Ahlfors-regularity for minimizers of a multiphase optimal design problem

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June 18, 2025

Abstract

We establish an Alhfors-regularity result for minimizers of a multiphase optimal design problem. It is a variant of the classical variational problem which involves a finite number of chambers $\mathcal{E}(i)$ of prescribed volume that partition a given domain $\Omega \subset \mathbb{R}^n$. The cost functional associated with a configuration $(\{\mathcal{E}(i)\}_i, u)$ is made up of the perimeter of the partition interfaces and a Dirichlet energy term, which is discontinuous across the interfaces. We prove that the union of the optimal interfaces is $(n-1)$-Alhfors-regular via a penalization method and decay estimates of the energy.

Keywords: free boundary problem, partition problem, Ahlfors-regularity, volume constraint

MSC: 49Q10, 49N60, 49Q20

1 Introduction

The problem of partitioning an open domain into regions with minimal interface has deep roots in both classical geometry and modern variational analysis. Formally, the goal is to partition an open set $\Omega \subset \mathbb{R}^n$ into a finite collection of disjoint subsets $\{\mathcal{E}(i)\}_{i=1}^N$ such that their union covers Ω (up to a set of measure zero) and they minimize the total interfacial energy, basically interpreted as the $(n-1)$-dimensional Hausdorff measure of the common boundaries. A celebrated example in two dimensions is the Honeycomb Conjecture, resolved by T. C. Hales [17], which states that the regular hexagonal tiling minimizes the total perimeter among all partitions of the plane into regions of equal area.

Such partition problems generalize classical isoperimetric inequalities and are intimately connected to the theory of minimal surfaces. They also arise naturally in various applications, including immiscible fluid separation (see [22, 29]), and image segmentation (see [1, 8, 28]).
Mathematically, the problem often involves minimizing an energy functional of the form

$$\mathcal{P}(\{\mathcal{E}(i)\}; \Omega) = \sum_{i<j} \mathcal{H}^{n-1}(\partial^*\mathcal{E}(i) \cap \partial^*\mathcal{E}(j) \cap \Omega),$$

subject to volume constraints $|\mathcal{E}(i)| = m_i$. Here, $\partial^*\mathcal{E}(i)$ denotes the reduced boundary of $\mathcal{E}(i)$ in the sense of geometric measure theory, which captures the essential structure of the interface between phases.

In this paper, we focus on functionals that depend not only on the interfacial energy of a partition, but also on a bulk energy term. To motivate this setting, we refer to the classical problem of liquid droplets subjected to an external electric field, where the equilibrium configuration is typically determined by the competition between interfacial and bulk energies. The interfacial energy, often modeled as proportional to the surface area of the droplet, reflects the action of surface tension and tends to favor compact shapes, such as spheres, that minimize the surface area for a given volume.

However, the presence of an external electric field introduces a nonlocal bulk energy that accounts for the interaction of the electric field with the dielectric properties of the droplet and the surrounding medium. This contribution is typically expressed through the Dirichlet energy of the electrostatic potential. The balance between bulk energy and interfacial energy gives rise to a free boundary variational problem in which the domain itself is an unknown to be optimized. For a comprehensive study of this model, the reader can refer to the work of Muratov and Novaga, who have extensively analyzed the variational problems associated with charged liquid droplets, see [10, 26, 27] and the references therein.

A prototype version of functionals involving bulk and perimeter energies is the following:

$$\int_\Omega \sigma_E(x)|\nabla u|^2\,dx + P(E;\Omega), \tag{1}$$

with $u = u_0$ prescribed on $\partial\Omega$ and $\sigma_E(x) = \beta\mathbb{1}_E + \alpha\mathbb{1}_{\Omega\setminus E}$, $0 < \alpha < \beta$.
This functional was formerly studied in 1993 in two papers by L. Ambrosio & G. Buttazzo and F.H. Lin (see [2, 23]). Later on, refined regularity results for functionals of type (1) have been established in [9, 16] and for dimension two in [19, 20, 21]. Furthermore, the same problem has been studied in the case where both the bulk and interfacial energies are of a more general nature (see [4, 5, 6, 7, 11, 13, 14, 15, 18, 24]).

In this paper, we study optimal partitions associated with functionals that also depend on a bulk Dirichlet energy, which is discontinuous across the partition interface. To our knowledge, there are no regularity results in the literature for this context. The presence of multiple chambers significantly complicates the study of regularity due to the possibility of triple points or, even worse, multiple intersections between the chambers.

Some notation is needed. Let $\Omega \subset \mathbb{R}^n$ a bounded connected open set and $N \in \mathbb{N}$ such that $N > 1$. An N-partition $\mathcal{E}$ of Ω is a family $\mathcal{E} = \{\mathcal{E}(i)\}_{i=1}^N$ of sets $\mathcal{E}(i)$ of finite perimeter with

$$|\mathcal{E}(i)| > 0, \quad \forall i \in \{1,\ldots,N\},$$
$$|\mathcal{E}(i) \cap \mathcal{E}(j)| = 0, \quad \forall i,j \in \{1,\ldots,N\},\ i<j.$$
$$\sum_{i=1}^N |\mathcal{E}(i)| = |\Omega|.$$

We introduce the following main functional associated to a partition $\mathcal{E}$:

$$\mathcal{F}(\mathcal{E},w) = \sum_{i=1}^N \int_{\mathcal{E}(i)} \alpha_i|\nabla w|^2\,dx + \frac{1}{2}\sum_{i=1}^N P(\mathcal{E}(i);\Omega), \tag{2}$$

where the vector $\alpha = \{\alpha_i\}_{i=1}^N$ is positive, i.e. $\alpha_i > 0$ for any $i \in \{1,\ldots N\}$.

The interfaces of the N-partition $\mathcal{E}$ of Ω are the $\mathcal{H}^{n-1}$-rectifiable sets

$$\mathcal{E}(h,k) = \partial^*\mathcal{E}(h) \cap \partial^*\mathcal{E}(k) \cap \Omega,$$

where $0 \leqslant h,k \leqslant N$ and $h \neq k$.

Given $\{d_i\}_{i=1}^N$ such that

$$d_i \in (0,|\Omega|),\ \forall i \in \{1,\ldots,N\} \quad \text{and} \quad \sum_{i=1}^N d_i = |\Omega|,$$

we consider the minimization of the functional (2) assuming that the measures of the chambers $\mathcal{E}(i)$ are equal to d_i and the function w is prescribed on the boundary of Ω. More precisely, given $u_0 \in H^1(\Omega)$, we consider the following constrained problem:

$$\min\left\{\mathcal{F}(\mathcal{E},v)\,:\,\mathcal{E} \text{ is an } N\text{-partition of } \Omega,\ |\mathcal{E}_i| = d_i,\ i=1,\ldots,N,\ v \in u_0 + H_0^1(\Omega)\right\}. \tag{P_c}$$

The aim of the paper is to prove the $(n-1)$-Ahlfors-regularity of the interfaces of the optimal chambers. We recall that a closed set $G \subset \mathbb{R}^n$ is said to be $(n-1)$-Ahlfors-regular if there exists a positive constant C_A such that

$$C_A^{-1}r^{n-1} \leqslant \mathcal{H}^{n-1}(G \cap B_r(x_0)) \leqslant C_A r^{n-1}, \quad \forall x_0 \in G,\ \forall r > 0.$$

In particular, we prove the following theorem.

Theorem 1.1. *Let $(\mathcal{E}, u)$ be a minimizer of the problem (P_c) and $U \Subset \Omega$ be an open set. Then, there exist a positive constant C_A such that, for every $x_0 \in \bigcup_{k=1}^N \partial\mathcal{E}(k) \cap \Omega$ and $B_r(x_0) \subset U$, it holds*

$$C_A^{-1} r^{n-1} \leqslant \sum_{k=1}^N P(\mathcal{E}(k); B_r(x_0)) \leqslant C_A r^{n-1}.$$

Moreover, $\mathcal{H}^{n-1}\Big(\Omega \cap \bigcup_{k=1}^N \partial\mathcal{E}(k) \setminus \bigcup_{k=1}^N \partial^*\mathcal{E}(k)\Big) = 0$ *and* $\bigcup_{k=1}^N \partial\mathcal{E}(k)$ *is $(n-1)$-Ahlfors-regular.*

The strategy of the proof of Theorem 1.1 follows a well-established path. First, we show that minimizers of (P_c) are indeed also minimizer for a penalized problem without constraint (see Theorem 2.1). Afterwards, the proof follows by combining the upper and lower density estimates for the minimizers of the penalized problem contained in Theorem 3.2 and Theorem 3.4.

2 From constrained to penalized problem

In the following theorem, we show that volume-constrained minimizers of (P_c) are, in fact, unconstrained Λ-minimizers of the functional $\mathcal{F}$ defined in (2) (see Definition 2.2 below). This type of relaxation of the volume constraint is standard in problems of this nature. To obtain this result, we employ a technique introduced in [12], which, in our setting, is more intricate and requires a suitable adaptation due to the presence of multiple chambers.

Theorem 2.1. *There exist $\Lambda_0 > 0$ such that if $(\mathcal{E}, u)$ is a minimizer of the functional*

$$\mathcal{F}_\Lambda(\mathcal{A}, w) = \sum_{i=1}^N \int_{\mathcal{A}(i)} \alpha_i |\nabla w|^2 \, dx + \frac{1}{2} \sum_{i=1}^N P(\mathcal{A}(i); \Omega) + \Lambda \sum_{i=1}^N \big| |\mathcal{A}(i)| - d_i \big|, \tag{3}$$

for some $\Lambda \geqslant \Lambda_0$, among all configurations $(\mathcal{A}, w)$ such that $w = u_0$ on $\partial\Omega$, then $|\mathcal{E}| = d$ and $(\mathcal{E}, u)$ is a minimizer of problem (P_c). Conversely, if $(\mathcal{E}, u)$ is a minimizer of problem (P_c) among all configurations $(\mathcal{A}, w)$ such that $w = u_0$ on $\partial\Omega$, then it is a minimizer of (3), for all $\Lambda \geqslant \Lambda_0$.

Proof. The first part of the theorem can be proved by contradiction. We assume that there exist a positive sequence $(\Lambda_h)_{h\in\mathbb{N}}$ such that $\Lambda_h \to +\infty$, as $h \to +\infty$, and a sequence of configurations $(\mathcal{E}_h, u_h)$ minimizing $\mathcal{F}_{\Lambda_h}$ and such that $u_h = u_0$ on $\partial\Omega$ and $|\mathcal{E}_h| \neq d$, for all $h \in \mathbb{N}$. We choose an arbitrary fixed partition $\mathcal{E}_0$ of Ω such that $|\mathcal{E}_0| = d$. We point out that

$$\mathcal{F}_{\Lambda_h}(\mathcal{E}_h, u_h) \leqslant \mathcal{F}(\mathcal{E}_0, u_0) := \Theta. \tag{4}$$

Our aim is to show that there exists a configuration $(\widetilde{\mathcal{E}}_h, \tilde{u}_h)$ such that, for h sufficiently large, $\mathcal{F}_{\Lambda_h}(\widetilde{\mathcal{E}}_h, \tilde{u}_h) < \mathcal{F}_{\Lambda_h}(\mathcal{E}_h, u_h)$, thus proving the result by contradiction.

