}$, where δ_ε is a constant such that (6.51) holds. By (6.29) we have $\mathcal{H}^1(\pi^\zeta(\check{E}^{k,\zeta}_{m_\varepsilon})) < \delta_\varepsilon$, for every $k \in H^\zeta_\varepsilon$ and $\omega = z_1\bar\zeta + z_2\zeta$ with $z_2 \in I^\zeta_\varepsilon$. Since $\mathcal{L}^1(I^\zeta_\varepsilon) \le \mathcal{L}^1(I^\zeta) \le \max\{\alpha, \beta\}$ by (6.27), from (6.51) and (6.52) we infer that

$$\int_{\mathbb{R}} \Big(\int_{\mathbb{R}} h^{k,\zeta}_{m_\varepsilon,\varepsilon}(z_1 \pi^\zeta(\bar\zeta), z_2) \mathrm{d}z_1 \Big) \mathrm{d}z_2 < \frac{\varepsilon^2 |\pi^\zeta(\bar\zeta)|}{4 M^\zeta c_2} \tag{6.53}$$

for every $k \in H^\zeta_\varepsilon$.

Let $\mathcal{P}^{k,\zeta} := \{j \in \mathbb{Z} : y^{k,\zeta}_j \text{ belongs to } \pi^\zeta(U)\}$. Since $\omega = z_1\bar\zeta + z_2\zeta$, by (5.4), (5.5), and (6.8) we have

$$y^{k,\zeta}_j = z_1 \pi^\zeta(\bar\zeta) + \tfrac{j}{k} \pi^\zeta(\bar\zeta) \quad \text{ for } j \in \mathbb{Z}. \tag{6.54}$$

We set $A^\zeta := \{(z_1, z_2) \in \mathbb{R}^2 : \omega = z_1\bar\zeta + z_2\zeta \in U\}$. Setting $z = (z_1, z_2)$, by the Fubini Theorem we obtain that

$$\begin{aligned} &\int_{A^\zeta} \Big(\frac{1}{k} \sum_{j \in \mathbb{Z}} h^{k,\zeta}_{m_\varepsilon,\varepsilon}(y^{k,\zeta}_j, z_2) \Big) \mathrm{d}z = \frac{1}{k} \sum_{j \in \mathcal{P}^{k,\zeta}} \int_{A^\zeta} h^{k,\zeta}_{m_\varepsilon,\varepsilon}((z_1 + \tfrac{j}{k}) \pi^\zeta(\bar\zeta), z_2)\, \mathrm{d}z \\ &\le \frac{1}{k} \sum_{j \in \mathcal{P}^{k,\zeta}} \int_{\mathbb{R}} \Big(\int_{\mathbb{R}} h^{k,\zeta}_{m_\varepsilon,\varepsilon}(z_1 \pi^\zeta(\bar\zeta), z_2)\, \mathrm{d}z_1 \Big) \mathrm{d}z_2 = \frac{\mathcal{H}^0(\mathcal{P}^{k,\zeta})}{k} \int_{\mathbb{R}} \Big(\int_{\mathbb{R}} h^{k,\zeta}_{m_\varepsilon,\varepsilon}(z_1 \pi^\zeta(\bar\zeta), z_2)\, \mathrm{d}z_1 \Big) \mathrm{d}z_2. \end{aligned}$$

Since $\mathcal{H}^0(\mathcal{P}^{k,\zeta}) \le k M^\zeta |\pi^\zeta(\bar\zeta)|^{-1}$, this inequality, together with (6.53), implies that

$$\int_{A^\zeta} \Big(\frac{1}{k} \sum_{j \in \mathbb{Z}} h^{k,\zeta}_{m_\varepsilon,\varepsilon}(y^{k,\zeta}_j, z_2) \Big) \mathrm{d}z \le \frac{\varepsilon^2}{4c_2} \tag{6.55}$$

for every $k \in H^\zeta_\varepsilon$.

We now set

$$V^{k,\zeta}_\varepsilon := \Big\{ z \in A^\zeta : \frac{1}{k} \sum_{j \in \mathbb{Z}} h^{k,\zeta}_{m_\varepsilon,\varepsilon}(y^{k,\zeta}_j, z_2) > \varepsilon \Big\} \cup F^{k,\zeta}, \tag{6.56}$$

where

$$F^{k,\zeta} := \Big\{ z \in A^\zeta : y^{k,\zeta}_j \in N_\zeta \text{ for some } j \in \mathcal{P}^{k,\zeta} \Big\}.$$

Since $\mathcal{H}^1(N_\zeta)=0$ by (6.54) we obtain that $\mathcal{L}^2(F^{k,\zeta})=0$. In light of Cebyšëv's inequality and (6.55), we infer that

$$\mathcal{L}^2(V_\varepsilon^{k,\zeta}) \leq \frac{\varepsilon}{4c_2}.$$

We also introduce the set

$$W_\varepsilon^{k,\zeta} := V_\varepsilon^{k,\zeta} \cup \Big\{ z \in A^\zeta : z_2 \in I^\zeta \setminus I_\varepsilon^\zeta \Big\} \tag{6.57}$$

and observe that, since $\mathcal{L}^2(\{z\in A^\zeta : z_2 \in I^\zeta\setminus I_\varepsilon^\zeta\}) \leq c_1 \mathcal{L}^1(I^\zeta \setminus I_\varepsilon^\zeta) \leq \varepsilon/(4c_2)$ by (6.48), we have

$$\mathcal{L}^2(W_\varepsilon^{k,\zeta}) \leq \frac{\varepsilon}{2c_2}. \tag{6.58}$$

It follows immediately from (6.49), (6.56), and (6.57) that

$$\frac{1}{k}\sum_{j\in\mathbb{Z}} |Du^\zeta_{y_j^{k,\zeta}}|(\check{E}^{k,\zeta}_{m_\varepsilon}(y_j^{k,\zeta}) \cap U^\zeta_{y_j^{k,\zeta}} \setminus J^\zeta_{y_j^{k,\zeta}}) = \frac{1}{k}\sum_{j\in\mathbb{Z}} h^{k,\zeta}_{m_\varepsilon,\varepsilon}(y_j^{k,\zeta}, z_2) \leq \varepsilon \tag{6.59}$$

for every $k\in H_\varepsilon^\zeta$ and for every $z \in A^\zeta \setminus W_\varepsilon^{k,\zeta}$.

For every $k,m\in\mathbb{N}$, $y\in\Pi^\zeta$, and $z_2\in\mathbb{R}$ we set

$$g_m^{k,\zeta}(y,z_2) := \begin{cases} |Du_y^\zeta|(\hat{E}_m^{k,\zeta}(y)\cap U_y^\zeta \setminus J_y^\zeta) & \text{if } y\in \Pi^\zeta\setminus N_\zeta \text{ and } z_2\in I^\zeta, \\ 0 & \text{if } N_\zeta \text{ or } z_2 \notin I^\zeta. \end{cases} \tag{6.60}$$

By Lemma A.4 the function $g_m^{k,\zeta}$ is Borel measurable on $\Pi^\zeta\times\mathbb{R}$. We also observe that by Lemma 6.5 for every $z_2\in\mathbb{R}$ and for $\mathcal{H}^1$-a.e. $y\in\pi^\zeta(U)$ the sequence $g_m^{k,\zeta}(y,z_2)$ converges to 0 as $k\to+\infty$. Arguing as before, we obtain that $g_m^{k,\zeta}(y,z_2)\leq g^\zeta(y)$ for every $y\in\Pi^\zeta$ and $z_2\in\mathbb{R}$, where g^ζ is the function defined by (6.50).

We set

$$f_\varepsilon^{k,\zeta}(z_1,z_2) := g^{k,\zeta}_{m_\varepsilon}(z_1\pi^\zeta(\bar\zeta), z_2) \tag{6.61}$$

for every $(z_1,z_2)\in\mathbb{R}^2$ and observe that $f_\varepsilon^{k,\zeta}=0$ out of a suitable bounded set. Using Lemma 3.2 and Remark 3.3 with $\mathbb{N}$ replaced by H_ε^ζ, we obtain an infinite set $K_\varepsilon^\zeta \subset H_\varepsilon^\zeta$ and a Borel set $B_\varepsilon^\zeta\subset A^\zeta$, with $\mathcal{L}^2(B_\varepsilon^\zeta)\leq \varepsilon/(2c_2)$, such that for every $m\in\mathbb{N}$

$$\lim_{\substack{k\to+\infty\\ k\in K_\varepsilon}} \frac{1}{k}\sum_{j\in\mathbb{Z}} f_\varepsilon^{k,\zeta}(z_1+\tfrac{j}{k}, z_2) = 0 \text{ uniformly for } (z_1,z_2)\in A^\zeta\setminus B_\varepsilon^\zeta. \tag{6.62}$$

We set $C_\varepsilon^\zeta := \{\omega\in U : \omega = z_1\bar\zeta + z_2\zeta \text{ with } (z_1,z_2)\in B_\varepsilon^\zeta\}$ and observe that from (6.47) it follows that

$$\mathcal{L}^2(C_\varepsilon^\zeta) \leq \frac{\varepsilon}{2}. \tag{6.63}$$

Moreover, from (6.60), (6.61), and (6.62) we deduce that

$$\lim_{\substack{k\to+\infty\\ k\in K_\varepsilon}} \frac{1}{k}\sum_{j\in\mathbb{Z}} |Du^\zeta_{y_j^{k,\zeta}}|(\hat{E}^{k,\zeta}_{m_\varepsilon}(y_j^{k,\zeta}) \cap U^\zeta_{y_j^{k,\zeta}} \setminus J^\zeta_{y_j^{k,\zeta}}) = 0 \text{ uniformly for } \omega\in U\setminus C_\varepsilon^\zeta. \tag{6.64}$$

We set $D_\varepsilon^{k,\zeta} = \{\omega\in U : \omega = z_1\bar\zeta + z_2\zeta \text{ with } (z_1,z_2)\in W_\varepsilon^{k,\zeta}\}$ and observe that by (6.47), (6.58), and (6.59) we have

$$\mathcal{L}^2(D_\varepsilon^{k,\zeta}) \leq \frac{\varepsilon}{2} \quad \text{for every } k\in K_\varepsilon^\zeta, \tag{6.65}$$

$$\frac{1}{k}\sum_{j\in\mathbb{Z}} |Du^\zeta_{y_j^{k,\zeta}}|(\check{E}^{k,\zeta}_{m_\varepsilon}(y_j^{k,\zeta}) \cap U^\zeta_{y_j^{k,\zeta}} \setminus J^\zeta_{y_j^{k,\zeta}}) \leq \varepsilon \quad \text{for every } k\in K_\varepsilon^\zeta \text{ and } \omega\in U\setminus D_\varepsilon^{k,\zeta}. \tag{6.66}$$

Let $U_\varepsilon^{k,\zeta} := U\setminus(C_\varepsilon^\zeta \cup D_\varepsilon^{k,\zeta})$. Combining (6.63) and (6.65) we obtain that $\mathcal{L}^2(U\setminus U_\varepsilon^{k,\zeta})\leq\varepsilon$. Finally, from (6.64) and (6.66) it follows that (6.45) and (6.46) hold for every sequence $(\omega_k)_{k\in K_\varepsilon^\zeta}$ such that $\omega_k\in U_\varepsilon^{k,\zeta}$ for every $k\in K_\varepsilon^\zeta$. This concludes the proof. ☐

We are finally ready to prove Theorem 6.1.

