THE SUPERPOSITION PRINCIPLE FOR THE CONTINUITY EQUATION WITH SINGULAR FLUX

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Abstract. Representation results for absolutely continuous curves $\mu\colon [0,T] \to \mathcal{P}_p(\mathbb{R}^d)$, $p > 1$, with values in the Wasserstein space $(\mathcal{P}_p(\mathbb{R}^d), W_p)$ of Borel probability measures in $\mathbb{R}^d$ with finite p-moment, provide a crucial tool to study evolutionary PDEs in a measure-theoretic setting. They are strictly related to the superposition principle for measure-valued solutions to the continuity equation.

This paper addresses the extension of these results to the case $p = 1$, *and* to curves $\mu\colon [0,+\infty) \to \mathcal{P}_1(\mathbb{R}^d)$ that are only of *bounded variation* in time: in the corresponding continuity equation, the flux measure $\boldsymbol{\nu} \in \mathcal{M}_{\mathrm{loc}}([0,+\infty) \times \mathbb{R}^d; \mathbb{R}^d)$ thus possesses a non-trivial singular part w.r.t. μ in addition to the absolutely continuous part featuring the velocity field.

Firstly, we carefully address the relation between curves in $\mathrm{BV}_{\mathrm{loc}}([0,+\infty); \mathcal{P}_1(\mathbb{R}^d))$ and solutions to the associated continuity equation, among which we select those with *minimal* singular (contribution to the) flux $\boldsymbol{\nu}$. We show that, with those distinguished solutions it is possible to associate an 'auxiliary' continuity equation, in an *augmented* phase space, solely driven by its velocity field. For that continuity equation, a standard version of the superposition principle can be thus obtained. In this way, we derive a first probabilistic representation of the pair $(\mu, \boldsymbol{\nu})$ solutions by projection over the time and space marginals. This representation involves Lipschitz trajectories in the augmented phase space, reparametrized in time and solving the characteristic system of ODEs. Finally, for the same pair $(\mu, \boldsymbol{\nu})$ we also prove a superposition principle in terms of BV curves on the *actual* time interval, providing a fine description of their behaviour at jump points.

Keywords: Continuity equation, BV curves, superposition principle.

Contents

1. Introduction

Representation results for Lipschitz (or even absolutely continuous) curves $\mu\colon [0,T]\to \mathcal{P}_p(\mathbb{R}^d)$, $p>1$, with values in the Wasserstein space $(\mathcal{P}_p(\mathbb{R}^d), W_p)$ of Borel probability measures in $\mathbb{R}^d$ with finite p-moment, metrized by the L^p-Kantorovich-Rubinstein-Wasserstein distance

$$W_p(\mu_0,\mu_1) := \min\Big\{ \int \|x-y\|^p \,\mathrm{d}\gamma(x,y) : \gamma\in \mathcal{P}(\mathbb{R}^d\times\mathbb{R}^d),\ \pi^1_\sharp\gamma=\mu_0,\ \pi^2_\sharp\gamma=\mu_1\Big\}, \tag{1.1}$$

provide a crucial tool to study evolutionary PDEs and geometric problems in a measure-theoretic setting.

Such results are strictly related to the corresponding representation formulae for measure-valued solutions to the continuity equation

$$\partial_t\mu + \operatorname{div}\boldsymbol{\nu} = 0 \qquad \text{in } \mathscr{D}'((0,T)\times\mathbb{R}^d), \quad \boldsymbol{\nu} = \boldsymbol{v}\mu \ll \mu, \tag{1.2}$$

as a superposition of absolutely continuous curves $\gamma\colon [0,T]\to\mathbb{R}^d$ solving the differential equation

$$\dot\gamma(t) = \boldsymbol{v}(t,\gamma(t)) \quad \text{a.e. in } (0,T) \tag{1.3}$$

in an integral sense. In fact, a curve $(\mu_t)_{t\in[0,T]}$ in $\mathcal{P}_p(\mathbb{R}^d)$ satisfies the p-absolute continuity property

$$W_p(\mu_s,\mu_t) \le \int_s^t L(r)\,\mathrm{d}r \quad \text{for every } 0\le s<t\le T, \text{ and some } L\in L^p(0,T), \tag{1.4}$$

if and only if [11, Sec. 8.1] the space-time measure $\mu = \mathscr{L}^1\otimes\mu_t = \int_0^T \delta_t\otimes\mu_t\,\mathrm{d}t$ solves the continuity equation (1.2) for a vector measure $\boldsymbol{\nu}\ll\mu$ whose density $\boldsymbol{v} = \frac{\mathrm{d}\boldsymbol{\nu}}{\mathrm{d}\mu}$ satisfies

$$\int_{\mathbb{R}^d} \|\boldsymbol{v}_t(x)\|^p\,\mathrm{d}\mu_t(x) \le L^p(t) \quad \text{for a.a. } t\in(0,T). \tag{1.5}$$

The measure $\boldsymbol{\nu}$ and its density $\boldsymbol{v}$ comply with the minimality condition

$$\Big(\int_{\mathbb{R}^d} \|\boldsymbol{v}_t(x)\|^p\,\mathrm{d}\mu_t(x)\Big)^{1/p} = \lim_{h\to 0}\frac{W_p(\mu_t,\mu_{t+h})}{|h|} \quad \text{for a.a. } t\in(0,T), \tag{1.6}$$

and are also uniquely determined by (1.6) if the norm $\|\cdot\|$ is strictly convex. On the other hand, if $(\mu_t)_{t\in[0,T]}$ is a continuous family of probability measures in $\mathbb{R}^d$ and $(\mu,\boldsymbol{\nu})$ is a solution to the (1.2) satisfying (1.5) for some $L\in L^p(0,T)$, then by [11, Sec. 8.2] there exists a Radon probability measure $\boldsymbol{\eta}$ in $\mathrm{C}([0,T];\mathbb{R}^d)$, concentrated on the subset of absolutely continuous curves satisfying (1.3), such that $\mu_t = (\mathsf{e}_t)_\sharp\boldsymbol{\eta}$ for every $t\in[0,T]$, where $\mathsf{e}_t\colon \gamma\mapsto\gamma(t)$ is the evaluation map. Equivalently,

$$\mu_t(A) = \boldsymbol{\eta}\Big(\{\gamma\in\mathrm{C}([0,T];\mathbb{R}^d) : \gamma(t)\in A\}\Big) \quad \text{for every } t\in[0,T],\ A \text{ Borel subset of } \mathbb{R}^d. \tag{1.7}$$

The characterization of AC^p curves in $\mathcal{P}_p(\mathbb{R}^d)$ and the lifting result from [11] have been extended to the case in which $\mathbb{R}^d$ is endowed with a non-flat Riemannian distance [35], or replaced by a separable complete metric space (X,d) [34]; cf. also [36] for the extension to spaces endowed with Wasserstein-Orlicz distances.

A first typical application of the above results concerns curves of measures in $\mathcal{P}_p(\mathbb{R}^d)$ arising from a suitable approximation method, when a priori estimates provide the bound (1.4) (e.g. in gradient flows, see [11], or in geometric problems, see [12]). In order to identify the PDE satisfied by the limit curve, one can start from the continuity equation (1.2) and then try to characterize the velocity field $\boldsymbol{v}$. In this respect, the minimality property (1.6) provides a particularly useful information (see e.g. [13]).

Another important application stems from problems where one tries to extract finer information from the continuity equation, using the representation given by the superposition principle. The latter in fact establishes a link between the Eulerian representation of the flow of measures solving the continuity equation, and its Lagrangian depiction that comes with the associated characteristic system of ODEs. This connection is at the core of the Young-measure type technique pioneered in [5] for transport and continuity equations featuring velocitiy fields with low regularity (see, e.g., [7, 6, 8, 15]). The recent [44] (see also [14]) has thoroughly investigated the correlation between these two facets of the superposition principle, i.e. as a bridge between the Eulerian and the Lagrangian standpoints, and as a tool to gain insight into the structure of curves of measures absolutely continuous to some Wasserstein distances, decomposed in simpler ones associated with rectifiable curves, cf. the cornerstone paper [43]. The approach from [44] has in particular led to an extension of the probabilistic representations previously obtained in [34, 36].

Eventually, let us mention that the superposition principle has also turned out to be a key tool for the well-posedness of mean-field particle systems, see, e.g., [9, 40, 3], as well as for the analysis and finite-particle approximation of mean-field optimal control problems [2, 4, 28, 32], where, compared to previous literature (see, for instance, [18, 33]), a priori regularity assumptions on the control variable can be dropped. Further applications include the formulation and convergence analysis of gradient methods for dynamic inverse problems [25, 26], which build upon an extension of the superposition principle to inhomogeneous continuity equations and on the characterization of extremal points of the Benamou-Brenier or Hellinger-Kantorovich type of energies [23, 24]. We also mention [16], where the superposition principle for absolutely continuous curves of measures has been leveraged to obtain uniqueness results for a transport equation. Finally, we recall that in [22] two counterexamples to the validity of the superposition principle have been exhibited in the case of *signed* measures.

The continuity equation with singular flux. The main aim of this paper is to investigate the validity and the appropriate formulation of the

- characterization of solutions to the continuity equation
- superposition principle

in the case $p = 1$, for the space of probability measures with finite moment $\mathcal{P}_1(\mathbb{R}^d)$, endowed with the metric W_1. Indeed, we will address curves of measures that have *bounded variation* as functions of time, and, above all, with respect to the W_1-metric. A crucial feature of this setting, that makes the interpretation of the differential equation (1.3) much more delicate, is the fact that the vector flux measure $\boldsymbol{\nu}$ in (1.2) may have a singular part with respect to μ. Consider, e.g., the simplest example (cf. [1, Ex. 1.1]) of the curves $(\mu_t)_{t\in[0,1]}$

$$\mu_t = (1-t)\delta_{x_0} + t\delta_{x_1}, \quad x_0 < x_1 \in \mathbb{R},\ t \in [0,1]. \tag{1.8}$$

It is immediate to check that

$$W_p^p(\mu_s, \mu_t) = |t-s| W_p^p(\delta_0, \delta_1) = |t-s| \qquad \text{for } s,\, t \in [0,1], \qquad p \in [1,\infty)\,.$$

Hence, while $\mu \notin \mathrm{AC}([0,1];\mathcal{P}_p(\mathbb{R}^d))$ if $p > 1$, we have $\mu \in \mathrm{Lip}([0,1];\mathcal{P}_1(\mathbb{R}^d))$. Now, it can be calculated (see Example 7.1 ahead), that μ satisfies the continuity equation together with the flux measure $\boldsymbol{\nu} = \mathcal{L}^1\big|_{[0,1]} \otimes \mathcal{L}^1\big|_{[x_0,x_1]}$.

The metric superposition principle for BV curves in $(\mathcal{P}_1(X), W_1)$ with X a (complete, separable) *metric* space, has been tackled in the recent [1], by a constructive argument carefully adapting the line of proof of [34, Thm. 5] to the BV setup. Therein, the superposition principle has indeed been obtained by lifting absolutely continuous curves to Càdlàg (i.e., right-continuous, left-limited) BV curves on $(0,T)$ with values in X by means of a probability measure $\boldsymbol{\eta}$. Additionally, in [1] it has been proved that the total variation of the measure μ can be reconstructed by averaging the variation of the BV curves via the path measure $\boldsymbol{\eta}$.

In this paper, while confining our analysis to the Euclidean setup $X = \mathbb{R}^d$, we will adopt a different approach. Indeed, we will primarily focus on the structure of the superposition measure and its link with the flux measure $\boldsymbol{\nu}$. In this way, we will shed more light into the properties of the continuity equation in the BV setup. More precisely,

(1) We will carefully address the relation between BV curves in $(\mathcal{P}_1(\mathbb{R}^d), W_1)$ and the continuity equation

$$\partial_t \mu + \operatorname{div} \boldsymbol{\nu} = 0 \qquad \text{in } \mathscr{D}'((0,+\infty)\times\mathbb{R}^d), \quad |\boldsymbol{\nu}|([0,T]\times\mathbb{R}^d) < \infty \quad \text{for every } T>0, \tag{1.9}$$

where the Lebesgue decomposition $\boldsymbol{\nu} = \boldsymbol{\nu}^a + \boldsymbol{\nu}^\perp$ of the flux measure $\boldsymbol{\nu}$ with respect to μ may feature a nontrivial *singular* part $\boldsymbol{\nu}^\perp$.

(2) Among solutions to (1.9) we will enucleate a particular class of flux measures, which we will call *minimal*, and we will show that starting from a non-minimal measure, it is always possible to replace the singular part $\boldsymbol{\nu}^\perp$ by a *minimal* one $\bar{\boldsymbol{\nu}}^\perp$ such that $\bar{\boldsymbol{\nu}} = \boldsymbol{\nu}^a + \bar{\boldsymbol{\nu}}^\perp$ satisfies

$$\partial_t \mu + \operatorname{div} \bar{\boldsymbol{\nu}} = 0, \quad \bar{\boldsymbol{\nu}}^\perp = \lambda \boldsymbol{\nu}^\perp \quad \text{for a Borel scalar map } \lambda\colon (0,+\infty)\times\mathbb{R}^d \to [0,1]. \tag{1.10}$$

(3) We will represent minimal solutions to (1.10) as marginals of solutions to an auxiliary continuity equation in the *augmented* phase space, driven by an autonomous bounded vector field.

(4) By applying the known superposition principle to the augmented equation we will obtain a first representation of the solutions to (1.9) by a measure on reparametrized 1-Lipschitz curves in the augmented phase space $[0,+\infty)\times\mathbb{R}^d$.

(5) Eventually, we will derive a superposition principle in the original space by a measure on a class of agumented BV curves, providing finer information on their jump transitions.

Let us explain some of the above points in more detail.

***Absolutely continuous and* BV *curves in* $(\mathcal{P}_1(\mathbb{R}^d), W_1)$.** Dealing with $p=1$, a first natural choice is to include the space $\mathrm{BV}_{\mathrm{loc}}([0,+\infty);\mathcal{P}_1(\mathbb{R}^d))$ of BV curves with values in $\mathcal{P}_1(\mathbb{R}^d)$ in the analysis, i.e. the curves $\mu\colon [0,+\infty)\to\mathcal{P}_1(\mathbb{R}^d)$ satisfying $\mathrm{Var}_{W_1}(\mu;[0,T]) < \infty$ for every $T>0$, where

$$\mathrm{Var}_{W_1}(\mu;[a,b]) := \sup\left\{ \sum_{i=1}^n W_1(\mu_{t_{i-1}}, \mu_{t_i}) \,:\, a = t_0 < t_1 < \ldots < t_n = b \right\}. \tag{1.11}$$

To avoid ambiguities at the jump points of μ and simplify this introductory exposition, we will also assume that μ is right continuous in $[0,+\infty)$. With every map $\mu \in \mathrm{BV}_{\mathrm{loc}}([0,+\infty);\mathcal{P}_1(\mathbb{R}^d))$ we will associate the increasing map $\mathrm{V}_\mu(t) := \mathrm{Var}_{W_1}(\mu;[0,t])$ and its distributional derivative

$$\mathrm{v}_\mu = \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{V}_\mu, \quad \text{a positive locally finite measure in } [0,+\infty). \tag{1.12}$$

First of all, in ***Theorem 3.4*** we will show that it is possible to associate with every curve $\mu \in \mathrm{BV}_{\mathrm{loc}}([0,+\infty);\mathcal{P}_1(\mathbb{R}^d))$ a "minimal" vector measure $\boldsymbol{\nu} \in \mathcal{M}_{\mathrm{loc}}([0,+\infty)\times\mathbb{R}^d;\mathbb{R}^d)$, with local-in-time finite total variation, such that the pair $(\mu,\boldsymbol{\nu})$ fulfills (1.9) and the push forward of the variation measure $|\boldsymbol{\nu}|$ (associated with the norm $\|\cdot\|$ in $\mathbb{R}^d$), with respect to the time projection map $\mathsf{t}\colon [0,+\infty)\times\mathbb{R}^d \ni (t,x)\mapsto t$ satisfies

$$\mathsf{t}_\sharp|\boldsymbol{\nu}| = \mathrm{v}_\mu \quad \text{i.e.} \quad |\boldsymbol{\nu}|\big((a,b]\times\mathbb{R}^d\big) = \mathrm{V}_\mu(b) - \mathrm{V}_\mu(a) \quad \text{for every } 0 \le a < b. \tag{1.13}$$

It is worth noticing that the disintegration $(\mu_t)_{t\ge 0}$ of a solution μ of (1.9) w.r.t. a vector measure $\boldsymbol{\nu}$ of local-in-time finite total variation admits a BV representation satisfying the further condition

$$\mathrm{v}_\mu \le \mathsf{t}_\sharp|\boldsymbol{\nu}|. \tag{1.14}$$

Therefore, condition (1.13) provides a variational characterization of $\boldsymbol{\nu}$ similar to (1.6) in the case $p>1$. Moreover, if the norm $\|\cdot\|$ is strictly convex, then $\boldsymbol{\nu}$ is uniquely characterized by (1.9) and (1.13).

The augmented continuity equation. A more difficult task is to establish the superposition principle for solutions to (1.9), also dealing with the case when the variational condition (1.13) is not satisfied. For this, we have drawn inspiration from the results by Smirnov on the representation of *solenoidal* (i.e., null-divergence) *charges* (currents) in terms of simpler ones associated with rectifiable curves, [43]. In fact, a pair $(\mu,\boldsymbol{\nu})$ solving the continuity equation (1.9) can be viewed as a "solenoidal charge", too, in the sense that (1.9) rewrites as

$$\mathrm{Div}_{(t,x)}(\mu,\boldsymbol{\nu}) = 0 \quad \text{in } \mathscr{D}'((0,+\infty)\times\mathbb{R}^d), \tag{1.15}$$

with the overall divergence operator $\mathrm{Div}_{(t,x)}(\star,\bullet) := \partial_t(\star) + \mathrm{div}(\bullet)$. This observation is, in fact, also at the core of the approach in [19, §3.1], where an alternative proof of the superposition principle from [5] has been developed, based on Smirnov's decomposition theorem in [43].

In the present BV setup, we have developed a self-contained approach, independent of Smirnov's result, which exploits the nonnegativity of μ and the irreversible direction of time. We have also been guided by the reparametrization technique that has been quite successful in the variational theory of rate-independent evolution (cf., e.g., [31, 37, 38, 39, 29]), where solution curves incorporate further information about jump transitions (not necessarily segments) and take values in the *augmented* phase space $\mathbb{R}^{d+1}_+ := [0,+\infty)\times\mathbb{R}^d$, including time.

Accordingly, our idea (cf. ***Theorem 4.7*** ahead), is to represent a minimal solution to (1.9) as the marginals $(\mu,\boldsymbol{\nu}) = \boldsymbol{\pi}_\sharp(\sigma^0,\boldsymbol{\sigma})$ (with $\boldsymbol{\pi}$ the projection, $\boldsymbol{\pi}(s,t,x) := (t,x)$), of a distinguished solution to the auxiliary continuity equation in the augmented phase space $\mathbb{R}^{d+1}_+ \ni (t,x)$, namely

$$\partial_s\sigma + \partial_t\sigma^0 + \mathrm{div}\,\boldsymbol{\sigma} = 0 \quad \text{in } \mathscr{D}'((0,+\infty)\times\mathbb{R}^{d+1}), \quad \sigma,\sigma^0 \geq 0, \quad \sigma_{s=0} = \delta_0\otimes\mu_0. \tag{1.16}$$

In (1.16), s is an artificial time-like parameter, the flux pair $(\sigma^0,\boldsymbol{\sigma})$ is absolutely continuous w.r.t. σ $(\sigma^0,\boldsymbol{\sigma}) = (\tau,\boldsymbol{v})\sigma$, and the autonomous velocity field $(\tau,\boldsymbol{v})$ is related to the original solution pair $(\mu,\boldsymbol{\nu})$ via

$$\tau = \frac{\mathrm{d}\mu}{\mathrm{d}|(\mu,\boldsymbol{\nu})|}, \qquad \boldsymbol{v} = \frac{\mathrm{d}\boldsymbol{\nu}}{\mathrm{d}|(\mu,\boldsymbol{\nu})|}.$$

In particular, the augmented norm of $(\tau,\boldsymbol{v})$ is 1 and $\sigma = |(\sigma^0,\boldsymbol{\sigma})|$. A simple modification (cf. Proposition 4.5) of the standard superposition principle can then be applied to the *augmented* continuity equation (1.16). It guarantees a representation of any solution σ in terms of a probability measure $\boldsymbol{\eta}$ supported on 1-Lipschitz curves y of the form

$$[0,+\infty) \ni s \mapsto \mathsf{y}(s) = (\mathsf{t}(s),\mathsf{x}(s)) \in [0,+\infty)\times\mathbb{R}^d, \tag{1.17}$$

solving the associated characteristic system

$$\dot{\mathsf{t}}(s) = \tau(\mathsf{t}(s),\mathsf{x}(s)), \quad \dot{\mathsf{x}}(s) = \boldsymbol{v}(\mathsf{t}(s),\mathsf{x}(s)).$$

We will then derive a probabilistic representation for the pair $(\mu,\boldsymbol{\nu})$ in terms of the trajectories γ. Specifically, in ***Theorem 5.1*** we will show that

$$(\mu,\boldsymbol{\nu}) = \mathsf{e}_\sharp(\dot{\mathsf{y}}\mathcal{L}^1\otimes\boldsymbol{\eta}), \qquad |(\mu,\boldsymbol{\nu})| = \mathsf{e}_\sharp(\|\dot{\mathsf{y}}\|\mathcal{L}^1\otimes\boldsymbol{\eta}), \tag{1.18}$$

where $\mathcal{L}^1$ denotes the Lebesgue measure in $[0,+\infty)$, and $\mathsf{e}\colon [0,+\infty)\times\mathrm{Lip}([0,+\infty);\mathbb{R}^{d+1}_+)$ is the evaluation map defined as $\mathsf{e}(s,\mathsf{y}) := \mathsf{y}(s)$.

Superposition by augmented* BV *curves. Our final contribution is to recover a superposition result for the original continuity equation (1.9) by a measure on the space of time dependent BV curves. If $\boldsymbol{\nu}$ does not satisfy the variational condition (1.13), one can expect the singular part $\boldsymbol{\nu}^\perp$ of $\boldsymbol{\nu}$ to carry crucial information about the jump transition of the curves (not necessarily along segments). Seemingly, such information cannot be fully captured by the usual descrition of a BV curve, which only characterizes the left and right limit of the curve at each jump point, but not the *actual* trajectory described along the jump.

To overcome this difficulty, we introduce the notion of *augmented* BV curves: they are maps $\mathsf{u}\colon \mathcal{Z}\to\mathbb{R}^d$ defined in the augmented parameter space $\mathcal{Z} := [0,+\infty)\times[0,1]$ such that

(1) the functions $u_-(t) := \mathsf{u}(t,0)$ (resp. $u_+(t) := \mathsf{u}(t,1)$) are left- (resp. right-) continuous (local) BV maps which coincide in the complement of their countable jump set $\mathfrak{J}_\mathsf{u}$;
(2) for every $t\notin\mathfrak{J}_\mathsf{u}$ the function $[0,1]\ni r\mapsto\mathsf{u}(t,r)$ is constant and coincides with $u_-(t) = u_+(t)$;
(3) for every $t\in\mathfrak{J}_\mathsf{u}$ the function $[0,1]\ni r\mapsto\mathsf{u}(t,r)$ is a Lipschitz (transition) map connecting $u_-(t)$ with $u_+(t)$ with constant (and strictly positive) velocity, thus equal to the length $\ell_\mathsf{u}(t)$ of the transition path.
(4) For every finite time interval $[0,T]$ we have

$$\mathrm{Var}(\mathsf{u};[0,T]\times[0,1]) = \mathrm{Var}(u_\pm;[0,T]) + \sum_{t\in\mathfrak{J}_\mathsf{u}}\Big(\ell_\mathsf{u}(t) - \|u_+(t)-u_-(t)\|\Big) < \infty. \tag{1.19}$$

We will denote such space by $\mathsf{ABV}([0,+\infty);\mathbb{R}^d)$ and endow it with a Lusin topology, for which the evaluation maps $\mathsf{e}(t,r,\mathsf{u}) := \mathsf{u}(t,r)$ are Borel.

In ***Theorem 6.5***, we will represent a minimal solution $(\mu,\boldsymbol{\nu})$ to (1.9) by means of a probability measure $\widehat{\boldsymbol{\eta}}$ on $\mathsf{ABV}([0,+\infty;\mathbb{R}^d)$ concentrated on curves u satisfying a suitable differential equation formulated in a BV sense. More precisely, we can write the Lebesgue decomposition $\boldsymbol{\nu}^a+\boldsymbol{\nu}^\perp$ of $\boldsymbol{\nu}$ as $\boldsymbol{\nu}^a = \boldsymbol{v}^a\mu$, $\boldsymbol{\nu}^\perp = \boldsymbol{v}^\perp|\boldsymbol{\nu}^\perp|$, and we observe that the curves $t\mapsto \mathsf{u}(t,r)$ coincide $\mathcal{L}^1$-a.e. on $(0,+\infty)$, so that their distributional time derivative $\partial_t\mathsf{u}(t,r)$ does not depend on r and can be decomposed in the sum of an absolutely continuous part $\partial_t^{\mathrm{L}}\mathsf{u}\,\mathcal{L}^1$, a Cantor part $\partial_t^{\mathrm{C}}\mathsf{u}$ and a jump part $\partial_t^{\mathrm{J}}\mathsf{u}$ concentrated on J_{u}:

$$\partial_t\mathsf{u} = \partial_t^{\mathrm{L}}\mathsf{u}\,\mathcal{L}^1 + \partial_t^{\mathrm{C}}\mathsf{u} + \partial_t^{\mathrm{J}}\mathsf{u}\,.$$

Thus, $\widehat{\boldsymbol{\eta}}$-a.e. curve u satisfy

$$\partial_t^{\mathrm{L}}\mathsf{u}(t) = \boldsymbol{v}^a(\mathsf{u}(t,\cdot)) \quad \text{for } \mathcal{L}^1\text{-a.a. } t\in(0,+\infty),$$
$$\partial_t^{\mathrm{C}}\mathsf{u} = \boldsymbol{v}^\perp(\mathsf{u}(t,\cdot))|\partial_t^{\mathrm{C}}\mathsf{u}|,$$

and

$$\partial_r\mathsf{u}(t,r) = \boldsymbol{v}^\perp(\mathsf{u}(t,r))\ell_{\mathsf{u}}(t) \quad \text{for a.a. } r\in(0,1) \text{ and every } t\in\mathfrak{J}_{\mathsf{u}}\,.$$

We then have

$$\mu_t^+ = (\mathsf{e}_{t,1})_\sharp\widehat{\boldsymbol{\eta}}, \quad \mu_t^- = (\mathsf{e}_{t,0})_\sharp\widehat{\boldsymbol{\eta}},$$

where the evaluation maps $\mathsf{e}_{t,0},\mathsf{e}_{t,1} : \mathsf{ABV}([0,+\infty);\mathbb{R}^d) \to \mathbb{R}^d$ are defined by $\mathsf{e}_{t,0}(\mathsf{u}) := \mathsf{u}(t,0)$ and $\mathsf{e}_{t,1}(\mathsf{u}) := \mathsf{u}(t,1)$ for every $\mathsf{u}\in\mathsf{ABV}([0,+\infty);\mathbb{R}^d)$. Whenever $\boldsymbol{\nu}$ is minimal, we can recover $\boldsymbol{\nu}$ by superimposing integration along u.

Plan of the paper. Our analysis is carried out as follows:

- In Section 2, after settling some notation and preliminary results from measure theory, we introduce an order relation between Radon measures and delve into the induced minimality concept, which will play a key role in the selection of distinguished solutions to the continuity equation with singular flux. We also lay the ground for the superposition principle by defining the function spaces that will come into play, and fixing their properties.
- Section 3 revolves around the relation between curves $\mu\in\mathrm{BV}_{\mathrm{loc}}([0,+\infty);\mathcal{P}_1(\mathbb{R}^d))$ and the continuity equation (1.9).
- In the main result of Section 4 we associate with (1.9) a new continuity equation in an augmented phase space, driven by a *non-singular* flux measure with bounded velocity field. Their relation is such that suitable marginals of the solutions to the augmented continuity equation provide, indeed, solutions to the continuity equation with minimal singular flux.
- Based on this, in Section 5 we derive our first, 'parametrized' version of the superposition principle. In fact, by leveraging the probabilistic representation for the solutions to the augmented continuity equation, we obtain a representation of the solutions to the original continuity equation, in terms of trajectories that are Lipschitz w.r.t. an artificial time-like parameter and solve the characteristic system in the extended phase space.
- Section 6 is devoted to establishing a probabilistic representation for solutions of the continuity equation in terms of BV curves depending on the 'true' process time. For this, we preliminarily carry out a thorough analysis of a distinguished class of BV curves that are 'attached' with their transitions at jump points. We use them as a bridge between the probabilistic representation in terms of reparametrized trajectories, and that involving BV curves.
- In Section 7 we discuss our assumptions, and illustrate our results (mostly focusing on the 'parametrized version' of the superposition principle) in a series of examples.
- Finally, in the Appendix we prove some technical results that have been employed at scattered spots in the paper.

2. Notation and preliminary results

The following table contains the main notation that we shall use throughout the paper:

$\Vert\cdot\Vert$	(generic) norm in $\mathbb{R}^h$
B_R, $\overline{B}_R$	open/closed ball of center 0 and radius $R>0$ in $\mathbb{R}^h$ (w.r.t. the norm $\Vert\cdot\Vert$)
$\mathcal{B}(\mathbb{R}^h), \mathcal{B}_b(\mathbb{R}^h)$	Borel (resp. bounded Borel) subsets of $\mathbb{R}^h$
$\mathbf{I}$	the positive half-line $[0,+\infty)$
$\mathcal{L}^1$	Lebesgue measure on $\mathbf{I}$
$\mathcal{P}(\mathbb{R}^h)$	Borel probability measures in $\mathbb{R}^h$
$\mathcal{P}_1(\mathbb{R}^h)$	probability measures in $\mathbb{R}^h$ with finite first moment, endowed with the
W_1	Wasserstein distance
$\mathcal{M}(A)$, $\mathcal{M}_{\rm loc}(A)$	finite (resp. Radon) Borel measures on $A\in\mathcal{B}(\mathbb{R}^h)$
$\mathcal{M}^+(A)$, $\mathcal{M}^+_{\rm loc}(A)$	finite (resp. Radon) nonnegative Borel measures on A
$\mathcal{M}(A;\mathbb{R}^m)$, $\mathcal{M}_{\rm loc}(A;\mathbb{R}^m)$	$\mathbb{R}^m$-valued Borel measures with finite total variation, (resp. $\mathbb{R}^m$-valued Radon meas.), on A
$\vert\boldsymbol{\lambda}\vert$	total variation of $\boldsymbol{\lambda}\in\mathcal{M}_{\rm loc}(A;\mathbb{R}^d)$
$\mathrm{C_c}(A), \mathrm{C}^k_\mathrm{c}(A)$	continuous (C^k, $k\geq 1$, resp.) real functions on A with compact support
$\mathrm{C_b}(A), \mathrm{C}^k_\mathrm{b}(A)$	continuous (C^k, $k\geq 1$, resp.) and bounded real functions on A
$\Vert\cdot\Vert_p$	norm on $L^p(A;\mathbb{R}^m)$ for some $p\geq 1$
$L^p_\theta(\mathbb{R}^h;\mathbb{R}^k), L^p_{{\rm loc},\theta}(\mathbb{R}^h;\mathbb{R}^k)$	L^p-spaces w.r.t. $\theta\in\mathcal{M}^+(\mathbb{R}^h)$
$\mathbb{R}^{d+1}_+$	the space-time domain $\mathbf{I}\times\mathbb{R}^d$.

2.1. Preliminaries of measure theory.

Finite and Radon vector measures. We denote by $\mathcal{M}(\mathbb{R}^h;\mathbb{R}^m)$ the space of Borel measures $\boldsymbol{\mu}\colon\mathcal{B}(\mathbb{R}^h)\to\mathbb{R}^m$ with finite total variation $\Vert\boldsymbol{\mu}\Vert_{\rm TV}:=|\boldsymbol{\mu}|(\mathbb{R}^h)<+\infty$, where for every $B\in\mathcal{B}(\mathbb{R}^h)$

$$|\boldsymbol{\mu}|(B):=\sup\left\{\sum_{i=0}^{+\infty}\Vert\boldsymbol{\mu}(B_i)\Vert\ :\ B_i\in\mathcal{B}(\mathbb{R}^h),\ B_i \text{ pairwise disjoint, } B=\bigcup_{i=0}^{+\infty}B_i\right\},$$

and $\Vert\cdot\Vert$ is a norm in $\mathbb{R}^m$. $(\mathcal{M}(\mathbb{R}^h;\mathbb{R}^m);\Vert\cdot\Vert_{\rm TV})$ is a Banach space. We recall that a Radon vector measure in $\mathcal{M}(\mathbb{R}^h;\mathbb{R}^m)$ is a set function $\boldsymbol{\lambda}\colon\mathcal{B}_b(\mathbb{R}^h)\to\mathbb{R}^m$ such that for every compact subset $K\Subset\mathbb{R}^h$ its restriction to $\mathcal{B}(K)$ is a (vector) measure with finite total variation.

We identify $\boldsymbol{\lambda}\in\mathcal{M}_{\rm loc}(\mathbb{R}^h;\mathbb{R}^m)$ with a vector $(\lambda_1,\lambda_2,\dots,\lambda_m)$ of m measures in $\mathcal{M}_{\rm loc}(\mathbb{R}^h)$, so that its integral with a continuous $\mathbb{R}^m$-valued function with compact support $\boldsymbol{\zeta}\in\mathrm{C_c}(\mathbb{R}^h;\mathbb{R}^m)$ is given by

$$\int_{\mathbb{R}^h}\boldsymbol{\zeta}(x)\mathrm{d}\boldsymbol{\lambda}(x):=\sum_{i=1}^m\int_{\mathbb{R}^h}\zeta_i(x)\mathrm{d}\lambda_i(x)\,. \tag{2.1}$$

By the above duality pairing, $\mathcal{M}_{\rm loc}(\mathbb{R}^h;\mathbb{R}^m)$ can be identified with the dual of $\mathrm{C_c}(\mathbb{R}^h;\mathbb{R}^m)$ and is thus endowed with the corresponding weak* topology; for the associated convergence notion we will use the symbol $\rightharpoonup^*$.

For every $\boldsymbol{\lambda}\in\mathcal{M}_{\rm loc}(\mathbb{R}^h;\mathbb{R}^m)$ and every open subset $O\subset\mathbb{R}^h$ we have that

$$|\boldsymbol{\lambda}|(O):=\sup\left\{\int_{\mathbb{R}^h}\boldsymbol{\zeta}(x)\mathrm{d}\boldsymbol{\lambda}(x)\ :\ \boldsymbol{\zeta}\in\mathrm{C_c}(\mathbb{R}^h;\mathbb{R}^m)\,,\ \mathrm{spt}(\boldsymbol{\zeta})\subset O,\ \sup_{x\in O}\Vert\boldsymbol{\zeta}(x)\Vert_*\leq 1\right\}$$

Clearly, the choice of the norm $\Vert\cdot\Vert$ on $\mathbb{R}^m$ (and its dual $\Vert\cdot\Vert_*$) affects the definition of the total variation measure $|\cdot|$, which depends on $\Vert\cdot\Vert$. The set function $|\boldsymbol{\lambda}|\colon\mathcal{M}_{\rm loc}(\mathbb{R}^h;\mathbb{R}^m)\to[0,+\infty]$ is a positive Radon measure and every $\boldsymbol{\lambda}\in\mathcal{M}(\mathbb{R}^h;\mathbb{R}^m)$ admits the *polar decomposition* $\boldsymbol{\lambda}=\boldsymbol{w}|\boldsymbol{\lambda}|$ for some Borel map $\boldsymbol{w}:\mathbb{R}^h\to\mathbb{R}^m$ with $\Vert\boldsymbol{w}\Vert\equiv 1$ $|\boldsymbol{\lambda}|$-a.e. in $\mathbb{R}^h$. It is trivial to check that the integral of (2.1) can also be written as

$$\int_{\mathbb{R}^h}\boldsymbol{\zeta}(x)\,\mathrm{d}\boldsymbol{\lambda}(x)=\int_{\mathbb{R}^h}\boldsymbol{\zeta}(x)\cdot\boldsymbol{w}(x)\,\mathrm{d}|\boldsymbol{\lambda}|(x) \tag{2.2}$$

and the previous formula can also be used to define a vector integral for a scalar function.

Weak* and narrow convergence. Every sequence $(\boldsymbol{\lambda}_k)_k \subset \mathcal{M}_{\rm loc}(\mathbb{R}^h;\mathbb{R}^m)$ such that

$$\sup_k |\boldsymbol{\lambda}_k|(B_R) < +\infty \qquad \text{for every } R>0$$

admits a subsequence $(\boldsymbol{\lambda}_{k_j})_j$ weakly*-converging to some $\boldsymbol{\lambda} \in \mathcal{M}_{\rm loc}(\mathbb{R}^h;\mathbb{R}^m)$; furthermore, the sequence $(|\boldsymbol{\lambda}_{k_j}|)_j$ weakly* converges to some $\lambda \in \mathcal{M}^+_{\rm loc}(\mathbb{R}^h)$ such that $\lambda \ge |\boldsymbol{\lambda}|$. If $\sup_k |\boldsymbol{\lambda}_k|(\mathbb{R}^h) < +\infty$, then up to a subsequence the measures $(\boldsymbol{\lambda}_k)_k$ weakly* converge to some $\boldsymbol{\lambda} \in \mathcal{M}(\mathbb{R}^h;\mathbb{R}^m)$.

We recall that a sequence $(\mu_k)_k \subset \mathcal{M}(\mathbb{R}^h)$ *narrowly* converges to $\mu \in \mathcal{M}(\mathbb{R}^h)$ if

$$\lim_{k\to\infty} \int_{\mathbb{R}^h} \varphi(x)\mathrm{d}\mu_k(x) = \int_{\mathbb{R}^h} \varphi(x)\mathrm{d}\mu(x) \qquad \text{for all } \varphi \in \mathrm{C_b}(\mathbb{R}^h).$$

Prokhorov's Theorem [30, III-59] asserts that a subset $M \subset \mathcal{M}(\mathbb{R}^h)$ has compact closure in this topology if and only if it is bounded in the total variation norm $|\cdot|$ and equally tight, namely

$$\forall\, \varepsilon>0 \ \ \exists\, K \Subset \mathbb{R}^h : \quad \sup_{\mu\in M} |\mu|(\mathbb{R}^h \backslash K) \le \varepsilon\,.$$

On $\mathcal{P}(\mathbb{R}^h)$ the narrow topology coincides with the weak* topology.

Restriction and push-forward of measures. For every $\boldsymbol{\mu} \in \mathcal{M}_{\rm loc}(\mathbb{R}^h;\mathbb{R}^m)$ and $A \in \mathcal{B}(\mathbb{R}^h)$ we denote by $\boldsymbol{\mu} \llcorner A \in \mathcal{M}_{\rm loc}(\mathbb{R}^h;\mathbb{R}^m)$ the restriction of $\boldsymbol{\mu}$ to A, i.e. $\boldsymbol{\mu}\llcorner A(B) := \boldsymbol{\mu}(B\cap A)$ for every $B \in \mathcal{B}_b(\mathbb{R}^h)$. We shall use that, whenever $\boldsymbol{\mu}_n \rightharpoonup^* \boldsymbol{\mu}$ in $\mathcal{M}_{\rm loc}(\mathbb{R}^h;\mathbb{R}^m)$ and $A \subset \mathbb{R}^h$ is open, then $\boldsymbol{\mu}_n \llcorner A \rightharpoonup^* \boldsymbol{\mu}\llcorner A$ in $\mathcal{M}_{\rm loc}(A;\mathbb{R}^m)$.

Let $\mathsf{p}\colon \mathbb{R}^h \to \mathbb{R}^k$ be a Borel map. For every $\boldsymbol{\mu} \in \mathcal{M}(\mathbb{R}^h;\mathbb{R}^m)$ we define the push-forward measure $\mathsf{p}_\sharp \boldsymbol{\mu}$ in $\mathcal{M}(\mathbb{R}^k;\mathbb{R}^m)$ via

$$\mathsf{p}_\sharp\boldsymbol{\mu}(B) := \boldsymbol{\mu}(\mathsf{p}^{-1}(B)) \qquad \text{for all } B \in \mathcal{B}(\mathbb{R}^k)\,.$$

In general, the above definition can be extended to define a measure in $\mathcal{M}_{\rm loc}(\mathbb{R}^k;\mathbb{R}^m)$ from a measure in $\mathcal{M}_{\rm loc}(\mathbb{R}^h;\mathbb{R}^m)$ if, in addition, the mapping $\mathsf{p}\colon \mathbb{R}^h \to \mathbb{R}^k$ is continuous and *proper*, namely for every compact subset $K \in \mathbb{R}^k$ we have that $\mathsf{p}^{-1}(K)$ is a compact subset of $\mathbb{R}^h$. Under this condition, we have that (cf., e.g., [10, Rmk. 1.7]) if $\boldsymbol{\mu}_n \rightharpoonup^* \boldsymbol{\mu}$ in $\mathcal{M}_{\rm loc}(\mathbb{R}^h;\mathbb{R}^m)$, then $\mathsf{p}_\sharp\boldsymbol{\mu}_n \rightharpoonup^* \mathsf{p}_\sharp\boldsymbol{\mu}$ in $\mathcal{M}_{\rm loc}(\mathbb{R}^k;\mathbb{R}^m)$. We further notice that if $(\boldsymbol{\mu}_n)_n, \boldsymbol{\mu} \in \mathcal{M}(\mathbb{R}^h;\mathbb{R}^m)$, $(\boldsymbol{\mu}_n)_n$ converges narrow to $\boldsymbol{\mu}$, and $\mathsf{p}\colon \mathbb{R}^k \to \mathbb{R}^h$ is continuous, then $(\mathsf{p}_\sharp\boldsymbol{\mu}_n)_n$ converges narrow to $\mathsf{p}_\sharp\boldsymbol{\mu}$.

The Wasserstein distance on $\mathcal{P}_1(\mathbb{R}^h)$. We recall that the distance W_1 on $\mathcal{P}_1(\mathbb{R}^h)$ is defined by

$$W_1(\mu_1,\mu_2) := \min\left\{ \int_{\mathbb{R}^h\times\mathbb{R}^h} \|x-y\|\mathrm{d}\gamma(x,y) \ : \ \gamma \in \mathcal{P}(\mathbb{R}^h\times\mathbb{R}^h),\ \pi^i_\sharp\gamma = \mu_i,\ i\in\{1,2\} \right\}. \tag{2.3}$$

Again, notice that the above definition depends on the choice of the norm $\|\cdot\|$ on $\mathbb{R}^h$. For a given curve $\mu\colon \mathbf{I} \to \mathcal{P}_1(\mathbb{R}^h)$ we will denote by Var_{W_1} its total variation w.r.t. W_1, defined on every $[a,b] \subset \mathbf{I}$ by

$$\mathrm{Var}_{W_1}(\mu;[a,b]) := \sup\left\{ \sum_{i=1}^n W_1(\mu_{t_{i-1}},\mu_{t_i}) \ : a=t_0<t_1<\ldots<t_n=b \right\}. \tag{2.4}$$

We will denote by $\mathrm{BV_{loc}}(\mathbf{I};\mathcal{P}_1(\mathbb{R}^h))$ the space of curves $\mu\colon \mathbf{I} \to \mathcal{P}_1(\mathbb{R}^h)$ such that $\mathrm{Var}_{W_1}(\mu;[a,b]) < +\infty$ for every $[a,b] \subset \mathbf{I}$. Finally, we recall (cf. [11, Thm. 1.1.2]) that for any $\mu \in \mathrm{AC_{loc}}(\mathbf{I};\mathcal{P}_1(\mathbb{R}^h))$ the limit

$$|\mu'_t|_{W_1} := \lim_{h\to 0} \frac{1}{h} W_1(\mu_t,\mu_{t+h}) \qquad \text{exists for a.a. } t \in (0,+\infty).$$

Metrizable spaces. Following [30, III-16] A topological space (X,τ) is called

- *Polish* if it can be endowed with a metric d inducing the topology τ such that (X,d) is a complete separable metric space;
- *Lusin* if it is the injective and continuous image of a Polish space.

2.2. **Submeasures and minimality.** In the spirit of the definition of *subcurrent* from [42, Def. 3.1], we introduce the concept of 'submeasure' and the induced order relation on $\mathcal{M}_{\mathrm{loc}}(O;\mathbb{R}^d)$, where O is some locally compact topological space.

Definition 2.1. Let $\boldsymbol{\theta},\boldsymbol{\zeta}\in\mathcal{M}_{\mathrm{loc}}(O;\mathbb{R}^k)$. We say that $\boldsymbol{\zeta}$ is a *submeasure* of $\boldsymbol{\theta}$ and write $\boldsymbol{\zeta}\prec\boldsymbol{\theta}$ if

$$\exists\,\lambda\in L^\infty_{|\boldsymbol{\theta}|}(O;[0,1])\quad\text{such that}\quad \boldsymbol{\zeta}=\lambda\boldsymbol{\theta}. \tag{2.5}$$

It can be immediately checked that $\prec$ is an order relation, and that it fulfills

$$(\boldsymbol{\zeta}\prec\boldsymbol{\theta}\text{ and }|\boldsymbol{\theta}|(O)\le|\boldsymbol{\zeta}|(O)<+\infty)\implies\boldsymbol{\zeta}=\boldsymbol{\theta}\,. \tag{2.6}$$

The relation $\prec$ can also be characterized by the following result.

Lemma 2.2. *Let $\boldsymbol{\theta},\boldsymbol{\zeta}\in\mathcal{M}_{\mathrm{loc}}(O;\mathbb{R}^k)$ and let $\boldsymbol{w}$ be the Lebesgue density of the polar decomposition of $\boldsymbol{\theta}$, i.e. $\boldsymbol{\theta}=\boldsymbol{w}|\boldsymbol{\theta}|$. The following properties are equivalent:*

(i) $\boldsymbol{\zeta}\prec\boldsymbol{\theta}$,

(ii) $|\boldsymbol{\zeta}|\le|\boldsymbol{\theta}|$ *and* $\boldsymbol{\zeta}=\boldsymbol{w}|\boldsymbol{\zeta}|$,

(iii) (assuming the norm $\|\cdot\|$ in $\mathbb{R}^k$ is strictly convex) there exists $\boldsymbol{\zeta}_C\in\mathcal{M}_{\mathrm{loc}}(O;\mathbb{R}^k)$ such that $\boldsymbol{\theta}=\boldsymbol{\zeta}+\boldsymbol{\zeta}_C$ and $|\boldsymbol{\theta}|=|\boldsymbol{\zeta}|+|\boldsymbol{\zeta}_C|$.

Proof. The implications *(i)* $\Rightarrow$ *(ii)*, *(iii)* are obvious.

In order to prove *(ii)* $\Rightarrow$ *(i)* we observe that $|\boldsymbol{\zeta}|=\lambda|\boldsymbol{\theta}|$ for some $\lambda\in L^\infty_{|\boldsymbol{\theta}|}(O,[0,1])$ since $|\boldsymbol{\zeta}|\le|\boldsymbol{\theta}|$, so that $\boldsymbol{\zeta}=\boldsymbol{w}\lambda|\boldsymbol{\theta}|=\lambda\boldsymbol{\theta}$.

As for *(iii)* $\Rightarrow$ *(i)* let $\boldsymbol{\zeta}_C\in\mathcal{M}_{\mathrm{loc}}(O,\mathbb{R}^k)$ be such that $\boldsymbol{\theta}=\boldsymbol{\zeta}+\boldsymbol{\zeta}_C$ and $|\boldsymbol{\theta}|=|\boldsymbol{\zeta}|+|\boldsymbol{\zeta}_C|$. Then, $\boldsymbol{\zeta},\boldsymbol{\zeta}_C\ll|\boldsymbol{\theta}|$. Denoting by $\boldsymbol{v}:=\frac{\mathrm{d}\boldsymbol{\zeta}}{\mathrm{d}|\boldsymbol{\theta}|}$ we may write $\boldsymbol{\zeta}=\boldsymbol{v}|\boldsymbol{\theta}|$ and $\boldsymbol{\zeta}_C=(\boldsymbol{w}-\boldsymbol{v})|\boldsymbol{\theta}|$. Now, the function $\boldsymbol{v}\in L^1_{|\boldsymbol{\theta}|}(O;\mathbb{R}^k)$ satisfies $1=\|\boldsymbol{w}\|=\|\boldsymbol{w}-\boldsymbol{v}\|+\|\boldsymbol{v}\|$ $|\boldsymbol{\theta}|$-a.e. in O. Since $\|\cdot\|$ is strictly convex, we deduce $\boldsymbol{v}=\lambda\,\boldsymbol{w}$ for some $\lambda\in L^\infty_{|\boldsymbol{\theta}|}(O;[0,1])$, i.e. (2.5) holds. □

We now consider the previous order relation in the particular case when O is an open subset of space-time Euclidean space $\mathbb{R}^{d+1}=\mathbb{R}\times\mathbb{R}^d$ (whose elements will be denoted by (t,x)), and measures have the same x-distributional divergence. The operator div is to be understood with respect to the 'spatial' variable $x\in\mathbb{R}^d$. This gives rise to the following definition.

Definition 2.3 (Minimal vector measures)**.** Let O be an open subset of $\mathbb{R}\times\mathbb{R}^d$ and let $\boldsymbol{\theta}\in\mathcal{M}_{\mathrm{loc}}(O;\mathbb{R}^d)$. We say that $\boldsymbol{\theta}$ is *minimal* if the following property holds:

$$\text{whenever }\boldsymbol{\zeta}\in\mathcal{M}_{\mathrm{loc}}(O;\mathbb{R}^d)\text{ fulfills}\quad\mathrm{div}\,\boldsymbol{\zeta}=\mathrm{div}\,\boldsymbol{\theta}\text{ and }\boldsymbol{\zeta}\prec\boldsymbol{\theta},\quad\text{then}\quad\boldsymbol{\zeta}=\boldsymbol{\theta}. \tag{2.7}$$

We illustrate this concept with the following example, where a *minimal* measure is constructed by juxtaposing the measures carried by finitely many regular and *injective* curves.

Example 2.4. *Let $(\varrho_i)_{i=1}^n$ be a family of regular injective curves in $\mathbb{R}^d$, with disjoint image sets. Let t_{ϱ_i}, $i=1,\dots,n$ be their tangent vector fields, and $r_{\varrho_i}:[0,L_{\varrho_i}]\to\mathbb{R}^d$ their arclength parametrizations. Let $(a,b)\subset\mathbb{R}$ be an arbitrary interval and $\lambda\in\mathcal{M}((a,b))$. Then, the measure $\boldsymbol{\theta}\in\mathcal{M}_{\mathrm{loc}}((a,b)\times\mathbb{R}^d;\mathbb{R}^d)$ defined by*

$$\boldsymbol{\theta}:=\lambda\otimes\boldsymbol{m}_\rho\qquad\text{with }\boldsymbol{m}_\rho:=\sum_{i=1}^n\mathbf{t}_{\varrho_i}\mathcal{H}^1\llcorner\varrho_i$$

is minimal.