By condition (4), it follows that the sequence $(u_h)_{h\in\mathbb{N}}$ is bounded in $H^1(\Omega)$, the perimeter of the partition $\mathcal{E}_h$ in Ω is uniformly bounded and $|\mathcal{E}_h(i)| \to d_i$, for any $i \in \{1, \ldots, N\}$. Therefore, possibly extracting a not relabelled subsequence, we may assume that there exists a configuration $(\mathcal{E}, u)$ such that u_h converges to u weakly in $H^1(\Omega)$, $\mathbb{1}_{\mathcal{E}_h(i)} \to \mathbb{1}_{\mathcal{E}(i)}$ a.e. in Ω, where the collection $\mathcal{E} = \{\mathcal{E}(i)\}_{i=1}^N$ is a partition of Ω and $|\mathcal{E}| = d$. The couple $(\mathcal{E}, u)$ will be used as a reference configuration for the definition of $(\widetilde{\mathcal{E}}_h, \tilde{u}_h)$.

By appropriately rearranging the order of the chambers, we can assume that there exists an $i \in \{1, \ldots, N\}$ such that $|\mathcal{E}_h(i)| < d_i$, for any $h \in \mathbb{N}$. Since

$$\sum_{j=1}^N |\mathcal{E}_h(j)| = |\Omega| = \sum_{j=1}^N d_j,$$

we can also assume that there exists $j \in \{1, \ldots, N\}$ such that $|\mathcal{E}_h(j)| > d_j$, for any $h \in \mathbb{N}$.
Let $0 \leqslant i, j \leqslant N$, $i \neq j$. We say that $\mathcal{E}(i)$ and $\mathcal{E}(j)$ are neighboring chambers, if $\mathcal{H}^{n-1}(\mathcal{E}(i,j)) > 0$. If there exist two neighboring chambers $\mathcal{E}(i)$ and $\mathcal{E}(j)$ with $|\mathcal{E}_h(i)| < d_i$ and $|\mathcal{E}_h(j)| > d_j$, then we can argue exactly as in [12] to construct the configuration $(\widetilde{\mathcal{E}}_h, \tilde{u}_h)$. Otherwise we will work

with the pair of chambers $\mathcal{E}(\bar{i})$ and $\mathcal{E}(\bar{j})$ with with $|\mathcal{E}_h(\bar{i})| < d_{\bar{i}}$ and $|\mathcal{E}_h(\bar{j})| > d_{\bar{j}}$ that are the *closest* in a suitable sense. More precisely, for $i, j \in \{1, \dots, N\}$, we denote by c_{ij} the order of link between the chambers $\mathcal{E}(i)$ and $\mathcal{E}(j)$ that is defined as the minimum number m such that there exist chambers $\mathcal{E}(k_1), \cdots \mathcal{E}(k_m)$, such that $\mathcal{E}(i)$ is neighboring $\mathcal{E}(k_1)$, $\mathcal{E}(j)$ is neighboring $\mathcal{E}(k_m)$ and $\mathcal{E}(k_l)$ is neighboring $\mathcal{E}(k_{l+1})$ for any $l \in \{1, \dots, m-1\}$. We identify $\bar{i}$ and $\bar{j}$ as the indices of two chambers such that

$$(\bar{i}, \bar{j}) = \operatorname{argmin}\{c_{ij} \,:\, (i,j) \in \{1, \dots, N\},\, |\mathcal{E}_h(i)| < d_i,\, |\mathcal{E}_h(j)| > d_j\}.$$

Therefore there exists $m \in \mathbb{N}$ such that $\mathcal{E}_h(i)$ and $\mathcal{E}_h(j)$ are linked through some chambers $\mathcal{E}_h(k_1), \dots, \mathcal{E}_h(k_m)$, where $k_\ell \in \{1, \dots, N\} \backslash \{\bar{i}, \bar{j}\}$ and $|\mathcal{E}_h(k_\ell)| = d_{k_\ell}$, for any $\ell \in \{1, \dots, N\}$.
We may also assume that there is only one intermediate chamber. i.e. $m = 1$ or equivalently that $c_{\bar{i}\bar{j}} = 1$ as in the other case the construction can be carried over in a similar way.

Then, for simplicity, up to relabeling the chambers, we will assume that $i = 1$, $j = 3$, $c_{13} = 1$ and $\mathcal{E}_h(2)$ is the linking chamber between $\mathcal{E}_h(1)$ and $\mathcal{E}_h(3)$, with

$$|\mathcal{E}_h(1)| < d_1, \quad |\mathcal{E}_h(2)| = d_2, \quad |\mathcal{E}_h(3)| > d_3.$$

Step 1. *Construction of* $(\tilde{\mathcal{E}}_h, \tilde{u}_h)$. Let us choose $(\sigma_1)_h \in \mathbb{R}$ and $(\sigma_2)_h$ such that

$$(\sigma_1)_h \in \left(0, \alpha \min\left\{\frac{1}{2^n}, \frac{|d_1 - |\mathcal{E}_h(1)||}{n 2^{n+1}}\right\}\right), \tag{5}$$

$$(\sigma_2)_h \in \left(0, \min\left\{\frac{1}{2^n}, \frac{|d_2 - |\mathcal{E}_h(2)||}{n 2^{n+1}}\right\}\right), \tag{6}$$

where $\alpha = \alpha(n, N) \in (0,1)$ is a constant that will be chosen later. We fix $\ell \in \{1, 2\}$. Since the three chambers are linked, the set $\partial^*\mathcal{E}(\ell) \cap \partial^*\mathcal{E}(\ell+1) \cap \Omega$ is not empty. Thus, we can take a point $x_\ell \in \partial^*\mathcal{E}(\ell) \cap \partial^*\mathcal{E}(\ell+1) \cap \Omega$ such that

$$\lim_{\rho \to 0^+} \frac{|\mathcal{E}(\ell) \cap B_\rho(x_\ell)|}{\omega_n \rho^n} = \lim_{\rho \to 0^+} \frac{|\mathcal{E}(\ell+1) \cap B_\rho(x_\ell)|}{\omega_n \rho^n} = \frac{1}{2}.$$

Since

$$1 = \frac{|B_\rho(x_\ell)|}{\omega_n \rho^n} = \frac{|\mathcal{E}(\ell) \cap B_\rho(x_\ell)|}{\omega_n \rho^n} + \frac{|\mathcal{E}(\ell+1) \cap B_\rho(x_\ell)|}{\omega_n \rho^n} + \sum_{\substack{k=1 \\ k \neq \ell, \ell+1}}^{N} \frac{|\mathcal{E}(k) \cap B_r(x_\ell)|}{\omega_n \rho^n},$$

there exists $\eta \in (0, \min\{\frac{\operatorname{dist}(x_1, x_2)}{2}, 1\})$ such that if $0 < r < \eta$, then

$$\sum_{\substack{j=1 \\ j \neq \ell, \ell+1}}^{N} |\mathcal{E}(j) \cap B_r(x_\ell)| < \kappa \min_{i=1,2} (\sigma_i)_h r^n, \tag{7}$$

for some positive constant κ that will be chosen later and h sufficiently large. By De Giorgi structure theorem for sets of finite perimeter, $\mathbb{1}_{\frac{\mathcal{E}(\ell) - x_\ell}{r}} \to \mathbb{1}_{H_\ell}$ in $L^1_{loc}(\mathbb{R}^n)$, as $r \to 0^+$, where $H_\ell = \{\langle z, \nu_{\mathcal{E}(\ell)}(x_l)\rangle < 0\} = \{\langle z, \nu_{\mathcal{E}(\ell+1)}(x_l)\rangle > 0\}$. Let $y_\ell \in B_1 \backslash H_\ell$ be the point

$$y_\ell = \frac{\nu_{\mathcal{E}(\ell)}(x_l)}{2} = -\frac{\nu_{\mathcal{E}(\ell+1)}(x_l)}{2}.$$

Therefore, there exists $0 < r < \eta$ such that (7) holds and

$$\left|\frac{\mathcal{E}(\ell) - x_\ell}{r} \cap B_{1/2}(y_\ell)\right| < \varepsilon, \quad \left|\frac{\mathcal{E}(\ell) - x_\ell}{r} \cap B_1(y_\ell)\right| > \frac{1}{2^{n+2}}.$$

Setting $x'_\ell := x_\ell + r y_\ell$, from the convergence of $\mathcal{E}_h$ to $\mathcal{E}$, we have that, for h sufficiently large,

$$|\mathcal{E}_h(\ell) \cap B_{r/2}(x'_\ell)| < \varepsilon r^n, \quad |\mathcal{E}_h(\ell) \cap B_r(x'_\ell)| > \frac{r^n}{2^{n+2}}, \quad \sum_{\substack{j=1 \\ j\neq \ell,\ell+1}}^{N} |\mathcal{E}_h(j) \cap B_r(x_\ell)| < \kappa \min_{i=1,2} (\sigma_i)_h r^n, \tag{8}$$

where κ will be chosen later. We remark that, since $r < \eta$, $B_r(x_1)$ and $B_r(x_2)$ are disjoint.

Now we define the following bi-Lipschitz map used in [12] which maps $B_r(0)$ into itself:

$$\Psi(x) := \begin{cases} \big[1 - (\sigma_\ell)_h \big(2^n - 1\big)\big] x & \text{if } |x| < \dfrac{r}{2}, \\ x + (\sigma_\ell)_h \left(1 - \dfrac{r^n}{|x|^n}\right) x & \text{if } \dfrac{r}{2} \leqslant |x| < r, \\ x & \text{if } |x| \geqslant r. \end{cases}$$

We denote the corresponding action localized in the ball $B_r(x'_\ell)$ by

$$\Phi_\ell(x) = x'_\ell + \Psi(x - x'_\ell), \quad \forall x \in \mathbb{R}^n.$$

In the sequel, we will use some estimates for the map Φ_ℓ that can be easily obtained by direct computations (see [12] for the explicit calculations). These estimates are trivial for $|x - x'_\ell| < r/2$, whereas they can be deduced by the explicit expression of $\nabla \Psi$ for $r/2 < |x - x'_\ell| < r$, that is

$$\frac{\partial (\Phi_\ell)_i}{\partial x_j}(x) = \delta_{ij} + (\sigma_\ell)_h \left[\left(1 - \frac{r^n}{|x - x'_\ell|^n}\right) \delta_{ij} + n r^n \frac{(x - x'_\ell)_i (x - x'_\ell)_j}{|x - x'_\ell|^{n+2}} \right], \quad \forall i, j \in \{1, \ldots, n\}.$$

There exists a positive constant $C = C(n)$ depending only on n such that,

$$\left\| \nabla \Phi_\ell^{-1}(y) - I \right\| \leqslant C(n) (\sigma_\ell)_h, \quad \forall y \in B_r(x'_\ell), \tag{9}$$

$$1 + C(n)(\sigma_\ell)_h \leqslant J\Phi_\ell(x) \leqslant 1 + n 2^n (\sigma_\ell)_h, \quad \forall x \in B_r(x'_\ell) \backslash B_{\frac{r}{2}}(x'_\ell). \tag{10}$$

$$1 - n(2^n - 1)(\sigma_\ell)_h \leqslant J\Phi_\ell(x) \leqslant 1 - (2^n - 1)(\sigma_\ell)_h, \quad \forall x \in B_{\frac{r}{2}}(x'_\ell). \tag{11}$$

