ON VARIATIONAL SCHEME MODELING THE ANISOTROPIC SURFACE DIFFUSION WITH ELASTICITY IN THE PLANE

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Abstract. In this paper, we prove the existence of classical solutions for the anisotropic surface diffusion with elasticity in the plane using a minimizing movements scheme, provided that the initial set is sufficiently regular. This scheme is inspired by the one introduced by Cahn-Taylor [15] to modeling the surface diffusion. Moreover, we prove that this scheme converges to the global solution of the equation.

Keywords: Geometric evolutions; variational methods; minimizing movement scheme.

AMS subject classifications: 53E10; 53E40; 49Q20; 37E35; 74G65.

Contents

1. Introduction

In this paper, we investigate the existence of solutions to the anisotropic surface diffusion equation with elasticity in the plane, employing the minimizing movements scheme.

We provide a brief overview of the physical and mathematical motivation for this equation. In recent years, there has been growing interest in the physics literature in energy functionals that involve a competition between surface interface energy and elastic energy. This interest is driven by the study of interface morphologies influenced by such energies. From a mathematical perspective, the problem is formulated as the analysis of local or global minimizers of a free energy functional, given by the sum of elastic energy and surface energy (typically modeled via isotropic or anisotropic perimeter terms). The static version of this problem has been extensively investigated in both the physical and numerical literature. In the mathematical literature, several works address this topic: [7, 9, 11, 27, 32, 35] present results on existence, regularity, and stability for variational models describing equilibrium configurations in two dimensions, while [10, 17] provide results in three dimensions. As previously mentioned, our focus is on the dynamic and evolutionary counterpart of such energy models. Before introducing the differential equation we are studying, we recall the Einstein–Nernst equation, as our equation represents a special case of it. This equation describes the evolution of an interface driven by surface mass transport under the influence of a chemical potential μ. In particular, the surface flux of atoms is proportional to the tangential gradient of the chemical potential, and the divergence of this flux corresponds to the rate at which material is either removed from or deposited onto the interface. Throughout the evolution, the volume is conserved, as bulk mass transport can be neglected due to its occurring on a much faster timescale (see [44]). Thus, the evolution law is

$$V_t = \Delta_\tau \mu_t \text{ on } \partial E_t \tag{1.1}$$

where V_t is the normal velocity, Δ_τ is the Laplace–Beltrami operator on ∂E_t, and μ_t is the chemical potential. The chemical potential μ is defined as the first variation of the free-energy functional. The prototypical free-energy functional we consider is given by:

$$J(F) = \int_{\partial F} \varphi(\nu_F)\, d\mathcal{H}^1 + \frac{1}{2}\int_{\Omega\setminus F} Q(E(u_F))\, dx, \tag{1.2}$$

where $\Omega \subset \mathbb{R}^2$ denotes the planar region in which the phenomena of interest occur (e.g., the region occupied by the elastic body), and $F \subset \Omega$ (e.g., represents the void that has formed within the elastic body). As previously mentioned, the minimizers of the functional $F \to J(F)$ under the volume constraint $|F| = m$ can be used to describe the equilibrium shapes of voids in elastically stressed solids; see [46]. We now clarify the various terms appearing in equation (1.2). The function u_F represents the elastic equilibrium in $\Omega \setminus F$ subject to the boundary condition $u_F = w_0$ on $\partial\Omega$, i.e.,

$$u_F \in \operatorname{argmin}\left\{\int_{\Omega\setminus F} Q(E(u))\, dx \colon u \in H^1(\Omega\setminus F, \mathbb{R}^2),\ u|_{\partial\Omega} = w_0\right\}. \tag{1.3}$$

The function Q is the quadratic form defined by $Q(A) := \frac{1}{2}\mathbb{C}A : A$ for all 2×2-symmetric matrices A, where $\mathbb{C}$ is the elasticity tensor. The quantity $E(u_F)$ denotes the symmetric part of the gradient ∇u_F, given by $E(u_F) = \frac{\nabla u_F + (\nabla u_F)^t}{2}$. Finally $\varphi(\nu_F)$ is the anisotropic surface energy density evaluated at the outer unit normal ν_F to F. The anisotropy considered in this work is regular and strictly convex; i.e., φ is one-homogeneous, $\varphi \in C^\infty(\mathbb{R}^2 \setminus \{0\})$ and

$$\exists J > 0 : D^2\varphi(\nu)\xi \cdot \xi \geq J|\xi|^2 \quad \forall \nu \in \mathcal{S}^1,\ \xi \in \mathbb{R}^2 \text{ such that } \nu \perp \xi. \tag{1.4}$$

The existence and regularity of minimizers of the energy functional $F \mapsto J(F)$ under a volume constraint on F have been studied in various works; see [16, 26] for the two-dimensional case. A relaxation result valid in all dimensions, concerning a variant of the energy (1.2), is provided in [12].

The equation studied in this work is derived from the Einstein–Nernst equation (see (1.1)), under the assumption that μ_t corresponds to the first variation of the free energy (1.2). As a result, we obtain the following system:

$$\begin{cases} V_t = \Delta_\tau\big(\kappa^{\varphi}_{E_t} - Q(E(u_{E_t}))\big), \text{ on } \partial E_t \\ E_0 \text{ initial datum,} \\ u_{E_t} \in \operatorname{argmin}\left\{\displaystyle\int_{\Omega\setminus E_t} Q(E(u))\, dx \colon u \in H^1(\Omega\setminus E_t, \mathbb{R}^2),\ u|_{\partial\Omega} = w_0\right\}, \end{cases} \tag{1.5}$$

where $\kappa^{\varphi}_{E_t}$ denotes the anisotropic curvature of ∂E_t. The existence of classical solutions to equation (1.5) and the asymptotic stability of strictly stable stationary sets are studied in [30]. In [31], the authors investigate the existence and asymptotic stability of solutions in three dimensions for the isotropic surface diffusion equation with elasticity.

The equation (1.5) can be viewed as a nonlocal perturbation of the surface diffusion equation, where the nonlocality arises from the elasticity term. In dimension n, the surface diffusion equation takes the form

$$\begin{cases} V_t = \Delta_\tau H_{E_t} \text{ on } \partial E_t, \\ E_0 \text{ initial datum,} \end{cases} \tag{1.6}$$

where $H_{E_t}(x)$ denotes the mean curvature of the hypersurface ∂E_t at the point x. The short-time existence of classical solutions to (1.6) was first established in the planar case in [6, 24, 33], and later extended to all dimensions by Escher, Mayer, and Simonett [25], for initial sets with $C^{2,\alpha}$-regularity. Remarkably, the result in [25] also applies to immersed surfaces, and the authors prove both global existence and exponential convergence for initial sets sufficiently close to a sphere. In the flat torus $\mathbb{T}^n$, similar long-time existence and convergence results near stable critical sets have been obtained: for $n = 3$ in [1], and for $n \geq 4$ in [19] and [23]. The equation (1.6) can be interpreted as the H^{-1}-gradient flow of the area functional; see [14]. This naturally leads to the question of whether a variational approach based on minimizing movements can be used to model the flow. In 1994, Cahn and Taylor [15] proposed such a scheme to describe surface diffusion. The proposed scheme is as follows: given any initial bounded set of finite perimeter $E_0 \subset \mathbb{R}^n$ and a small time step $h > 0$, one defines $E^h_0 = E_0$ and then constructs E^h_{hk} for $k = 1, 2, \ldots$ inductively as a minimizer of the functional

$$P(F) + \frac{d_{H^{-1}}(F; E^h_{h(k-1)})^2}{2h}, \tag{1.7}$$

where

$$d_{H^{-1}}(F; E) := \sup_{\|\nabla_{\partial E} f\|_{L^2(\partial E)} \leq 1} \int_{\mathbb{R}^n} f(\pi_{\partial E}(x))(\chi_F(x) - \chi_E(x))\, dx.$$

Above, $P(F)$ denotes the De-Giorgi perimeter of the set F, $\nabla_{\partial E}$ denotes the tangential gradient, χ_E the characteristic function of E and $\pi_{\partial E}$ the projection on the boundary ∂E. Only recently, however, has it been rigorously shown in [18] that this scheme indeed models surface diffusion. In particular [18] proves that the scheme produces classical solutions and converges to the classical solution of (1.1) throughout the full interval of existence in dimension 3.

Our work focuses on the implementation of the minimizing movements scheme employed in [18] for the case of equation (1.5). In the literature, minimizing movement-type schemes have previously been used to model the H^{-1} gradient flow of a variant of the energy (1.2), although these schemes differ from the one in [18]. Specifically, the variant energy considered includes a curvature regularization term added to the original energy (1.2). It is worth noting that such variants have been extensively studied in physical and mathematical literature; see, for example, [5, 13, 21, 36, 37, 45, 46]. For instance, the authors of [28] study the H^{-1} gradient flow of the functional

$$F \to \int_{\partial F} \varphi(\nu_F)\, d\mathcal{H}^1 + \frac{1}{2}\int_{\Omega\setminus F} Q(E(u_F))\, dx + \frac{\varepsilon}{2}\int_{\partial F} \kappa_F^2\, d\mathcal{H}^1,$$

where $\varepsilon > 0$ and κ_F denotes the curvature. Their analysis focuses on periodic graph models describing the evolution of epitaxially strained elastic films in two dimensions. The corresponding flow is governed by the area-preserving evolution equation

$$\begin{cases} V_t = \Delta_\tau\big(\kappa^\varphi_{E_t} - Q(E(u_{E_t})) - \varepsilon(\Delta_\tau \kappa_{E_t} + \frac{1}{2}\kappa^3_{E_t})\big) \text{ on } \partial E_t, \\ E_0 \text{ initial datum,} \end{cases}$$

They prove a local existence result even when φ does not satisfy condition (1.4). As previously mentioned, their approach is based on the minimizing movements scheme, which differs from that of [18]. It is well defined only when the sets have boundaries that can be represented as the graph of a function, unlike the method in [18], which applies to general sets of finite perimeter. Moreover, their approach crucially depends on the curvature regularization term. In fact, all estimates derived in their work are ε-dependent and degenerate as $\varepsilon \to 0^+$, even when φ satisfies (1.4). A similar analysis was carried out in the three-dimensional setting in [29].