Indeed, let $\boldsymbol{\zeta}\in\mathcal{M}_{\mathrm{loc}}((a,b)\times\mathbb{R}^d;\mathbb{R}^d)$ fulfill $\boldsymbol{\zeta}\prec\boldsymbol{\theta}$ and $\mathrm{div}\,\boldsymbol{\zeta}=\mathrm{div}\,\boldsymbol{\theta}$. Then, there exists $\ell\in L^\infty_{|\boldsymbol{\theta}|}((a,b)\times\mathbb{R}^d;[0,1])$ such that $\boldsymbol{\zeta}=\ell\boldsymbol{\theta}$. Moreover, for every $\varphi\in\mathrm{C}^1_{\mathrm{c}}((a,b)\times\mathbb{R}^d)$ we have that

$$\begin{aligned}\int_{(a,b)}\sum_{i=1}^n\int_0^{L_{\varrho_i}}\frac{\mathrm{d}}{\mathrm{d}s}\varphi(t,r_{\varrho_i}(s))\,\mathrm{d}s\,\mathrm{d}\lambda(t)&=\int_{(a,b)}\sum_{i=1}^n\int_{\mathbb{R}^d}\mathrm{D}_x\varphi(t,x)\cdot\mathbf{t}_{\varrho_i}(x)\,\mathrm{d}(\mathcal{H}^1\llcorner\varrho_i)(x)\,\mathrm{d}\lambda(t)\\&=\iint_{(a,b)\times\mathbb{R}^d}\mathrm{D}_x\varphi(t,x)\,\mathrm{d}\boldsymbol{\theta}(t,x)=\iint_{(a,b)\times\mathbb{R}^d}\mathrm{D}_x\varphi(t,x)\,\mathrm{d}\boldsymbol{\zeta}(t,x)\end{aligned} \tag{2.8}$$

$$= \int_{(a,b)} \sum_{i=1}^{n} \int_{\mathbb{R}^d} \ell(t,x) \mathrm{D}_x \varphi(t,x) \cdot \mathbf{t}_{\varrho_i}(x) \, \mathrm{d}(\mathcal{H}^1 \llcorner \varrho_i)(x) \, \mathrm{d}\lambda(t)$$
$$= \int_{(a,b)} \sum_{i=1}^{n} \int_0^{L_{\varrho_i}} \ell(t, r_{\varrho_i}(s)) \frac{\mathrm{d}}{\mathrm{d}s} \varphi(t, r_{\varrho_i}(s)) \, \mathrm{d}s \, \mathrm{d}\lambda(t) \, .$$

Taking $\varphi(t,x) = \psi_1(t)\psi_2(x)$ for $\psi_1 \in \mathrm{C}^1_{\mathrm{c}}((a,b))$ and $\psi_2 \in \mathrm{C}^1_{\mathrm{c}}(\mathbb{R}^d)$, we deduce from (2.8) that for λ-a.a. $t \in (a,b)$

$$\sum_{i=1}^{n} \int_0^{L_{\varrho_i}} \frac{\mathrm{d}}{\mathrm{d}s} \psi_2(r_{\varrho_i}(s)) \, \mathrm{d}s = \sum_{i=1}^{n} \int_0^{L_{\varrho_i}} \ell(t, r_{\varrho_i}(s)) \frac{\mathrm{d}}{\mathrm{d}s} \psi_2(r_{\varrho_i}(s)) \, \mathrm{d}s \, .$$

Since each ϱ_i is regular and injective, and their image sets are disjoint, we infer that for every $\psi \in \mathrm{C}^1([0, L_{\rho_i}])$, for λ-a.a. $t \in (a,b)$ and every $i = 1, \ldots, n$ it holds

$$\int_0^{L_{\varrho_i}} \Big(1 - \ell(t, r_{\varrho_i}(s))\Big) \frac{\mathrm{d}}{\mathrm{d}s} \psi(s) \, \mathrm{d}s = 0. \tag{2.9}$$

Choosing a countable set of test functions strongly dense in $\mathrm{C}^1([0,L])$, with $L := \max_i L_{\varrho_i}$, then for every $i = 1, \ldots, n$ we have that $1 - \ell(t, r_{\varrho_i}(\cdot)) = 0$ a.e. in $[0, L_{\varrho_i}]$ for λ-a.a. $t \in (a,b)$. Thus, $\ell \equiv 1$ $|\boldsymbol{\theta}|$-a.e., hence $\boldsymbol{\zeta} = \boldsymbol{\theta}$.

In the next two statements we discuss the existence of minimal submeasures. We start with the case of bounded Radon measures.

Proposition 2.5. *Let O be an open subset of $\mathbb{R}^{d+1}$ and $\boldsymbol{\theta} \in \mathcal{M}(O; \mathbb{R}^d)$. Then, the problem*

$$\min \{ |\boldsymbol{\zeta}|(O) : \boldsymbol{\zeta} \prec \boldsymbol{\theta} \text{ and } \operatorname{div} \boldsymbol{\zeta} = \operatorname{div} \boldsymbol{\theta} \} \tag{2.10}$$

admits a solution. Moreover, every solution to (2.10) is minimal.

We point out for later use that, by Lemma 2.2, the minimum problem (2.10) can be reformulated in terms of densities:

$$\min \left\{ \int_O \lambda \, \mathrm{d}|\boldsymbol{\theta}| : \lambda \in L^\infty_{|\boldsymbol{\theta}|}(O; [0,1]), \int_O (1-\lambda) \mathrm{D}_x \varphi \, \mathrm{d}\boldsymbol{\theta} = 0 \text{ for every } \varphi \in \mathrm{C}^1_{\mathrm{c}}(O) \right\}. \tag{2.11}$$

Proof. A solution to the minimum problem (2.11), and thus to (2.10), exists, since the constraint is convex and weakly*-compact and the functional is weakly*-continuous.

Let $\boldsymbol{\zeta} \in \mathcal{M}(O; \mathbb{R}^d)$ be a solution of (2.10) and let $\widetilde{\boldsymbol{\zeta}} \in \mathcal{M}(O; \mathbb{R}^d)$ be such that $\operatorname{div} \widetilde{\boldsymbol{\zeta}} = \operatorname{div} \boldsymbol{\zeta}$ and $\widetilde{\boldsymbol{\zeta}} \prec \boldsymbol{\zeta}$. Then, $\operatorname{div} \widetilde{\boldsymbol{\zeta}} = \operatorname{div} \boldsymbol{\theta}$ and $\widetilde{\boldsymbol{\zeta}} \prec \boldsymbol{\theta}$, and $\widetilde{\boldsymbol{\zeta}}$ is a competitor for (2.10), so that $|\boldsymbol{\zeta}|(O) \le |\widetilde{\boldsymbol{\zeta}}|(O)$. Hence, by (2.6) we conclude that $\widetilde{\boldsymbol{\zeta}} = \boldsymbol{\zeta}$. □

With our following result we show that the existence of minimal submeasures extends to the case in which $\boldsymbol{\theta}$ is just a *Radon* measure in a cylindrical open set $(a,b) \times \mathbb{R}^d$ (we again emphasize that the divergence operator is only considered w.r.t. the variable $x \in \mathbb{R}^d$).

Corollary 2.6. *Let $\boldsymbol{\theta} \in \mathcal{M}_{\mathrm{loc}}((a,b) \times \mathbb{R}^d; \mathbb{R}^d)$ be such that $|\boldsymbol{\theta}|([c,d] \times \mathbb{R}^d) < +\infty$ for every $[c,d] \subset (a,b)$. Then there exists $\boldsymbol{\zeta} \in \mathcal{M}_{\mathrm{loc}}((a,b) \times \mathbb{R}^d; \mathbb{R}^d)$ minimal such that $\boldsymbol{\zeta} \prec \boldsymbol{\theta}$ and $\operatorname{div} \boldsymbol{\zeta} = \operatorname{div} \boldsymbol{\theta}$.*

Proof. We consider two sequences $(a_j)_j$, $(b_j)_j \subset (a,b)$ with $a_j \searrow a$ and $b_j \nearrow b$ as $j \to \infty$, and set $O_j := (a_j, b_j) \times \mathbb{R}^d$. By assumption, for every $j \in \mathbb{N}$ the restriction $\boldsymbol{\theta}_j := \boldsymbol{\theta} \llcorner O_j$ belongs to $\mathcal{M}(O_j; \mathbb{R}^d)$; we denote by $\operatorname{div}|_{O_j}$ the divergence operator relative to the open set O_j, i.e., restricted to test functions with a compact support in O_j, and observe that $\operatorname{div}|_{O_j}(\boldsymbol{\theta}_j) = \operatorname{div}|_{O_j}(\boldsymbol{\theta})$. We can apply Proposition 2.5 and find, for every $j \in \mathbb{N}$, a minimal measure

$$\boldsymbol{\zeta}_j \in \mathcal{M}(O_j; \mathbb{R}^d) \text{ such that } \boldsymbol{\zeta}_j \prec \boldsymbol{\theta}_j \text{ and } \operatorname{div}|_{O_j}(\boldsymbol{\zeta}_j) = \operatorname{div}|_{O_j}(\boldsymbol{\theta}_j) \, .$$

We now show that it is not restrictive to assume that

$$\boldsymbol{\zeta}_j \llcorner O_\ell = \boldsymbol{\zeta}_\ell \qquad \text{if } \ell \le j \, . \tag{2.12}$$

Indeed, since $\boldsymbol{\zeta}_j \prec \boldsymbol{\theta}_j$ and $\boldsymbol{\zeta}_\ell \prec \boldsymbol{\theta}_\ell$, there exist $\lambda_j \in L^\infty_{|\boldsymbol{\theta}_j|}(O_j;[0,1])$ and $\lambda_\ell \in L^\infty_{|\boldsymbol{\theta}_\ell|}(O_\ell;[0,1])$ such that $\boldsymbol{\zeta}_j = \lambda_j \boldsymbol{\theta}_j$ and $\boldsymbol{\zeta}_\ell = \lambda_\ell \boldsymbol{\theta}_\ell$. Recall that λ_j and λ_ℓ solve the minimum problem (2.11) on O_j and on O_ℓ, respectively. Define now $\widehat{\lambda}_j : O_j \to [0,1]$ via

$$\widehat{\lambda}_j(t,x) := \begin{cases} \lambda_\ell(t,x) & \text{if } (t,x) \in O_\ell, \\ \lambda_j(t,x) & \text{if } (t,x) \in O_j \setminus O_\ell\,. \end{cases}$$

Then, $\widehat{\lambda}_j \in L^\infty_{|\boldsymbol{\theta}_j|}(O_j;[0,1])$ and the measure $\widehat{\boldsymbol{\zeta}}_j := \widehat{\lambda}_j \boldsymbol{\theta}_j$ clearly fulfills $\widehat{\boldsymbol{\zeta}}_j \prec \boldsymbol{\theta}_j$, $\operatorname{div}|_{O_j}(\widehat{\boldsymbol{\zeta}}_j) = \operatorname{div}|_{O_j}(\boldsymbol{\theta}_j)$, and $\widehat{\boldsymbol{\zeta}}_j \llcorner O_\ell = \boldsymbol{\zeta}_\ell$. By minimality of $\boldsymbol{\zeta}_\ell$ on O_ℓ, we have that

$$|\widehat{\boldsymbol{\zeta}}_j|(O_j) = |\widehat{\boldsymbol{\zeta}}_j|(O_\ell) + |\widehat{\boldsymbol{\zeta}}_j|(O_j \setminus O_\ell) = |\boldsymbol{\zeta}_\ell|(O_\ell) + |\boldsymbol{\zeta}_j|(O_j \setminus O_\ell) \le |\boldsymbol{\zeta}_j|(O_j)\,.$$

Hence, $\widehat{\boldsymbol{\zeta}}_j$ solves the minimum problem (2.11) on O_j and is minimal. This implies that, up to replacing $\boldsymbol{\zeta}_j$ with $\widehat{\boldsymbol{\zeta}}_j$, we may assume (2.12).

Let us trivially extend each $\boldsymbol{\zeta}_j$ to $O = (a,b) \times \mathbb{R}^d$. Since $|\boldsymbol{\zeta}_j| \le |\boldsymbol{\theta}_j|$ for every $j \in \mathbb{N}$, we find that there exists $\boldsymbol{\zeta} \in \mathcal{M}_{\text{loc}}(O;\mathbb{R}^d)$ such that, up to a subsequence, $\boldsymbol{\zeta}_j \rightharpoonup^* \boldsymbol{\zeta}$ in $\mathcal{M}_{\text{loc}}(O;\mathbb{R}^d)$. Thus, $\operatorname{div} \boldsymbol{\zeta} = \operatorname{div} \boldsymbol{\theta}$. By the lower semicontinuity of the total variation and by the relation $\boldsymbol{\zeta}_j \prec \boldsymbol{\theta}_j$ we deduce that $\boldsymbol{\zeta} \prec \boldsymbol{\theta}$.

Let us now show that $\boldsymbol{\zeta}$ is minimal. Indeed, let $\boldsymbol{\xi} \in \mathcal{M}_{\text{loc}}(O;\mathbb{R}^d)$ be such that $\operatorname{div} \boldsymbol{\xi} = \operatorname{div} \boldsymbol{\zeta}$ and $\boldsymbol{\xi} \prec \boldsymbol{\zeta}$. In particular, $\boldsymbol{\xi} \llcorner O_j$ satisfies $\operatorname{div}|_{O_j}(\boldsymbol{\xi} \llcorner O_j) = \operatorname{div}|_{O_j}(\boldsymbol{\zeta}_j)$ and $\boldsymbol{\xi} \llcorner O_j \boldsymbol{\zeta}_j \prec \boldsymbol{\zeta}_j$. Thus, $\boldsymbol{\xi} \llcorner O_j = \boldsymbol{\zeta}_j$ for every $j \in \mathbb{N}$ and $\boldsymbol{\xi} = \boldsymbol{\zeta}$.

□

A crucial step in the proof of Theorem 4.7 ahead will consist in relating the weak* limits of the projections of (weakly* converging) sequences of positive and vector-valued measures, with the push forward of their weak* limits through the projection $\boldsymbol{\pi}(s,t,x) := (t,x)$, $\mathbb{R} \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R} \times \mathbb{R}^d$, which is not proper.

Now, the last result of this section addresses this issue in general, for the push forward through a generic continuous map. It provides sufficient conditions under which the push forward of a weakly* converging sequence of measures is a *submeasure* of the weak*-limits of their push forwards.

Lemma 2.7. *Let O, G be open subsets of some Euclidean spaces, let $\mathsf{p} : O \to G$ be a continuous map, let $\mathbb{R}^k$ be endowed with a strictly convex norm $\|\cdot\|$, and let $(\boldsymbol{\zeta}_n)_n \subset \mathcal{M}_{\text{loc}}(O;\mathbb{R}^k)$ satisfy*

$$\sup_{n\in\mathbb{N}} |\boldsymbol{\zeta}_n|(\mathsf{p}^{-1}(K)) < +\infty \quad \textit{for every compact subset } K \subset G\,,$$

so that $\boldsymbol{\lambda}_n = \mathsf{p}_\sharp \boldsymbol{\zeta}_n$ is a well defined measure in $\mathcal{M}_{\text{loc}}(G;\mathbb{R}^k)$. Let us assume that

$$\boldsymbol{\zeta}_n \rightharpoonup^* \boldsymbol{\zeta} \textit{ in } \mathcal{M}_{\text{loc}}(O;\mathbb{R}^k), \qquad \boldsymbol{\lambda}_n = \mathsf{p}_\sharp \boldsymbol{\zeta}_n \rightharpoonup^* \boldsymbol{\lambda} \textit{ in } \mathcal{M}_{\text{loc}}(G;\mathbb{R}^k), \tag{2.13}$$

for some $\boldsymbol{\zeta} \in \mathcal{M}_{\text{loc}}(O;\mathbb{R}^d)$, and $\boldsymbol{\lambda} \in \mathcal{M}_{\text{loc}}(G;\mathbb{R}^k)$. If

$$\mathsf{p}_\sharp |\boldsymbol{\zeta}_n| \rightharpoonup^* |\boldsymbol{\lambda}| \qquad \textit{in } \mathcal{M}^+_{\text{loc}}(G), \tag{2.14}$$

then

$$\mathsf{p}_\sharp \boldsymbol{\zeta} \prec \boldsymbol{\lambda}. \tag{2.15}$$

Proof. Let $\eta^j \in \mathrm{C_c}(O)$ form an increasing sequence such that $0 \le \eta^j \le 1$ for all $j \in \mathbb{N}$ and $\eta^j(x) \uparrow 1$ as $j \to \infty$ for every $x \in O$. For every $n \in \mathbb{N}$ and $j \ge 1$ we set

$$\boldsymbol{\zeta}_n^j := \eta^j \boldsymbol{\zeta}_n, \quad \widehat{\boldsymbol{\zeta}}_n^j := (1-\eta^j)\boldsymbol{\zeta}_n, \quad \boldsymbol{\lambda}_n := \mathsf{p}_\sharp \boldsymbol{\zeta}_n, \quad \boldsymbol{\lambda}_n^j := \mathsf{p}_\sharp \boldsymbol{\zeta}_n^j, \quad \widehat{\boldsymbol{\lambda}}_n^j := \mathsf{p}_\sharp \widehat{\boldsymbol{\zeta}}_n^j,$$

so that

$$\boldsymbol{\zeta}_n = \boldsymbol{\zeta}_n^j + \widehat{\boldsymbol{\zeta}}_n^j, \quad \boldsymbol{\lambda}_n = \boldsymbol{\lambda}_n^j + \widehat{\boldsymbol{\lambda}}_n^j. \tag{2.16}$$

Since the functions η^j have compact support, if we pass to the limit as $n \to \infty$ while keeping $j \ge 1$ fixed, we get

$$\boldsymbol{\zeta}_n^j \rightharpoonup^* \boldsymbol{\zeta}^j = \eta^j \boldsymbol{\zeta}, \quad \boldsymbol{\lambda}_n^j \rightharpoonup^* \boldsymbol{\lambda}^j = \mathsf{p}_\sharp(\boldsymbol{\zeta}^j) \quad \text{as } n \to \infty, \tag{2.17}$$

and we correspondingly deduce the convergence of $\widehat{\boldsymbol{\zeta}}^j_n$ and $\widehat{\boldsymbol{\lambda}}^j_n$ to measures $\widehat{\boldsymbol{\zeta}}^j$ and $\widehat{\boldsymbol{\lambda}}^j$ respectively, satisfying the decomposition

$$\boldsymbol{\zeta} = \boldsymbol{\zeta}^j + \widehat{\boldsymbol{\zeta}}^j, \quad \boldsymbol{\lambda} = \boldsymbol{\lambda}^j + \widehat{\boldsymbol{\lambda}}^j. \tag{2.18}$$

(Notice, however, that in general $\widehat{\lambda}^j$ does not coincide with $\mathsf{p}_\sharp \boldsymbol{\zeta}^j$). We can now consider similar decompositions on the level of the total variations

$$\alpha_n := |\boldsymbol{\zeta}_n|, \quad \alpha^j_n := \eta^j |\boldsymbol{\zeta}_n|, \quad \widehat{\alpha}^j_n := (1-\eta^j)|\boldsymbol{\zeta}_n|, \quad \beta_n := \mathsf{p}_\sharp \alpha_n, \quad \beta^j_n := \mathsf{p}_\sharp \alpha^j_n, \quad \widehat{\beta}^j_n := \mathsf{p}_\sharp \widehat{\alpha}^j_n,$$

which satisfy

$$\alpha_n = \alpha^j_n + \widehat{\alpha}^j_n, \quad \beta_n = \beta^j_n + \widehat{\beta}^j_n, \quad \beta_n \geq |\boldsymbol{\lambda}_n|, \quad \beta^j_n \geq |\boldsymbol{\lambda}^j_n|, \quad \widehat{\beta}^j_n \geq |\widehat{\boldsymbol{\lambda}}^j_n|, \quad \beta_n \rightharpoonup^* \beta = |\boldsymbol{\lambda}| \quad \text{as } n \to \infty. \tag{2.19}$$

By a possible extraction of a (not relabeled) subsequence, it is not restrictive to assume that there exists $\alpha \in \mathcal{M}^+_{\rm loc}(O)$ such that $\alpha_n \rightharpoonup^* \alpha$ as $n \to \infty$, so that

$$\alpha^j_n \rightharpoonup^* \alpha^j = \eta^j \alpha, \quad \widehat{\alpha}^j_n \rightharpoonup^* \widehat{\alpha}^j = (1-\eta^j)\alpha, \quad \beta^j_n \rightharpoonup^* \beta^j = \mathsf{p}_\sharp \alpha^j, \quad \widehat{\beta}^j_n \rightharpoonup^* \widehat{\beta}^j = \beta - \beta^j. \tag{2.20}$$

By Cantor's diagonal argument, it is possible to extract an increasing subsequence $m \mapsto n(m)$ and to find limit measures $\lambda^j, \widehat{\lambda}^j \in \mathcal{M}^+_{\rm loc}(G)$ such that for every $j \in \mathbb{N}$

$$|\boldsymbol{\lambda}^j_{n(m)}| \rightharpoonup^* \lambda^j \geq |\boldsymbol{\lambda}^j|, \quad |\widehat{\boldsymbol{\lambda}}^j_{n(m)}| \rightharpoonup^* \widehat{\lambda}^j \geq |\widehat{\boldsymbol{\lambda}}^j| \quad \text{as } m \to \infty. \tag{2.21}$$

Since

$$|\boldsymbol{\lambda}_n| \leq |\boldsymbol{\lambda}^j_n| + |\widehat{\boldsymbol{\lambda}}^j_n|$$

we deduce

$$|\boldsymbol{\lambda}| \leq \lambda^j + \widehat{\lambda}^j. \tag{2.22}$$

On the other hand, the inequalities

$$\lambda^j_n \leq \beta^j_n, \quad \widehat{\lambda}^j_n \leq \widehat{\beta}^j_n$$

yield

$$\lambda^j \leq \beta^j, \quad \widehat{\lambda}^j \leq \widehat{\beta}^j, \tag{2.23}$$

and since $\beta^j + \widehat{\beta}^j = \beta = |\boldsymbol{\lambda}|$ we conclude that

$$|\boldsymbol{\lambda}| = \lambda^j + \widehat{\lambda}^j. \tag{2.24}$$

Similarly, the inequalities $\lambda^j \geq |\boldsymbol{\lambda}^j|$, $\widehat{\lambda}^j \geq |\widehat{\boldsymbol{\lambda}}^j|$ and $|\boldsymbol{\lambda}| \leq |\boldsymbol{\lambda}^j| + |\widehat{\boldsymbol{\lambda}}^j|$ yield

$$\lambda^j = |\boldsymbol{\lambda}^j|, \quad \widehat{\lambda}^j = |\widehat{\boldsymbol{\lambda}}^j|, \quad |\boldsymbol{\lambda}| = |\boldsymbol{\lambda}^j| + |\widehat{\boldsymbol{\lambda}}^j|, \quad \boldsymbol{\lambda} = \boldsymbol{\lambda}^j + \widehat{\boldsymbol{\lambda}}^j. \tag{2.25}$$

We deduce that $\boldsymbol{\lambda}^j \prec \boldsymbol{\lambda}$. For every $\varphi \in \mathrm{C}_c(G;\mathbb{R}^k)$ we easily check that

$$\int_G \boldsymbol{\varphi} \cdot \mathrm{d}\boldsymbol{\lambda}^j = \int_O \eta^j \boldsymbol{\varphi}(\mathsf{p}(x)) \cdot \mathrm{d}\boldsymbol{\zeta}(x) \longrightarrow \int_O \boldsymbol{\varphi}(\mathsf{p}(x)) \cdot \mathrm{d}\boldsymbol{\zeta}(x) = \int_G \boldsymbol{\varphi} \cdot \mathrm{d}(\mathsf{p}_\sharp \boldsymbol{\zeta}) \quad \text{as } j \to \infty,$$

i.e. $\boldsymbol{\lambda}^j \rightharpoonup^* \mathsf{p}_\sharp \boldsymbol{\zeta}$. We eventually conclude that $\mathsf{p}_\sharp \boldsymbol{\zeta} \prec \boldsymbol{\lambda}$ by (iii) of Lemma 2.2. □

2.3. **Function spaces for the superposition principle.** Recall that $\mathbf{I}$ denotes the interval $[0,+\infty)$; if (X, d) is a complete and separable metric space, we will endow the pathspace $\mathrm{C}(\mathbf{I};\mathrm{X})$ with the Polish topology of uniform convergence on the compact subsets of $\mathbf{I}$ (see Lemma B.1). We introduce the spaces

- $\mathrm{Lip}_k(\mathbf{I};\mathrm{X})$, of k-Lipschitz paths, $k \geq 0$ which is a (closed, thus Polish) subset of $\mathrm{C}(\mathbf{I};\mathrm{X})$;
- $\mathrm{Lip}(\mathbf{I};\mathrm{X})$ of Lipschitz paths; since $\mathrm{Lip}(\mathbf{I};\mathrm{X}) = \bigcup_{k\in\mathbb{N}} \mathrm{Lip}_k(\mathbf{I};\mathrm{X})$, $\mathrm{Lip}(\mathbf{I};\mathrm{X})$ is a F_σ (namely, a countable union of closed sets), thus a Borel subset of $\mathrm{C}(\mathbf{I};\mathrm{X})$.

We introduce a few more subsets of $\mathrm{C}(\mathbf{I};\mathbb{R}^{d+1})$: first of all, the set

$$\mathrm{C}^{\uparrow}(\mathbf{I};\mathbb{R}^{d+1}) := \Big\{\mathsf{y} = (\mathsf{t},\mathsf{x}) \in \mathrm{C}(\mathbf{I};\mathbb{R}^{d+1}) :\ \mathsf{t}(0)=0,\ \mathsf{t} \text{ is non-decreasing},\ \lim_{s\uparrow\infty} \mathsf{t}(s) = +\infty\Big\}. \tag{2.26}$$

$\mathrm{C}^{\uparrow}(\mathbf{I};\mathbb{R}^{d+1})$ is a Polish space, in particular a Borel subset of $\mathrm{C}(\mathbf{I};\mathbb{R}^{d+1})$. In fact, it can be written as the intersection $A\cap B$ where

$$A := \bigcap_{n\in\mathbb{N}} \Big\{\mathsf{y}\in \mathrm{C}(\mathbf{I};\mathbb{R}^{d+1}) :\ \sup_{s\in\mathbf{I}} \mathsf{t}(s) > n\Big\},$$

$$B := \Big\{\mathsf{y} = (\mathsf{t},\mathsf{x}) \in \mathrm{C}(\mathbf{I};\mathbb{R}^{d+1})\ :\ \mathsf{t}(0)=0,\ \mathsf{t} \text{ is non-decreasing}\Big\}.$$

Since the map $\mathsf{y}\mapsto \sup_{s\in\mathbf{I}}\mathsf{t}(s)$ is lower semicontinuous in $\mathrm{C}(\mathbf{I};\mathbb{R}^{d+1})$, A is a G_δ set (namely, the countable intersection of open sets). Since B is closed, $\mathrm{C}^{\uparrow}(\mathbf{I};\mathbb{R}^{d+1})$ is a G_δ as well, and thus also Polish.

We further set

$$\mathrm{Lip}_k^{\uparrow}(\mathbf{I};\mathbb{R}^{d+1}) := \mathrm{Lip}_k(\mathbf{I};\mathbb{R}^{d+1}) \cap \mathrm{C}^{\uparrow}(\mathbf{I};\mathbb{R}^{d+1}), \quad \mathrm{Lip}^{\uparrow}(\mathbf{I};\mathbb{R}^{d+1}) := \bigcup_{k\in\mathbb{N}} \mathrm{Lip}_k^{\uparrow}(\mathbf{I};\mathbb{R}^{d+1}), \tag{2.27}$$

which are respectively a Polish and a F_σ subset of $\mathrm{C}(\mathbf{I};\mathbb{R}^{d+1})$. Finally, we define

$$\mathrm{ArcLip}(\mathbf{I};\mathbb{R}^{d+1}) := \{\mathsf{y}\in \mathrm{Lip}_1^{\uparrow}(\mathbf{I};\mathbb{R}^{d+1}) :\ \|\mathsf{y}'(s)\| = 1 \text{ for a.e. } s\in\mathbf{I}\}.$$

We notice that

$$\mathrm{ArcLip}(\mathbf{I};\mathbb{R}^{d+1}) = \bigcap_{m\in\mathbb{N}}\bigcap_{n\in\mathbb{N}} \Big\{\mathsf{y}\in \mathrm{Lip}_1^{\uparrow}(\mathbf{I};\mathbb{R}^{d+1}) :\ \int_0^m \|\mathsf{y}'(s)\|\,\mathrm{d}s > m - \frac{1}{n}\Big\},$$

so that $\mathrm{ArcLip}(\mathbf{I};\mathbb{R}^{d+1})$ is a G_δ, thus Polish and Borel, subset of $\mathrm{C}(\mathbf{I};\mathbb{R}^{d+1})$.

3. BV curves and the 'relaxed' continuity equation

The main result of this section, Theorem 3.4 below, will unveil the relation between bounded-variation curves with values in $\mathcal{P}_1(\mathbb{R}^d)$, and the continuity equation (3.1), which, throughout the paper, will be formulated as in Definition 3.1 below. Recall that we denote by $\mathbb{R}^{d+1}_+$ the space-time domain $\mathbf{I}\times\mathbb{R}^d$, which we can consider as a subset or $\mathbb{R}^{d+1}$.

Definition 3.1 (Distributional and $\mathcal{P}_1$-solutions to the continuity equation)**.** We call a pair $(\mu,\boldsymbol{\nu}) \in \mathcal{M}^+_{\mathrm{loc}}(\mathbb{R}^{d+1}_+)\times \mathcal{M}_{\mathrm{loc}}(\mathbb{R}^{d+1}_+;\mathbb{R}^d)$ a (forward, distributional) solution to the continuity equation

$$\partial_t\mu + \mathrm{div}\,\boldsymbol{\nu} = 0 \quad \text{in } \mathbb{R}^{d+1}_+, \quad \mu\geq 0, \quad \text{with initial datum } \mu_0\in\mathcal{M}^+_{\mathrm{loc}}(\mathbb{R}^d), \tag{3.1}$$

if

$$\iint_{\mathbb{R}^{d+1}} \partial_t\varphi(t,x)\,\mathrm{d}\mu(t,x) + \iint_{\mathbb{R}^{d+1}} \mathrm{D}\varphi(t,x)\,\mathrm{d}\boldsymbol{\nu}(t,x) = -\int_{\mathbb{R}^d}\varphi(0,x)\,\mathrm{d}\mu_0(x) \tag{3.2}$$

for every $\varphi\in \mathrm{C}^1_{\mathrm{c}}(\mathbb{R}^{d+1})$. We say that $(\mu,\boldsymbol{\nu})$ is a $\mathcal{P}_1$-solution if

$$\mu_0\in\mathcal{P}_1(\mathbb{R}^d), \qquad |\boldsymbol{\nu}|([0,T]\times\mathbb{R}^d) < +\infty \qquad \text{for every } T>0. \tag{3.3}$$

Observe that, since $\mu,\boldsymbol{\nu}$ are supported in $\mathbb{R}^{d+1}_+$, we could restrict the integrals in (3.2) to $\mathbb{R}^{d+1}_+$. We have integrated on $\mathbb{R}^{d+1}$, and thus considered test functions in $\mathrm{C}^1_{\mathrm{c}}(\mathbb{R}^{d+1})$, to be consistent with the usual distributional formulation in $\mathscr{D}'(\mathbb{R}^{d+1})$.

In this paper we will mainly focus on $\mathcal{P}_1$-solutions; we will also consider an important subclass characterized by a minimality condition.

Definition 3.2 (Minimal $\mathcal{P}_1$-solutions)**.** Let $(\mu,\boldsymbol{\nu})$ be a $\mathcal{P}_1$-solution to the continuity equation (3.1) and let us consider the Lebesgue decomposition of $\boldsymbol{\nu}$ as

$$\boldsymbol{\nu} = \boldsymbol{\nu}^a + \boldsymbol{\nu}^\perp, \qquad \boldsymbol{\nu}^a \ll \mu, \quad \boldsymbol{\nu}^\perp \perp \mu. \tag{3.4}$$

We say that $(\mu,\boldsymbol{\nu})$ is a *minimal* $\mathcal{P}_1$-solution if $\boldsymbol{\nu}^\perp$ is *minimal* in the sense of Definition 2.3.

Let now $\bar{\boldsymbol{\nu}} \in \mathcal{M}_{\rm loc}(\mathbb{R}^{d+1}_+;\mathbb{R}^d)$. We say that $(\mu,\bar{\boldsymbol{\nu}})$ is a *minimal pair induced* by $(\mu,\boldsymbol{\nu})$ if

$$\bar{\boldsymbol{\nu}} = \boldsymbol{\nu}^a + \bar{\boldsymbol{\nu}}^\perp, \quad \bar{\boldsymbol{\nu}}^\perp \prec \boldsymbol{\nu}^\perp, \quad \bar{\boldsymbol{\nu}}^\perp \text{ is minimal according to Definition 2.3,} \tag{3.5}$$

so that in particular $(\mu,\bar{\boldsymbol{\nu}})$ is a $\mathcal{P}_1$-solution of (3.1) as well.

Remark 3.3. The existence of a minimal pair induced by $(\mu,\boldsymbol{\nu})$ is guaranteed by Corollary 2.6. Notice that the pair $(\mu,\bar{\boldsymbol{\nu}})$ in (3.5) satisfies

$$|(\mu,\bar{\boldsymbol{\nu}})| = \theta|(\mu,\boldsymbol{\nu})|, \quad (\mu,\bar{\boldsymbol{\nu}}) = \theta(\mu,\boldsymbol{\nu}) \prec (\mu,\boldsymbol{\nu}) \qquad \text{for } \theta:\mathbb{R}^{d+1}_+ \to [0,1] \text{ Borel}, \quad \theta = 1 \ \mu\text{-a.e.} \tag{3.6}$$

We will see that minimality can also be characterized directly in terms of $\boldsymbol{\nu}$.

We now establish the analogue of [11, Thm. 8.3.1].

Theorem 3.4. *(1) Let $\mu \in \mathrm{BV}_{\rm loc}(\mathbf{I};\mathcal{P}_1(\mathbb{R}^d))$ and let $\mu^\pm$ be the left- and right-continuous representatives of μ, respectively. Then, there exists a Borel measure $\boldsymbol{\nu} \in \mathcal{M}_{\rm loc}(\mathbb{R}^{d+1}_+;\mathbb{R}^d)$ such that for every $T \in [0,+\infty)$*

$$|\boldsymbol{\nu}|([0,T)\times\mathbb{R}^d) = \mathrm{Var}_{W_1}(\mu^-;[0,T]), \qquad |\boldsymbol{\nu}|((0,T]\times\mathbb{R}^d) = \mathrm{Var}_{W_1}(\mu^+;[0,T]), \tag{3.7}$$

and the pair $(\mu,\boldsymbol{\nu})$ is a minimal $\mathcal{P}_1$-solution to the continuity equation (3.1) in the sense of Definition 3.2. 2.3.

(2) Conversely, if $(\mu,\boldsymbol{\nu})$ is a $\mathcal{P}_1$-solution to the continuity equation in the sense of Definition 3.1 with initial datum $\mu_0 \in \mathcal{P}_1(\mathbb{R}^d)$, then

(a) $\pi^0_\sharp\mu = \mathcal{L}^1$ (with $\pi^0\colon \mathbb{R}^{d+1}_+ \to \mathbf{I}$ the projection $(t,x)\mapsto t$), in particular $\mu([0,T)\times\mathbb{R}^d) = T$ for every $T\in[0,+\infty)$;

(b) there exists a curve $t\mapsto\mu_t \in \mathrm{BV}_{\rm loc}(\mathbf{I};\mathcal{P}_1(\mathbb{R}^d))$ such that $\mu = \mathcal{L}^1\otimes\mu_t$. The curve admits a narrowly left-continuous representative μ^- (a right-continuous representative μ^+, respectively), such that $\mu^-(0) := \mu_0$, the functions $t\mapsto\mu^\pm_t$ belong to $\mathrm{BV}_{\rm loc}(\mathbf{I};\mathcal{P}_1(\mathbb{R}^d))$, and there holds

$$\mathrm{Var}_{W_1}(\mu^-;[a,b]) \le |\boldsymbol{\nu}|([a,b)\times\mathbb{R}^d), \quad \mathrm{Var}_{W_1}(\mu^+;[a,b]) \le |\boldsymbol{\nu}|((a,b]\times\mathbb{R}^d) \text{ for all } [a,b]\subset\mathbf{I}. \tag{3.8}$$

Furthermore, for every $0\le a<b<+\infty$ and $\varphi\in \mathrm{C}^1_{\rm c}(\mathbb{R}^{d+1}_+)$, there holds

$$\int_{\mathbb{R}^d}\varphi(b,x)\,\mathrm{d}\mu^-_b(x) - \int_{\mathbb{R}^d}\varphi(a,x)\,\mathrm{d}\mu^+_a(x) = \int_a^b\int_{\mathbb{R}^d}\partial_t\varphi(t,x)\,\mathrm{d}\mu_t(x)\,\mathrm{d}t + \iint_{(a,b)\times\mathbb{R}^d}\mathrm{D}\varphi(t,x)\,\mathrm{d}\boldsymbol{\nu}(t,x)\,, \tag{3.9a}$$

$$\int_{\mathbb{R}^d}\varphi(b,x)\,\mathrm{d}\mu^-_b(x) - \int_{\mathbb{R}^d}\varphi(0,x)\,\mathrm{d}\mu_0(x) = \int_0^b\int_{\mathbb{R}^d}\partial_t\varphi(t,x)\,\mathrm{d}\mu_t(x)\,\mathrm{d}t + \iint_{[0,b)\times\mathbb{R}^d}\mathrm{D}\varphi(t,x)\,\mathrm{d}\boldsymbol{\nu}(t,x)\,. \tag{3.9b}$$

In particular, for every $\varphi\in\mathrm{C}^1_{\rm c}(\mathbb{R}^d)$ and $b\in\mathbf{I}$ there holds

$$\int_{\mathbb{R}^d}\varphi(x)\,\mathrm{d}\mu^+_b(x) - \int_{\mathbb{R}^d}\varphi(x)\,\mathrm{d}\mu^-_b(x) = \iint_{\{b\}\times\mathbb{R}^d}\mathrm{D}\varphi(x)\,\mathrm{d}\boldsymbol{\nu}(t,x)\,. \tag{3.10}$$

In fact, a partial analogue of part (1) of the statement has been established for BV curves of currents in [21, Thm. 6.1, Prop. 6.4] (see also [20, Theorems 6.1 and 6.2]). We will develop the *proof* of Theorem 3.4 in the ensuing subsections, starting from the second part of the statement.

Remark 3.5. The definition $\mu^-(0) := \mu_0$ for the left-continuous representative of the curve $t\mapsto\mu_t$ associated with a solution to the continuity equation, reflects the fact that, if $|\boldsymbol{\nu}|(\{0\}\times\mathbb{R}^d)>0$ the curve $t\mapsto\mu_t$ has a jump at $t=0$. Hence, $\mu^+(0)\ne\mu^-(0)$ and it is meaningful to set $\mu^-(0):=\mu_0$.

Remark 3.6 (Continuity equation in $[a,b]\times\mathbb{R}^d$). Let $0\le a<b$, $\mu_a,\mu_b\in\mathcal{P}_1(\mathbb{R}^d)$, and $(\mu,\boldsymbol{\nu})\in\mathcal{M}^+([a,b]\times\mathbb{R}^d)\times\mathcal{M}([a,b]\times\mathbb{R}^d;\mathbb{R}^d)$ (recall that $\mathcal{M}(A;\mathbb{R}^m)$ denotes the space of $\mathbb{R}^m$-valued Borel measures with finite total variation) satisfy

$$\iint_{[a,b]\times\mathbb{R}^d}\partial_t\varphi(t,x)\,\mathrm{d}\mu(t,x) + \iint_{[a,b]\times\mathbb{R}^d}\mathrm{D}\varphi(t,x)\,\mathrm{d}\boldsymbol{\nu}(t,x) = \int_{\mathbb{R}^d}\varphi(b,x)\,\mathrm{d}\mu_b(x) - \int_{\mathbb{R}^d}\varphi(a,x)\,\mathrm{d}\mu_a(x)$$

for every $\varphi \in \mathrm{C}^1_\mathrm{c}([a,b]\times\mathbb{R}^d)$. It can be immediately checked that the extensions $\widetilde{\mu}, \widetilde{\boldsymbol{\nu}}$ defined for every Borel set $A \subset \mathbb{R}^{d+1}_+$ by

$$\widetilde{\mu}(A) := \mu\Big(A\cap\big([a,b]\times\mathbb{R}^d\big)\Big) + (\mathcal{L}^1\otimes\mu_a)\Big(A\cap\big([0,a)\times\mathbb{R}^d\big)\Big) + (\mathcal{L}^1\otimes\mu_b)\Big(A\cap\big(b,+\infty)\times\mathbb{R}^d\big)\Big),$$
$$\widetilde{\boldsymbol{\nu}}(A) := \boldsymbol{\nu}\Big(A\cap\big([a,b]\times\mathbb{R}^d\big)\Big)$$

solve (3.2) in the sense of Definition 3.1.

3.1. **Proof of Part (2) of Thm. 3.4.** The proof is carried out in several steps. First of all, we show that, if μ solves (3.1), then its marginal w.r.t. the time variable coincides with the 1-dimensional Lebesgue measure on $\mathbf{I}$. We also provide a useful chain-rule formula.

Lemma 3.7 (Time marginals and distributional chain rule for $\mathcal{P}_1$-solutions)**.** *Let $\mu_0 \in \mathcal{P}_1(\mathbb{R}^d)$ and let $(\mu,\boldsymbol{\nu})$ be a $\mathcal{P}_1$-solution of the continuity equation in the sense of Definition 3.1. Then, $\mu([0,T)\times\mathbb{R}^d) = T$ for all $T>0$, $\pi^0_\sharp\mu = \mathcal{L}^1$ and $\mu = \mathcal{L}^1\otimes\mu_t$ for a family of probability measures $(\mu_t)_{t\in\mathbf{I}}$ in $\mathcal{P}_1(\mathbb{R}^d)$ with finite first moment. Furthermore, for every Lipschitz function $\varphi \in \mathrm{C}^1(\mathbb{R}^{d+1}_+)$ the map $t \mapsto \int_{\mathbb{R}^d}\varphi(t,x)\,\mathrm{d}\mu_t(x)$ (trivially extended to 0 for $t<0$) has distributional derivative*

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d}\varphi(t,\cdot)\,\mathrm{d}\mu_t = \int_{\mathbb{R}^d}\partial_t\varphi(t,\cdot)\,\mathrm{d}\mu_t + \pi^0_\sharp(\mathrm{D}\varphi\cdot\boldsymbol{\nu}) + \delta_0\int_{\mathbb{R}^d}\varphi(0,\cdot)\,\mathrm{d}\mu_0 \quad \text{in } \mathcal{D}'(\mathbb{R}), \tag{3.11}$$

(where the scalar product $\mathrm{D}\varphi\cdot\boldsymbol{\nu}$ has to be understood in the sense of (2.1)), and in particular it has essential bounded variation in every bounded interval $(0,T)$, $T>0$.

Proof. Let us first observe that selecting $\zeta \in \mathrm{C}^1_c(\mathbf{I})$ and $\varphi_c \in \mathrm{C}^1(\mathbb{R}^{d+1}_+)$ with support in $\mathbf{I}\times B_R(0)$ for some $R>0$, (3.2) yields

$$\iint_{\mathbb{R}^{d+1}_+}(\zeta'\varphi_c + \zeta\partial_t\varphi_c)\,\mathrm{d}\mu + \iint_{\mathbb{R}^{d+1}_+}\zeta\mathrm{D}\varphi_c\,\mathrm{d}\boldsymbol{\nu} = -\int_{\mathbb{R}^d}\zeta(0)\varphi_c(0,x)\,\mathrm{d}\mu_0(x). \tag{3.12}$$

In order to evaluate $\mu([0,T)\times\mathbb{R}^d)$, we consider a regularization of the function $\zeta(t) := t^- = \max(-t,0)$, for instance

$$\zeta_\varepsilon(t) := \begin{cases} t+\frac{\varepsilon}{2} & \text{if } t\le -\varepsilon,\\ -\frac{t^2}{2\varepsilon} & \text{if } -\varepsilon < t \le 0,\\ 0 & \text{if } t>0 \end{cases}$$

and we set $\zeta_{\varepsilon,T}(t) := \zeta_\varepsilon(t-T)$. We also take a function $\theta \in \mathrm{C}^\infty_\mathrm{c}(\mathbb{R}^d)$ fulfilling

$$0\le\theta\le 1, \quad \theta\equiv 1 \text{ in } B_1(0), \quad \theta\equiv 0 \text{ in } \mathbb{R}^d\setminus B_2(0), \quad \|\mathrm{D}\theta\|_\infty \le 2,$$

and for $\psi \in \mathrm{C}^1(\mathbb{R}^d)$ we set

$$\theta_R(x) := \theta(x/R), \quad \psi_R(x) := \psi(x)\theta_R(x).$$

Choosing $0<\varepsilon<T$ and $\varphi_c = \psi_R$, (3.12) yields

$$\iint_{[0,T-\varepsilon]\times\mathbb{R}^d}\psi_R\,\mathrm{d}\mu + \iint_{(T-\varepsilon,T)\times\mathbb{R}^d}\frac{T-t}{\varepsilon}\psi_R\,\mathrm{d}\mu = \left(T-\frac{\varepsilon}{2}\right)\int_{\mathbb{R}^d}\psi_R\,\mathrm{d}\mu_0 - \iint_{\mathbb{R}^{d+1}_+}\zeta_{\varepsilon,T}\mathrm{D}\psi_R\,\mathrm{d}\boldsymbol{\nu}\,. \tag{3.13}$$

We first take $\psi\equiv 1$, so that $\psi_R = \theta_R$. Since $\zeta_{\varepsilon,T} \to \zeta_T = \zeta(\cdot - T)$ as $\varepsilon\downarrow 0$, uniformly in $\mathbf{I}$, and $\|\zeta_{\varepsilon,T}\|_\infty \le T+\varepsilon/2$, $\|\mathrm{D}\theta_R\|_\infty \le \frac{2}{R}$, and $|\boldsymbol{\nu}|([0,T]\times\mathbb{R}^d) < +\infty$, we find that

$$\lim_{\varepsilon\downarrow 0}\iint_{\mathbb{R}^{d+1}_+}\zeta_{\varepsilon,T}(t)\mathrm{D}\theta_R(x)\,\mathrm{d}\boldsymbol{\nu}(t,x) = \iint_{[0,T]\times\mathbb{R}^d}(T-t)_+\mathrm{D}\theta_R(x)\,\mathrm{d}\boldsymbol{\nu}(t,x)\,.$$

Clearly,

$$\lim_{\varepsilon\downarrow 0}\iint_{[0,T-\varepsilon]\times\mathbb{R}^d}\theta_R(x)\,\mathrm{d}\mu(t,x) = \iint_{[0,T)\times\mathbb{R}^d}\theta_R(x)\,\mathrm{d}\mu(t,x)\,.$$

Finally,

$$\left|\iint_{(T-\varepsilon,T)\times\mathbb{R}^d} \frac{T-t}{\varepsilon}\theta_R(x)\,\mathrm{d}\mu(t,x)\right| \le |\mu|((T-\varepsilon,T)\times\mathbb{R}^d) \longrightarrow 0 \quad \text{as } \varepsilon\downarrow 0\,.$$

Passing to the limit as $\varepsilon\downarrow 0$ in (3.13) we get

$$\iint_{[0,T)\times\mathbb{R}^d} \theta_R(x)\,\mathrm{d}\mu(t,x) = T\int_{\mathbb{R}^d}\theta_R(x)\,\mathrm{d}\mu_0(x) - \iint_{[0,T]\times\mathbb{R}^d}(T-t)_+\mathrm{D}\theta_R(x)\,\mathrm{d}\boldsymbol{\nu}(t,x)\,.$$

We now take the limit as $R\to\infty$ in both sides of the above equality, recalling that $\theta_R\to 1$ and that $\|\mathrm{D}\theta_R\|_\infty\to 0$. Hence, we conclude that $\mu([0,T)\times\mathbb{R}^d) = T\mu_0(\mathbb{R}^d) = T$, i.e. $\pi^0_\sharp\mu([0,T)) = T$ and therefore $\pi^0_\sharp\mu([a,b)) = b-a$ for every $0\le a<b$. This implies that $\pi^0_\sharp\mu$ is the Lebesgue measure $\mathcal{L}^1$.

By the disintegration theorem (cf., e.g., [45, Cor. A.5]), we can disintegrate the measure μ with respect to the projection $\pi^0\colon\mathbb{R}^{d+1}_+\to\mathbf{I}$, so that there exists a Borel family $\{\mu_t\}_{t\in\mathbf{I}}$ of probability measures on $\mathbb{R}^d$ such that $\mu=\mathcal{L}^1\otimes\mu_t$.

We now use (3.13) by choosing $\psi(x):=\sqrt{1+\|x\|^2}$; since

$$|\mathrm{D}\psi_R(x)| \le |\mathrm{D}\theta_R(x)\psi(x)| + |\theta_R(x)\mathrm{D}\psi(x)| \le \frac{2}{R}\sqrt{1+(2R)^2}+1 \le 5 \quad \text{if } R\ge 2,$$

we obtain the uniform estimate for $R\ge 2$

$$\int_{[0,T-\varepsilon]\times\mathbb{R}^d}\psi_R\,\mathrm{d}\mu \le T\int_{\mathbb{R}^d}\psi_R\,\mathrm{d}\mu_0 + 5T|\boldsymbol{\nu}|([0,T]\times\mathbb{R}^d).$$

Passing to the limit as $R\uparrow+\infty$ we deduce that

$$\int_0^T\int_{\mathbb{R}^d}\sqrt{1+\|x\|^2}\,\mathrm{d}\mu_t\,\mathrm{d}t \le CT, \quad C:=\int_{\mathbb{R}^d}\sqrt{1+\|x\|^2}\,\mathrm{d}\mu_0 + 5|\boldsymbol{\nu}|([0,T]\times\mathbb{R}^d), \tag{3.14}$$

so that $\mu_t\in\mathcal{P}_1(\mathbb{R}^d)$ for $\mathcal{L}^1$-a.a. $t>0$.

Eventually, we write (3.12) for an arbitrary $\zeta\in\mathrm{C}^1_c(\mathbf{I})$ and $\varphi_c=\varphi\theta_R$ (with $\varphi\in\mathrm{C}^1(\mathbb{R}^{d+1}_+)$ Lipschitz) and we get

$$\iint_{\mathbb{R}^{d+1}_+}(\zeta'\varphi+\zeta\partial_t\varphi)\theta_R\,\mathrm{d}\mu + \iint_{\mathbb{R}^{d+1}_+}\zeta\mathrm{D}\varphi\theta_R\mathrm{d}\boldsymbol{\nu} = -\int_{\mathbb{R}^d}\zeta(0)\varphi(0,\cdot)\theta_R\,\mathrm{d}\mu_0 - E_R, \tag{3.15}$$

where

$$E_R = \iint_{\mathbb{R}^{d+1}_+}\zeta\varphi\mathrm{D}\theta_R\,\mathrm{d}\boldsymbol{\nu}\,.$$

Choosing constants a,L,T such that $|\zeta(t)|\le a$, $\mathrm{supp}(\zeta)\subset[0,T]$, $|\varphi(t,x)|\le L(1+\|x\|)$ whenever $0\le t\le T$, we obtain

$$|E_R| \le 2aL\frac{(1+2R)}{R}\,|\boldsymbol{\nu}|\Big([0,T]\times(B_{2R}(0)\setminus B_R(0))\Big),$$

so that $\lim_{R\to\infty}|E_R|=0$. Passing to the limit in (3.15) as $R\uparrow+\infty$ using the fact that φ has linear growth and $\partial_t\varphi$ is bounded, we get

$$\int_{\mathbf{I}}\Big[\zeta'\Big(\int_{\mathbb{R}^d}\varphi_t\,\mathrm{d}\mu_t\Big)+\zeta\Big(\int_{\mathbb{R}^d}\partial_t\varphi_t\,\mathrm{d}\mu_t\Big)\Big]\,\mathrm{d}t + \iint_{\mathbb{R}^{d+1}_+}\zeta\mathrm{D}\psi\,\mathrm{d}\boldsymbol{\nu} = -\int_{\mathbb{R}^d}\zeta(0)\varphi_0\,\mathrm{d}\mu_0, \tag{3.16}$$

which in particular yields (3.11). □

We now show the existence of the left- and right-continuous representatives. The following result extends [11, Lemma 8.1.2] and concludes **the proof of Part** 2. **of Thm. 3.4**.

Lemma 3.8 (Left- and right- continuous representatives)**.** *Let $\mu_0\in\mathcal{P}_1(\mathbb{R}^d)$ and let $(\mu,\boldsymbol{\nu})$ be a $\mathcal{P}_1$-solution to the continuity equation in the sense of Definition 3.1. Then, there exists a narrowly left- (resp. narrowly right-)continuous representative $\mathbf{I}\ni t\mapsto\mu^-_t\in\mathcal{P}(\mathbb{R}^d)$ (resp. $\mathbf{I}\ni t\mapsto\mu^+_t\in\mathcal{P}(\mathbb{R}^d)$) of the curve $t\mapsto\mu_t$, such that, setting $\mu^-_0:=\mu_0$, for every $0\le a\le b<+\infty$ the following estimates hold*

$$W_1(\mu^-_a,\mu^-_b)\le|\boldsymbol{\nu}|([a,b)\times\mathbb{R}^d), \tag{3.17}$$

$$W_1(\mu_a^+, \mu_b^+) \le |\boldsymbol{\nu}|((a,b] \times \mathbb{R}^d), \tag{3.18}$$

as well as estimates (3.8) *and relations* (3.9) *and* (3.10).

Proof. We combine the argument of [11, Lemma 8.1.2] with the duality characterization of the Kantorovich-Rubinstein-Wasserstein metric.