Accordingly, for $j \neq \ell, \ell+1$, we estimate,

$$\begin{aligned} &\Big| |\Phi_\ell(\mathcal{E}_h(j) \cap B_r(x'_\ell))| - |\mathcal{E}_h(j) \cap B_r(x'_\ell)| \Big| \\ &\leqslant \int_{\mathcal{E}_h(j) \cap B_r(x'_\ell)} |J\Phi_\ell - 1| \, dx \leqslant n 2^n (\sigma_\ell)_h |\mathcal{E}_h(j) \cap B_r(x'_\ell)| \leqslant n 2^n \kappa \min_{j \in \{1,2\}} (\sigma_j)_h r^n, \end{aligned} \tag{12}$$

whenever, for $j = \ell, \ell+1$,

$$\begin{aligned} &\Big| |\Phi_\ell(\mathcal{E}_h(j) \cap B_r(x'_\ell))| - |\mathcal{E}_h(j) \cap B_r(x'_\ell)| \Big| \\ &\leqslant \int_{\mathcal{E}_h(j) \cap B_r(x'_\ell)} |J\Phi_\ell - 1| \, dx \leqslant n 2^n (\sigma_\ell)_h |\mathcal{E}_h(j) \cap B_r(x'_\ell)| \leqslant n 2^n (\sigma_\ell)_h r^n. \end{aligned} \tag{13}$$

We define

$$\Phi := \Phi_1 \circ \Phi_2, \quad \tilde{\mathcal{E}}_h := \{\Phi(\mathcal{E}_h(i))\}_{i=1}^N, \quad \tilde{u}_h := u_h \circ \Phi^{-1}, \tag{14}$$

and remark that Φ act in not trivial way only in $B_r(x'_1)$ and $B_r(x'_2)$, leaving anything unchanged outside these balls.

Step 2. *Φ does not modify too much $|\mathcal{E}_h(1)|$ and $|\mathcal{E}_h(3)|$.*
Let us show that under conditions (5) , (6) on $(\sigma_1)_h$ and $(\sigma_2)_h$, it results

$$|\Phi(\mathcal{E}_h(1))| < d_1, \tag{15}$$

$$|\Phi(\mathcal{E}_h(3))| > d_3. \tag{16}$$

Since the application of the map Φ leaves the measure unchanged outside $B_r(x_1') \cup B_r(x_2')$, we can evaluate the differences in measure solely within $B_r(x_1') \cup B_r(x_2')$. We have that

$$\begin{aligned}&\Big||\Phi(\mathcal{E}_h(1))| - |\mathcal{E}_h(1)|\Big|\\ &\leqslant \Big||\Phi_1(\mathcal{E}_h(1) \cap B_r(x_1'))| - |\mathcal{E}_h(1) \cap B_r(x_1')|\Big| + \Big||\Phi_2(\mathcal{E}_h(1) \cap B_r(x_2'))| - |\mathcal{E}_h(1) \cap B_r(x_2')|\Big|.\end{aligned}$$

Applying (12) and (13), by the choice of $(\sigma_1)_h$ we deduce

$$|\Phi(\mathcal{E}_h(1))| \leqslant |\mathcal{E}_h(1)| + n2^{n+1}(\sigma_1)_h < d_1.$$

The same argument can be applied to chamber $\mathcal{E}_h(3)$ under the transformation by the map Φ; that is, we have

$$\begin{aligned}&\Big||\Phi(\mathcal{E}_h(3))| - |\mathcal{E}_h(3)|\Big|\\ &\leqslant \Big||\Phi_1(\mathcal{E}_h(3) \cap B_r(x_1'))| - |\mathcal{E}_h(3) \cap B_r(x_1')|\Big| + \Big||\Phi_2(\mathcal{E}_h(3) \cap B_r(x_2'))| - |\mathcal{E}_h(3) \cap B_r(x_2')|\Big|.\end{aligned}$$

Using again (12) and (13), by the choice of $(\sigma_2)_h$ we deduce

$$|\Phi(\mathcal{E}_h(3))| \geqslant |\mathcal{E}_h(3)| - n2^{n+1}(\sigma_2)_h > d_3.$$

Step 3. *The choice of* $(\sigma_2)_h$. In this step we prove that for every $(\sigma_1)_h$ as in (5) there exists $(\sigma_2)_h$ as in (6) such that

$$|\Phi(\mathcal{E}_h(2))| = d_2. \tag{17}$$

To this end, we prove that the action of the map Φ results in an increase in the measure of $\mathcal{E}_h(2) \cap B_r(x_2')$ and a decrease in the measure of $\mathcal{E}_h(2) \cap B_r(x_1')$, modulated by the parameters $(\sigma_2)_h$ and $(\sigma_1)_h$. Precisely we prove that there exist two positive constants $C_g = C_g(n)$ and $C_l = C_l(n)$ such that

$$\mathcal{G}_h := |\Phi_2(\mathcal{E}_h(2)) \cap B_r(x_2')| - |\mathcal{E}_h(2) \cap B_r(x_2')| \geqslant C_g(n)(\sigma_2)_h r^n, \tag{18}$$

$$\mathcal{L}_h := |\mathcal{E}_h(2) \cap B_r(x_1')| - |\Phi_1(\mathcal{E}_h(2)) \cap B_r(x_1')| \geqslant C_l(n)(\sigma_1)_h r^n. \tag{19}$$

With this notations (17) can be rephrased

$$\mathcal{G}_h = \mathcal{L}_h$$

Inequality (18) is the simplest to prove. Indeed, taking the estimates (10) and (11) on $J\Phi_2$ and (8) into account, we have that

$$\begin{aligned}\mathcal{G}_h &= |\Phi_2(\mathcal{E}_h(2)) \cap B_r(x_2')| - |\mathcal{E}_h(2) \cap B_r(x_2')| = \int_{\mathcal{E}_h(2)\cap B_r(x_2')} (J\Phi_2 - 1)\,dx \qquad (20)\\ &= \int_{\mathcal{E}_h(2)\cap B_{\frac{r}{2}}(x_2')} (J\Phi_2 - 1)\,dx + \int_{\mathcal{E}_h(2)\cap (B_r(x_2')\setminus B_{\frac{r}{2}}(x_2'))} (J\Phi_2 - 1)\,dx\\ &\geqslant -\varepsilon n\big(2^n - 1\big)(\sigma_2)_h r^n + C(n)\left(\frac{\omega_n}{2^{n+2}} - \varepsilon\right)(\sigma_2)_h r^n\\ &= \left(\big[-n\big(2^n-1\big) - C(n)\big]\varepsilon + C(n)\frac{\omega_n}{2^{n+2}}\right)(\sigma_2)_h r^n = C_g(n)(\sigma_2)_h r^n,\end{aligned}$$

where ε is chosen small enough in such a way that $C_g(n) > 0$.

To prove (19), we first note that by employing exactly the same computations used to establish (18), we deduce that $\mathcal{E}_h(1)$ increases in measure within $B_r(x_1')$. Specifically, we have that

$$|\Phi_1(\mathcal{E}_h(1)) \cap B_r(x_1')| - |\mathcal{E}_h(1) \cap B_r(x_1')| \geqslant C_g(n)(\sigma_1)_h r^n. \tag{21}$$

Since the total measure of the ball $B_r(x_1')$ is preserved by the map Φ_1 we have that

$$\sum_{j=1}^{N} |\Phi_1(\mathcal{E}_h(j)) \cap B_r(x_1')| = \sum_{j=1}^{N} |\mathcal{E}_h(j)) \cap B_r(x_1')|,$$

consequently we deduce

$$\begin{aligned}
&|\mathcal{E}_h(2) \cap B_r(x_1')| - |\Phi_1(\mathcal{E}_h(2)) \cap B_r(x_1')| \\
&= |\Phi_1(\mathcal{E}_h(1)) \cap B_r(x_1')| - |\mathcal{E}_h(1) \cap B_r(x_2')| + \sum_{\substack{j=1 \\ j\neq 1,2}}^{N} \big[|\Phi_1(\mathcal{E}_h(j)) \cap B_r(x_1')| - |\mathcal{E}_h(j) \cap B_r(x_1')|\big] \\
&\geqslant C_g(n)(\sigma_1)_h r^n - \kappa n(N-2)2^n(\sigma_1)_h r^n = [C_g(n) - \kappa n(N-2)2^n](\sigma_1)_h r^n,
\end{aligned}$$

where we used (21) and (12). Therefore (19) is proved with $C_l(n) = C_g(n)/2$ if we chose $\kappa = C_g(n)/\big[n(N-2)2^{n+1}\big]$. If we denote with C_g^m and C_l^m the greatest constant such that (18) and (19) holds true we deduce that

$$\begin{aligned}
|\Phi_2(\mathcal{E}_h(2)) \cap B_r(x_2')| - |\mathcal{E}_h(2) \cap B_r(x_2')| &= C_g^m(n)(\sigma_2)_h r^n \\
|\mathcal{E}_h(2) \cap B_r(x_1')| - |\Phi_1(\mathcal{E}_h(2)) \cap B_r(x_1')| &= C_l^m(n)(\sigma_1)_h r^n.
\end{aligned}$$

Finally we can conclude observing that

$$\mathcal{G}_h - \mathcal{L}_h = [C_g^m(n)(\sigma_2)_h - C_l^m(n)(\sigma_1)_h]r^n.$$

Then (17) is proved if we choose $(\sigma_2)_h = \frac{C_l^m(n)}{C_g^m(n)}(\sigma_1)_h$.