Proof of Theorem 6.1. Recalling (6.1) and (6.17) for every $k \in \mathbb{N}$ we have

$$\mathcal{B}^k = \big\{(i,j) \in \mathcal{J}_2^k : (x_{i,j}^k + \tfrac{1}{k}S) \cap J \neq \emptyset\big\}. \tag{6.67}$$

Since by (6.15) we have

$$z^{k,\zeta}(x_{i,j}^k) = x_{i,j}^k \tag{6.68}$$

(see also the comments after (6.16)), from (6.18) and (6.67) we deduce that for every $\zeta \in \{\xi, \eta, \xi+\eta, \xi-\eta\}$ we have

$$\mathcal{B}^k = \{(i,j) \in \mathcal{J}_2^k : x_{i,j}^k \in E^{k,\zeta}\}.$$

For every $k, m \in \mathbb{N}$ and $\zeta \in \{\xi, \eta, \xi+\eta, \xi-\eta\}$ we set

$$\begin{aligned} \hat{\mathcal{B}}_m^{k,\zeta} &:= \{(i,j) \in \mathcal{J}_2^k : x_{i,j}^k \in \hat{E}_m^{k,\zeta}\}, \\ \check{\mathcal{B}}_m^{k,\zeta} &:= \{(i,j) \in \mathcal{J}_2^k : x_{i,j}^k \in \check{E}_m^{k,\zeta}\}, \end{aligned} \tag{6.69}$$

and observe that $\mathcal{B}^k = \hat{\mathcal{B}}_m^{k,\zeta} \cup \check{\mathcal{B}}_m^{k,\zeta}$.

Let us fix $0 < \varepsilon < \mathcal{L}^2(U)/2$. For every $n \in \mathbb{N}$ we apply Lemma 5.3 with ε replaced by $\varepsilon/2^n$ and obtain an infinite set H_n and a Borel set U_n, with $\mathcal{L}^2(U \setminus U_n) \leq \varepsilon/2^n$, such that all conditions of Lemma 5.3 hold with U_ε and K_ε replaced by U_n and H_n. In the step $n+1$ we can replace $\mathbb{N}$ in Lemma 5.3 by H_n, obtaining $H_{n+1} \subset H_n$ for every $n \in \mathbb{N}$. Then, we apply Lemma 6.8, with ε replaced by $\varepsilon/2^n$ and $\mathbb{N}$ replaced by H_n, and we obtain $m_n \in \mathbb{N}$, with $m_{n+1} \geq m_n$, an infinite set $K_n \subset H_n$, and for every $k \in K_n$ a Borel set $U_n^k \subset U$, with $\mathcal{L}^2(U \setminus U_n^k) \leq \varepsilon/2^n$, with the following property: for every sequence $(\omega_k)_{k \in K_n}$ such that $\omega_k \in U_n^k$ for every $k \in K_n$ and for every $\zeta \in \{\xi, \eta, \xi+\eta, \xi-\eta\}$ we have

$$\frac{1}{k}\sum_{j \in \mathbb{Z}} |Du^\zeta_{y_j^{k,\zeta}}|(\check{E}_{m_n}^{k,\zeta}(y_j^{k,\zeta}) \cap U^\zeta_{y_j^{k,\zeta}} \setminus J^\zeta_{y_j^{k,\zeta}}) \leq \frac{\varepsilon}{2^n} \quad \text{for every } k \in K_n, \tag{6.70}$$

$$\lim_{\substack{k \to +\infty \\ k \in K_n}} \frac{1}{k}\sum_{j \in \mathbb{Z}} |Du^\zeta_{y_j^{k,\zeta}}|(\hat{E}_{m_n}^{k,\zeta}(y_j^{k,\zeta}) \cap U^\zeta_{y_j^{k,\zeta}} \setminus J^\zeta_{y_j^{k,\zeta}}) = 0. \tag{6.71}$$

Replacing U_n^k by $U_n^k \cap U_n$ we obtain that $U_n^k \subset U_n$ and $\mathcal{L}^2(U_n \setminus U_n^k) \leq \varepsilon/2^n$. Moreover, repeating the same argument used when we pass from n to $n+1$, it is not restrictive to assume that $K_{n+1} \subset K_n$ for every $n \in \mathbb{N}$.

By a diagonal argument, we can find an infinite set $K \subset \mathbb{N}$ and a strictly increasing sequence $(k_n)_n \subset \mathbb{N}$ such that $K \cap [k_n, +\infty) \subset K_n$ for every $n \in \mathbb{N}$. For every $k \in K$ we also define $n_k \in \mathbb{N}$ as the largest integer such that $k_n \leq k$ and observe that $n_k \to +\infty$ as $k \to +\infty$. For $k \in K$ consider the set $U^k := \bigcap_{n=1}^{n_k} U_n^k$ and observe that $\mathcal{L}^2(U^k) \geq \mathcal{L}^2(U) - 2\varepsilon$ for every $k \in K$, hence $U^k \neq \emptyset$. We fix $(\omega_k)_{k \in K} \subset U$ such that $\omega_k \in U^k$ for every $k \in K$. By (6.71) we have

$$\lim_{\substack{k \to +\infty \\ k \in K}} \frac{1}{k}\sum_{j \in \mathbb{Z}} |Du^\zeta_{y_j^{k,\zeta}}|(\hat{E}_{m_n}^{k,\zeta}(y_j^{k,\zeta}) \cap U^\zeta_{y_j^{k,\zeta}} \setminus J^\zeta_{y_j^{k,\zeta}}) = 0 \quad \text{for every } n \in \mathbb{N}. \tag{6.72}$$

We now prove (6.5) for $\zeta = \xi$. Let us fix $(i,j) \in \mathcal{B}^k$ and $t \in I_{i,j}^{k,\xi}$. By (6.67) we have

$$(x_{i,j}^k + \tfrac{1}{k}S) \cap J \neq \emptyset. \tag{6.73}$$

Recalling (5.16), we also have $t_{i,j}^{k,\xi} \leq t < t_{i,j}^{k,\xi} + \frac{1}{k}$, where $t_{i,j}^{k,\xi}$ is defined in (5.6). Since $x_{i,j}^k \in \{\omega_k + \frac{i}{k}\xi + s\eta : s \in \mathbb{R}\}$, by (5.6) and (6.9) we have $t_i^{k,\xi}(y_j^{k,\xi}) = t_{i,j}^{k,\xi}$, so that $t_i^{k,\xi}(y_j^{k,\xi}) \leq t < t_i^{k,\xi}(y_j^{k,\xi}) + \frac{1}{k}$. From (6.16) and (6.68) we deduce that

$$z^{k,\xi}(y_j^{k,\xi} + t\xi) = z^{k,\xi}(y_j^{k,\xi} + t_i^{k,\xi}(y_j^{k,\xi})\xi) = z^{k,\xi}(y_j^{k,\xi} + t_{i,j}^{k,\xi}\xi) = z^{k,\xi}(x_{i,j}^k) = x_{i,j}^k.$$

In light of (6.73), this implies that

$$(z^{k,\xi}(y_j^{k,\xi} + t\xi) + \tfrac{1}{k}S) \cap J \neq \emptyset,$$

which, by (6.18), implies that $y_j^{k,\xi} + t\xi \in E^{k,\xi}$. Recalling (6.20), this is equivalent to $t \in E^{k,\xi}(y_j^{k,\xi})$. Since $(i,j) \in \mathcal{J}_2^k$, this shows that

$$I_{i,j}^{k,\xi} \subset E^{k,\xi}(y_j^{k,\xi}) \cap U^{\xi}_{y_j^{k,\xi}}. \tag{6.74}$$

If $(i,j) \in \hat{\mathcal{B}}_m^{k,\xi}$ for some $m \in \mathbb{N}$, it follows from (6.24) and (6.69) that $\mathcal{N}^{k,\xi}(x_{i,j}^k) \le m$, where $\mathcal{N}^{k,\xi}$ is defined by (6.19). By (6.21) we have $\mathcal{N}^{k,\xi}(x_{i,j}^k) = \mathcal{N}^{k,\xi}(y_j^k + t\xi)$ for every $t \in \mathbb{R}$, so that from (6.74) we also deduce that

$$I_{i,j}^{k,\xi} \subset \hat{E}_m^{k,\xi}(y_j^{k,\xi}) \cap U^{\xi}_{y_j^{k,\xi}}. \tag{6.75}$$

In a similar way we prove that if $(i,j) \in \check{\mathcal{B}}_m^{k,\xi}$ for some $m \in \mathbb{N}$ then

$$I_{i,j}^{k,\xi} \subset \check{E}_m^{k,\xi}(y_j^{k,\xi}) \cap U^{\xi}_{y_j^{k,\xi}}. \tag{6.76}$$

For every $k, m \in \mathbb{N}$ and $j \in \mathbb{Z}$ we set $\mathcal{B}^k(j) := \{i \in \mathbb{Z} : (i,j) \in \mathcal{B}^k\}$, $\hat{\mathcal{B}}_m^{k,\xi}(j) := \{i \in \mathbb{Z} : (i,j) \in \hat{\mathcal{B}}_m^{k,\xi}\}$, and $\check{\mathcal{B}}_m^{k,\xi}(j) := \{i \in \mathbb{Z} : (i,j) \in \check{\mathcal{B}}_m^{k,\xi}\}$. From (6.75) and (6.76) we deduce that for every $k, m \in \mathbb{N}$ and $j \in \mathbb{Z}$ we have

$$\bigcup_{i \in \hat{\mathcal{B}}_m^{k,\xi}(j)} I_{i,j}^{k,\xi} \subset \hat{E}_m^{k,\xi}(y_j^{k,\xi}) \cap U^{\xi}_{y_j^{k,\xi}} \quad \text{and} \quad \bigcup_{i \in \check{\mathcal{B}}_m^{k,\xi}(j)} I_{i,j}^{k,\zeta} \subset \check{E}_m^{k,\xi}(y_j^{k,\xi}) \cap U^{\xi}_{y_j^{k,\xi}}.$$

Therefore, for every $k, m \in \mathbb{N}$ we have

$$\sum_{i \in \hat{\mathcal{B}}_m^{k,\xi}(j)} |Du^{\xi}_{y_j^{k,\xi}}|(I_{i,j}^{k,\xi} \setminus J^{\xi}_{y_j^{k,\xi}}) \le |Du^{\xi}_{y_j^{k,\xi}}|(\hat{E}_m^{k,\xi}(y_j^{k,\xi}) \cap U^{\xi}_{y_j^{k,\xi}} \setminus J^{\xi}_{y_j^{k,\xi}}), \tag{6.77}$$

$$\sum_{i \in \check{\mathcal{B}}_m^{k,\xi}(j)} |Du^{\xi}_{y_j^{k,\xi}}|(I_{i,j}^{k,\xi} \setminus J^{\xi}_{y_j^{k,\xi}}) \le |Du^{\xi}_{y_j^{k,\xi}}|(\check{E}_m^{k,\xi}(y_j^{k,\xi}) \cap U^{\xi}_{y_j^{k,\xi}} \setminus J^{\xi}_{y_j^{k,\xi}}). \tag{6.78}$$