We set $\nu := \pi^0_\sharp(|\boldsymbol{\nu}|)$ and $D_\nu := \{t \in \mathbf{I} : \nu(\{t\}) = 0\}$, whose complement is at most countable; we also select a Borel set D_μ such that $\mathcal{L}^1(\mathbf{I} \setminus D_\mu) = 0$ and $\int_{\mathbb{R}^d} |x| \,\mathrm{d}\mu_t(x) < +\infty$ for every $t \in D_\mu$. Finally, we select a countable set $Z \subset \mathrm{C}^1_c(\mathbb{R}^d)$ such that every function $\psi \in Z$ is 1-Lipschitz and Z provides the representation

$$W_1(\mu', \mu'') = \sup\Big\{ \int_{\mathbb{R}^d} \psi \,\mathrm{d}(\mu' - \mu'') : \psi \in Z \Big\}. \tag{3.19}$$

For such ψ, let us still denote by $\mu_t(\psi)$ a good representative [10, Theorem 3.28] of the map $t \mapsto \int_{\mathbb{R}^d} \psi \,\mathrm{d}\mu_t$, and let us denote with D_ψ the set of continuity points of the function $t \mapsto \mu_t(\psi)$. We eventually set $D := D_\nu \cap D_\mu \cap \bigcap_{\psi \in Z} D_\psi$. Then, $\mathcal{L}^1(\mathbf{I} \setminus D) = 0$ and $t \mapsto \mu_t(\psi)$ is continuous at $t \in D$ for every $\psi \in Z$.

For every $r < s \in D$ and every $\psi \in Z$ we have

$$|\mu_s(\psi) - \mu_r(\psi)| \le \iint_{[r,s] \times \mathbb{R}^d} |\mathrm{D}\psi| \,\mathrm{d}|\boldsymbol{\nu}|(t,x) \le |\boldsymbol{\nu}|([r,s] \times \mathbb{R}^d) = \nu((r,s)), \tag{3.20}$$

so that (3.19) yields

$$W_1(\mu_r, \mu_s) \le \nu((r,s]) = \nu((r,s)) \quad \text{for every } r, s \in D,\ r < s. \tag{3.21}$$

Thus, the map $D \ni r \mapsto \mu_r$ has pointwise bounded variation in every bounded subset of D. Since $(\mathcal{P}_1(\mathbb{R}^d), W_1)$ is complete, by a standard density argument we deduce that the limits

$$\mu_t^- := \lim_{s \in D, s \uparrow t} \mu_s, \quad \mu_t^+ := \lim_{s \in D, s \downarrow t} \mu_t \tag{3.22}$$

exist in $(\mathcal{P}_1(\mathbb{R}^d), W_1)$ for every $t \in \mathbf{I}$ and define a left-continuous and a right continuous map respectively satisfying (3.8).

Then, relations (3.9) and (3.10) immediately follow from (3.11) by observing that for every Lipschitz function $\varphi \in \mathrm{C}^1(\mathbb{R}^{d+1}_+)$ the map $t \mapsto \mu^-(\varphi_t)$ (resp. $t \mapsto \mu^+(\varphi_t)$) is left- (resp. right-)continuous and thus provides the unique left- (resp. right-) continuous representative of $t \mapsto \mu_t(\varphi_t)$. □

3.2. Proof of Part (1) of Thm. 3.4.

Let $\mu \in \mathrm{BV}_{\mathrm{loc}}(\mathbf{I}; \mathcal{P}_1(\mathbb{R}^d))$ and let $\mu^\pm$ be the left- and right-continuous representatives of μ, respectively. We define

$$V_\mu(t) := \mathrm{Var}_{W_1}(\mu^-; [0,t]) = \mathrm{Var}_{W_1}(\mu^-; [0,t)). \tag{3.23}$$

We prove the following claim: *there exists a Borel measure $\boldsymbol{\nu} \in \mathcal{M}_{\mathrm{loc}}(\mathbb{R}^{d+1}_+; \mathbb{R}^d)$ such that for every $T \in [0,+\infty)$ relations* (3.7) *hold, and the pair $(\mu, \boldsymbol{\nu})$ satisfies the continuity equation* (3.1) *in the sense of Definition 3.1. Moreover, writing $\boldsymbol{\nu} = \boldsymbol{\nu}^a \mu + \boldsymbol{\nu}^\perp$ with $\boldsymbol{\nu}^a \mu \ll \mu$ and $\boldsymbol{\nu}^\perp \perp \mu$, we have that $\boldsymbol{\nu}^\perp$ is minimal in the sense of Definition 2.3.*

Indeed, let us consider two continuity points $a < b$ for V_μ (and thus for μ) and let us define the linear functional $\ell_{a,b} \colon \mathrm{C}^1_c([a,b] \times \mathbb{R}^d) \to \mathbb{R}$ by

$$\ell_{a,b}(\zeta) := \int_a^b \int_{\mathbb{R}^d} \partial_t \zeta \,\mathrm{d}\mu_t(x) \,\mathrm{d}t + \int_{\mathbb{R}^d} \zeta(a,x) \,\mathrm{d}\mu_a(x) - \int_{\mathbb{R}^d} \zeta(b,x) \,\mathrm{d}\mu_b(x). \tag{3.24}$$

Observe that, by continuity of μ at a, b, and continuity of ζ, we have

$$\begin{aligned} \ell_{a,b}(\zeta) &= \lim_{h \downarrow 0} \frac{1}{h} \int_a^{b-h} \int_{\mathbb{R}^d} \Big(\zeta(t+h,x) - \zeta(t,x) \Big) \,\mathrm{d}\mu_t(x) \,\mathrm{d}t \\ &\qquad + \lim_{h \downarrow 0} \frac{1}{h} \int_a^{a+h} \int_{\mathbb{R}^d} \zeta(t,x) \,\mathrm{d}\mu_t(x) \,\mathrm{d}t - \lim_{h \downarrow 0} \frac{1}{h} \int_{b-h}^b \int_{\mathbb{R}^d} \zeta(t,x) \,\mathrm{d}\mu_t(x) \,\mathrm{d}t \\ &= \lim_{h \downarrow 0} \frac{1}{h} \int_{a+h}^b \int_{\mathbb{R}^d} \zeta(s,x) \,\mathrm{d}(\mu_{s-h} - \mu_s)(x) \,\mathrm{d}s = \lim_{h \downarrow 0} \frac{1}{h} \int_{a+h}^b \int_{\mathbb{R}^d} \zeta(s,x) \,\mathrm{d}(\mu^-_{s-h} - \mu^-_s)(x) \,\mathrm{d}s. \end{aligned}$$

The duality formula for the Wasserstein metric yields

$$\left|\frac{1}{h}\int_{\mathbb{R}^d}\zeta_s\,\mathrm{d}(\mu^-_{s-h}-\mu^-_s)\right|\le\frac{1}{h}W_1(\mu^-_{s-h},\mu^-_s)\sup_{\mathbb{R}^d}\|\mathrm{D}\zeta_s\|\le\frac{1}{h}\Big(V_\mu(s)-V_\mu(s-h)\Big)\sup_{\mathbb{R}^d}\|\mathrm{D}\zeta_s\|,$$

so that

$$\left|\frac{1}{h}\int_{a+h}^b\int_{\mathbb{R}^d}\zeta_s\,\mathrm{d}(\mu_{s-h}-\mu_s)\,\mathrm{d}s\right|\le\frac{1}{h}\|\mathrm{D}\zeta\|_\infty\Big(\int_{b-h}^b V_\mu\,\mathrm{d}s-\int_a^{a+h}V_\mu\,\mathrm{d}s\Big)\,.$$

Therefore, taking the limit as $h\downarrow 0$ we obtain

$$|\ell_{a,b}(\zeta)|\le\|\mathrm{D}\zeta\|_\infty\Big(V_\mu(b)-V_\mu(a)\Big)\,.$$

Therefore, the linear functional $L_{a,b}$ defined on the space $\mathbf{V}_{a,b}:=\{\mathrm{D}\zeta\ :\ \zeta\in\mathrm{C}^1_{\mathrm{c}}([a,b]\times\mathbb{R}^d)\}$ by $L_{a,b}(\boldsymbol{\xi}):=\ell_{a,b}(\zeta)$ whenever $\boldsymbol{\xi}=\mathrm{D}\zeta$ is well defined and it satisfies

$$\|L_{a,b}\|=\sup_{\boldsymbol{\xi}\in\mathbf{V}_{a,b},\|\boldsymbol{\xi}\|_\infty\le1}L_{a,b}(\boldsymbol{\xi})\le V_\mu(b)-V_\mu(a)\,.$$

By the Hahn-Banach and Riesz representation theorems, we can find a vector measure $\boldsymbol{\nu}_{a,b}$ on $[a,b]\times\mathbb{R}^d$, which satisfies

$$\iint_{[a,b]\times\mathbb{R}^d}\mathrm{D}\zeta\,\mathrm{d}\boldsymbol{\nu}_{a,b}=\langle\boldsymbol{\nu}_{a,b},\mathrm{D}\zeta\rangle=\ell_{a,b}(\zeta),\tag{3.25}$$

$$|\boldsymbol{\nu}_{a,b}|([a,b]\times\mathbb{R}^d)=\|L_{a,b}\|\le V_\mu(b)-V_\mu(a)\,.\tag{3.26}$$

Now, from (3.25) we deduce that the pair $(\mu,\boldsymbol{\nu}_{a,b})$ satisfies the continuity equation on $[a,b]\times\mathbb{R}^d$ in the sense of Remark 3.6. Since μ is Lipschitz, estimate (3.8) then yields

$$V_\mu(\beta)-V_\mu(\alpha)\le|\boldsymbol{\nu}_{a,b}|([\alpha,\beta)\times\mathbb{R}^d)\quad\text{for every }a\le\alpha<\beta\le b,\tag{3.27}$$

so that we derive

$$|\boldsymbol{\nu}_{a,b}|(\{a\}\times\mathbb{R}^d)=|\boldsymbol{\nu}_{a,b}|(\{b\}\times\mathbb{R}^d)=0,\quad|\boldsymbol{\nu}_{a,b}|([\alpha,\beta)\times\mathbb{R}^d)=V_\mu(\beta)-V_\mu(\alpha)\,.\tag{3.28}$$

Possibly extending μ to $(-\infty,0)$ by setting $\mu_t:=\mu_0$ and selecting a diverging sequence $(a_n)_{n\in\mathbb{N}}$ of continuity points for V_μ with $a_0\le 0$, we can now apply the above results to a sequence of intervals $[a_n,a_{n+1}]$, $n\in\mathbb{N}$, and we define the vector measure $\boldsymbol{\nu}$ whose restriction to $[a_n,a_{n+1}]\times\mathbb{R}^d$ coincides with $\boldsymbol{\nu}_{a_n,a_{n+1}}$. By (3.28) such a gluing process is well defined and it is easy to check that $(\mu,\boldsymbol{\nu})$ satisfies the continuity equation and (3.7).

Finally, we decompose $\boldsymbol{\nu}=\boldsymbol{\nu}^a+\boldsymbol{\nu}^\perp$ into its absolutely continuous part $\boldsymbol{\nu}^a$ and singular part $\boldsymbol{\nu}^\perp$ w.r.t. μ, and show that $\boldsymbol{\nu}^\perp$ is minimal. Let $\boldsymbol{\rho}\in\mathcal{M}_{\mathrm{loc}}(\mathbb{R}^{d+1}_+;\mathbb{R}^d)$ fulfill $\operatorname{div}\boldsymbol{\rho}=\operatorname{div}\boldsymbol{\nu}^\perp$ and $\boldsymbol{\rho}\prec\boldsymbol{\nu}^\perp$. Setting $\boldsymbol{\theta}=\boldsymbol{\nu}^a+\boldsymbol{\rho}$, the pair $(\mu,\boldsymbol{\theta})$ satisfies the continuity equation

$$\partial_t\mu+\operatorname{div}\boldsymbol{\theta}=0\qquad\text{in }(0,+\infty)\times\mathbb{R}^d$$

with initial datum μ_0, in the sense of Definition 3.1. In particular, by (3.7) and (3.8) we have that for $T\in[0,+\infty)$

$$\begin{aligned}|\boldsymbol{\nu}^a|([0,T)\times\mathbb{R}^d)+|\boldsymbol{\nu}^\perp|([0,T)\times\mathbb{R}^d)&\le\mathrm{Var}_{W_1}(\mu,[0,T))\\&\le|\boldsymbol{\theta}|([0,T)\times\mathbb{R}^d)=|\boldsymbol{\nu}^a|([0,T)\times\mathbb{R}^d)+|\boldsymbol{\rho}|([0,T)\times\mathbb{R}^d)\,.\end{aligned}\tag{3.29}$$

Thus, $|\boldsymbol{\nu}^\perp|([0,T)\times\mathbb{R}^d)\le|\boldsymbol{\rho}|([0,T)\times\mathbb{R}^d)$. Combining this with the fact that $\boldsymbol{\rho}\prec\boldsymbol{\nu}^\perp$ and recalling property (2.6), we conclude that $\boldsymbol{\nu}^\perp=\boldsymbol{\rho}$. Thus, $\boldsymbol{\nu}^\perp$ is minimal. ■

4. The augmented continuity equation

This section revolves around the result at the core of our approach to the superposition principle for the continuity equation (3.1). The main idea is to lift a pair $(\mu, \boldsymbol{\nu})$ solving (3.1), to a solution of an augmented continuity equation in $\mathbb{R}^{d+2}_+ = \mathbf{I} \times \mathbb{R}^{d+1}$ exhibiting distinguished properties.

Throughout this section we will denote by

$$(t, x) \text{ any element in } \mathbb{R}^{d+1} = \mathbb{R} \times \mathbb{R}^d, \quad \boldsymbol{\pi} : \mathbb{R} \times \mathbb{R}^{d+1} \to \mathbb{R}^{d+1}, \ \boldsymbol{\pi}(s; t, x) := (t, x). \tag{4.1}$$

and indicate by the symbol $\|\cdot\|$ a norm in $\mathbb{R}^{d+1}$, whose restriction on $\{0\} \times \mathbb{R}^d$ induces a norm on $\mathbb{R}^d$ which will be denoted by the same symbol. As previously observed, a choice of the norm in $\mathbb{R}^{d+1}$ affects the W_1-distance on $\mathcal{P}_1(\mathbb{R}^{d+1})$, cf. (2.3).

Theorem 4.7 ahead associates with a solution $(\mu, \boldsymbol{\nu})$ to the continuity equation (in the sense of Definition 3.1), a curve of measures $(\sigma_s)_{s \in \mathbf{I}} \subset \mathcal{P}_1(\mathbb{R}^{d+1})$ and a vector measure $(\sigma^0, \boldsymbol{\sigma}) \in \mathcal{M}_{\text{loc}}(\mathbf{I} \times \mathbb{R}^{d+1}; \mathbb{R}^{d+1})$ that turn out to solve the 'augmented' continuity equation in $\mathbb{R}^{d+2}_+ = \mathbf{I} \times \mathbb{R}^{d+1}$,

$$\begin{cases} \partial_s \sigma + \partial_t \sigma^0 + \operatorname{div} \boldsymbol{\sigma} = 0 & \text{in } \mathbb{R}^{d+2}_+, \\ \sigma, \sigma^0 \geq 0 & \text{in } \mathbb{R}^{d+2}_+, \\ \sigma_0 = \delta_0 \otimes \mu_0 & \text{in } \mathbb{R}^{d+1}. \end{cases} \tag{4.2}$$

In this connection, we mention that, hereafter, with slight abuse of notation we shall denote by the same symbol *both* the curve $\sigma : \mathbf{I} \to \mathcal{P}_1(\mathbb{R}^{d+1})$ *and* the measure $\sigma = \mathcal{L}^1 \otimes \sigma_s \in \mathcal{M}^+_{\text{loc}}(\mathbf{I} \times \mathbb{R}^{d+1})$. In equation (4.2), the 'augmented' operator $(\partial_t, \operatorname{div})$ plays the role that the 'spatial divergence' had for the continuity equation (3.1). Thanks to the construction carried out in the proof of Theorem 4.7 ahead, the vector measure $(\sigma^0, \boldsymbol{\sigma})$ will be induced by an autonomous (i.e. independent of s) velocity field given by a pair of bounded Borel maps $(\tau, \boldsymbol{v}) : \mathbb{R}^{d+1} \to \mathbb{R}^{d+1}$ that satisfy

$$\sigma^0 = (\tau \circ \boldsymbol{\pi})\sigma, \quad \boldsymbol{\sigma} = (\boldsymbol{v} \circ \boldsymbol{\pi})\sigma, \quad \tau \geq 0. \tag{4.3}$$

We formalize the above properties in the next Definition 4.2, after recalling an equivalence relation between positive measures.

Definition 4.1 (Uniformly equivalent measures)**.** We say that two measures $\varrho, \vartheta \in \mathcal{M}^+_{\text{loc}}(\mathbb{R}^h)$ are k-uniformly equivalent (for some $k \geq 1$), and we write $\varrho \sim_k \vartheta$, if

$$k^{-1} \varrho \leq \vartheta \leq k \varrho. \tag{4.4}$$

We write $\varrho \sim \vartheta$ if there exists $k \geq 1$ such that $\varrho \sim_k \vartheta$.

Clearly, two uniformly equivalent measures are mutually absolutely continuous (and thus equivalent, sharing the same collection of null sets). Moreover, their mutual Lebesgue densities are bounded and uniformly bounded away from 0.

With this notion at hand, we can introduce 'qualified' solutions $(\sigma, \sigma^0, \boldsymbol{\sigma})$ of the augmented continuity equation. In particular, the second and third properties below establish a relation between σ and the pair $(\sigma^0, \boldsymbol{\sigma})$.

Definition 4.2 (Solutions of the augmented continuity equation)**.** Let $\mu_0 \in \mathcal{P}_1(\mathbb{R}^d)$ and let $\sigma = \mathcal{L}^1 \otimes \sigma_s \in \mathcal{M}^+_{\text{loc}}(\mathbb{R}^{d+2}_+)$, $\sigma^0 \in \mathcal{M}^+_{\text{loc}}(\mathbb{R}^{d+2}_+)$, and $\boldsymbol{\sigma} \in \mathcal{M}_{\text{loc}}(\mathbb{R}^{d+2}_+; \mathbb{R}^d)$ be such that $(\sigma, \sigma^0, \boldsymbol{\sigma})$ is a $\mathcal{P}_1$-solution to the augmented continuity equation (4.2) according to Definition 3.1. We say that

(1) $(\sigma, \sigma^0, \boldsymbol{\sigma})$ *has locally finite* $\boldsymbol{\pi}$*-marginals* if

$$|(\sigma, \sigma^0, \boldsymbol{\sigma})|\Big(\mathbf{I} \times [0, T] \times \mathbb{R}^d\Big) < +\infty \qquad \text{for every } T > 0, \tag{4.5}$$

so that, in particular, $\boldsymbol{\pi}_\sharp(\sigma, \sigma^0, \boldsymbol{\sigma})$ is a Radon vector measure in $\mathbb{R}^{d+1}_+$.

(2) $(\sigma, \sigma^0, \boldsymbol{\sigma})$ *is* k*-adapted* if $\sigma \sim_k |(\sigma^0, \boldsymbol{\sigma})|$ and *normalized* if $\sigma = |(\sigma^0, \boldsymbol{\sigma})|$ (i.e. $k = 1$).

(3) $(\sigma, \sigma^0, \boldsymbol{\sigma})$ is $\boldsymbol{\pi}$*-autonomous* if there exists a pair of *Borel maps* $(\tau, \boldsymbol{v}) : \mathbb{R}^{d+1}_+ \to \mathbb{R}^{d+1}_+$ such that the autonomous density condition (4.3) holds.

An *adapted* (resp. *normalized*) solution $(\sigma,\sigma^0,\boldsymbol{\sigma})$ which has *locally finite* $\boldsymbol{\pi}$*-marginals* and is $\boldsymbol{\pi}$*-autonomous* will be called $\boldsymbol{\pi}$*-adapted* (resp. $\boldsymbol{\pi}$-normalized).

Lemma 4.3 (Elementary properties of augmented solutions). *Let $(\sigma,\sigma^0,\boldsymbol{\sigma})$ be a $\mathcal{P}_1$-solution to the augmented continuity equation (4.2) with $\mu_0\in\mathcal{P}_1(\mathbb{R}^d)$ and locally finite time marginals.*

(1) If $(\sigma,\sigma^0,\boldsymbol{\sigma})$ is k-adapted, then the disintegration $(\sigma_s)_{s\ge0}$ of σ w.r.t. the Lebesgue measure in $\mathbf{I}$ admits a representation which belongs to the space $\mathrm{Lip}_k(\mathbf{I};\mathcal{P}_1(\mathbb{R}^{d+1}))$ (it is in fact Lipschitz with values in $\mathcal{P}_p(\mathbb{R}^{d+1})$ if $\mu_0\in\mathcal{P}_p(\mathbb{R}^d)$, $p\ge1$).

(2) If $(\sigma,\sigma^0,\boldsymbol{\sigma})$ is $\boldsymbol{\pi}$-autonomous, then

$$|(\sigma^0,\boldsymbol{\sigma})| = \|(\tau,\boldsymbol{v})\|\sigma \ll \sigma, \quad \boldsymbol{\pi}_\sharp(\sigma^0,\boldsymbol{\sigma}) = (\tau,\boldsymbol{v})\boldsymbol{\pi}_\sharp\sigma, \tag{4.6}$$

$$\boldsymbol{\pi}_\sharp|(\sigma^0,\boldsymbol{\sigma})| = \boldsymbol{\pi}_\sharp\big(\|(\tau,\boldsymbol{v})\|\sigma\big) = \|(\tau,\boldsymbol{v})\|\boldsymbol{\pi}_\sharp\sigma = |\boldsymbol{\pi}_\sharp(\sigma^0,\boldsymbol{\sigma})|. \tag{4.7}$$

(3) If $(\sigma,\sigma^0,\boldsymbol{\sigma})$ is $\boldsymbol{\pi}$-normalized then $\|(\tau,\boldsymbol{v})\|\equiv1$ σ-a.e. and

$$\sigma = |(\sigma^0,\boldsymbol{\sigma})|, \quad \boldsymbol{\pi}_\sharp\sigma = |(\boldsymbol{\pi}_\sharp\sigma^0,\boldsymbol{\pi}_\sharp\boldsymbol{\sigma})|\,. \tag{4.8}$$

Moreover, the map $s\mapsto\sigma_s$ belongs to $\mathrm{Lip}_1(\mathbf{I};\mathcal{P}_1(\mathbb{R}^{d+1}))$.

(4) Conversely, if $\|\cdot\|$ is a strictly convex norm of $\mathbb{R}^{d+1}$ and $(\sigma,\sigma^0,\boldsymbol{\sigma})$ satisfies (4.8) then it is a $\boldsymbol{\pi}$-normalized solution.

Proof. Claim (1) immediately follows from Theorem 3.4 since the Lebesgue density of $(\sigma^0,\boldsymbol{\sigma})$ w.r.t. σ is uniformly bounded.

Claim (2) is an immediate consequence of the autonomous property. Claim (3) follows from Claim (2) and the normalization condition.

Concerning the last Claim (4), the normalized property is obious since $\sigma=|(\sigma^0,\boldsymbol{\sigma})|$. The autonomous condition is a consequence of Lemma A.1 ahead and the strict convexity of the norm. □

Let us now establish a first easy link between the solutions to the continuity equation (3.1) and those to its *augmented* counterpart (4.2).

Lemma 4.4 (Marginals of sutonomous solutions). *Let $(\sigma,\sigma^0,\boldsymbol{\sigma})$ be an adapted solution to the augmented continuity equation (4.2) with locally finite $\boldsymbol{\pi}$-marginals according to Definition 4.2, and an initial datum $\mu_0\in\mathcal{P}_1(\mathbb{R}^d)$. Then, setting*

$$\mu := \boldsymbol{\pi}_\sharp\sigma^0, \qquad \boldsymbol{\nu} := \boldsymbol{\pi}_\sharp\boldsymbol{\sigma}\,,$$

the pair $(\mu,\boldsymbol{\nu})$ is a $\mathcal{P}_1$-solution to the continuity equation (3.1), with initial datum μ_0, in the sense of Definition 3.1.

Proof. Recall that $\mu,\boldsymbol{\nu}$ are well defined Radon measures thanks to (4.5). As in Theorem 3.4(2), we have that $\sigma=\mathcal{L}^1\otimes\sigma_s$, where $\sigma_s\in\mathcal{P}(\mathbb{R}^{d+1}_+)$ for $s\in\mathbf{I}$ thanks to the next Theorem 4.5.

Let us fix $\zeta\in\mathrm{C}^1_\mathrm{c}([0,2))$ such that $0\le\zeta\le1$ and $\zeta\equiv1$ in $[0,1]$, and let us define the sequence $\zeta_n\in\mathrm{C}^1_\mathrm{c}(\mathbf{I})$ by $\zeta_n(s) := \zeta(s/n)$, so that $\zeta_n(s)\to1$ for every $s\in\mathbf{I}$ as $n\to\infty$, $0\le\zeta_n\le1$, and $\|\zeta_n'\|_\infty\le\frac{\|\zeta'\|_\infty}{n}$. For every n and every $\varphi\in\mathrm{C}^1_\mathrm{c}(\mathbb{R}\times\mathbb{R}^d)$ it holds $\zeta_n\varphi\in\mathrm{C}^1_\mathrm{c}(\mathbf{I}\times\mathbb{R}\times\mathbb{R}^d)$. Since the triple $(\sigma,\sigma^0,\boldsymbol{\sigma})$ solves the Cauchy problem (4.2) and $\mathrm{spt}(\sigma^0)$, $\mathrm{spt}(\boldsymbol{\sigma})\subset\mathbf{I}\times[0,+\infty)\times\mathbb{R}^d$, we have that

$$\begin{aligned}\int_{\mathbf{I}}\zeta_n'(s)\iint_{\mathbb{R}^{d+1}_+}\varphi(t,x)\,\mathrm{d}\sigma_s(t,x)\,\mathrm{d}s + \iiint_{\mathbf{I}\times\mathbb{R}^{d+1}_+}\zeta_n(s)\partial_t\varphi(t,x)\,\mathrm{d}\sigma^0(s,t,x)\\ + \iint_{\mathbf{I}\times\mathbb{R}^{d+1}_+}\zeta_n(s)\mathrm{D}\varphi(t,x)\,\mathrm{d}\boldsymbol{\sigma}(s,t,x) = -\int_{\mathbb{R}^d}\varphi(0,x)\,\mathrm{d}\mu_0(x)\,.\end{aligned} \tag{4.9}$$

Since φ has compact support, there exists $T_\varphi<+\infty$ such that $\mathrm{spt}(\varphi)\subseteq[-T_\varphi,T_\varphi]\times\mathbb{R}^d$. Hence, the first integral in (4.9) can be estimated by

$$\left|\int_{\mathbf{I}}\zeta_n'(s)\iint_{\mathbb{R}^{d+1}_+}\varphi(t,x)\,\mathrm{d}\sigma_s(t,x)\,\mathrm{d}s\right| \le \frac{\|\varphi\|_\infty\|\zeta'\|_\infty}{n}\,\sigma([n,+\infty)\times[0,T_\varphi]\times\mathbb{R}^d),$$

and therefore it tends to 0 as $n \to \infty$. By dominated convergence, we can take the limit in the second and third integrals of (4.9), thus obtaining

$$
\begin{aligned}
-\int_{\mathbb{R}^d} \varphi(0,x)\,\mathrm{d}\mu_0(x) &= \iiint_{\mathbf{I}\times\mathbb{R}^{d+1}_+} \partial_t\varphi(t,x)\,\mathrm{d}\sigma^0(s,t,x) + \iiint_{\mathbf{I}\times\mathbb{R}^{d+1}_+} \mathrm{D}\varphi(t,x)\,\mathrm{d}\boldsymbol{\sigma}(s,t,x) \\
&= \iint_{\mathbb{R}^{d+1}_+} \partial_t\varphi(t,x)\,\mathrm{d}\mu(t,x) + \iint_{\mathbb{R}^{d+1}_+} \mathrm{D}\varphi(t,x)\,\mathrm{d}\boldsymbol{\nu}(t,x)
\end{aligned}
$$

for every $\varphi \in \mathrm{C}^1_\mathrm{c}(\mathbb{R}\times\mathbb{R}^d)$, which concludes the proof. □

4.1. **Superposition results for the augmented continuity equation.** Since $\boldsymbol{\pi}$-adapted solutions of the augmented continuity equations are driven by a *bounded* Borel velocity field, it is easy to state a *superposition principle* in the spirit of [11, Theorem 8.2.1].

In order to formulate the probabilistic representation of the curve σ, we introduce the *evaluation* and the *augmented evaluation* maps

$$
\begin{aligned}
&\mathfrak{e}\colon \mathbf{I}\times \mathrm{C}(\mathbf{I};\mathbb{R}^{d+1}) \to \mathbb{R}^{d+1}, && \mathfrak{e}(s,\mathbf{y}) := \mathbf{y}(s) = (\mathsf{t}(s),\mathbf{x}(s)),\\
&\mathfrak{e}_s\colon \mathrm{C}(\mathbf{I};\mathbb{R}^{d+1}) \to \mathbb{R}^{d+1}, && \mathfrak{e}_s(\mathbf{y}) := \mathbf{y}(s) = (\mathsf{t}(s),\mathbf{x}(s)),\\
&\mathfrak{a}\colon \mathbf{I}\times \mathrm{C}(\mathbf{I};\mathbb{R}^{d+1}) \to \mathbf{I}\times\mathbb{R}^{d+1}, && \mathfrak{a}(s,\mathbf{y}) := (s,\mathbf{y}(s)) = (s,\mathsf{t}(s),\mathbf{x}(s)).
\end{aligned}
\tag{4.10}
$$

Clearly, $\mathfrak{a}$, $\mathfrak{e}$ and $\mathfrak{e}_s$, $s \ge 0$, are continuous maps.

We also introduce the Borel maps $\mathfrak{y}'\colon \mathbf{I}\times \mathrm{Lip}(\mathbf{I};\mathbb{R}^{d+1}) \to \mathbb{R}^{d+1}$ defined by

$$
\mathfrak{y}'(s,\mathbf{y}) := \mathbf{y}'(s), \quad \text{where} \quad (\mathbf{y}')_i(s) := \limsup_{h\to 0} \frac{\mathbf{y}_i(s+h)-\mathbf{y}_i(s)}{h},\ i = 0,\cdots,d. \tag{4.11}
$$

Clearly, $\mathfrak{y}'(s,\mathbf{y})$ coincides with the usual pointwise derivative of $\mathbf{y}(s)$ for $\mathcal{L}^1$-a.a. $s \in \mathbf{I}$. We will also denote by $\mathfrak{t}'$ (resp $\mathfrak{x}'$) the first (corresponding to the index $i = 0$) component (resp. the vector of the last d components) of $\mathfrak{y}'$, $\mathfrak{y}' = (\mathfrak{t}',\mathfrak{x}')$:

$$
\begin{aligned}
&\mathfrak{t}'\colon \mathbf{I}\times \mathrm{Lip}(\mathbf{I};\mathbb{R}^{d+1}) \to \mathbb{R}, && \mathfrak{t}'(s,\mathbf{y}) := \mathsf{t}'(s),\\
&\mathfrak{x}'\colon \mathbf{I}\times \mathrm{Lip}(\mathbf{I};\mathbb{R}^{d+1}) \to \mathbb{R}^{d}, && \mathfrak{x}'(s,\mathbf{y}) := \mathbf{x}'(s)\,.
\end{aligned}
\tag{4.12}
$$

Notice that the restriction of $\mathfrak{y}'$ to $\mathrm{Lip}^\uparrow_k(\mathbf{I};\mathbb{R}^{d+1})$ is a bounded Borel vector field whose image is contained in $\mathbb{R}^{d+1}_+$.

Now, for every $\mathbf{y} = (\mathsf{t},\mathbf{x}) \in \mathrm{C}^\uparrow(\mathbf{I};\mathbb{R}^{d+1})$ the set

$$
\{s \ge 0 : \mathsf{t}(s) \in [0,T]\} = \mathbf{y}^{-1}\big([0,T]\times\mathbb{R}^d\big) \quad \text{is a compact interval.}
$$

For every time interval $[0,T]$ we consider the domain

$$
\mathrm{E}(T) := \Big\{(s,\mathbf{y}) \in \mathbf{I}\times \mathrm{C}^\uparrow(\mathbf{I};\mathbb{R}^{d+1}) : \mathsf{t}(s) \in [0,T]\Big\} = \mathfrak{e}^{-1}(\mathbf{I}\times[0,T]\times\mathbb{R}^d). \tag{4.13}
$$

If $\boldsymbol{\eta}$ is a probability measure on $\mathrm{Lip}^\uparrow(\mathbf{I};\mathbb{R}^{d+1})$ we set

$$
\boldsymbol{\eta}_\mathcal{L} := \mathcal{L}^1 \otimes \boldsymbol{\eta} \in \mathcal{M}^+_{\mathrm{loc}}(\mathbf{I}\times \mathrm{Lip}^\uparrow(\mathbf{I};\mathbb{R}^{d+1})) \tag{4.14}
$$

and we observe that

$$
\int_{\mathrm{E}(T)} \mathfrak{t}'(s,\mathbf{y})\,\mathrm{d}\boldsymbol{\eta}_\mathcal{L}(s,\mathbf{y}) = T, \tag{4.15}
$$

so that

$$
\int_{\mathrm{E}(T)} \|\mathfrak{x}'(s,\mathbf{y})\|\,\mathrm{d}\boldsymbol{\eta}_\mathcal{L}(s,\mathbf{y}) \le \int_{\mathrm{E}(T)} \|\mathfrak{y}'(s,\mathbf{y})\|\,\mathrm{d}\boldsymbol{\eta}_\mathcal{L}(s,\mathbf{y}) \le T + \int_{\mathrm{E}(T)} \|\mathfrak{x}'(s,\mathbf{y})\|\,\mathrm{d}\boldsymbol{\eta}_\mathcal{L}(s,\mathbf{y}). \tag{4.16}
$$

We are now in a position to state our result on the probabilistic representation of the solutions to the augmented continuity equation.

Theorem 4.5 (Superposition principle for solutions to the augmented continuity equation)**.**

(1) Let $k > 0$ and $\boldsymbol{\eta}$ be a probability measure in $\mathrm{Lip}^{\uparrow}_k(\mathbf{I};\mathbb{R}^{d+1})$ such that

$$\int \|\mathfrak{e}_0\|\,\mathrm{d}\boldsymbol{\eta} < +\infty, \qquad \int_{\mathrm{E}(T)} \|\mathfrak{x}'\|\,\mathrm{d}\boldsymbol{\eta}_{\mathcal{L}} < \infty \quad \textit{for every } T > 0. \tag{4.17}$$

Setting

$$\sigma_s := (\mathfrak{e}_s)_\sharp\boldsymbol{\eta}, \quad \sigma := \mathfrak{a}_\sharp\boldsymbol{\eta}_{\mathcal{L}}, \quad \sigma_0 := \mathfrak{a}_\sharp\big(\mathfrak{t}'\,\boldsymbol{\eta}_{\mathcal{L}}\big), \quad \boldsymbol{\sigma} := \mathfrak{a}_\sharp\big(\mathfrak{x}'\,\boldsymbol{\eta}_{\mathcal{L}}\big) \tag{4.18}$$

then the curve $s \mapsto \sigma_s$ is k-Lipschitz with values in $\mathcal{P}_1(\mathbb{R}^{d+1}_+)$, $(\sigma,\sigma^0,\boldsymbol{\sigma})$ is a $\mathcal{P}_1$-solution to the augmented continuity equation with locally finite $\boldsymbol{\pi}$-marginals, and

$$|(\sigma^0,\boldsymbol{\sigma})| \le k\sigma. \tag{4.19}$$

If moreover there exists a Borel vector field $\boldsymbol{w}\colon \mathbb{R}^{d+1}_+ \to \mathbb{R}^{d+1}_+$ such that $\|\boldsymbol{w}\| \ge k^{-1}$ and

$$\mathfrak{y}' = \boldsymbol{w}(\mathfrak{e}) \quad \boldsymbol{\eta}_{\mathcal{L}}\textit{-a.e.}, \tag{4.20}$$

then $(\sigma,\sigma^0,\boldsymbol{\sigma})$ is a $\boldsymbol{\pi}$-adapted solution (with constant k; it is $\boldsymbol{\pi}$-normalized if $k=1$) and

$$(\sigma_0,\boldsymbol{\sigma}) = \boldsymbol{w}\sigma. \tag{4.21}$$

(2) Conversely, let $(\sigma,\sigma^0,\boldsymbol{\sigma})$ be a $\boldsymbol{\pi}$-adapted solution (with constant $k \ge 1$) to the augmented continuity equation (4.2) according to Definition 4.2, and let it be driven by the (autonomous) Borel vector field $\boldsymbol{w} := (\tau,\boldsymbol{v})$. Then, the support of $(\sigma,\sigma^0,\boldsymbol{\sigma})$ is contained in $\mathbf{I}\times\mathbb{R}^{d+1}_+$ and there exists $\boldsymbol{\eta} \in \mathcal{P}(\mathrm{Lip}^{\uparrow}_k(\mathbf{I};\mathbb{R}^{d+1}_+))$ that satisfies (4.17), (4.18), and is concentrated on the curves solving the Cauchy problem

$$\begin{cases} \dot{\mathsf{y}}(s) = \boldsymbol{w}(\mathsf{y}(s)), & s \in (0,\infty), \\ \mathsf{y}(0) = (0,x), & x \in \mathrm{spt}(\mu_0). \end{cases} \tag{4.22}$$

If moreover $(\sigma,\sigma^0,\boldsymbol{\sigma})$ is $\boldsymbol{\pi}$-normalized, then $\boldsymbol{\eta} \in \mathcal{P}(\mathrm{Lip}^{\uparrow}_1(\mathbf{I};\mathbb{R}^{d+1}_+))$.

Proof. The first claim is well known (see e.g. the second part of [11, Theorem 8.2.1] and can be easily checked by a direct computation. Condition (4.17) ensures that $(\sigma,\sigma^0,\boldsymbol{\sigma})$ has locally finite $\boldsymbol{\pi}$-marginals. Property (4.21) easily follows from Lemma A.1.

In order to prove the second claim, we can still rely on [11, Theorem 8.2.1] (which corresponds to the case of a finite interval), applied to the restrictions of $(\sigma,\sigma^0,\boldsymbol{\sigma})$ to the intervals $[i,i+1]$. We find measures $\boldsymbol{\eta}^i$ concentrated on $\mathrm{Lip}_k([i,i+1];\mathbb{R}^{d+1}_+)$ and corresponding measures $\widetilde{\boldsymbol{\eta}}^i = \mathcal{L}^1\otimes\boldsymbol{\eta}^i$ satisfying (4.18) in $[i,i+1]$ together with

$$\int_{\mathrm{E}_{[i,i+1]}(T)} \|\mathfrak{x}'\|\,\mathrm{d}\widetilde{\boldsymbol{\eta}}^i = |\boldsymbol{\sigma}|([i,i+1]\times[0,T]\times\mathbb{R}^d), \tag{4.23}$$

where

$$\mathrm{E}_{[i,i+1]}(T) := \Big\{(s,\mathsf{y}) \in [i,i+1]\times \mathrm{C}^{\uparrow}(\mathbf{I};\mathbb{R}^{d+1}) : \mathsf{t}(s) \in [0,T]\Big\} = \mathfrak{e}^{-1}([i,i+1]\times[0,T]\times\mathbb{R}^d)\,.$$

We can then apply the glueing Lemma C.1: it is sufficient to use

$$X := \mathrm{C}(\mathbf{I};\mathbb{R}^{d+1}_+), \quad X^i := \mathrm{C}([i,i+1];\mathbb{R}^{d+1}_+), \quad Y^j := \mathbb{R}^{d+1}_+,$$

and choose $\mathsf{p}^i\colon X \to X^i$ as the operators mapping a continuous curve defined in $\mathbf{I}$ into its restriction to the interval $[i,i+1]$. We eventually set $\mathsf{R}^i = \mathsf{L}^{i+1} := \mathfrak{e}_{i+1}$ and we thus find a measure $\boldsymbol{\eta}$ such that $\mathsf{p}^i_\sharp\boldsymbol{\eta} = \boldsymbol{\eta}^i$ for every $i \in \mathbb{N}$. It is easy to check that $\boldsymbol{\eta}$ satisfies all the properties stated in Claim (2). The second estimate in (4.17) can be derived from (4.23). □

We can now state a useful rescaling property, which is strongly related to the fact that the velocity vector field is autonomous according to (4.3).

Lemma 4.6 (Rescaling). *Let $(\sigma, \sigma^0, \boldsymbol{\sigma})$ be a $\boldsymbol{\pi}$-adapted solution to the augmented continuity equation* (4.2) *driven by the (autonomous) Borel vector field $\boldsymbol{w} = (\tau, \boldsymbol{v})\colon \mathbb{R}^{d+1}_+ \to \mathbb{R}^{d+1}_+$, let*

$$\theta\colon \mathbb{R}^{d+1}_+ \to (0, +\infty) \text{ be a Borel map satisfying } c^{-1} \le \theta \le c < +\infty \text{ in } \mathbb{R}^{d+1}_+ \text{ for some constant } c \ge 1, \tag{4.24}$$

and let

$$\hat{\tau} := \theta\tau, \quad \hat{\boldsymbol{v}} := \theta\boldsymbol{v}, \quad \hat{\boldsymbol{w}} = (\hat{\tau}, \hat{\boldsymbol{v}}) = \theta\boldsymbol{w}. \tag{4.25}$$

There exists an autonomous solution $(\hat{\sigma}, \hat{\sigma}^0, \hat{\boldsymbol{\sigma}})$ satisfying

$$\hat{\sigma}^0 = \hat{\tau}\,\hat{\sigma}, \quad \hat{\boldsymbol{\sigma}} = \hat{\boldsymbol{v}}\,\hat{\sigma}, \quad \boldsymbol{\pi}_\sharp(\hat{\sigma}^0, \hat{\boldsymbol{\sigma}}) = \boldsymbol{\pi}_\sharp(\sigma^0, \boldsymbol{\sigma}). \tag{4.26}$$

Proof. By Theorem 4.5 there exists $\boldsymbol{\eta} \in \mathcal{P}(\mathrm{Lip}^{\uparrow}_k(\mathbf{I}; \mathbb{R}^{d+1}_+))$ providing the representation formulae (5.6) and supported on solutions of the Cauchy problem (4.22), with autonomous velocity field $\boldsymbol{w} = (\tau, \boldsymbol{v})$ satisfying $k^{-1} \le \|\boldsymbol{w}\| \le k$. For every Lipschitz curve $\mathsf{y}\colon \mathbf{I} \to \mathbb{R}^{d+1}_+$ we consider the solution ℓ_{y} to the differential equation

$$\dot{\ell}_{\mathsf{y}}(r) = (\theta \circ \mathsf{y})(\ell_{\mathsf{y}}(r)), \quad \ell_{\mathsf{y}}(0) = 0. \tag{4.27}$$

Indeed, ℓ_{y} can be easily obtained as the inverse of the bi-Lipschitz map

$$\Theta_{\mathsf{y}}(s) := \int_0^s \frac{1}{\theta(\mathsf{y}(r))}\,\mathrm{d}r, \quad c^{-1} \le \Theta'_{\mathsf{y}} \le c. \tag{4.28}$$

Thus, we may define the function $\mathfrak{R} : \mathrm{Lip}^{\uparrow}_k(\mathbf{I}; \mathbb{R}^{d+1}) \to \mathrm{Lip}^{\uparrow}_{\hat{k}}(\mathbf{I}; \mathbb{R}^{d+1})$, $\hat{k} := ck$, that associates with every Lipschitz curve y the rescaled curve

$$\mathfrak{R}(\mathsf{y}) := \mathsf{y} \circ \ell_{\mathsf{y}}. \tag{4.29}$$

Notice that $\hat{\mathsf{y}} := \mathfrak{R}(\mathsf{y})$ satisfies the system

$$\hat{\mathsf{y}}'(r) = \theta(\hat{\mathsf{y}}(r))\boldsymbol{w}(\hat{\mathsf{y}}(r)) = \hat{\boldsymbol{w}}(\hat{\mathsf{y}}(r)), \quad \hat{\mathsf{y}}(0) = \mathsf{y}(0). \tag{4.30}$$

By Lemma D.2 in Appendix D ahead, $\mathfrak{R}$ is a Borel map. Let us set $\hat{\boldsymbol{\eta}} := \mathfrak{R}_\sharp(\boldsymbol{\eta}) \in \mathcal{P}(\mathrm{Lip}^{\uparrow}(\mathbf{I}; \mathbb{R}^{d+1}_+))$. A further application of Theorem 4.5 yields the thesis. □

4.2. **A representation result by the augmented continuity equation.** We can now apply the previous results to get a first representation for $\mathcal{P}_1$-solutions to the continuity equation (3.1).

Theorem 4.7 (Augmented representations of $\mathcal{P}_1$-solutions). *Let $(\mu, \boldsymbol{\nu}) \in \mathcal{M}^+_{\mathrm{loc}}(\mathbb{R}^{d+1}_+) \times \mathcal{M}_{\mathrm{loc}}(\mathbb{R}^{d+1}_+; \mathbb{R}^d)$ be a $\mathcal{P}_1$-solution to the continuity equation in the sense of Definition 3.1, with initial condition $\mu_0 \in \mathcal{P}_1(\mathbb{R}^d)$, let $(\mu, \bar{\boldsymbol{\nu}}) = \theta(\mu, \boldsymbol{\nu})$ be a minimal pair induced by $(\mu, \boldsymbol{\nu})$ according to Definition 3.2 and fulfilling* (3.6)*, let $\varrho \sim |(\mu, \boldsymbol{\nu})|$, and let $(\tau, \boldsymbol{v})$ be bounded Borel vector field representing the density of $(\mu, \boldsymbol{\nu})$ w.r.t. ϱ, i.e.*

$$\mu = \tau\varrho, \qquad \boldsymbol{\nu} = \boldsymbol{v}\varrho \qquad \varrho\text{-a.e. in } \mathbb{R}^{d+1}_+. \tag{4.31}$$

Then, there exists a Lipschitz continuous curve $\sigma \in \mathrm{Lip}(\mathbf{I}; \mathcal{P}_1(\mathbb{R}^{d+1}_+))$ satisfying the following properties:

(1) the associated measure $\sigma = \mathcal{L}^1 \otimes \sigma_s \in \mathcal{M}^+_{\mathrm{loc}}(\mathbf{I} \times \mathbb{R}^{d+1}_+)$ has marginal

$$\boldsymbol{\pi}_\sharp\sigma = \bar{\varrho} = \theta\varrho. \tag{4.32}$$

(2) The measures

$$\sigma^0 := (\tau \circ \boldsymbol{\pi})\sigma, \qquad \boldsymbol{\sigma} := (\boldsymbol{v} \circ \boldsymbol{\pi})\sigma \tag{4.33}$$

have marginals

$$\boldsymbol{\pi}_\sharp\sigma^0 = \mu, \qquad \boldsymbol{\pi}_\sharp\boldsymbol{\sigma} = \bar{\boldsymbol{\nu}}. \tag{4.34}$$

(3) The triple $(\sigma, \sigma^0, \boldsymbol{\sigma})$ is a $\boldsymbol{\pi}$-adapted solution to the augmented continuity equation (4.2)*, in the sense of Definition 4.2.*

In particular, when $\varrho = |(\mu, \boldsymbol{\nu})|$ then $(\sigma, \sigma^0, \boldsymbol{\sigma})$ is also a $\boldsymbol{\pi}$-normalized solution.

Remark 4.8. When $(\mu, \boldsymbol{\nu})$ is a minimal $\mathcal{P}_1$-solution, then $\boldsymbol{\nu}^\perp = \bar{\boldsymbol{\nu}}^\perp$ is minimal and (4.32) holds for $\boldsymbol{\nu} = \bar{\boldsymbol{\nu}}$.

Proof. Thanks to Lemma 4.6 it is not restrictive to assume that $\varrho = |(\mu, \boldsymbol{\nu})|$, $\|\cdot\|$ is the Euclidean norm (so that it is strictly convex), and $\theta \equiv 1$. Therefore, in the remainder of the proof we shall use that $\bar{\boldsymbol{\nu}} = \boldsymbol{\nu}$. We will split the *proof* in the following steps:

(1) Regularization of the pair $(\mu, \boldsymbol{\nu})$ via convolution;
(2) Analysis of the 'augmented', regularized system;
(3) Passage to the limit in the regularization parameter;
(4) Proof of property (4.31).

Step 1: regularization. It follows from Theorem 3.4 that μ admits a left-continuous representative w.r.t. narrow convergence. Therefore, from now on, without loss of generality we shall suppose that $t \mapsto \mu_t$ is (narrowly) left-continuous. We now extend the measures μ and $\boldsymbol{\nu}$ to the whole $\mathbb{R}^{d+1}$ by setting

$$\mu_t := \begin{cases} \mu_0 & \text{if } t < 0, \\ \mu_t & \text{if } t \geq 0, \end{cases} \qquad \text{and } \boldsymbol{\nu} = 0 \text{ on } (-\infty, 0) \times \mathbb{R}^d.$$

Let us now consider convolution kernels $\kappa^0 \in \mathrm{C}_c^\infty(\mathbb{R})$ $\kappa^1 \in \mathrm{C}^\infty(\mathbb{R}^d)$ satisfying

$$\kappa^0 \geq 0, \quad \mathrm{spt}(\kappa^0) \subset [0,1], \quad \int_0^1 \kappa^0 \,\mathrm{d}t = 1, \tag{4.35}$$

$$0 < \kappa^1 \leq 1, \quad \int_{\mathbb{R}^d} \kappa^1 \,\mathrm{d}x = 1, \quad \int_{\mathbb{R}^d} \|x\| \,\kappa^1(x) \,\mathrm{d}x = M^1 < \infty. \tag{4.36}$$

Let us set

$$\kappa^0_\varepsilon(t) := \varepsilon^{-1}\kappa(t/\varepsilon), \quad \kappa^1_\varepsilon(x) := \varepsilon^{-d}\kappa^1(x/\varepsilon)\,\mathrm{d}x, \quad \kappa_\varepsilon(t,x) := \kappa^0_\varepsilon(t)\kappa^1_\varepsilon(x). \tag{4.37}$$

For $(t,x) \in \mathbb{R}^{d+1}$ we define

$$\begin{aligned} \mu^\varepsilon(t,x) = (\mu \star \kappa_\varepsilon)(t,x) &= \int_{\mathbb{R}^{d+1}} \kappa_\varepsilon(t-\tau, x-y)\mathrm{d}\mu(\tau,y), \\ \boldsymbol{\nu}^\varepsilon(t,x) := (\boldsymbol{\nu} \star \kappa_\varepsilon)(t,x) &= \int_{\mathbb{R}^{d+1}} \kappa_\varepsilon(t-\tau, x-y)\mathrm{d}\boldsymbol{\nu}(\tau,y)\,. \end{aligned} \tag{4.38}$$

Since $(\mu, \boldsymbol{\nu})$ is a $\mathcal{P}_1$-solution to the continuity equation, the functions $\mu^\varepsilon \in \mathrm{C}^\infty(\mathbb{R}^{d+1})$ and $\boldsymbol{\nu}^\varepsilon \in \mathrm{C}^\infty(\mathbb{R}^{d+1};\mathbb{R}^d)$ are smooth solutions to the continuity equation

$$\partial_t \mu^\varepsilon + \mathrm{div}\, \boldsymbol{\nu}^\varepsilon = 0 \quad \text{in } \mathbb{R}^{d+1}, \tag{4.39}$$

with

$$\mu^\varepsilon(t,x) = \int_{\mathbb{R}^d} \kappa^1_\varepsilon(x-y)\mathrm{d}\mu_0(y) \doteq \bar{\mu}^\varepsilon_0(x) \quad \text{for every } t \leq 0.$$

It is easy to check that

$$\int_{\mathbb{R}^d} \mu_\varepsilon(t,x)\,\mathrm{d}x = 1 \quad \text{for every } t \in \mathbb{R}. \tag{4.40}$$

Moreover

$$\begin{aligned} \int_{\mathbb{R}^d} \|x\|\, \bar{\mu}^\varepsilon_0(x)\,\mathrm{d}x &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \|x\| \kappa^1_\varepsilon(x-y)\mathrm{d}\mu_0(y)\,\mathrm{d}x \leq \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \Big(\|x-y\| + \|y\|\Big)\kappa^1_\varepsilon(x-y)\mathrm{d}\mu_0(y)\,\mathrm{d}x \\ &\leq \varepsilon M^1 + \int_{\mathbb{R}^d} \|y\|\,\mathrm{d}\mu_0(y). \end{aligned}$$

With slight abuse of notation, we shall denote by $\mu^\varepsilon = (\mu^\varepsilon_t)_\varepsilon$ and $\boldsymbol{\nu}^\varepsilon = (\boldsymbol{\nu}^\varepsilon_t)$ (where $\boldsymbol{\nu}^\varepsilon_t := \boldsymbol{\nu}^\varepsilon(t,\cdot)$) also the measures with densities μ^ε and $\boldsymbol{\nu}^\varepsilon$, respectively. Due to [10, Thm. 2.2] and the previous estimate, we have the following convergences as $\varepsilon \downarrow 0$:

$$\begin{aligned} &\mu^\varepsilon \rightharpoonup^* \mu \quad \text{in } \mathcal{M}^+_{\mathrm{loc}}(\mathbb{R}^{d+1}_+), \quad \boldsymbol{\nu}^\varepsilon \rightharpoonup^* \boldsymbol{\nu} \quad \text{in } \mathcal{M}_{\mathrm{loc}}(\mathbb{R}^{d+1}_+;\mathbb{R}^d), \quad \bar{\mu}^\varepsilon_0 \to \mu_0 \text{ in } \mathcal{P}_1(\mathbb{R}^d), \\ &\mu^\varepsilon \llcorner [0,T] \times \mathbb{R}^d \rightharpoonup \mu \llcorner [0,T] \times \mathbb{R}^d \quad \text{narrowly in } \mathcal{M}^+([0,T]\times\mathbb{R}^d) \quad \text{for every } T > 0. \end{aligned} \tag{4.41a}$$

We also have by [10, Thm. 2.2] that

$$|(\mu^\varepsilon,\boldsymbol{\nu}^\varepsilon)| \rightharpoonup^* |(\mu,\boldsymbol{\nu})|, \quad |\boldsymbol{\nu}^\varepsilon| \rightharpoonup^* |\boldsymbol{\nu}| \qquad \text{in } \mathcal{M}^+_{\mathrm{loc}}(\mathbb{R}^{d+1}_+). \tag{4.41b}$$

In a similar way, we can show that

$$|(\mu^\varepsilon,\boldsymbol{\nu}^\varepsilon)|([0,T]\times\mathbb{R}^d) \le |(\mu,\boldsymbol{\nu})|([0,T]\times\mathbb{R}^d), \quad |\boldsymbol{\nu}^\varepsilon|([0,T]\times\mathbb{R}^d) \le |\boldsymbol{\nu}|([0,T]\times\mathbb{R}^d)\,, \tag{4.42}$$

which implies that

$$\begin{cases} |\boldsymbol{\nu}^\varepsilon| \llcorner [0,T]\times\mathbb{R}^d \rightharpoonup |\boldsymbol{\nu}| \llcorner [0,T]\times\mathbb{R}^d, \\ |(\mu^\varepsilon,\boldsymbol{\nu}^\varepsilon)| \llcorner [0,T]\times\mathbb{R}^d \rightharpoonup |(\mu,\boldsymbol{\nu})| \llcorner [0,T]\times\mathbb{R}^d \end{cases} \qquad \text{narrowly in } \mathcal{M}([0,T]\times\mathbb{R}^d) \tag{4.43}$$

for every $T\in[0,+\infty)$ such that $|\boldsymbol{\nu}|(\{T\}\times\mathbb{R}^d)=0$.