The main results of this work are the proof of the existence of a solution to equation (1.5), and the prove of the consistency of the minimizing movements scheme. We briefly outline the strategy of the proof. The first step is to introduce a constrained elastic equilibrium by modifying the original problem (1.3). To this end, we fix two constants $K_{el} > 0$ and $h > 0$, where h plays the role of a time discretization parameter, as in formula (1.7). Given a set $F \subset \Omega$, we consider the following constrained minimization problem:

$$\min \left\{ \int_{\Omega \setminus F} Q(E(u))\, dx \colon \|u\|_{C^{3,\frac{1}{4}}(\Omega)} \le K_{el},\ \|\nabla^4 u\|_{C^{0,\frac{1}{4}}(\Omega)} \le \frac{K_{el}}{h^{\frac{1}{4}}},\ u|_{\partial\Omega} = \omega_0 \right\}.$$

We denote by $u_F^{K_{el},h}$ a minimizer of this problem. Accordingly, the constrained elastic energy is defined as

$$\mathcal{E}(E(u_F^{K_{el},h})) := \int_{\Omega\setminus F} Q(E(u_F^{K_{el},h}))\, dx.$$

As a second step, we implemented the minimizing movement algorithm as described in [18]. Let $E_0 \Subset \Omega$ be an open, connected set of class C^5 , which serves as the initial datum. We fix a small parameter $\beta > 0$, set $E_0^{h,\beta} = E_0$ and define $E_{hk}^{h,\beta}$ as a minimizer of the following incremental minimization problem:

(1.8)

$$\inf \Big\{ \int_{\partial F} \varphi(\nu_F)\, d\mathcal{H}^1 + \mathcal{E}(E(u_F^{K_{el},h})) + \frac{d_{H^{-1}}(F; E^{h,\beta}_{h(k-1)})^2}{2h} : F \Delta E^{h,\beta}_{h(k-1)} \subset \mathcal{I}_\beta(\partial E^{h,\beta}_{h(k-1)}) \Big\},$$

where $\mathcal{I}_\beta(\Gamma)$ denotes the tubular neighborhood of a set $\Gamma \subset \mathbb{R}^2$ (see (2.2) for its definition). Due to the constraint condition $F\Delta E^{h,\beta}_{hk} \subset \mathcal{I}_\beta(\partial E^{h,\beta}_{h(k-1)})$, the existence of a minimizer for the above problem follows readily from the direct methods of the Calculus of Variations. Using quantitative geometric estimates we show that any minimizer $E^{\beta,h}_k$ of (1.8) satisfies

$$E^{h,\beta}_k \Delta E^{h,\beta}_{h(k-1)} \subset \mathcal{I}_{\frac{\beta}{2}}(\partial E^{h,\beta}_{h(k-1)}),$$

when h is sufficiently small, provided $\partial E^{h,\beta}_{h(k-1)}$ is sufficiently regular. This shows that the additional constraint in (1.8) it is not touched. We define $E^{h,\beta}_t = E^{h,\beta}_k$ for $t \in [kh, (k+1)h)$ for all $k \in \mathbb{N}$. The family $\{E^{h,\beta}_t\}_{t\ge 0}$ is called a constrained discrete flat flow with initial datum E_0 and time step h (see definition 3.7).

Our main result is the short time regularity and the consistency of the minimizing movement scheme defined above.

Theorem 1.1. *There exist constants $K_{el}, T, \beta_0, \sigma_1$ with the following property: for every $\beta < \beta_0$ there exists $h_0 > 0$ such that the family $\{E^{h,\beta}_t\}_{t\in[0,T]}$ satisfies*

$$\partial E^{h,\beta}_t = \{x + f^{h,\beta}(t,x)\nu_{E_0}(x) \colon x \in \partial E_0\},\ \|f^{h,\beta}\|_{H^4(\partial E_0)} \le C_0,\ \|f^{h,\beta}\|_{L^\infty(\partial E_0)} \le \sigma_1,$$

for all $t \in [0,T]$ *and* $0 < h \le h_0$. *The function* $f^{h,\beta}$ *converge in* $L^\infty([0,T], H^4(\partial E_0))$ *to a function* f^β, *such that the family* $(E^\beta(t))_{t\in[0,T]}$ *have the properties*

$$\partial E_t^\beta = \{x + f^\beta(t,x)\nu_{E_0}(x) : x \in \partial E_0\},$$

and $(E_t^\beta)_{t\in[0,T]}$ *is a solution to* (1.5) *with initial datum* E_0 *on the interval* $[0,T]$.

We briefly outline the strategy used to prove the main theorem. The key ingredients for establishing the main result—whose proof is presented in the final section of the paper (see Section 6)—are the preliminary estimates (see Section 4) and an iteration argument (see Section 5).

The main goal of the preliminary estimates is to show that the minimizers of each incremental problem (1.8) satisfy suitable regularity estimates. By regularity estimates, we mean that the minimizer $E_{hk}^{h,\beta}$ of problem (1.8) satisfies the following properties:

- the boundary of $E_{hk}^{h,\beta}$ can be written as a normal graph over the previous step, i.e.,

$$\partial E_{hk}^{h,\beta} = \{x + \psi_k(x)\nu_{E_{h(k-1)}^{h,\beta}}(x) : x \in \partial E_{hk}^{h,\beta}\}, \text{ with } \psi_k \in C^1(\partial E_{h(k-1)}^{h,\beta}), \tag{1.9}$$

- the boundary of $E_{hk}^{h,\beta}$ does not intersect the constraint, i.e.,

$$\partial E_{hk}^{h,\beta} \Subset \mathcal{I}_\beta(\partial E_{h(k-1)}^{h,\beta}), \tag{1.10}$$

- the function ψ_k satisfies the bounds

$$\|\psi_k\|_{L^2(\partial E_{hk}^{h,\beta})} \le Ch,\ \|\psi_k\|_{H^4(\partial E_{hk}^{h,\beta})} \le C,\ \|\kappa_{E_{hk}^{h,\beta}}\|_{H^3(\partial E_{hk}^{h,\beta})} \le Ch^{-\frac{1}{4}}, \tag{1.11}$$

where the constant C depends only on the H^2-norm of the curvature of $E_{h(k-1)}^{h,\beta}$. We explain here how to obtain the estimates for the case $k = 1$, since the subsequent steps will be proved by induction and iteration. The idea is to show that the minimizer $E_h^{h,\beta}$ is a Λ-minimizer of the φ-perimeter, for some constant Λ independent of h, but depending only on the H^2-norm of the curvature of E_0 (see Lemma 4.7). This allows us to apply a variant of the ε-regularity theorem for Λ-minimizer of the φ-perimeter, namely Lemma 4.3, from which we deduce the existence of a function $\psi_1 : \partial E_0 \to \mathbb{R}$ such that

$$\partial E_h^{h,\beta} = \{x + \psi_1(x)\nu_{E_0}(x) \colon x \in \partial E_0\}, \text{ with } \psi \in C^1(\partial E_0).$$

To carry out all of this, we need to show that the discrete velocity in H^{-1} is bounded, namely

$$\frac{d_{H^{-1}}(E_h^{h,\beta}, E_0)}{h} \le C,$$

where the constant C depends only on the H^2-norm of the curvature of E_0. This inequality follows from the minimality of $E_h^{h,\beta}$ and the regularity of E_0; see Lemmas 4.6 and 4.7. Furthermore, using Lemma 4.6, we obtain that $\partial E_h^{h,\beta} \Subset \mathcal{I}_\beta(\partial E_0)$, so the constraint $\partial\mathcal{I}_\beta(\partial E_0)$ is never touched. Thanks to these results, we can compute the first variation of the energy

$$F \to \int_{\partial F} \varphi(\nu_F)\, d\mathcal{H}^1 + \mathcal{E}(E(u_F^{K_{el},h})) + \frac{d_{H^{-1}}(F; E_0)^2}{2h}$$

at the minimizer $E_h^{h,\beta}$. This leads to a differential equation for the unknown function ψ_1; see equation (4.58). Using the Euler–Lagrange equation, we also obtain another estimate for the discrete velocity, this time in L^2:

$$\frac{\|\psi_1\|_{L^2(\partial E_0)}}{h} \le C,$$

where the constant C depends only on the H^2-norm of the curvature of E_0. Moreover, we obtain a bound in the H^4-norm, namely $\|\psi_1\|_{H^4(\partial E_0)} \leq C$, while the curvature satisfies $\|\kappa_{E_h^{h,\beta}}\|_{H^3(\partial E_0)} \leq Ch^{-\frac{1}{4}}$, where the constant C depends only on the H^2-norm of the curvature of E_0. All of this is proved in Theorem 4.1.

In Section 5, the goal is to establish a connection between the steps $k-1, k, k+1$ in such a way that the validity of formulas (1.9), (1.10), and (1.11) can be ensured for every admissible k. The main idea is to relate the Euler–Lagrange equation satisfied by the set $E^{h,\beta}_{h(k+1)}$ with the one satisfied by the set $E^{h,\beta}_{hk}$. To achieve this, we use the expansion of the φ-curvature given in formula (4.47). Indeed, the Euler–Lagrange equation for $E^{h,\beta}_{h(k+1)}$ involves the φ-curvature of the set $E^{h,\beta}_{hk}$, which also appears in the Euler–Lagrange equation satisfied by $E^{h,\beta}_{hk}$ itself. Therefore, by substituting the latter equation into the former, we derive the desired iteration—see Lemma 5.1 and Proposition 5.2.