From (6.70) and (6.77) for every $n \in \mathbb{N}$ we obtain that

$$\frac{1}{k} \sum_{j \in \mathbb{Z}} \sum_{i \in \check{\mathcal{B}}_{mn}^{k,\xi}(j)} |Du^{\xi}_{y_{i,j}^{k,\xi}}|(I_{i,j}^{k,\xi} \setminus J^{\xi}_{y_{i,j}^{k,\xi}}) \le \frac{\varepsilon}{2^n} \quad \text{for every } k \in K \text{ with } k \ge k_n, \tag{6.79}$$

while (6.72) and (6.78) give

$$\lim_{\substack{k \to +\infty \\ k \in K}} \frac{1}{k} \sum_{j \in \mathbb{Z}} \sum_{i \in \hat{\mathcal{B}}_{mn}^{k,\xi}(j)} |Du^{\xi}_{y_{i,j}^{k,\xi}}|(I_{i,j}^{k,\xi} \setminus J^{\xi}_{y_{i,j}^{k,\xi}}) = 0 \quad \text{for every } n \in \mathbb{N}. \tag{6.80}$$

Combining (6.79) and (6.80), we deduce that

$$\limsup_{\substack{k \to +\infty \\ k \in K}} \frac{1}{k} \sum_{(i,j) \in \mathcal{B}^k} |Du^{\xi}_{y_{i,j}^{k,\xi}}|(I_{i,j}^{k,\xi} \setminus J^{\xi}_{y_{i,j}^{k,\xi}}) \le \frac{\varepsilon}{2^n} \quad \text{for every } n \in \mathbb{N},$$

which concludes the proof of (6.5) for $\zeta = \xi$. The proof of (6.5) for $\zeta \in \{\eta, \xi - \eta, \xi + \eta\}$, as well as the proof of (6.6) and (6.7), is analogous.

Equalities (6.2)-(6.4) can then be obtained by difference from (5.20). This concludes the proof of Theorem 6.1. □

7. The case of dimension $d > 2$

We now show that Theorem 5.1 can be extended to the general case $d > 2$. This is done by means of a Fubini-type argument. To this aim, we present and prove a short lemma that shows that the measure λ_u introduced in (2.20) does not charge Borel sets that are σ-finite with respect to $\mathcal{H}^{d-1}$ and do not intersect the jump set.

Lemma 7.1. *Let $d \ge 1$, let $u \in GBD(\Omega)$, and let $B \subset \Omega$ be a Borel set that is σ-finite with respect to $\mathcal{H}^{d-1}$. Then $\lambda_u(B \setminus J_u) = 0$.*

Proof. It is not restrictive to assume that $\mathcal{H}^{d-1}(B) < +\infty$. By (2.20) to prove the claim it is enough to show that for every $\xi \in \mathbb{S}^{d-1}$ the measure λ_u^ξ defined by (2.19) satisfies $\lambda_u^\xi(B \setminus J_u) = 0$. Let us fix $\xi \in \mathbb{S}^{d-1}$. Since $J_{u_y^\xi} \subset (J_u)_y^\xi$ for $\mathcal{H}^{d-1}$-a.e. $y \in \Pi^\xi$ by Theorem 2.12, it follows from (2.19) that

$$\lambda_u^\xi(B \setminus J_u) \leq \int_{\Pi^\xi} |Du_y^\xi|(B_y^\xi \setminus J_{u_y^\xi}) \, \mathrm{d}\mathcal{H}^{d-1}(y). \tag{7.1}$$

Recalling that by assumption $\mathcal{H}^{d-1}(B) < +\infty$, Lemma 2.4 implies that for $\mathcal{H}^{d-1}$-a.e. $y \in \Pi^\xi$ the slice B_y^ξ is a finite set. By well-known properties of BV functions of one variable (see [7, Corollary 3.33]), this implies that for $\mathcal{H}^{d-1}$-a.e. $y \in \Pi^\xi$ we have $|Du_y^\xi|(B_y^\xi \setminus J_{u_y^\xi}) = 0$. By (7.1) this equality gives $\lambda_u^\xi(B \setminus J_u) = 0$, concluding the proof. □

Theorem 7.2. *Let $d > 2$, let $u \in GBD(\Omega)$, and let $B \subset \Omega$ be a Borel set. Then the function $\xi \mapsto \sigma_u^\xi(B)$ is quadratic.*

Proof. By Proposition 2.2 it is enough to show that $\xi \mapsto \sigma_u^\xi(B)$ is 2-homogeneous, satisfies the parallelogram identity, and is lower bounded in the sense of (c) of Proposition 2.2. Since by Proposition 4.4 the function $\xi \mapsto \sigma_u^\xi(B)$ is 2-homogeneous and by Remark 4.2 it satisfies the correct lower bound, we only need to prove the parallelogram identity.

We decompose $\sigma_u^\xi(B)$ as

$$\sigma_u^\xi(B) = \sigma_u^\xi(B \setminus J_u) + \sigma_u^\xi(B \cap J_u)$$

and observe that by Propositions 4.5 the function $\xi \mapsto \sigma_u^\xi(B \cap J_u)$ is quadratic. Thus, to conclude we only need to prove that $\xi \mapsto \sigma_u^\xi(B \setminus J_u)$ satisfies the parallelogram identity.

Let $\xi, \eta \in \mathbb{R}^d$ be two linearly independent vectors and consider the 2-dimensional vector space V generated by ξ and η. Let $\pi_V \colon \mathbb{R}^d \to V$ be the orthogonal projection onto V. For $z \in \mathbb{R}^d$ and $E \subset \mathbb{R}^d$ let $E_z^V := \{y \in V : z + y \in E\} = V \cap (E - z)$ and let $u_z^V \colon \Omega_z^V \to V$ be the function defined for every $y \in \Omega_z^V$ by $u_z^V(y) := \pi_V(u(z + y))$. By (a) of Theorem 2.14 for $\mathcal{H}^{d-2}$-a.e. $z \in V^\perp$ we have $u_z^V \in GBD(\Omega_z^V)$.

We observe that for every $\zeta \in \{\xi, \eta, \xi + \eta, \xi - \eta\} \subset V$ and for every $E \subset \mathbb{R}^d$ we have

$$(E_z^V)_y^\zeta = E_{z+y}^\zeta \quad \text{for every } z \in V^\perp \text{ and } y \in V \cap \Pi^\zeta. \tag{7.2}$$

Moreover, for every $z \in V^\perp$, $y \in V \cap \Pi^\zeta$, and $t \in (\Omega_z^V)_y^\zeta = \Omega_{z+y}^\zeta$ we have

$$(u_z^V)_y^\zeta(t) = u_{z+y}^\zeta(t). \tag{7.3}$$

This implies that for $\mathcal{H}^{d-2}$-a.e. $z \in V^\perp$ and for $\mathcal{H}^1$-a.e. $y \in V \cap \Pi^\zeta$ we have

$$D(u_z^V)_y^\zeta = Du_{z+y}^\zeta \tag{7.4}$$

as Borel measures on $(\Omega_z^V)_y^\zeta = \Omega_{y+z}^\zeta$.

We then apply Theorem 2.14 to deduce that for $\mathcal{H}^{d-2}$-a.e. $z \in V^\perp$ there exists a Borel set $N_z \subset V$ such that

$$\mathcal{H}^1(N_z) = 0 \qquad \text{and} \qquad J_{u_z^V} \subset (J_u)_z^V \cup N_z. \tag{7.5}$$

By Remark 2.10 the set J_u is σ-finite with respect to $\mathcal{H}^{d-1}$. Therefore, by [28, Theorem 2.10.25] we have that $\mathcal{H}^1((J_u)_z^V)$ is σ-finite with respect to $\mathcal{H}^1$ for $\mathcal{H}^{d-2}$-a.e. $z \in V^\perp$. We can then apply Lemma 7.1, with $d = 2$, Ω replaced by Ω_z^V, u replaced by u_z^V, and $B = (J_u)_z^V$, and we obtain that $\lambda_{u_z^V}((J_u)_z^V \setminus J_{u_z^V}) = 0$ for $\mathcal{H}^{d-2}$-a.e. $z \in V^\perp$. By Definition 2.5, applied to $u_z^V \in GBD(\Omega_z^V)$, and from this equality it follows that

$$\int_{V \cap \Pi^\zeta} |D(u_z^V)_y^\zeta|(((J_u)_z^V \setminus J_{u_z^V})_y^\zeta) \, \mathrm{d}\mathcal{H}^1(y) \leq \lambda_{u_z^V}((J_u)_z^V \setminus J_{u_z^V}) = 0 \tag{7.6}$$

for $\mathcal{H}^{d-2}$-a.e. $z \in V^\perp$.

Since for every triple of sets A_1, A_2, A_3 we have $(A_1 \setminus A_2) \setminus (A_1 \setminus A_3) \subset A_3 \setminus A_2$, we obtain that

$$(B_z^V \setminus J_{u_z^V}) \setminus (B_z^V \setminus (J_u)_z^V) \subset (J_u)_z^V \setminus J_{u_z^V},$$

hence by (7.6)

$$|D(u_z^V)_y^\zeta|((B_z^V \setminus J_{u_z^V})_y^\zeta \setminus (B_z^V \setminus (J_u)_z^V)_y^\zeta) = 0 \tag{7.7}$$

for $\mathcal{H}^{d-2}$-a.e. $z \in V^\perp$ and for $\mathcal{H}^1$-a.e. $y \in V \cap \Pi^\zeta$.

The inclusion in (7.5) implies that for $\mathcal{H}^{d-2}$-a.e. $z \in V^\perp$ and for every $y \in V \cap \Pi^\zeta \setminus \pi^\zeta(N_z)$ we have $(J_{u_z^V})_y^\zeta \subset ((J_u)_z^V)_y^\zeta$, and hence $(B_z^V \setminus (J_u)_z^V)_y^\zeta \subset (B_z^V \setminus J_{u_z^V})_y^\zeta$. Observing that $\mathcal{H}^1(\pi^\zeta(N_z)) = 0$ by the equality in (7.5), we deduce from this inclusion and from (7.7) that

$$D(u_z^V)_y^\zeta((B_z^V \setminus J_{u_z^V})_y^\zeta) = D(u_z^V)_y^\zeta((B_z^V \setminus (J_u)_z^V)_y^\zeta)$$

for $\mathcal{H}^{d-2}$-a.e. $z \in V^\perp$ and for $\mathcal{H}^1$-a.e. $y \in V \cap \Pi^\zeta$.