Step 4: *Reaching a contradiction.* Finally, we prove that the perturbation defined in (14) leads to a decrease in energy, namely

$$\mathcal{F}_{\Lambda_h}(\mathcal{E}_h, u_h) - \mathcal{F}_{\Lambda_h}(\tilde{\mathcal{E}}_h, \tilde{u}_h) > 0,$$

which contradicts the minimality of $(\mathcal{E}_h, u_h)$. For convenience of notation, we reformulate the energy difference by setting

$$\begin{aligned}
&\mathcal{F}_{\Lambda_h}(\mathcal{E}_h, u_h) - \mathcal{F}_{\Lambda_h}(\tilde{\mathcal{E}}_h, \tilde{u}_h) \\
&= \sum_{i=1}^{N}\sum_{\ell=1}^{2}\Big[\int_{B_r(x_\ell')} \sigma_{\mathcal{E}_h(i)}|\nabla u_h|^2\,dx - \int_{B_r(x_\ell')} \sigma_{\Phi_\ell(\mathcal{E}_h(i))}|\nabla(u_h\circ\Phi_\ell^{-1})|^2\,dx\Big] \\
&+ \frac{1}{2}\sum_{i=1}^{N}\sum_{\ell=1}^{2}\big[P(\mathcal{E}_h(i), \overline{B}_r(x_\ell')) - P(\Phi_\ell(\mathcal{E}_h(i)), \overline{B}_r(x_\ell'))\big] \\
&+ \Lambda_h\sum_{i=1}^{N}[||\mathcal{E}_h(i)| - d_i| - ||\Phi(\mathcal{E}_h(i))| - d_i|] = I_{1,h} + I_{2,h} + I_{3,h}. \qquad (22)
\end{aligned}$$

Substep 1. *Estimate of* $I_{1,h}$. Performing the change of variables $y = \Phi_\ell(x)$, and observing that $\mathbb{1}_{\Phi(\mathcal{E}_h(i))} \circ \Phi = \mathbb{1}_{\mathcal{E}_h(i)}$, we get

$$I_{1,h} = \sum_{i=1}^{N}\sum_{\ell=1}^{2}\int_{B_r(x_\ell')} \sigma_{\mathcal{E}_h(i)}(x)\big[|\nabla u_h(x)|^2 - \big|\nabla u_h(x)\circ\nabla\Phi_\ell^{-1}(\Phi_\ell(x))\big|^2 J\Phi_\ell(x)\big]\,dx = A_{1,h} + A_{2,h},$$

where $A_{1,h}$ stands for the above integral over $B_{\frac{r}{2}}(x_\ell')$ and $A_{2,h}$ for the same integral over $B_r(x_\ell')\backslash B_{\frac{r}{2}}(x_\ell')$. Using the explicit expressions of $\nabla\Phi_\ell^{-1}$ and $J\Phi$, which remain constant inside $B_{\frac{r}{2}}(x_\ell')$, we easily get

$$A_{1,h} = \sum_{i=1}^{N}\sum_{\ell=1}^{2}\int_{B_{\frac{r}{2}}(x_\ell')} \sigma_{\mathcal{E}_h(i)}|\nabla u_h|^2\Big\{1 - \big[1 - (\sigma_\ell)_h(2^n-1)\big]^{n-2}\Big\}\,dx \geqslant 0.$$

Inside $B_r(x'_\ell)\backslash B_{\frac{r}{2}}(x'_\ell)$, even though $\nabla\Phi_\ell^{-1}$ is not constant, we can use (9) and (10) to obtain

$$\begin{aligned}A_{1,h} &\geqslant \sum_{i=1}^{N}\sum_{\ell=1}^{2}\int_{B_r(x'_\ell)\backslash B_{\frac{r}{2}}(x'_\ell)} \sigma_{\mathcal{E}_h(i)}|\nabla u_h|^2\Big\{1-\big[1-(\sigma_\ell)_h\big(2^n-1\big)\big]^{-2}\big(1+2^n n(\sigma_\ell)_h\big)\Big\}\,dx\\ &\geqslant -c(n)\sum_{i=1}^{N}\sum_{\ell=1}^{2}(\sigma_\ell)_h\int_{B_r(x'_\ell)\backslash B_{\frac{r}{2}}(x'_\ell)} \sigma_{\mathcal{E}_h(i)}|\nabla u_h|^2\,dx\\ &\geqslant -c(n)\Theta((\sigma_1)_h+(\sigma_2)_h),\end{aligned}$$

where we also used (4), thus getting

$$I_{1,h} \geqslant -\overline{C}_1\Theta((\sigma_1)_h+(\sigma_2)_h), \tag{23}$$

for some positive constant $\overline{C}_1=\overline{C}_1(n)$.

Substep 2. *Estimate of* $I_{2,h}$. In order to estimate $I_{2,h}$, we can use the area formula for maps between rectifiable sets. If we denote by $T^{\ell,i}_{h,x}$ the tangential gradient of Φ_ℓ along the approximate tangent space to $\partial^*\mathcal{E}_h(i)$ in x and $\big(T^{\ell,i}_{h,x}\big)^*$ is the adjoint of the map $T^{\ell,i}_{h,x}$, the $(n-1)$-dimensional jacobian of $T^{\ell,i}_{h,x}$ is given by

$$J_{n-1}T^{\ell,i}_{h,x}=\sqrt{\det\big(\big(T^{\ell,i}_{h,x}\big)^*\circ T^{\ell,i}_{h,x}\big)}.$$

Thereafter we can estimate

$$J_{n-1}T^{\ell,i}_{h,x}\leqslant 1+(\sigma_\ell)_h+2^n(n-1)(\sigma_\ell)_h. \tag{24}$$

We address the reader to [12] where explicit calculations are given. In order to estimate $I_{2,h}$, we use the area formula for maps between rectifiable sets ([3, Theorem 2.91]), thus getting

$$\begin{aligned}I_{2,h} &= \sum_{i=1}^{N}\sum_{\ell=1}^{2}\big[P(\mathcal{E}_h(i);\overline{B}_r(x'_\ell))-P(\Phi_\ell(\mathcal{E}_h);\overline{B}_r(x'_\ell))\big]\\ &=\sum_{i=1}^{N}\sum_{\ell=1}^{2}\bigg[\int_{\partial^*\mathcal{E}_h(i)\cap\overline{B}_r(x'_\ell)} d\mathcal{H}^{n-1}-\int_{\partial^*\mathcal{E}_h(i)\cap\overline{B}_r(x'_\ell)} J_{n-1}T^{\ell,i}_{h,x}\,d\mathcal{H}^{n-1}\bigg]\\ &=\sum_{i=1}^{N}\sum_{\ell=1}^{2}\bigg[\int_{\partial^*\mathcal{E}_h(i)\cap\overline{B}_r(x'_\ell)\backslash B_{\frac{r}{2}}(x'_\ell)}\Big(1-J_{n-1}T^{\ell,i}_{h,x}\Big)\,d\mathcal{H}^{n-1}\\ &\quad+\int_{\partial^*\mathcal{E}_h(i)\cap\overline{B}_{\frac{r}{2}}(x'_\ell)}\Big(1-J_{n-1}T^{\ell,i}_{h,x}\Big)\,d\mathcal{H}^{n-1}\bigg].\end{aligned}$$

Notice that the last integral in the above formula is non-negative since Φ_ℓ is a contraction in $B_{\frac{r}{2}}$, hence $J_{n-1}T^{\ell,i}_{h,x}<1$ in $B_{r/2}$, while from (24) we have

$$\begin{aligned}&\sum_{i=1}^{N}\sum_{\ell=1}^{2}\int_{\partial^*\mathcal{E}_h(i)\cap\overline{B}_r(x'_\ell)\backslash B_{\frac{r}{2}}(x'_\ell)}\Big(1-J_{n-1}T^{\ell,i}_{h,x}\Big)\,d\mathcal{H}^{n-1}\\ &\geqslant -2^n n((\sigma_1)_h+(\sigma_2)_h)\sum_{i=1}^{N}P(\mathcal{E}_h(i);\overline{B}_r(x'_\ell))\geqslant -2^n n\Theta((\sigma_1)_h+(\sigma_2)_h)\\ &=:-\overline{C}_2(n,\Theta)((\sigma_1)_h+(\sigma_2)_h),\end{aligned}$$

thus concluding that

$$I_{2,h}\geqslant -\overline{C}_2(n,\Theta)((\sigma_1)_h+(\sigma_2)_h). \tag{25}$$

Substep 3. *Estimate of* $I_{3,h}$. Taking (15), (16) and (17) into account, we split $I_{3,h}$ in three addends as follows:

$$\Lambda_h^{-1}I_{3,h}=\big[|\Phi(\mathcal{E}_h(1))|-\mathcal{E}_h(1)|\big]+\big[|\mathcal{E}_h(3)|-|\Phi(\mathcal{E}_h(3))|\big]$$

$$+ \sum_{i=4}^{N} \left[\left| |\mathcal{E}_h(i)| - d_i \right| - \left| |\Phi(\mathcal{E}_h(i))| - d_i \right| \right] = B_{1,h} + B_{2,h} + B_{3,h}. \tag{26}$$

We estimate the addends separately. For $i > 3$, by (12) we have

$$\begin{aligned}
&\left| |\mathcal{E}_h(i)| - d_i \right| - \left| |\Phi(\mathcal{E}_h(i))| - d_i \right| \leqslant \left| |\mathcal{E}_h(i)| - |\Phi(\mathcal{E}_h(i))| \right| \\
&\leqslant \left| |\mathcal{E}_h(i) \cap B_r(x_1')| - |\Phi_1(\mathcal{E}_h(i) \cap B_r(x_1'))| \right| + \left| |\mathcal{E}_h(i) \cap B_r(x_2')| - |\Phi_2(\mathcal{E}_h(i) \cap B_r(x_2'))| \right| \\
&\leqslant n2^{n+1} \kappa \min_{j \in \{1,2\}} (\sigma_j)_h r^n.
\end{aligned}$$

Therefore, we have

$$B_{3,h} \geqslant -n(N-3)2^{n+1} \kappa \min_{j \in \{1,2\}} (\sigma_j)_h r^n. \tag{27}$$

Regarding $B_{1,h}$, we use (21) and (12) to get

$$\begin{aligned}
B_{1,h} &= |\Phi(\mathcal{E}_h(1))| - |\mathcal{E}_h(1)| \\
&= \left[|\Phi_1(\mathcal{E}_h(1)) \cap B_r(x_1')| - |\mathcal{E}_h(1) \cap B_r(x_1')| \right] + \left[|\Phi_2(\mathcal{E}_h(1)) \cap B_r(x_2')| - |\mathcal{E}_h(1) \cap B_r(x_2')| \right] \\
&\geqslant C_g(n)(\sigma_1)_h r^n - n2^n (\sigma_2)_h \kappa r^n.
\end{aligned} \tag{28}$$

Similarly, we can estimate $B_{2,h}$ in $B_r(x_2')$ observing that the total measure of the ball $B_r(x_2')$ is preserved by the map Φ_2, that is

$$\sum_{j=1}^{N} |\Phi_2(\mathcal{E}_h(j)) \cap B_r(x_2')| = \sum_{j=1}^{N} |\mathcal{E}_h(j)) \cap B_r(x_2')|.$$

Accordingly, using (20) and (12), we deduce that

$$\begin{aligned}
&|\mathcal{E}_h(3) \cap B_r(x_2')| - |\Phi_2(\mathcal{E}_h(3)) \cap B_r(x_2')| \\
&= \left[|\Phi_2(\mathcal{E}_h(2) \cap B_r(x_2'))| - |\mathcal{E}_h(2) \cap B_r(x_2')| \right] + \sum_{j \neq 2,3} \left[|\Phi_2(\mathcal{E}_h(j)) \cap B_r(x_2')| - \mathcal{E}_h(j) \cap B_r(x_2')| \right] \\
&\geqslant C_g(n)(\sigma_2)_h r^n - n2^n (N-2) \kappa \min_{i \in \{1,2\}} (\sigma_i)_h r^n.
\end{aligned}$$

Therefore, we can conclude, using again (12),

$$\begin{aligned}
B_{2,h} &= |\mathcal{E}_h(3)| - |\Phi(\mathcal{E}_h(3))| \\
&= \left[|\mathcal{E}_h(3) \cap B_r(x_2')| - |\Phi_2(\mathcal{E}_h(3)) \cap B_r(x_2')| \right] + \left[|\mathcal{E}_h(3) \cap B_r(x_1')| - |\Phi_1(\mathcal{E}_h(3)) \cap B_r(x_1')| \right] \\
&\geqslant C_g(n)(\sigma_2)_h r^n - n2^n (N-2) \kappa \min_{i \in \{1,2\}} (\sigma_i)_h r^n - n2^n \kappa (\sigma_1)_h r^n \\
&= C_g(n)(\sigma_2)_h r^n - n2^n (N-1) \kappa \min_{i \in \{1,2\}} (\sigma_i)_h r^n.
\end{aligned} \tag{29}$$

Finally, combining (26), (27), (28) and (29), we conclude that

$$I_{3,h} \geqslant \Lambda_h \left[C_g(n) - n2^n (N+1) \kappa \right] ((\sigma_1)_h + (\sigma_2)_h) r^n. \tag{30}$$

Substep 4. *The contradiction.* Inserting (23), (25), (30) in (22), we conclude that

$$I_{1,h} + I_{2,h} + I_{3,h} \geqslant [(\sigma_1)_h + (\sigma_2)_h] \left[-\overline{C}_1 \Theta - \overline{C}_2 + \Lambda_h \left(C_g(n) r^n - n2^n (N+1) \kappa r^n \right) \right] > 0,$$

if $\kappa = \kappa(n, N)$ is sufficiently small and $\Lambda_h \geqslant \Lambda_0 = \Lambda_0(n, N, \Theta)$. This contradicts the minimality of $(\mathcal{E}_h, u_h)$, thus concluding the proof. □

The previous theorem motivates the following definition.