Since $\mu^\varepsilon(t,x)>0$ (see [11, Lemma 8.1.9]) we may introduce the velocity field

$$\boldsymbol{w}^\varepsilon(t,x) := \frac{\boldsymbol{\nu}^\varepsilon_t(x)}{\mu^\varepsilon_t(x)} \qquad \text{for all } (t,x)\in\mathbb{R}^{d+1}. \tag{4.44}$$

The velocity field $\boldsymbol{w}^\varepsilon$ fulfills the local regularity conditions of [11, Prop. 8.1.8], which we may therefore apply to the continuity equation (4.39). We can introduce the characteristic system

$$\begin{cases} \dot X^\varepsilon_t = \boldsymbol{w}^\varepsilon_t(X^\varepsilon_t), \\ X^\varepsilon_0 = x\,, \end{cases} \tag{4.45}$$

and we denote by D^ε the subset of $x\in\mathbb{R}^d$ for which the unique maximal solution is globally defined. We know that $\mathbb{R}^d\setminus D^\varepsilon$ is $\bar\mu^\varepsilon_0$-negligible (equivalently $\mathcal{L}^d(\mathbb{R}^d\setminus D^\varepsilon)=0)$) and (4.45) defines a flow $X^\varepsilon_t\colon D^\varepsilon\to D^\varepsilon$, $t\ge 0$, inducing the representation formula

$$\mu^\varepsilon_t = (X^\varepsilon_t)_\sharp \bar\mu^\varepsilon_0. \tag{4.46}$$

We finally notice that by (4.44), (4.46), and (4.42), we get for every $T\in[0,+\infty)$ the bound

$$\int_{\mathbb{R}^d}\int_0^T \|\boldsymbol{w}^\varepsilon(t,X^\varepsilon_t(x))\|\,\mathrm{d}t\,\mathrm{d}\bar\mu^\varepsilon_0(x) \le |\boldsymbol{\nu}^\varepsilon|([0,T]\times\mathbb{R}^d) \le |\boldsymbol{\nu}|([0,T]\times\mathbb{R}^d)\,. \tag{4.47}$$

Step 2: Analysis of the augmented system. We define

$$\tau^\varepsilon\colon\mathbb{R}^{d+1}\to\mathbb{R}, \qquad \tau^\varepsilon(t,x) := \frac{1}{\|(1,\boldsymbol{w}^\varepsilon(t,x))\|}, \tag{4.48a}$$

$$\boldsymbol{v}^\varepsilon\colon\mathbb{R}^{d+1}\to\mathbb{R}^d, \qquad \boldsymbol{v}^\varepsilon(t,x) := \tau^\varepsilon(t,x)\boldsymbol{w}^\varepsilon(t,x) = \frac{\boldsymbol{w}^\varepsilon(t,x)}{\|(1,\boldsymbol{w}^\varepsilon(t,x))\|}\,. \tag{4.48b}$$

By construction we have that

$$\|(\tau^\varepsilon(t,x),\boldsymbol{v}^\varepsilon(t,x))\| \equiv 1 \qquad \text{for every } (t,x)\in\mathbb{R}^{d+1}. \tag{4.49}$$

For each $\varepsilon>0$ the functions τ^ε and $\boldsymbol{v}^\varepsilon$ are locally Lipschitz and globally bounded.

We now consider the following 'augmented' characteristic system, in the unknowns $T\colon\mathbf{I}\to\mathbb{R}$ and $Y\colon\mathbf{I}\to\mathbb{R}^d$

$$\begin{cases} \dot T_s = \tau^\varepsilon(T_s,Y_s), \\ \dot Y_s = \boldsymbol{v}^\varepsilon(T_s,Y_s), \\ T_0 = t, \\ Y_0 = x\,. \end{cases} \tag{4.50}$$

For every $(t,x)\in\mathbb{R}^{d+1}$, the Cauchy problem possesses a unique solution $s\mapsto(T^\varepsilon_s(t,x),Y^\varepsilon_s(t,x))$ which is globally defined. Clearly, $s\mapsto T^\varepsilon_s$ is an increasing map and in particular $T^\varepsilon_s(t,x)\ge 0$ if $(t,x)\in\mathbb{R}^{d+1}_+$. The following result relates the flow map $(T^\varepsilon,Y^\varepsilon)\colon\mathbf{I}\times\mathbb{R}^{d+1}\to\mathbb{R}^{d+1}$, defined by $T^\varepsilon(s,t,x):=T^\varepsilon_s(t,x)$, $Y^\varepsilon(s,t,x):=Y^\varepsilon_s(t,x)$, with the flow map $X^\varepsilon\colon\mathbf{I}\times D^\varepsilon\to D^\varepsilon$ of the ODE system (4.45).

Lemma 4.9. *For every $x \in D^\varepsilon$, let $(\widehat{T}^\varepsilon_s(x))_{s\in\mathbf{I}}$ solve the Cauchy problem*

$$\begin{cases} \widehat{T}_s' = \tau^\varepsilon(\widehat{T}_s, X^\varepsilon(\widehat{T}_s, x)), \\ \widehat{T}_0 = 0, \end{cases} \tag{4.51}$$

and thus define a map $\widehat{T}^\varepsilon : \mathbf{I}\times D^\varepsilon \to \mathbf{I}$. Define $\widehat{Y}^\varepsilon : \mathbf{I}\times D^\varepsilon \to D^\varepsilon$ via $\widehat{Y}^\varepsilon(s,x) := X^\varepsilon(\widehat{T}^\varepsilon(s,x),x)$. Then,

$$\widehat{T}^\varepsilon(s,x) = T^\varepsilon(s,0,x), \qquad \widehat{Y}^\varepsilon(s,x) = Y^\varepsilon(s,0,x) \qquad \text{for all } (s,x)\in \mathbf{I}\times D^\varepsilon, \tag{4.52}$$

and $\widehat{T}^\varepsilon(\cdot,x)$ is a (strictly increasing and surjective) diffeomorphism of $\mathbf{I}$ for every $x\in D^\varepsilon$. In particular,

$$Y^\varepsilon(s,0,x) = X^\varepsilon(T^\varepsilon(s,0,x),x) \qquad \text{for all } (s,x)\in \mathbf{I}\times\mathbb{R}^d. \tag{4.53}$$

Furthermore, if $S^\varepsilon : \mathbf{I}\times D^\varepsilon \to \mathbf{I}$ is defined by

$$S^\varepsilon(t,x) := \int_0^t \|(1, \boldsymbol{w}^\varepsilon(\tau, X^\varepsilon(\tau,x)))\|\,\mathrm{d}\tau, \tag{4.54}$$

then

$$\int_{\mathbb{R}^d} S^\varepsilon(T,x)\,\mathrm{d}\bar{\mu}^\varepsilon_0(x) \le |(\mu,\boldsymbol{\nu})|([0,T]\times\mathbb{R}^d) \tag{4.55}$$

and

$$S^\varepsilon(\cdot,x) = (\widehat{T}^\varepsilon)^{-1}(\cdot,x) \qquad \text{for every } x\in D^\varepsilon. \tag{4.56}$$

Proof. We observe that the functions $\widehat{T}^\varepsilon$ and $\widehat{Y}^\varepsilon$ satisfy

$$\partial_s \widehat{T}^\varepsilon(s,x) \overset{(4.51)}{=} \tau^\varepsilon(\widehat{T}^\varepsilon(s,x), X^\varepsilon(\widehat{T}^\varepsilon(s,x),x)) = \tau^\varepsilon(\widehat{T}^\varepsilon(s,x), \widehat{Y}^\varepsilon(s,x)),$$

as well as

$$\begin{aligned} \partial_s \widehat{Y}^\varepsilon(s,x) &= \partial_t X^\varepsilon(\widehat{T}^\varepsilon(s,x),x)\partial_s\widehat{T}^\varepsilon(s,x) \\ &\overset{(4.45)}{=} \boldsymbol{w}^\varepsilon(\widehat{T}^\varepsilon(s,x), X^\varepsilon(\widehat{T}^\varepsilon(s,x),x))\,\tau^\varepsilon(\widehat{T}^\varepsilon(s,x), X^\varepsilon(\widehat{T}^\varepsilon(s,x),x)) \\ &\overset{(4.48b)}{=} \boldsymbol{v}^\varepsilon(\widehat{T}^\varepsilon(s,x), \widehat{Y}^\varepsilon(s,x)). \end{aligned}$$

Since we also have that $\widehat{Y}^\varepsilon(0,x) = X^\varepsilon(T^\varepsilon(0,0,x),x) = X^\varepsilon(0,x) = x$, we conclude that the pair $(\widehat{T}^\varepsilon, \widehat{Y}^\varepsilon)$ solves system (4.50) for $t=0$. By uniqueness, (4.52) follows.

The function S^ε defined in (4.54) is finite (since the integrand is a continuous function w.r.t. τ) and it is clearly strictly increasing. Estimate (4.55) follows immediately by

$$\int_{\mathbb{R}^d} S^\varepsilon(T,x)\,\mathrm{d}\bar{\mu}^\varepsilon_0(x) = \int_{\mathbb{R}^d}\int_0^T \|(1,\boldsymbol{w}^\varepsilon(t,X^\varepsilon(t,x)))\|\,\mathrm{d}t\,\mathrm{d}\bar{\mu}^\varepsilon_0(x) \overset{(4.47)}{\le} |(\mu,\boldsymbol{\nu})|\big([0,T]\times\mathbb{R}^d\big) < +\infty.$$

Finally, we observe that for every $x\in D^\varepsilon$

$$\begin{aligned} \partial_t(S^\varepsilon\circ\widehat{T}^\varepsilon)(s,x) &= \partial_t S^\varepsilon(\widehat{T}^\varepsilon(s,x),x)\,\partial_s\widehat{T}^\varepsilon(s,x) \\ &\overset{(1)}{=} \|(1,\boldsymbol{w}^\varepsilon(\widehat{T}^\varepsilon(s,x), X^\varepsilon(\widehat{T}^\varepsilon(s,x),x)))\|\cdot\tau^\varepsilon(\widehat{T}^\varepsilon(s,x), X^\varepsilon(\widehat{T}^\varepsilon(s,x),x)) \overset{(2)}{\equiv} 1, \end{aligned}$$

where (1) is due to (4.54) and (4.51), while (2) is a consequence of (4.48). Hence, (4.56) follows, whence we conclude that $\widehat{T}^\varepsilon(\cdot,x)$ is a strictly increasing diffeomorphism of $\mathbf{I}$ for every $x\in D^\varepsilon$. □

Let us now consider the continuity equation with the vector field $(\tau^\varepsilon, \boldsymbol{v}^\varepsilon)$ and initial datum $\sigma^\varepsilon_0 = \delta_0\otimes\bar{\mu}^\varepsilon_0$. Since σ^ε_0 is supported in $\mathbb{R}^{d+1}_+$ then the family of measures

$$\sigma^\varepsilon_s := (T^\varepsilon_s, Y^\varepsilon_s)_\sharp(\delta_0\otimes\bar{\mu}^\varepsilon_0) \qquad \text{for all } s\in\mathbf{I}, \tag{4.57}$$

are supported in $\mathbb{R}^{d+1}_+$ as well. Moreover, (4.52) shows that

$$\sigma^\varepsilon_s = (\widehat{T}^\varepsilon_s, \widehat{Y}^\varepsilon_s)_\sharp\bar{\mu}^\varepsilon_0 \qquad \text{for all } s\in\mathbf{I}. \tag{4.58}$$

It follows from [11, Lemma 8.1.6, Prop. 8.1.8] that the curve σ^ε belongs to $\mathrm{Lip}(\mathbf{I};\mathcal{P}_1(\mathbb{R}^{d+1}_+))$ (it is in fact 1-Lipschitz) and fulfills

$$\begin{cases}\partial_s\sigma^\varepsilon+\partial_t(\tau^\varepsilon\sigma^\varepsilon)+\mathrm{div}(\boldsymbol{v}^\varepsilon\sigma^\varepsilon)=0 & \text{in } \mathbf{I}\times\mathbb{R}^{d+1},\\ \sigma^\varepsilon_0=\delta_0\otimes\bar\mu^\varepsilon_0\,. \end{cases} \tag{4.59}$$

From now on, we will use the short-hand notation

$$\sigma^{\varepsilon,0}:=\tau^\varepsilon\sigma^\varepsilon,\qquad \boldsymbol{\sigma}^\varepsilon:=\boldsymbol{v}^\varepsilon\sigma^\varepsilon.$$

Observe that, in view of (4.49), the measures $\sigma^{\varepsilon,0}$ and $\boldsymbol{\sigma}^\varepsilon$ satisfy

$$|(\sigma^{\varepsilon,0},\boldsymbol{\sigma}^\varepsilon)|=\sigma^\varepsilon\qquad \text{in } \mathcal{M}^+_{\mathrm{loc}}(\mathbf{I}\times\mathbb{R}^{d+1}_+). \tag{4.60}$$

In the following lemma the relation between μ^ε, $\boldsymbol{\nu}^\varepsilon$, and σ^ε is established in terms of the projection operator $\boldsymbol{\pi}\colon\mathbb{R}\times\mathbb{R}\times\mathbb{R}^d\to\mathbb{R}\times\mathbb{R}^d$, $\boldsymbol{\pi}(s,t,x):=(t,x)$ from (4.1).

Lemma 4.10. *There holds*

$$\mu^\varepsilon=\boldsymbol{\pi}_\sharp\sigma^{\varepsilon,0}\,,\qquad \boldsymbol{\nu}^\varepsilon=\boldsymbol{\pi}_\sharp\boldsymbol{\sigma}^\varepsilon\,,\qquad |(\mu^\varepsilon,\boldsymbol{\nu}^\varepsilon)|=\boldsymbol{\pi}_\sharp\sigma^\varepsilon\qquad \textit{in } \mathbf{I}\times\mathbb{R}^d\,. \tag{4.61}$$

Moreover for every $S,T>0$

$$\sigma^{\varepsilon,0}((S,+\infty)\times[0,T]\times\mathbb{R}^d)\le\frac{T}{S}\,|(\mu,\boldsymbol{\nu})|\big([0,T]\times\mathbb{R}^d\big). \tag{4.62}$$

Proof. For every $\varphi_0\in\mathrm{C_c}(\mathbb{R}^{d+1})$ we have

$$\begin{aligned}\int_{\mathbb{R}^{d+1}}\varphi_0(t,x)\,\mathrm{d}\mu^\varepsilon(t,x)&\overset{(1)}{=}\int_{\mathbf{I}}\int_{\mathbb{R}^d}\varphi_0(t,X^\varepsilon(t,x))\,\mathrm{d}\bar\mu^\varepsilon_0(x)\,\mathrm{d}t\\ &\overset{(2)}{=}\int_{D^\varepsilon}\int_{\mathbf{I}}\varphi_0(\widehat{T}^\varepsilon(s,x),X^\varepsilon(\widehat{T}^\varepsilon(s,x),x))\,\partial_s\widehat{T}^\varepsilon(s,x)\,\mathrm{d}s\,\mathrm{d}\bar\mu^\varepsilon_0(x)\\ &\overset{(3)}{=}\int_{D^\varepsilon}\int_{\mathbf{I}}\varphi_0(\widehat{T}^\varepsilon(s,x),\widehat{Y}^\varepsilon(s,x))\,\tau^\varepsilon(\widehat{T}^\varepsilon(s,x),\widehat{Y}^\varepsilon(s,x))\,\mathrm{d}s\,\mathrm{d}\bar\mu^\varepsilon_0(x)\\ &=\int_{\mathbf{I}}\Big(\int_{D^\varepsilon}\varphi_0(\widehat{T}^\varepsilon(s,x),\widehat{Y}^\varepsilon(s,x))\,\tau^\varepsilon(\widehat{T}^\varepsilon(s,x),\widehat{Y}^\varepsilon(s,x))\,\mathrm{d}\bar\mu^\varepsilon_0(x)\Big)\,\mathrm{d}s\\ &\overset{(4)}{=}\int_{\mathbf{I}}\Big(\int_{\mathbb{R}^{d+1}}\varphi_0(t,x)\tau^\varepsilon(t,x)\,\mathrm{d}\sigma^\varepsilon_s(t,x)\Big)\,\mathrm{d}s=\int_{\mathbb{R}^{d+2}_+}\varphi_0(t,x)\,\mathrm{d}\sigma^{\varepsilon,0}(s,t,x)\end{aligned}$$

where (1) follows from (4.46), (2) and (3) from the change of variables $t=\widehat{T}^\varepsilon(s,x)$ (see Lemma 4.9), (4) from (4.58), and we have repeatedly applied Fubini's Theorem.

The second of (4.61) follows from the fact that $\boldsymbol{\nu}^\varepsilon=\boldsymbol{w}^\varepsilon\mu^\varepsilon$ and $\boldsymbol{v}^\varepsilon=\tau^\varepsilon\boldsymbol{w}^\varepsilon$, so that for all test functions $\boldsymbol{\varphi}\in\mathrm{C_c}(\mathbb{R}^{d+1};\mathbb{R}^d)$

$$\begin{aligned}\int_{\mathbb{R}^{d+1}}\boldsymbol{\varphi}(t,x)\,\mathrm{d}\boldsymbol{\nu}^\varepsilon(t,x)&=\int_{\mathbf{I}}\int_{\mathbb{R}^d}\boldsymbol{\varphi}(t,X^\varepsilon(t,x))\cdot\boldsymbol{w}^\varepsilon(t,X^\varepsilon(t,x))\,\mathrm{d}\bar\mu^\varepsilon_0(x)\,\mathrm{d}t\\ &=\int_{\mathbb{R}^d}\int_{\mathbf{I}}\boldsymbol{\varphi}(\widehat{T}^\varepsilon(s,x),X^\varepsilon(\widehat{T}^\varepsilon(s,x),x))\cdot\boldsymbol{w}^\varepsilon(\widehat{T}^\varepsilon(s,x),X^\varepsilon(\widehat{T}^\varepsilon(s,x),x))\,\partial_s\widehat{T}^\varepsilon(s,x)\,\mathrm{d}s\,\mathrm{d}\bar\mu^\varepsilon_0(x)\\ &=\int_{\mathbb{R}^d}\int_{\mathbf{I}}\boldsymbol{\varphi}(\widehat{T}^\varepsilon(s,x),\widehat{Y}^\varepsilon(s,x))\cdot\tau^\varepsilon(\widehat{T}^\varepsilon(s,x),\widehat{Y}^\varepsilon(s,x))\boldsymbol{w}^\varepsilon(\widehat{T}^\varepsilon(s,x),\widehat{Y}^\varepsilon(s,x))\,\mathrm{d}s\,\mathrm{d}\bar\mu^\varepsilon_0(x)\\ &=\int_{\mathbf{I}}\Big(\int_{\mathbb{R}^d}\boldsymbol{\varphi}(\widehat{T}^\varepsilon(s,x),\widehat{Y}^\varepsilon(s,x))\cdot\boldsymbol{v}^\varepsilon(\widehat{T}^\varepsilon(s,x),\widehat{Y}^\varepsilon(s,x))\,\mathrm{d}\bar\mu^\varepsilon_0(x)\Big)\,\mathrm{d}s\\ &=\int_{\mathbf{I}}\Big(\int_{\mathbb{R}^{d+1}}\boldsymbol{\varphi}(t,x)\cdot\boldsymbol{v}^\varepsilon(t,x)\,\mathrm{d}\sigma^\varepsilon_s(t,x)\Big)\,\mathrm{d}s=\int_{\mathbb{R}^{d+2}_+}\boldsymbol{\varphi}(t,x)\,\mathrm{d}\boldsymbol{\sigma}^\varepsilon(s,t,x)\,.\end{aligned}$$

Finally, for every open subset A of $\mathbb{R}^{d+1}_+$ we have that

$$|(\mu^\varepsilon,\boldsymbol{\nu}^\varepsilon)|(A)=\int_A\|(1,\boldsymbol{w}^\varepsilon)(t,x)\|\,\mathrm{d}\mu^\varepsilon(t,x)\overset{(1)}{=}\int_{\mathbf{I}\times A}\|(1,\boldsymbol{w}^\varepsilon)(t,x)\|\,\mathrm{d}\sigma^{\varepsilon,0}(s,t,x)$$

$$\overset{(2)}{=} \int_{\mathbf{I}\times A} \|(1,\boldsymbol{w}^\varepsilon)(t,x)\|\, \tau^\varepsilon(t,x)\,\mathrm{d}\sigma^\varepsilon(s,t,x) \overset{(3)}{=} \int_{\mathbf{I}\times A} \mathrm{d}\sigma^\varepsilon(s,t,x) = \boldsymbol{\pi}_\sharp\sigma^\varepsilon(A)\,,$$

where (1) follows from the previously proved fact that $\mu^\varepsilon = \boldsymbol{\pi}_\sharp\sigma^{\varepsilon,0}$; for (2) we have used that $\sigma^{\varepsilon,0} = \tau^\varepsilon\sigma^\varepsilon$, while (3) is a consequence of (4.48a). This concludes the proof of (4.61).

In order to check the tightness estimate (4.62), let us denote by $\iota_{S,T}$ the characteristic function of $(S,+\infty)\times[0,T]$ and by $j^\varepsilon_{S,T}(s,x)$ the characteristic function of $(S,S^\varepsilon(T,x)]$ (which is identically 0 if $S \geq S^\varepsilon(T,x)$). We first observe that

$$\iota_{S,T}(s,\widehat{T}^\varepsilon(s,x)) = j^\varepsilon_{S,T}(s,x)$$

so that

$$\begin{aligned}
\sigma^{\varepsilon,0}((S,+\infty)\times[0,T]\times\mathbb{R}^d) &= \int_{\mathbb{R}^{d+2}} \iota_{S,T}(s,t)\tau^\varepsilon(t,x)\,\mathrm{d}\sigma^\varepsilon(s,t,x)\\
&= \int_{\mathbb{R}^d}\int_{\mathbf{I}} \iota_{S,T}(s,\widehat{T}^\varepsilon(s,x))\tau^\varepsilon(\widehat{T}^\varepsilon(s,x),\widehat{Y}^\varepsilon(s,x))\,\mathrm{d}s\,\mathrm{d}\bar\mu^\varepsilon_0(x)\\
&= \int_{\mathbb{R}^d}\int_{\mathbf{I}} j^\varepsilon_{S,T}(s,x)\partial_s\widehat{T}^\varepsilon(s,x)\,\mathrm{d}s\,\mathrm{d}\bar\mu^\varepsilon_0(x)\\
&= \int_{\mathbb{R}^d} \big(T-\widehat{T}^\varepsilon(S,x)\big)_+\,\mathrm{d}\bar\mu^\varepsilon_0(x) \leq T\,\bar\mu^\varepsilon_0\Big\{x\in\mathbb{R}^d\,:\,\widehat{T}^\varepsilon(S,x)<T\Big\}\\
&= T\,\bar\mu^\varepsilon_0\Big\{x\in D^\varepsilon\,:\,S^\varepsilon(T,x)>S\Big\}\,.
\end{aligned}$$

Estimate (4.62) then follows by (4.55) and by the Chebyschev inequality. □

Step 3: Passage to the limit as $\varepsilon\downarrow 0$. Since the curves of measures $(\sigma^\varepsilon)_\varepsilon \subset \mathrm{Lip}(\mathbf{I};\mathcal{P}_1(\mathbb{R}^{d+1}))$ are 1-Lipschitz continuous for every $\varepsilon>0$, and bounded sets in $\mathcal{P}_1(\mathbb{R}^{d+1})$ are narrowly compact in $\mathcal{P}(\mathbb{R}^{d+1})$, we can find a limit curve $\sigma\in\mathrm{Lip}(\mathbf{I};\mathcal{P}_1(\mathbb{R}^{d+1}))$, 1-Lipschitz and supported in $\mathbb{R}^{d+1}_+$, and a vanishing subsequence $(\varepsilon_k)_k$ such that $\sigma^{\varepsilon_k}_s \rightharpoonup \sigma_s$ narrowly in $\mathcal{P}_1(\mathbb{R}^{d+1})$ for every $s\in\mathbf{I}$. As usual we will also denote by σ the Radon measure $\mathcal{L}^1\otimes\sigma_s$ in $\mathcal{M}^+_{\mathrm{loc}}(\mathbf{I}\times\mathbb{R}^{d+1})$ satisfying

$$\sigma^{\varepsilon_k}\llcorner[0,S]\times\mathbb{R}^{d+1}_+ \rightharpoonup \sigma\llcorner[0,S]\times\mathbb{R}^{d+1}_+ \qquad \text{narrowly in } \mathcal{M}^+(\mathbb{R}^{d+2}) \quad \text{for every } S>0. \tag{4.63}$$

Clearly, the above convergence also yields $\sigma^{\varepsilon_k}\rightharpoonup^*\sigma$ in $\mathcal{M}^+_{\mathrm{loc}}([0,+\infty)\times\mathbb{R}^{d+1}_+)$ as $k\uparrow\infty$. Moreover, we deduce from (4.43) and from (4.61) that

$$\boldsymbol{\pi}_\sharp\sigma \leq |(\mu,\boldsymbol{\nu})|\,. \tag{4.64}$$

From (4.60) and the projection relations (4.61) it follows that the restrictions to $[0,S]\times\mathbb{R}^{d+1}_+$ of the families of measures $(\sigma^{\varepsilon,0})_\varepsilon$ and $(\boldsymbol{\sigma}^\varepsilon)_\varepsilon$ are uniformly tight, so that there exist $\sigma^0\in\mathcal{M}^+_{\mathrm{loc}}(\mathbb{R}^{d+2}_+)$ and $\boldsymbol{\sigma}\in\mathcal{M}_{\mathrm{loc}}(\mathbb{R}^{d+2}_+;\mathbb{R}^d)$ such that up to a further (not relabeled) subsequence

$$(\sigma^{\varepsilon_k,0},\boldsymbol{\sigma}^{\varepsilon_k})\llcorner[0,S]\times\mathbb{R}^{d+1}_+ \rightharpoonup (\sigma^0,\boldsymbol{\sigma})\llcorner[0,S]\times\mathbb{R}^{d+1}_+ \quad \text{in } \mathcal{M}(\mathbb{R}^{d+2};\mathbb{R}^{d+1}) \quad \text{for every } S>0. \tag{4.65}$$

Therefore, also thanks to the third of (4.41a) and (4.63) we gather that the triple $(\sigma,\sigma^0,\boldsymbol{\sigma})$ satisfies the Cauchy problem (4.2) in the sense of Definition 3.1, with the operator $(\partial_t,\mathrm{div})$ playing the role of the 'spatial divergence' in (3.2). Moreover, it follows from (4.49), (4.63), and the lower semicontinuity of the total variation functional that

$$|(\sigma^0,\boldsymbol{\sigma})| \leq \sigma \qquad \text{in } \mathbb{R}^{d+2}, \tag{4.66}$$

so that in particular $\sigma^0,\boldsymbol{\sigma}\ll\sigma$. Hence, since σ is concentrated in $\mathbf{I}\times\mathbb{R}^{d+1}_+$, the Radon-Nikodým derivatives can be represented by bounded Borel fields

$$\hat\tau := \frac{\mathrm{d}\sigma^0}{\mathrm{d}\sigma} : \mathbf{I}\times\mathbb{R}^{d+1}_+ \to \mathbf{I} \qquad \hat{\boldsymbol{v}} := \frac{\mathrm{d}\boldsymbol{\sigma}}{\mathrm{d}\sigma} : \mathbf{I}\times\mathbb{R}^{d+1}_+ \to \mathbb{R}^d$$

satisfying

$$\|(\hat\tau,\hat{\boldsymbol{v}})\| \leq 1 \qquad \sigma\text{-a.e. in } \mathbf{I}\times\mathbb{R}^{d+1}_+. \tag{4.67}$$

In terms of $\hat{\tau}$ and $\hat{\boldsymbol{v}}$, we rewrite the continuity equation (4.2) as

$$\begin{cases} \partial_s \sigma + \partial_t(\hat{\tau}\sigma) + \operatorname{div}(\hat{\boldsymbol{v}}\sigma) = 0 & \text{in } \mathbf{I} \times \mathbb{R}^{d+1} \\ \sigma_0 = \delta_0 \otimes \mu_0 \,. \end{cases} \tag{4.68}$$

It remains to prove the projection properties (4.34). We start by showing that

$$\mu = \boldsymbol{\pi}_\sharp \sigma^0. \tag{4.69}$$

For this, the tightness estimate (4.62) implies that $\sigma^{\varepsilon,0} \llcorner \mathbf{I} \times [0,T] \times \mathbb{R}^d$ narrowly converge to $\sigma^0 \llcorner \mathbf{I} \times [0,T] \times \mathbb{R}^d$ for every $T > 0$. This shows that $\mu([0,T] \times \mathbb{R}^d) = \sigma^0(\mathbf{I} \times [0,T] \times \mathbb{R}^d)$ for every $T > 0$. Since $\boldsymbol{\pi}_\sharp \sigma^0 \le \mu$, the above equality yields (4.69).

Let us now show that

$$\boldsymbol{\nu} = \boldsymbol{\pi}_\sharp \boldsymbol{\sigma} \,. \tag{4.70}$$

With this aim, we set $\tilde{\boldsymbol{\nu}} := \boldsymbol{\pi}_\sharp \boldsymbol{\sigma}$. In order to show that $\tilde{\boldsymbol{\nu}} = \boldsymbol{\nu}$, we argue in the following way. On the one hand, it follows from the previously proved Lemma 4.4 that the pair $(\mu, \tilde{\boldsymbol{\nu}})$ solves the continuity equation

$$\partial_t \mu + \operatorname{div} \tilde{\boldsymbol{\nu}} = 0 \qquad \text{in } \mathbb{R}^{d+1}_+ \,, \tag{4.71}$$

with initial condition μ_0, in the sense of Definition 3.1. On the other hand, applying Lemma 2.7 with the choices $\mathsf{p} = \boldsymbol{\pi} \colon \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R} \times \mathbb{R}^d$, $\boldsymbol{\zeta}_k = (\sigma^{\varepsilon_k,0}, \boldsymbol{\sigma}^{\varepsilon_k})$, $\boldsymbol{\zeta} = (\sigma^0, \boldsymbol{\sigma})$, $\boldsymbol{\lambda}_k = \boldsymbol{\pi}_\sharp(\sigma^{\varepsilon_k,0}, \boldsymbol{\sigma}^{\varepsilon_k}) = (\mu^{\varepsilon_k}, \boldsymbol{\nu}^{\varepsilon_k})$, $\boldsymbol{\lambda} = (\mu, \boldsymbol{\nu})$, we show that

$$(\mu, \tilde{\boldsymbol{\nu}}) \overset{(4.69)}{=} (\boldsymbol{\pi}_\sharp \sigma^0, \boldsymbol{\pi}_\sharp \boldsymbol{\sigma}) \prec (\mu, \boldsymbol{\nu}).$$

Then, by Lemma 2.2 there exists $\lambda \in L^\infty_{|(\mu,\boldsymbol{\nu})|}(\mathbb{R}^{d+1}_+; [0,1])$ such that $(\mu, \tilde{\boldsymbol{\nu}}) = \lambda(\mu, \boldsymbol{\nu})$. Since the first components coincide, from that equality we infer that $\tilde{\boldsymbol{\nu}} = \lambda \boldsymbol{\nu}$ and $\lambda \equiv 1$ μ-a.e., as well. We decompose $\boldsymbol{\nu}$ and $\tilde{\boldsymbol{\nu}}$ into their absolutely continuous and singular part w.r.t. μ, namely,

$$\boldsymbol{\nu} = \boldsymbol{\nu}^a + \boldsymbol{\nu}^\perp \qquad \tilde{\boldsymbol{\nu}} = \tilde{\boldsymbol{\nu}}^a + \tilde{\boldsymbol{\nu}}^\perp \qquad \text{with } \boldsymbol{\nu}^\perp, \tilde{\boldsymbol{\nu}}^\perp \perp \mu \,.$$

Since $\tilde{\boldsymbol{\nu}} = \lambda \boldsymbol{\nu}$, we have

$$\tilde{\boldsymbol{\nu}}^a + \tilde{\boldsymbol{\nu}}^\perp = (\lambda \boldsymbol{\nu}^a) + \lambda \boldsymbol{\nu}^\perp \,.$$

As $\lambda \equiv 1$ μ-a.e. in $\mathbb{R}^{d+1}_+$, we have that $\tilde{\boldsymbol{\nu}}^a = \boldsymbol{\nu}^a$, $\tilde{\boldsymbol{\nu}}^\perp = \lambda \boldsymbol{\nu}^\perp$. Therefore, $\tilde{\boldsymbol{\nu}}^\perp \prec \boldsymbol{\nu}^\perp$. Moreover, by Lemma 4.4 it holds $\operatorname{div} \tilde{\boldsymbol{\nu}}^\perp = \operatorname{div} \boldsymbol{\nu}^\perp$. By hypothesis, $\boldsymbol{\nu}^\perp$ is minimal. Therefore, we conclude that $\tilde{\boldsymbol{\nu}}^\perp = \boldsymbol{\nu}^\perp$ and, since $\tilde{\boldsymbol{\nu}}^a \mu = \boldsymbol{\nu}^a \mu$, we ultimately have $\tilde{\boldsymbol{\nu}} = \boldsymbol{\nu}$. Then, (4.70) follows.

Lastly, we may conclude (4.32) by observing that

$$|(\mu, \boldsymbol{\nu})| \overset{(4.64)}{\ge} \boldsymbol{\pi}_\sharp \sigma \overset{(4.66)}{\ge} \boldsymbol{\pi}_\sharp |(\sigma^0, \boldsymbol{\sigma})| \ge |\boldsymbol{\pi}_\sharp(\sigma^0, \boldsymbol{\sigma})| = |(\mu, \boldsymbol{\nu})| \,.$$

This finishes the proof.

□

5. The parametrized superposition principle

The main result of this section, Theorem 5.1, provides a probabilistic representation of the solutions of the continuity equation, which in fact extends the representation obtained in [11, Thm. 8.2.1] for absolutely continuous solutions. We will derive it from the probabilistic representation (4.18), provided by Theorem 4.5, for the solutions to the 'augmented' system (4.2). For that, we will need to take into account that σ and the measures μ and $\boldsymbol{\nu}$ are related via (4.34). Fine properties of our probabilistic representation will be proved in Proposition 5.2 and in Theorem 5.3.

In the spirit of [11, Thm. 8.2.1], the statement of Theorem 5.1 below consists of two parts:

(1) First of all, starting from a measure $\boldsymbol{\eta} \in \mathcal{P}(\mathrm{Lip}^\uparrow(\mathbf{I}; \mathbb{R}^{d+1}_+))$, we construct the measures $\mathfrak{t}'\mathcal{L}^1 \otimes \boldsymbol{\eta}$ and $\boldsymbol{\mathfrak{x}}'\mathcal{L}^1 \otimes \boldsymbol{\eta}$ (recall the definition (4.12) of $\mathfrak{t}'$ and $\boldsymbol{\mathfrak{x}}'$), and show that their push-forwards through the evaluation map $\mathfrak{e}$ from (4.10), cf. (5.2) below, solve the Cauchy problem (5.3). In fact, the Cauchy condition is expressed in terms of the push-forward $(\mathfrak{e}_0)_\sharp \boldsymbol{\eta}$ (where $\mathfrak{e}_0$ stands for $\mathfrak{e}(0, \cdot)$): in this respect, let us specify that, while $\mathfrak{e}_0$ is evaluated along curves $\mathsf{y} \in \mathrm{C}^\uparrow(\mathbf{I}; \mathbb{R}^{d+1})$, so that

$\mathfrak{e}_0(\mathsf{y}) = (\mathsf{t}(0), \mathsf{x}(0)) = (0, \mathsf{x}(0))$, in (5.3) with slight abuse of notation we will consider $\mathfrak{e}_0$ as valued in $\mathbb{R}^d$.

(2) Conversely, we prove that any solution of the continuity equation admits the probabilistic representation (5.2) below in terms of a probability measure $\boldsymbol{\eta}$ on the space $\mathrm{Lip}_1^{\uparrow}(\mathbf{I}; \mathbb{R}_+^{d+1})$ from (2.27). In addition, we show that for $\boldsymbol{\eta}$-a.a. curve y, the velocity $\mathsf{y}'(r)$ at a given time $r \in \mathbf{I}$ does not depend explicitly on r but only on the position $\mathsf{y}(r)$, see (5.7a) ahead.

Theorem 5.1. *The following facts hold:*

(1) Let $\boldsymbol{\eta} \in \mathcal{P}(\mathrm{Lip}_k^{\uparrow}(\mathbf{I}; \mathbb{R}_+^{d+1}))$ and $\boldsymbol{\eta}_{\mathcal{L}} := \mathcal{L}^1 \otimes \boldsymbol{\eta}$ fulfill

$$\int \|\mathfrak{e}_0\| \, \mathrm{d}\boldsymbol{\eta} < +\infty, \qquad \int_{\mathrm{E}(T)} \|\mathfrak{x}'\| \, \mathrm{d}\boldsymbol{\eta}_{\mathcal{L}} < \infty \quad \text{for every } T > 0. \tag{5.1}$$

Then, the pair $(\mu, \boldsymbol{\nu}) \in \mathcal{M}_{\mathrm{loc}}^+(\mathbb{R}_+^{d+1}) \times \mathcal{M}_{\mathrm{loc}}(\mathbb{R}_+^{d+1}; \mathbb{R}^d)$ defined by

$$\mu := \mathfrak{e}_\sharp(\mathfrak{t}' \boldsymbol{\eta}_{\mathcal{L}}) \qquad \boldsymbol{\nu} := \mathfrak{e}_\sharp(\mathfrak{x}' \boldsymbol{\eta}_{\mathcal{L}}) \tag{5.2}$$

is a $\mathcal{P}_1$-solution to the continuity equation,

$$\partial_t \mu + \operatorname{div} \boldsymbol{\nu} = \mu_0 \quad \text{in } (0, +\infty) \times \mathbb{R}^d, \qquad \text{with } \mu_0 = (\mathfrak{e}_0)_\sharp \boldsymbol{\eta} \in \mathcal{P}_1(\mathbb{R}^d), \tag{5.3}$$

in the sense of Definition 3.1. Moreover,

$$|(\mu, \boldsymbol{\nu})| \le \mathfrak{e}_\sharp(\|\mathfrak{y}'\| \boldsymbol{\eta}_{\mathcal{L}}). \tag{5.4}$$

(2) Conversely, let $(\mu, \boldsymbol{\nu}) \in \mathcal{M}_{\mathrm{loc}}^+(\mathbb{R}_+^{d+1}) \times \mathcal{M}_{\mathrm{loc}}(\mathbb{R}_+^{d+1}; \mathbb{R}^d)$ be a $\mathcal{P}_1$-solution to the continuity equation in the sense of Definition 3.1, with initial condition $\mu_0 \in \mathcal{P}_1(\mathbb{R}^d)$; let $(\mu, \bar{\boldsymbol{\nu}}) = \theta(\mu, \boldsymbol{\nu})$ be a minimal pair induced by $(\mu, \boldsymbol{\nu})$ according to Definition 3.2 and (3.6), let $\varrho \sim_k |(\mu, \boldsymbol{\nu})|$ for some $k > 0$, and let $(\tau, \boldsymbol{v})$ be a bounded Borel vector field representing the density of $(\mu, \boldsymbol{\nu})$ w.r.t. ϱ, namely

$$\mu = \tau \varrho, \qquad \boldsymbol{\nu} = \boldsymbol{v} \varrho \qquad \varrho\text{-a.e. in } \mathbb{R}_+^{d+1}. \tag{5.5}$$

Then, there exists a measure $\boldsymbol{\eta} \in \mathcal{P}(\mathrm{Lip}_k^{\uparrow}(\mathbf{I}; \mathbb{R}_+^{d+1}))$ such that the representation

$$\mu = \mathfrak{e}_\sharp(\mathfrak{t}' \boldsymbol{\eta}_{\mathcal{L}}) \qquad \bar{\boldsymbol{\nu}} = \mathfrak{e}_\sharp(\mathfrak{x}' \boldsymbol{\eta}_{\mathcal{L}}), \qquad |(\mu, \bar{\boldsymbol{\nu}})| = \mathfrak{e}_\sharp(\|\mathfrak{y}'\| \boldsymbol{\eta}_{\mathcal{L}}), \qquad \varrho = \mathfrak{e}_\sharp \boldsymbol{\eta}_{\mathcal{L}}. \tag{5.6a}$$

holds, and $\boldsymbol{\eta}$ is supported on curves solving the Cauchy problem

$$\begin{cases} \dot{\mathsf{y}}(s) = (\tau(\mathsf{y}(s)), \boldsymbol{v}(\mathsf{y}(s))) & \text{for a.a. } s \in (0, +\infty) \\ \mathsf{y}(0) = (0, x), & x \in \operatorname{spt}(\mu_0). \end{cases} \tag{5.7a}$$

Choosing in particular $\varrho = |(\mu, \boldsymbol{\nu})|$ we get $\boldsymbol{\eta} \in \mathcal{P}(\mathrm{Lip}_1^{\uparrow}(\mathbf{I}; \mathbb{R}_+^{d+1}))$ is concentrated on $\mathrm{ArcLip}(\mathbf{I}; \mathbb{R}_+^{d+1})$ *and*

$$|(\mu, \bar{\boldsymbol{\nu}})| = \mathfrak{e}_\sharp(\|\mathfrak{y}'\| \boldsymbol{\eta}_{\mathcal{L}}) = \mathfrak{e}_\sharp \boldsymbol{\eta}_{\mathcal{L}}. \tag{5.8}$$

Proof. Recalling the definitions of the evaluation maps (4.10), we observe that

$$\mathfrak{e} = \boldsymbol{\pi} \circ \mathfrak{a}. \tag{5.9}$$

Claim (1) follows by Claim (1) of Theorem 4.5 and Lemma 4.4.

Claim (2) follows by the augmented representation of $(\mu, \boldsymbol{\nu})$ given in Theorem 4.7, and by Claim (2) of Theorem 4.5. □

5.1. **Fine properties of the representing measure.** With the upcoming Proposition 5.2 we unveil some refined properties of the probabilistic representation provided by Theorem 5.1 for the solutions of the continuity equation. Namely, we give a finer representation formula for μ and specify the probabilistic representation of the measures $\boldsymbol{\nu}^a \ll \mu$ and $\boldsymbol{\nu}^\perp$ featuring in the decomposition $\boldsymbol{\nu} = \boldsymbol{\nu}^a + \boldsymbol{\nu}^\perp$, cf. (5.11) below. To this purpose, we need to introduce some further notation. For every $\mathsf{y} = (\mathsf{t}, \mathsf{x}) \in \mathrm{Lip}^{\uparrow}(\mathbf{I}; \mathbb{R}_+^{d+1})$ we define the sets $D_+[\mathsf{y}]$, $D_0[\mathsf{y}]$, $D_c[\mathsf{y}] \subseteq \mathbf{I}$ as

$$\begin{aligned} D_+[\mathsf{y}] &:= \{s \in \mathbf{I} : \mathsf{t}'(s) > 0\}, \\ D_0[\mathsf{y}] &:= \{s \in \mathbf{I} : \mathsf{t}'(s) = 0\}, \\ D_c[\mathsf{y}] &:= \{s \in \mathbf{I} : \mathsf{t}(\cdot) \text{ is constant in a neighborhood of } s\}, \end{aligned} \tag{5.10}$$

where $\mathfrak{t}'$ denotes the upper derivative defined in (4.11)–(4.12). Its usage is motivated by the fact that $\mathfrak{t}'$ exists at every $s \in \mathbf{I}$ and it is a bounded nonnegative Borel map. We further set

$$D_+ := \{(s,\mathbf{y}) \in \mathbf{I} \times \mathrm{Lip}^{\uparrow}(\mathbf{I};\mathbb{R}^{d+1}_+) : s \in D_+[\mathbf{y}]\}\,,$$
$$D_0 := \{(s,\mathbf{y}) \in \mathbf{I} \times \mathrm{Lip}^{\uparrow}(\mathbf{I};\mathbb{R}^{d+1}_+) : s \in D_0[\mathbf{y}]\}\,,$$
$$D_c := \{(s,\mathbf{y}) \in \mathbf{I} \times \mathrm{Lip}^{\uparrow}(\mathbf{I};\mathbb{R}^{d+1}_+) : s \in D_c[\mathbf{y}]\}\,.$$

Since we have $D_c[\mathbf{y}] \subseteq D_0[\mathbf{y}]$ for every $\mathbf{y}$, there holds $D_c \subseteq D_0$.

We are now in a position to provide the probabilistic representation of $\boldsymbol{\nu}^a$ and $\boldsymbol{\nu}^\perp$ in terms of the measure $\boldsymbol{\eta} \in \mathcal{P}(\mathrm{Lip}^{\uparrow}(\mathbf{I};\mathbb{R}^{d+1}_+))$ featuring in (5.6a) and (5.8).

Proposition 5.2 (Decomposition property of the probabilistic representation)**.** *Let $(\mu,\boldsymbol{\nu})$ be a $\mathcal{P}_1$ solution to the continuity equation (3.1) in the sense of Definition 3.1, let $(\mu,\bar{\boldsymbol{\nu}})$ be a minimal solution induced by $(\mu,\boldsymbol{\nu})$ with Lebesgue decomposition $\bar{\boldsymbol{\nu}} = \boldsymbol{\nu}^a + \bar{\boldsymbol{\nu}}^\perp$. Suppose that the representation formulae (5.6a) and (5.8) hold with $\boldsymbol{\eta} \in \mathcal{P}(\mathrm{Lip}^{\uparrow}(\mathbf{I};\mathbb{R}^{d+1}_+))$ and $\boldsymbol{\eta}_{\mathcal{L}} = \mathcal{L}^1 \otimes \boldsymbol{\eta}$. Then, we have that*

$$\mu = \mathfrak{e}_\sharp(\mathfrak{t}'\,\boldsymbol{\eta}_{\mathcal{L}} \llcorner D_+)\,, \qquad \boldsymbol{\nu}^a = \mathfrak{e}_\sharp(\mathfrak{x}'\,\boldsymbol{\eta}_{\mathcal{L}} \llcorner D_+)\,, \qquad \bar{\boldsymbol{\nu}}^\perp = \mathfrak{e}_\sharp(\mathfrak{x}'\,\boldsymbol{\eta}_{\mathcal{L}} \llcorner D_0)\,, \tag{5.11}$$
$$|(\mu,\boldsymbol{\nu}^a)| = \mathfrak{e}_\sharp(\|\mathfrak{y}'\|\,\boldsymbol{\eta}_{\mathcal{L}} \llcorner D_+)\,, \qquad |\bar{\boldsymbol{\nu}}^\perp| = \mathfrak{e}_\sharp(\|\mathfrak{x}'\|\,\boldsymbol{\eta}_{\mathcal{L}} \llcorner D_0). \tag{5.12}$$

Proof. Since $\mu = \mathfrak{e}_\sharp(\mathfrak{t}'\boldsymbol{\eta}_{\mathcal{L}})$, the first equality in (5.11) holds. Being $\boldsymbol{\nu}^a \ll \mu$ and $D_0 \cap D_+ = \emptyset$, the second and the third equalities in (5.11) follow.

Concerning (5.12), we have

$$|(\mu,\boldsymbol{\nu})| = |(\mu,\boldsymbol{\nu}^a)| + |\boldsymbol{\nu}^\perp| \le \mathfrak{e}_\sharp(\|\mathfrak{y}'\|\,\boldsymbol{\eta}_{\mathcal{L}} \llcorner D_+) + \mathfrak{e}_\sharp(\|\mathfrak{y}'\|\,\boldsymbol{\eta}_{\mathcal{L}} \llcorner D_0) = \mathfrak{e}_\sharp(\|\mathfrak{y}'\|\,\boldsymbol{\eta}_{\mathcal{L}}) = |(\mu,\boldsymbol{\nu})|$$

so that all inequalities are in fact equalities and (5.12) follows. □

With the next result, we use the representation formulae (5.11) provided by Proposition 5.2, to show that the superposition measure $\boldsymbol{\eta} \in \mathcal{P}(\mathrm{Lip}^{\uparrow}(\mathbf{I};\mathbb{R}^{d+1}))$ is supported on injective curves. In the proof, we will identify the pair $(\mu,\boldsymbol{\nu})$ with an associated mininimal solution $(\mu,\bar{\boldsymbol{\nu}})$.

Theorem 5.3. *Let $(\mu,\boldsymbol{\nu})$ be a minimal $\mathcal{P}_1$-solution to the continuity equation (3.1) and let the measure $\boldsymbol{\eta} \in \mathcal{P}(\mathrm{Lip}^{\uparrow}_k(\mathbf{I};\mathbb{R}^{d+1}_+))$ provide the representation formulae (5.6a). Then, $\boldsymbol{\eta}$-a.e. curve $\mathbf{y} \in \mathrm{Lip}^{\uparrow}(\mathbf{I};\mathbb{R}^{d+1}_+)$ is injective.*

Proof. We prove the theorem by contradiction. Let us assume that there exist $S > 0$ and a set

$$\Lambda \subseteq \mathrm{Lip}^{\uparrow}(\mathbf{I};\mathbb{R}^{d+1}_+) \text{ with } \boldsymbol{\eta}(\Lambda) > 0 \text{ s.t. every } \mathbf{y} \in \Lambda \text{ is not injective in the interval } [0,S]. \tag{5.13}$$

Since $\boldsymbol{\eta}$ is a Radon measure, it is not restrictive to assume that Λ is compact.