In Section 6, we provide the proof of Theorem 1.1. This proof follows from Theorems 6.2 and 6.4. In the first of these, we show that for any fixed $K_{el} > 0$, there exists a time $T > 0$ such that the family $\{E^{h,\beta}_t\}_{t\in[0,T]}$ satisfies

$$\partial E^{h,\beta}_t = \{x + f^{h,\beta}(t,x)\nu_{E_0}(x) \colon x \in \partial E_0\},\ \|f^{h,\beta}\|_{H^4(\partial E_0)} \leq C_0,\ \|f^{h,\beta}\|_{L^\infty(\partial E_0)} \leq \sigma_1,$$

for all $t \in [0,T]$. The function $f^{h,\beta}$ converge as $h \to 0^+$ in $L^\infty([0,T], H^4(\partial E_0))$ to a function f^β, with $f^\beta \in \mathrm{Lip}([0,T], L^2(\partial E_0))$, such that the family $(E^\beta(t))_{t\in[0,T]}$ satisfies

$$\partial E^\beta_t = \{x + f^\beta(t,x)\nu_{E_0}(x) : x \in \partial E_0\},$$

and

$$\|f^\beta(t,\cdot)\|_{C^{3,\frac{1}{4}}(\partial E_0)} \leq Ct^{\frac{1}{21}} \tag{1.12}$$

where $C = C(K_{el})$. Formula (1.12) will be sufficient to prove the existence of classical solutions. Indeed, by fixing a sufficiently large K_{el}, one can show that there exists a small time $T > 0$ such that the constraint K_{el} is not active for the minimizer of the constrained elasticity problem associated with E^β_t, for every $t \in [0,T]$. Therefore, thanks to the regularity of ∂E^β_t, this minimizer coincides with the one for the unconstrained elasticity problem (1.3). This will allow us to prove that the family $\{E^\beta_t\}_{t\in[0,T]}$ satisfies the equation (1.5).

In the final section, namely Section 7, we prove that the minimizing movements scheme converges to the solution of problem (1.5) throughout the entire interval of existence.

The paper is organized as follows. In Section 2, we introduce the notation used throughout the paper, along with some useful formulas, the functional spaces involved, and interpolation inequalities. In Section 3, we define the function $d_{H^{-1}}(F,E)$, discuss some of its properties including the computation of its first variation, introduce both the free and constrained elasticity problems, and finally present the minimizing movement scheme used in the analysis. In Section 4, we prove the Λ-minimality property for the minimizer of the incremental problem and establish a regularity estimate for the heightfunction. Section 5 is devoted to the proof of the iteration argument. In Section 6, we prove Theorem 1.1. Finally, in Section 7, we prove the convergence to the global solution of equation (1.5).

2. Notation of the paper and useful formulas

In this paper, we work in the 2-dimensional Euclidean space $\mathbb{R}^2$. We denote with $\{e_1, e_2\}$ the canonical basis of $\mathbb{R}^2$, by $|\cdot|$ the Euclidean norm, and by $\cdot$ the inner product in $\mathbb{R}^2$. Let $r > 0$ we set $B_r(x) = \{y \in \mathbb{R}^2 : |x-y| < r\}$ when $x = 0$, we simply write $B_r := B_r(0)$. For every $A \subset \mathbb{R}^2$ we denote by $\mathrm{cl}(A)$ $(\mathrm{int}(A))$ its topological closure (respectively its topological interior) with respect to the Euclidean topology. Given $A \subset \mathbb{R}^2$ and $x \in \mathbb{R}^2$, we denote by $\mathrm{dist}(x, A)$ the distance between x and A. The Lebesgue measure of a Borel set $A \subset \mathbb{R}^2$ is denoted by $|A|$. We denote by $\mathcal{H}^1$ the 1-dimensional Hausdorff measure and by $\mathrm{dist}_{\mathcal{H}}$ the Hausdorff distance between sets. In what follows we denote with φ a regular strictly convex norm; i.e., $\varphi \in C^\infty(\mathbb{R}^2 \setminus \{0\})$ and

$$\exists J > 0 : D^2\varphi(\nu)\xi \cdot \xi \geq J|\xi|^2 \quad \forall \nu \in \mathcal{S}^1,\, \xi \in \mathbb{R}^2 \text{ such that } \nu \perp \xi.$$

We denote by m_φ, M_φ the constants

$$m_\varphi := \min_{|\nu|=1} \varphi(\nu), \quad M_\varphi := \max_{|\nu|=1} \varphi(\nu). \tag{2.1}$$

The dual norm φ^0 is defined as $\varphi^0(\xi) = \sup_{\eta \in \mathbb{R}^2 \setminus \{0\}} \frac{\xi \cdot \eta}{\varphi(\eta)}$. Given $a, b \in \mathbb{R}^2$, we denote by $a \otimes b : \mathbb{R}^2 \to \mathbb{R}^2$ the linear map defined as $a \otimes b(x) := (x \cdot b)a$. We denote by $\mathbb{R}^{2\times 2}$ the space of the 2×2 matrices. Given $A \subset \mathbb{R}^2$ we denote by $A^c = \mathbb{R}^2 \setminus A$. Given $P, C \in \mathbb{R}^{2\times 2}$, we set $P : C = \sum_{i,j=1}^2 p_{ij}c_{ij}$. The standard gradient in $\mathbb{R}^2$ is denoted by ∇ and the Laplace operator in $\mathbb{R}^2$ is denoted by $\Delta_{\mathbb{R}^2}$. Throughout the paper, we write $C(*, \cdots, *)$ to indicate a generic positive constant that depends only on $*, \cdots, *$ and that may change from line to line.

2.1. **Regular sets and useful formulas.** Let $E \subset \mathbb{R}^2$ be a bounded open set of class C^2. The derivative of a function f or of a vector field X along ∂E is denoted by $\partial_\tau f$ and $\partial_\tau X$, respectively. In cases of ambiguity, we use $\partial_{\partial E} f$ and $\partial_{\partial E} X$. The Laplace–Beltrami operator on ∂E is denoted by ∂_τ^2 (or Δ_τ) and the tangential divergence on ∂E is denoted by div_τ. If necessary for clarity, we also write these as $\partial^2_{\partial E}$ or $(\Delta_{\partial E})$ and $\mathrm{div}_{\partial E}$. We recall that the second fundamental form $B_E : \partial E \mapsto \mathbb{R}^{2\times 2}$ and the curvature $\kappa_E : \partial E \to \mathbb{R}$ are given by

$$\kappa_E = \mathrm{div}_\tau \nu_E \qquad B_E = \kappa_E\, \tau_E \otimes \tau_E,$$

where $\nu_E : \partial E \to \mathbb{R}^2$ is the outer normal vector field on ∂E and $\tau_E : \partial E \to \mathbb{R}^2$ is the tangent vector field on ∂E, obtained by rotating ν_E by $\frac{\pi}{2}$ clockwise. We denote the tangential gradient on ∂E by ∇_τ (or $\nabla_{\partial E}$), so that $\nabla_\tau f = \partial_\tau f \tau_E = \nabla f - (\nabla f \cdot \nu_E)\nu_E$ for a function f. Let $A \subset \mathbb{R}^2$ and given $\delta > 0$, define the tubular neighborhood

$$\mathcal{I}_\delta(A) := \{x \in \mathbb{R}^2 : \mathrm{dist}(x, A) < \delta\}. \tag{2.2}$$

We define the signed distance function to ∂E by

$$d_E(x) := \begin{cases} \mathrm{dist}(x, \partial E) & \text{for } x \in \mathbb{R}^2 \setminus E\,, \\ -\mathrm{dist}(x, \partial E) & \text{for } x \in E\,. \end{cases}$$

Let $E \subset \mathbb{R}^2$ be a open and bounded set of class C^2. We define

$$\sigma_E := \frac{1}{2\|\kappa_E\|_{L^\infty(\partial E)}},$$

It is known (see [34, Chapter 14.6]) that $d_E \in C^2(\mathcal{I}_{\sigma_E}(\partial E))$. The projection onto ∂E is define for all $x \in \mathbb{R}^2$ where exists $\nabla d_E(x)$, and is denoted by $\pi_E(x)$. For all $x \in I_{\sigma_E}(\partial E)$ the projection satisfies

$$x = \pi_E(x) + d_E(x)\nabla d_E(x).$$

As shown in [38, formula (2.31)], for all $x \in \mathcal{I}_{\sigma_E}(\partial E)$, it holds that

$$\begin{aligned} &\nabla \pi_{\partial E}(x) \\ &\quad = I - \nu_E \circ \pi_{\partial E}(x) \otimes \nu_E \circ \pi_{\partial E}(x) - d_E(x)(B_E \circ \pi_{\partial E}(x))(I + d_E(x)B_E \circ \pi_{\partial E}(x))^{-1}. \end{aligned} \tag{2.3}$$

Definition 2.1. *Let $E \subset \mathbb{R}^2$ be an open bounded set of class C^2 and let $0 < \sigma \le \sigma_E$. Let $F \subset \mathbb{R}^2$ be another open bounded set. We say that ∂F is a normal graph over ∂E if there exists a function $\psi : \partial E \to [-\sigma, \sigma]$, called the height function, such that*

$$\partial F = \{x + \psi(x)\nu_E(x) \colon x \in \partial E\} \text{ and } E\Delta F \subset \mathrm{cl}\big(I_\sigma(\partial E)\big).$$

Let E, F as in the above definition with $\partial F = \{x + \psi(x)\nu_E(x) \colon x \in \partial E\}$, and let $f \in C^1(\partial E)$. Then, for all $y \in \partial F$,

$$\nabla_{\partial F}(f \circ \pi_{\partial E})(y) = \nabla_{\partial E} f(\pi_{\partial E}(y))\nabla_{\partial F}\pi_{\partial E}(y),$$

where

$$\nabla_{\partial F}\pi_{\partial E}(y) = \nabla\pi_{\partial E}(y) - \nabla\pi_{\partial E}(y)\nu_F(y) \otimes \nu_F(y). \tag{2.4}$$

If $\psi \in C^1(\partial E)$, then the following formulas hold (see [30, formulas (2.5), (2.6), (2.7)]). For $x \in \partial E$:

$$\tau_F(x + \psi(x)\nu_F(x)) = \frac{(1 + \psi(x)\kappa_E(x))\tau_E(x) + \partial_\tau\psi(x)\nu_E(x)}{\sqrt{(1 + \psi(x)\kappa_E(x))^2 + |\nabla_\tau\psi(x)|^2}}$$

and

$$\nu_F(x + \psi(x)\nu_E(x)) = \frac{-\nabla_\tau\psi(x) + (1 + \psi(x)\kappa_E(x))\nu_E(x)}{\sqrt{(1 + \psi(x)\kappa_E(x))^2 + |\nabla_\tau\psi(x)|^2}}. \tag{2.5}$$

If $\psi \in C^2(\partial E)$, then the curvature expands as (see [30, formulas (2.7)], [18, Lemma 2.5]):

$$\kappa_F(x + \psi(x)\nu_E(x)) = -\Delta_\tau\psi(x) + \kappa_E(x) + R_0(x), \; x \in \partial E \tag{2.6}$$

where the error term R_0 is given by

$$R_0 = a_0(\psi, \partial_\tau\psi, \kappa_E) + a_1(\psi\kappa_E, \partial_\tau\psi)\Delta_\tau\psi + a_2(\psi\kappa_E, \partial_\tau\psi)\partial_\tau(\psi\kappa_E) \tag{2.7}$$

with a_0, a_1, a_2 smooth functions satisfying $a_0(0, 0, \cdot) = a_1(0, 0) = a_2(0, 0) = 0$.