Definition 2.2 (Λ-minimizers)**.** *The energy pair $(\mathcal{E}, u)$ is a Λ-minimizer in Ω of the functional $\mathcal{F}$, defined in (2), if and only if for every $B_r(x_0) \subset \Omega$ it holds that*

$$\mathcal{F}(\mathcal{E}, u; B_r(x_0)) \leqslant \mathcal{F}(\mathcal{G}, v; B_r(x_0)) + \Lambda \sum_{i=1}^{N} |\mathcal{G}(k) \Delta \mathcal{E}(k)|,$$

whenever $(\mathcal{G}, v)$ is an admissible test pair, namely, $\mathcal{G}$ is an N-partion of Ω such that $\mathcal{G}(k) \Delta \mathcal{E}(k) \subset\subset B_r(x_0)$ and $v - u \in H_0^1(B_r(x_0))$.

3 Energy decay estimates

We start by proving a fundamental lemma which establishes that, for any ball that is either almost entirely contained within a single chamber or lies at the interface of only two chambers, the Dirichlet part of the functional satisfies a favorable decay estimate.

Lemma 3.1. *Let $(\mathcal{E}, u)$ be a Λ-minimizer of the functional $\mathcal{F}$ defined in* (2)*. There exists $\tau_0 \in (0,1)$ such that the following statement is true: for all $\tau \in (0,\tau_0)$ there exists $\varepsilon_1 = \varepsilon_1(\tau) > 0$ such that if $B_r(x_0) \subset\subset \Omega$ and one of the following conditions holds:*

(i) *There exists $i \in \{1,\dots,N\}$ such that $\frac{|\mathcal{E}(k)\cap B_r(x_0)|}{|B_r(x_0)|} < \varepsilon_1$, for any $k \neq i$,*

(ii) *There exist $i,j \in \{1,\dots,N\}$ and a half-space H such that $\frac{|(\mathcal{E}(i)\setminus H)\cap B_r(x_0)|}{|B_r(x_0)|} < \varepsilon_1$, $\frac{|(\mathcal{E}(j)\cap H)\cap B_r(x_0)|}{|B_r(x_0)|} < \varepsilon_1$ and $\frac{|\mathcal{E}(k)\cap B_r(x_0)|}{|B_r(x_0)|} < \varepsilon_1$, for any $k \neq i,j$,*

then

$$\int_{B_{\tau r}(x_0)} |\nabla u|^2\,dx \leqslant C_1\tau^n \int_{B_r(x_0)} |\nabla u|^2\,dx,$$

for some positive constant $C_1 = C_1(n,N,\alpha_k)$.

Proof. Let us fix $B_r(x_0) \subset\subset \Omega$ and $0 < \tau < 1$. Without loss of generality, we may assume that $\tau < 1/2$ and $x_0 = 0$. We start proving (i). We denote by v the harmonic function in $B_{\frac{r}{2}}$ satisfying the condition $v = u$ on $B_r \setminus B_{\frac{r}{2}}$. Let $\phi \in H^1_0(B_{\frac{r}{2}})$. It holds that

$$\int_{B_{\frac{r}{2}}} \langle \nabla v, \nabla\phi\rangle\,dx = 0. \tag{31}$$

On the other hand, u solves the following equation:

$$\alpha_i \int_{B_{\frac{r}{2}}\cap\mathcal{E}(i)} \langle \nabla u, \nabla\phi\rangle\,dx = -\sum_{k\neq i}\alpha_k \int_{B_{\frac{r}{2}}\cap\mathcal{E}(k)} \langle \nabla u, \nabla\phi\rangle\,dx. \tag{32}$$

Adding the equation

$$\alpha_i \int_{B_{\frac{r}{2}}\setminus\mathcal{E}(i)} \langle \nabla u, \nabla\phi\rangle\,dx = \alpha_i\sum_{k\neq i} \int_{B_{\frac{r}{2}}\cap\mathcal{E}(k)} \langle \nabla u, \nabla\phi\rangle\,dx,$$

to (32) and dividing by α_i, we obtain

$$\int_{B_{\frac{r}{2}}} \langle \nabla u, \nabla\phi\rangle\,dx = \frac{1}{\alpha_i}\sum_{k\neq i}(\alpha_i-\alpha_k) \int_{B_{\frac{r}{2}}\cap\mathcal{E}(k)} \langle \nabla u, \nabla\phi\rangle\,dx.$$

Now we choose $\phi := v - u$ and we subtract (31) from the previous equation, getting

$$\begin{aligned}
\int_{B_{\frac{r}{2}}} |\nabla(v-u)|^2\,dx &= \frac{1}{\alpha_i}\sum_{k\neq i}(\alpha_i-\alpha_k) \int_{B_{\frac{r}{2}}\cap\mathcal{E}(k)} \langle \nabla u, \nabla\phi\rangle\,dx \\
&\leqslant \frac{\alpha}{\alpha_i}\sum_{k\neq i} \int_{B_{\frac{r}{2}}\cap\mathcal{E}(k)} |\langle \nabla u, \nabla\phi\rangle|\,dx \\
&\leqslant \frac{\alpha}{\alpha_i}\left(\sum_{k\neq i} \int_{B_{\frac{r}{2}}\cap\mathcal{E}(k)} |\nabla u|^2\,dx\right)^{\frac{1}{2}} \left(\int_{B_{\frac{r}{2}}} |\nabla(v-u)|^2\,dx\right)^{\frac{1}{2}},
\end{aligned}$$

where we have denoted $\alpha := \max_k |\alpha_i - \alpha_k|$. Thus we infer that

$$\int_{B_{\tau r}} |\nabla(v-u)|^2\,dx \leqslant \left(\frac{\alpha}{\alpha_i}\right)^2 \sum_{k\neq i} \int_{B_{\frac{r}{2}}\cap\mathcal{E}(k)} |\nabla u|^2\,dx.$$

By the higher integrability for quadratic functional, (see for example [16, Lemma 2.2], where an explicit calculation of the constants is provided)

$$\fint_{B_{\frac{r}{2}}} |\nabla u|^{2p} dx \leqslant C\Big(\fint_{B_r} |\nabla u|^2 dx\Big)^p,$$

for some p and C both depending on n, α. Then using Hölder inequality, we deduce

$$\int_{B_{\tau r}} |\nabla(v-u)|^2\,dx \leqslant C^{\frac{1}{p}}\Big(\frac{\alpha}{\alpha_i}\Big)^2 \sum_{k\neq i}\Big(\frac{|\mathcal{E}(k)\cap B_r|}{|B_r|}\Big)^{1-\frac{1}{p}} \int_{B_r} |\nabla u|^2\,dx.$$

We choose ε_1 such that $\varepsilon_1^{1-\frac{1}{p}} = \tau^n$. Since v is harmonic, we finally get

$$\begin{aligned}\int_{B_{\tau r}} |\nabla u|^2\,dx &\leqslant 2\int_{B_{\tau r}} |\nabla(v-u)|^2\,dx + 2\int_{B_{\tau r}} |\nabla v|^2\,dx\\ &\leqslant C^{\frac{1}{p}}\Big(\frac{\alpha}{\alpha_i}\Big)^2 \tau^n \int_{B_r} |\nabla u|^2\,dx + 2^{n+1}\tau^n \int_{B_r} |\nabla v|^2\,dx\\ &\leqslant C(n,N,\alpha,\alpha_i)\tau^n \int_{B_r} |\nabla u|^2\,dx,\end{aligned}$$

where we have also used the minimality of v.

We are left with case (ii). We denote by v the minimizer of the energy

$$\int_{B_{\frac{r}{2}}} (\alpha_i \mathbb{1}_H + \alpha_j \mathbb{1}_{B_{\frac{r}{2}}\setminus H})|\nabla v|^2\,dx,$$

with the condition $v = u$ on $B_r \setminus B_{\frac{r}{2}}$. Let $\phi \in H_0^1(B_{\frac{r}{2}})$. It holds that

$$\alpha_i \int_{B_{\frac{r}{2}}\cap H} \langle \nabla v, \nabla \phi\rangle\,dx + \alpha_j \int_{B_{\frac{r}{2}}\setminus H} \langle \nabla v, \nabla\phi\rangle\,dx = 0. \tag{33}$$

Now we rewrite the equation (32).

$$\alpha_i \int_{B_{\frac{r}{2}}\cap \mathcal{E}(i)} \langle \nabla u, \nabla\phi\rangle\,dx + \alpha_j \int_{B_{\frac{r}{2}}\cap\mathcal{E}(j)} \langle \nabla u, \nabla\phi\rangle\,dx + \sum_{k\neq i,j} \alpha_k \int_{B_{\frac{r}{2}}\cap\mathcal{E}(k)} \langle\nabla u, \nabla\phi\rangle\,dx = 0. \tag{34}$$

We decompose

$$\begin{aligned}&\alpha_i \int_{B_{\frac{r}{2}}\cap\mathcal{E}(i)} \langle\nabla u,\nabla\phi\rangle\,dx = \alpha_i \int_{B_{\frac{r}{2}}\cap\mathcal{E}(i)\cap H} \langle\nabla u,\nabla\phi\rangle\,dx + \alpha_i \int_{B_{\frac{r}{2}}\cap\mathcal{E}(i)\setminus H} \langle\nabla u,\nabla\phi\rangle\,dx\\ &= \alpha_i \int_{B_{\frac{r}{2}}\cap H} \langle\nabla u,\nabla\phi\rangle\,dx + \alpha_i\Big(\int_{B_{\frac{r}{2}}\cap\mathcal{E}(i)\setminus H} \langle\nabla u,\nabla\phi\rangle\,dx\\ &- \int_{B_{\frac{r}{2}}\cap\mathcal{E}(j)\cap H} \langle\nabla u,\nabla\phi\rangle\,dx - \sum_{k\neq i,j} \int_{B_{\frac{r}{2}}\cap\mathcal{E}(k)\cap H} \langle\nabla u,\nabla\phi\rangle\,dx\Big)\end{aligned} \tag{35}$$

and similarly

$$\begin{aligned}\alpha_j \int_{B_{\frac{r}{2}}\cap\mathcal{E}(j)} \langle\nabla u,\nabla\phi\rangle\,dx &= \alpha_j \int_{B_{\frac{r}{2}}\setminus H} \langle\nabla u,\nabla\phi\rangle\,dx + \alpha_j\Big(\int_{B_{\frac{r}{2}}\cap\mathcal{E}(j)\cap H} \langle\nabla u,\nabla\phi\rangle\,dx\\ &- \int_{B_{\frac{r}{2}}\cap\mathcal{E}(i)\setminus H} \langle\nabla u,\nabla\phi\rangle\,dx - \sum_{k\neq i,j} \int_{B_{\frac{r}{2}}\cap\mathcal{E}(k)\setminus H} \langle\nabla u,\nabla\phi\rangle\,dx\Big).\end{aligned} \tag{36}$$