For $\mathbf{y} \in \mathrm{Lip}^{\uparrow}(\mathbf{I};\mathbb{R}^{d+1}_+)$ and $s \in [0,S]$ we define

$$r(s,\mathbf{y}) := \max\{\sigma \in [0,S] : \mathbf{y}(\sigma) = \mathbf{y}(s)\}\,, \qquad r_\flat(s,\mathbf{y}) := r(s,\mathbf{y}) - s\,, \qquad \mathfrak{r}(\mathbf{y}) := \max_{s\in[0,S]} r_\flat(s,\mathbf{y}),$$

and we notice that the maps $(s,\mathbf{y}) \mapsto r_\flat(s,\mathbf{y})$ and $\mathbf{y} \mapsto \mathfrak{r}(\mathbf{y})$ are upper semicontinuous in $[0,S] \times \Lambda$ and in Λ respectively. Since $\mathfrak{r}(\mathbf{y}) > 0$ for every $\mathbf{y} \in \Lambda$, we may find $h \in \mathbb{N}$ such that the (closed, thus compact) set

$$\Lambda' := \Big\{\mathbf{y} \in \Lambda : \mathfrak{r}(\mathbf{y}) \ge \frac{1}{h}\Big\}$$

satisfies $\boldsymbol{\eta}(\Lambda') > 0$. For every $\mathbf{y} \in \Lambda'$ we now define

$$\Xi(\mathbf{y}) := \Big\{s \in [0,S] : r_\flat(s,\mathbf{y}) \ge 1/h\Big\}, \quad \mathfrak{s}_{\mathrm{l}}(\mathbf{y}) := \min \Xi(\mathbf{y}),$$

where l stands for 'lower'. Notice that $\mathfrak{s}_{\mathrm{l}}$ is well defined since $\Xi(\mathbf{y})$ is a closed nonempty subset of $[0,S]$; one can easily check that is lower semicontinuous: if $(\mathbf{y}_n)_n \subset \Lambda'$ converges to $\mathbf{y}$ and $s_n = \mathfrak{s}_{\mathrm{l}}(\mathbf{y}_n)$ is converging (up to the extraction of a subsequence) to s, we know that $r_\flat(s,\mathbf{y}) \ge 1/h$ (by the upper semicontinuity of $r_\flat$) so that $s \in \Xi(\mathbf{y})$ and $\mathfrak{s}_{\mathrm{l}}(\mathbf{y}) \le s$.

We set $\mathfrak{s}_{\rm u}(\mathbf{y}) := r(\mathfrak{s}_{\rm l}(\mathbf{y}), \mathbf{y})$ (with u for 'upper'); the function $\mathbf{y} \mapsto \big(\mathfrak{s}_{\rm l}(\mathbf{y}), \mathfrak{s}_{\rm u}(\mathbf{y})\big)$ is Borel measurable and for every $\mathbf{y} \in \Lambda'$ we have by construction

$$\mathbf{y}(\mathfrak{s}_{\rm l}(\mathbf{y})) = \mathbf{y}(\mathfrak{s}_{\rm u}(\mathbf{y})), \qquad \mathfrak{s}_{\rm u}(\mathbf{y}) - \mathfrak{s}_{\rm l}(\mathbf{y}) = r_\flat(\mathfrak{s}_{\rm l}(\mathbf{y}), \mathbf{y}) \geq \frac{1}{h}. \tag{5.14}$$

Since t is non-decreasing we conclude that t is constant in $(\mathfrak{s}_{\rm l}(\mathbf{y}), \mathfrak{s}_{\rm u}(\mathbf{y}))$, hence for the interval $(\mathfrak{s}_{\rm l}(\mathbf{y}),\, \mathfrak{s}_{\rm u}(\mathbf{y}))$ we have

$$(\mathfrak{s}_{\rm l}(\mathbf{y}),\, \mathfrak{s}_{\rm u}(\mathbf{y})) \subseteq D_c[\mathbf{y}] \subseteq D_0[\mathbf{y}]\,. \tag{5.15}$$

We introduce the function $\mathcal{T}\colon \mathrm{Lip}^\uparrow(\mathbf{I}; \mathbb{R}^{d+1}_+) \to \mathrm{Lip}^\uparrow(\mathbf{I}; \mathbb{R}^{d+1}_+)$ defined by $\mathcal{T}(\mathbf{y}) = \mathbf{y}$ for $\mathbf{y} \notin \Lambda'$ and

$$\mathcal{T}(\mathbf{y})(s) = \begin{cases} \mathbf{y}(s) & \text{for } s \leq \mathfrak{s}_{\rm l}(\mathbf{y})\,, \\ \mathbf{y}(s + r_\flat(\mathfrak{s}_{\rm l}(\mathbf{y}), \mathbf{y})) & \text{for } s > \mathfrak{s}_{\rm l}(\mathbf{y}) \end{cases}$$

for $\mathbf{y} \in \Lambda'$. By construction, the map $\mathcal{T}$ is Borel measurable and satisfies $\mathcal{T}(\mathbf{y}) \in \mathrm{Lip}^\uparrow_k(\mathbf{I}; \mathbb{R}^{d+1})$ for every $\mathbf{y} \in \mathrm{Lip}^\uparrow_k(\mathbf{I}; \mathbb{R}^{d+1})$, so that the push-forward $\boldsymbol{\eta}_\flat := \mathcal{T}_\sharp \boldsymbol{\eta}$ belongs to $\mathcal{P}(\mathrm{Lip}^\uparrow_k(\mathbf{I}; \mathbb{R}^{d+1}_+))$.

After these preliminary definitions, we are in a position to carry out the contradiction, which will be essentially based on the fact that the measure $\boldsymbol{\eta}_\flat$ also provides a probabilistic representation of the pair $(\mu, \boldsymbol{\nu})$, cf. the upcoming Claims 1, 2, and 3. For later use, let us set $\boldsymbol{\nu}_\flat := \mathfrak{e}_\sharp(\mathbf{x}'\mathcal{L}^1 \otimes \boldsymbol{\eta}_\flat)$. Let us write $\boldsymbol{\nu} = \boldsymbol{\nu}^a + \boldsymbol{\nu}^\perp$ and $\boldsymbol{\nu}_\flat = \boldsymbol{\nu}_\flat^a + \boldsymbol{\nu}_\flat^\perp$, where $\boldsymbol{\nu}^a, \boldsymbol{\nu}_\flat^a \ll \mu$ and $\boldsymbol{\nu}^\perp, \boldsymbol{\nu}_\flat^\perp \perp \mu$.

Claim 1: *we have*

$$\mu = \mathfrak{e}_\sharp(\mathsf{t}'\mathcal{L}^1 \otimes \boldsymbol{\eta}_\flat). \tag{5.16}$$

This follows from the finer representation of μ provided by (5.11), and by the definition of $\mathcal{T}$. Indeed, using the place-holder $\mu_\flat := \mathfrak{e}_\sharp(\mathsf{t}'\mathcal{L}^1 \otimes \boldsymbol{\eta}_\flat)$, for every $\varphi_0 \in \mathrm{C_c}(\mathbb{R}^{d+1}_+)$ we have

$$\begin{aligned} &\int_{\mathbb{R}^{d+1}_+} \varphi_0(t,x)\,\mathrm{d}(\mu - \mu_\flat)(t,x) \\ &= \int_{\Lambda'} \int_{\mathbf{I}} \varphi_0(\mathbf{y}(s))\mathsf{t}'(s)\,\mathrm{d}s\,\mathrm{d}\boldsymbol{\eta}(\mathbf{y}) - \int_{\Lambda'} \int_{\mathbf{I}} \varphi_0(\mathbf{y}(s))\mathsf{t}'(s)\,\mathrm{d}s\,\mathrm{d}\boldsymbol{\eta}_\flat(\mathbf{y}) \\ &\overset{(1)}{=} \int_{\Lambda'} \int_{\mathfrak{s}_{\rm l}(\mathbf{y})}^{+\infty} \varphi_0(\mathbf{y}(s))\mathsf{t}'(s)\,\mathrm{d}s\,\mathrm{d}\boldsymbol{\eta}(\mathbf{y}) - \int_{\Lambda'} \int_{\mathfrak{s}_{\rm l}(\mathbf{y})}^{+\infty} \varphi_0(\mathbf{y}(s{+}r_\flat(\mathfrak{s}_{\rm l}(\mathbf{y}),\mathbf{y}))\mathsf{t}'(s{+}r_\flat(\mathfrak{s}_{\rm l}(\mathbf{y}),\mathbf{y}))\,\mathrm{d}s\,\mathrm{d}\boldsymbol{\eta}(\mathbf{y}) \\ &\overset{(2)}{=} \int_{\Lambda'} \int_{\mathfrak{s}_{\rm u}(\mathbf{y})}^{+\infty} \varphi_0(\mathbf{y}(s))\mathsf{t}'(s)\,\mathrm{d}s\,\mathrm{d}\boldsymbol{\eta}(\mathbf{y}) - \int_{\Lambda'} \int_{\mathfrak{s}_{\rm u}(\mathbf{y})}^{+\infty} \varphi_0(\mathbf{y}(\tau))\mathsf{t}'(\tau)\,\mathrm{d}\tau\,\mathrm{d}\boldsymbol{\eta}(\mathbf{y}) = 0\,. \end{aligned} \tag{5.17}$$

In the above chain of equalities, (1) follows from the fact that $\mathcal{T}$ is the identity in the complement of Λ' and modifies a curve $\mathbf{y} \in \Lambda'$ only on $(\mathfrak{s}_{\rm l}(\mathbf{y}), +\infty)$, while (2) ensues from the fact that $\mathsf{t}' \equiv 0$ on $(\mathfrak{s}_{\rm l}(\mathbf{y}), \mathfrak{s}_{\rm u}(\mathbf{y}))$ and from a change of variable in the second integral.

Claim 2: *we have*

$$\boldsymbol{\nu}^a = \boldsymbol{\nu}_\flat^a\,. \tag{5.18}$$

With the very same arguments as in (5.17) we check that

$$\int_{\mathbb{R}^{d+1}_+} \boldsymbol{\varphi}(t,x)\,\mathrm{d}(\boldsymbol{\nu} - \boldsymbol{\nu}_\flat)(t,x) = \int_{\Lambda'} \Big(\int_{\mathfrak{s}_{\rm l}(\mathbf{y})}^{\mathfrak{s}_{\rm u}(\mathbf{y})} \boldsymbol{\varphi}(\mathbf{y}(s)) \cdot \mathbf{x}'(s)\,\mathrm{d}s \Big)\,\mathrm{d}\boldsymbol{\eta}(\mathbf{y}) \tag{5.19}$$

for every $\boldsymbol{\varphi} \in \mathrm{C_c}(\mathbb{R}^{d+1}_+; \mathbb{R}^d)$. By (5.11) and by (5.15) we therefore have that $(\boldsymbol{\nu}{-}\boldsymbol{\nu}_\flat) \perp \mu$ and $\boldsymbol{\nu}^a = \boldsymbol{\nu}_\flat^a$. Thus, $\boldsymbol{\nu}_\flat = \boldsymbol{\nu}^a + \boldsymbol{\nu}_\flat^\perp$.

Claim 3: *we have*

$$\mathrm{div}\,\boldsymbol{\nu}_\flat^\perp = \mathrm{div}\,\boldsymbol{\nu}^\perp\,. \tag{5.20}$$

Denoting $\boldsymbol{\theta} := \boldsymbol{\nu}^\perp - \boldsymbol{\nu}_\flat^\perp = \boldsymbol{\nu} - \boldsymbol{\nu}_\flat$, it follows from (5.19) and Theorem 5.1(2), that for every test function $\boldsymbol{\varphi} \in \mathrm{C_c}(\mathbb{R}^{d+1}_+; \mathbb{R}^d)$ there holds

$$\int_{\mathbb{R}^{d+1}_+} \boldsymbol{\varphi}(t,x)\,\mathrm{d}\boldsymbol{\theta}(t,x) = \int_{\Lambda'} \int_{\mathfrak{s}_{\rm l}(\mathbf{y})}^{\mathfrak{s}_{\rm u}(\mathbf{y})} \boldsymbol{\varphi}(\mathbf{y}(s)) \cdot \mathbf{x}'(s)\,\mathrm{d}s\,\mathrm{d}\boldsymbol{\eta}(\mathbf{y}), \tag{5.21}$$

i.e.

$$\boldsymbol{\theta} = \mathfrak{e}_\sharp\big(\mathfrak{x}'\,\boldsymbol{\eta}_{\mathcal{L}}\,\llcorner\,\Theta\big), \quad \Theta := \Big\{(s,\mathbf{y}) : \mathbf{y}\in\Lambda',\ \mathfrak{s}_{\rm l}(\mathbf{y}) \le s \le \mathfrak{s}_{\rm u}(\mathbf{y})\Big\}. \tag{5.22}$$

It is sufficient to select $\boldsymbol{\varphi} = \mathrm{D}\varphi$ for $\varphi \in \mathrm{C}^1_{\rm c}(\mathbb{R}^{d+1}_+)$ in (5.21) and to observe that for every $\mathbf{y}\in\Lambda'$ the inner integral

$$\int_{\mathfrak{s}_{\rm l}(\mathbf{y})}^{\mathfrak{s}_{\rm u}(\mathbf{y})} \mathrm{D}\varphi(\mathbf{y}(s))\cdot\mathbf{x}'(s)\,\mathrm{d}s = \int_{\mathfrak{s}_{\rm l}(\mathbf{y})}^{\mathfrak{s}_{\rm u}(\mathbf{y})} \mathrm{D}\varphi(t,\mathbf{x}(s))\cdot\mathbf{x}'(s)\,\mathrm{d}s = \varphi(t,\mathbf{x}(\mathfrak{s}_{\rm u}(\mathbf{y}))) - \varphi(t,\mathbf{x}(\mathfrak{s}_{\rm l}(\mathbf{y}))) = 0$$

since $\mathsf{t}(s)\equiv t$ is constant in the interval $(\mathfrak{s}_{\rm l}(\mathbf{y}),\mathfrak{s}_{\rm l}(\mathbf{y}))$. Plugging this formula in (5.21), with $\boldsymbol{\varphi} = \mathrm{D}\varphi$ for an arbitrary $\varphi\in\mathrm{C}^1_c(\mathbb{R}^{d+1}_+)$, and we get $\operatorname{div}\boldsymbol{\theta} = 0$.

Claim 4: *we have*

$$\boldsymbol{\nu}^\perp_\flat \prec \boldsymbol{\nu}^\perp . \tag{5.23}$$

Let us define $\boldsymbol{\eta}^0_\flat := \boldsymbol{\eta}_\flat \llcorner D_0$, $\boldsymbol{\vartheta} = \mathfrak{x}'\boldsymbol{\eta}^0_\flat = (\boldsymbol{v}\circ\mathfrak{e})\boldsymbol{\eta}^0_\flat$; let us consider the Borel function

$$\lambda(s,\mathbf{y}) := \begin{cases} 0 & \text{if } (s,\mathbf{y})\in\Theta,\\ 1 & \text{otherwise.}\end{cases}$$

and the measure $\boldsymbol{\zeta} = \lambda\boldsymbol{\vartheta}$. The representation formula (5.11) yields $\boldsymbol{\nu}^\perp = \mathfrak{e}_\sharp\boldsymbol{\vartheta}$, whereas (5.21) yields $\boldsymbol{\nu}^\perp_\flat = \mathfrak{e}_\sharp\boldsymbol{\zeta}$. Since $\boldsymbol{\zeta}\prec\boldsymbol{\vartheta}$ and $\boldsymbol{\vartheta}$ satisfies (A.2) (w.r.t. the measure $\alpha := \boldsymbol{\eta}^0_\flat$ and the map $\mathsf{p} := \mathfrak{e}$), we infer from Lemma A.1 ahead that $\boldsymbol{\nu}^\perp_\flat \prec \boldsymbol{\nu}^\perp$.

Conclusion of the proof: Since $\boldsymbol{\nu}$ is minimal, we deduce that $\boldsymbol{\nu}^\perp_\flat = \boldsymbol{\nu}^\perp$; since $\|\boldsymbol{v}\|\ge 1/k$ $\boldsymbol{\eta}^0_\flat$-a.e., we have $\lambda\equiv 1$ a.e. in Θ, i.e. $\boldsymbol{\eta}_\flat(\Theta) = 0$. This implies the for $\boldsymbol{\eta}_\flat$-a.e. $\mathbf{y}$ $\mathfrak{s}_{\rm l}(\mathbf{y}) = \mathfrak{s}_{\rm u}(\mathbf{y})$, a contradiction. □

The next result provides another property for any measure $\boldsymbol{\eta}\in\mathcal{P}(\mathrm{Lip}^\uparrow_1(\mathbf{I};\mathbb{R}^{d+1}_+))$ representing the solutions to the continuity equation, when the pair $(\mu,\boldsymbol{\nu})$ satisfies condition (3.7) (in which case, $\boldsymbol{\nu}$ is minimal, cf. Theorem 3.4). In this situation, we show that the measure $\boldsymbol{\eta}$ is concentrated on curves $\mathbf{y} = (\mathsf{t},\mathbf{x})\in\mathrm{Lip}^\uparrow_1(\mathbf{I};\mathbb{R}^{d+1})$ given by segments on intervals where $\mathsf{t}'\equiv 0$. The proof mimics the contradiction argument carried out for Theorem 5.3.

Theorem 5.4. *Let $(\mu,\boldsymbol{\nu})\in\mathcal{M}^+_{\rm loc}(\mathbb{R}^{d+1}_+)\times\mathcal{M}_{\rm loc}(\mathbb{R}^{d+1}_+;\mathbb{R}^d)$ be a minimal $\mathcal{P}_1$-solution of the continuity equation (3.1) with initial datum $\mu_0\in\mathcal{P}_1(\mathbb{R}^d)$, in the sense of Definition 3.2. Suppose that $(\mu,\boldsymbol{\nu})$ comply with (3.7).*

Let $\boldsymbol{\eta}\in\mathcal{P}(\mathrm{Lip}^\uparrow_1(\mathbf{I};\mathbb{R}^{d+1}_+))$ satisfy the representation formulae (5.2) and (5.8). Then, $\boldsymbol{\eta}$-almost every curve $\mathbf{y}\in\mathrm{Lip}^\uparrow_1(\mathbf{I};\mathbb{R}^{d+1}_+)$ enjoys the following property:

$$\begin{gathered}\text{if } s_1 < s_2\in[0,+\infty) \text{ are such that } \mathsf{t}'\equiv 0 \text{ in } (s_1,s_2), \text{ then}\\ \mathbf{x}(s) = \mathbf{x}(s_1) + (s-s_1)\frac{\mathbf{x}(s_2)-\mathbf{x}(s_1)}{\|\mathbf{x}(s_2)-\mathbf{x}(s_1)\|} \qquad \text{for all } s\in[s_1,s_2].\end{gathered} \tag{5.24}$$

Proof of Theorem 5.4. Assume by contradiction that the thesis is false. Then, there exist $S>0$ and $\Lambda\subseteq\mathrm{Lip}^\uparrow_1(\mathbf{I};\mathbb{R}^{d+1}_+)$ such that $\boldsymbol{\eta}(\Lambda)>0$ and for every $\mathbf{y} = (\mathsf{t},\mathbf{x})\in\Lambda$ there exist $s_1<s_2\in[0,S]$ such that $\mathsf{t}(s_1) = \mathsf{t}(s_2)$ and the the restriction of $\mathbf{x}$ to the interval $[s_1,s_2]$ is not of the form (5.24). In particular, since $\mathbf{y}$ is 1-Lipschitz continuous, we have that $\|\mathbf{x}(s_1)-\mathbf{x}(s_2)\| < s_2 - s_1$. Arguing similarly as in the proof of Theorem 5.3, for $\mathbf{y}\in\mathrm{Lip}^\uparrow_1(\mathbf{I};\mathbb{R}^{d+1}_+)$ and $s\in[0,S]$ we set

$$\rho(s,\mathbf{y}) := \sup\{\sigma\in[0,S] : \mathsf{t}(\sigma) = \mathsf{t}(s)\}, \qquad \rho_\flat(s,\mathbf{y}) := \rho(s,\mathbf{y}) - s .$$

Then, the maps $(s,\mathbf{y})\mapsto\rho(s,\mathbf{y})$ and $(s,\mathbf{y})\mapsto\rho_\flat(s,\mathbf{y})$ are upper semicontinuous.

Let us show that also the map

$$(s,\mathbf{y}) \mapsto \rho_\flat(s,\mathbf{y}) - \|\mathbf{x}(s) - \mathbf{x}(\rho(s,\mathbf{y}))\| \tag{5.25}$$

is upper semicontinuous. Let $s_n\to s$ and $\mathbf{y}_n\to\mathbf{y}$ uniformly on compact subsets of $[0,+\infty)$, and assume that $(s_n)_n,\ s\in[0,S]$. We need to show that

$$\limsup_{n\to\infty}\left(\rho_\flat(s_n,\mathbf{y}_n) - \|\mathbf{x}_n(s_n) - \mathbf{x}_n(\rho(s_n,\mathbf{y}_n))\|\right) \le \rho_\flat(s,\mathbf{y}) - \|\mathbf{x}(s) - \mathbf{x}(\rho(s,\mathbf{y}))\| . \tag{5.26}$$

Up to a not relabeled subsequence, we may assume that

$$\overline{s} = \lim_{n\to\infty} \rho(s_n, \mathsf{y}_n) = \limsup_{n\to\infty} \rho(s_n, \mathsf{y}_n) \le \rho(s, \mathsf{y}) ,$$
$$\limsup_{n\to\infty} \|\mathsf{x}_n(s_n) - \mathsf{x}_n(\rho(s_n, \mathsf{y}_n))\| = \lim_{n\to\infty} \|\mathsf{x}_n(s_n) - \mathsf{x}_n(\rho(s_n, \mathsf{y}_n))\| = \|\mathsf{x}(s) - \mathsf{x}(\overline{s})\| .$$

Hence, we deduce that

$$(\overline{s} - s) - \|\mathsf{x}(s) - \mathsf{x}(\overline{s})\| = \lim_{n\to\infty} \big(\rho_\flat(s_n, \mathsf{y}_n) - \|\mathsf{x}_n(s_n) - \mathsf{x}_n(\rho(s_n, \mathsf{y}_n))\|\big) . \tag{5.27}$$

If $\overline{s} = \rho(s,\mathsf{y})$, equality (5.27) proves the upper semicontinuity. If $\overline{s} < \rho(s,\mathsf{y})$, since $\mathsf{y} \in \mathrm{Lip}_1^\uparrow(\mathbf{I};\mathbb{R}_+^{d+1})$ and $\mathsf{t}' \equiv 0$ in $(\overline{s}, \rho(s,\mathsf{y}))$, we have that

$$\|\mathsf{x}(\overline{s}) - \mathsf{x}(\rho(s,\mathsf{y}))\| \le \rho(s,\mathsf{y}) - \overline{s}. \tag{5.28}$$

Therefore, we deduce from (5.27) and (5.28) that

$$\begin{aligned}
\rho_\flat(s,\mathsf{y}) - \|\mathsf{x}(s) - \mathsf{x}(\rho(s,\mathsf{y}))\| &\ge (\rho(s,\mathsf{y}) - \overline{s}) + (\overline{s} - s) - \|\mathsf{x}(s) - \mathsf{x}(\overline{s})\| - \|\mathsf{x}(\overline{s}) - \mathsf{x}(\rho(s,\mathsf{y}))\| \\
&\ge (\overline{s} - s) - \|\mathsf{x}(s) - \mathsf{x}(\overline{s})\| \\
&= \limsup_{n\to\infty} \big(\rho_\flat(s_n, \mathsf{y}_n) - \|\mathsf{x}_n(s_n) - \mathsf{x}_n(\rho(s_n, \mathsf{y}_n))\|\big),
\end{aligned}$$

whence (5.26).

Since $\boldsymbol{\eta}(\Lambda) > 0$, there exists $k \in \mathbb{N}$ such that the set

$$\Lambda' := \Big\{\mathsf{y} \in \mathrm{Lip}_1^\uparrow(\mathbf{I};\mathbb{R}_+^{d+1}) \ : \ \max_{s\in[0,S]} \big(\rho_\flat(s,\mathsf{y}) - \|\mathsf{x}(s) - \mathsf{x}(\rho(s,\mathsf{y}))\|\big) \ge \frac{1}{k}\Big\}$$

is Borel measurable and satisfies $\boldsymbol{\eta}(\Lambda') > 0$. We define the multifunction $\Delta\colon \Lambda' \to 2^{[0,S]}$ by $\Delta(\mathsf{y}) := \{s \in [0,S] \ : \ \rho_\flat(s,\mathsf{y}) - \|\mathsf{x}(s) - \mathsf{x}(\rho(s,\mathsf{y})\| \ge \frac{1}{k}\}$. Since the map (5.25) is upper semicontinuous, the multifunction Δ is upper semicontinuous. Thanks to [27, Theorem III.6] we find a Borel measurable selection of Δ, namely, a Borel function $\mathfrak{l}\colon \Lambda' \to [0,S]$ such that $\mathfrak{l}(\mathsf{y}) \in \Delta(\mathsf{y})$ for every $\mathsf{y} \in \Lambda'$. By construction, we have that

$$\rho_\flat(\mathfrak{l}(\mathsf{y}),\mathsf{y}) - \|\mathsf{x}(\mathfrak{l}(\mathsf{y})) - \mathsf{x}(\rho(\mathfrak{l}(\mathsf{y}),\mathsf{y}))\| \ge \frac{1}{k} , \tag{5.29}$$

and the function $\mathsf{y} \mapsto (\mathfrak{l}(\mathsf{y}), \rho(\mathfrak{l}(\mathsf{y}),\mathsf{y}))$ is Borel measurable. Moreover, for every $\mathsf{y} \in \Lambda'$ we have $\mathsf{t}(\mathfrak{l}(\mathsf{y})) = \mathsf{t}(\rho(\mathfrak{l}(\mathsf{y}),\mathsf{y}))$ and $\mathsf{t}'(s) \equiv 0$ for every s in the interval $(\mathfrak{l}(\mathsf{y}), \rho(\mathfrak{l}(\mathsf{y}),\mathsf{y}))$. Hence,

$$(\mathfrak{l}(\mathsf{y}), \rho(\mathfrak{l}(\mathsf{y}),\mathsf{y})) \subseteq D_c[\mathsf{y}] \subseteq D_0[\mathsf{y}] . \tag{5.30}$$

For $\mathsf{y} \in \Lambda'$ we further define

$$\mathfrak{r}(\mathsf{y}) := \mathfrak{l}(\mathsf{y}) + \|\mathsf{x}(\mathfrak{l}(\mathsf{y})) - \mathsf{x}(\rho(\mathfrak{l}(\mathsf{y}),\mathsf{y}))\|$$

and $\mathsf{x}_\mathsf{y}\colon [\mathfrak{l}(\mathsf{y}), \mathfrak{r}(\mathsf{y})] \to \mathbb{R}^d$ as the segment

$$\mathsf{x}_\mathsf{y}(s) := \mathsf{x}(\mathfrak{l}(\mathsf{y})) + (s - \mathfrak{l}(\mathsf{y}))\frac{\mathsf{x}(\rho(\mathfrak{l}(\mathsf{y}),\mathsf{y})) - \mathsf{x}(\mathfrak{l}(\mathsf{y}))}{\|\mathsf{x}(\rho(\mathfrak{l}(\mathsf{y}),\mathsf{y})) - \mathsf{x}(\mathfrak{l}(\mathsf{y}))\|} \qquad \text{for } s \in [\mathfrak{l}(\mathsf{y}), \mathfrak{r}(\mathsf{y})] . \tag{5.31}$$

We introduce the function $\mathcal{G}\colon \mathrm{Lip}_1^\uparrow(\mathbf{I};\mathbb{R}_+^{d+1}) \to \mathrm{Lip}_1^\uparrow(\mathbf{I};\mathbb{R}_+^{d+1})$ defined by $\mathcal{G}(\mathsf{y}) := \mathsf{y}$ for $\mathsf{y} \notin \Lambda'$ and, for $\mathsf{y} \in \Lambda'$, we set

$$\mathcal{G}(\mathsf{y})(s) := \begin{cases} \mathsf{y}(s) & \text{for } s < \mathfrak{l}(\mathsf{y}) , \\ \mathsf{z}(s) & \text{for } s \in [\mathfrak{l}(\mathsf{y}), \mathfrak{r}(\mathsf{y})] , \qquad \text{with } \mathsf{z}(s) = (\mathsf{t}(\mathfrak{l}(\mathsf{y}))), \mathsf{x}_\mathsf{y}(s)) \\ \mathsf{y}(s - \mathfrak{r}(\mathsf{y}) + \rho(\mathfrak{l}(\mathsf{y}),\mathsf{y})) & \text{for } s > \mathfrak{r}(\mathsf{y}) \end{cases}$$

for $\mathsf{y} \in \Lambda'$. By construction, the map $\mathcal{G}$ is Borel measurable, so that the push-forward $\boldsymbol{\eta}_\flat := \mathcal{G}_\sharp\boldsymbol{\eta}$ belongs to $\mathcal{P}(\mathrm{Lip}_1^\uparrow(\mathbf{I};\mathbb{R}_+^{d+1}))$.

After these preparations, we are in a position to carry out the proof by contradiction, based on the properties of the pair

$$\mu_\flat := \mathfrak{e}_\sharp(\mathfrak{t}'\mathcal{L}^1 \otimes \boldsymbol{\eta}_\flat), \qquad \boldsymbol{\nu}_\flat := \mathfrak{e}_\sharp(\mathfrak{x}'\mathcal{L}^1 \otimes \boldsymbol{\eta}_\flat)$$

stated in the following Claims 1, 2, and 3.

Claim 1: *we have that*

$$\mu = \mu_\flat \,. \tag{5.32}$$

This follows from the definition of $\mathcal{G}$ (in particular, the fact that $\mathcal{G}$ is the identity in $\mathrm{Lip}_1^{\uparrow}(\mathbf{I};\mathbb{R}_+^{d+1}) \setminus \Lambda'$ and modifies curves in Λ' only where $\mathsf{t}' \equiv 0$, cf. (5.30)), also taking into account the representation of μ provided by Proposition 5.2.

Claim 2: *the pair* $(\mu, \boldsymbol{\nu}_\flat)$ *solves the continuity equation* (3.1) *with initial datum* $\mu_0 \in \mathcal{P}_1(\mathbb{R}^d)$*, in the sense of Definition 3.1.* For this, it is enough to show that $\operatorname{div}(\boldsymbol{\nu} - \boldsymbol{\nu}_\flat) = 0$ in $\mathbb{R}_+^{d+1}$. Since $\mathcal{G}$ is the identity map in $\mathrm{Lip}_1^{\uparrow}(\mathbf{I};\mathbb{R}_+^{d+1}) \setminus \Lambda'$ and modifies a curve $\mathbf{y} \in \Lambda'$ only on $(\mathfrak{l}(\mathbf{y}), \rho(\mathfrak{l}(\mathbf{y}), \mathbf{y}))$, for every $\varphi \in \mathrm{C_c}(\mathbb{R}_+^{d+1})$ we have that

$$\begin{aligned}
\int_{\mathbf{I}}\int_{\mathbb{R}^d} \mathrm{D}\varphi(t,x)\,\mathrm{d}(\boldsymbol{\nu} - \boldsymbol{\nu}_\flat)(t,x) &= \int_{\Lambda'}\int_{\mathfrak{l}(\mathbf{y})}^{\rho(\mathfrak{l}(\mathbf{y}),\mathbf{y})} \mathrm{D}\varphi(\mathsf{t}(\mathfrak{l}(\mathbf{y})), \mathbf{x}(s)) \cdot \mathbf{x}'(s)\,\mathrm{d}s\,\mathrm{d}\boldsymbol{\eta}(\mathbf{y}) \\
&\quad - \int_{\Lambda'}\int_{\mathfrak{l}(\mathbf{y})}^{\mathfrak{r}(\mathbf{y})} \mathrm{D}\varphi(\mathsf{t}(\mathfrak{l}(\mathbf{y})), \mathbf{x}_{\mathbf{y}}(s)) \cdot \frac{\mathbf{x}(\rho(\mathfrak{l}(\mathbf{y}),\mathbf{y})) - \mathbf{x}(\mathfrak{l}(\mathbf{y}))}{\|\mathbf{x}(\rho(\mathfrak{l}(\mathbf{y}),\mathbf{y})) - \mathbf{x}(\mathfrak{l}(\mathbf{y}))\|}\,\mathrm{d}s\,\mathrm{d}\boldsymbol{\eta}(\mathbf{y}) \\
&\overset{(\star)}{=} \int_{\Lambda'} \big(\varphi(\mathsf{t}(\mathfrak{l}(\mathbf{y})), \mathbf{x}(\rho(\mathfrak{l}(\mathbf{y}),\mathbf{y}))) - \varphi(\mathsf{t}(\mathfrak{l}(\mathbf{y})), \mathbf{x}(\mathfrak{l}(\mathbf{y})))\big)\,\mathrm{d}\boldsymbol{\eta}(\mathbf{y}) \\
&\quad - \int_{\Lambda'} \big(\varphi(\mathsf{t}(\mathfrak{l}(\mathbf{y})), \mathbf{x}_{\mathbf{y}}(\mathfrak{r}(\mathbf{y}))) - \varphi(\mathsf{t}(\mathfrak{l}(\mathbf{y})), \mathbf{x}_{\mathbf{y}}(\mathfrak{l}(\mathbf{y})))\big)\,\mathrm{d}\boldsymbol{\eta}(\mathbf{y}) = 0
\end{aligned}$$

by definition of $\mathbf{x}_{\mathbf{y}}$ for $\mathbf{y} \in \Lambda'$, and with $(\star)$ due to the chain rule and to (5.30). Hence, $\operatorname{div}(\boldsymbol{\nu} - \boldsymbol{\nu}_\flat) = 0$ as desired.

Since $\boldsymbol{\eta}(\Lambda') > 0$ and $\Lambda' = \bigcup_{T>0}\{\mathbf{y} \in \Lambda' : \mathsf{t}(\mathfrak{l}(\mathbf{y})) < T\}$ with

$$\{\mathbf{y} \in \Lambda' : \mathsf{t}(\mathfrak{l}(\mathbf{y})) < T_1\} \subseteq \{\mathbf{y} \in \Lambda' : \mathsf{t}(\mathfrak{l}(\mathbf{y})) < T_2\} \qquad \text{if } T_1 \le T_2 \,,$$

we find $T \in [0, +\infty)$ such that

$$\boldsymbol{\eta}(\{\mathbf{y} \in \Lambda' : \mathsf{t}(\mathfrak{l}(\mathbf{y})) < T\}) > 0 \,. \tag{5.33}$$

Claim 3: *we have that*

$$|\boldsymbol{\nu}|([0,T) \times \mathbb{R}^d) > |\boldsymbol{\nu}_\flat|([0,T) \times \mathbb{R}^d) \,. \tag{5.34}$$

Indeed, recalling the representation $\boldsymbol{\nu} = \mathfrak{e}_\sharp(\mathfrak{r}'\boldsymbol{\eta}_{\mathcal{L}})$ and the fact that the curves $\mathbf{y} \in \operatorname{spt}(\boldsymbol{\eta})$ solve the Cauchy problem (4.22) with velocity field $(\tau, \boldsymbol{v})$ independent of s, we rewrite the left-hand side of (5.34) as

$$\begin{aligned}
|\boldsymbol{\nu}|([0,T) \times \mathbb{R}^d) &= \int_{\mathrm{Lip}_1^{\uparrow}} \int_{\{s \in \mathbf{I}\,:\,\mathsf{t}(s) < T\}} \|\boldsymbol{v}(\mathsf{t}(s), \mathbf{x}(s))\|\,\mathrm{d}s\,\mathrm{d}\boldsymbol{\eta}(\mathbf{y}) \\
&= \int_{\mathrm{Lip}_1^{\uparrow} \setminus \Lambda'} \int_{\{s \in \mathbf{I}\,:\,\mathsf{t}(s) < T\}} \|\boldsymbol{v}(\mathsf{t}(s), \mathbf{x}(s))\|\,\mathrm{d}s\,\mathrm{d}\boldsymbol{\eta}(\mathbf{y}) \\
&\quad + \int_{\Lambda'} \int_{\{s \in [0, \mathfrak{l}(\mathbf{y}))\,:\,\mathsf{t}(s) < T\}} \|\boldsymbol{v}(\mathsf{t}(s), \mathbf{x}(s))\|\,\mathrm{d}s\,\mathrm{d}\boldsymbol{\eta}(\mathbf{y}) \\
&\quad + \int_{\Lambda'} \int_{\{s \in [\mathfrak{l}(\mathbf{y}), \rho(\mathfrak{l}(\mathbf{y}),\mathbf{y})]\,:\,\mathsf{t}(s) < T\}} \|\boldsymbol{v}(\mathsf{t}(s), \mathbf{x}(s))\|\,\mathrm{d}s\,\mathrm{d}\boldsymbol{\eta}(\mathbf{y}) \\
&\quad + \int_{\Lambda'} \int_{\{s \in (\rho(\mathfrak{l}(\mathbf{y}),\mathbf{y}), +\infty)\,:\,\mathsf{t}(s) < T\}} \|\boldsymbol{v}(\mathsf{t}(s), \mathbf{x}(s))\|\,\mathrm{d}s\,\mathrm{d}\boldsymbol{\eta}(\mathbf{y}) \,.
\end{aligned} \tag{5.35}$$

Since the pair $(\tau, \boldsymbol{v})$ fulfills $\|(\tau, \boldsymbol{v})(t,x)\| \equiv 1$ $|(\mu, \boldsymbol{\nu})|$-almost everywhere in $\mathbb{R}_+^{d+1}$, $\boldsymbol{\eta}$-a.e. curve $\mathbf{y} \in \mathrm{Lip}_1^{\uparrow}(\mathbf{I};\mathbb{R}_+^{d+1})$ satisfies

$$\|\mathbf{y}'(s)\| = \|(\mathsf{t}'(s), \mathbf{x}'(s))\| = \|(\tau(\mathbf{y}(s)), \boldsymbol{v}(\mathbf{y}(s)))\| = 1 \qquad \text{for a.a. } s \in \mathbf{I}.$$

In particular, since $\mathsf{t}' \equiv 0$ in $(\mathfrak{l}(\mathbf{y}), \rho(\mathfrak{l}(\mathbf{y}), \mathbf{y}))$ for $\boldsymbol{\eta}$-a.a. $\mathbf{y} \in \Lambda'$, it must be $\|\boldsymbol{v}(\mathbf{y}(s))\| \equiv 1$ for a.a. $s \in (\mathfrak{l}(\mathbf{y}), \rho(\mathfrak{l}(\mathbf{y}), \mathbf{y}))$, for $\boldsymbol{\eta}$-a.e. $\mathbf{y} \in \Lambda'$. Moreover, if $\mathsf{t}(\mathfrak{l}(\mathbf{y})) < T$, then $\mathsf{t}(s) < T$ for every $s \in [\mathfrak{l}(\mathbf{y}), \rho(\mathfrak{l}(\mathbf{y}), \mathbf{y})]$.

Thus, by construction of $\mathcal{G}$ and by (5.33) we may continue in (5.35) with

$$
\begin{aligned}
|\boldsymbol{\nu}|([0,T)\times\mathbb{R}^d) \geq & \int_{\mathrm{Lip}_1^{\uparrow}\setminus\Lambda'} \int_{\{s\in\mathbf{I}\,:\,\mathsf{t}(s)<T\}} \|\boldsymbol{v}(\mathcal{G}(\mathbf{y})(s))\|\,\mathrm{d}s\,\mathrm{d}\boldsymbol{\eta}(\mathbf{y}) \\
& + \int_{\Lambda'} \int_{\{s\in[0,\mathfrak{l}(\mathbf{y}))\colon \mathsf{t}(s)<T\}} \|\boldsymbol{v}(\mathcal{G}(\mathbf{y})(s))\|\,\mathrm{d}s\,\mathrm{d}\boldsymbol{\eta}(\mathbf{y}) \\
& + \int_{\Lambda'} \int_{\mathfrak{l}(\mathbf{y})}^{\mathfrak{r}(\mathbf{y})} \|\mathbf{x}_{\mathbf{y}}'(s)\|\,\mathrm{d}s\,\mathrm{d}\boldsymbol{\eta}(\mathbf{y}) + \frac{\boldsymbol{\eta}(\{\mathbf{y}\in\Lambda'\,:\,\mathsf{t}(\mathfrak{l}(\mathbf{y}))<T\})}{k} \\
& + \int_{\Lambda'} \int_{\{s\in(\mathfrak{r}(\mathbf{y}),+\infty)\colon \mathsf{t}(s-\mathfrak{r}(\mathbf{y})+\rho(\mathfrak{l}(\mathbf{y}),\mathbf{y})))<T\}} \|\boldsymbol{v}(\mathcal{G}(\mathbf{y})(s))\|\,\mathrm{d}s\,\mathrm{d}\boldsymbol{\eta}(\mathbf{y})\,. \\
& \geq |\boldsymbol{\nu}_\flat|([0,T)\times\mathbb{R}^d) + \frac{\boldsymbol{\eta}(\{\mathbf{y}\in\Lambda'\,:\,\mathsf{t}(\mathfrak{l}(\mathbf{y}))\leq T\})}{k} > |\boldsymbol{\nu}_\flat|([0,T)\times\mathbb{R}^d)\,,
\end{aligned}
$$

where we have also used that $\rho(\mathfrak{l}(\mathbf{y}),\mathbf{y})-\mathfrak{r}(\mathbf{y})\geq\frac{1}{k}$ due to (5.29), as well as (5.33).
Conclusion of the proof: From the assumed (3.7) and (5.34) we further deduce that

$$
|\boldsymbol{\nu}_\flat|([0,T)\times\mathbb{R}^d) < \mathrm{Var}_{W_1}(\mu,[0,T]) \tag{5.36}
$$

for every $T\in\mathbf{I}$ such that (5.33) holds. Since $(\mu,\boldsymbol{\nu}_\flat)$ solves the continuity equation (3.1), with initial datum $\mu_0\in\mathcal{P}_1(\mathbb{R}^d)$, in the sense of Definition 3.1, (5.36) contradicts (3.8) of Theorem 3.4. Hence, the assertion with which we started the proof is false. This concludes the proof of the theorem. □

6. Superposition by BV curves

The superposition principle obtained in Theorem 5.1 offers a probabilistic representation of a solution $(\mu,\boldsymbol{\nu})$ to the continuity equation in terms of a measure $\boldsymbol{\eta}$ concentrated on curves $\mathbf{y}=\mathbf{y}(s)=(\mathsf{t}(s),\mathbf{x}(s))$ in the augmented space $\mathbb{R}^{d+1}=\mathbb{R}\times\mathbb{R}^d$. We may in fact think of them as 'parametrized' trajectories, and accordingly regard s as an 'artificial' time variable.

With the main result of this section, Theorem 6.5 ahead, we provide an *alternative* probabilistic representation involving a probability measure $\widehat{\boldsymbol{\eta}}$ on the space of curves with locally bounded variation, that are functions of the process time t.

The bridge between the superposition principles of Theorem 5.1 and Theorem 6.5 will be provided by a suitable class of curves that represent BV curves *augmented* by their transition at jumps. The next section revolves around them.

6.1. Augmented reparametrized BV curves.

Preliminaries. We denote by $\mathcal{Z}$ the set $\mathbf{I}\times[0,1]$ endowed with the *dictionary* order relation:

$$
(t_1,r_1),(t_2,r_2)\in\mathcal{Z}:\ (t_1,r_1)\lll(t_2,r_2)\ \Leftrightarrow\ t_1<t_2 \text{ or } (t_1=t_2 \text{ and } r_1<r_2)\,.
$$

We can define on $\mathcal{Z}$ the *order topology* (cf., e.g., [41, II.14]): a basis for such a topology is the collection of all the open intervals $(z_1,z_2):=\{z\in\mathcal{Z}:z_1\lll z\lll z_2\}$ and all the intervals of the form $[(0,0),z_2):=\{z\in\mathcal{Z}:z\lll z_2\}$. Notice that, while with this topology $\mathcal{Z}$ is neither separable nor metrizable, it satisfies the first axiom of countability.

We use the symbol $\rightharpoonup$ for the associated notion of convergence. Observe that, for any $t\in\mathbf{I}$ we have that

$$
\forall\,(r_n)_n\subset[0,1]: \qquad (t_n,r_n)\rightharpoonup\begin{cases}(t,0) & \text{if } t_n<t \text{ and } t_n\to t,\\ (t,1) & \text{if } t_n>t \text{ and } t_n\to t.\end{cases}
$$

In turn, for any $(t_n,r_n)_n,(t,r)\in\mathcal{Z}$,

$$
(t_n,r_n)\rightharpoonup(t,r) \implies \text{for } n \text{ big enough, we have } \begin{cases} t_n\equiv t & \text{if } r\in(0,1),\\ t_n\leq t & \text{if } r=0,\\ t_n\geq t & \text{if } r=1.\end{cases}
$$

Therefore, let $\mathsf{v}\colon \mathcal{Z} \to \mathbb{R}^d$ be a continuous curve. Necessarily,

$$\begin{aligned} &\forall\, t \in [0,+\infty) : && \text{the curve } [0,1] \ni r \mapsto \mathsf{v}(t,r) \text{ is continuous} \\ &\forall\, t \in [0,+\infty)\ \forall (r_n)_n \subset [0,1] : && \mathsf{v}(t_n,r_n) \to \begin{cases} \mathsf{v}(t,0) & \text{if } t_n < t \text{ and } t_n \to t, \\ \mathsf{v}(t,1) & \text{if } t_n > t \text{ and } t_n \to t. \end{cases} \end{aligned} \tag{6.1}$$

ABV **curves.** We are now in a position to introduce the class of curves on $\mathcal{Z}$ we shall employ to 'bridge' the Lipschitz continuous trajectories $\mathbf{y}$ to their BV counterpart.

Definition 6.1. We call *augmented* BV curve any $\mathsf{u}\colon \mathcal{Z} \to \mathbb{R}^d$ such that

(1) u is continuous in $\mathcal{Z}$;
(2) $[0,1] \ni r \mapsto \mathsf{u}(t,r)$ is Lipschitz continuous for all $t \in [0,+\infty)$;
(3) $[0,1] \ni r \mapsto \|\partial_r \mathsf{u}(t,r)\|$ is constant for all $t \in [0,+\infty)$;
(4) for all $T \in (0,+\infty)$ we have that

$$\sup_{\mathcal{P} \text{ partition of } [0,T]\times[0,1]} \sum_{(t_k,r_k)\in\mathcal{P}} \|\mathsf{u}(t_k,r_k) - \mathsf{u}(t_{k-1},r_{k-1})\| < +\infty\,,$$

where the partition $\mathcal{P}$ is constructed using the order relation $\lll$ in $\mathcal{Z}$.

We denote by $\mathsf{ABV}(\mathcal{Z};\mathbb{R}^d)$ the set of all such curves. For $t \in [0,+\infty)$, we further denote by $\ell_{\mathsf{u}}(t)$ the length of the curve $r \mapsto \mathsf{u}(t,r)$.

Loosely speaking, a curve in $\mathsf{ABV}(\mathcal{Z};\mathbb{R}^d)$ may be interpreted as an *augmented* version of a curve $u \in \mathrm{BV}_{\mathrm{loc}}(\mathbf{I};\mathbb{R}^d)$, to which we attach Lipschitz continuous transition curves at the jump points. We may indeed associate with any $u \in \mathrm{BV}_{\mathrm{loc}}(\mathbf{I};\mathbb{R}^d)$ (which is, in particular, *regulated*, with left and right limits $u^-(t)$ and $u^+(t)$ at each $t \in \mathbf{I}$), with *jump set* J_u, a curve $\mathsf{u} \in \mathsf{ABV}(\mathcal{Z};\mathbb{R}^d)$ by setting

$$\begin{aligned} &\text{if } t \in \mathbf{I}\backslash\mathrm{J}_u && \mathsf{u}(t,r) \equiv u(t) \text{ for all } r \in [0,1], \\ &\text{if } t \in \mathrm{J}_u && \mathsf{u}(t,r) := \begin{cases} u^-(t) & \text{if } r = 0 \\ \text{a transition curve with constant velocity joining } u^-(t) \text{ to } u^+(t) & \text{if } r \in (0,1) \\ u^+(t) & \text{if } r = 1 \end{cases}. \end{aligned}$$

Because of (6.1), the resulting u is indeed continuous on $\mathcal{Z}$, even when originating from a curve u with jumps. Notice that, at any $t \in \mathrm{J}_u$ the transition curve $r \mapsto \mathsf{u}(t,r)$ has constant speed on $[0,1]$, equal to its length $\ell_{\mathsf{u}}(t)$ (observe that this property may be obtained through *reparametrization*). Moreover, we remark that, for all $r_1, r_2 \in [0,1]$ the mappings $t \mapsto \mathsf{u}(t,r_1)$ and $t \mapsto \mathsf{u}(t,r_2)$ coincide a.e. in $\mathbf{I}$ (namely, outside J_u). Hence, the mappings $t \mapsto \mathsf{u}(t,r)$ share the same distributional derivative $\partial_t \mathsf{u}$, i.e. $\partial_t \mathsf{u}(t,r) = \partial_t \mathsf{u}(t,0)$ for all $r \in [0,1]$.

Conversely, we may consider the mapping

$$\mathfrak{V}\colon \mathsf{ABV}(\mathcal{Z};\mathbb{R}^d) \to \mathrm{BV}_{\mathrm{loc}}(\mathbf{I};\mathbb{R}^d), \quad \mathsf{u} \mapsto \mathfrak{v}_{\mathsf{u}} \quad \text{where } \mathfrak{v}_{\mathsf{u}}(t) := \mathsf{u}(t,0)\,. \tag{6.2}$$

Loosely speaking, $\mathfrak{v}_{\mathsf{u}}$ is the "BV skeleton" of u. We notice that $\mathfrak{v}_{\mathsf{u}}$ is left-continuous, i.e., at each $t \in (0,+\infty)$ its left limit $\mathfrak{v}_{\mathsf{u}}^-(t)$ coincides with $\mathfrak{v}_{\mathsf{u}}(t) = \mathsf{u}(t,0)$, while its right limit $\mathfrak{v}_{\mathsf{u}}^+(t)$ is $\mathsf{u}(t,1)$.

Augmented reparametrized BV **curves and Lipschitz trajectories.** For $\mathsf{u} \in \mathsf{ABV}(\mathcal{Z};\mathbb{R}^d)$, $t \in [0,+\infty)$, and $r \in [0,1]$ we define

$$\begin{aligned} L_{\mathsf{u}}^-(t) &:= \sup_{\mathcal{P} \text{ partition of } [0,t)\times[0,1]} \sum_{(t_k,r_k)\in\mathcal{P}} \|(t_k,\mathsf{u}(t_k,r_k)) - (t_{k-1},\mathsf{u}(t_{k-1},r_{k-1}))\|\,, \\ L_{\mathsf{u}}^+(t) &:= \sup_{\mathcal{P} \text{ partition of } [0,t]\times[0,1]} \sum_{(t_k,r_k)\in\mathcal{P}} \|(t_k,\mathsf{u}(t_k,r_k)) - (t_{k-1},\mathsf{u}(t_{k-1},r_{k-1}))\|\,, \\ L_{\mathsf{u}}(t,r) &:= L_{\mathsf{u}}^-(t) + r\ell_{\mathsf{u}}(t)\,, \end{aligned}$$

(recall that $\|\partial_r \mathsf{u}(t,r)\| \equiv \ell_{\mathsf{u}}(t)$ for all $r \in [0,1]$). For later use, we further introduce the set

$$\mathfrak{J}_{\mathsf{u}} := \{t \in \mathbf{I} : \|\partial_r \mathsf{u}(t,\cdot)\| \equiv \ell_{\mathsf{u}}(t) \neq 0\}. \tag{6.3}$$

With each curve $\mathsf{u} \in \mathsf{ABV}(\mathcal{Z};\mathbb{R}^d)$ we may associate a trajectory $\mathbf{y} \in \mathrm{Lip}_1^{\uparrow}(\mathbf{I};\mathbb{R}^{d+1})$ as follows. For $s \in \mathbf{I}$ we define

$$\mathsf{t}(s) := \inf\{t \in \mathbf{I} :\ L_{\mathsf{u}}^{+}(t) > s\}\,, \tag{6.4}$$

$$\mathsf{r}(s) := \begin{cases} \dfrac{s - L_{\mathsf{u}}^{-}(\mathsf{t}(s))}{L_{\mathsf{u}}^{+}(\mathsf{t}(s)) - L_{\mathsf{u}}^{-}(\mathsf{t}(s))} & \text{if } L_{\mathsf{u}}^{+}(\mathsf{t}(s)) \neq L^{-}(\mathsf{t}(s))\,, \\ 0 & \text{otherwise.} \end{cases} \tag{6.5}$$

In particular, notice that $L_{\mathsf{u}}^{-}(\mathsf{t}(s)) \le s \le L_{u}^{+}(\mathsf{t}(s))$ for every $s \in \mathbf{I}$. We define the curve $\mathbf{y}$ by

$$\mathbf{y}(s) := (\mathsf{t}(s), \mathsf{u}(\mathsf{t}(s), \mathsf{r}(s))), \qquad s \in \mathbf{I}. \tag{6.6}$$

Then, $\mathbf{y}$ is 1-Lipschitz continuous: for $s_1 < s_2$ it holds

$$\begin{aligned} \|\mathbf{y}(s_2) - \mathbf{y}(s_2)\| &= \big\|\big(\mathsf{t}(s_2), \mathsf{u}(\mathsf{t}(s_2), \mathsf{r}(s_2))\big) - \big(\mathsf{t}(s_1), \mathsf{u}(\mathsf{t}(s_1), \mathsf{r}(s_1))\big)\big\| \\ &\le L_{\mathsf{u}}(\mathsf{t}(s_2), \mathsf{r}(s_2)) - L_{\mathsf{u}}(\mathsf{t}(s_1), \mathsf{r}(s_1)) \le s_2 - s_1\,. \end{aligned}$$

Moreover, notice that $\|\mathbf{y}'(s)\| \equiv 1$ for a.e. $s \in \mathbf{I}$. We denote by $\mathscr{T} \colon \mathsf{ABV}(\mathcal{Z};\mathbb{R}^d) \to \mathrm{ArcLip}(\mathbf{I};\mathbb{R}^{d+1})$ the map that associates with any $\mathsf{u} \in \mathsf{ABV}(\mathcal{Z};\mathbb{R}^d)$ the curve $\mathbf{y} \in \mathrm{ArcLip}(\mathbf{I};\mathbb{R}^{d+1})$ from (6.6).