Integrating this equality with respect to y we obtain that

$$\int_{V\cap\Pi^\zeta} D(u_z^V)_y^\zeta((B \setminus J_u)_z^V)_y^\zeta)\,\mathrm{d}\mathcal{H}^1(y) = \int_{V\cap\Pi^\zeta} D(u_z^V)_y^\zeta((B_z^V \setminus J_{u_z^V})_y^\zeta)\,\mathrm{d}\mathcal{H}^1(y)$$

for $\mathcal{H}^{d-2}$- a.e. $z \in V^\perp$, so that, setting $y' = z + y$, by (7.2)-(7.4) and the Fubini Theorem we have

$$\int_{\Pi^\zeta} Du_{y'}^\zeta((B \setminus J_u)_{y'}^\zeta)\,\mathrm{d}\mathcal{H}^{d-1}(y')$$
$$= \int_{V^\perp} \Big(\int_{V\cap\Pi^\zeta} D(u_z^V)_y^\zeta((B_z^V \setminus J_{u_z^V})_y^\zeta)\,\mathrm{d}\mathcal{H}^1(y)\Big)\mathrm{d}\mathcal{H}^{d-2}(z).$$

Taking into account the definition of σ_u^ζ (see (4.1)), this last equality can be written as

$$\sigma_u^\zeta(B \setminus J_u) = \int_{V^\perp} \sigma_{u_z^V}^\zeta(B_z^V \setminus J_{u_z^V})\,\mathrm{d}\mathcal{H}^{d-2}(z). \tag{7.8}$$

We may now apply Theorem 5.1 to the function $u_z^V \in GBD(\Omega_z^V)$ to obtain

$$\sigma_{u_z^V}^{\xi+\eta}(B_z^V \setminus J_{u_z^V}) + \sigma_{u_z^V}^{\xi-\eta}(B_z^V \setminus J_{u_z^V}) = 2\sigma_{u_z^V}^{\xi}(B_z^V \setminus J_{u_z^V}) + 2\sigma_{u_z^V}^{\eta}(B_z^V \setminus J_{u_z^V})$$

for $\mathcal{H}^{d-2}$-a.e. $z \in V^\perp$. Integrating this equality with respect to z and exploiting (7.8) we deduce that

$$\sigma_u^{\xi+\eta}(B \setminus J_u) + \sigma_u^{\xi-\eta}(B \setminus J_u) = 2\sigma_u^\xi(B \setminus J_u) + 2\sigma_u^\eta(B \setminus J_u).$$

This shows that the function $\xi \mapsto \sigma_u^\xi(B \setminus J_u)$ satisfies the parallelogram identity, concluding the proof. □

8. A matrix-valued measure associated to a *GBD* function

In this section, for every $u \in GBD(\Omega)$ we introduce a matrix-valued measure μ_u that generalises the distributional symmetric gradient Eu of $BD(\Omega)$ functions. We then analyse some of its properties and deduce some useful consequences.

Theorem 8.1. *Let $d \geq 1$ and $u \in GBD(\Omega)$. Then there exists a measure $\mu_u \in \mathcal{M}_b(\Omega; \mathbb{R}^{d\times d}_{\mathrm{sym}})$ such that for every $\xi \in \mathbb{S}^{d-1}$ we have*

$$\mu_u(B)\xi \cdot \xi = \sigma_u^\xi(B) = \lim_{R\to+\infty} D_\xi(\tau_R(u\cdot\xi))(B \setminus J_u^1) \quad \text{for every Borel set } B \subset \Omega, \tag{8.1}$$

where σ_u^ξ is the measure defined in (4.1) *and τ_R are the truncation functions defined in* (4.3)*. Moreover, the variation $|\mu_u|$ with respect to the operator norm in $\mathbb{R}^{d\times d}_{\mathrm{sym}}$ satisfies*

$$|\mu_u|(J_u^1) = 0, \tag{8.2}$$

$$|\mu_u| = \lambda_u \llcorner (\Omega \setminus J_u^1) \quad \text{as Borel measures on } \Omega, \tag{8.3}$$

where λ_u is the positive measure defined by (2.20).

Proof. If $d = 1$ these results follow from Remarks 4.1, 4.2, and Proposition 4.4. We may thus assume that $d \geq 2$. From Theorems 5.1 and 7.2 it follows that the function $\xi \mapsto \sigma_u^\xi(B)$ is quadratic for every Borel set $B \subset \Omega$. Thus, there exists a set function μ_u defined on the σ-algebra of all Borel subsets of Ω and with values in $\mathbb{R}^{d\times d}_{\rm sym}$ such that

$$\sigma_u^\xi(B) = \mu_u(B)\xi \cdot \xi \tag{8.4}$$

for every Borel set $B \subset \Omega$ and $\xi \in \mathbb{R}^d \setminus \{0\}$. Observing that $\sigma_u^\xi(B \cap J_u^1) = 0$ by (4.1), we may apply Proposition 4.3 to $B \setminus J_u^1$ and we obtain

$$\sigma_u^\xi(B) = \lim_{R\to+\infty} D_\xi(\tau_R(u\cdot\xi))(B \setminus J_u^1)$$

for every Borel set $B \subset \Omega$ and $\xi \in \mathbb{S}^{d-1}$. As (8.2) is an obvious consequence of (8.3), we are left with proving that $\mu_u \in \mathcal{M}_b(\Omega; \mathbb{R}^{d\times d}_{\rm sym})$ and that equality (8.3) holds.

Since $B \mapsto \sigma_u^\xi(B)$ is a bounded scalar-valued Radon measure for every $\xi \in \mathbb{R}^d \setminus \{0\}$, it follows from (8.4) that the same property holds for $B \mapsto \mu_u(B)\xi\cdot\xi$. The polarisation identity then implies that $B \mapsto \mu_u(B)\xi \cdot \eta$ belongs to $\mathcal{M}_b(\Omega)$ for every $\xi, \eta \in \mathbb{R}^d$, hence $\mu_u \in \mathcal{M}_b(\Omega; \mathbb{R}^{d\times d}_{\rm sym})$.

To prove (8.3) let us first show that

$$|\mu_u| \leq \lambda_u \llcorner (\Omega \setminus J_u^1) \quad \text{ as Borel measures on } \Omega. \tag{8.5}$$

To this aim, we observe that, since μ_u takes values in $\mathbb{R}^{d\times d}_{\rm sym}$, for every Borel set $B \subset \Omega$ the operator norm $|\mu_u(B)|$ satsfies

$$|\mu_u(B)| = \sup_{\xi\in\mathbb{S}^{d-1}} |(\mu_u(B)\xi\cdot\xi)| = \sup_{\xi\in\mathbb{S}^{d-1}} |\sigma_u^\xi(B)|,$$

so that by (2.20) and (4.2) we have

$$|\mu_u|(B) = \sup \sum |\sigma_u^{\xi_i}(B_i)| \leq \sup \sum \lambda_u^{\xi_i}(B_i \setminus J_u^1) = \lambda_u(B \setminus J_u^1),$$

where the *supremum* is taken over all finite Borel partitions $(B_i)_i$ of B and all finite collections of vectors $(\xi_i)_i \subset \mathbb{S}^{d-1}$. This shows (8.5).

To prove the inequality

$$|\mu_u| \geq \lambda_u \llcorner (\Omega \setminus J_u^1) \quad \text{ as Borel measures on } \Omega, \tag{8.6}$$

we argue as follows. Consider the measure $\lambda \in \mathcal{M}_b^+(\Omega)$ defined for Borel set $B \subset \Omega$ by

$$\lambda(B) := |\mu_u|(B) + \mathcal{H}^{d-1}(B \cap J_u^1).$$

Thanks to (8.5), Lemma 2.4, Remark 2.10, and Theorem 2.12 it follows that λ satisfies (2.10). Since λ_u is the minimal measure that satisfies (2.10) it follows that $\lambda_u(B) \leq \lambda(B)$ for every Borel set $B \subset \Omega$, which implies (8.6). □

Remark 8.2. By Remark 4.1 it follows immediately from (8.4) that if $u \in BD(\Omega)$, then $\mu_u = (Eu) \llcorner (\Omega \setminus J_u^1)$ as Borel measures on Ω.

Given $u \in GBD(\Omega)$, the Lebesgue Decomposition Theorem allows us to decompose the measure μ_u as the sum of a measure μ_u^a, which is absolutely continuous with respect to $\mathcal{L}^d$, and a measure μ_u^s, which is singular with respect to $\mathcal{L}^d$. In the following definition, we introduce a further decomposition of μ_u, which closely resembles the classical decomposition (1.2) of Eu for a function $u \in BD(\Omega)$.

Definition 8.3. For $u \in GBD(\Omega)$ we introduce the measures $\mu_u^c, \mu_u^j \in \mathcal{M}_b(\Omega; \mathbb{R}^{d\times d}_{\rm sym})$, called the *Cantor* part and the *jump* part of μ_u, defined for every Borel set $B \subset \Omega$ by

$$\mu_u^c(B) := \mu_u^s(B \setminus J_u),$$
$$\mu_u^j(B) := \mu_u^s(B \cap J_u) = \mu_u(B \cap J_u).$$

Since $\mu_u = \mu_u^a + \mu_u^s$, we have $\mu_u = \mu_u^a + \mu_u^c + \mu_u^j$ as Borel measures on Ω.

Remark 8.4. It follows from Remark 8.2 that if $u \in BD(\Omega)$ then the measure μ_u^c of Definition 8.3 coincides with the Cantor part $E^c u$ (see [6, Definition 4.1]) of the symmetrised gradient Eu.

We recall that for every $\xi \in \mathbb{R}^d \setminus \{0\}$ and every $y \in \Pi^\xi$ such that $u_y^\xi \in BV_{\rm loc}(\Omega_y^\xi)$ and $Du_y^\xi \in \mathcal{M}_b(\Omega_y^\xi)$ we can consider the measures $D^a u_y^\xi \in \mathcal{M}_b(\Omega_y^\xi)$ and $D^s u_y^\xi \in \mathcal{M}_b(\Omega_y^\xi)$, defined as the absolutely continuous and the singular part of Du_y^ξ with respect to the one-dimensional Lebesgue measure, and the measures $D^c u_y^\xi \in \mathcal{M}_b(\Omega_y^\xi)$ and $D^j u_y^\xi \in \mathcal{M}_b(\Omega_y^\xi)$, defined for every Borel set $B \subset \Omega_y^\xi$ by

$$D^c u_y^\xi(B) := D^s u_y^\xi(B \setminus J_{u_y^\xi}),$$

$$D^j u_y^\xi(B) := D^s u_y^\xi(B \cap J_{u_y^\xi}) = Du_y^\xi(B \cap J_{u_y^\xi}).$$

Since $Du_y^\xi = D^a u_y^\xi + D^s u_y^\xi$, we have $Du_y^\xi = D^a u_y^\xi + D^c u_y^\xi + D^j u_y^\xi$ as Borel measures on Ω_y^ξ.

We now show that the measures μ_u^a and μ_u^j can be expressed as suitable integrals depending on the approximate symmetric gradient $\mathcal{E}u$ (see Theorem 2.12) and $[u]$, respectively, and that μ_u^c an be expressed by means of $D^c u_y^\xi$.