Inserting (35) and (36) in (34), we deduce that

$$
\begin{aligned}
&\alpha_i \int_{B_{\frac{r}{2}} \cap H} \langle \nabla u, \nabla \phi \rangle \, dx + \alpha_j \int_{B_{\frac{r}{2}} \backslash H} \langle \nabla u, \nabla \phi \rangle \, dx \\
&= (\alpha_i - \alpha_j) \int_{B_{\frac{r}{2}} \cap \mathcal{E}(j) \cap H} \langle \nabla u, \nabla \phi \rangle \, dx + (\alpha_j - \alpha_i) \int_{B_{\frac{r}{2}} \cap \mathcal{E}(i) \backslash H} \langle \nabla u, \nabla \phi \rangle \, dx \\
&+ \sum_{k \neq i,j} \left(\alpha_i \int_{B_{\frac{r}{2}} \cap \mathcal{E}(k) \cap H} \langle \nabla u, \nabla \phi \rangle \, dx + \alpha_j \int_{B_{\frac{r}{2}} \cap \mathcal{E}(k) \backslash H} \langle \nabla u, \nabla \phi \rangle \, dx - \alpha_k \int_{B_{\frac{r}{2}} \cap \mathcal{E}(k)} \langle \nabla u, \nabla \phi \rangle \, dx \right).
\end{aligned}
$$

Choosing $\phi := v - u$, subtracting (33) from the previous equation and applying Hölder's inequality, we get

$$
\begin{aligned}
&\min\{\alpha_i, \alpha_j\} \int_{B_{\frac{r}{2}}} |\nabla(u-v)|^2 \, dx \leqslant \max_{k=1\ldots N} \alpha_k \Bigg[\int_{B_{\frac{r}{2}} \cap \mathcal{E}(j) \cap H} |\nabla u|^2 \, dx + \int_{B_{\frac{r}{2}} \cap \mathcal{E}(i) \backslash H} |\nabla u|^2 \, dx \\
&+ \sum_{k \neq i,j} \left(\int_{B_{\frac{r}{2}} \cap \mathcal{E}(k) \cap H} |\nabla u|^2 \, dx + \int_{B_{\frac{r}{2}} \cap \mathcal{E}(k) \backslash H} |\nabla u|^2 \, dx + \int_{B_{\frac{r}{2}} \cap \mathcal{E}(k)} |\nabla u|^2 \, dx \right) \Bigg]^{\frac{1}{2}} \left(\int_{B_{\frac{r}{2}}} |\nabla(u-v)|^2 \, dx \right)^{\frac{1}{2}}.
\end{aligned}
$$

Thus

$$
\int_{B_{\tau r}} |\nabla(u-v)|^2 \, dx \leqslant C(\alpha) \left(\int_{B_{\frac{r}{2}} \cap (\mathcal{E}(j) \cap H)} |\nabla u|^2 \, dx + \int_{B_{\frac{r}{2}} \cap (\mathcal{E}(i) \backslash H)} |\nabla u|^2 \, dx + \sum_{k \neq i,j} \int_{B_{\frac{r}{2}} \cap \mathcal{E}(k)} |\nabla u|^2 \, dx \right).
$$

Arguing as above, the higher integrability of ∇u and Hölder's inequality yield

$$
\int_{B_{\frac{r}{2}} \cap (\mathcal{E}(j) \cap H)} |\nabla u|^2 \, dx \leqslant C^{\frac{1}{p}} \left(\frac{|(\mathcal{E}(j) \cap H) \cap B_{\frac{r}{2}}|}{|B_{\frac{r}{2}}|} \right)^{1-\frac{1}{p}} \int_{B_r} |\nabla u|^2 \, dx,
$$

$$
\int_{B_{\frac{r}{2}} \cap (\mathcal{E}(i) \backslash H)} |\nabla u|^2 \, dx \leqslant C^{\frac{1}{p}} \left(\frac{|(\mathcal{E}(i) \backslash H) \cap B_{\frac{r}{2}}|}{|B_{\frac{r}{2}}|} \right)^{1-\frac{1}{p}} \int_{B_r} |\nabla u|^2 \, dx,
$$

$$
\int_{B_{\frac{r}{2}} \cap \mathcal{E}(k)} |\nabla u|^2 \, dx \leqslant C^{\frac{1}{p}} \left(\frac{|\mathcal{E}(k) \cap B_{\frac{r}{2}}|}{|B_{\frac{r}{2}}|} \right)^{1-\frac{1}{p}} \int_{B_r} |\nabla u|^2 \, dx, \quad \forall k \neq i,j.
$$

Choosing ε_1 such that $N\varepsilon_1^{1-\frac{1}{p}} = \tau^n$ and making the same computations as before we obtain the thesis. □

Theorem 3.2 (Density upper bound)**.** *Let $(\mathcal{E}, u)$ be a Λ-minimizer of the functional $\mathcal{F}$ defined in* (2)*. Then, for any open set $U \Subset \Omega$ there exists a positive constant $C_2 = C_2(n, N, \Lambda, \alpha)$ such that*

$$
\mathcal{F}(\mathcal{E}, u; B_r(x_0)) \leqslant C_2 r^{n-1}, \tag{37}
$$

for any $B_r(x_0) \subset U$.

Proof. Let $B_r(x_0) \subset U \Subset \Omega$. Let $i \in \{1, \ldots, N\}$ such that $\alpha_i := \min_{j \in \{1,\ldots,N\}} \alpha_j$. We may assume that $x_0 = 0$ and $i = 1$. We define:

$$
\begin{aligned}
&\mathcal{F}(1) := \mathcal{E}(1) \cup B_r, \\
&\mathcal{F}(h) := \mathcal{E}(h) \backslash B_r, \quad \forall h \in \{2, \ldots, N\}.
\end{aligned}
$$

It holds that

$$
P(\mathcal{F}(1); \Omega) = P(\mathcal{E}(1) \backslash \overline{B_r}; \Omega) + \mathcal{H}^{n-1}(\partial B_r \backslash \mathcal{E}(1)),
$$

$$\sum_{h=2}^{N} P(\mathcal{F}(h);\Omega) = \sum_{h=2}^{N}[P(\mathcal{E}(h)\backslash\overline{B_r};\Omega) + \mathcal{H}^{n-1}(\partial B_r \cap \mathcal{E}(h))] = \sum_{h=2}^{N} P(\mathcal{E}(h)\backslash\overline{B_r};\Omega) + \mathcal{H}^{n-1}(\partial B_r\backslash\mathcal{E}(1)).$$

Using the previous equalities, the minimality of $(\mathcal{E}, u)$ with respect to $(\mathcal{F}, u)$ yields

$$\sum_{k=1}^{N} \alpha_k \int_{B_r\cap\mathcal{E}(k)} |\nabla u|^2\,dx + \frac{1}{2}\sum_{k=1}^{N} P(\mathcal{E}(k);B_r) \leqslant \alpha_1 \int_{B_r} |\nabla u|^2\,dx + \mathcal{H}^{n-1}(\partial B_r\backslash\mathcal{E}(1)) + 2\Lambda|B_r|.$$

It follows that

$$\delta\alpha \int_{B_r\backslash\mathcal{E}(1)} |\nabla u|^2\,dx + \frac{1}{2}\sum_{k=1}^{N} P(\mathcal{E}(k);B_r) \leqslant c(n,\Lambda)r^{n-1}.$$

where $\delta\alpha = \min_{k=2,\dots N}(\alpha_k - \alpha_1)$. We deduce that

$$\int_{B_r\backslash\mathcal{E}(1)} |\nabla u|^2\,dx + \sum_{k=1}^{N} P(\mathcal{E}(k);B_r) \leqslant c(n,\Lambda,\alpha)r^{n-1}. \tag{38}$$

This concludes the proof of (37) in $B_r\backslash\mathcal{E}(1)$.

It remains to prove that the Dirichlet integral decays in the right way also in $B_r \cap \mathcal{E}(1)$. This can be proved by contradiction using a quite standard blow-up argument that we present for the reader's convenience.

We prove that there exist $\tau \in \left(0, \frac{1}{2}\right)$ and $M > 0$ such that for every $\delta \in (0,1)$ there exists $h_0 \in \mathbb{N}$ such that, for any $B_r(x_0) \subset U$, we have

$$\int_{B_r(x_0)} |\nabla u|^2 \leqslant h_0 r^{n-1} \quad \text{or} \quad \int_{B_{\tau r}(x_0)} |\nabla u|^2\,dx \leqslant M\tau^{n-\delta}\int_{B_r(x_0)} |\nabla u|^2\,dx. \tag{39}$$

Arguing by contradiction, for every $\tau \in \left(0, \frac{1}{2}\right)$ and $M > 0$ there exists $\delta \in (0,1)$ such that for every $h \in \mathbb{N}$ there exists a ball $B_{r_h}(x_h) \subset U$ such that

$$\int_{B_{r_h}(x_h)} |\nabla u|^2\,dx > h r_h^{n-1} \tag{40}$$

and

$$\int_{B_{\tau r_h}(x_h)} |\nabla u|^2\,dx > M\tau^{n-\delta}\int_{B_{r_h}(x_h)} |\nabla u|^2\,dx. \tag{41}$$

We choose $M > 1$. Note that estimates (38) and (40) yield

$$\sum_{k=2}^{N} \int_{B_{r_h}(x_h)\cap\mathcal{E}(k)} |\nabla u|^2\,dx + P(\mathcal{E}(k);B_{r_h}(x_h)) \leqslant c_0 r_h^{n-1} < \frac{c_0}{h}\int_{B_{r_h}(x_h)} |\nabla u|^2\,dx, \tag{42}$$

and so

$$\sum_{k=2}^{N} \int_{B_{r_h}(x_h)\cap\mathcal{E}(k)} |\nabla u|^2\,dx < \frac{c_0}{h}\int_{B_{r_h}(x_h)} |\nabla u|^2\,dx, \tag{43}$$

for some positive constant c_0.