We also introduce a map $\mathscr{S} \colon \mathrm{Lip}_1^{\uparrow}(\mathbf{I};\mathbb{R}^{d+1}) \to \mathsf{ABV}(\mathcal{Z};\mathbb{R}^d)$ as follows. For $\mathbf{y} = (\mathsf{t}, \mathbf{x}) \in \mathrm{Lip}_1^{\uparrow}(\mathbf{I};\mathbb{R}^{d+1})$ we set

$$\mathsf{s}_{\mathbf{y}}^{-}(t) := \sup\{s \in \mathbf{I} :\ \mathsf{t}(s) < t\} \qquad \text{for every } t \in \mathbf{I}\,, \tag{6.7}$$

$$\mathsf{s}_{\mathbf{y}}^{+}(t) := \inf\{s \in \mathbf{I} :\ \mathsf{t}(s) > t\} \qquad \text{for every } t \in \mathbf{I}\,. \tag{6.8}$$

Then, $\mathsf{s}_{\mathbf{y}}^{\pm} \colon \mathbf{I} \to \mathbf{I}$ satisfy $\mathsf{t}(\mathsf{s}_{\mathbf{y}}^{\pm}(t)) = t$ for every $t \in \mathbf{I}$, and

$$\mathsf{s}_{\mathbf{y}}^{-}(\mathsf{t}(s)) \le s \le \mathsf{s}^{+}(\mathsf{t}(s)) \qquad \text{for every } s \in \mathbf{I}. \tag{6.9}$$

Moreover, if $\mathsf{t}'(s) > 0$ at some $s \in \mathbf{I}$, then $\mathsf{s}_{\mathbf{y}}^{\pm}(\mathsf{t}(s)) = s$. Indeed, if $\mathsf{s}_{\mathbf{y}}^{-}(\mathsf{t}(s)) < s$ or $\mathsf{s}^{+}(\mathsf{t}(s)) > s$, then we would have $\mathsf{t}(\sigma) = \mathsf{t}(s)$ for $\sigma \in (\mathsf{s}_{\mathbf{y}}^{-}(\mathsf{t}(s)), \mathsf{s}_{\mathbf{y}}^{+}(\mathsf{t}(s)))$, which would contradict the assumption $\mathsf{t}'(s) > 0$. Since $\mathbf{y}$ is 1-Lipschitz continuous and t is monotone non-decreasing with $\mathsf{t}(s) \to +\infty$ for $s \to +\infty$, we have that the functions $\mathsf{s}_{\mathbf{y}}^{\pm}$ are monotone non-decreasing, $\mathsf{s}_{\mathbf{y}}^{\pm}(t) \to +\infty$ as $t \to +\infty$. We define $\mathscr{S}(\mathbf{y}) \in \mathsf{ABV}(\mathcal{Z};\mathbb{R}^d)$ via

$$\mathscr{S}(\mathbf{y})(t,r) := \mathbf{x}\big(\mathsf{s}_{\mathbf{y}}^{-}(t) + r(\mathsf{s}_{\mathbf{y}}^{+}(t) - \mathsf{s}_{\mathbf{y}}^{-}(t))\big)\,. \tag{6.10}$$

We will prove in Appendix E the following.

Lemma 6.2. *For every* $\mathbf{y} \in \mathrm{ArcLip}(\mathbf{I};\mathbb{R}^{d+1})$ *and every* $\mathsf{u} \in \mathsf{ABV}(\mathcal{Z};\mathbb{R}^d)$ *we have*

$$\mathscr{T}(\mathscr{S}(\mathbf{y})) = \mathbf{y}\,, \qquad \mathscr{S}(\mathscr{T}(\mathsf{u})) = \mathsf{u}\,. \tag{6.11}$$

Recall that $\mathrm{Lip}_1^{\uparrow}(\mathbf{I};\mathbb{R}^{d+1})$ is endowed with the following distance, which metrizes the uniform convergence on compact subintervals of $\mathbf{I}$ (see Appendix B):

$$D(\mathbf{y}_1, \mathbf{y}_2) := \sum_{n=1}^{\infty} 2^{-n} \sup_{s \in [0,n]} \big(\min\{\|\mathbf{y}_1(s) - \mathbf{y}_2(s)\|\ ,\ 1\}\big) \qquad \text{for } \mathbf{y}_1, \mathbf{y}_2 \in \mathrm{Lip}_1^{\uparrow}(\mathbf{I};\mathbb{R}^{d+1})\,. \tag{6.12}$$

We further define the distance D_{ABV} on $\mathsf{ABV}(\mathcal{Z};\mathbb{R}^d)$ via

$$D_{\mathsf{ABV}}(\mathsf{u}_1, \mathsf{u}_2) := D(\mathscr{T}(\mathsf{u}_1), \mathscr{T}(\mathsf{u}_2))\,.$$

Then, the map $\mathscr{T} \colon \mathsf{ABV}(\mathcal{Z};\mathbb{R}^d) \to \mathrm{Lip}_1^{\uparrow}(\mathbf{I};\mathbb{R}^{d+1})$ is trivially continuous. Likewise, the restriction of $\mathscr{S}$ to $\mathrm{ArcLip}(\mathbf{I};\mathbb{R}^{d+1})$ is continuous with values in $\mathsf{ABV}(\mathcal{Z};\mathbb{R}^d)$.

6.2. **Bridging the probabilistic representations, from Lipschitz to ABV curves.** In order to obtain our superposition principle by ABV curves, we need to first revisit the probabilistic representation for solutions $(\mu,\boldsymbol{\nu})$ of the continuity equation guaranteed by Theorem 5.1. Recall (5.6a) and (5.8), (which in fact involved the minimal pair $(\mu,\bar{\boldsymbol{\nu}})$ associated with $(\mu,\boldsymbol{\nu})$), namely

$$\mu = \mathfrak{e}_\sharp\big(\mathfrak{t}'\,\boldsymbol{\eta}_{\mathcal{L}}\big), \qquad \bar{\boldsymbol{\nu}} = \mathfrak{e}_\sharp\big(\mathfrak{x}'\,\boldsymbol{\eta}_{\mathcal{L}}\big), \qquad |(\mu,\bar{\boldsymbol{\nu}})| = \mathfrak{e}_\sharp(\|\mathfrak{y}'\|\boldsymbol{\eta}_{\mathcal{L}}) = \mathfrak{e}_\sharp\boldsymbol{\eta}_{\mathcal{L}}\,, \tag{6.13}$$

(where $\mathfrak{e}\colon \mathbf{I}\times \mathrm{C}(\mathbf{I};\mathbb{R}^{d+1}) \to \mathbb{R}^{d+1}$ is the evaluation mapping, and the Borel maps $\mathfrak{t}'$ and $\mathfrak{x}'$ are defined on $\mathbf{I}\times\mathrm{Lip}(\mathbf{I};\mathbb{R}^{d+1})$ by $\mathfrak{t}'(s,\mathsf{y}) := \mathsf{t}'(s)$, $\mathfrak{x}'(s,\mathsf{y}) := \mathsf{x}'(s)$). The above representation brings into play a measure $\boldsymbol{\eta}\in\mathcal{P}(\mathrm{Lip}_1^{\uparrow}(\mathbf{I};\mathbb{R}^{d+1}))$ supported on curves $\mathsf{y}\in\mathrm{ArcLip}(\mathbf{I};\mathbb{R}^{d+1})$ solving the Cauchy problem (5.7).

Let us introduce a general construction that provides an equivalent formulation of (6.13). Firstly, observe that every $\mathsf{y}\in\mathrm{Lip}_1^{\uparrow}(\mathbf{I};\mathbb{R}^{d+1})$ induces the vector measure

$$\begin{aligned} &\omega_{\mathsf{y}} := \mathsf{y}_\sharp((\mathsf{t}',\mathsf{x}')\mathcal{L}^1) \in \mathcal{M}_{\mathrm{loc}}(\mathbb{R}^{d+1}_+;\mathbb{R}^{d+1}) \\ &\text{i.e.,}\quad \langle\omega_{\mathsf{y}},\varphi\rangle = \int_{\mathbf{I}} \varphi(\mathsf{t}(s),\mathsf{x}(s))\cdot(\mathsf{t}'(s),\mathsf{x}'(s))\mathrm{d}s \text{ for all } \varphi=(\varphi_0,\boldsymbol{\varphi})\in\mathrm{C_c}(\mathbb{R}^{d+1}_+;\mathbb{R}^{d+1})\,. \end{aligned} \tag{6.14}$$

Namely, ω_{y} is the integration measure along the curve y. Now, for a given probability measure $\boldsymbol{\lambda}\in\mathcal{P}(\mathrm{Lip}_1^{\uparrow}(\mathbf{I};\mathbb{R}^{d+1}))$ fulfilling the integrability conditions (5.1), we may consider the measure

$$\Omega^{\boldsymbol{\lambda}} := \int_{\mathrm{Lip}_1^{\uparrow}} \omega_{\mathsf{y}}\,\mathrm{d}\boldsymbol{\lambda}(\mathsf{y}) \in \mathcal{M}_{\mathrm{loc}}(\mathbb{R}^{d+1}_+;\mathbb{R}^{d+1})\,.$$

Let now $\boldsymbol{\lambda}$ be the measure $\boldsymbol{\eta}\in\mathcal{P}(\mathrm{Lip}_1^{\uparrow}(\mathbf{I};\mathbb{R}^{d+1}))$ supported on $\mathrm{ArcLip}(\mathbf{I};\mathbb{R}^{d+1})$ and providing the representation formulae (6.13). It follows from such representation that for $\boldsymbol{\eta}$-almost every $\mathsf{y}\in\mathrm{Lip}_1^{\uparrow}(\mathbf{I};\mathbb{R}^{d+1})$ we have $|\omega_{\mathsf{y}}| = \mathsf{y}_\sharp(\|\mathsf{y}'\|\mathcal{L}^1)$ and

$$|\Omega^{\boldsymbol{\eta}}| = \int_{\mathrm{Lip}_1^{\uparrow}} |\omega_{\mathsf{y}}|\,\mathrm{d}\boldsymbol{\eta}(\mathsf{y})\,.$$

Hence, (6.13) reformulates in compact form as

$$(\mu,\boldsymbol{\nu}) = \Omega^{\boldsymbol{\eta}}, \qquad |(\mu,\boldsymbol{\nu})| = |\Omega^{\boldsymbol{\eta}}| = \int_{\mathrm{Lip}_1^{\uparrow}} |\omega_{\mathsf{y}}|\,\mathrm{d}\boldsymbol{\eta}(\mathsf{y}). \tag{6.15}$$

This observation is at the core of the representation provided by Theorem 6.5 ahead.

Indeed, with any $\mathsf{u}\in\mathsf{ABV}(\mathcal{Z};\mathbb{R}^d)$ we associate the measure $\vartheta_{\mathsf{u}}\in\mathcal{M}_{\mathrm{loc}}(\mathbb{R}^{d+1}_+;\mathbb{R}^{d+1})$ defined by

$$\vartheta_{\mathsf{u}} := \omega_{\mathscr{T}(\mathsf{u})} \quad \text{i.e.,}\quad \int_{\mathbb{R}^{d+1}} \varphi(t,x)\,\mathrm{d}\vartheta_{\mathsf{u}}(t,x) = \int_{\mathbf{I}} \varphi(\mathsf{y}(s))\cdot\mathsf{y}'(s)\,\mathrm{d}s \text{ with } \mathsf{y}=\mathscr{T}(\mathsf{u}),$$

for $\varphi=(\varphi_0,\boldsymbol{\varphi})\in\mathrm{C_c}(\mathbb{R}^{d+1}_+;\mathbb{R}^{d+1})$. By construction, we have that for every $\varphi\in\mathrm{C_c}(\mathbb{R}^{d+1}_+;\mathbb{R}^{d+1})$ the map $\mathsf{u}\mapsto\int_{\mathbb{R}^{d+1}}\varphi\,\mathrm{d}\vartheta_{\mathsf{u}}$ is continuous with respect to the convergence in $\mathsf{ABV}(\mathcal{Z};\mathbb{R}^d)$. Hence, for $A\subseteq\mathbb{R}^{d+1}_+$ open we infer that $\mathsf{u}\mapsto\vartheta_{\mathsf{u}}(A)$ is Borel. The following result allows us to express the measure $\Omega^{\boldsymbol{\eta}}$ in terms of a probability measure $\widehat{\boldsymbol{\eta}}$ on curves in $\mathsf{ABV}(\mathcal{Z};\mathbb{R}^d)$.

Lemma 6.3. *For every $\boldsymbol{\lambda}\in\mathcal{P}(\mathrm{Lip}_1^{\uparrow}(\mathbf{I};\mathbb{R}^{d+1}))$ concentrated in $\mathrm{ArcLip}(\mathbf{I};\mathbb{R}^{d+1})$, let $\widehat{\boldsymbol{\lambda}} := \mathscr{S}_\sharp\boldsymbol{\lambda}$. Then,*

(1) $\widehat{\boldsymbol{\lambda}}$ is a Borel probability measure over $\mathsf{ABV}(\mathcal{Z};\mathbb{R}^d)$;

(2) if $\widehat{\boldsymbol{\lambda}}$ satisfies the integrability condition (5.1), then

$$\Theta^{\widehat{\boldsymbol{\lambda}}} := \int_{\mathsf{ABV}} \vartheta_{\mathsf{u}}\,\mathrm{d}\widehat{\boldsymbol{\lambda}}(\mathsf{u}) \textit{ is a measure in } \mathcal{M}_{\mathrm{loc}}(\mathbb{R}^{d+1}_+;\mathbb{R}^{d+1})\,. \tag{6.16}$$

(3) Let $\boldsymbol{\lambda}=\boldsymbol{\eta}$ fulfill (6.13). Then,

$$\Theta^{\widehat{\boldsymbol{\eta}}} = \Omega^{\boldsymbol{\eta}} = (\mu,\boldsymbol{\nu}) \quad \textit{and} \quad |\Theta^{\widehat{\boldsymbol{\eta}}}| = \int_{\mathsf{ABV}} |\vartheta_{\mathsf{u}}|\,\mathrm{d}\widehat{\boldsymbol{\eta}}(\mathsf{u}) = |(\mu,\boldsymbol{\nu})|\,. \tag{6.17}$$

Proof. Since the map $\mathscr{S}$ is continuous on $\mathrm{ArcLip}(\mathbf{I};\mathbb{R}^{d+1})$, we have that $\widehat{\boldsymbol{\lambda}} = \mathscr{S}_\sharp \boldsymbol{\lambda}$ is a Borel probability measure over $\mathsf{ABV}(\mathcal{Z};\mathbb{R}^d)$.

Let $\boldsymbol{\lambda}$ additionally satisfy (5.1). To show that $\Theta^{\widehat{\boldsymbol{\lambda}}} \in \mathcal{M}^+_{\mathrm{loc}}(\mathbb{R}^{d+1}_+;;\mathbb{R}^{d+1})$, it is enough to notice that for every $T \in [0,+\infty)$

$$\Theta^{\widehat{\boldsymbol{\lambda}}}([0,T]\times\mathbb{R}^d) = \int_{\mathrm{E}(T)} \|\mathbf{y}'(s)\|\,\mathrm{d}s\,\mathrm{d}\widehat{\boldsymbol{\lambda}}(\mathbf{y}) < +\infty$$

(where $\mathrm{E}(T)$ is from (4.13)).

When $\boldsymbol{\lambda} = \boldsymbol{\eta}$ fulfills (6.13), (6.17) obviously follows from (6.15). □

In the upcoming Proposition 6.4, we give an alternative formula for ϑ_{u}. It brings into play

- the mapping $\mathfrak{V}\colon \mathsf{ABV}(\mathcal{Z};\mathbb{R}^d) \to \mathrm{BV}_{\mathrm{loc}}(\mathbf{I};\mathbb{R}^d)$ from (6.2) that associates with each $\mathsf{u} \in \mathsf{ABV}(\mathcal{Z};\mathbb{R}^d)$ its BV 'skeleton' $\mathfrak{v}_{\mathsf{u}}(t) := \mathsf{u}(t,0)$,
- as well as the jump transitions $[0,1] \ni r \mapsto \mathsf{u}(t,r)$ at the discontinuity points t of $\mathfrak{v}_{\mathsf{u}}$.

To prepare the statement of Proposition 6.4, let us recall that the distributional derivative $(\mathfrak{v}_{\mathsf{u}})'_{\mathrm{d}}$ is a Radon vector measure on $\mathbf{I}$ that can be decomposed into the sum of three mutually singular measures

$$(\mathfrak{v}_{\mathsf{u}})'_{\mathrm{d}} = (\mathfrak{v}_{\mathsf{u}})'_{\mathscr{L}^1} + (\mathfrak{v}_{\mathsf{u}})'_{\mathrm{C}} + (\mathfrak{v}_{\mathsf{u}})'_{\mathrm{J}}.$$

Now,

- $(\mathfrak{v}_{\mathsf{u}})'_{\mathscr{L}^1}$ is the absolutely continuous part with respect to $\mathscr{L}^1$, is given by $(\mathfrak{v}_{\mathsf{u}})'_{\mathscr{L}^1} = \mathfrak{v}'_{\mathsf{u}}\mathscr{L}^1$ (with $\mathfrak{v}'_{\mathsf{u}}$ the a.e.-defined pointwise derivative of $\mathfrak{v}_{\mathsf{u}}$). In turn, since the a.e. defined derivative of $t \mapsto \partial_t \mathsf{u}(t,r)$ does not, in fact, depend on r, we have that $\mathfrak{v}'_{\mathsf{u}} = \partial_t\mathsf{u}(\cdot,0) = \partial_t\mathsf{u}(\cdot,r)$ $\mathscr{L}^1$-a.e. in $\mathbf{I}$ for every $r\in[0,1]$. Therefore, hereafter we will use the more evocative notation
$$\partial_t^{\mathrm{L}}\mathsf{u} \quad \text{in place of} \quad \mathfrak{v}'_{\mathsf{u}}. \tag{6.18}$$
- $(\mathfrak{v}_{\mathsf{u}})'_{\mathrm{C}}$ is the so-called Cantor part of $\mathfrak{v}_{\mathsf{u}}$, still satisfying $(\mathfrak{v}_{\mathsf{u}})'_{\mathrm{C}}(\{t\}) = 0$ for all $t \in [0,+\infty)$. In accordance with (6.18), we will write
$$\partial_t^{\mathrm{C}}\mathsf{u} \quad \text{in place of} \quad (\mathfrak{v}_{\mathsf{u}})'_{\mathrm{C}}. \tag{6.19}$$
- $(\mathfrak{v}_{\mathsf{u}})'_{\mathrm{J}}$ is a discrete measure concentrated on the (at most countable) jump set of $\mathfrak{v}_{\mathsf{u}}$, i.e.
$$\mathrm{J}_{\mathfrak{v}_{\mathsf{u}}} := \{t\in\mathbf{I}\,:\,\mathfrak{v}^+_{\mathsf{u}}(t) \neq \mathfrak{v}^-_{\mathsf{u}}(t)\} = \{t\in\mathbf{I}\,:\,\mathsf{u}(t,0)\neq\mathsf{u}(t,1)\} \subseteq \mathfrak{J}_{\mathsf{u}}.$$

Observe that $\partial_t^{\mathrm{L}}\mathsf{u}\mathscr{L}^1 + \partial_t^{\mathrm{C}}\mathsf{u}$ does not charge $\mathfrak{J}_{\mathsf{u}}$ (it is indeed known as the diffuse part of $(\mathfrak{v}_{\mathsf{u}})'_{\mathrm{d}}$). Clearly, it is concentrated on

$$C_{\mathfrak{v}_{\mathsf{u}}} := \mathbf{I}\setminus\mathrm{J}_{\mathfrak{v}_{\mathsf{u}}}. \tag{6.20}$$

We further denote by

$$C_{\mathsf{u}} := \mathbf{I}\setminus\mathfrak{J}_{\mathsf{u}} \tag{6.21}$$

and notice that $C_{\mathsf{u}} \subseteq C_{\mathfrak{v}_{\mathsf{u}}}$ and $|\partial_t^{\mathrm{L}}\mathsf{u}\mathscr{L}^1 + \partial_t^{\mathrm{C}}\mathsf{u}|(C_{\mathfrak{v}_{\mathsf{u}}}\setminus C_{\mathsf{u}}) = 0$. While $\partial_t^{\mathrm{L}}\mathsf{u}\mathscr{L}^1$ and $\partial_t^{\mathrm{C}}\mathsf{u}$ will be encompassed in the alternative representation of ϑ_{u}, the jump contribution will feature, in place of $(\mathfrak{v}_{\mathsf{u}})'_{\mathrm{J}}$, the measures

$$\begin{aligned} &\boldsymbol{j}_{t,\mathsf{u}} \in \mathcal{M}_{\mathrm{loc}}(\mathbb{R}^{d+1}_+;\mathbb{R}^{d+1}),\ t\in\mathfrak{J}_{\mathfrak{v}_{\mathsf{u}}},\ \text{given by} \\ &\langle \boldsymbol{j}_{t,\mathsf{u}},\boldsymbol{\varphi}\rangle := \int_0^1 \boldsymbol{\varphi}(t,\mathsf{u}(t,r))\cdot\partial_r\mathsf{u}(t,r)\,\mathrm{d}r \text{ for all } \boldsymbol{\varphi}\in\mathrm{C_c}(\mathbb{R}^{d+1}_+;\mathbb{R}^d). \end{aligned} \tag{6.22}$$

Finally, for fixed $\mathsf{u}\in\mathsf{ABV}(\mathcal{Z};\mathbb{R}^d)$, we introduce the map (where the gothic letter G stands for 'graph')

$$\mathfrak{G}_{\mathsf{u}}\colon \mathbf{I}\to\mathbb{R}^{d+1}_+, \quad \mathfrak{G}_{\mathsf{u}}(t) := (t,\mathfrak{v}_{\mathsf{u}}(t)) = (t,\mathsf{u}(t,0)),$$

and observe that it is Borel measurable, since may rewrite it as the composition $\gamma_2\circ\gamma_1$ of the following functions:

$$\begin{aligned} &\gamma_1\colon \mathbf{I}\to\mathbf{I}\times\mathbf{I} \qquad \gamma_1(t) := (t,\mathsf{s}^-_{\mathscr{T}(\mathsf{u})}(t)), \\ &\gamma_2\colon \mathbf{I}\times\mathbf{I}\to\mathbb{R}^{d+1}_+ \qquad \gamma_2(t,\sigma) = (t,\pi(\mathscr{T}(\mathsf{u})(\sigma))), \end{aligned}$$

where $\pi\colon\mathbb{R}^{d+1}_+\to\mathbb{R}^d$ is the projection $\pi(t,x) := x$. Then, γ_2 is continuous, while γ_1 is Borel measurable, as the second component is lower-semicontinuous. In fact, $\mathfrak{G}_{\mathsf{u}}$ will come into play in the representation formula (6.23) for ϑ_{u}, where ϑ_{u} results from the sum of three contributions involving the *absolutely continuous*, the *Cantor* and the *jump* parts of the distributional derivative $(\mathfrak{v}_{\mathsf{u}})'_{\mathrm{d}}$; the corresponding mappings $\mathscr{A}$, $\mathscr{C}$, and $\mathscr{J}$, cf. (6.25), will be shown to be Borel measurable.

Let us emphasize that, the 'jump contribution' to ϑ_{u} features the jump transitions $r \mapsto \mathsf{u}(t,r)$ at the transition times $t \in \mathfrak{J}_{\mathsf{u}}$, i.e., those times t such that $\ell_{\mathsf{u}}(t) = \|\partial_r \mathsf{u}(t,r)\| \neq 0$, by means of the measures $\boldsymbol{j}_{t,\mathsf{u}}$ of (6.22). We notice that we cannot, in principle, only consider the jump points $t \in \mathrm{J}_{\mathfrak{v}_{\mathsf{u}}}$ in the jump contribution $\boldsymbol{j}_{t,\mathsf{u}}$ of ϑ_{u}, as it may happen that $\mathsf{u}(t,0) = \mathsf{u}(t,1)$ but a transition occurs. Nevertheless, in the representation result of Theorem 6.5 there will come into play curves u such that $\mathfrak{J}_{\mathsf{u}}$ may be replaced by $\mathrm{J}_{\mathfrak{v}_{\mathsf{u}}}$.

Proposition 6.4. *For any* $\mathsf{u} \in \mathsf{ABV}(\mathcal{Z};\mathbb{R}^d)$, *the measure* ϑ_{u} *is given by*

$$\vartheta_{\mathsf{u}} := (\mathfrak{G}_{\mathsf{u}})_\sharp \left((1, \partial_t^{\mathrm{L}}\mathsf{u})\mathcal{L}^1 + (0, \partial_t^{\mathrm{C}}\mathsf{u})\right) + \sum_{t\in\mathfrak{J}_{\mathsf{u}}} \delta_t \otimes \boldsymbol{j}_{t,\mathsf{u}}\,. \tag{6.23}$$

The above measures can be expressed in terms of $\mathbf{y} = \mathscr{T}(\mathsf{u})$ *as*

$$\left\langle (\mathfrak{G}_{\mathsf{u}})_\sharp\left((1, \partial_t^{\mathrm{L}}\mathsf{u})\mathcal{L}^1 + (0, \partial_t^{\mathrm{C}}\mathsf{u})\right), \varphi\right\rangle = \int_{\mathscr{C}_{\mathbf{y}}} \varphi_0(\mathbf{y}(s))\,\mathsf{t}'(s)\,\mathrm{d}s + \int_{\mathscr{C}_{\mathbf{y}}} \boldsymbol{\varphi}(\mathbf{y}(s)) \cdot \mathbf{x}'(s)\,\mathrm{d}s\,, \tag{6.24a}$$

$$\left\langle \boldsymbol{j}_{t,\mathsf{u}}, \boldsymbol{\varphi}\right\rangle = \int_{\mathsf{s}_{\mathbf{y}}^-(t)}^{\mathsf{s}_{\mathbf{y}}^+(t)} \boldsymbol{\varphi}(\mathbf{y}(s)) \cdot \mathbf{x}'(s)\,\mathrm{d}s \tag{6.24b}$$

for all $\varphi = (\varphi_0, \boldsymbol{\varphi}) \in \mathrm{C_c}(\mathbb{R}^{d+1}_+;\mathbb{R}^{d+1})$, *where* $\mathscr{C}_{\mathbf{y}} := \mathsf{t}^{-1}(C_{\mathsf{u}})$ *and* C_{u} *is from* (6.21).

Finally, the mappings

$$\begin{cases} \mathscr{A}\colon \mathsf{ABV}(\mathcal{Z};\mathbb{R}^d) \to \mathcal{M}(\mathbf{I}\times\mathbb{R}^d;\mathbb{R}^{d+1}), & \mathsf{u} \mapsto (\mathfrak{G}_{\mathsf{u}})_\sharp(1, \partial_t^{\mathrm{L}}\mathsf{u})\mathcal{L}^1 \\ \mathscr{C}\colon \mathsf{ABV}(\mathcal{Z};\mathbb{R}^d) \to \mathcal{M}(\mathbf{I}\times\mathbb{R}^d;\mathbb{R}^{d+1}), & \mathsf{u} \mapsto (\mathfrak{G}_{\mathsf{u}})_\sharp(0, \partial_t^{\mathrm{C}}\mathsf{u}) \\ \mathscr{J}\colon \mathsf{ABV}(\mathcal{Z};\mathbb{R}^d) \to \mathcal{M}(\mathbf{I}\times\mathbb{R}^d;\mathbb{R}^{d+1}), & \mathsf{u} \mapsto \sum_{t\in\mathfrak{J}_{\mathsf{u}}} \delta_t \otimes \boldsymbol{j}_{t,\mathsf{u}} \end{cases} \quad \textit{are Borel.} \tag{6.25}$$

We postpone the *proof* to Appendix F.

6.3. **The** BV **probabilistic representation.** We are now in a position to state the 'BV version' to Theorem 5.1. In the same way as the latter result, Theorem 6.5 also provides information on the curves $\mathsf{u} \in \mathsf{ABV}(\mathcal{Z};\mathbb{R}^d)$ on which the representation measure $\widehat{\boldsymbol{\eta}}$ is concentrated. Recall that in Theorem 5.1 the superposition principle involved trajectories solving the characteristic system. Now, relations (6.27) ahead, which involve $\partial_t^{\mathrm{L}}\mathsf{u}$ (6.18), $\partial_t^{\mathrm{C}}\mathsf{u}$ (6.19), and $\partial_r\mathsf{u}(t,\cdot)$ at each $t \in \mathfrak{J}_{\mathsf{u}} = \mathrm{J}_{\mathfrak{v}_{\mathsf{u}}}$, may be interpreted as a counterpart to the Cauchy problem (4.22).

Without loss of generality, in what follows we will identify a given solution to the continuity equation with an induced minimal pair, and thus suppose minimality straight away.

Theorem 6.5. *Let* $(\mu, \boldsymbol{\nu}) \in \mathcal{M}^+_{\mathrm{loc}}(\mathbb{R}^{d+1}_+) \times \mathcal{M}_{\mathrm{loc}}(\mathbb{R}^{d+1}_+;\mathbb{R}^d)$ *be a minimal* $\mathcal{P}_1$*-solution to the continuity equation in the sense of Definition 3.2, with initial condition* $\mu_0 \in \mathcal{P}_1(\mathbb{R}^d)$. *Let* $(\tau, \boldsymbol{v})$ *be bounded Borel vector field representing the density of* $(\mu, \boldsymbol{\nu})$ *w.r.t.* $|(\mu, \boldsymbol{\nu})|$.

Then, there exists a Borel probability measure $\widehat{\boldsymbol{\eta}}$ *on* $\mathsf{ABV}(\mathcal{Z};\mathbb{R}^d)$ *that provides the probabilistic representation*

$$(\mu, \boldsymbol{\nu}) = \Theta^{\widehat{\boldsymbol{\eta}}}, \qquad |(\mu, \boldsymbol{\nu})| = |\Theta^{\widehat{\boldsymbol{\eta}}}|\,, \tag{6.26}$$

with $\Theta^{\widehat{\boldsymbol{\eta}}}$ *defined as in* (6.16). *The measure* $\widehat{\boldsymbol{\eta}}$ *is concentrated on curves* $\mathsf{u} \in \mathsf{ABV}(\mathcal{Z};\mathbb{R}^d)$ *fulfilling* $\mathsf{u}(0) \in \mathrm{spt}(\mu_0)$ *and*

$$\partial_t^{\mathrm{L}}\mathsf{u}(t) = \frac{\boldsymbol{v}(t,\mathsf{u}(t,r)))}{\tau(t,\mathsf{u}(t,r)))} = \frac{\mathrm{d}\boldsymbol{\nu}^a}{\mathrm{d}\mu}(t,\mathsf{u}(t,r)) \qquad \textit{for } \mathcal{L}^1\textit{-a.a. } t \in \mathbf{I} \textit{ and } \mathcal{L}^1\textit{-a.a. } r \in [0,1], \tag{6.27a}$$

$$\partial_t^{\mathrm{C}}\mathsf{u} = \boldsymbol{v}(t,\mathsf{u}(t,r))|\partial_t^{\mathrm{C}}\mathsf{u}| = \frac{\mathrm{d}\boldsymbol{\nu}^\perp}{\mathrm{d}|\boldsymbol{\nu}^\perp|}(t,\mathsf{u}(t,r))|\partial_t^{\mathrm{C}}\mathsf{u}| \quad \textit{for } |\partial_t^{\mathrm{C}}\mathsf{u}|\textit{-a.a. } t \in \mathbf{I} \textit{ and } \mathcal{L}^1\textit{-a.a. } r \in [0,1], \tag{6.27b}$$

$$\partial_r\mathsf{u}(t,r) = \frac{\mathrm{d}\boldsymbol{\nu}^\perp}{\mathrm{d}|\boldsymbol{\nu}^\perp|}(t,\mathsf{u}(t,r))\,\|\partial_r\mathsf{u}(t,r)\| \qquad \textit{for a.a. } r \in [0,1] \textit{ and all } t \in \mathrm{J}_{\mathfrak{v}_{\mathsf{u}}}\,, \tag{6.27c}$$

with $\boldsymbol{\nu} = \boldsymbol{\nu}^a + \boldsymbol{\nu}^\perp$ the Lebesgue decomposition of $\boldsymbol{\nu}$ into $\boldsymbol{\nu}^a \ll \mu$ and $\boldsymbol{\nu}^\perp \perp \mu$.

As a consequence of Theorem 6.5 and of the Borel measurability in (6.25) we can rewrite the representation formulae (6.26) and (6.27) as follows.

Corollary 6.6. *Under the assumptions of Theorem 6.5, the representation formulae (6.26) and (6.27) rephrase as*

$$\iint_{\mathbb{R}^{d+1}_+} \varphi_0(t,x)\,\mathrm{d}\mu(t,x) = \int_{\mathsf{ABV}} \int_{\mathbf{I}} \varphi_0(t,\mathfrak{v}_{\mathsf{u}}(t))\,\mathrm{d}t\,\mathrm{d}\widehat{\boldsymbol{\eta}}(\mathsf{u}) \quad \text{for all } \varphi_0 \in \mathrm{C_c}(\mathbb{R}^{d+1}_+), \tag{6.28a}$$

while for $\boldsymbol{\nu}^a$ and $\boldsymbol{\nu}^\perp$ we have

$$\begin{aligned}\iint_{\mathbb{R}^{d+1}_+} \boldsymbol{\varphi}(t,x)\,\mathrm{d}\boldsymbol{\nu}^a(t,x) &= \int_{\mathsf{ABV}} \int_{\mathbf{I}} \boldsymbol{\varphi}(t,\mathfrak{v}_{\mathsf{u}}(t)) \frac{\boldsymbol{v}(t,\mathfrak{v}_{\mathsf{u}}(t))}{\tau(t,\mathfrak{v}_{\mathsf{u}}(t)))}\,\mathrm{d}t\,\mathrm{d}\widehat{\boldsymbol{\eta}}(\mathsf{u}), \\ \iint_{\mathbb{R}^{d+1}_+} \boldsymbol{\varphi}(t,x)\,\mathrm{d}\boldsymbol{\nu}^\perp(t,x) &= \int_{\mathsf{ABV}} \int_{\mathbf{I}} \boldsymbol{\varphi}(t,\mathfrak{v}_{\mathsf{u}}(t)) \boldsymbol{v}(t,\mathfrak{v}_{\mathsf{u}}(t)))\,\mathrm{d}|\partial_t^C \mathsf{u}|(t)\,\mathrm{d}\widehat{\boldsymbol{\eta}}(\mathsf{u}) \\ &\quad + \int_{\mathsf{ABV}} \sum_{t\in \mathrm{J}_{\mathfrak{v}_{\mathsf{u}}}} \int_0^1 \boldsymbol{\varphi}(t,\mathsf{u}(t,r))\cdot \partial_r \mathsf{u}(t,r)\,\mathrm{d}r\,\mathrm{d}\widehat{\boldsymbol{\eta}}(\mathsf{u})\end{aligned} \tag{6.28b}$$

for every $\boldsymbol{\varphi} \in \mathrm{C_c}(\mathbb{R}^{d+1}_+;\mathbb{R}^d)$.

Finally, the left and right representatives $\mu_t^- = \mu_t$ and μ_t^+ of μ fulfill

$$\begin{cases} \displaystyle\int_{\mathbb{R}^d} \psi(x)\mathrm{d}\mu_t(x) = \int_{\mathbb{R}^d} \psi(x)\mathrm{d}\mu_t^-(x) = \int_{\mathsf{ABV}} \psi(\mathfrak{v}_{\mathsf{u}}(t))\,\mathrm{d}\widehat{\boldsymbol{\eta}}(\mathsf{u}) \\ \displaystyle\int_{\mathbb{R}^d} \psi(x)\mathrm{d}\mu_t^+(x) = \int_{\mathsf{ABV}} \psi(\mathfrak{v}_{\mathsf{u}}^+(t))\,\mathrm{d}\widehat{\boldsymbol{\eta}}(\mathsf{u}) \end{cases} \quad \text{for all } \psi \in \mathrm{C_c}(\mathbb{R}^d). \tag{6.29}$$

In the proof of Theorem 6.5 we will also resort to some measure-theoretic results in Appendix G.

Proof of Theorem 6.5. We divide the proof in three steps, proving (6.26)–(6.29) separately.
Step 1: proof of (6.26). Since $\boldsymbol{\nu}$ fulfills the minimality condition by Theorem 5.1 there exists $\boldsymbol{\eta} \in \mathcal{P}(\mathrm{Lip}_1^\uparrow(\mathbf{I};\mathbb{R}^{d+1}))$ concentrated on $\mathrm{ArcLip}(\mathbf{I};\mathbb{R}^{d+1})$ and such that the representation formulae (6.15) hold. In view of Lemma 6.3, we conclude that the Borel measure $\widehat{\boldsymbol{\eta}} := \mathscr{S}_\sharp \boldsymbol{\eta}$ fulfills (6.26).
Step 2: proof of (6.27). From (6.26) we gather in particular that $|\Theta^{\widehat{\boldsymbol{\eta}}}| = \int_{\mathsf{ABV}} |\vartheta_{\mathsf{u}}|\,\mathrm{d}\widehat{\boldsymbol{\eta}}(\mathsf{u})$. Therefore, we are in a position to apply Proposition G.1, thus concluding that

$$\vartheta_{\mathsf{u}} = \mathsf{f}|\vartheta_{\mathsf{u}}| \qquad \text{for } \widehat{\boldsymbol{\eta}}\text{-a.a. } \mathsf{u} \in \mathsf{ABV}(\mathcal{Z};\mathbb{R}^d) \qquad \text{with } \mathsf{f} = \frac{\mathrm{d}\Theta^{\widehat{\boldsymbol{\eta}}}}{\mathrm{d}|\Theta^{\widehat{\boldsymbol{\eta}}}|} = \frac{\mathrm{d}(\mu,\boldsymbol{\nu})}{\mathrm{d}|(\mu,\boldsymbol{\nu})|} = (\tau,\boldsymbol{v}).$$

Combining this with (6.23), we thus obtain

$$\mathscr{A}(\mathsf{u}) + \mathscr{C}(\mathsf{u}) + \mathscr{J}(\mathsf{u}) = \vartheta_{\mathsf{u}} = (\tau,\boldsymbol{v})|\vartheta_{\mathsf{u}}| \quad \text{with } \begin{cases} \mathscr{A}(\mathsf{u}) = (\mathfrak{G}_{\mathsf{u}})_\sharp(1,\partial_t^{\mathrm{L}}\mathsf{u})\mathcal{L}^1, \\ \mathscr{C}(\mathsf{u}) = (\mathfrak{G}_{\mathsf{u}})_\sharp(0,\partial_t^{\mathrm{C}}\mathsf{u}), \\ \mathscr{J}(\mathsf{u}) = \sum_{t\in\mathrm{J}_{\mathfrak{v}_{\mathsf{u}}}} \delta_t \otimes \boldsymbol{j}_{t,\mathsf{u}}. \end{cases} \tag{6.30}$$

Notice that in the last equality in (6.30) we have used that $\mathrm{J}_{\mathfrak{v}_{\mathsf{u}}} = \mathfrak{J}_{\mathsf{u}}$, which is a consequence of Theorem 5.3. Namely, a transition at time t may occur if and only if $\mathsf{u}(t,0) \neq \mathsf{u}(t,1)$, since $\boldsymbol{\eta}$ is supported on injective curves in $\mathrm{ArcLip}(\mathbf{I};\mathbb{R}^{d+1})$. We shall obtain (6.27) by restricting (6.30) to the support of each of the three mutually singular contributions $\mathscr{A}(\mathsf{u})$, $\mathscr{C}(\mathsf{u})$, $\mathscr{J}(\mathsf{u})$.

Indeed, restricting (6.30) to $\mathrm{spt}(\mathscr{A}(\mathsf{u}))$ we infer

$$\begin{aligned}(1,\partial_t^{\mathrm{L}}\mathsf{u}(t)) &= (\tau(t,\mathfrak{v}_{\mathsf{u}}(t)),\boldsymbol{v}(t,\mathfrak{v}_{\mathsf{u}}(t)))|(1,\partial_t^{\mathrm{L}}\mathsf{u}(t))| \\ &= (\tau(t,\mathsf{u}(t,r)),\boldsymbol{v}(t,\mathsf{u}(t,r)))|(1,\partial_t^{\mathrm{L}}\mathsf{u}(t))| \qquad \text{for } \mathcal{L}^1\text{-a.a. } t\in\mathbf{I} \text{ and every } r\in[0,1].\end{aligned}$$

Hence, for $\mathcal{L}^1$-a.a. $t \in \mathbf{I}$ and every $r \in [0,1]$ we have

$$\tau(t,\mathsf{u}(t,r)) = \frac{1}{|(1,\partial_t^{\mathrm{L}}\mathsf{u}(t))|}, \qquad \boldsymbol{v}(t,\mathsf{u}(t,r)) = \frac{\partial_t^{\mathrm{L}}\mathsf{u}(t)}{|(1,\partial_t^{\mathrm{L}}\mathsf{u}(t))|}.$$

Ultimately, we deduce (6.27a).

Analogously, restricting (6.30) to $\mathrm{spt}(\mathscr{C}(\mathsf{u}))$ we obtain

$$(0,\partial_t^{\mathrm{C}}\mathsf{u}) = (\tau(\cdot,\mathfrak{v}_{\mathsf{u}}(\cdot)),\boldsymbol{v}(t,\mathfrak{v}_{\mathsf{u}}(\cdot)))|(0,\partial_t^{\mathrm{C}}\mathsf{u})| \qquad |\partial_t^{\mathrm{C}}\mathsf{u}|\text{-a.e. in } \mathbf{I}\,.$$

Therefore, we obtain

$$\tau(t,\mathsf{u}(t,r)) \equiv 0, \qquad \boldsymbol{v}(t,\mathsf{u}(t,r)) = \frac{\mathrm{d}\partial_t^{\mathrm{C}}\mathsf{u}}{\mathrm{d}|\partial_t^{\mathrm{C}}\mathsf{u}|}(t) \qquad \text{for } |\partial_t^{\mathrm{C}}\mathsf{u}|\text{-a.a. } t\in\mathbf{I} \text{ and every } r\in[0,1].$$

Observing that $\mathrm{spt}(\partial_t^{\mathrm{C}}\mathsf{u})$ coincides with the image set $\mathsf{t}(D_0[\mathbf{y}]\backslash D_c[\mathbf{y}])$ and recalling the representation formula (5.11) for $\boldsymbol{\nu}^\perp$, we deduce that $\boldsymbol{v}(t,\mathsf{u}(t,r)) = \frac{\mathrm{d}\boldsymbol{\nu}^\perp}{\mathrm{d}|\boldsymbol{\nu}^\perp|}(t)$ for $|\partial_t^{\mathrm{C}}\mathsf{u}|$-a.a. $t\in\mathbf{I}$ and every $r\in[0,1]$, and (6.27b) ensues.

Finally, restricting (6.30) to $\mathrm{spt}(\mathscr{J}(\mathsf{u}))$ we deduce that for $t\in\mathrm{J}_{\mathfrak{v}_{\mathsf{u}}}$ and $r\in[0,1]$

$$(0,\partial_r\mathsf{u}(t,r)) = (\tau(t,\mathsf{u}(t,r)),\boldsymbol{v}(t,\mathsf{u}(t,r)))\|\partial_r\mathsf{u}(t,r)\| = (\tau(t,\mathsf{u}(t,r)),\boldsymbol{v}(t,\mathsf{u}(t,r)))\ell_{\mathsf{u}}(t)\,, \tag{6.31}$$

with $\ell_{\mathsf{u}}(t)$ the length of the curve connecting $\mathsf{u}(t,0)$ to $\mathsf{u}(t,1)$. Hence, at every $t\in\mathrm{J}_{\mathfrak{v}_{\mathsf{u}}}$ and $\mathcal{L}^1$-a.a. $r\in[0,1]$ there holds

$$\tau(t,\mathsf{u}(t,r)) \equiv 0, \qquad \boldsymbol{v}(t,\mathsf{u}(t,r)) = \frac{\partial_r\mathsf{u}(t,r)}{\|\partial_r\mathsf{u}(t,r)\|} = \frac{\mathrm{d}\boldsymbol{\nu}^\perp}{\mathrm{d}|\boldsymbol{\nu}^\perp|}(t,\mathsf{u}(t,r))\,,$$

whence (6.27c). □

We conclude with the proof of Corollary 6.6.

Proof of Corollary 6.6. The representation formulae in (6.28) are a consequence of Theorem 6.5. Let us plug in (6.28a) the test function $\varphi_\varepsilon(r,x) = \eta_\varepsilon(r)\psi(x)$, with $\psi\in\mathrm{C_c}(\mathbb{R}^d)$ and $\eta_\varepsilon\in\mathrm{C_c}(\mathbf{I})$, $0<\varepsilon\ll 1$, such that

$$\begin{cases} \mathrm{spt}(\eta_\varepsilon)\subset(t-2\varepsilon-\varepsilon^2,t-\varepsilon+\varepsilon^2), \\ \eta_\varepsilon(r)\equiv\frac{1}{\varepsilon}=\max_{[t-2\varepsilon-\varepsilon^2,t-\varepsilon+\varepsilon^2]}\eta_\varepsilon \quad \text{for all } r\in[t-2\varepsilon,t-\varepsilon], \end{cases} \qquad \text{for any fixed } t\in(0,+\infty)\,.$$

On the one hand, we have

$$\begin{aligned} &\iint_{\mathbb{R}^{d+1}_+}\varphi_\varepsilon(r,x)\,\mathrm{d}\mu_r(x)\,\mathrm{d}r \\ &\quad = \iint_{(t-2\varepsilon-\varepsilon^2,t-2\varepsilon)\times\mathbb{R}^d}\eta_\varepsilon(r)\psi(x)\,\mathrm{d}\mu_r(x)\,\mathrm{d}r + \frac{1}{\varepsilon}\iint_{(t-2\varepsilon,t-\varepsilon)\times\mathbb{R}^d}\psi(x)\,\mathrm{d}\mu_r(x)\,\mathrm{d}r \\ &\qquad + \iint_{[t-\varepsilon,t-\varepsilon+\varepsilon^2)\times\mathbb{R}^d}\eta_\varepsilon(r)\psi(x)\,\mathrm{d}\mu_r(x)\,\mathrm{d}r \doteq I_{1,\varepsilon}+I_{2,\varepsilon}+I_{3,\varepsilon}\,. \end{aligned}$$

We observe that

$$|I_{1,\varepsilon}| \le \frac{1}{\varepsilon}\|\psi\|_\infty\varepsilon^2 = \varepsilon\|\psi\|_\infty \longrightarrow 0 \text{ as } \varepsilon\downarrow 0,$$

and with analogous calculations we have $I_{3,\varepsilon}\to 0$, while

$$I_{2,\varepsilon} = \frac{1}{\varepsilon}\int_{t-2\varepsilon}^{t-\varepsilon}\int_{\mathbb{R}^d}\psi(x)\,\mathrm{d}\mu_r(x)\,\mathrm{d}r \longrightarrow \int_{\mathbb{R}^d}\psi(x)\,\mathrm{d}\mu_t^-(x) = \int_{\mathbb{R}^d}\psi(x)\,\mathrm{d}\mu_t(x)$$

where we have used the assumed left-continuity of $t \mapsto \mu_t$. On the other hand,

$$
\begin{aligned}
&\int_{\mathsf{ABV}} \int_0^{+\infty} \eta_\varepsilon(r)\psi(\mathfrak{v}_{\mathsf{u}}(r))\,\mathrm{d}r\,\mathrm{d}\widehat{\boldsymbol{\eta}}(\mathsf{u}) \\
&\qquad = \int_{\mathsf{ABV}} \int_{t-2\varepsilon-\varepsilon^2}^{t-2\varepsilon} \eta_\varepsilon(r)\psi(\mathfrak{v}_{\mathsf{u}}(r))\,\mathrm{d}r\,\mathrm{d}\widehat{\boldsymbol{\eta}}(\mathsf{u}) + \frac{1}{\varepsilon}\int_{\mathsf{ABV}} \int_{t-2\varepsilon}^{t-\varepsilon} \psi(\mathfrak{v}_{\mathsf{u}}(r))\,\mathrm{d}r\,\mathrm{d}\widehat{\boldsymbol{\eta}}(\mathsf{u}) \\
&\qquad\quad + \int_{\mathsf{ABV}} \int_{t-\varepsilon}^{t-\varepsilon+\varepsilon^2} \eta_\varepsilon(r)\psi(\mathfrak{v}_{\mathsf{u}}(r))\,\mathrm{d}r\,\mathrm{d}\widehat{\boldsymbol{\eta}}(\mathsf{u}) \\
&\qquad \doteq I_{4,\varepsilon} + I_{5,\varepsilon} + I_{6,\varepsilon}\,.
\end{aligned}
$$

Arguing in the same way as in the above lines we conclude that $I_{4,\varepsilon} \to 0$ and $I_{6,\varepsilon} \to 0$ as $\varepsilon \downarrow 0$ while (recall that $\mathfrak{v}_{\mathsf{u}}$ is assumed to be left-continuous)

$$
\lim_{\varepsilon\downarrow 0} \frac{1}{\varepsilon} \int_{\mathsf{ABV}} \int_{t-2\varepsilon}^{t-\varepsilon} \psi(\mathfrak{v}_{\mathsf{u}}(r))\,\mathrm{d}r\,\mathrm{d}\widehat{\boldsymbol{\eta}}(\mathsf{u}) = \int_{\mathsf{ABV}} \psi(\mathfrak{v}_{\mathsf{u}}(t))\,\mathrm{d}\widehat{\boldsymbol{\eta}}(\mathsf{u})\,.
$$

We thus have the first of (6.29). The very same argument yields the second of (6.29).