Lemma 2.2. *Let $E \subset \mathbb{R}^2$ be a open and bounded set of class C^2 and let $F \subset \mathbb{R}^2$ be open and bounded set of class C^1 such that ∂F is a normal graph over ∂E given by $\partial F = \{x + \psi(x)\nu_E(x) \colon x \in \partial E\}$. Let $g \in C^1(\partial F)$, and define $\hat{g} : \partial E \to \mathbb{R}$ by $\hat{g}(x) = g(x + \psi(x)\nu_E(x))$. Then,*

$$\int_{\partial F} |\nabla_{\partial F} g|^2 \, d\mathcal{H}^1 = \int_{\partial E} \frac{|\nabla_{\partial E}\hat{g}|^2}{\sqrt{(1 + \psi\kappa_E)^2 + |\nabla_{\partial E}\psi|^2}} \, d\mathcal{H}^1. \tag{2.8}$$

Proof. Let $x \in \partial E$ and let $y \in \partial F$ such that $\pi_{\partial E}(y) = x$. Hence $d_E(y) = \psi(x)$. Without loss of generality, we may assume that $\tau_E(x) = (1, 0)$ and $\nu_E(x) = (0, 1)$, and we write $\nu_F(y) = (\nu_1, \nu_2)$. Using formula (2.3) we get

$$\begin{aligned}\nabla\pi_{\partial E}(y) &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} - d_E(y)\kappa_E(x) \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 + d_E(y)\kappa_E(x) & 0 \\ 0 & 0 \end{bmatrix}^{-1} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - \frac{\psi(x)\kappa_E(x)}{1 + \psi(x)\kappa_E(x)} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{1+\psi(x)\kappa_E(x)} & 0 \\ 0 & 0 \end{bmatrix}.\end{aligned}$$

Therefore, by (2.4), we have

$$\nabla_{\partial F}\pi_{\partial E}(y) = \begin{bmatrix} \frac{1-\nu_1\nu_1}{1+\psi(x)\kappa_E(x)} & -\frac{\nu_1\nu_2}{1+\psi(x)\kappa_E(x)} \\ 0 & 0 \end{bmatrix}.$$

Using the formula above, we deduce that

$$|\tau_E(x)\nabla_{\partial F}\pi_{\partial E}(y)|^2 = \frac{|\nu_E(x) \cdot \nu_F(y)|^2}{(1 + d_E(y)\kappa_E(x))^2} \quad \forall (x, y) \in \partial E \times \partial F : \, x = \pi_{\partial E}(y).$$

Hence, from the previous expression, we obtain

$$
\begin{aligned}
\int_{\partial F}|\nabla_{\partial F} g|^2 d\mathcal{H}^1 &= \int_{\partial F}|\nabla_{\partial F}(\hat{g}\circ\pi_{\partial E})|^2 d\mathcal{H}^1\\
&= \int_{\partial F}|\nabla_{\partial E}\hat{g}(\pi_{\partial E}(y))\nabla_{\partial F}\pi_{\partial E}(y)|^2 d\mathcal{H}^1_y \qquad (2.9)\\
&= \int_{\partial F}\frac{|\nabla_{\partial E}\hat{g}(\pi_{\partial E}(x))|^2|\nu_E(\pi_{\partial E}(x))\cdot\nu_F(x)|^2}{|1+d_E(x)\kappa_E(\pi_{\partial E}(x))|^2}d\mathcal{H}^1_x.
\end{aligned}
$$

Using formula (2.5), we have that for all $y\in\partial F$

$$
\nu_E(\pi_{\partial E}(y))\cdot\nu_F(y)=\frac{1+d_E(y)\kappa_E(\pi_{\partial E}(y))}{\sqrt{(1+\psi(\pi_{\partial E}(y)))\kappa_E(\pi_{\partial E}(y)))^2+|(\nabla_{\partial E}\psi)(\pi_{\partial E}(y))|^2}}. \tag{2.10}
$$

Let us define $\Psi:\partial E\to\partial F$ as $\Psi(x):=x+\psi(x)\nu_E(x)$. Recalling that the tangential Jacobian of Ψ is

$$
J_\tau\Psi(x)=\sqrt{(1+\psi(x)\kappa_E(x))^2+|(\nabla_{\partial E}\psi)(x)|^2},
$$

we deduce formula (2.8), from (2.9) and (2.10), indeed

$$
\begin{aligned}
\int_{\partial F}|\nabla_{\partial F} g|^2 d\mathcal{H}^1 &= \int_{\partial F}\frac{|\nabla_{\partial E}\hat{g}(\pi_{\partial E}(y))|^2|\nu_E(\pi_{\partial E}(y))\cdot\nu_F(y)|^2}{|1+d_E(y)\kappa_E(\pi_{\partial E}(y))|^2}d\mathcal{H}^1_y\\
&= \int_{\Psi(\partial E)}\frac{|\nabla_{\partial E}\hat{g}(\pi_{\partial E}(y))|^2}{(1+\psi(\pi_{\partial E}(y)))\kappa_E(\pi_{\partial E}(y)))^2+|(\nabla_{\partial E}\psi)(\pi_{\partial E}(y))|^2}d\mathcal{H}^1_y\\
&= \int_{\partial E}\frac{|\nabla_{\partial E}\hat{g}(x)|^2}{\sqrt{(1+\psi(x))\kappa_E(x)^2+|(\nabla_{\partial E}\psi)(x)|^2}}d\mathcal{H}^1_x.
\end{aligned}
$$

□

2.2. **Spaces of functions.** In what follows, we denote by $\Omega\subset\mathbb{R}^2$ an open and bounded set of class C^5. Let $E_0\Subset\Omega$ be open and connected set of class C^5 such that $|E_0|=1$. We denote by σ_0 a constant such that

$$
\sigma_0<\min\{\sigma_{E_0},\mathrm{dist}_{\mathcal{H}}(\partial E_0,\partial\Omega)\}.
$$

Given $1\le k\le 5$, $\alpha\in[0,1]$, and $K>0$ we define

$$
\begin{aligned}
\mathfrak{C}^{k,\alpha}_{K,\sigma_0}(E_0):=\big\{E\subset\mathbb{R}^2:\ & E\Delta E_0\subset\mathrm{cl}\big(\mathcal{I}_{\sigma_0}(\partial E_0)\big),\ \partial E=\{y+\varphi_E(y)\nu_{E_0}(y)\colon y\in\partial E_0\},\\
&\|\varphi_E\|_{L^\infty(\partial E_0)}\le\sigma_0,\ \|\varphi_E\|_{C^{k,\alpha}(\partial E_0)}\le K\big\}.
\end{aligned}
$$

For every $k\in\{1,\cdots,5\}$, we define the set $\mathfrak{H}^{k,\alpha}_{K,\sigma_0}(E_0)$ in the same way of $\mathfrak{C}^{k,\alpha}_{K,\sigma_0}(E_0)$ by replacing $\|\varphi_E\|_{C^{k,\alpha}(\partial E_0)}$ with $\|\varphi_E\|_{H^k(\partial E_0)}$. Let $\{E_n\}_{n\in\mathbb{N}}$ and E be such that $E_n\in\mathfrak{C}^{k,\alpha}_{K,\sigma_0}(E_0)$ (respectively $\mathfrak{H}^{k,\alpha}_K(E_0)$) for all $n\in\mathbb{N}$. We say that $E_n\to E$ in $\mathfrak{C}^{k,\alpha}_{K,\sigma_0}(E_0)$ (respectively in $\mathfrak{H}^{k,\alpha}_{K,\sigma_0}(E_0)$) if φ_{E_n} is uniformly bounded by σ_0 in $L^\infty(\partial E_0)$ and it is a Cauchy sequence in $C^{k,\alpha}(\partial E_0)$ (respectively $H^k(\partial E_0)$).
Let $F\subset\Omega$ be an open set. Given $k\in\mathbb{N}$, $\alpha\in[0,1]$, and $M>0$ we define

$$
\mathfrak{C}^{k,\alpha}_M(F,\mathbb{R}^2):=\left\{f\in C^{k,\alpha}(F,\mathbb{R}^2)\colon\|f\|_{C^{k,\alpha}(F)}\le M\right\}.
$$

2.3. **Sets of finite φ-perimeter and anisotropic curvature.** Let $E\subset\mathbb{R}^2$ be a Borel set. We define the De Giorgi φ-perimeter of E as

$$
P_\varphi(E)=\sup\left\{\int_E\mathrm{div}X\,dx\colon X\in C^1_c(\mathbb{R}^2,\mathbb{R}^2),\ \sup_{x\in\mathbb{R}^2}\varphi^0(X)\le 1\right\}.
$$