Now we employ a typical blow-up argument. We set

$$\varsigma_h^2 := \fint_{B_{r_h}(x_h)} |\nabla u|^2\,dx$$

and, for $y \in B_1$, we introduce the sequence of rescaled functions defined as

$$v_h(y) := \frac{u(x_h + r_h y) - a_h}{\varsigma_h r_h}, \quad \text{with} \quad a_h := \fint_{B_{r_h}(x_h)} u\,dx.$$

We have $\nabla u(x_h + r_h y) = \varsigma_h \nabla v_h(y)$ and a change of variable yields

$$\fint_{B_1} |\nabla v_h(y)|^2 \, dy = \frac{1}{\varsigma_h^2} \fint_{B_{r_h}(x_h)} |\nabla u(x)|^2 \, dx = 1.$$

Therefore, there exist a (not relabeled) subsequence of $\{v_h\}_{h\in\mathbb{N}}$ and $v \in H^1(B_1)$ such that $v_h \rightharpoonup v$ in $H^1(B_1)$ and $v_h \to v$ in $L^2(B_1)$. Moreover, the semicontinuity of the norm implies

$$\fint_{B_1} |\nabla v(y)|^2 \, dy \leqslant \liminf_{h\to\infty} \fint_{B_1} |\nabla v_h(y)|^2 \, dy = 1.$$

Let us define the sets

$$\mathcal{E}_h^*(k) := \frac{\mathcal{E}(k) - x_h}{r_h} \cap B_1.$$

We rewrite the inequalities (40), (41) and (43). They become, respectively,

$$\varsigma_h^2 > \frac{h}{r_h},$$

$$\fint_{B_\tau} |\nabla v_h(y)|^2 \, dy > M\tau^{-\delta}, \tag{44}$$

$$\sum_{k=2}^{N} \int_{B_1\cap\mathcal{E}^*(k)} |\nabla v_h(y)|^2 \, dy < \frac{c_0}{h} \int_{B_1} |\nabla v_h(y)|^2 \, dy = \frac{c_0\omega_n}{h}. \tag{45}$$

To achieve the desired contradiction, it remains to show that the sequence v_h cannot fulfill (44) e (45) because its limit v minimizes the Dirichlet functional. Nevertheless, to establish a connection between the decay properties of ∇v and ∇v_h we must prove that the L^2-norm of v_h converges to the L^2-norm of v. Observe that (44) implies that $\varsigma_h \to \infty$, as $h \to \infty$.

Since $r_h^{n-1} P(\mathcal{E}_h^*(k); B_1) = P(\mathcal{E}(k); B_{r_h}(x_h))$, by (42), we have that the sequence $\{P(\mathcal{E}_h^*(k); B_1)\}_{h\in\mathbb{N}}$ is bounded for any $k \in \{2, \ldots, N\}$. Therefore, up to a not relabeled subsequence, $\mathbb{1}_{\mathcal{E}_h^*(k)} \to \mathbb{1}_{\mathcal{E}^*(k)}$ in $L^1(B_1)$, for some set $\mathcal{E}^*(k) \subset B_1$ of locally finite perimeter. By semicontinuity we deduce that

$$\begin{aligned} \int_{B_1} \mathbb{1}_{\mathcal{E}^*(k)} |\nabla v|^2 \, dy &\leqslant \liminf_{h\to\infty} \int_{B_1} \mathbb{1}_{\mathcal{E}^*(k)} |\nabla v_h|^2 \, dy \\ &\leqslant \liminf_{h\to\infty} \Big(\int_{B_1} \mathbb{1}_{\mathcal{E}_h^*(k)} |\nabla v_h|^2 dy + \int_{B_1} \mathbb{1}_{\mathcal{E}^*(k)\setminus\mathcal{E}_h^*(k)} |\nabla v_h|^2 dy \Big) = 0, \end{aligned} \tag{46}$$

for any $k \in \{2, \ldots, N\}$, where we used (45) and the equi-integrability of $\big(|\nabla v_h|^2\big)_{h\in\mathbb{N}}$.

By Λ-minimality of $(\mathcal{E}, u)$ with respect to $(\mathcal{E}, u + \phi)$ we get, for $\phi \in H_0^1(B_{r_h}(x_h))$,

$$\sum_{k=1}^{N} \alpha_k \int_{B_{r_h}(x_h)\cap\mathcal{E}(k)} |\nabla u|^2 \, dx \leqslant \sum_{k=1}^{N} \alpha_k \int_{B_{r_h}(x_h)\cap\mathcal{E}(k)} |\nabla u + \nabla\phi|^2 \, dx$$

Using the change of variable $x = x_h + r_h y$, we deduce for every $\psi \in H_0^1(B_1)$

$$\sum_{k=1}^{N} \alpha_k \int_{B_1\cap\mathcal{E}_h^*(k)} |\varsigma_h \nabla v_h|^2 \, dx \leqslant \sum_{k=1}^{N} \alpha_k \int_{B_1\cap\mathcal{E}_h^*(k)} |\varsigma_h \nabla v_h + \nabla\psi|^2 \, dx.$$

Let $\eta \in C_c^\infty(B_1)$ such that $0 \leqslant \eta \leqslant 1$. Choosing as a test function $\psi_h = \varsigma_h \eta(v - v_h)$, we deduce that

$$\begin{aligned} \int_{B_1\cap\mathcal{E}_h^*(1)} \eta |\nabla v_h|^2 \, dx &\leqslant \int_{B_1\cap\mathcal{E}_h^*(1)} \eta |\nabla v|^2 \, dx \\ &+ C(\alpha) \Bigg[\sum_{k=2}^{N} \int_{B_1\cap\mathcal{E}_h^*(k)} \Big(|\nabla v_h|^2 + |\nabla v|^2 \Big) \, dx + \int_{B_1} |\nabla\eta||v - v_h| \, dx \Bigg]. \end{aligned}$$

By (45), (46) and the strong convergence of v_h to v in L^2 we deduce that the last term in the previous inequality tends to zero as $h \to \infty$, thus obtaining

$$\limsup_{h\to+\infty} \int_{B_1} \eta |\nabla v_h|^2\, dx \leqslant \int_{B_1} \eta |\nabla v|^2\, dx.$$

By the arbitrariness of η and lower semicontinuity we conclude that $\nabla v_h \to \nabla v$ in L^2 and v is harmonic. Thus, since $M > 1$, by the harmonicity of v and (44) we get

$$\fint_{B_\tau} |\nabla v|^2\, dy \leqslant \fint_{B_1} |\nabla v|^2\, dy \leqslant 1 \leqslant \frac{\tau^\delta}{M} \lim_{h\to\infty} \fint_{B_\tau} |\nabla v_h|^2\, dy < \lim_{h\to\infty} \fint_{B_\tau} |\nabla v_h|^2\, dy = \fint_{B_\tau} |\nabla v|^2\, dy,$$

which is a contradiction. Once (39) is in force, the conclusion follows in a standard manner by applying an iteration argument.

□

We are now in position to prove that in the neighborhoods where the perimeters of the chambers are small, the energy $\mathcal{F}$ decays in an appropriate manner.

Proposition 3.3. *Let $(\mathcal{E}, u)$ be a Λ-minimizer of the functional $\mathcal{F}$ defined in* (2)*. Then, for any $\tau \in (0,1)$ there exists $\varepsilon_2 = \varepsilon_2(\tau) > 0$ such that, if for $B_r(x_0) \subset \Omega$*

$$P(\mathcal{E}(k); B_r(x_0)) < \varepsilon_2 r^{n-1}, \quad \forall k \in \{1, \dots, N\}, \tag{47}$$

then

$$\mathcal{F}(\mathcal{E}, u; B_{\tau r}(x_0)) \leqslant C_3 \tau^n (\mathcal{F}(\mathcal{E}, u; B_r(x_0)) + \Lambda r^n),$$

for some positive constant $C_3 = C_3(n, N, \Lambda, \alpha)$.

Proof. Let $B_r(x_0) \subset \Omega$ such that (47) holds and $\tau \in (0,1)$. Setting

$$\mathcal{E}_r(k) := \frac{\mathcal{E}(k) - x_0}{r}, \quad \mathcal{E}_r := \{\mathcal{E}_r(k)\}_{k=1}^N, \quad u_r(y) := r^{-\frac{1}{2}} u(x_0 + ry), \quad \forall y \in B_1,$$

scaling (47) we obtain

$$P(\mathcal{E}_r(k); B_1) < \varepsilon_2, \quad \forall k \in \{1, \dots, N\}.$$

We need to prove that

$$\mathcal{F}(\mathcal{E}_r, u_r; B_\tau) \leqslant \tau^n (\mathcal{F}(\mathcal{E}_r, u_r; B_1) + \Lambda r).$$

In what follows, for simplicity of notation, we will still denote $\mathcal{E}_r(k)$ by $\mathcal{E}(k)$ and u_r by u and we explicitly observe that $(\mathcal{E}(k), u)$ is a Λr minimizer of $\mathcal{F}$.

We choose $\varepsilon_2 < \min\left\{ \left(\frac{\omega_n}{2c_{IP}N}\right)^{\frac{n-1}{n}}, \tau^{(n+1)(n-1)}, \left(\frac{\omega_n}{c_{IP}}\varepsilon_1\right)^{\frac{n}{n-1}} \right\}$, where $c_{IP} = c_{IP}(n)$ is the constant of the relative isoperimetric inequality and $\varepsilon_1 = \varepsilon_1(\tau)$ is as in Lemma 3.1. We first show that there exists $i \in \{1, \dots, N\}$ such that

$$|B_1 \backslash \mathcal{E}(i)| \leqslant c_{IP} P(\mathcal{E}(i); B_1)^{\frac{n}{n-1}}. \tag{48}$$

Let us assume by contradiction that the previous inequality is false for any $i \in \{1, \dots, N\}$. By the isoperimetric inequality, we have that

$$|\mathcal{E}(k) \cap B_1| \leqslant c_{IP} P(\mathcal{E}(k); B_1)^{\frac{n}{n-1}}, \quad \forall k \in \{1, \dots, N\}.$$

Thus, by the choice of ε_2, we get the following contradiction:

$$|B_1| = \sum_{k=1}^N |\mathcal{E}(k) \cap B_1| \leqslant c_{IP} \sum_{k=1}^N P(\mathcal{E}(k); B_1)^{\frac{n}{n-1}} \leqslant c_{IP} N \varepsilon_2^{\frac{n}{n-1}} < \frac{\omega_n}{2}.$$

Therefore, (48) is proved. We may assume that $i = 1$ for simplicity. As a consequence we have:

$$\sum_{k=2}^{N} |\mathcal{E}(k) \cap B_1| = |B_1| - |\mathcal{E}(1) \cap B_1| = |B_1 \backslash \mathcal{E}(1)| \leqslant c_{IP} P(\mathcal{E}(1); B_1)^{\frac{n}{n-1}} < \frac{\omega_n}{2}.$$

The isoperimetric inequality yields

$$|\mathcal{E}(k) \cap B_1| \leqslant c_{IP} P(\mathcal{E}(k); B_1)^{\frac{n}{n-1}}, \quad \forall k \in \{2, \dots, N\}.$$

Since

$$c_{IP} P(\mathcal{E}(1); B_1)^{\frac{n}{n-1}} \geqslant |B_1 \backslash \mathcal{E}(1)| \geqslant \int_{\tau}^{2\tau} \mathcal{H}^{n-1}(\partial B_\rho \backslash \mathcal{E}(1))\, d\rho,$$

we can choose $\rho \in (\tau, 2\tau)$ such that

$$\mathcal{H}^{n-1}(\partial B_\rho \backslash \mathcal{E}(1)) \leqslant \frac{c_{IP}}{\tau} \varepsilon_2^{\frac{1}{n-1}} P(\mathcal{E}(1); B_1),$$

$$\mathcal{H}^{n-1}(\partial^* \mathcal{E}(h) \cap \partial B_\rho) = 0, \quad \forall h \in \{1, \dots, N\}.$$