Proposition 8.5. *Let $u \in GBD(\Omega)$. Then*

$$\mu_u^a(B) = \int_B \mathcal{E}u \,\mathrm{d}x, \tag{8.7}$$

$$\mu_u^j(B) = \int_{(J_u \setminus J_u^1)\cap B} [u] \odot \nu_u \,\mathrm{d}\mathcal{H}^{d-1}, \tag{8.8}$$

for every Borel set $B \subset \Omega$. Moreover, for every $\xi \in \mathbb{R}^d \setminus \{0\}$ and every Borel set $B \subset \Omega$ we have

$$\mu_u^c(B)\xi \cdot \xi = |\xi| \int_{\Pi^\xi} D^c u_y^\xi(B_y^\xi) \,\mathrm{d}\mathcal{H}^{d-1}(y). \tag{8.9}$$

Proof. Let us fix a Borel set $B \subset \Omega$. By definition of μ_u^a and of μ_u^j, it follows from the polarisation identity and from (8.4) that

$$\mu_u^a(B)\xi \cdot \eta = \frac{1}{4}((\sigma_u^{\xi+\eta})^a(B) - (\sigma_u^{\xi-\eta})^a(B)), \tag{8.10}$$

$$\mu_u^j(B)\xi \cdot \eta = \frac{1}{4}(\sigma_u^{\xi+\eta}(B \cap J_u) - \sigma_u^{\xi-\eta}(B \cap J_u)), \tag{8.11}$$

for every $\xi, \eta \in \mathbb{R}^d \setminus \{0\}$, where for a vector $\zeta \in \mathbb{R}^d \setminus \{0\}$ the measure $(\sigma_u^\zeta)^a$ is the absolutely continuous part of σ_u^ζ with respect to $\mathcal{L}^d$. By Lemma A.2 for every $\zeta \in \mathbb{R}^d \setminus \{0\}$ we have

$$(\sigma_u^\zeta)^a(B) = |\zeta| \int_{\Pi^\zeta} D^a u_y^\zeta(B_y^\zeta) \,\mathrm{d}\mathcal{H}^{d-1}(y), \tag{8.12}$$

$$(\sigma_u^\zeta)^s(B) = |\zeta| \int_{\Pi^\zeta} D^s u_y^\zeta((B \setminus J_u^1)_y^\zeta) \,\mathrm{d}\mathcal{H}^{d-1}(y), \tag{8.13}$$

where $(\sigma_u^\zeta)^s$ is the singular part of σ_u^ζ with respect to $\mathcal{L}^d$. In light of Theorem 2.12, using (8.12) and the Fubini Theorem, we deduce that

$$(\sigma_u^\zeta)^a(B) = \int_\Omega \mathcal{E}u\, \zeta \cdot \zeta \,\mathrm{d}x$$

for every $\zeta \in \mathbb{R}^d \setminus \{0\}$. Combining this equality with (8.10), we obtain that

$$\mu_u^a(B)\xi \cdot \eta = \int_B \mathcal{E}u\, \xi \cdot \eta \,\mathrm{d}x = \Big(\int_B \mathcal{E}u \,\mathrm{d}x\Big)\xi \cdot \eta$$

for every $\xi, \eta \in \mathbb{R}^d \setminus \{0\}$, which proves (8.7).

To prove (8.8), we observe that by Proposition 4.5 for every $\zeta \in \mathbb{R}^d \setminus \{0\}$ we have

$$\sigma_u^\zeta(B \cap J_u)z = \int_{(J_u \setminus J_u^1)\cap B} ([u] \odot \nu_u)\zeta \cdot \zeta \, \mathrm{d}\mathcal{H}^{d-1}.$$

Together with (8.11), this equality shows that

$$\mu_u^j(B)\xi \cdot \eta = \int_{(J_u \setminus J_u^1)\cap B} ([u] \odot \nu_u)\xi \cdot \eta \, \mathrm{d}\mathcal{H}^{d-1} = \Big(\int_{(J_u \setminus J_u^1)\cap B} ([u] \odot \nu_u) \, \mathrm{d}\mathcal{H}^{d-1}\Big)\xi \cdot \eta$$

for every $\xi, \eta \in \mathbb{R}^d \setminus \{0\}$. This proves (8.8).

To conclude, we fix $\xi \in \mathbb{R}^d \setminus \{0\}$ and a Borel set $B \subset \Omega$ and show that (8.9) holds. We observe that for every $y \in \Pi^\xi$ such that (2.21) holds we have $|D^j u_y^\xi|((B \setminus J_u)_y^\xi) = 0$, hence

$$D^s u_y^\xi((B \setminus J_u)_y^\xi) = D^c u_y^\xi((B \setminus J_u)_y^\xi).$$

Since (2.21) holds for $\mathcal{H}^{d-1}$-a.e. $y \in \Pi^\xi$, from this equality, (8.4), (8.13), and Definition 8.3 it follows that

$$\mu_u^c(B)\xi \cdot \xi = (\sigma_u^\xi)^s(B \setminus J_u) = |\xi| \int_{\Pi^\xi} D^c u_y^\xi((B \setminus J_u)_y^\xi) \, \mathrm{d}\mathcal{H}^{d-1}(y). \tag{8.14}$$

Recalling Lemma 2.4, by Remark 2.10 for $\mathcal{H}^{d-1}$-a.e. $y \in \Pi^\xi$ the set $(J_u)_y^\xi$ is finite or countable. Recalling the properties of the derivatives of BV functions in dimension one, this implies that $D^c u_y^\xi((J_u)_y^\xi) = 0$ for $\mathcal{H}^{d-1}$-a.e. $y \in \Pi^\xi$. Therefore (8.14) implies (8.9). □

The following corollary shows that in the previous results we can replace J_u^1 by J_u^r (see (2.4)) for an arbitrary $r > 0$ and that the absolutely continuous and the Cantor part of the corresponding measure $\mu_{u,r}$ do not depend on r.

Corollary 8.6. *Let $d \geq 1$, $u \in GBD(\Omega)$, and $r > 0$. Then there exists a measure $\mu_{u,r} \in \mathcal{M}_b(\Omega; \mathbb{R}^{d\times d}_{\mathrm{sym}})$ such that for every $\xi \in \mathbb{S}^{d-1}$ we have*

$$\mu_{u,r}(B)\xi \cdot \xi = \lim_{R\to+\infty} D_\xi(\tau_R(u \cdot \xi))(B \setminus J_u^r) \quad \text{for every Borel set } B \subset \Omega, \tag{8.15}$$

where τ_R are the truncation functions defined in (4.3)*. Moreover, setting $\mu_{u,r}^j := \mu_{u,r} \llcorner J_u$, we have*

$$\mu_{u,r}^j(B) = \int_{(J_u \setminus J_u^r)\cap B} [u] \odot \nu_u \, \mathrm{d}\mathcal{H}^{d-1} \quad \text{for every Borel set } B \subset \Omega. \tag{8.16}$$

Finally, we have $\mu_{u,r} = \mu_u^a + \mu_u^c + \mu_{u,r}^j$ as Borel measures on Ω.

Proof. Let $v := u/r$ and $\mu_{u,r} := r\mu_v$. Using the equalities

$$\tau_R(v \cdot \xi) = \frac{1}{r}\tau_{(rR)}(u \cdot \xi) \quad \text{and} \quad J_v^1 = J_u^r, \tag{8.17}$$

from (8.1) we obtain (8.15). Using the equalities $J_v^1 = J_u^r$ and $J_v = J_u$, from (8.8) we deduce (8.16).

Let $\mu_{u,r}^a$ and $\mu_{u,r}^s$ be the absolutely continuous and the singular part of $\mu_{u,r}$ with respect to $\mathcal{L}^d$. By (8.1) and (8.15), with B replaced by $B \setminus J_u$, we obtain $\mu_{u,r}(B \setminus J_u) = \mu_u(B \setminus J_u)$. This implies that $\mu_{u,r}^a(B) = \mu_u^a(B)$ and $\mu_{u,r}^s(B \setminus J_u) = \mu_u^s(B \setminus J_u) = \mu_u^c(B)$ for every Borel set $B \subset \Omega$. Hence, $\mu_{u,r}(B) = \mu_{u,r}^a(B) + \mu_{u,r}^s(B \setminus J_u) + \mu_{u,r}^s(B \cap J_u) = \mu_u^a(B) + \mu_u^c(B) + \mu_{u,r}^j(B)$ for every Borel set $B \subset \Omega$. □

Remark 8.7. Using the function $v := u/r$, it follows from Remark 8.2 that, if $u \in BD(\Omega)$, then $\mu_{u,r} = (Eu) \llcorner (\Omega \setminus J_u^r)$ as Borel measures on Ω.

The following result shows that, in analogy with $E^c u$, the measure μ_u^c does not charge Borel sets which are σ-finite with respect to $\mathcal{H}^{d-1}$.

Proposition 8.8. *Let $u \in GBD(\Omega)$ and let B be a Borel set that is σ-finite with respect to $\mathcal{H}^{d-1}$. Then $|\mu_u^c|(B) = 0$.*

Proof. It is not restrictive to assume that $\mathcal{H}^{d-1}(B) < +\infty$. Let us fix $\xi \in \mathbb{S}^{d-1}$. Thanks to Lemma 2.4, we have $\mathcal{H}^0(B^\xi_y) < +\infty$ for $\mathcal{H}^{d-1}$-a.e. $y \in \Pi^\xi$. By the properites of one-dimensional BV functions this implies that $D^c u^\xi_y(B^\xi_y) = 0$ for $\mathcal{H}^{d-1}$-a.e. $y \in \Pi^\xi$. Hence, by (8.9) we have $\mu^c_u(B)\xi\cdot\xi = 0$. Since this is true for every $\xi \in \mathbb{S}^{d-1}$, we obtain $\mu^c_u(B) = 0$. As this property holds also for every Borel subset of B, we deduce that $|\mu^c_u|(B) = 0$. □

The definition and properties of μ_u allow us to give a new characterisation of the space $GSBD(\Omega)$, originally defined by slicing. We recall that $GSBD(\Omega)$ (see [20, Definition 4.2]) is the space of all $u \in GBD(\Omega)$ such that for every $\xi \in \mathbb{S}^{d-1}$ and for $\mathcal{H}^{d-1}$-a.e. $y \in \Pi^\xi$ we have

$$u^\xi_y \in SBV_{\rm loc}(\Omega^\xi_y). \tag{8.18}$$

Theorem 8.9. *Let $u \in GBD(\Omega)$. Then $u \in GSBD(\Omega)$ if and only if $\mu^c_u = 0$.*

Proof. Assume that $\mu^c_u = 0$ as a Borel measure on Ω. Recalling the uniqueness of the disintegration of measures (see [7, Theorem 2.28]), it follows from (8.9) that for every $\xi \in \mathbb{S}^{d-1}$ and for $\mathcal{H}^{d-1}$-a.e. $y \in \Pi^\zeta$ we have $D^c u^\xi_y = 0$ as a Borel measure on Ω^ξ_y, i.e., (8.18) holds. By definition this implies that $u \in GSBD(\Omega)$.