When $\varphi(\cdot) = |\cdot|$ (the Euclidean norm), we write $P(E)$ instead of $P_{|\,|}(\cdot)$. We say that a Borel set $E \subset \mathbb{R}^2$ has finite perimeter if $P(E) < +\infty$. Given the assumptions we made on the function φ, it is easy to verify that

$$P_\varphi(E) < +\infty \iff P(E) < +\infty.$$

For every set $E \subset \mathbb{R}^2$ with finite perimeter, the set $\partial^* E \subset \mathbb{R}^2$ identifies the reduced boundary of E and the Borel measurable map $\nu_E : \partial^* E \to \mathbb{R}^2$ the measure theoretic outer normal vector field (see, for instance, [4, Definition 3.53] for the definitions of these objects). By De Giorgi's structure theorem (see, for instance, [4, Theorem 3.59], [42, Theorem 15.19]), for every set $E \subset \mathbb{R}^2$ of finite perimeter, we have that

$$P(E) = \mathcal{H}^1(\partial^* E).$$

Let $E \subset \mathbb{R}^2$ be a set of finite perimeter. A straightforward computation gives

$$P_\varphi(E) := \int_{\partial^* E} \varphi(\nu_E)\, d\mathcal{H}^1.$$

Now we recall the well-known first variation formula for the anisotropic perimeter. Let $E \subset \mathbb{R}^2$ of class C^2. For any vector field $X \in C^1_c(\mathbb{R}^2, \mathbb{R}^2)$, let $(\Phi(t,\cdot))_{t\in(-\varepsilon,\varepsilon)}$ be the unique solution of the Cauchy problem

$$\begin{cases} \frac{\partial}{\partial t}\Phi(t,x) = X \circ \Phi(t,x) & \forall x \in \mathbb{R}^2 \\ \Phi(0,x) = x \quad \forall x \in \mathbb{R}^2. \end{cases}$$

Then we have

$$\frac{d}{dt}\bigg|_{t=0} \int_{\partial\Phi(t,E)} \varphi(\nu_{\Phi(t,E)})\, d\mathcal{H}^1 = \int_{\partial E} \kappa^\varphi_E X \cdot \nu_E \, d\mathcal{H}^1 \tag{2.11}$$

where the anisotropic curvature κ^φ_E of ∂E is given by

$$\kappa^\varphi_E := \mathrm{div}_{\partial E}(\nabla\varphi(\nu_E))$$

and can also be written as

$$\begin{aligned} \kappa^\varphi_E &= \mathrm{div}_{\partial E}(\nabla\varphi(\nu_E)) = \nabla(\nabla\varphi(\nu_E))\tau_E \cdot \tau_E - \nabla(\nabla\varphi(\nu_E))\nu_E \cdot \nu_E \\ &= (\nabla^2\varphi(\nu_E)\tau_E \cdot \tau_E)\kappa_E := g(\nu_E)\kappa_E \end{aligned} \tag{2.12}$$

where

$$g \in C^\infty(\mathbb{R}^2 \setminus \{0\}),\ C_g = \min_{|\nu|=1} g(\nu) > 0. \tag{2.13}$$

We recall an anisotropic version of the Gauss–Bonnet theorem for curves (see [39] for a proof).

Lemma 2.3. *Let $\varphi \in C^\infty(\mathbb{R}^n \setminus \{0\})$ be a regular, strictly convex norm. There exists a constant $C_\varphi > 0$, depending only on φ, such that for all open, bounded sets $E \subset \mathbb{R}^2$ of class C^2, the following holds:*

$$\int_{\partial E} \kappa^\varphi_E(x)\varphi(\nu_E(x))\, d\mathcal{H}^1_x = C_\varphi.$$

2.4. **Interpolation inequality.** We recall the interpolation inequalities involving Sobolev norms on embedded surfaces. We use the result from [43, Proposition 6.5] (see also [23, Proposition 4.3]).

Proposition 2.4. *Let $E \in \mathfrak{C}^m_{K,\sigma_0}(E_0)$ for some $m \geq 2$. Then for integers $0 \leq k < l \leq m$ and numbers $p \in [1,\infty)$, $q, r \in [1,\infty]$ there is $\theta \in [k/l, 1]$ such that for every function f of class C^l on ∂E it holds*

$$\|\partial^k_\tau f\|_{L^p(\partial E)} \leq C\|f\|^\theta_{W^{l,q}(\partial E)}\|f\|^{1-\theta}_{L^r(\partial E)}$$

for a constant $C = C(k, l, p, q, r, \theta, C_0)$, provided that the following condition is satisfied

$$\frac{1}{p} = k + \theta\left(\frac{1}{q} - l\right) + \frac{1}{r}(1-\theta).$$

Moreover, if $f : \partial E \to \mathbb{R}$ is a smooth function with $\int_{\partial E} f \, d\mathcal{H}^1 = 0$ the above inequality can be written as

$$\|\partial_\tau^k f\|_{L^p(\partial E)} \le C \|\partial_\tau^l f\|^\theta_{L^q(\partial E)} \|f\|^{1-\theta}_{L^r(\partial E)}.$$

If $E \in \mathfrak{C}^{k,\alpha}_{K,\sigma_0}(E_0)$ for some $1 \le k \le 5$ and $\alpha \in [0,1]$, then the classical interpolation inequality in Hölder norms holds, i.e., for $0 < \beta < \alpha \le 1$ and $0 \le l \le m \le k$ it holds

$$\|f\|_{C^{l,\beta}(\partial E)} \le C \|f\|^\theta_{C^{m,\alpha}(\partial E)} \|f\|^{1-\theta}_{C^0(\partial E)} \quad \theta = \frac{l+\beta}{m+\alpha}, \tag{2.14}$$

where C depend on K, l, m, α, β. This result follows from the Euclidean case; see, for example [41, Example 1.9]. The interpolation inequality in Proposition 2.4 implies the following useful estimate. The proof is standard, and we refer to [38, Proposition 2.3]. Note that the argument is similar to that used in the Euclidean case; see [48, Proposition 3.7]. We denote the sum of the components of an index vector $\alpha \in \mathbb{N}^l$ by

$$|\alpha| = \alpha_1 + \cdots + \alpha_l.$$

Lemma 2.5. *Let $E \in \mathfrak{C}^m_{K,\sigma_0}(E_0)$ for some $m \ge 2$ and let $f_1, \cdots, f_l$ be function of class C^m. Then for an index vector $\alpha \in \mathbb{N}^l$ with norm $|\alpha| \le k \le m$ it holds*

$$\| |\partial_\tau^{\alpha_1} f_1| \cdots |\partial_\tau^{\alpha_l} f_l| \|_{L^2(\partial E)} \le C(K) \sum_{\sigma \in S_l} \|f_{\sigma(1)}\|_{L^\infty(\partial E)} \cdots \|f_{\sigma(l-1)}\|_{L^\infty(\partial E)} \|f_{\sigma(l)}\|_{H^k(\partial E)}$$

where S_l is the group of permutation of l object. In particular,

$$\|\partial_\tau^k (f_1 f_2)\|_{L^2(\partial E)} \le C(K)\big[\|f_1\|_{L^\infty(\partial E)} \|f_2\|_{H^k(\partial E)} + \|f_2\|_{L^\infty(\partial E)} \|f_1\|_{H^k(\partial E)}\big].$$

3. Setting of the problem

3.1. Pseudo-pseudo-H^{-1} metric.

In this subsection, we recall the definition and some basic properties of the pseudo-pseudo-H^{-1} distance introduced in [15] to model surface diffusion.

Definition 3.1 (Pseudo-pseudo-H^{-1} metric)**.** *Let $E \subset \mathbb{R}^2$ be a set of finite perimeter, and let $F \subset \mathbb{R}^2$ be a measurable set. We define the function $d_{H^{-1}}(F, E)$ as*

$$d_{H^{-1}}(F, E) := \sup_{\|\nabla f\|_{L^2(\partial^* E)} \le 1} \int_{\mathbb{R}^2} f \circ \pi_{\partial^* E}(x)(\chi_F(x) - \chi_E(x)) \, dx. \tag{3.1}$$

Remark 3.2. *Let $E \subset \mathbb{R}^2$ be a set of finite perimeter $|E| < +\infty$, and let $F \subset \mathbb{R}^2$ be a measurable set. We observe that if $|E| \ne |F|$ then $d_{H^{-1}}(F, E) = +\infty$. Indeed, for every $a \in \mathbb{R}$ we define $f : \partial E \to \mathbb{R}$ by $f(x) = a$. Then, by* (3.1)*, we have*

$$d_{H^{-1}}(F, E) \ge \sup_{a \in \mathbb{R}} a(|F| - |E|) = +\infty.$$

Lemma 3.3. *Let $E \subset \mathbb{R}^2$ be a open bounded set of class C^2. Fix $\sigma > 0$ be such that $\sigma < \sigma_E$. Let $F \subset \mathbb{R}^2$ such that $|F| = |E|$ and $F \Delta E \subset \mathrm{cl}(\mathcal{I}_\sigma(\partial E))$. We define*

$$\xi_{F,E} : \partial E \to \mathbb{R} \quad \xi_{F,E}(x) := \int_{-\sigma}^{\sigma} (\chi_F(x + t\nu_E(x)) - \chi_E(x + t\nu_E(x)))(1 + t\kappa_E(x)) \, dt. \tag{3.2}$$

Then,

$$\int_{\partial E} \xi_{F,E} \, d\mathcal{H}^1 = 0, \quad d_{H^{-1}}(F, E) = \|\xi_{F,E}\|_{H^{-1}(\partial E)}.$$

Moreover, if ∂F is a normal graph respect ∂E, i.e., $\partial F = \{x + \psi(x)\nu_E(x)\colon x \in \partial E\}$, then we have

$$\xi_{F,E} = \psi + \kappa_E \frac{\psi^2}{2}, \quad d_{H^{-1}}(F,E) = \|\psi + \kappa_E \frac{\psi^2}{2}\|_{H^{-1}(\partial E)}. \tag{3.3}$$

Proof. Let $t \in [-\sigma, \sigma]$, we define

$$\Psi_t : \partial E \to \{x\colon d_E(x) = t\}, \ \Psi_t(x) := x + t\nu_E(x).$$