This concludes the proof. □

7. Examples

In this final section we illustrate our results, and discuss our assumptions, in the context of some examples. In what follows we will often work with time-dependent measures $(\ell_t^{p_0,p_1})_{t\in\mathbf{I}}$ on $\mathbb{R}^h$ given by the linear combination of two Dirac masses at $p_0,\, p_1 \in \mathbb{R}^h$, i.e.

$$
\ell_t^{p_0,p_1} := \max\{1-t,0\}\delta_{p_0} + \min\{t,1\}\delta_{p_1} = \begin{cases} t\delta_{p_1} + (1-t)\delta_{p_0} & t \in [0,1]\,, \\ \delta_{p_1} & t \in (1,+\infty)\,. \end{cases} \tag{7.1}
$$

Example 7.1. *We consider the curve $(\mu_t)_{t\in\mathbf{I}}$ of probability measures on $\mathbb{R}$*

$$
\mu_t := \ell_t^{x_0,x_1} \ \textit{for some } x_0 < x_1 \in \mathbb{R}\,,
$$

so that the corresponding measure on the time-space cylinder $\mathbf{I}\times\mathbb{R}$ is $\mu = \mathcal{L}^1 \otimes \mu_t$. Let $\boldsymbol{\nu}$ be the measure on $\mathbf{I}\times\mathbb{R}$, concentrated on $[0,1]\times[x_0,x_1]$, given by

$$
\boldsymbol{\nu} := (\mathcal{L}^1 \llcorner [0,1]) \otimes (\mathcal{L}^1 \llcorner [x_0,x_1])
$$

(although in this spatially one-dimensional case $\boldsymbol{\nu}$ is a scalar measure, we will stick to the vectorial notation used throughout the paper for better reference). For simplicity of notation, let us set $d_0 := x_1 - x_0 > 0$. The pair $(\mu,\boldsymbol{\nu})$ fulfills the continuity equation on $(0,+\infty)\times\mathbb{R}$, with initial datum $\mu_0 = \delta_{x_0}$, in the sense of Definition 3.1 (cf. Remark 3.6), since for every $\varphi \in \mathrm{C}^1_{\mathrm{c}}(\mathbf{I}\times\mathbb{R})$ there holds

$$
\begin{aligned}
\iint_{\mathbf{I}\times\mathbb{R}} \partial_t\varphi(t,x)\,\mathrm{d}\mu(t,x) &= \int_0^1 ((1-t)\partial_t\varphi(t,x_0) + t\partial_t\varphi(t,x_1))\,\mathrm{d}t + \int_1^{+\infty} \partial_t\varphi(t,x_1)\,\mathrm{d}t \\
&= -\int_0^1 (\varphi(t,x_1) - \varphi(t,x_0))\,\mathrm{d}t + [(1-t)\varphi(t,x_0) + t\varphi(t,x_1)]_0^1 - \varphi(1,x_1) \\
&= -\int_0^1 \int_{x_0}^{x_1} \partial_x\varphi(t,x)\,\mathrm{d}x\,\mathrm{d}t - \varphi(0,x_0) \\
&= -\iint_{\mathbf{I}\times\mathbb{R}} \partial_x\varphi(t,x)\,\mathrm{d}\boldsymbol{\nu}(t,x) - \int_{\mathbb{R}} \varphi(0,x)\,\mathrm{d}\delta_{x_0}(x)\,.
\end{aligned}
$$

In order to illustrate the superposition principle from Theorem 5.1, let us consider the fields $\tau\colon \mathbf{I}\times\mathbb{R} \to \mathbb{R}$, $\boldsymbol{v}\colon \mathbf{I}\times\mathbb{R} \to \mathbb{R}$ associated with the pair $(\mu,\boldsymbol{\nu})$ via (5.5). Since the measures μ and $\boldsymbol{\nu}$ are mutually singular, we have that

$$
|(\mu,\boldsymbol{\nu})| = \mathcal{L}^1|_{[0,1]} \otimes (\mu_t + \mathcal{L}^1|_{[x_0,x_1]}) + \mathcal{L}^1|_{(1,+\infty)} \otimes \mu_t\,,
$$

so that

$$\tau(t,x)=\frac{\mathrm{d}\mu}{\mathrm{d}|(\mu,\boldsymbol{\nu})|}(t,x)=\begin{cases}1 & \text{if } t\in[0,1) \text{ and } x\in\{x_0,x_1\}\,,\\ 1 & \text{if } t\in[1,+\infty) \text{ and } x=x_1\,,\\ 0 & \text{otherwise,}\end{cases}$$

$$\boldsymbol{v}(t,x)=\frac{\mathrm{d}\boldsymbol{\nu}}{\mathrm{d}|(\mu,\boldsymbol{\nu})|}(t,x)=\begin{cases}1 & \text{if } t\in[0,1) \text{ and } x\in[x_0,x_1]\,,\\ 0 & \text{otherwise,}\end{cases}$$

for $\mathcal{L}^1$-almost all $t\in(0,+\infty)$. The measure $\boldsymbol{\eta}$ involved in the representation (5.2) of $(\mu,\boldsymbol{\nu})$ is concentrated on the curves $\mathbf{y}$ solving the Cauchy system

$$\begin{cases}\dot{\mathbf{y}}(s)=(\tau(\mathbf{y}(s)),\boldsymbol{v}(\mathbf{y}(s))),\ \ s\in\mathbf{I},\\ \mathbf{y}(0)=(0,x),\ \ x\in\operatorname{spt}(\mu_0)=\{x_0\}\,.\end{cases}\tag{7.2}$$

Now, for every $\bar t\in[0,1]$ the curves $\mathbf{y}^{\bar t}\colon\mathbf{I}\to\mathbb{R}^2$ defined by

$$\mathbf{y}^{\bar t}(s):=(s,x_0)\chi_{[0,\bar t]}(s)+\left(\bar t,\frac{\bar t+d_0-s}{d_0}\,x_0+\frac{s-\bar t}{d_0}\,x_1\right)\chi_{[\bar t,\bar t+d_0]}(s)+(s-d_0,x_1)\chi_{[\bar t+d_0,+\infty)}(s)$$

solve (7.2). Loosely speaking, each curve $\mathbf{y}^{\bar t}$ can decide to move time till $\bar t$, then it 'fills in' the jump from x_0 to x_1, and then moves time again. Let us now consider the mapping $\Upsilon\colon[0,1]\to\mathrm{Lip}_1^{\uparrow}(\mathbf{I};\mathbb{R}^2)$ that with each $\bar t\in[0,1]$ associates the curve $\mathbf{y}^{\bar t}$, and let us consider the probability measure on $\mathrm{Lip}_1^{\uparrow}(\mathbf{I};\mathbb{R}^2)$ defined by

$$\boldsymbol{\eta}:=\Upsilon_\sharp(\mathcal{L}^1\llcorner[0,1]),\qquad \boldsymbol{\eta}_{\mathcal{L}}:=\mathcal{L}^1\otimes\boldsymbol{\eta}\,.\tag{7.3}$$

We will now check that $\boldsymbol{\eta}$ provides the probabilistic representation (5.2) of the pair $(\mu,\boldsymbol{\nu})$. Indeed, for every $\varphi_0\in\mathrm{C_b}(\mathbb{R}^2)$ there holds

$$\begin{aligned}\langle\mathfrak{e}_\sharp(\mathrm{t}'\boldsymbol{\eta}_{\mathcal{L}}),\varphi_0\rangle&=\int_{\mathrm{Lip}_1^{\uparrow}}\int_{\mathbf{I}}\varphi_0(\mathbf{y}(s))\mathrm{t}'(s)\,\mathrm{d}s\,\mathrm{d}\boldsymbol{\eta}(\mathbf{y})\\&\overset{(1)}{=}\int_0^1\int_{\mathbf{I}}\varphi_0(\mathbf{y}^{\bar t}(s))\tau(\mathbf{y}^{\bar t}(s))\,\mathrm{d}s\,\mathrm{d}\bar t\\&\overset{(2)}{=}\int_0^1\left(\int_0^{\bar t}\varphi_0(s,x_0)\,\mathrm{d}s+\int_{\bar t+d_0}^{+\infty}\varphi_0(s-d_0,x_1)\,\mathrm{d}s\right)\mathrm{d}\bar t\\&=\int_0^1\left(\int_0^{\bar t}\varphi_0(s,x_0)\,\mathrm{d}s+\int_{\bar t}^{+\infty}\varphi_0(s,x_1)\,\mathrm{d}s\right)\mathrm{d}\bar t\\&\overset{(3)}{=}\int_0^1\int_s^1\varphi_0(s,x_0)\,\mathrm{d}\bar t\,\mathrm{d}s+\int_0^{+\infty}\int_0^{\min\{s,1\}}\varphi_0(s,x_1)\,\mathrm{d}\bar t\,\mathrm{d}s\\&=\int_0^1(1-s)\varphi_0(s,x_0)\,\mathrm{d}s+\int_0^1 s\varphi_0(s,x_1)\,\mathrm{d}s+\int_1^{+\infty}\varphi_0(s,x_1)\,\mathrm{d}s\\&=\int_{\mathbf{I}}\int_{\mathbb{R}}\varphi_0(s,x)\,\mathrm{d}\mu_s(x)\,\mathrm{d}s,\end{aligned}$$

where (1) follows from (7.2), (2) from the fact that $\tau(t,\cdot)\equiv 0$ on $\mathbb{R}\setminus\{x_0,x_1\}$, and (3) from the Fubini theorem. Analogously, we easily check that, for $\boldsymbol{\eta}$ given by (7.3) there holds $\boldsymbol{\nu}=\mathfrak{e}_\sharp(\mathfrak{x}'\boldsymbol{\eta}_{\mathcal{L}})$.

In our next example the measures μ and $\boldsymbol{\nu}$ are again mutually singular and the minimality of $\boldsymbol{\nu}^\perp$ does not hold. In this case, the pair $(\mu,\boldsymbol{\nu})$ lacks a probabilistic representation.

Example 7.2. *Let $\mathbf{x}_0=(0,0)$, Λ be the unitary circle centered at $\mathbf{x}_0$ with tangent vector $\mathbf{t}_\Lambda$, and*

$$\begin{cases}\mu:=\mathcal{L}^1\otimes\mu_t \text{ with } \mu_t=\delta_{\mathbf{x}_0}\\ \boldsymbol{\nu}:=\delta_{t_0}\otimes\mathbf{t}_\Lambda\mathcal{H}^1\llcorner\Lambda\end{cases}$$

with any $t_0 \in \mathbf{I}$. *The pair* $(\mu, \boldsymbol{\nu})$ *solves the continuity equation with initial datum* $\mu_0 = \delta_{\mathbf{x}_0}$, *since for any* $\varphi \in \mathrm{C}^1_{\mathrm{c}}(\mathbf{I}\times\mathbb{R}^2)$ *there holds*

$$\begin{aligned}\iint_{\mathbf{I}\times\mathbb{R}^2} \partial_t\varphi(t,x)\,\mathrm{d}\mu(t,x) &= \int_{\mathbf{I}} \partial_t\varphi(t,\mathbf{x}_0)\mathrm{d}t = -\varphi(0,\mathbf{x}_0) \\ &= -\iint_{\mathbf{I}\times\mathbb{R}^2} \nabla\varphi(t,x)\cdot\mathbf{t}_\Lambda(t,x)\mathrm{d}(\mathcal{H}^1 \llcorner \Lambda)(x)\mathrm{d}t - \varphi(0,\mathbf{x}_0)\,.\end{aligned} \tag{7.4}$$

Since the measures μ *and* $\boldsymbol{\nu}$ *are mutually singular (and, in particular,* $\mathrm{spt}(\mu)\cap\mathrm{spt}(\boldsymbol{\nu}) = \emptyset$*), we have that* $|(\mu,\boldsymbol{\nu})| = \mu + |\boldsymbol{\nu}| = \mathcal{L}^1\otimes\delta_{\mathbf{x}_0} + \delta_{t_0}\otimes(\mathcal{H}^1\llcorner\Lambda)$ *and*

$$\begin{aligned}\tau(t,x) &= \frac{\mathrm{d}\mu}{\mathrm{d}|(\mu,\boldsymbol{\nu})|}(t,x) = \begin{cases}1 & \text{if } x = \mathbf{x}_0, \\ 0 & \text{otherwise}\end{cases} \quad \text{for } \mathcal{L}^1\text{-a.a. } t\in\mathbf{I}, \\ \boldsymbol{v}(t,x) &= \frac{\mathrm{d}\boldsymbol{\nu}}{\mathrm{d}|(\mu,\boldsymbol{\nu})|}(t,x) = \begin{cases}\mathbf{t}_\Lambda(x) & \text{if } x\in\Lambda,\ t = t_0, \\ 0 & \text{otherwise}\,.\end{cases}\end{aligned} \tag{7.5}$$

Now, any probabilistic representation of the pair $(\mu,\boldsymbol{\nu})$ *would involve a measure* $\boldsymbol{\eta}$ *supported on the solutions of the Cauchy problem*

$$\begin{cases}\dot{\mathbf{y}}(s) = (\tau(\mathbf{y}(s)), \boldsymbol{v}(\mathbf{y}(s))),\ s\in\mathbf{I}, \\ \mathbf{y}(0) = (0,x),\ x\in\mathrm{spt}(\mu_0) = \{\mathbf{x}_0\}\,.\end{cases} \tag{7.6}$$

However, it is immediate to check that the solution to (7.6) *is given by* $\mathbf{y}(s) = (s,\mathbf{x}_0)$ *for all* $s\in\mathbf{I}$ *so that, in particular,* $\boldsymbol{v}(\mathbf{y}(s))\equiv 0$ *for all* $s\in\mathbf{I}$ *and* $\mathbf{y}$ *never intersects* $\mathrm{spt}(\boldsymbol{\nu})$. *Hence, no probability measure supported on the solution trajectories of* (7.6) *could represent the measure* $\boldsymbol{\nu}$ *in the sense of Theorem* 5.1.

A modification of Example 7.2 provides a situation in which the measures $(\mu,\boldsymbol{\nu})$ are still mutually singular and the minimality of $\boldsymbol{\nu}^\perp$ does not hold, but it is still possible to provide a probabilistic representation. Hence, minimality is not a necessary condition for the validity of the representation from Theorem 5.1.

Example 7.3. *Let* $\mathbf{x}_1 = (1,0)$ *and*

$$\mu = \mathcal{L}^1\otimes\delta_{\mathbf{x}_1}, \qquad \boldsymbol{\nu} = \delta_{t_0}\otimes\mathbf{t}_\Lambda\mathcal{H}^1\llcorner\Lambda\,.$$

with the same notation as in Example 7.2. *The very same calculations as in* (7.4) *show that the pair* $(\mu,\boldsymbol{\nu})$ *solves the continuity equation with initial datum* $\mu_0 = \delta_{\mathbf{x}_1}$. *In this case,*

$$\tau(t,x) = \frac{\mathrm{d}\mu}{\mathrm{d}|(\mu,\boldsymbol{\nu})|}(t,x) = \begin{cases}1 & \text{if } x = \mathbf{x}_1, \\ 0 & \text{otherwise}\end{cases} \quad \text{for } \mathcal{L}^1\text{-a.e. } t\in\mathbf{I}\,, \tag{7.7}$$

and $\boldsymbol{v}$ *is as in* (7.5). *Let us now examine the Cauchy problem* (7.6). *Its solution is provided by the curve* $\mathbf{y}^{t_0}\colon\mathbf{I}\to\mathbb{R}^3$ *defined by*

$$\mathbf{y}^{t_0}(s) := \begin{cases}(s,\mathbf{x}_1) & \text{if } 0\le s< t_0\,, \\ (t_0, r(s-t_0)) & \text{if } t_0\le s\le t_0+2\pi\,, \\ (s-2\pi,\mathbf{x}_1) & \text{if } s > t_0+2\pi\,,\end{cases} \tag{7.8}$$

where $r\colon[0,2\pi]\to\mathbb{R}^2$ *is the arclength parametrization of* Λ, $r(\tau) := (\cos(\tau),\sin(\tau))$. *The measure*

$$\boldsymbol{\eta} := \delta_{\mathbf{y}^{t_0}} \in \mathcal{P}(\mathrm{Lip}_1^{\uparrow}(\mathbf{I};\mathbb{R}^3))$$

gives the probabilistic representation of the pair $(\mu,\boldsymbol{\nu})$*. Indeed, for every* $\varphi_0 \in \mathrm{C}(\mathbb{R}^3)$ *we have (with the notation* $\mathbf{y}^{t_0} = (\bar{\mathfrak{t}},\bar{\mathbf{x}})$*)*

$$\begin{aligned}\langle \mathfrak{e}_\sharp(\mathfrak{t}'\boldsymbol{\eta}_{\mathcal{L}}),\varphi_0\rangle &= \int_{\mathbf{I}} \varphi_0(\mathbf{y}^{t_0}(s))\bar{\mathfrak{t}}'(s)\,\mathrm{d}s \\ &\overset{(7.8)}{=} \int_0^{t_0} \varphi_0(s,\mathbf{x}_1)\mathrm{d}s + \int_{t_0+2\pi}^{+\infty} \varphi_0(s-2\pi,\mathbf{x}_1)\mathrm{d}s \\ &= \int_0^{t_0} \varphi_0(s,\mathbf{x}_1)\mathrm{d}s + \int_{t_0}^{+\infty} \varphi_0(s,\mathbf{x}_1)\mathrm{d}s = \langle \mu,\varphi_0\rangle\,,\end{aligned}$$

and for all $\boldsymbol{\varphi} \in \mathrm{C_c}(\mathbb{R}^3;\mathbb{R}^2)$ *there holds*

$$\begin{aligned}\langle \mathfrak{e}_\sharp(\mathfrak{x}'\boldsymbol{\eta}_{\mathcal{L}}),\boldsymbol{\varphi}\rangle &= \int_{\mathbf{I}} \boldsymbol{\varphi}(\mathbf{y}^{t_0}(s))\cdot\bar{\mathbf{x}}'(s)\,\mathrm{d}s \\ &\overset{(7.8)}{=} \int_{t_0}^{t_0+2\pi} \boldsymbol{\varphi}(t_0,r(s-t_0))\cdot r'(s-t_0)\mathrm{d}s = \int_0^{2\pi} \boldsymbol{\varphi}(t_0,r(s))\cdot r'(s)\mathrm{d}s = \langle \boldsymbol{\nu},\boldsymbol{\varphi}\rangle\,.\end{aligned}$$

Another instance of a pair $(\mu,\boldsymbol{\nu})$ *for which the minimality of* $\boldsymbol{\nu}^\perp$ *does not hold, but a probabilistic representation still exists, is offered by a variant of the above example in which* $\boldsymbol{\nu}$ *is diffuse in time, i.e.*

$$\mu = \mathcal{L}^1 \otimes \delta_{\mathbf{x}_1}, \qquad \boldsymbol{\nu} = \mathcal{L}^1 \otimes \mathbf{t}_\Lambda \mathcal{H}^1 \llcorner \Lambda\,.$$

In this case, τ *is still given by* (7.7) *while*

$$\boldsymbol{v}(t,x) = \begin{cases}\mathbf{t}_\Lambda(x) & \text{if } x \in \Lambda\setminus\{\mathbf{x}_1\}\\ 0 & \text{otherwise}\end{cases} \qquad \text{for } \mathcal{L}^1\text{-a.a. } t \in \mathbf{I}\,,$$

so that $\tau^2 + |\boldsymbol{v}|^2 \equiv 1$ $|(\mu,\boldsymbol{\nu})|$*-a.e. in* $\mathbf{I}\times\mathbb{R}^2$*. The solutions to the Cauchy problem* (7.6) *are provided by the family of curves* $\mathbf{y}^{\bar t}\colon \mathbf{I}\to\mathbb{R}^3$*,* $\bar t \in [0,1]$ *defined by* (7.8) *(with* $\bar t$ *in place of* t_0*). Let us consider the measure* $\boldsymbol{\eta} := \Upsilon_\sharp(\mathcal{L}^1\llcorner[0,1])$*, where in this case the mapping* $\Upsilon\colon [0,1]\to \mathrm{Lip}_1^\uparrow(\mathbf{I};\mathbb{R}^2)$ *associates with each* $\bar t\in[0,1]$ *the curve* $\mathbf{y}^{\bar t}$*. A straightforward adaptation of the above calculations show that* $\boldsymbol{\eta}$ *represents* $(\mu,\boldsymbol{\nu})$ *in the sense of* (5.2)*.*

The following example shows that the representation provided by Theorem. 5.1 is not in general stable for weak* convergence.

Example 7.4. *Recall the notation* $\mathbf{x}_0 = (0,0)$ *and* $\mathbf{x}_1 = (1,0)$ *from Examples 7.2 and 7.3. For every* $n \geq 1$ *consider the probability measures on* $(0,1)\times\mathbb{R}^2$

$$\mu^n = \mathcal{L}^1 \otimes \left(\left(1-\tfrac{1}{n}\right)\delta_{\mathbf{x}_0} + \tfrac{1}{n}\delta_{\mathbf{x}_1}\right)$$

and $\boldsymbol{\nu} = \delta_{t_0}\otimes\mathbf{t}_\Lambda\mathcal{H}^1\llcorner\Lambda$*. In this case as well, the minimality condition is not satisfied, but for each* $n\geq 1$ *the pair* $(\mu^n,\boldsymbol{\nu})$ *admits a probabilistic representation. Indeed, the associated fields* $(\tau_n,\boldsymbol{v}_n)$ *are*

$$\tau_n(t,x) = \begin{cases}1 & \text{if } x = \mathbf{x}_0,\mathbf{x}_1\,,\\ 0 & \text{otherwise}\end{cases} \qquad \mathcal{L}^1\text{-a.a. in } \mathbf{I}\,,$$

while $\boldsymbol{v}_n \equiv \boldsymbol{v}$ *with* $\boldsymbol{v}$ *as in* (7.5)*. The Cauchy problem* (7.6) *featuring the fields* $(\tau_n,\boldsymbol{v})$ *is solved by the curves*

$$\mathbf{y}_0(s) = (s,\mathbf{x}_0) \qquad \text{for all } s\in\mathbb{R}$$

and

$$\mathbf{y}_1^n(s) = \begin{cases}(s,\mathbf{x}_1) & \text{if } s < t_0\,,\\ (t_0,r(s-t_0)) & \text{if } s\in[t_0,t_0+2\pi n]\,,\\ (s-2\pi n,\mathbf{x}_1) & \text{if } s\in[t_0+2\pi n,+\infty]\,.\end{cases}$$

We then consider the probability measure

$$\boldsymbol{\eta}^n = \left(1-\frac{1}{n}\right)\delta_{\gamma_0} + \frac{1}{n}\delta_{\gamma_1^n}\,.$$

For every $\varphi_0 \in \mathrm{C_c}(\mathbf{I}\times\mathbb{R}^2)$ *we have that*

$$\begin{aligned}\langle \mathfrak{e}_\sharp(\mathfrak{t}'\boldsymbol{\eta}^n_{\mathcal{L}}), \varphi_0\rangle &= \frac{1}{n}\int_0^{t_0} \varphi_0(s,\mathbf{x}_1)\,\mathrm{d}s + \frac{1}{n}\int_{t_0+2\pi n}^{+\infty} \varphi_0(s-2\pi n,\mathbf{x}_1)\,\mathrm{d}s + \left(1-\frac{1}{n}\right)\int_0^{+\infty}\varphi_0(s,\mathbf{x}_0)\,\mathrm{d}s \\ &= \frac{1}{n}\int_0^{+\infty}\varphi_0(s,\mathbf{x}_1)\,\mathrm{d}s + \left(1-\frac{1}{n}\right)\int_0^{+\infty}\varphi_0(s,\mathbf{x}_0)\,\mathrm{d}s = \langle \mu^n, \varphi_0\rangle\,.\end{aligned}$$

In a similar way, for $\boldsymbol{\varphi} \in \mathrm{C_c}(\mathbf{I}\times\mathbb{R}^2;\mathbb{R}^2)$ *we have that*

$$\begin{aligned}\langle \mathfrak{e}_\sharp(\mathfrak{x}'\boldsymbol{\eta}^n_{\mathcal{L}}), \boldsymbol{\varphi}\rangle &= \frac{1}{n}\int_{t_0}^{t_0+2\pi n} \boldsymbol{\varphi}(t_0, r(s-t_0))\cdot \mathbf{t}_\Lambda(r(s-t_0))\,\mathrm{d}s \\ &= \int_0^{2\pi}\boldsymbol{\varphi}(t_0,r(s))\cdot\mathbf{t}_\Lambda(r(s))\,\mathrm{d}s = \langle \boldsymbol{\nu},\boldsymbol{\varphi}\rangle\ .\end{aligned}$$

Nonetheless, it turns out that, as $n\to\infty$, $\mu^n \rightharpoonup^* \mu^\infty = \mathcal{L}^1\otimes\delta_{\mathbf{x}_0}$ *and* $\boldsymbol{\eta}^n \rightharpoonup^* \boldsymbol{\eta}^\infty = \delta_{\mathbf{y}_0}$, *which represents* μ^∞ *but no longer provides a representation for* $\boldsymbol{\nu}^\infty = \boldsymbol{\nu}$.

Example 7.5. *Let* ϱ *be a regular and injective curve connecting* $\mathbf{x}_0$ *to* $\mathbf{x}_1$, $r_\varrho\colon[0,L_\varrho]\to\mathbb{R}^2$ *be its arclength parametrization and* $\mathbf{t}_\varrho$ *its tangent vector. Consider the measures*

$$\begin{aligned}&\mu = \mathcal{L}^1\otimes\mu_t \qquad \text{with } \mu_t = \ell_t^{\mathbf{x}_0,\mathbf{x}_1},\\ &\boldsymbol{\nu} = (\mathcal{L}^1 \llcorner [0,1])\otimes \mathbf{t}_\varrho \mathcal{H}^1\big|_\varrho\,.\end{aligned}$$

Here, $\boldsymbol{\nu}^\perp = \boldsymbol{\nu}$ *and the minimality condition is satisfied, cf. Example 2.4. In order to illustrate the probabilistic representation of the pair* $(\mu,\boldsymbol{\nu})$, *we consider the fields defined for* $\mathcal{L}^1$-a.e. $t\in\mathbf{I}$ *by*

$$\tau(t,x) = \frac{\mathrm{d}\mu}{\mathrm{d}|(\mu,\boldsymbol{\nu})|}(t,x) = \begin{cases}1 & \text{if } x\in\{\mathbf{x}_0,\mathbf{x}_1\},\\ 0 & \text{otherwise,}\end{cases} \qquad \boldsymbol{v}(t,x) = \frac{\mathrm{d}\boldsymbol{\nu}}{\mathrm{d}|(\mu,\boldsymbol{\nu})|}(t,x) = \begin{cases}\mathbf{t}_\varrho(x) & \text{if } x\in\varrho\,,\\ 0 & \text{otherwise.}\end{cases}$$

The family of curves $(\mathbf{y}_{\bar t})_{\bar t\in[0,1]}$ *defined by*

$$\mathbf{y}_{\bar t}(s) := \begin{cases}(s,\mathbf{x}_0) & \text{if } 0\le s<\bar t\,,\\ (\bar t, r_\varrho(s-\bar t)) & \text{if } \bar t\le s\le \bar t+L_\varrho\,,\\ (s-L_\varrho,\mathbf{x}_1) & \text{if } \bar t+L_\varrho < s<+\infty\end{cases} \tag{7.9}$$

provide the solutions to the Cauchy problem (7.6). Let $\Upsilon\colon[0,1]\to \mathrm{Lip}_1^\uparrow(\mathbf{I};\mathbb{R}^2)$ *associate with each* $\bar t\in[0,1]$ *the corresponding curve* $\mathbf{y}_{\bar t}$. *It can be easily checked that the measure* $\boldsymbol{\eta} = \Upsilon_\sharp(\mathcal{L}^1\llcorner[0,1])$ *fulfills* $\mu = \mathfrak{e}_\sharp(\mathfrak{t}'\boldsymbol{\eta}_{\mathcal{L}})$ *and* $\boldsymbol{\nu} = \mathfrak{e}_\sharp(\mathfrak{x}'\boldsymbol{\eta}_{\mathcal{L}})$.

In our last example we consider a solution pair $(\mu,\boldsymbol{\nu})$ such that $\boldsymbol{\nu}\ll\mu$. Therefore, in this absolutely continuous case [11, Theorem 8.2.1] applies. We show that representation from our Theorem 5.1 follows from that provided by [11, Theorem 8.2.1] via a reparametrization. Hence, Theorem 5.1 is consistent with the classical result.

Example 7.6. *Let us consider the scalar measures*

$$\mu = \mathcal{L}^1\otimes\left(\frac{1}{2}\ell_t^{0,1} + \frac{1}{2}\mathcal{L}^1\big|_{(0,1)}\right), \qquad \boldsymbol{\nu} = \mathcal{L}^1\otimes\frac{1}{2}\mathcal{L}^1\big|_{(0,1)}\,.$$

Then, $\boldsymbol{\nu}\ll\mu$ *and* $\boldsymbol{\nu} = \boldsymbol{w}\mu$ *with*

$$\boldsymbol{w}(t,x) := \begin{cases}1 & \text{for } t\in\mathbf{I} \text{ and } x\in(0,1),\\ 0 & \text{elsewhere.}\end{cases}$$

The representation of Theorem 5.1(2) follows, for instance, from [11, Theorem 8.2.1] *by an arc-length reparametrization, arguing as in (4.48). In particular, we may write* $\mu = \mathfrak{e}_\sharp(\mathfrak{t}'\boldsymbol{\eta}_{\mathcal{L}})$ *and* $\boldsymbol{\nu} = \mathfrak{e}_\sharp\boldsymbol{\eta}_{\mathcal{L}}$, *where the measure* $\boldsymbol{\eta}\in\mathcal{P}(\mathrm{Lip}_1^\uparrow(\mathbf{I};\mathbb{R}^2))$ *is supported on the set of curves* $\mathbf{y}\in\mathrm{Lip}_1^\uparrow(\mathbf{I};\mathbb{R}^2)$ *solving the Cauchy problem*

$$\begin{cases}\dot{\mathbf{y}}(s) = (\tau,\boldsymbol{v})(\mathbf{y}(s))\,,\\ \mathbf{y}(0) = (0,x_0),\ x_0\in[0,1]\,,\end{cases}$$

where

$$\tau(t,x)=\begin{cases}1 & \text{for } t\in[0,1) \text{ and } x\in\{0,1\},\\ \frac{1}{\sqrt{2}} & \text{for } t\in[0,+\infty) \text{ and } x\in(0,1),\\ 1 & \text{for } t\in[1,+\infty) \text{ and } x=1,\\ 0 & \text{elsewhere,}\end{cases} \tag{7.10}$$

$$\boldsymbol{v}(t,x)=\begin{cases}\frac{1}{\sqrt{2}} & \text{for } t\in[0,+\infty) \text{ and } x\in(0,1)\,,\\ 0 & \text{elsewhere.}\end{cases} \tag{7.11}$$

On the other hand, we may write $\mu=\frac{1}{2}\mu_1+\frac{1}{2}\mu_2$ *with*

$$\mu_1:=\mathcal{L}^1\otimes\ell_t^{0,1}\,,\qquad \mu_2:=\mathcal{L}^1\otimes(\mathcal{L}^1\llcorner(0,1))\,.$$

The pairs $(\mu_1,2\boldsymbol{\nu})$ *and* $(\mu_2,0)$ *solve the continuity equation in the sense of Definition 3.1 and both admit a representation in the form (5.2) satisfying the conditions of Theorem 5.1(2). Precisely, we take* $\boldsymbol{\eta}_1\in\mathcal{P}_1(\mathrm{Lip}_1^{\uparrow}(\mathbf{I};\mathbb{R}^2))$ *as in Example 7.1 (with the obvious modifications) and write* $\mu_1=\mathfrak{e}_\sharp(\mathfrak{t}'\mathcal{L}^1\otimes\boldsymbol{\eta}_1)$ *and* $\boldsymbol{\nu}=\mathfrak{e}_\sharp(\mathfrak{x}'\mathcal{L}^1\otimes\boldsymbol{\eta}_1)$. *As for* μ_2, *we consider the measure* $\boldsymbol{\eta}_2\in\mathcal{P}_1(\mathrm{Lip}_1^{\uparrow}(\mathbf{I};\mathbb{R}^2))$ *of the form* $\boldsymbol{\eta}_2=\Upsilon_\sharp(\mathcal{L}^1\llcorner[0,1])$ *where* $\Upsilon\colon(0,1)\to\mathcal{P}_1(\mathrm{Lip}_1^{\uparrow}(\mathbf{I};\mathbb{R}^2))$ *is defined as* $\Upsilon(x):=\mathbf{y}_x$ *with* $\mathbf{y}_x(s):=(s,x)$ *for every* $s\in\mathbf{I}$ *and every* $x\in(0,1)$. *Then, it is easy to see that* $\mu_2=\mathfrak{e}_\sharp(\mathfrak{t}'\mathcal{L}^1\otimes\boldsymbol{\eta}_2)$.

As a consequence, we obtain the alternative representation

$$\mu=\mathfrak{e}_\sharp\left(\mathfrak{t}'\mathcal{L}^1\otimes\left(\frac{1}{2}\boldsymbol{\eta}_1+\frac{1}{2}\boldsymbol{\eta}_2\right)\right),\qquad \boldsymbol{\nu}=\mathfrak{e}_\sharp\left(\mathfrak{x}'\mathcal{L}^1\otimes\left(\frac{1}{2}\boldsymbol{\eta}_1+\frac{1}{2}\boldsymbol{\eta}_2\right)\right).$$

However, we notice that this second representation does not fulfill the conditions of Theorem 5.1(2). Indeed, the curves contained in $\mathrm{spt}(\boldsymbol{\eta}_1)\cup\mathrm{spt}(\boldsymbol{\eta}_2)$ *do not solve* $\mathbf{y}'(s)=(\tau(\mathbf{y}(s)),\boldsymbol{v}(\mathbf{y}(s)))$ *with* $\tau,\boldsymbol{v}$ *as in (7.10)–(7.11). This shows that the superposition of two representations from Theorem 5.1 does not, in general, yield a representation in the sense Theorem 5.1.*

Appendix A. Push forward of vector measures

Let $\|\cdot\|$ be a strictly convex norm on $\mathbb{R}^h$ and let $\|\cdot\|_*$ denote its dual norm. The corresponding duality (multivalued) map $J\colon\mathbb{R}^h\twoheadrightarrow\mathbb{R}^h$ is defined by

$$J_1(\boldsymbol{x}):=\Big\{\boldsymbol{y}\in\mathbb{R}^h:\|\boldsymbol{y}\|_*\le 1,\ \boldsymbol{y}\cdot\boldsymbol{x}=\|\boldsymbol{x}\|\Big\}.$$

Notice that if $\|\cdot\|$ is strictly convex then for every $\boldsymbol{x}_1,\boldsymbol{x}_2\in\mathbb{R}^h$

$$\|\boldsymbol{x}_1\|=\|\boldsymbol{x}_2\|,\quad \boldsymbol{y}\in J(\boldsymbol{x}_1)\cap J(\boldsymbol{x}_2)\quad\Longrightarrow\quad \boldsymbol{x}_1=\boldsymbol{x}_2. \tag{A.1}$$

In fact, setting $\bar{\boldsymbol{x}}:=\frac{1}{2}\boldsymbol{x}_1+\frac{1}{2}\boldsymbol{x}_2$, we get

$$\|\bar{\boldsymbol{x}}\|\ge\boldsymbol{y}\cdot\bar{\boldsymbol{x}}=\frac{1}{2}\boldsymbol{y}\cdot\boldsymbol{x}_1+\frac{1}{2}\boldsymbol{y}\cdot\boldsymbol{x}_2=\frac{1}{2}\|\boldsymbol{x}_1\|+\frac{1}{2}\|\boldsymbol{x}_2\|$$

which implies $\boldsymbol{x}_1=\boldsymbol{x}_2$ by the strict convexity of the norm. In the following statement, X and Y are two locally compact topological spaces.

Lemma A.1. *Let* $\boldsymbol{\vartheta}\in\mathcal{M}_{\mathrm{loc}}(X;\mathbb{R}^h)$, *let* $\mathsf{p}\colon X\to Y$ *be a* $|\boldsymbol{\vartheta}|$*-proper map (i.e.* $|\boldsymbol{\vartheta}|(\mathsf{p}^{-1}(K))<+\infty$ *for every compact subset* $K\subset Y$*).*

(1) *If there exists* $\alpha\in\mathcal{M}^+_{\mathrm{loc}}(X)$ *and a Borel map* $\boldsymbol{f}\colon Y\to\mathbb{R}^h$ *such that*
$$\boldsymbol{\vartheta}=(\boldsymbol{f}\circ\mathsf{p})\,\alpha, \tag{A.2}$$
then
$$|\mathsf{p}_\sharp\boldsymbol{\vartheta}|=\mathsf{p}_\sharp|\boldsymbol{\vartheta}|. \tag{A.3}$$

(2) *If the norm* $\|\cdot\|$ *on* $\mathbb{R}^h$ *is strictly convex and (A.3) holds, then (A.2) holds with respect to* $\alpha:=|\boldsymbol{\vartheta}|$ *and* $\boldsymbol{f}$ *the density of the polar decomposition of* $\mathsf{p}_\sharp\boldsymbol{\vartheta}$, *i.e.*
$$\mathsf{p}_\sharp\boldsymbol{\vartheta}=\boldsymbol{f}|\mathsf{p}_\sharp\boldsymbol{\vartheta}|,\quad \boldsymbol{\vartheta}=(\boldsymbol{f}\circ\mathsf{p})\,|\boldsymbol{\vartheta}|. \tag{A.4}$$

(3) If (A.2) *holds and* $\boldsymbol{\zeta} \prec \boldsymbol{\vartheta}$ *then*

$$\mathsf{p}_\sharp \boldsymbol{\zeta} \prec \mathsf{p}_\sharp \boldsymbol{\vartheta}. \tag{A.5}$$

Proof. First of all, we observe that (A.2) implies a similar identity for $|\boldsymbol{\vartheta}|$ up to rescaling $\boldsymbol{f}$ by a suitable positive Borel function, therefore it is not restrictive to assume that $\alpha = |\boldsymbol{\vartheta}|$ and therefore $\|\boldsymbol{f}\| = 1$ $\mathsf{p}_\sharp|\boldsymbol{\vartheta}|$-a.e. in Y.

In order to prove **Claim 1**, notice that, by (A.2), we have that $\mathsf{p}_\sharp\boldsymbol{\vartheta} = \mathsf{p}_\sharp(\boldsymbol{f}\circ\mathsf{p})|\boldsymbol{\vartheta}| = \boldsymbol{f}\mathsf{p}_\sharp|\boldsymbol{\vartheta}|$. Since $\|\boldsymbol{f}\| = 1$ $\mathsf{p}_\sharp|\boldsymbol{\vartheta}|$-a.e. in Y, we immediately deduce that $|\mathsf{p}_\sharp\boldsymbol{\vartheta}| \le \mathsf{p}_\sharp|\boldsymbol{\vartheta}|$. On the other hand, let us select $\boldsymbol{\varphi} \in L^\infty_{\mathsf{p}_\sharp|\boldsymbol{\vartheta}|}(Y;\mathbb{R}^h)$ so that $\boldsymbol{\varphi}(y) \in J(\boldsymbol{f}(y))$ for $\mathsf{p}_\sharp|\boldsymbol{\vartheta}|$-a.a. $y \in Y$; for every K compact in Y we have

$$|\mathsf{p}_\sharp\boldsymbol{\vartheta}|(K) \ge \int_K \boldsymbol{\varphi}\cdot \mathrm{d}\mathsf{p}_\sharp\boldsymbol{\vartheta} = \int_K \boldsymbol{\varphi}\cdot\boldsymbol{f}\,\mathrm{d}\mathsf{p}_\sharp|\boldsymbol{\vartheta}| = \mathsf{p}_\sharp|\boldsymbol{\vartheta}|(K).$$

This implies that $\mathsf{p}_\sharp|\boldsymbol{\vartheta}| = |\mathsf{p}_\sharp\boldsymbol{\vartheta}|$.

In order to prove **Claim 2** (which is also well known, see e.g. [13, Lemma 2.4] for a similar statement) we observe that for $\boldsymbol{\varphi}$ as above it holds

$$\begin{aligned}|\mathsf{p}_\sharp\boldsymbol{\vartheta}|(K) &= \int_K \|\boldsymbol{f}\|\mathrm{d}|\mathsf{p}_\sharp\boldsymbol{\vartheta}| = \int_K \boldsymbol{\varphi}\cdot\boldsymbol{f}\mathrm{d}|\mathsf{p}_\sharp\boldsymbol{\vartheta}| = \int_K \boldsymbol{\varphi}\cdot\mathrm{d}\mathsf{p}_\sharp\boldsymbol{\vartheta} \\ &= \int_{\mathsf{p}^{-1}(K)} \boldsymbol{\varphi}\circ\mathsf{p}\cdot\mathrm{d}\boldsymbol{\vartheta} = \int_{\mathsf{p}^{-1}(K)} \boldsymbol{\varphi}\circ\mathsf{p}\cdot\boldsymbol{g}\mathrm{d}|\boldsymbol{\vartheta}| \le \int_{\mathsf{p}^{-1}(K)} \|\boldsymbol{\varphi}\circ\mathsf{p}\|_*\,\|\boldsymbol{g}\|\mathrm{d}|\boldsymbol{\vartheta}| \le \mathsf{p}_\sharp|\boldsymbol{\vartheta}|(K),\end{aligned} \tag{A.6}$$

where $\boldsymbol{\vartheta} = \boldsymbol{g}|\boldsymbol{\vartheta}|$. Then, (A.3) and (A.6) yield that $(\boldsymbol{\varphi}\circ\mathsf{p})\cdot\boldsymbol{g} = \|\boldsymbol{g}\|$ holds $|\boldsymbol{\vartheta}|$-a.e. on $\mathsf{p}^{-1}(K)$ so that $\boldsymbol{\varphi}(\mathsf{p}(x)) \in J(\boldsymbol{g}(x))$ for $|\boldsymbol{\vartheta}|$-a.a. $x \in \mathsf{p}^{-1}(K)$. On the other hand, by construction $\boldsymbol{\varphi}(\mathsf{p}(x)) \in J(\boldsymbol{f}(\mathsf{p}(x)))$ so that (A.1) yields $\boldsymbol{f}(\mathsf{p}(x)) = \boldsymbol{g}(x)$ for $|\boldsymbol{\vartheta}|$-a.e. $x \in \mathsf{p}^{-1}(K)$. Exhausting Y with a countable sequence of compact sets, we conclude.

Let us eventually consider **Claim 3**: we can write $\boldsymbol{\zeta} = \lambda\boldsymbol{\vartheta}$ for a Borel map λ with values in $[0,1]$. We can select the Euclidean norm and we thus have

$$|\boldsymbol{\zeta}| = \lambda\,|\boldsymbol{\vartheta}|, \quad \boldsymbol{\zeta} = \lambda\boldsymbol{\vartheta} = \lambda\boldsymbol{f}\circ\mathsf{p}\,|\boldsymbol{\vartheta}| = \boldsymbol{f}\circ\mathsf{p}\,|\boldsymbol{\zeta}|$$

and therefore, by Claim 1,

$$\mathsf{p}_\sharp|\boldsymbol{\zeta}| = |\mathsf{p}_\sharp\boldsymbol{\zeta}|.$$

Similarly $\mathsf{p}_\sharp|\boldsymbol{\vartheta}-\boldsymbol{\zeta}| = \mathsf{p}_\sharp|(1-\lambda)\boldsymbol{\vartheta}| = |\mathsf{p}_\sharp((1-\lambda)\boldsymbol{\vartheta})| = |\mathsf{p}_\sharp(\boldsymbol{\vartheta}-\boldsymbol{\zeta})|$. We deduce that

$$|\mathsf{p}_\sharp\boldsymbol{\zeta}| + |\mathsf{p}_\sharp(\boldsymbol{\vartheta}-\boldsymbol{\zeta})| = \mathsf{p}_\sharp\big(|\boldsymbol{\zeta}| + |\boldsymbol{\vartheta}-\boldsymbol{\zeta}|\big) = \mathsf{p}_\sharp|\boldsymbol{\vartheta}| = |\mathsf{p}_\sharp\boldsymbol{\vartheta}|$$

so that $\mathsf{p}_\sharp\boldsymbol{\zeta} \prec \mathsf{p}_\sharp\boldsymbol{\vartheta}$. □

Appendix B. Topological properties of functions spaces

Recall that $\mathrm{C}(\mathbf{I};\mathbb{R}^h)$ denotes the space of $\mathbb{R}^h$-valued continuous paths endowed with the topology of uniform convergence on compact sets of $\mathbf{I}$.

Lemma B.1. *The metric*

$$D(\mathbf{y}_1,\mathbf{y}_2) := \sum_{n=0}^{\infty} 2^{-n}(\|\mathbf{y}_1-\mathbf{y}_2\|_{\infty,n}\wedge 1), \qquad \textit{with } \|\mathbf{y}_1-\mathbf{y}_2\|_{\infty,n} = \max_{s\in[0,n]}\|\mathbf{y}_1(s)-\mathbf{y}_2(s)\|, \tag{B.1}$$

makes the topological space $\mathrm{C}(\mathbf{I};\mathbb{R}^h)$ *complete, separable, and induces on* $\mathrm{C}(\mathbf{I};\mathbb{R}^h)$ *the topology of uniform convergence on compact sets. In particular* $\mathrm{C}(\mathbf{I};\mathbb{R}^h)$ *is Polish.*

Proof. It is easy to check that $(\mathrm{C}(\mathbf{I};\mathbb{R}^{d+1}),D)$ is complete. It is also separable: indeed, for every $n\ge 1$ the space $\mathrm{C}([0,n];\mathbb{R}^{d+1})$ has a countable and dense subset $(\mathbf{y}_i)_{i\in I_n}$; we then extend each γ_i to a function $\tilde{\mathbf{y}}_i \in \mathrm{C}(\mathbf{I};\mathbb{R}^{d+1})$ by setting $\tilde{\mathbf{y}}_i(x) \equiv \mathbf{y}_i(n)$ for $x\in(n,+\infty)$. Then, the set $\bigcup_{n\ge1}(\mathbf{y}_i)_{i\in I_n}$ is countable and dense in $(\mathrm{C}(\mathbf{I};\mathbb{R}^{d+1}),D)$. □

Appendix C. Glueing properties

We establish a useful generalization of the glueing Lemma [11, Lemma 5.3.2, 5.3.4].

Lemma C.1. *Let* $N \in \mathbb{N} \cup \{\infty\}$, $\mathcal{I}(N) := \{1, 2, \cdots, N\}$ *if* $N \in \mathbb{N}$ *and* $\mathcal{I}(\infty) := \mathbb{N}$ *(*$N = \infty$*), let* X, X^i, Y^j, *be Polish spaces and let* $\mathsf{p}^i \colon X \to X_i$, $\mathsf{R}^i \colon X^i \to Y^i$, $\mathsf{L}^{j+1} \colon X^{j+1} \to Y^j$, *be Borel maps for* $i \in \mathcal{I}(N)$ *and* $j \in \mathcal{I}(N-1)$. *We set* $\boldsymbol{X} := \Pi_{i\in I} X^i$, $\boldsymbol{X}_0 := \big\{\boldsymbol{x} = (x_i)_{i\in I} \in \boldsymbol{X} : \mathsf{R}^i(x_i) = \mathsf{L}^{i+1}(x_{i+1}),\ i \in \mathcal{I}(N-1)\big\}$, *and we suppose that the image of the map* $\mathbf{p} : X \to \boldsymbol{X}$, $\mathbf{p}(x) := (\mathsf{p}^i(x))_{i\in\mathbb{N}}$ *contains* $\boldsymbol{X}_0$.

If $\mu^i \in \mathcal{P}(X^i)$, $i \in \mathcal{I}(N)$, *satisfy the compatibility conditions*

$$\mathsf{R}^i_\sharp \mu^i = \mathsf{L}^{i+1}_\sharp \mu^{i+1}, \quad i \in \mathcal{I}(N-1), \tag{C.1}$$

then there exists $\mu \in \mathcal{P}(X)$ *such that* $\mathsf{p}^i_\sharp \mu = \mu^i$ *for every* $i \in \mathcal{I}(N)$.

Proof. We consider the case $N = \infty$, $\mathcal{I}(N) = \mathcal{I}(N-1) = \mathbb{N}$; the argument in the finite case is even simpler.

Let d_i be a metric inducing the topology of Y^i taking values in $[0,1]$. Let us set $\nu^i := \mathsf{R}^i_\sharp \mu^i = \mathsf{L}^{i+1}_\sharp \mu^{i+1} \in \mathcal{P}(Y^i)$, $\hat\mu^{i\to} := (\mathsf{i}_{X^i}, \mathsf{R}^i)_\sharp \mu^i \in \Gamma(\mu^i, \nu) \subset \mathcal{P}(X^i \times Y)$, $\hat\mu^{i\leftarrow} := (\mathsf{i}_{X^{i+1}}, \mathsf{L}^{i+1})_\sharp \mu^{i+1} \in \Gamma(\mu^{i+1}, \nu) \subset \mathcal{P}(X^{i+1} \times Y)$. Notice that

$$\int_{X^i\times Y} d_i(\mathsf{R}^i(x_i), y)\,\mathrm{d}\hat\mu^{i\to}(x_i, y) = 0, \quad \int_{X^{i+1}\times Y} d_i(\mathsf{L}^{i+1}(x_{i+1}), y)\,\mathrm{d}\hat\mu^{i\leftarrow}(x_{i+1}, y) = 0, \quad i \in \mathbb{N}. \tag{C.2}$$

By the standard glueing Lemma (see e.g. [11, Lemma 5.3.2]) there exist $\beta^i \in \mathcal{P}(X^i \times X^{i+1} \times Y^i)$ such that

$$\pi^{i\to}_\sharp \beta^i = \hat\mu^{i\to}, \quad \pi^{i\leftarrow}_\sharp \beta^i = \hat\mu^{i\leftarrow}, \quad \text{where} \quad \pi^{i\to}(x_i, x_{i+1}, y) := (x_i, y), \quad \pi^{i\leftarrow}(x_i, x_{i+1}, y) := (x_{i+1}, y).$$

In particular, using (C.2) we deduce

$$\begin{aligned}\int d_i(\mathsf{R}^i(x_i), \mathsf{L}^{i+1}(x_{i+1}))\,\mathrm{d}\beta^i &\le \int d_i(\mathsf{R}^i(x_i), y)\,\mathrm{d}\beta^i + \int d_i(y, \mathsf{L}^{i+1}(x_{i+1}))\,\mathrm{d}\beta^i \\ &= \int d_i(\mathsf{R}^i(x_i), y)\,\mathrm{d}\hat\mu^{i\to} + \int d_i(\mathsf{L}^{i+1}(x_{i+1}), y)\,\mathrm{d}\hat\mu^{i\leftarrow} = 0,\end{aligned}$$

so that $\alpha^i := \pi^i_\sharp \gamma$ (where $\pi^i(x_i, x_{i+1}, y) = (x_i, x_{i+1})$) is concentrated on $\{(x_i, x_{i+1}) \in X^i \times X^{i+1} : \mathsf{R}^i(x_i) = \mathsf{L}^{i+1}(x_{i+1})\}$.

We can then use the glueing Lemma [11, 5.3.4] to find a probability measure $\boldsymbol{\alpha} \in \mathcal{P}(\boldsymbol{X})$ such that $\boldsymbol{\pi}^i_\sharp \boldsymbol{\alpha} = \alpha^i$ for every $i \in \mathbb{N}$, where $\boldsymbol{\pi}^i(\boldsymbol{x}) = (x_i, x_{i+1})$. Clearly, $\boldsymbol{\alpha}$ is concentrated on $\boldsymbol{X}_0$; since the image of $\mathbf{p}$ contains $\boldsymbol{X}_0$, we can find $\mu \in \mathcal{P}(X)$ such that $\mathbf{p}_\sharp \mu = \boldsymbol{\alpha}$, so that $\mathsf{p}^i_\sharp \mu = \mu^i$. ☐

Appendix D. A measurability result

In this section we prove the measurability of the mapping $\mathfrak{R}$ from (4.29), coming into play in the proof of Lemma 4.6. In the proof we shall resort to the following functional version of the *monotone class theorem*, which we record here (in a shortened and simplified version, adapted to our usage) for the reader's convenience. We refer to [17, Thm. 2.12.9] for the general statement.

Theorem D.1 (Functional monotone class theorem)**.** *Let* $\boldsymbol{H}$ *be a class of real functions on a set* $O \subset \mathbb{R}^k$ *containing* $f \equiv 1$. *Suppose that* $\boldsymbol{H}$ *is closed with respect to the formation of uniform and monotone limits and that* $f \equiv 1 \in \boldsymbol{H}$. *Let* $\boldsymbol{H}_0 \subset \boldsymbol{H}$ *be a subclass closed with respect to multiplication (i.e.,* $fg \in \boldsymbol{H}_0$ *for every* $f, g \in \boldsymbol{H}_0$*).*

Then, $\boldsymbol{H}$ *contains all bounded functions measurable with respect to the* σ*-algebra generated by* $\boldsymbol{H}_0$.

We will then prove the following result.

Lemma D.2. *The mapping* $\mathfrak{R}\colon \mathrm{Lip}^{\uparrow}_k(\mathbf{I}; \mathbb{R}^{d+1}) \to \mathrm{Lip}^{\uparrow}_{\hat k}(\mathbf{I}; \mathbb{R}^{d+1})$ *from* (4.29) *is Borel.*

Proof. Let θ be a Borel measurable function $\theta\colon \mathbb{R}^{d+1} \to (0,+\infty)$ such that $c^{-1} \le \theta \le c$ for some $c \in [1,+\infty)$ (cf. (4.24)). For every function $\zeta\colon \mathbb{R}^{d+1} \to (0,+\infty)$ such that $\zeta \ge c_\zeta > 0$ we consider the mapping

$$\mathsf{F}_\zeta\colon \mathrm{Lip}_k^\uparrow(\mathbf{I};\mathbb{R}^{d+1}) \to L^1_{\mathrm{loc}}(\mathbf{I}), \qquad \mathbf{y} \mapsto \frac{1}{\zeta(\mathbf{y})}\,.$$

We consider $L^1_{\mathrm{loc}}(\mathbf{I})$ endowed with the (Fréchet, hence metrizable) topology that induces the L^1 convergence on the compact subsets of $\mathbf{I}$, whereas we recall that $\mathrm{Lip}_k^\uparrow(\mathbf{I};\mathbb{R}^{d+1})$ is with the (metrizable) topology of the convergence on compact subsets. We claim that for every ζ as above we have

$$\mathsf{F}_\zeta \text{ is Borel.} \tag{D.1}$$

To show this, we introduce the class $\boldsymbol{H}$ as

$$\zeta \in \boldsymbol{H} \quad \Longleftrightarrow \quad \begin{cases} \zeta\colon \mathbb{R}^{d+1} \to (0,+\infty) \text{ is Borel,} \\ \exists\, c_\zeta > 0 \ \ \zeta \ge c_\zeta \text{ in } \mathbb{R}^{d+1}, \\ \mathsf{F}_\zeta\colon \mathrm{Lip}_k^\uparrow(\mathbf{I};\mathbb{R}^{d+1}) \to L^1_{\mathrm{loc}}(\mathbf{I}) \text{ is Borel.} \end{cases}$$

Now, we clearly have that $\zeta \equiv 1$ belongs to $\boldsymbol{H}$ and that $\boldsymbol{H}$ is closed w.r.t. monotone limits of uniformly bounded sequences. Moreover, it is immediate to check that $\boldsymbol{H}$ contains the set

$$\boldsymbol{H}_0 := \{\zeta \in \mathrm{C}(\mathbb{R}^{d+1}) : \exists\, c > 0 \ \ \zeta \ge c \text{ in } \mathbb{R}^{d+1}\}\,.$$

Furthermore, $\boldsymbol{H}_0$ is closed with respect to multiplication. Hence, by the monotone class theorem the family $\boldsymbol{H}$ contains all positive Borel functions bounded away from 0. In particular, $\theta \in \boldsymbol{H}$ and (D.1) follows.