Conversely, if $u \in GSBD(\Omega)$ it follows from (8.9) that $\mu^c_u(B)\xi\cdot\xi = 0$ for every $\xi \in \mathbb{S}^{d-1}$ and every Borel set $B \subset \Omega$. This implies that $\mu^c_u = 0$. □

Combining this result with recent results of [13], where new characterisations of the spaces $GBD(\Omega)$ and $GSBD^p(\Omega)$, for $p > 1$, are obtained, we can give an analogous characterisation for $GSBD(\Omega)$. More precisely, we show that an $\mathcal{L}^d$-measurable function $u\colon \Omega \to \mathbb{R}^d$ belongs to $GSBD(\Omega)$ if and only if (2.10) and (8.18) hold for a suitable finite number of directions $\xi \in \mathbb{S}^{d-1}$.

Theorem 8.10. *Let $u\colon \Omega \to \mathbb{R}^d$ be an $\mathcal{L}^d$-measurable function. Assume that there exists an orthonormal basis $\{\xi_i : i = 1, ..., d\}$ such that for every $\xi \in \Xi := \{\xi_i : i = 1, ..., d\} \cup \{\xi_i + \xi_j : 1 \le i \le j \le d\}$ the two following conditions hold:*

$$u^\xi_y \in SBV_{\rm loc}(\Omega^\xi_y) \quad \textit{for } \mathcal{H}^{d-1}\textit{-a.e. } y \in \Pi^\xi, \tag{8.19}$$

$$\Lambda^\xi_u := \int_{\Pi^\xi} |Du^\xi_y|(\Omega^\xi_y \setminus J^1_{u^\xi_y}) + \mathcal{H}^0(J^1_{u^\xi_y})\,\mathrm{d}\mathcal{H}^{d-1}(y) < +\infty.$$

Then $u \in GSBD(\Omega)$ and, setting $\Lambda := \sum_{\xi\in\Xi} \Lambda^\xi_u$, there exists a constant $C_d > 0$, depending only on d, such that

$$\lambda_u(\Omega) \le C_d\Lambda, \tag{8.20}$$

where λ_u is the measure defined by (2.20).

Proof. Since the inclusion $u \in GBD(\Omega)$ and inequality (8.20) follow directly from [13, Theorem 1, Corollary 1], to conclude we only need to show that $u \in GSBD(\Omega)$.

To prove this, we observe that from (8.9) it follows that given a Borel set $B \subset \Omega$ we have that

$$\mu^c_u(B)\xi\cdot\xi = |\xi| \int_{\Pi^\xi} D^c u^\xi_y(B^\xi_y)\,\mathrm{d}\mathcal{H}^{d-1}(y) = 0$$

for every $\xi \in \Xi$. By (8.19) this equality implies that $\mu^c_u(B)\xi\cdot\xi = 0$ for every $\xi \in \Xi$. From the polarisation identity we obtain $\mu^c_u(B)\xi_i\cdot\xi_j = 0$ for every $i, j = 1, ..., d$. Recalling that $\{\xi_i\}_i$ is a basis of $\mathbb{R}^d$, we deduce that $\mu^c_u(B) = 0$. Since this property holds for every Borel set $B \subset \Omega$, from Theorem 8.9 we obtain that $u \in GSBD(\Omega)$, concluding the proof. □

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A. Auxiliary results

The purpose of this section is to show that the functions $g_m^{k,\zeta}$ and $h_{m,\varepsilon}^{k,\zeta}$ defined in (6.60) and (6.49) are Borel measurable, and prove some general properties of measures defined by integration. Results similar to those we present here are already well-established in the existing literature. However, given the specific form of the functions we study, it is not easy to apply them directly to our case. For this reason, we give here a precise statement and a complete proof of the results we need.

A.1. **Lebesgue decomposition of measures defined by integration.** In this subsection we consider measures defined on the slices of a set, depending on a parameter $\omega \in \mathbb{R}^k$, and the measures that can be obtained by integrating with respect to the parameters corresponding to the slices. We are interested in a formula for the Lebesgue decomposition of these measures.

We begin by a lemma concerning measurability conditions with respect to these parameters. Given $h, k \in \mathbb{N}$, a Borel set $B \subset \mathbb{R}^h \times \mathbb{R}^k$, and $\omega \in \mathbb{R}^k$ we set

$$B(\omega) := \{x \in \mathbb{R}^h : (x, \omega) \in B\}. \tag{A.1}$$

Lemma A.1. *Let $k \in \mathbb{N}$ and let $\zeta \in \mathbb{R}^d \setminus \{0\}$. For every $y \in \Pi^\zeta$ and $\omega \in \mathbb{R}^k$ let $\mu_y^\omega \in \mathcal{M}_b(\mathbb{R})$ be a signed measure. The following three measurability conditions are equivalent:*

(a) for every $\psi \in C_c^0(\mathbb{R} \times \mathbb{R}^k)$

$$\text{the function } (y,\omega) \mapsto \int_{-\infty}^{+\infty} \psi(t,\omega)\,\mathrm{d}\mu_y^\omega(t) \text{ is Borel measurable on } \Pi^\zeta \times \mathbb{R}^k;$$

(b) for every $\varphi \in C_c^0(\mathbb{R}^d \times \mathbb{R}^k)$

$$\text{the function } (y,\omega) \mapsto \int_{-\infty}^{+\infty} \varphi(y + t\zeta,\omega)\,\mathrm{d}\mu_y^\omega(t) \text{ is Borel measurable on } \Pi^\zeta \times \mathbb{R}^k;$$

(c) for every Borel set $B \subset \mathbb{R}^d \times \mathbb{R}^k$ the function $(y,\omega) \mapsto \mu_y^\omega(B(\omega)_y^\zeta)$ is Borel measurable on $\Pi^\zeta \times \mathbb{R}^k$.

Moreover, if the previous conditions are satisfied, then for every Borel set $B \subset \mathbb{R}^d \times \mathbb{R}^k$ the function $(y,\omega) \mapsto |\mu_y^\omega|(B(\omega)_y^\zeta)$ is Borel measurable on $\Pi^\zeta \times \mathbb{R}^k$.

Proof. For simplicity of notation we consider only the case $\zeta = e_d$, the d-th vector of the canonical basis, the proof in the other cases being analogous. For $x \in \mathbb{R}^d$ we set $x = (x', x_d)$, where $x' := (x_1, ..., x_{d-1})$, and we identify $\Pi^{e_d} = \mathbb{R}^{d-1} \times \{0\}$ with $\mathbb{R}^{d-1}$. Therefore the measure $\mu_{(x',0)}^\omega$ is denoted simply by $\mu_{x'}^\omega$ and for the slicing of sets we use the notation $B_{x'}$ instead of $B_{(x',0)}^{e_d}$.

Assume (a). For every pair of functions $w \in C_c^0(\mathbb{R}^{d-1} \times \mathbb{R}^k)$ and $\psi \in C_c^0(\mathbb{R} \times \mathbb{R}^k)$ it follows from (a) that the function

$$(x',\omega) \mapsto \int_{-\infty}^{+\infty} w(x',\omega)\psi(x_d,\omega)\,\mathrm{d}\mu_{x'}^\omega(x_d)$$

is Borel measurable on $\mathbb{R}^{d-1} \times \mathbb{R}^k$.

By an argument based on partition of unities corresponding to coverings by open sets with small diameter we can prove that the class of linear combinations of functions $\varphi \in C_c^0(\mathbb{R}^d \times \mathbb{R}^k)$ of the form $\varphi(x,\omega) = w(x',\omega)\psi(x_d,\omega)$ is dense in $C_c^0(\mathbb{R}^d \times \mathbb{R}^k)$ with respect to the uniform convergence. This implies that the function

$$(x',\omega) \mapsto \int_{-\infty}^{+\infty} \varphi((x',x_d),\omega)\,\mathrm{d}\mu_{x'}^\omega(x_d)$$

is Borel measurable on $\mathbb{R}^{d-1} \times \mathbb{R}^k$ for every $\varphi \in C_c^0(\mathbb{R}^d \times \mathbb{R}^k)$, which is condition (b) in the case $\zeta = e_d$.

Assume (b) and consider the class $\mathcal{F}$ of bounded Borel functions $f\colon \mathbb{R}^d\times\mathbb{R}^k\to\mathbb{R}$ such that

$$(x',\omega)\mapsto\int_{-\infty}^{+\infty} f((x',x_d),\omega)\,\mathrm{d}\mu^\omega_{x'}(x_d) \text{ is Borel measurable on } \mathbb{R}^{d-1}\times\mathbb{R}^k.$$

It is easy to check that $\mathcal{F}$ is a monotone class, that is,

(i) if $(f_n)_n\subset\mathcal{F}$, with $f_n\le g$, for some $g\in\mathcal{F}$, and $f_n\nearrow f$, then $f\in\mathcal{F}$;
(ii) if $(f_n)_n\subset\mathcal{F}$, with $f_n\ge g$, for some $g\in\mathcal{F}$, and $f_n\searrow f$, then $f\in\mathcal{F}$.

Moreover, thanks to (b), we have that $C^0_c(\mathbb{R}^d\times\mathbb{R}^k)\subset\mathcal{F}$. Hence, from the Monotone Class Theorem (see [42, Section 3.14]) we deduce that for every bounded Borel function $f\colon\mathbb{R}^d\times\mathbb{R}^k\to\mathbb{R}$ the function

$$(x',\omega)\mapsto\int_{-\infty}^{+\infty} f((x',x_d),\omega)\,\mathrm{d}\mu^\omega_{x'}(x_d)$$

is Borel measurable on $\mathbb{R}^{d-1}\times\mathbb{R}^k$. By taking as f the characteristic function of a Borel set $B\subset\mathbb{R}^d\times\mathbb{R}^k$ we obtain that the function

$$(x',\omega)\mapsto\mu^\omega_{x'}(B(\omega)_{x'})$$

is Borel measurable on $\mathbb{R}^{d-1}\times\mathbb{R}^k$, which is condition (c) in the case $\zeta=e_d$.

Assume now (c). Let $E\subset\mathbb{R}\times\mathbb{R}^k$ be a Borel set and let $B:=\mathbb{R}^{d-1}\times E\subset\mathbb{R}^d\times\mathbb{R}^k$. Clearly, $B(\omega)_{x'}=E(\omega)$ for every $x'\in\mathbb{R}^{d-1}$, so that by (c) the function $(x',\omega)\mapsto\mu^\omega_{x'}(E(\omega))$ is $\mathbb{R}^{d-1}\times\mathbb{R}^k$ measurable. Hence,

$$(x',\omega)\mapsto\int_{-\infty}^{+\infty}\chi_E(x_d,\omega)\,\mathrm{d}\mu^\omega_{x'}(x_d)$$

is Borel measurable on $\mathbb{R}^{d-1}\times\mathbb{R}^k$. By linearity we obtain that

$$(x',\omega)\mapsto\int_{-\infty}^{+\infty} g(x_d,\omega)\,\mathrm{d}\mu^\omega_{x'}(x_d)$$

is Borel measurable on $\mathbb{R}^{d-1}\times\mathbb{R}^k$ for every simple function $g\colon\mathbb{R}\times\mathbb{R}^k\to\mathbb{R}$. Since every function $\psi\in C^0_c(\mathbb{R}\times\mathbb{R}^k)$ can be approximated by a uniformly bounded sequence of simple functions, an application of the Dominated Convergence Theorem yields (a).