We set

$$\begin{aligned} &\mathcal{G}(1) := \mathcal{E}(1) \cup B_\rho, \\ &\mathcal{G}(k) := \mathcal{E}(h) \backslash B_\rho, \quad \forall k \in \{2, \dots, N\}. \end{aligned}$$

We remark that, $(\mathcal{G}, u)$ is an admissible test pair to test the minimality of $(\mathcal{E}, u)$ in B_1 because $\mathcal{E}(k) = \mathcal{G}(k)$ outside of B_ρ for any $k \in \{1, \dots, N\}$. Thus

$$\mathcal{F}(\mathcal{E}, u; B_1) \leqslant \mathcal{F}(\mathcal{G}, u; B_1) + \Lambda r \rho^n. \tag{49}$$

To eliminate the common contribution in $B_1 \backslash \overline{B}_\rho$ in the previous equation we use the following equalities for the perimeter term:

$$P(\mathcal{E}(k); B_1) = P(\mathcal{E}(k); B_\rho) + P(\mathcal{E}(k); B_1 \backslash \overline{B}_\rho),$$

$$P(\mathcal{G}(1); B_1) = P(\mathcal{E}(1); B_1 \backslash \overline{B}_\rho) + \mathcal{H}^{n-1}(\partial B_\rho \backslash \mathcal{E}(1)),$$

$$\begin{aligned} \sum_{h=2}^{N} P(\mathcal{G}(h); B_1) &= \sum_{h=2}^{N} [P(\mathcal{E}(h); B_1 \backslash \overline{B}_\rho) + \mathcal{H}^{n-1}(\partial B\rho \cap \mathcal{E}(h))] \\ &= \sum_{h=2}^{N} P(\mathcal{E}(h); B_1 \backslash \overline{B}_\rho) + \mathcal{H}^{n-1}(\partial B_\rho \backslash \mathcal{E}(1)). \end{aligned}$$

Moreover we observe that the choice of ε_2 implies that we are in position to apply Lemma 3.1, since

$$|\mathcal{E}(h) \cap B_1| \leqslant c_{IP} P(\mathcal{E}(h); B_1)^{\frac{n-1}{n}} \leqslant c_{IP} \varepsilon_2^{\frac{n-1}{n}} \leqslant \omega_n \varepsilon_1, \quad \forall h \in \{2, \dots, N\}.$$

Deleting the common term in (49) and applying Lemma 3.1 we conclude:

$$\begin{aligned} \mathcal{F}(\mathcal{E}, u; B_\tau) &\leqslant \mathcal{F}(\mathcal{E}, u; B_\rho) \\ &\leqslant \alpha_1 \int_{B_\rho} |\nabla u|^2\, dx + \mathcal{H}^{n-1}(\partial B_\rho \backslash \mathcal{E}(1)) + c(n) \Lambda r \rho^n \\ &\leqslant \alpha_1 \int_{B_{2\tau}} |\nabla u|^2\, dx + \frac{c_{IP}}{\tau} \varepsilon_2^{\frac{1}{n-1}} P(\mathcal{E}(1); B_1) + c(n) \Lambda r \tau^n \\ &\leqslant c(n, N, \alpha) \tau^n \int_{B_1} |\nabla u|^2\, dx + c(n) \tau^n P(\mathcal{E}(1); B_1) + c(n) \Lambda r \tau^n \\ &\leqslant c(n, N, \alpha) \tau^n (\mathcal{F}(\mathcal{E}, u; B_1) + \Lambda r). \end{aligned}$$

□

Building upon the previous proposition, we proceed to establish the following theorem, which states a lower bound estimate for the perimeter of the interfaces of optimal chambers.

Theorem 3.4 (Density lower bound). *Let $(\mathcal{E}, u)$ be a Λ-minimizer of the functional $\mathcal{F}$ defined in (2) and $U \Subset \Omega$ be an open set. Then, there exists a positive constant $C_4 = C_4(n, N, \Lambda, \alpha)$ such that, for every $x_0 \in \bigcup_{k=1}^N \partial\mathcal{E}(k) \cap \Omega$ ad $B_r(x_0) \subset U$, it holds*

$$\sum_{k=1}^{N} P(\mathcal{E}(k); B_r(x_0)) \geqslant C_4 r^{n-1}. \tag{50}$$

Moreover, $\mathcal{H}^{n-1}\Big(\Omega \cap \bigcup_{k=1}^N \partial\mathcal{E}(k) \backslash \bigcup_{k=1}^N \partial^*\mathcal{E}(k)\Big) = 0$.

Proof. Since $\overline{\partial^*\mathcal{E}(k)} = \partial\mathcal{E}(k)$, for any $k \in \{1, \ldots, N\}$, it is not restrictive to set $x_0 = 0 \in \bigcup_{k=1}^N \partial^*\mathcal{E}(k) \cap \Omega$. Fix $\tau \in (0,1)$ such that $2C_3\tau^{1/2} < 1$, and fix $\sigma \in (0,1)$ such that

$$2C_3C_2\sigma < \frac{\varepsilon_2(\tau)}{2}, \tag{51}$$

and let r_0 be such that

$$\Lambda r_0 < \min\{\varepsilon_2(\tau), C_2\}, \tag{52}$$

where C_2, C_3 and ε_2 are the constants from Theorem 3.2 and Proposition 3.3. Assume by contradiction that for some $B_r \subset U$ with $r < r_0$, we have

$$\sum_{k=1}^{N} P(\mathcal{E}(k); B_r(x_0)) \leqslant \varepsilon_2(\sigma) r^{n-1}. \tag{53}$$

By induction, it is straightforward to show that

$$\mathcal{F}(\mathcal{E}, u; B_{\sigma\tau^h r}) \leqslant \varepsilon_2(\tau)\tau^{\frac{h}{2}}\left(\sigma\tau^h r\right)^{n-1}, \tag{54}$$

for every $h \geqslant 0$. Indeed, for $h = 0$, using Proposition 3.3 and Theorem 3.2 and conditions (51), (52) and (53)

$$\mathcal{F}(\mathcal{E}, u; B_{\sigma r}) \leqslant C_3\sigma^n(C_2 r^{n-1} + \Lambda r^n) \leqslant 2C_3C_2\sigma(\sigma r)^{n-1} \leqslant \varepsilon_2(\tau)(\sigma r)^{n-1}.$$

Assuming that (54) holds for some h, to prove it also holds $h+1$ it suffices to apply Proposition 3.3 again and observe that $2C_2\tau^{1/2} < 1$, together with (52). Indeed, we get

$$\begin{aligned}
\mathcal{F}(\mathcal{E}, u; B_{\sigma\tau^{h+1}r}) &\leqslant C_3\tau^n\left[\varepsilon_2(\tau)\tau^{\frac{h}{2}}\left(\sigma\tau^h r\right)^{n-1} + \Lambda(\sigma\tau^h r)^n\right] \\
&\leqslant C_3\varepsilon_2(\tau)\tau^n\left[\tau^{\frac{h}{2}}\left(\sigma\tau^h r\right)^{n-1} + \sigma\tau^h\left(\sigma\tau^h r\right)^{n-1}\right] \\
&\leqslant \varepsilon_2(\tau)2C_3\tau^{\frac{1}{2}}\tau^{\frac{1}{2}}\tau^{n-1}\tau^{\frac{h}{2}}\left(\sigma\tau^h r\right)^{n-1} \leqslant \varepsilon_2(\tau)\tau^{\frac{h+1}{2}}\left(\sigma\tau^{h+1}r\right)^{n-1}
\end{aligned}$$

It follows that

$$\frac{1}{2}\sum_{k=1}^{N} P(\mathcal{E}(k); B_{\sigma\tau^h r}) \leqslant \varepsilon_2(\tau)\tau^h\left(\sigma\tau^h r\right)^{n-1}.$$

Finally, it holds:

$$\limsup_{\rho\to 0^+}\frac{\sum_{k=1}^N P(\mathcal{E}(k); B_\rho)}{\rho^{n-1}} = \limsup_{h\to+\infty}\frac{\sum_{k=1}^N P(\mathcal{E}(k); B_{\sigma\tau^h r})}{\left(\sigma\tau^h r\right)^{n-1}} \leqslant 2\lim_{h\to+\infty}\varepsilon_2(\tau)\tau^h = 0,$$

which is a contradiction.

We are left to prove that

$$\mathcal{H}^{n-1}\Big(\Omega \cap \bigcup_{k=1}^N \partial\mathcal{E}(k) \backslash \bigcup_{k=1}^N \partial^*\mathcal{E}(k)\Big) = 0. \tag{55}$$

By the lower bound (50), we get

$$\limsup_{r\to 0^+} \frac{\mathcal{H}^{n-1}\big(\big(\bigcup_{k=1}^N \partial^*\mathcal{E}(k)\big)\cap B_r(x)\big)}{r^{n-1}}$$
$$= \limsup_{r\to 0^+} \sum_{k=1}^N \frac{\mathcal{H}^{n-1}(\partial^*\mathcal{E}(k)\cap B_r(x))}{r^{n-1}} = \limsup_{r\to 0^+} \sum_{k=1}^N \frac{P(\mathcal{E}(k);B_r(x))}{r^{n-1}} > 0, \quad \forall x \in \bigcup_{k=1}^N \partial\mathcal{E}(k)\cap\Omega.$$

On the other hand, by density property of $\mathcal{H}^{n-1}$-measurable sets with finite measure, see [3, (2.42)],

$$\limsup_{r\to 0^+} \frac{\mathcal{H}^{n-1}\big(\big(\bigcup_{k=1}^N \partial^*\mathcal{E}(k)\big)\cap B_r(x)\big)}{r^{n-1}} = 0, \quad \text{for } \mathcal{H}^{n-1}\text{-a.e. } x \notin \bigcup_{k=1}^N \partial^*\mathcal{E}(k)\cap\Omega,$$

thus (55) follows. □

3.1 Conclusion

Theorem 1.1 summarizes the results obtained in Theorems 3.2 and 3.4, and is a direct consequence of them. We emphasize that the result concerns the union of the interfaces, but does not necessarily imply any regularity for the boundary of a single chamber. We believe that such a result could be achieved provided a suitable infiltration lemma were available. Specifically, if certain chambers of a minimizing partition occupy most of the ball $B_{2r}(x)$, then they must entirely fill the smaller ball $B_r(x)$ (cf. [25, Lemma 30.2] in the case of clusters). Establishing such a lemma, however, appears to be non-trivial due to the lack of sufficiently strong decay estimates for the Dirichlet bulk energy. However, its validity would represent a major breakthrough, potentially leading to more significant geometric insights into the chambers and, hopefully, to the regularity of the interface.

Acknowledgements. The authors are members of the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM). The authors wish to acknowledge financial support from INdAM GNAMPA Project 2024 "Regolarità per problemi a frontiera libera e disuguaglianze funzionali in contesto finsleriano". The work of GP was partially supported by the project "Harmony" (code MATE.IP.DR111.2024.HARMONY) within the program of the University "Luigi Vanvitelli".

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