We have that $J_\tau \Psi_t(x) = 1 + t\kappa_E(x)$. Let $f \in H^1(\partial E)$. Using the coarea formula and a change of variables, we get

$$\begin{aligned}\int_{\mathbb{R}^2} f \circ \pi_{\partial E}(\chi_F - \chi_E)\, dx &= \int_{-\sigma}^{\sigma} \int_{\{x\colon d_E(x)=t\}} f \circ \pi_{\partial E}(x)(\chi_F(x) - \chi_E(x))\, d\mathcal{H}^1_x\, dt \\ &= \int_{-\sigma}^{\sigma} \int_{\Psi_t(\partial E)} f \circ \pi_{\partial E}(x)(\chi_F(x) - \chi_E(x))\, d\mathcal{H}^1_x\, dt \\ &= \int_{-\sigma}^{\sigma} \int_{\partial E} f(y)(\chi_F(y + t\nu_E(y)) - \chi_E(y + t\nu_E(y))J_\tau \Psi_t(y)\, d\mathcal{H}^1_y\, dt.\end{aligned} \tag{3.4}$$

By (3.2) and (3.4) we obtain

$$\int_{\mathbb{R}^2} f \circ \pi_{\partial E}(x)(\chi_F(x) - \chi_E(x))\, dx = \int_{\partial E} f(y)\xi_{F,E}(y)\, d\mathcal{H}^1_y. \tag{3.5}$$

In particular, for $f = 1$, we find

$$0 = |F| - |E| = \int_{\partial E} \xi_{F,E}\, d\mathcal{H}^1. \tag{3.6}$$

Hence, from (3.5) and the definition of $d_{H^{-1}}(E,F)$, we obtain

$$d_{H^{-1}}(E,F) = \|\xi_{F,E}\|_{H^{-1}(\partial E)}.$$

In the case where ∂F is a normal graph over ∂E, we compute

$$\xi_{F,E} = \int_0^\psi 1 + t\kappa_E\, dt = \psi + \kappa_E \frac{\psi^2}{2}.$$

□

Therefore, under the assumptions of the above lemma, we have

$$d^2_{H^{-1}}(F,E) = \int_{\partial E} |\nabla_\tau v_{F,E}|^2\, d\mathcal{H}^1 \tag{3.7}$$

where $v_{F,E}$ is the unique solution to the equation

$$\begin{cases} \Delta_{\partial E} v_{F,E} = \xi_{F,E} & \text{on } \partial E, \\ \displaystyle\int_{\partial E} v_{F,E}\, d\mathcal{H}^1 = 0. \end{cases} \tag{3.8}$$

The function f that realize the supremum in formula (3.1) is given by

$$f = \frac{v_{F,E}}{d_{H^{-1}}(F,E)}.$$

In the next proposition, we compute the first variation of the function $F \to d_{H^{-1}}(F,E)$.

Proposition 3.4. *Let $E \subset \mathbb{R}^2$ be a bounded open set of class C^2, and let $\sigma < \sigma_E$. Let $F \Subset \mathbb{R}^2$ be a set of class C^1 such that $F \Delta E \subset \mathcal{I}_\sigma(\partial E)$. Let $X \in C^1_c(\mathbb{R}^2, \mathbb{R}^2)$ be such that* $\mathrm{div} X = 0$, *and let $\Psi : (-\varepsilon, \varepsilon) \times \mathbb{R}^2 \to \mathbb{R}^2$ be the solution of the Cauchy problem*

$$\begin{cases} \frac{\partial}{\partial t}\Psi(t,x) = X \circ \Psi(t,x) & \forall x \in \mathbb{R}^2 \\ \Psi(0,x) = x \quad \forall x \in \mathbb{R}^2. \end{cases}$$

Finally, let $f_0 \in H^1(\partial E)$ with $\int_{\partial F} f_0\, d\mathcal{H}^1 = 0$ be the function that realizes the supremum in the definition of $d_{H^{-1}}(F,E)$. Then

$$\frac{d}{dt} d_{H^{-1}}(\Psi(t,F),E)|_{t=0} = \int_{\partial F} f_0(\pi_{\partial E}(x)) X(x)\cdot \nu_F(x) d\mathcal{H}^1_x.$$

Proof. We fix $\varepsilon > 0$ such that $\Psi(t,F) \triangle F \Subset \mathcal{I}_\sigma(\partial E)$ for all $t \in (-\varepsilon,\varepsilon)$. Set $F_t := \Psi(t,F)$ for $t \in (-\varepsilon,\varepsilon)$. Define $\xi_t : \partial E \to \mathbb{R}$ for $t \in (-\varepsilon,\varepsilon)$, by

$$\xi_t(y) := \int_{-\sigma}^{\sigma} (\chi_{F_t}(y + s\nu_E(y)) - \chi_E(y + s\nu_E(y)) J_\tau \Phi_s(y)\, ds, \qquad y \in \partial E,$$

where $\Phi_s(x) = x + s\nu_E(x)$. For every $t \in (-\varepsilon,\varepsilon)$ we have

$$d^2_{H^{-1}}(F_t,E) = \int_{\partial E} |\nabla_\tau v_t|^2 d\mathcal{H}^1, \tag{3.9}$$

where

$$\begin{cases} -\Delta_\tau v_t = \xi_t & \text{on } \partial E, \\ \displaystyle\int_{\partial E} v_t\, d\mathcal{H}^1 = 0. \end{cases}$$

We note that $\xi_t \in L^\infty(\partial E)$ with $\|\xi_t\|_\infty \le C(\|\kappa_E\|_\infty)$.
Claim: $\xi_t \to \xi_0$ in $L^p(\partial E)$ for all $p \ge 1$.

Indeed, since $\partial F_t \subset \mathcal{I}_{Ct}(\partial F)$ for some $C > 0$ depending only on $\|X\|_\infty$, we have

$$\begin{aligned} \|\xi_t - \xi_0\|^p_{L^p(\partial E)} &= \int_{\partial E} \Big| \int_{-\sigma}^{\sigma} (\chi_{F_t}(y + s\nu_E(y)) - \chi_E(y + s\nu_E(y)) J\Phi_s(y)\, ds\Big|^p d\mathcal{H}^1_x \\ &= \int_{B(z,Ct)\cap\partial E} \Big| \int_{-Ct}^{Ct} (\chi_{F_t}(y + s\nu_E(y)) - \chi_E(y + s\nu_E(y)) J\Phi_s(y)\, ds\Big|^p d\mathcal{H}^1_x \le Ct^p. \end{aligned}$$

In particular this implies that $v_t \to v_0$ in $W^{2,p}(\partial E)$ for all $p \ge 1$, hence also uniformly on ∂E. Therefore we have

$$v_t \circ \pi_{\partial E} \to v_0 \circ \pi_{\partial E} \quad \text{ as } t \to 0 \text{ in } C^0(\mathrm{cl}(\mathcal{I}_\sigma(\partial E))). \tag{3.10}$$

Recall now that, see [42, Proposition 17.8], that for all $\varphi \in C_c(\mathbb{R}^2)$

$$\lim_{t\to 0} \frac{1}{t}\Big(\int_{F_t} \varphi\, dx - \int_F \varphi\, dx \Big) = \int_{\partial F} \varphi X \cdot \nu_F\, d\mathcal{H}^1,$$

that is

$$\frac{1}{t}(\chi_{F_t} - \chi_F)\mathcal{L}^2 \overset{*}{\rightharpoonup} X \cdot \nu_F \mathcal{H}^1 \llcorner \partial F \qquad \text{in the sense of measures.} \tag{3.11}$$

Now, using the divergence theorem, formula (3.10) and coarea formula, we have that

$$\begin{aligned} &\int_{\partial E} (|\nabla_\tau v_t|^2 - |\nabla_\tau v_0|^2)\, d\mathcal{H}^1 = \int_{\partial E} (\nabla_\tau v_t - \nabla_\tau v_0)\cdot(\nabla_\tau v_t + \nabla_\tau v_0)\, d\mathcal{H}^1 \\ &= \int_{\partial E} (-\Delta_\tau v_t + \Delta_\tau v_0)(v_t + v_0)\, d\mathcal{H}^1 = \int_{\partial E} (\xi_t - \xi_0)(v_t + v_0)\, d\mathcal{H}^1 \\ &= \int_{\partial E}\int_{-\sigma}^{\sigma} (\chi_{F_t}(y + s\nu_E(y)) - \chi_F(y + s\nu_E(y)) J\Phi_s(y)(v_t(y) + v_0(y))\, ds\, d\mathcal{H}^1_y \\ &= \int_{\mathbb{R}^2} (v_t(\pi_{\partial E}(x)) + v_0(\pi_{\partial E}(x)))(\chi_{F_t}(x) - \chi_F(x))\, dx. \end{aligned} \tag{3.12}$$

Therefore, by (3.9), (3.10), (3.11), (3.12), we obtain

$$\lim_{t\to 0}\frac{d_{H^{-1}}(F_t,E)^2-d_{H^{-1}}(F,E)^2}{t}=\lim_{t\to 0}\frac{\int_{\partial E}|\nabla_\tau v_t|^2\,d\mathcal{H}^1-\int_{\partial E}|\nabla_\tau v_0|^2\,d\mathcal{H}^1}{t}$$
$$=\lim_{t\to 0}\frac{\int_{F_t}(v_t(\pi_{\partial E}(x))+v_0(\pi_{\partial E}(x)))\,dx-\int_F(v_t(\pi_{\partial E}(x))+v_0(\pi_{\partial E}(x)))\,dx}{t}$$
$$=\int_{\partial F}2v_0(\pi_{\partial E}(y))X(y)\cdot\nu_F(y)\,d\mathcal{H}^1_y.$$

Hence by above formula and recalling $f_0=v_0/\|\nabla_\tau v_0\|_{L^2(\partial E)}$, we obtain the desired result. □