We now show that, if (a)-(c) hold, then the last part of the statement holds. By the equivalence of (a)-(c) for $|\mu^\omega_{x'}|$, to conclude the proof it is enough to show that for every function $\psi\in C^0_c(\mathbb{R}\times\mathbb{R}^k)$ the function

$$(x',\omega)\mapsto\int_{-\infty}^{+\infty}\psi(x_d,\omega)\,\mathrm{d}|\mu^\omega_{x'}|(x_d) \tag{A.2}$$

is Borel measurable on $\mathbb{R}^{d-1}\times\mathbb{R}^k$. Assume for a moment that $\psi\ge 0$. By definition of total variation of a measure we have

$$\int_{-\infty}^{+\infty}\psi(x_d,\omega)\,\mathrm{d}|\mu^\omega_{x'}|(x_d)=\sup_{\substack{\varphi\in C^0_c(\mathbb{R}\times\mathbb{R}^k)\\ |\varphi|\le\psi}}\int_{-\infty}^{+\infty}\varphi(x_d,\omega)\,\mathrm{d}\mu^\omega_{x'}(x_d).$$

Since the supremum above can be reduced to a countable dense subset of $C^0_c(\mathbb{R}\times\mathbb{R}^k)$, this equality, together with (a) for $\mu^\omega_{x'}$, implies that the function in (A.2) is Borel measurable on $\mathbb{R}^{d-1}\times\mathbb{R}^k$ when $\psi\ge 0$. In the general case, one can split ψ into its positive and negative part. □

Let $\zeta\in\mathbb{R}^d\setminus\{0\}$ and for every $y\in\Pi^\zeta$ let $\mu_y\in\mathcal{M}_b(\Omega^\zeta_y)$. For every $y\in\Pi^\zeta$ let μ^a_y and μ^s_y be the absolutely continuous part and the singular part of μ_y with respect to the one dimensional Lebesgue measure. In the following lemma we consider some general conditions on $(\mu_y)_{y\in\Pi^\zeta}$ which allow us to define a measure μ on Ω by integrating μ_y with respect to y. We then show that the absolutely continuous part μ^a and its singular part μ^s of μ with respect to $\mathcal{L}^d$ can be obtained by integrating μ^a_y and μ^s_y with respect to y.

Lemma A.2. *Let $\zeta \in \mathbb{R}^d \setminus \{0\}$ and for every $y \in \Pi^\zeta$ let $\mu_y \in \mathcal{M}_b(\Omega^\zeta_y)$. Assume that for every Borel set $B \subset \Omega$*

$$\text{the function } y \mapsto \mu_y(B^\zeta_y) \text{ is Borel measurable on } \Pi^\zeta \tag{A.3}$$

and that there exists $g \in L^1(\Pi^\zeta, \mathcal{H}^{d-1})$ such that

$$|\mu_y|(\Omega^\zeta_y) \leq g(y) \text{ for } \mathcal{H}^{d-1}\text{-a.e. } y \in \Pi^\zeta. \tag{A.4}$$

Consider the measure defined for every Borel set $B \subset \Omega$ by

$$\mu(B) := \int_{\Pi^\zeta} \mu_y(B^\zeta_y) \, \mathrm{d}\mathcal{H}^{d-1}(y). \tag{A.5}$$

Le μ^a and μ^s be the absolutely continuous part and the singular part of μ with respect to the Lebesgue measure $\mathcal{L}^d$. Then for every Borel set $B \subset \Omega$ the functions

$$y \mapsto \mu^a_y(B^\zeta_y) \text{ and } y \mapsto \mu^s_y(B^\zeta_y) \text{ are Borel measurable and } \mathcal{H}^{d-1}\text{-integrable on } \Pi^\zeta \tag{A.6}$$

and we have

$$\mu^a(B) = \int_{\Pi^\zeta} \mu^a_y(B^\zeta_y) \, \mathrm{d}\mathcal{H}^{d-1}(y) \quad \text{and} \quad \mu^s(B) = \int_{\Pi^\zeta} \mu^s_y(B^\zeta_y) \, \mathrm{d}\mathcal{H}^{d-1}(y) \tag{A.7}$$

for every Borel set $B \subset \Omega$.

Proof. As in the previous lemma we consider only the case $\zeta = e_d$ and use the notation of the proof of the previous lemma. We also drop the hypothesis that Ω is bounded and assume that $\Omega = \mathbb{R}^d$, as the result for a general Ω then easily follows.

We now prove (A.6). To this aim let

$$S := \Big\{ x \in \mathbb{R}^d : x = (x', x_d) \text{ and } \limsup_{\rho \to 0^+} \frac{|\mu_{x'}|((x_d - \rho, x_d + \rho))}{2\rho} = +\infty \Big\} \tag{A.8}$$

and, for every $x' \in \mathbb{R}^{d-1}$, let $S_{x'}$ be the corresponding slice. By the Besicovitch Derivation Theorem (see [7, Theorem 2.22]) for every $x' \in \mathbb{R}^{d-1}$ we have

$$\mu^s_{x'}(B_{x'}) = \mu_{x'}(B_{x'} \cap S_{x'}) \quad \text{and} \quad \mu^a_{x'}(B_{x'}) = \mu_{x'}(B_{x'} \setminus S_{x'}) \tag{A.9}$$

for every Borel set $B \subset \mathbb{R}^d$.

Therefore, by (A.3) to prove (A.6) it is enough to show that the set S is Borel measurable. To this end, we note that, as the function $\rho \mapsto |\mu_{x'}|((x_d - \rho, x_d + \rho))$ is left-continuous, in the lim sup in (A.8) we can reduce to considering ρ varying in a countable dense set. Hence, to conclude that S is Borel measurable we only need to prove that for every $\rho > 0$ the function

$$(x', x_d) \mapsto |\mu_{x'}|((x_d - \rho, x_d + \rho)) = \int_{\mathbb{R}} \chi_{(-\rho,\rho)}(x_d - t) \, \mathrm{d}|\mu_{x'}|(t) \tag{A.10}$$

is Borel measurable on $\mathbb{R}^{d-1} \times \mathbb{R}$, where $\chi_{(-\rho,\rho)}$ denotes the characteristic function of $(-\rho, \rho)$. Let $(\psi_n)_n \subset C^0_c(\mathbb{R})$ be a sequence of functions with $\psi_n \leq \psi_{n+1}$ for every $n \in \mathbb{N}$ and converging pointwise to $\chi_{(-\rho,\rho)}$ as $n \to +\infty$. By the Monotone Convergence Theorem for every $x' \in \mathbb{R}^{d-1}$ we have

$$\int_{\mathbb{R}} \chi_{(-\rho,\rho)}(x_d - t) \, \mathrm{d}|\mu_{x'}|(t) = \lim_{n \to +\infty} \int_{\mathbb{R}} \psi_n(x_d - t) \, \mathrm{d}|\mu_{x'}|(t). \tag{A.11}$$

For every $n \in \mathbb{N}$ the function

$$(x', x_d) \mapsto \int_{\mathbb{R}} \psi_n(x_d - t) \, \mathrm{d}|\mu_{x'}|(t)$$

is Borel measurable in x' for x_d fixed, thanks to (A.3) and Lemma A.1, and continuous in x_d for x' fixed. Thus, it is Borel measurable in the product space $\mathbb{R}^{d-1} \times \mathbb{R}$. Thanks to (A.11), this implies that the function (A.10) is Borel measurable, which proves that the set S is Borel measurable and concludes the proof of the measurability property in (A.6). The integrability follows from (A.4).

Thanks to (A.6) we can define two bounded Radon measures on $\mathbb{R}^d$ by setting

$$\nu_1(B) := \int_{\mathbb{R}^{d-1}} \mu^a_{x'}(B_{x'})\, \mathrm{d}\mathcal{H}^{d-1}(x') \quad \text{and} \quad \nu_2(B) := \int_{\mathbb{R}^{d-1}} \mu^s_{x'}(B_{x'})\, \mathrm{d}\mathcal{H}^{d-1}(x')$$

for every Borel set $B \subset \mathbb{R}^d$. It follows from the Fubini Theorem that ν_1 is absolutely continuous with respect to $\mathcal{L}^d$ and that $\mathcal{L}^d(S) = 0$. By (A.9) we also deduce that $\mu^s_{x'}(B_{x'}) = \mu^s_{x'}(B_{x'} \cap S_{x'})$, hence $\nu_2(B) = \nu_2(B \cap S)$ for every Borel set $B \subset \Omega$. This shows that ν_2 is singular with respect to the Lebesgue measure. Since $\mu = \nu_1 + \nu_2$, the equalities in (A.7) follow from the uniqueness of the Lebesgue decomposition. □

A.2. **Measurability of the auxiliary functions used in Section 6.** In this subsection we prove the the measurability of the functions $g^{k,\zeta}_m$ and $h^{k,\zeta}_{m,\varepsilon}$ defined in (6.60) and (6.49).

As in the proof of Theorem 5.1, Ω is a bounded open set of $\mathbb{R}^2$, u is a function in $GBD(\Omega)$ with $J^1_u = J_u$, ξ and η are two linearly independent vectors in $\mathbb{R}^2$ and U is the parallelogram defined by (5.2). We keep u, ξ, η, and U fixed throughout the rest of the subsection.

We also recall that $J \subset U$ is the set defined in (5.9), and that the sets $E^{k,\zeta}$, $\hat{E}^{k,\zeta}_m$, and $\check{E}^{k,\zeta}_m$ are defined in (6.18) and (6.23). Since it will be important to keep track of the dependence of such sets on ω, in the following we underline their dependence on ω by writing $E^{k,\zeta,\omega}$, $\hat{E}^{k,\zeta,\omega}_m$, and $\check{E}^{k,\zeta,\omega}_m$.

We introduce some sets which will play a crucial role in our arguments. For every $k \in \mathbb{N}$, $m \in \mathbb{N}$, and $\zeta \in \{\xi, \eta, \xi+\eta, \xi-\eta\}$ we set

$$\begin{aligned} \mathfrak{E}^{k,\zeta} &:= \{(x,\omega) \in \mathbb{R}^2 \times \mathbb{R}^2 : x \in E^{k,\zeta,\omega}\} \subset \mathbb{R}^4, \\ \hat{\mathfrak{E}}^{k,\zeta}_m &:= \{(x,\omega) \in \mathbb{R}^2 \times \mathbb{R}^2 : x \in \hat{E}^{k,\zeta,\omega}_m\} \subset \mathbb{R}^4, \\ \check{\mathfrak{E}}^{k,\zeta}_m &:= \{(x,\omega) \in \mathbb{R}^2 \times \mathbb{R}^2 : x \in \check{E}^{k,\zeta,\omega}_m\} \subset \mathbb{R}^4, \\ \mathfrak{J} &:= \{(x,\omega) \in \mathbb{R}^2 \times \mathbb{R}^2 : x \in J\} \subset \mathbb{R}^4. \end{aligned} \tag{A.12}$$

The following lemma addresses the Borel measurability of these sets. The proof is very similar to that of Lemma 6.2, but for the sake of completeness we give here all details.

Lemma A.3. *The sets* $\mathfrak{E}^{k,\zeta}$, $\hat{\mathfrak{E}}^{k,\zeta}_m$, $\check{\mathfrak{E}}^{k,\zeta}_m$, *and* $\mathfrak{J}$ *are Borel measurable subsets of* $\mathbb{R}^4$.