We now consider the mapping

$$\mathsf{A}_\theta\colon \mathrm{Lip}_k^\uparrow(\mathbf{I};\mathbb{R}^{d+1}) \to \mathrm{C}(\mathbf{I}), \qquad \mathsf{A}_\theta(\mathbf{y})(t) := \Theta_{\mathbf{y}}(t) = \int_0^t \frac{1}{\theta(\mathbf{y}(r))}\mathrm{d}r = \int_0^t \mathsf{F}_\theta(\mathbf{y}(r))\mathrm{d}r\,.$$

In particular, we notice that for every $\mathbf{y} \in \mathrm{Lip}_k^\uparrow(\mathbf{I};\mathbb{R}^{d+1})$ we have

$$\mathsf{A}_\theta(\mathbf{y}) \in \mathrm{biLip}_{c,c^{-1}}(\mathbf{I}) := \left\{ g : \mathbf{I} \to [0,+\infty) \ : \ g(0) = 0, \ g \text{ is bi-Lipschitz, with } \ \frac{1}{c} \le g' \le c \text{ in } \mathbf{I} \right\},$$

where $c \ge 1$ the constant from (4.24). Recalling that $\mathrm{C}(\mathbf{I})$ is endowed with the (metrizable) topology that induced the uniform convergence on compact sets, we have that the map A_θ is the composition of the Borel mapping F_θ with the function

$$A\colon L^1_{\mathrm{loc}}(\mathbf{I}) \to \mathrm{C}(\mathbf{I}), \qquad f \mapsto A(f) \text{ with } A(f)(t) := \int_0^t f(r)\mathrm{d}r\,.$$

Since A is continuous, we have that A_θ is Borel.

Finally, we show that the mapping

$$\mathsf{L}_\theta\colon \mathrm{Lip}_k^\uparrow(\mathbf{I};\mathbb{R}^{d+1}) \to \mathrm{C}(\mathbf{I})\,, \qquad \mathbf{y} \mapsto \ell_{\mathbf{y}} = \Theta_{\mathbf{y}}^{-1}\,,$$

is Borel measurable. We notice that $\mathsf{L}_\theta(\mathbf{y})$ is well defined, as $\mathsf{A}_\theta(\mathbf{y})$ is invertible for every $\mathbf{y} \in \mathrm{Lip}_k^\uparrow(\mathbf{I};\mathbb{R}^{d+1})$. Moreover, $\mathsf{L}_\theta(\mathbf{y})$ is the composition of A_θ with the inversion operator $\mathsf{I}\colon \mathrm{biLip}_{c,c^{-1}}(\mathbf{I}) \to \mathrm{biLip}_{c,c^{-1}}(\mathbf{I})$. Now, I is continuous: indeed, let $(g_n)_n, g \in \mathrm{biLip}_{c,c^{-1}}(\mathbf{I})$ with $g_n \to g$ uniformly con compact subsets of $\mathbf{I}$. Consider $g^{-1}, (g_n^{-1})_n \subset \mathrm{biLip}_{c,c^{-1}}(\mathbf{I})$. Taking into account that $(g_n^{-1})_n$ is bounded in $L^\infty_{\mathrm{loc}}(\mathbf{I})$ with $\frac{1}{c} \le (g_n^{-1})' \le c$, in order to check that $g_n^{-1} \to g^{-1}$ on compact subsets of $\mathbf{I}$ it is sufficient to show that $g_n^{-1} \to g^{-1}$ pointwise in $\mathbf{I}$. Hence, let $r \in \mathbf{I}$ and $s_n := g_n^{-1}(r)$, i.e. $g_n(s_n) = r$. Since $(s_n)_n$ is bounded, it admits a subsequence $(s_{n_k})_k$ converging to some s^*. Recalling that $(g_n)_n$ converges uniformly to g on the compact subsets of $\mathbf{I}$, we gather that $r = g_{n_k}(s_{n_k}) \to g(s^*)$. Hence, $s^* = g^{-1}(r)$. As the limit does not depend on the extracted subsequence, then we have that the *whole* sequence $(s_n = g_n^{-1}(r))_n$ converges to $g^{-1}(r)$. Therefore, $\mathsf{L}_\theta = \mathsf{I} \circ \mathsf{A}_\theta$ is the composition of a Borel and of a continuous mapping: a fortiori, it is Borel.

Ultimately, the map $\mathfrak{R}\colon \mathrm{Lip}_k^\uparrow(\mathbf{I};\mathbb{R}^{d+1}) \to \mathrm{Lip}_{\hat{k}}^\uparrow(\mathbf{I};\mathbb{R}^{d+1})$ defined by $\mathbf{y} \mapsto \mathbf{y} \circ \ell_{\mathbf{y}} = \mathbf{y} \circ \mathsf{L}_\theta(\mathbf{y})$ is Borel. $\square$

Appendix E. Proof of Lemma 6.2

We divide the proof in 2 steps, proving the two equalities in (6.11) separately.
Step 1: $\mathscr{S}(\mathscr{T}(\mathsf{u})) = \mathsf{u}$ for $\mathsf{u} \in \mathrm{ABV}(\mathcal{Z};\mathbb{R}^d)$. Let us denote by $\mathbf{y} = (\mathsf{t}, \mathbf{x}) = \mathscr{T}(\mathsf{u})$, constructed according to (6.4)–(6.6). Then, we have that for $t \in \mathbf{I}$

$$\mathsf{s}_{\mathbf{y}}^-(t) := \sup\{s \in \mathbf{I} : \mathsf{t}(s) < t\} = \sup\{s \in \mathbf{I} : \inf\{\tau \in \mathbf{I} : L_{\mathsf{u}}^+(\tau) > s\} < t\}.$$

For every $s < L_{\mathsf{u}}^-(t)$ we have that $\inf\{\tau \in \mathbf{I} : L_{\mathsf{u}}^+(\tau) > s\} < t$, since $L_{\mathsf{u}}^+(\tau) < L_{\mathsf{u}}^-(t)$ for $\tau < t$ and $L_{\mathsf{u}}^+(\tau) \to L_{\mathsf{u}}^-(t)$ as $\tau \nearrow t$. Hence, $L_{\mathsf{u}}^-(t) \le \mathsf{s}_{\mathbf{y}}^-(t)$. On the other hand, for $s > L_{\mathsf{u}}^-(t)$ we get that

$$\inf\{\tau \in \mathbf{I} : L_{\mathsf{u}}^+(\tau) > s\} \ge \inf\{\tau \in \mathbf{I} : L_{\mathsf{u}}^+(\tau) > L_{\mathsf{u}}^-(t)\} \ge t.$$

Hence, $L_{\mathsf{u}}^-(t) = \mathsf{s}_{\mathbf{y}}^-(t)$. With a similar argument we infer $L_{\mathsf{u}}^+(t) = \mathsf{s}_{\mathbf{y}}^+(t)$.

Recalling (6.10), we write for $(t, r) \in \mathcal{Z}$

$$\begin{aligned}\mathscr{S}(\mathbf{y})(t,r) = \mathscr{S}(\mathscr{T}(\mathsf{u}))(t,r) &= \mathsf{u}\Big(\mathsf{t}\big(\mathsf{s}_{\mathbf{y}}^-(t) + r(\mathsf{s}_{\mathbf{y}}^+(t) - \mathsf{s}_{\mathbf{y}}^-(t))\big), \mathsf{r}\big(\mathsf{s}_{\mathbf{y}}^-(t) + r(\mathsf{s}_{\mathbf{y}}^+(t) - \mathsf{s}_{\mathbf{y}}^-(t))\big)\Big)\\ &= \mathsf{u}\Big(t, \mathsf{r}\big(L_{\mathsf{u}}^-(t) + r(L_{\mathsf{u}}^+(t) - L_{\mathsf{u}}^-(t))\big)\Big) = \mathsf{u}(t,r),\end{aligned}$$

where we have used that $\mathsf{t}(s) = t$ for every $s \in [\mathsf{s}_{\mathbf{y}}^-(t), \mathsf{s}_{\mathbf{y}}^+(t)]$ and the definition of r in (6.5).
Step 2: $\mathscr{T}(\mathscr{S}(\mathbf{y})) = \mathbf{y}$ for $\mathbf{y} = (\mathsf{t}, \mathbf{x}) \in \mathrm{ArcLip}(\mathbf{I};\mathbb{R}^{d+1})$. For $(t,r) \in \mathcal{Z}$ we notice that

$$\mathscr{S}(\mathbf{y})(t,r) = \mathbf{x}(\mathsf{s}_{\mathbf{y}}^-(t) + r(\mathsf{s}_{\mathbf{y}}^+(t) - \mathsf{s}_{\mathbf{y}}^-(t))). \tag{E.1}$$

To shorten the notation, we set $\mathsf{u} = \mathscr{S}(\mathbf{y})$ and $(\mathsf{t}_{\mathsf{u}}, \mathbf{x}_{\mathsf{u}}) = \mathscr{T}(\mathscr{S}(\mathbf{y}))$. Since $\|\mathbf{y}'\| = 1$ a.e. in $\mathbf{I}$, it holds

$$L_{\mathsf{u}}^{\pm}(t) = \int_0^{\mathsf{s}_{\mathbf{y}}^{\pm}(t)} \|\mathbf{y}'(s)\| \, \mathrm{d}s = \mathsf{s}_{\mathbf{y}}^{\pm}(t) \qquad \text{for } t \in \mathbf{I}, \tag{E.2}$$

$$L_{\mathsf{u}}(t,r) = \mathsf{s}_{\mathbf{y}}^-(t) + r(\mathsf{s}_{\mathbf{y}}^+(t) - \mathsf{s}_{\mathbf{y}}^-(t)) \qquad \text{for } (t,r) \in \mathcal{Z}. \tag{E.3}$$

By definition of $\mathscr{T}$ and by the characterization of $L_{\mathsf{u}}^{\pm}$ above we have that for $s \in \mathbf{I}$

$$\mathsf{t}_{\mathsf{u}}(s) = \inf\{t \in \mathbf{I} : s_{\mathbf{y}}^+(t) > s\}.$$

In particular, it is immediate to see that $\mathsf{t}(s) \ge \mathsf{t}_{\mathsf{u}}(s)$. By contradiction, if $\mathsf{t}(s) > \mathsf{t}_{\mathsf{u}}(s)$, then it must be $\mathsf{s}_{\mathbf{y}}^+(t) > s$ for every $t \in (\mathsf{t}_{\mathsf{u}}(s), \mathsf{t}(s))$, which implies $\mathsf{s}_{\mathbf{y}}^-(\mathsf{t}(s)) > s$, whence (6.9). Thus, $\mathsf{t} = \mathsf{t}_{\mathsf{u}}$ in $\mathbf{I}$.

We now consider the second component $\mathbf{x}_{\mathsf{u}}$ of $\mathscr{T}(\mathscr{S}(\mathbf{y}))$. We recall that, in view of (6.5) and (E.2), it holds for $s \in \mathbf{I}$

$$\mathsf{r}(s) = \frac{s - \mathsf{s}_{\mathbf{y}}^-(\mathsf{t}(s))}{\mathsf{s}_{\mathbf{y}}^+(\mathsf{t}(s)) - \mathsf{s}_{\mathbf{y}}^-(\mathsf{t}(s))} \qquad \text{if } \mathsf{s}_{\mathbf{y}}^+(\mathsf{t}(s)) \ne \mathsf{s}_{\mathbf{y}}^-(\mathsf{t}(s)), \tag{E.4}$$

$$\begin{aligned}\mathbf{x}_{\mathsf{u}}(s) &= \mathsf{u}(\mathsf{t}_{\mathsf{u}}(s), \mathsf{r}(s)) = \mathsf{u}(\mathsf{t}(s), \mathsf{r}(s))\\ &= \mathbf{x}(\mathsf{s}_{\mathbf{y}}^-(\mathsf{t}(s))) + (\mathsf{s}_{\mathbf{y}}^+(\mathsf{t}(s)) - \mathsf{s}_{\mathbf{y}}^-(\mathsf{t}(s))) \int_0^{\mathsf{r}(s)} \mathbf{x}'(\mathsf{s}_{\mathbf{y}}^-(\mathsf{t}(s)) + \ell(\mathsf{s}_{\mathbf{y}}^+(\mathsf{t}(s)) - \mathsf{s}_{\mathbf{y}}^-(\mathsf{t}(s)))) \, \mathrm{d}\ell.\end{aligned} \tag{E.5}$$

Hence, if $\mathsf{s}_{\mathbf{y}}^+(\mathsf{t}(s)) = \mathsf{s}_{\mathbf{y}}^-(\mathsf{t}(s))$, from (6.9) we immediately conclude that $\mathbf{x}_{\mathsf{u}}(s) = \mathbf{x}(s)$. If $\mathsf{s}_{\mathbf{y}}^+(\mathsf{t}(s)) > \mathsf{s}_{\mathbf{y}}^-(\mathsf{t}(s))$, (E.1) and (E.4) yield $\mathbf{x}_{\mathsf{u}}(s) = \mathbf{x}(s)$. Hence, $\mathscr{T}(\mathscr{S}(\mathbf{y})) = \mathbf{y}$.

Appendix F. Proof of Proposition 6.4

▷ (6.23) & (6.24). We will prove the representation formula (6.23) for ϑ_{u} by showing (6.24). To do so, we need to relate the curve $\mathfrak{v}_{\mathsf{u}}$ to the trajectory $\mathcal{T}(\mathsf{u}) = \mathbf{y} = (\mathsf{t}, \mathbf{x})$. In fact,

$$\mathfrak{v}_{\mathsf{u}}(t) = \mathsf{u}(t, 0) = \mathbf{x}(\mathsf{s}_{\mathbf{y}}^-(t)) \qquad \text{for all } t \in \mathbf{I}.$$

Moreover, $\mathfrak{J}_{\mathsf{u}} = \mathrm{J}_{\mathsf{s}_{\mathbf{y}}^{\pm}}$, and $\mathscr{C}_{\mathbf{y}} := \mathsf{t}^{-1}(C_{\mathsf{u}}) = \mathsf{t}^{-1}(\mathbf{I} \backslash \mathfrak{J}_{\mathsf{u}})$ is the set where $\mathsf{t}(\cdot)$ is injective. In particular, $\mathbf{I} \setminus \mathscr{C}_{\mathbf{y}}$ is union of the intervals $[\mathsf{s}_{\mathbf{y}}^-(t), \mathsf{s}_{\mathbf{y}}^+(t)]$.

Now, for every $\boldsymbol{\varphi} \in \mathrm{C_c}(\mathbb{R}^{d+1}_+;\mathbb{R}^d)$ we have

$$\langle \sum_{t\in\mathfrak{J}_{\mathsf{u}}} \delta_t \otimes \boldsymbol{j}_{t,\mathsf{u}}, \boldsymbol{\varphi}\rangle = \sum_{t\in\mathfrak{J}_{\mathsf{u}}} \int_0^1 \boldsymbol{\varphi}(t,\mathsf{u}(t,r))\cdot\partial_r \mathsf{u}(t,r)\,\mathrm{d}r = \sum_{t\in\mathfrak{J}_{\mathsf{u}}} \int_{\mathsf{s}^-_{\mathsf{y}}(t)}^{\mathsf{s}^+_{\mathsf{y}}(t)} \boldsymbol{\varphi}(t,\mathsf{u}(t,r(s)))\cdot\mathbf{x}'(s)\,\mathrm{d}s \tag{F.1}$$

$$= \sum_{t\in\mathfrak{J}_{\mathsf{u}}} \int_{\mathsf{s}^-_{\mathsf{y}}(t)}^{\mathsf{s}^+_{\mathsf{y}}(t)} \boldsymbol{\varphi}(\mathsf{t}(s),\mathsf{u}(\mathsf{t}(s),r(s)))\cdot\mathbf{x}'(s)\,\mathrm{d}s$$

$$= \sum_{t\in\mathfrak{J}_{\mathsf{u}}} \int_{\mathsf{s}^-_{\mathsf{y}}(t)}^{\mathsf{s}^+_{\mathsf{y}}(t)} \boldsymbol{\varphi}(\mathbf{y}(s))\cdot\mathbf{x}'(s)\,\mathrm{d}s\,,$$

where, in the last equality, we have used the fact that $\mathsf{t}(s)\equiv t$ for every $s\in[\mathsf{s}^-_{\mathsf{y}}(t),\mathsf{s}^+_{\mathsf{y}}(t)]$. Hence, (6.24b) follows.

In order to show (6.24a), we start by recalling that for every $\zeta\in\mathrm{C_c}(\mathbf{I})$ and $\boldsymbol{\zeta}\in\mathrm{C_c}(\mathbf{I};\mathbb{R}^d)$ we further have (see, e.g., [38, Proposition 6.11])

$$\int_{\mathbf{I}} \zeta(t)\,\mathrm{d}t = \int_{\mathscr{C}_{\mathsf{y}}} \zeta(\mathsf{t}(s))\,\mathsf{t}'(s)\,\mathrm{d}s\,, \tag{F.2}$$

$$\int_{C_{\mathsf{u}}} \boldsymbol{\zeta}(t)\,\mathrm{d}((\mathfrak{v}_{\mathsf{u}})'_{\mathcal{L}^1}+(\mathfrak{v}_{\mathsf{u}})'_{\mathrm{C}})(t) = \int_{C_{\mathfrak{v}_{\mathsf{u}}}} \boldsymbol{\zeta}(t)\,\mathrm{d}((\mathfrak{v}_{\mathsf{u}})'_{\mathcal{L}^1}+(\mathfrak{v}_{\mathsf{u}})'_{\mathrm{C}})(t) = \int_{\mathscr{C}_{\mathsf{y}}} \boldsymbol{\zeta}(\mathsf{t}(s))\cdot\mathbf{x}'(s)\,\mathrm{d}s \tag{F.3}$$

(in (F.3) and in what follows in this proof, we use the more compact notation $\mathfrak{v}'_{\mathsf{u}}$, $(\mathfrak{v}_{\mathsf{u}})'_{\mathcal{L}^1}$, and $(\mathfrak{v}_{\mathsf{u}})'_{\mathrm{C}}$, in place of (6.18) and (6.19)). For $\varphi_0\in\mathrm{C_c}(\mathbb{R}^{d+1}_+)$ and $\boldsymbol{\varphi}\in\mathrm{C_c}(\mathbb{R}^{d+1}_+;\mathbb{R}^d)$, we test (F.2) and (F.3) with $\zeta_\varepsilon := \varphi_0(\cdot,\mathfrak{v}_{\mathsf{u}}(\cdot))*\rho_\varepsilon$ and $\boldsymbol{\zeta}_\varepsilon := \boldsymbol{\varphi}(\cdot,\mathfrak{v}_{\mathsf{u}}(\cdot))*\rho_\varepsilon$, for a mollifier ρ_ε supported in $[0,\varepsilon]$. Since $\zeta_\varepsilon(t)\to\varphi_0(t,\mathfrak{v}_{\mathsf{u}}(t))$ and $\boldsymbol{\zeta}_\varepsilon(t)\to\boldsymbol{\varphi}(t,\mathfrak{v}_{\mathsf{u}}(t))$ for every $t\in\mathbf{I}$, we infer that

$$\int_{\mathbf{I}} \varphi_0(t,\mathfrak{v}_{\mathsf{u}}(t))\,\mathrm{d}t = \int_{\mathscr{C}_{\mathsf{y}}} \varphi_0(\mathsf{t}(s),\mathfrak{v}_{\mathsf{u}}(\mathsf{t}(s)))\,\mathsf{t}'(s)\,\mathrm{d}s\,, \tag{F.4}$$

$$\int_{C_{\mathfrak{v}_{\mathsf{u}}}} \boldsymbol{\varphi}(t,\mathfrak{v}_{\mathsf{u}}(t))\,\mathrm{d}((\mathfrak{v}_{\mathsf{u}})'_{\mathcal{L}^1}+(\mathfrak{v}_{\mathsf{u}})'_{C})(t) = \int_{\mathscr{C}_{\mathsf{y}}} \boldsymbol{\varphi}(\mathsf{t}(s),\mathfrak{v}_{\mathsf{u}}(\mathsf{t}(s)))\cdot\mathbf{x}'(s)\,\mathrm{d}s\,. \tag{F.5}$$

Since $\mathfrak{v}_{\mathsf{u}}(\mathsf{t}(s)) = \mathsf{u}(\mathsf{t}(s),0)$ and, for $s\in\mathscr{C}_{\mathsf{y}}$, $\mathsf{u}(\mathsf{t}(s),0)=\mathsf{u}(\mathsf{t}(s),r)$ for every $r\in[0,1]$, we rewrite (F.4)–(F.5) as

$$\int_{\mathbf{I}} \varphi_0(t,\mathfrak{v}_{\mathsf{u}}(t))\,\mathrm{d}t = \int_{\mathscr{C}_{\mathsf{y}}} \varphi_0(\mathbf{y}(s))\,\mathsf{t}'(s)\,\mathrm{d}s\,, \tag{F.6}$$

$$\int_{C_{\mathfrak{v}_{\mathsf{u}}}} \boldsymbol{\varphi}(t,\mathfrak{v}_{\mathsf{u}}(t))\,\mathrm{d}((\mathfrak{v}_{\mathsf{u}})'_{\mathcal{L}^1}+(\mathfrak{v}_{\mathsf{u}})'_{C})(t) = \int_{\mathscr{C}_{\mathsf{y}}} \boldsymbol{\varphi}(\mathbf{y}(s))\cdot\mathbf{x}'(s)\,\mathrm{d}s\,, \tag{F.7}$$

whence (6.24a).

Combining (F.6)–(F.7) with (F.1) we conclude that for all $\varphi=(\varphi_0,\boldsymbol{\varphi})\in\mathrm{C_c}(\mathbb{R}^{d+1}_+;\mathbb{R}^{d+1})$

$$\langle\omega_{\mathsf{y}},\varphi\rangle = \big\langle(\mathfrak{G}_{\mathsf{u}})_\sharp\left((1,\mathfrak{v}'_{\mathsf{u}})\mathcal{L}^1+(0,(\mathfrak{v}_{\mathsf{u}})'_{\mathrm{C}})\right)+\sum_{t\in\mathfrak{J}_{\mathsf{u}}}\delta_t\otimes\boldsymbol{j}_{t,\mathsf{u}},\varphi\big\rangle\,,$$

and (6.23) follows.

▷ (6.25) : We will show the measurability of $\mathscr{A}$, $\mathscr{C}$, and $\mathscr{J}$ by proving that the following mappings

$$\begin{cases} \mathsf{ABV}(\mathcal{Z};\mathbb{R}^d)\ni\mathsf{u}\mapsto\langle\mathscr{A}(\mathsf{u}),\varphi\rangle \\ \mathsf{ABV}(\mathcal{Z};\mathbb{R}^d)\ni\mathsf{u}\mapsto\langle\mathscr{J}(\mathsf{u}),\varphi\rangle \\ \mathsf{ABV}(\mathcal{Z};\mathbb{R}^d)\ni\mathsf{u}\mapsto\langle\mathscr{C}(\mathsf{u}),\varphi\rangle \end{cases} \qquad \text{are Borel for every test function } \varphi=(\varphi_0,\boldsymbol{\varphi})\in\mathrm{C_c}(\mathbb{R}^{d+1}_+;\mathbb{R}^{d+1})\,.$$

In turn, this will be shown via the representation formulae (6.24). We start by observing that the mappings

$$\begin{cases} \mathrm{Lip}^{\uparrow}_1(\mathbf{I};\mathbb{R}^{d+1})\ni\mathbf{y}\mapsto\mathrm{I}_0(\mathbf{y}) := \int_{\mathbf{I}}\varphi_0(\mathbf{y}(s))\mathsf{t}'(s)\,\mathrm{d}s \\ \mathrm{Lip}^{\uparrow}_1(\mathbf{I};\mathbb{R}^{d+1})\ni\mathbf{y}\mapsto\mathrm{I}(\mathbf{y}) := \int_{\mathbf{I}}\boldsymbol{\varphi}(\mathbf{y}(s))\cdot\mathbf{x}'(s)\,\mathrm{d}s \end{cases} \qquad \text{are } \textit{Borel}. \tag{F.8}$$

Indeed, they are continuous with respect to the topology of uniform convergence on compact sets of $\mathbf{I}$ induced by the metric D from (B.1): to check this, it suffices to take $(\mathsf{y}_n)_n$, $\mathsf{y} \in \mathrm{Lip}_1^{\uparrow}(\mathbf{I};\mathbb{R}^{d+1})$ with $D(\mathsf{y}_n,\mathsf{y}) \to 0$ as $n\to\infty$, and and observe that, since $\|(\mathsf{t}_n',\mathsf{x}_n')\| \le 1$ a.e. in $\mathbf{I}$, we may suppose (up to a not relabeled subsequence), that $\mathsf{t}_n' \rightharpoonup^* \mathsf{t}'$ and $\mathsf{x}_n' \rightharpoonup^* \mathsf{x}$ in $L^\infty(\mathbf{I})$. Then, the convergences

$$\lim_{n\to\infty}\int_{\mathbf{I}} \varphi_0(\mathsf{y}_n(s))\mathsf{t}_n'(s)\,\mathrm{d}s = \int_{\mathbf{I}} \varphi_0(\mathsf{y}(s))\mathsf{t}'(s)\,\mathrm{d}s\,,$$
$$\lim_{n\to\infty}\int_{\mathbf{I}} \boldsymbol{\varphi}(\mathsf{y}_n(s))\cdot\mathsf{x}_n'(s)\,\mathrm{d}s = \int_{\mathbf{I}} \boldsymbol{\varphi}(\mathsf{y}(s))\cdot\mathsf{x}'(s)\,\mathrm{d}s$$

follow by dominated convergence. Now, recalling (F.4) and (F.6), we conclude that the mapping

$$\mathrm{ABV}(\mathcal{Z};\mathbb{R}^d) \ni \mathsf{u} \mapsto \mathrm{I}_0(\mathscr{T}(\mathsf{u})) = \int_{\mathbf{I}} \varphi_0(t,\mathfrak{v}_{\mathsf{u}}(t))\,\mathrm{d}t \qquad \text{is Borel,} \tag{F.9}$$

as it is given by the composition of two Borel mappings.

In turn, we observe that for all $\mathsf{y} \in \mathrm{Lip}_1^{\uparrow}(\mathbf{I};\mathbb{R}^{d+1})$ we have

$$\mathrm{I}(\mathsf{y}) = \mathrm{I}_{\mathrm{cont}}(\mathsf{y}) + \mathrm{I}_{\mathrm{sing}}(\mathsf{y}) \qquad \text{with } \begin{cases} \mathrm{I}_{\mathrm{cont}}(\mathsf{y}) = \int_{\mathbf{I}\cap\{\mathsf{t}'>0\}} \boldsymbol{\varphi}(\mathsf{y}(s))\cdot\mathsf{x}'(s)\,\mathrm{d}s, \\ \mathrm{I}_{\mathrm{sing}}(\mathsf{y}) = \int_{\mathbf{I}\cap\{\mathsf{t}'=0\}} \boldsymbol{\varphi}(\mathsf{y}(s))\cdot\mathsf{x}'(s)\,\mathrm{d}s. \end{cases}.$$

Indeed, the pedices cont and sing refer to the fact that $\mathrm{I}_{\mathrm{cont}}$ and $\mathrm{I}_{\mathrm{sing}}$ represent the contributions to ϑ_{u} involving the measures $(\mathfrak{v}_{\mathsf{u}})'_{\mathcal{L}^1}$ and $(\mathfrak{v}_{\mathsf{u}})'_{\mathrm{C}} + (\mathfrak{v}_{\mathsf{u}})'_{\mathrm{J}}$, respectively, as

$$\begin{cases} \mathrm{I}_{\mathrm{cont}}(\mathsf{y}) = \int_{\mathbf{I}} \boldsymbol{\varphi}(t,\mathfrak{v}_{\mathsf{u}}(t))\cdot\mathfrak{v}_{\mathsf{u}}'(t)\,\mathrm{d}t, \\ \mathrm{I}_{\mathrm{sing}}(\mathsf{y}) = \int_{\mathbf{I}} \boldsymbol{\varphi}(t,\mathfrak{v}_{\mathsf{u}}(t))\,\mathrm{d}(\mathfrak{v}_{\mathsf{u}})'_{\mathrm{C}}(t) + \langle \sum_{t\in\mathfrak{J}_{\mathsf{u}}} \delta_t \otimes \boldsymbol{j}_{t,\mathsf{u}}, \boldsymbol{\varphi}\rangle. \end{cases} \tag{F.10}$$

Now, we claim that the mapping

$$\mathrm{ArcLip}(\mathbf{I};\mathbb{R}^{d+1}) \ni \mathsf{y} \mapsto \mathrm{I}_{\mathrm{sing}}(\mathsf{y}) = \int_{\mathbf{I}} \mathbf{1}_{\{s\in\mathbf{I}\,:\,\mathsf{t}'(s)=0\}}(r)\boldsymbol{\varphi}(\mathsf{y}(r))\cdot\mathsf{x}'(r)\,\mathrm{d}r \qquad \text{is Borel.} \tag{F.11}$$

Since we can write $\mathrm{I}_{\mathrm{sing}}(\mathsf{y}) = \mathrm{I}^+_{\mathrm{sing}}(\mathsf{y}) - \mathrm{I}^-_{\mathrm{sing}}(\mathsf{y})$ with

$$\mathrm{I}^{\pm}_{\mathrm{sing}}(\mathsf{y}) := \int_{\mathbf{I}} \mathbf{1}_{\{s\in\mathbf{I}\,:\,\mathsf{t}'(s)=0\}}(r)(\boldsymbol{\varphi}(\mathsf{y}(r))\cdot\mathsf{x}'(r))_{\pm}\,\mathrm{d}r \quad \text{with } \begin{cases} r_+ = \max\{r,0\}, \\ r_- = \max\{-r,0\}\,, \end{cases}$$

it is enough to prove that $\mathsf{y}\mapsto \mathrm{I}^{\pm}_{\mathrm{sing}}(\mathsf{y})$ are Borel measurable. We proceed with the proof for $\mathrm{I}^+_{\mathrm{sing}}$. The very same argument applies to $\mathrm{I}^-_{\mathrm{sing}}$.

For $n,k\in\mathbb{N}\setminus\{0\}$ we consider the map

$$\mathrm{ArcLip}(\mathbf{I};\mathbb{R}^{d+1}) \ni \mathsf{y} \mapsto J_{n,k}(\mathsf{y}) = \int_{\mathbf{I}} \mathbf{1}_{S_{n,k}(\mathsf{t})}(r)(\boldsymbol{\varphi}(\mathsf{y}(r))\cdot\mathsf{x}'(r))_+\,\mathrm{d}r\,,$$

where $\mathbf{1}_{S_{n,k}}$ is the characteristic function of the set $S_{n,k}(\mathsf{t}) := \{s\in\mathbf{I}\ :\ k(\mathsf{t}(s+\frac{1}{k})-\mathsf{t}(s)) \le \frac{1}{n}\}$. Then, $J_{n,k}$ is Borel measurable, as it is upper semicontinuous with respect to the uniform convergence on compact subsets of $\mathbf{I}$. Notice that here we are also using that, whenever $\mathsf{y}_m, \mathsf{y}\in\mathrm{ArcLip}(\mathbf{I};\mathbb{R}^{d+1})$ are such that $D(\mathsf{y}_m,\mathsf{y})\to 0$ as $m\to\infty$, then we also have $\mathsf{y}_m'\to\mathsf{y}'$ in $L^p_{\mathrm{loc}}(\mathbf{I};\mathbb{R}^{d+1})$ for every $1\le p<+\infty$.

The Borel measurability of $J_{n,k}$ implies that also the maps

$$\mathrm{ArcLip}(\mathbf{I};\mathbb{R}^{d+1}) \ni \mathsf{y} \mapsto \liminf_{k\to\infty} J_{n,k}(\mathsf{y})\,, \qquad \mathrm{ArcLip}(\mathbf{I};\mathbb{R}^{d+1}) \ni \mathsf{y} \mapsto \limsup_{k\to\infty} J_{n,k}(\mathsf{y})$$

are Borel measurable for every $n\in\mathbb{N}$. By Fatou lemma, we further notice that for every $\mathsf{y}\in\mathrm{ArcLip}(\mathbf{I};\mathbb{R}^{d+1})$ it holds

$$\begin{aligned} \mathrm{I}^+_{\mathrm{sing}}(\mathsf{y}) &\le \int_{\mathbf{I}} \liminf_{k\to\infty} \mathbf{1}_{S_{n,k}(\mathsf{t})}(r)(\boldsymbol{\varphi}(\mathsf{y}(r))\cdot\mathsf{x}'(r))_+\,\mathrm{d}r \le \liminf_{k\to\infty} J_{n,k}(\mathsf{y}) \\ &\le \limsup_{k\to\infty} J_{n,k}(\mathsf{y}) \le \int_{\mathbf{I}} \limsup_{k\to\infty} \mathbf{1}_{S_{n,k}(\mathsf{t})}(r)(\boldsymbol{\varphi}(\mathsf{y}(r))\cdot\mathsf{x}'(r))_+\,\mathrm{d}r \end{aligned} \tag{F.12}$$

$$\leq \int_{\mathbf{I}} \mathbf{1}_{\{s\in\mathbf{I}:\, \mathsf{t}'(s)\leq \frac{1}{n}\}}(r)(\boldsymbol{\varphi}(\mathbf{y}(r))\cdot \mathbf{x}'(r))_+ \,\mathrm{d}r\,.$$

Taking the limit as $n\to\infty$ in the chain of inequalities (F.12), we infer that

$$\mathrm{I}^+_{\mathrm{sing}}(\mathbf{y}) = \lim_{n\to\infty}\liminf_{k\to\infty} J_{n,k}(\mathbf{y})\,.$$

This implies that $\mathrm{I}^+_{\mathrm{sing}}$ is a Borel map. This concludes the proof of (F.11).

Combining (F.8) and (F.11) we deduce that also $\mathrm{I}_{\mathrm{cont}}$ is Borel measurable. Thus, in view of (F.10) we have that

$$\mathsf{ABV}(\mathcal{Z};\mathbb{R}^d)\ni \mathsf{u}\mapsto \mathrm{I}_{\mathrm{cont}}(\mathscr{T}(\mathsf{u})) = \int_{\mathbf{I}} \varphi(t,\mathfrak{v}_{\mathsf{u}}(t))\cdot \mathfrak{v}'_{\mathsf{u}}(t)\,\mathrm{d}t \qquad \text{is Borel.} \tag{F.13}$$

From (F.9) and (F.13) we then have that the mapping

$$\mathsf{ABV}(\mathcal{Z};\mathbb{R}^d)\ni \mathsf{u}\mapsto \langle \mathscr{A}(\mathsf{u}),\varphi\rangle \text{ is Borel for every } \varphi\in \mathrm{C_c}(\mathbb{R}^{d+1}_+;\mathbb{R}^{d+1})\,.$$

Now, Lemma F.1 ahead ensures that the mapping $\mathsf{ABV}(\mathcal{Z};\mathbb{R}^d)\ni\mathsf{u}\mapsto \langle \mathscr{J}(\mathsf{u}),\boldsymbol{\varphi}\rangle$, is Borel for all test functions $\boldsymbol{\varphi}$. Ultimately,

$$\mathsf{ABV}(\mathcal{Z};\mathbb{R}^d)\ni \mathsf{u}\mapsto \mathrm{I}_{\mathrm{cont}}(\mathscr{T}(\mathsf{u})) - \langle \mathscr{J}(\mathsf{u}),\boldsymbol{\varphi}\rangle = \langle \mathscr{C}(\mathsf{u}),\boldsymbol{\varphi}\rangle \qquad \text{is Borel for all } \boldsymbol{\varphi}\in \mathrm{C_c}(\mathbb{R}^{d+1}_+;\mathbb{R}^d)\,.$$

We have thus proven (6.25). ∎

The last result of this section addresses the measurability of the mapping $\mathscr{J}$.

Lemma F.1. *For all* $\boldsymbol{\varphi}\in \mathrm{C_c}(\mathbb{R}^{d+1}_+;\mathbb{R}^d)$ *we consider the mapping*

$$\mathrm{I}_{\mathrm{jump}}\colon \mathrm{ArcLip}(\mathbf{I};\mathbb{R}^{d+1})\to\mathbb{R} \qquad \mathbf{y}\mapsto \sum_{t\in \mathrm{L}(\mathbf{y})}\int_{\mathsf{s}^-_{\mathbf{y}}(t)}^{\mathsf{s}^+_{\mathbf{y}}(t)} \boldsymbol{\varphi}(\mathbf{y}(s))\cdot\mathbf{x}'(s)\,\mathrm{d}s \tag{F.14}$$

with the short-hand notation $L_{\mathbf{y}} := \{s\in\mathbf{I}:\ \mathsf{s}^+_{\mathbf{y}}(s)-\mathsf{s}^-_{\mathbf{y}}(s)>0\}$. *Then,*

$$\forall\,\boldsymbol{\varphi}\in \mathrm{C_c}(\mathbf{I}\times\mathbb{R}^d;\mathbb{R}^d) \qquad \mathsf{u}\mapsto \mathrm{I}_{\mathrm{jump}}(\mathscr{T}(\mathsf{u})) = \langle \mathscr{J}(\mathsf{u}),\boldsymbol{\varphi}\rangle \quad \textit{is Borel.}$$

Proof. First of all, observe that

$$\mathrm{I}_{\mathrm{jump}}(\mathbf{y}) = \mathrm{I}^+_{\mathrm{jump}}(\mathbf{y}) - \mathrm{I}^-_{\mathrm{jump}}(\mathbf{y})$$

with

$$\mathrm{I}^\pm_{\mathrm{jump}}(\mathbf{y}) := \sum_{t\in\mathrm{L}_{\mathbf{y}}}\int_{\mathsf{s}^-_{\mathbf{y}}(t)}^{\mathsf{s}^+_{\mathbf{y}}(t)} (\boldsymbol{\varphi}(\mathbf{y}(s))\cdot\mathbf{x}'(s))_\pm\,\mathrm{d}s \quad \text{with} \quad \begin{cases} r_+ = \max\{r,0\},\\ r_- = \max\{-r,0\}\,.\end{cases}$$

Hence, we can show the measurability property for the functions $\mathrm{I}^\pm_{\mathrm{jump}}$. We provide the full proof for $\mathrm{I}^+_{\mathrm{jump}}$. A similar argument applies to $\mathrm{I}^-_{\mathrm{jump}}$.

Let us now introduce the continuous function

$$A\colon \mathbf{I}\to\mathbb{R}^+, \qquad A(r) := \int_0^r (\boldsymbol{\varphi}(\mathbf{y}(s))\cdot\mathbf{x}'(s))_+\,\mathrm{d}s,$$

so that

$$\int_{\mathsf{s}^-_{\mathbf{y}}(t)}^{\mathsf{s}^+_{\mathbf{y}}(t)} (\boldsymbol{\varphi}(\mathbf{y}(s))\cdot\mathbf{x}'(s))\,\mathrm{d}s = A(\mathsf{s}^+_{\mathbf{y}}(t)) - A(\mathsf{s}^-_{\mathbf{y}}(t)) \tag{F.15}$$

We will thus prove that the function

$$\mathcal{A}\colon \mathrm{ArcLip}(\mathbf{I};\mathbb{R}^{d+1})\to\mathbb{R}^+ \qquad \mathbf{y}\mapsto \sum_{t\in\mathrm{L}(\mathbf{y})} [A(\mathsf{s}^+_{\mathbf{y}}(t)) - A(\mathsf{s}^-_{\mathbf{y}}(t))] \qquad \text{is Borel.} \tag{F.16}$$

We split the argument for (F.16) in the following steps.

Claim 1:

$$\begin{cases} \mathbf{I}\times\mathrm{ArcLip}(\mathbf{I};\mathbb{R}^{d+1})\ni(t,\mathbf{y})\mapsto A(\mathsf{s}^+_{\mathbf{y}}(t)) & \text{is upper semicontinuous.}\\ \mathbf{I}\times\mathrm{ArcLip}(\mathbf{I};\mathbb{R}^{d+1})\ni(t,\mathbf{y})\mapsto A(\mathsf{s}^-_{\mathbf{y}}(t)) & \text{is lower semicontinuous.}\end{cases} \tag{F.17}$$

As for the first property, it suffices to observe that $(t,\mathbf{y}) \mapsto \mathsf{s}^+_{\mathbf{y}}(t)$ is upper semicontinuous. Hence, since every $\mathbf{y} \in \mathrm{ArcLip}(\mathbf{I};\mathbb{R}^{d+1})$ satisfies $\|\mathbf{y}'(s)\| = 1$ for a.e. $s \in \mathbf{I}$, we have that $A \circ \mathsf{s}^+_{\mathbf{y}}$ is upper semicontinuous. Indeed, the constraint $\|\mathbf{y}'(s)\| = 1$ implies that

$$\mathbf{y} \mapsto \int_0^r (\varphi(\mathbf{y}(s)) \cdot \mathbf{x}'(s))_+ \,\mathrm{d}s$$

is continuous in $\mathrm{ArcLip}(\mathbf{I};\mathbb{R}^{d+1})$ for every $r \in \mathbf{I}$. This yields the desired upper semicontinuity. Analogously, the second statement follows from the lower semicontinuity of $(t,\mathbf{y}) \mapsto \mathsf{s}^-_{\mathbf{y}}(t)$.

Claim 2: *for every $T > 0$, $S > 0$, and $\zeta > 0$, the mapping*

$$\mathcal{A}^{T,S}_\zeta \colon \mathrm{ArcLip}(\mathbf{I};\mathbb{R}^{d+1}) \to \mathbb{R}^+, \qquad \mathbf{y} \mapsto \begin{cases} \sum_{t \in \mathrm{L}^{T,S}_\zeta(\mathbf{y})} [A(\mathsf{s}^+_{\mathbf{y}}(t)) - A(\mathsf{s}^-_{\mathbf{y}}(t))] & \text{if } \mathsf{s}^-_{\mathbf{y}}(T) \le S, \\ 0 & \text{if } \mathsf{s}^-_{\mathbf{y}}(T) > S, \end{cases} \tag{F.18}$$

$$\text{with } \mathrm{L}^T_\zeta(\mathbf{y}) := \{t \in \mathrm{L}(\mathbf{y}) \;:\; t \in [0,T],\ |\mathsf{s}^+_{\mathbf{y}}(t) - \mathsf{s}^-_{\mathbf{y}}(t)| \ge \zeta\}$$

is upper semicontinuous. Let us consider $(\mathbf{y}_j)_j$, $\mathbf{y} \in \mathrm{ArcLip}(\mathbf{I};\mathbb{R}^{d+1})$ such that $D(\mathbf{y}_j,\mathbf{y}) \to 0$ and show that

$$\limsup_{j\to\infty} \mathcal{A}^{T,S}_\zeta(\mathbf{y}_j) \le \mathcal{A}^{T,S}_\zeta(\mathbf{y}).$$

Up to a subsequence, we may assume that the limsup is a limit. If $\mathsf{s}^-_{\mathbf{y}_j}(T) > S$ definitely for j large enough, there is nothing to prove. Let us therefore assume that for every $j \in \mathbb{N}$ we have $\mathsf{s}^-_{\mathbf{y}_j}(T) \le S$. By lower-semicontinuity, observe that $\mathsf{s}^-_{\mathbf{y}}(T) \le S$. Moreover, we observe that, in correspondence with the sequence $(\mathbf{y}_j)_j$ there exists $N \in \mathbb{N}$ such that for every $j \in \mathbb{N}$ there exist at most N times $t^j_1 < \ldots < t^j_N \in [0,T]$ such that $|\mathsf{s}^+_{\mathbf{y}_j}(t^j_i) - \mathsf{s}^-_{\mathbf{y}_j}(t^j_i)| \ge \zeta$ for all $i = 1,\ldots,N$. In fact, by definition of $\mathsf{s}^\pm_{\mathbf{y}_j}$, for every $j \in \mathbb{N}$ it holds that

$$S \ge \mathsf{s}^-_{\mathbf{y}_j}(T) \ge \sum_{t \in \mathrm{L}^T_\zeta(\mathbf{y}) \cap [0,T)} |\mathsf{s}^+_{\mathbf{y}}(t) - \mathsf{s}^-_{\mathbf{y}}(t)| \ge \big(\#\big[\mathrm{L}^T_\zeta(\mathbf{y}) \cap [0,T)\big]\big)\,\zeta.$$

This implies that $\# L^T_\zeta(\mathbf{y}_j) \le \frac{S}{\zeta} + 1$ for every $j \in \mathbb{N}$. Then,

$$\mathcal{A}^{T,S}_\zeta(\mathbf{y}_j) = \sum_{i=1}^N [A(\mathsf{s}^+_{\mathbf{y}_j}(t^j_i)) - A(\mathsf{s}^-_{\mathbf{y}_j}(t^j_i))].$$

Now, up to a non-relabeled subsequence we have that there exist $(t_i)_{i=1}^N \subset [0,T]$ such that $t^j_i \to t_i$ as $j \to \infty$. In particular, it holds $|\mathsf{s}^+_{\mathbf{y}}(t_i) - \mathsf{s}^-_{\mathbf{y}}(t_i)| \ge \zeta$ for every $i = 1,\ldots,N$, so that $(t_i)_{i=1}^N \subset \mathrm{L}^T_\zeta(\mathbf{y})$. We notice that some of the t_i's may coincide. With a slight abuse of notation, we denote by t_k, for $k = 1,\ldots,M \le N$ the distinct limit points of t^j_i. By (F.17) we have that, whenever $t^j_i \to t_k$ as $j \to \infty$, then

$$A(\mathsf{s}^+_{\mathbf{y}}(t_k)) - A(\mathsf{s}^-_{\mathbf{y}}(t_k)) \ge \limsup_{j\to\infty} A(\mathsf{s}^+_{\mathbf{y}_j}(t^j_i)) - A(\mathsf{s}^-_{\mathbf{y}_j}(t^j_i)). \tag{F.19}$$

If we have that $t^j_i,\ldots,t^j_{i+\ell} \to t_k$ for some $\ell > 0$ and some $k = 1,\ldots,M$, then

$$\begin{aligned} A(\mathsf{s}^+_{\mathbf{y}}(t_k)) - A(\mathsf{s}^-_{\mathbf{y}}(t_k)) &\ge \limsup_{j\to\infty} A(\mathsf{s}^+_{\mathbf{y}_j}(t^j_{i+\ell})) - A(\mathsf{s}^-_{\mathbf{y}_j}(t^j_i)) \\ &\ge \limsup_{j\to\infty} \sum_{n=0}^{\ell} A(\mathsf{s}^+_{\mathbf{y}_j}(t^j_{i+n})) - A(\mathsf{s}^-_{\mathbf{y}_j}(t^j_{i+n})). \end{aligned} \tag{F.20}$$

Combining (F.19)–(F.20) we conclude that

$$\mathcal{A}^{T,S}_\zeta(\mathbf{y}) \ge \sum_{k=1}^M [A(\mathsf{s}^+_{\mathbf{y}}(t_k)) - A(\mathsf{s}^-_{\mathbf{y}}(t_k))] \ge \limsup_{j\to\infty} \mathcal{A}^{T,S}_\zeta(\mathbf{y}_j).$$

Conclusion: Clearly, we have that

$$\forall\, \mathbf{y} \in \mathrm{ArcLip}(\mathbf{I};\mathbb{R}^{d+1}) \;: \qquad \mathcal{A}(\mathbf{y}) = \lim_{T\uparrow\infty,\, S\uparrow\infty,\, \zeta\downarrow 0} \mathcal{A}^{T,S}_\zeta(\mathbf{y}).$$

Then, $\mathcal{A}$ is the pointwise limit of Borel mappings. Thus, (F.16) follows. This finishes the proof. □

Appendix G. Auxiliary measure-theoretic tools

Let $\mathbf{X}$ be a Polish metric space and $\mathcal{M}(\mathbf{X};\mathbb{R}^h)$ the space of $\mathbb{R}^h$-valued Borel measures on $\mathbf{X}$ with finite total variation, endowed with the weak* topology. Let $\boldsymbol{\Xi}$ also be a Polish space, and let $(\lambda_{\boldsymbol{\xi}})_{\boldsymbol{\xi}\in\boldsymbol{\Xi}} \subset \mathcal{M}(\mathbf{X};\mathbb{R}^h)$ be a Borel family. With any given $\mathsf{m} \in \mathcal{P}(\boldsymbol{\Xi})$ with

$$\int_{\boldsymbol{\Xi}} |\lambda_{\boldsymbol{\xi}}|(\mathbf{X})\,\mathrm{d}\mathsf{m}(\boldsymbol{\xi}) < +\infty$$

we may associate the measures

$$\Lambda^{\mathsf{m}} := \int_{\boldsymbol{\Xi}} \lambda_{\boldsymbol{\xi}}\,\mathrm{d}\mathsf{m}(\boldsymbol{\xi}) \in \mathcal{M}(\mathbf{X};\mathbb{R}^h) \quad \text{and} \quad \Upsilon^{\mathsf{m}} := \int_{\boldsymbol{\Xi}} |\lambda_{\boldsymbol{\xi}}|\,\mathrm{d}\mathsf{m}(\boldsymbol{\xi}) \in \mathcal{M}^+_{\mathrm{loc}}(\mathbf{X})\,.$$

Clearly, we have that $|\Lambda^{\mathsf{m}}| \leq \Upsilon^{\mathsf{m}}$. The following result provides a useful property of the 'generating' measures $(\lambda_{\boldsymbol{\xi}})_{\boldsymbol{\xi}\in\boldsymbol{\Xi}} \subset \mathcal{M}(\mathbf{X};\mathbb{R}^d)$ in the case when the measures $|\Lambda^{\mathsf{m}}|$ and Υ^{m} coincide.

Proposition G.1. *Let $\mathsf{f} : X \to \mathbb{R}^h$ be a Borel density of Λ^{m} w.r.t. Υ^{m}, with $|\mathsf{f}(x)| \leq 1$ for all $x \in \mathbf{X}$. Suppose that $|\Lambda^{\mathsf{m}}| = \Upsilon^{\mathsf{m}}$. Then*

$$\lambda_{\boldsymbol{\xi}} = \mathsf{f}|\lambda_{\boldsymbol{\xi}}| \qquad \textit{for } \mathsf{m}\textit{-a.e. } \boldsymbol{\xi} \in \boldsymbol{\Xi}\,. \tag{G.1}$$

Proof. Observe that, for a given measure $\lambda \in \mathcal{M}(\mathbf{X};\mathbb{R}^h)$, a Borel function $\boldsymbol{h}\colon X \to \mathbb{R}^h$ is the density of λ w.r.t. $|\lambda|$ if and only if

$$|\boldsymbol{h}| \leq 1 \ |\lambda|\text{-a.e. in } \mathbf{X}, \quad \text{and} \qquad \int_{\mathbf{X}} \boldsymbol{h}(x)\mathrm{d}\lambda(x) = |\lambda|(\mathbf{X})\,. \tag{G.2}$$

Now, from $|\Lambda^{\mathsf{m}}| = \Upsilon^{\mathsf{m}}$ we have that $|\mathsf{f}(x)| \equiv 1$ for Υ^{m}-a.e. $x \in \mathbf{X}$. Therefore, we have the following chain of identities

$$\int_{\boldsymbol{\Xi}} |\lambda_{\boldsymbol{\xi}}|(\mathbf{X})\,\mathrm{d}\mathsf{m}(\boldsymbol{\xi}) = \int_{\mathbf{X}} |\mathsf{f}(x)|^2\,\mathrm{d}\Upsilon^{\mathsf{m}}(x) \overset{(1)}{=} \int_{\mathbf{X}} \mathsf{f}(x)\mathrm{d}\Lambda^{\mathsf{m}}(x) \overset{(2)}{=} \int_{\boldsymbol{\Xi}} \left(\int_{\mathbf{X}} \mathsf{f}(x)\,\mathrm{d}\lambda_{\boldsymbol{\xi}}(x)\right)\mathrm{d}\mathsf{m}(\boldsymbol{\xi})$$

where (1) follows from the fact that $\Lambda^{\mathsf{m}} = \mathsf{f}\Upsilon^{\mathsf{m}}$ and (2) from the Fubini theorem. Then, we immediately conclude that

$$|\lambda_{\boldsymbol{\xi}}|(\mathbf{X}) = \int_{\mathbf{X}} \mathsf{f}(x)\,\mathrm{d}\lambda_{\boldsymbol{\xi}}(x) \qquad \text{for } \mathsf{m}\text{-a.e. } \boldsymbol{\xi} \in \boldsymbol{\Xi}\,.$$

Thus, on account of (G.2) we obtain (G.1).

□

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