3.2. **Elastic energy.** Let $F\Subset\Omega$, and let $u\colon\Omega\setminus F\to\mathbb{R}^2$ be an elastic displacement. We define $E(u)$, the symmetric part of ∇u, as

$$E(u):=\frac{\nabla u+(\nabla u)^T}{2}.$$

Throughout this work, $\mathbb{C}$ denotes a fourth-order elasticity tensor acting on symmetric 2×2 matrices A, satisfying the coercivity condition

$$\mathbb{C}A\colon A>0 \text{ for all } A\neq 0.$$

We define the elastic energy density as

$$Q(A):=\frac{1}{2}\mathbb{C}A\colon A.$$

3.2.1. *Constrained elastic energy.* Let $K_{el}>0$ and $h>0$ be fixed. Given a boundary displacement $w_0\in C^{3,\frac14}(\partial\Omega)$, we define the minimization problem

(3.13)
$$u_F^{K_{el},h}\in\operatorname{argmin}\left\{\int_{\Omega\setminus F}Q(E(u))\,dx\colon u\in\mathfrak{C}^{3,\frac14}_{K_{el}}(\Omega,\mathbb{R}^2),\ \|\nabla^4 u\|_{C^{0,\frac14}(\Omega)}\le\frac{K_{el}}{h^{\frac14}},\ u|_{\partial\Omega}=\omega_0\right\}.$$

We then define the constrained elastic energy as

$$\mathcal{E}(E(u_F^{K_{el},h})):=\int_{\Omega\setminus F}Q(E(u_F^{K_{el},h}))\,dx. \tag{3.14}$$

Remark 3.5. *The existence of an minimizier for the problem* (3.13) *follows from the Arzela-Ascoli Theorem. Hence, the energy functional in* (3.14) *is well-defined.*

In what follow, we omit the explicit dependence of $u_F^{K_{el},h}$ on h, and we write $u_F^{K_{el}}$ for brevity.

3.2.2. *Elastic energy.* Now, fix a boundary displacement $w_0\in C^{3,\frac14}(\partial\Omega)$ we define the (unconstrained) elastic problem as:

$$u_F\in\operatorname{argmin}\left\{\int_{\Omega\setminus F}Q(E(u))\,dx\colon u\in H^1(\Omega\setminus F,\mathbb{R}^2)\right\} \tag{3.15}$$

and we define the corresponding energy as

$$\mathcal{E}(E(u_F)):=\int_{\Omega\setminus F}Q(E(u_F))\,dx.$$

More precisely, u_F is the unique solution in $H^1(\Omega\setminus F,\mathbb{R}^2)$ to the following elliptic system:

$$\begin{cases}\operatorname{div}\mathbb{C}E(u_F)=0 & \text{in } \Omega\setminus F,\\ \mathbb{C}E(u_F)[\nu_F]=0 & \text{on } \partial F,\\ u_F=w_0 & \text{on } \partial\Omega.\end{cases} \tag{3.16}$$

We recall that if $w_0 \in C^{3,\frac{1}{4}}(\partial\Omega)$ and F is of class $C^{3,\frac{1}{4}}$, then the solution $u_F \in C^{3,\frac{1}{4}}(\Omega \setminus F)$ by standard elliptic regularity theory (see [2], [32, Proposition 8.9]). Moreover, the following estimate holds:

$$\|u_F\|_{C^{3,\frac{1}{4}}(\Omega\setminus F)} \le C(\|w_0\|_{C^{3,\frac{1}{4}}(\partial\Omega)} + \|u_F\|_{C^{3,\frac{1}{4}}(\partial F)})$$

where C is an universal constant. Thank to this observation, we have that for K_{el} sufficiently large, the minimization problems (3.13) and (3.15) are equivalent, so that $u_F = u_F^{K_{el},0}$.

In the next proposition, we compute the first variation of the function

$$F \to \mathcal{E}(E(u_F^{K_{el}})).$$

Proposition 3.6. *Let $F \Subset \Omega$ be a set of class C^1, let $X \in C_c^1(\Omega, \mathbb{R}^2)$, and let $(\Phi(t,\cdot))_{t\in(-\varepsilon,\varepsilon)}$ be the unique solution of the Cauchy problem:*

$$\begin{cases} \dfrac{\partial}{\partial t}\Phi(t,x) = X \circ \Phi(t,x) & \forall x \in \mathbb{R}^2, \\ \Phi(0,x) = x & \forall x \in \mathbb{R}^2. \end{cases}$$

We define $F_t = \Phi(t,F)$. Then the following identity holds:

$$\left.\frac{d}{dt}\right|_{t=0} \mathcal{E}(u_{F_t}^{K_{el}}) = -\int_{\partial F} Q(E(u_F^{K_{el}}))X\cdot\nu_F \, d\mathcal{H}^1. \tag{3.17}$$

Proof. Without loss of generality, we can assume that $F_t \Subset \Omega$ for all $t \in (-\varepsilon,\varepsilon)$. Note that the symmetric difference $F_t\Delta F$ is contained in a tubular neighborhood $\mathcal{I}_{tC}(\partial F)$, where $C = C(\|X\|_\infty)$. As a result,

$$|F_t\Delta F| \to 0 \text{ as } t \to 0. \tag{3.18}$$

For every $t \in (-\varepsilon,\varepsilon)$, let $u_{F_t}^{K_{el}} \in \mathfrak{C}_{K_{el}}^{3,\frac{1}{4}}(\Omega,\mathbb{R}^2)$ be a minimizer of the problem (3.13) for $F = F_t$. Then using the Arezela-Ascoli Theorem, up to a subsequece, we have that

$$u_{F_t}^{K_{el}} \to u \text{ in } \mathfrak{C}_{K_{el}}^{3,\frac{1}{4}}(\Omega,\mathbb{R}^2). \tag{3.19}$$

Claim $u = u_F^{K_{el}}$.
Combining (3.18) and (3.19), we deduce:

$$\int_{\Omega\setminus F} Q(E(u))\,dx = \lim_{t\to 0}\int_{\Omega\setminus F_t} Q(E(u_{F_t}^{K_{el}}))\,dx \le \lim_{t\to 0}\int_{\Omega\setminus F_t} Q(E(v))\,dx = \int_{\Omega\setminus F} Q(E(v))\,dx$$

for every admissible test function $v \in \mathfrak{C}_{K_{el}}^{3,\frac{1}{4}}(\Omega,\mathbb{R}^2)$. Therefore, u must be a minimizer $u_F^{K_{el}}$, and hence:

$$u_{F_t}^{K_{el}} \to u_F^{K_{el}} \text{ in } \mathfrak{C}_{K_{el}}^{3,\frac{1}{4}}(\Omega,\mathbb{R}^2). \tag{3.20}$$

Finally, recall [42, Proposition 17.8] that:

$$\frac{1}{t}(\chi_{F_t} - \chi_F)\mathcal{L}^2 \overset{*}{\rightharpoonup} X\cdot\nu_F\mathcal{H}^1 \llcorner \partial F,$$

in the sense of measures. Combining this with formula (3.20), we get the desired derivative formula, i.e., (3.17). □

3.3. **Minimizing movement scheme and flat solution.** Fix $h > 0$ be a fixed time step discretization. Let $K_{el} > 0$ be fixed. Let $E \Subset \Omega$ be a bounded open set of class C^2. For every set $F \subset \mathbb{R}^2$ sufficiently close to E, we define the functional

$$\mathcal{F}_h(F,E) := \mathcal{G}(F) + \frac{1}{2h}d_{H^{-1}}^2(F,E) \tag{3.21}$$

where

$$\mathcal{G}(F) = P_\varphi(F) + \mathcal{E}(E(u_F^{K_{el}})). \tag{3.22}$$

Definition 3.7 (Constrained discrete flat flow). *Let*

$$\beta < \min\{\sigma_{E_0}, \operatorname{dist}(\partial\Omega, \partial E_0)\} \text{ and } K_{el} \text{ be fixed.}$$

Let $h > 0$ be the time step discretization. Define the family of sets $\{E_{hk}^{h,\beta}\}_{k\in\mathbb{N}}$ iteratively by setting $E_0^{h,\beta} := E_0$ and,

$$E_{hk}^{h,\beta} \in \operatorname{arg\,min}\Big\{\mathcal{F}_h(F,\, E_{h(k-1)}^{h,\beta}) \colon\ F\Delta E_{h(k-1)}^{h} \subset \operatorname{cl}(I_\beta(\partial E_{h(k-1)}^{h,\beta}))\Big\} \quad k \geq 1,$$

where the functional $\mathcal{F}_h$ is defined in (3.21). *We define*

$$E_t^{h,\beta} := E_{hk}^{h,\beta} \quad \text{for any } t \in [kh,\, (k+1)h). \tag{3.23}$$

The family $\{E_t^{h,\beta}\}_{t\geq 0}$ is called a constrained discrete flat flow with initial datum E_0 and time step h.

We define a flat flow solution $\{E_t^{\beta}\}_{t\geq 0}$ of the anisotropic surface diffusion with elasticity as any cluster point when we let $h \to 0^+$ of $\{E_t^{h,\beta}\}_{t\geq 0}$.

4. Preliminary estimates

The aim of this section is to establish a regularity estimate for the set F that minimizes the incremental problem

$$\min\{\mathcal{F}_h(A, E) \colon\ A\Delta E \subset \operatorname{cl}(\mathcal{I}_\eta(\partial E))\} \tag{4.1}$$

where $E \in \mathfrak{H}^4_{K,\sigma_0}(E_0)$ and $\eta(K, K_{el}) > 0$. We recall that $E_0 \Subset \Omega$ be open and connected set of class C^5 such that $|E_0| = 1$. The main result of this section is the following:

Theorem 4.1. *Let E be a set of class C^5 such that $E \in \mathfrak{H}^4_{K,\sigma_0}(E_0)$ and $\|\partial_\tau^3 \kappa_E^\varphi\|_{L^2(\partial E)} \leq \frac{K}{h^{\frac14}}$. Then there exist constants $\eta_0 = \eta_0(K, K_{el})$, $C_1 = C_1(K, K_{el})$, and $C_2 = C_2(K, K_{el})$, such that, for every $\eta < \eta_0$, there exits h_0 with the following property: if $0 < h \leq h_0$ and F is a minimizer of* (4.1)*, then $\partial F \Subset \mathcal{I}_\eta(\partial E)$ and coincides with a graph of a smooth function $\psi \colon \partial E \to \mathbb{R}$ satisfying*

$$\|\psi\|_{L^2(\partial E)} \leq C_1 h, \quad \|\psi\|_{H^4(\partial E)} \leq C_1 \tag{4.2}$$

and

$$\|\kappa_F^\varphi\|_{H^2(\partial F)} \leq C_2, \qquad \|\partial^3_{\partial F} \kappa_F^\varphi\|_{L^2(\partial F)} \leq \frac{C_2}{h^{\frac14}}. \tag{4.3}$$

Moreover, there exist constants $\hat\sigma = \hat\sigma(K, K_{el})$, and $K_1 = K_1(K, K_{el})$ such that $F \in \mathfrak{H}^4_{K_1,\hat\sigma}(E_0)$.