Proof. The property for $\mathfrak{J}$ is trivial. To prove the result for the other sets we consider the map $z^{k,\zeta} : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$ defined by (6.14) (see Figure 3), where the dependence on the variable ω is made clear by (5.3). By elementary geometrical arguments, it follows that $(x,\omega) \mapsto z^{k,\zeta}(x,\omega)$ is Borel measurable.

For every set $B \subset \mathbb{R}^2$ we define

$$\begin{aligned} \mathfrak{E}^{k,\zeta}_B &:= \{(x,\omega) \in \mathbb{R}^2 \times \mathbb{R}^2 : (z^{k,\zeta}(x,\omega) + \tfrac{1}{k}S) \cap B \neq \emptyset\}, \\ F_B &:= \{z \in \mathbb{R}^2 : (z + \tfrac{1}{k}S) \cap B \neq \emptyset\}, \end{aligned}$$

where S is defined by (6.17). For a compact set $K \subset \mathbb{R}^2$ the set F_K is closed, so that, observing that $\mathfrak{E}^{k,\zeta}_K = \{(x,\omega) \in \mathbb{R}^2 \times \mathbb{R}^2 : z^{k,\zeta}(x,\omega) \in F_K\}$ and recalling that $z^{k,\zeta}$ is Borel measurable, we conclude that $\mathfrak{E}^{k,\zeta}_K$ is Borel measurable. By definition, $\mathfrak{E}^{k,\zeta} = \mathfrak{E}^{k,\zeta}_J$. Since $J = \bigcup_{n\in\mathbb{N}} K_n$ with K_n compact, this gives that $\mathfrak{E}^{k,\zeta} = \bigcup_{n\in N} \mathfrak{E}^{k,\zeta}_{K_n}$. Since the sets $\mathfrak{E}^{k,\zeta}_{K_n}$ are Borel measurable, so is $\mathfrak{E}^{k,\zeta}$.

To prove that $\hat{\mathfrak{E}}^{k,\zeta}_m$ is Borel measurable, we observe that by (6.22) a pair (x,ω) belongs to $\hat{\mathfrak{E}}^{k,\zeta}_m$ if and only if the number of indices $i \in \mathbb{Z}$ such that $x \in E^{k,\zeta,\omega} - \frac{i}{k}\zeta$ is less than or equal to m. Hence, setting $\mathfrak{E}^{k,\zeta}_i := \{(x,\omega) \in \mathbb{R}^2 \times \mathbb{R}^2 : x \in E^{k,\zeta,\omega} - \frac{i}{k}\zeta\}$, we have that

$$\hat{\mathfrak{E}}^{k,\zeta}_m = \big\{(x,\omega) \in \mathbb{R}^2 \times \mathbb{R}^2 : \textstyle\sum_i \chi_{\mathfrak{E}^{k,\zeta}_i}(x,\omega) \leq m\big\},$$

where $\chi_{\mathfrak{E}_i^{k,\zeta}}$ is the characteristic function of $\mathfrak{E}_i^{k,\zeta}$. Since the sets $\mathfrak{E}_i^{k,\zeta}$ are Borel measurable, we deduce that $\hat{\mathfrak{E}}_m^{k,\zeta}$ is a Borel set as well. From the equality $\check{\mathfrak{E}}_m^{k,\zeta} = \mathfrak{E}^{k,\zeta} \setminus \hat{\mathfrak{E}}_m^{k,\zeta}$, we deduce that $\check{\mathfrak{E}}_m^{k,\zeta}$ is Borel measurable too, concluding the proof. □

We are now ready to state and prove the final result of this subsection.

Lemma A.4. *Let $k, m \in \mathbb{N}$, let $\zeta \in \{\xi, \eta, \xi+\eta, \xi-\eta\}$, and let $N_\zeta \subset \Pi^\zeta$ be a Borel set such that $\mathcal{H}^{d-1}(N_\zeta) = 0$ and $u_y^\zeta \in BV(U_y^\zeta)$ for every $y \in \Pi^\zeta \setminus N_\zeta$. Then the functions $g_m^{k,\zeta} \colon \Pi^\zeta \times \mathbb{R}^2 \to [0,+\infty)$ and $h_m^{k,\zeta} \colon \Pi^\zeta \times \mathbb{R}^2 \to [0,+\infty)$ defined by*

$$g_m^{k,\zeta}(y,\omega) := \begin{cases} |Du_y^\zeta|(\hat{E}_m^{k,\zeta,\omega}(y) \cap U_y^\zeta \setminus J_y^\zeta) & \text{if } y \in \Pi^\zeta \setminus N_\zeta, \\ 0 & \text{if } y \in N_\zeta, \end{cases} \tag{A.13}$$

$$h_m^{k,\zeta}(y,\omega) := \begin{cases} |Du_y^\zeta|(\check{E}_m^{k,\zeta,\omega}(y) \cap U_y^\zeta \setminus J_y^\zeta) & \text{if } y \in \Pi^\zeta \setminus N_\zeta, \\ 0 & \text{if } y \in N_\zeta, \end{cases} \tag{A.14}$$

are Borel measurable on $\Pi^\zeta \times \mathbb{R}^2$.

Proof. We begin by observing that by (A.1) and (A.12) for every $y \in \Pi^\zeta$ and $\omega \in \mathbb{R}^2$ we have

$$\hat{E}_m^{k,\zeta,\omega}(y) \cap U_y^\zeta \setminus J_y^\zeta = \hat{\mathfrak{E}}(\omega)_y^\zeta \cap U_y^\zeta \setminus \mathfrak{J}(\omega)_y^\zeta, \tag{A.15}$$

$$\check{E}_m^{k,\zeta,\omega}(y) \cap U_y^\zeta \setminus J_y^\zeta = \check{\mathfrak{E}}(\omega)_y^\zeta \cap U_y^\zeta \setminus \mathfrak{J}(\omega)_y^\zeta. \tag{A.16}$$

By Lemma A.3, the sets $\hat{\mathfrak{E}}_m^{k,\zeta}$, $\check{\mathfrak{E}}_m^{k,\zeta}$, and $\mathfrak{J}$ are Borel measurable subsets of $\mathbb{R}^2 \times \mathbb{R}^2$, so that the same property holds for $\hat{\mathfrak{E}}_m^{k,\zeta} \setminus \mathfrak{J}$ and $\check{\mathfrak{E}}_m^{k,\zeta} \setminus \mathfrak{J}$.

We want to apply Lemma A.1 with the measure $\mu_y^\omega \in \mathcal{M}_b(\mathbb{R})$ defined for every Borel set $B \subset \mathbb{R}$ by

$$\mu_y^\omega(B) := \begin{cases} Du_y^\zeta(B \cap U_y^\zeta) & \text{if } y \in \Pi^\zeta \setminus N_\zeta, \\ 0 & y \in N_\zeta. \end{cases} \tag{A.17}$$

To show that condition (b) of Lemma A.1 holds we fix a function $\varphi \in C_c^\infty(\mathbb{R}^2 \times \mathbb{R}^2)$ and observe that

$$\int_{-\infty}^{+\infty} \varphi(y+t\zeta,\omega)\,\mathrm{d}\mu_y^\omega(t) = \int_{U_y^\zeta} \varphi(y+t\zeta,\omega)\,\mathrm{d}Du_y^\zeta(t)$$

for every $y \in \Pi^\zeta \setminus N_\zeta$. Let us consider a bounded sequence $(\rho_n) \subset C_c^\infty(U)$ converging to 1 on Ω as $n \to +\infty$ and set $\varphi_n := \rho_n \varphi \in C_c^\infty(U \times \mathbb{R}^2)$. For every $y \in \Pi^\zeta \setminus N_\zeta$ by the Dominated Convergence Theorem we have

$$\int_{-\infty}^{+\infty} \varphi(y+t\zeta,\omega)\,\mathrm{d}\mu_y^\omega(t) = \lim_{n\to+\infty} \int_{U_y^\zeta} \varphi_n(y+t\zeta,\omega)\,\mathrm{d}Du_y^\zeta(t). \tag{A.18}$$

Moreover, integrating by parts, we have

$$\int_{U_y^\zeta} \varphi_n(y+t\zeta,\omega)\,\mathrm{d}Du_y^\zeta(t) = -\int_{U_y^\zeta} u_y^\zeta(t)\nabla_x\varphi_n(y+t\zeta,\omega)\cdot\zeta\,\mathrm{d}t, \tag{A.19}$$

where for a given $\omega \in \mathbb{R}^2$, we denote by $\nabla_x\varphi_n(y+t\zeta,\omega) \in \mathbb{R}^2$ the vector whose two components are the partial derivatives of φ_n with respect to x at the point $(y+t\zeta,\omega)$. By the Fubini Theorem, it follows from (A.19) that

$$(y,\omega) \mapsto \int_{U_y^\zeta} \varphi_n(y+t\zeta,\omega)\,\mathrm{d}Du_y^\zeta(t)$$

is Borel measurable on $(\Pi^\zeta \setminus N_\zeta) \times \mathbb{R}^2$. It follows from (A.18) that the same property holds for

$$(y,\omega) \mapsto \int_{-\infty}^{+\infty} \varphi(y+t\zeta,\omega)\,\mathrm{d}\mu_y^\omega(t). \tag{A.20}$$

Since by definition $\mu_y^\omega(B) = 0$ for $y \in N_\zeta$, it follows that the function (A.20) is Borel measurable on $\Pi^\zeta \times \mathbb{R}^2$. As every function in $C_c^0(\mathbb{R}^2 \times \mathbb{R}^2)$ can be approximated uniformly by a sequence of functions in $C_c^\infty(\mathbb{R}^2 \times \mathbb{R}^2)$, the function (A.20) is Borel measurable on $\Pi^\zeta \times \mathbb{R}^2$ for every $\varphi \in C_c^0(\mathbb{R}^2 \times \mathbb{R}^2)$. Hence, μ_y^ω satisfies condition (b) of Lemma A.1.

By Lemma A.1 applied to the measure μ_y^ω defined by (A.17) and with $B = \hat{\mathfrak{E}}_m^{k,\zeta} \setminus \mathfrak{J}$ and $B = \check{\mathfrak{E}}_m^{k,\zeta} \setminus \mathfrak{J}$ we obtain that the functions $(y, \omega) \mapsto |Du_y^\zeta|(\hat{\mathfrak{E}}(\omega)_y^\zeta \cap U_y^\zeta \setminus \mathfrak{J}(\omega)_y^\zeta)$ and $(y, \omega) \mapsto |Du_y^\zeta|(\check{\mathfrak{E}}(\omega)_y^\zeta \cap U_y^\zeta \setminus \mathfrak{J}(\omega)_y^\zeta)$ are Borel measurable on $(\Pi^\zeta \setminus N_\zeta) \times \mathbb{R}^2$. By (A.13)-(A.16) this implies that the functions $g_m^{k,\zeta}$ and $h_m^{k,\zeta}$ are Borel measurable on $\Pi^\zeta \times \mathbb{R}^2$. $\square$

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