4.1. **Λ-minimality estimate.** In this subsection, we prove that any minimizer F of (4.1) is a Λ-minimizer of the φ-perimeter, with Λ independent from h.

We begin by recalling the definition of a Λ-minimizer of the φ-perimeter.

Definition 4.2. *Let $E \subset \mathbb{R}^2$ be a set of finite perimeter. We say that E is a Λ-minimizer of the φ-perimeter if there exists $\Lambda > 0$ such that*

$$P_\varphi(E) \leq P_\varphi(G) + \Lambda|G\Delta E|$$

for every $G \subset \mathbb{R}^2$.

It is known that if $E \subset \mathbb{R}^2$ is a Λ-minimizer of the φ-perimeter, then ∂E is of class $C^{1,\eta}$ for all $\eta \in [0, \frac12)$ see [3], [8] and [20]. In the case where φ is the Euclidean norm, see also [47, Theorem 1.9].

We will use the following lemma. The proof is similar to those in [18, Lemma 2.8] and [40], but we include it here for the reader's convenience.

Lemma 4.3. *Assume that $E \in \mathfrak{H}^3_{K,\sigma_0}(E_0)$ and let F be an Λ-minimizer of the φ-perimeter. Then for every $\gamma \le \frac{1}{4}$, there exists $\delta_0 = \delta_0(K, \Lambda, \gamma)$ such that if*

$$F \Delta E \subset \mathrm{cl}(\mathcal{I}_{\delta_0}(\partial E)),$$

then there exists a function $\psi \in C^{1,\gamma}(\partial E)$ such that

$$\partial F = \{x + \psi(x)\nu_E(x) : x \in \partial E\}.$$

Moreover, for every $\varepsilon > 0$ there exists $\delta_0 = \delta_0(\varepsilon)$ such that $\|\psi\|_{C^{1,\gamma'}(\partial E)} \le \varepsilon$ for $\gamma' < \gamma$.

Proof. By assumption, $E\Delta F \subset \mathrm{cl}(\mathcal{I}_{\delta_0}(\partial E))$ we have that for every $x \in \partial E$,

$$\mathrm{cl}\big(B_{\delta_0}(x)\big) \cap \partial F \ne \emptyset.$$

Let $\varepsilon > 0$ be fixed, and let $C(K) \ge K > 0$ be a constant that we will choose later.
Claim: For all $\delta_0 \in (0, \frac{\varepsilon}{100C(K)})$ if E and F satisfy the assumption, then

$$|\nu_E(x) - \nu_F(y)| \le \varepsilon \quad \text{for all } y \in \partial F \cap \mathrm{cl}(B_{\delta_0}(x)). \tag{4.4}$$

We argue by contradiction. Suppose the claim fails. Then there exist $\varepsilon > 0$, sequences $\{E_n\}_{n\in\mathbb{N}}$, $\{F_n\}_{n\in\mathbb{N}}$ such that

(1) $E_n \in \mathfrak{H}^3_{K,\sigma_0}(E_0)$ for all $n \in \mathbb{N}$,
(2) F_n is a Λ-minimizer of the φ-perimeter for all $n \in \mathbb{N}$,
(3) $E_n\Delta F_n \subset \mathrm{cl}(\mathcal{I}_{\delta_0}(\partial E))$ for all $n \in \mathbb{N}$,
(4) exist $x_n \in \partial E_n$, $y_n \in B_{\frac{1}{n}}(x_n) \cap \partial F_n$ such that

$$|\nu_{E_n}(x_n) - \nu_{F_n}(y_n)| \ge \varepsilon \text{ for all } n \in \mathbb{N}. \tag{4.5}$$

Without loss of generality and up to extracting a subsequence, we have $x_n, y_n \to x$ as $n \to +\infty$,

$$E_n \to E \text{ in } \mathfrak{H}^3_{K,\sigma_0}(E_0),\ F_n \to F \text{ in Hausdorff distance},$$

where F is a Λ-minimizer of the φ-perimeter. Therefore, we have

$$\nu_{E_n}(x_n) \to \nu_E(x) \text{ as } n \to +\infty.$$

Now using the Λ-minimality of F_n, we obtain $\nu_{F_n}(y_n) \to \nu_E(x)$; see [8]. This contradicts (4.5).

The conclusion of the lemma follows from (4.4) and using a standard regularity argument. Indeed, let $x_0 \in \partial E$. We may assume, without loss of generality, that $x_0 = 0$ and $\nu_E(0) = e_2$. Since $E \in \mathfrak{H}^3_{K,\sigma_0}(E_0)$, there exists $r_0 = r_0(K) \le \frac{1}{C(K)}$ such that $E \cap B_{r_0/2}$ coincides with the subgraph of a function $f : (-\frac{r_0}{2}, \frac{r_0}{2}) \to \mathbb{R}$, with

$$\|f\|_{C^{1,\frac{1}{4}}((-\frac{r_0}{2},\frac{r_0}{2}))} \le \frac{10}{r_0},$$

provided $r_0 \le 1$. It then follows that $|\nu_E(x) - e_2| < \varepsilon$ for all $x \in \partial E \cap B_{r_\varepsilon}$, where $r_\varepsilon = \frac{r_0}{20}\varepsilon$. Observe that $\delta_0 < \frac{r_0}{80}\varepsilon$ implies $\delta_0 < \frac{r_\varepsilon}{4}$. Then, by (4.4), we obtain $|\nu_F(y) - e_2| < 2\varepsilon$ for all $y \in \partial F \cap B_{\frac{3r_\varepsilon}{4}}$. Choose any point $y_0 \in \partial F \cap B_{\delta_0}$ and using the previous inequality and the perimeter density estimates for Λ-minimizers of φ-perimeter, we conclude that the excess satisfies

$$\mathbf{e}\big(F, y_0, \frac{r_\varepsilon}{2}\big) = \min_{\omega\in\mathcal{S}^1} \frac{1}{r_\varepsilon} \int_{\partial F\cap B_{\frac{r_\varepsilon}{2}}(y_0)} |\nu_F(y) - \omega|^2 \, d\mathcal{H}^1_y \le C\varepsilon^2,$$

provided $r_\varepsilon < r_1 = r_1(\Lambda, K)$, for some constant $C = C(\Lambda, K)$. Then, by the ε-regularity theorem (see [8]), and since $B_{r_\varepsilon/4} \subset B_{r_\varepsilon/2}(y_0)$ there exists a function $\varphi : (-r_\varepsilon/4, r_\varepsilon/4) \to \mathbb{R}$ such that

$$F \cap B_{r_\varepsilon/4} = \{(y_1, y_2) \in \mathbb{R}^2 : y_2 < \varphi(y_1)\} \cap B_{r_\varepsilon/4}$$

with $\|\varphi\|_{C^{1,\gamma}((-r_\varepsilon/4,r_\varepsilon/4))} \le C$ and $\gamma \le \frac{1}{4}$. The existence of the heightfunction $\psi \in C^{1,\gamma}(\partial E)$ follows from the assumption that $E \in \mathfrak{H}^3_{K,\sigma_0}(E_0)$, see [22, Section 1.2].

Finally, the smallness of the norm $\|\psi\|_{C^{1,\gamma'}(\partial E)}$ for $\gamma' < \gamma$,when δ_0 is small, follows from interpolation inequality (2.14), using that $\|\psi\|_{L^\infty(\partial E)} \le \delta_0$. $\square$

We proceed to prove a technical lemma that will be instrumental at various stages of the article.

Lemma 4.4. *Let $E \in \mathfrak{H}^3_{K,\sigma_0}(E_0)$ be such that $|E| = 1$. Then there exist constants σ, C depending only on K such that the following holds: if $F \subset \mathbb{R}^2$ with $\partial F = \{x + \psi(x)\nu_E(x) \colon x \in \partial E\}$ for some function $\|\psi\|_{C^1(\partial E)} \le \sigma$ with $|F| = 1$, then*

$$\frac{1}{C}\|\nabla_{\partial E}\psi\|_{L^2(\partial E)} \le \|\nabla_{\partial E}\xi_{F,E}\|_{L^2(\partial E)} \le C\|\nabla_{\partial E}\psi\|_{L^2(\partial E)} \tag{4.6}$$

where $\xi_{F,E}$ is defined in Lemma 3.3.

Proof. By Lemma 3.3, we have $\xi_{F,E} = \psi + \frac{\psi^2}{2}\kappa_E$. For σ sufficiently small, we obtain

$$\frac{\psi^2}{2} \le \big(\psi + \frac{\psi^2}{2}\kappa_E\big)^2 = \xi^2_{E,F}. \tag{4.7}$$

Computing the tangential gradient of $\xi_{F,E}$, we find

$$\nabla_{\partial E}\xi_{F,E} = \nabla_{\partial E}\psi + \psi\kappa_E\nabla_{\partial E}\psi + \frac{\psi^2}{2}\nabla_{\partial E}\kappa_E. \tag{4.8}$$

Since $|E| = |F|$, we have $\int_{\partial E}\xi_{F,E}\, d\mathcal{H}^1 = 0$ (see formula (3.6)). Therefore, using (4.7), (4.8), and the Hölder inequality, we obtain

$$\begin{aligned} \|\nabla_{\partial E}\xi_{F,E}\|_{L^2(\partial E)} &\le (1+\sigma K)\|\nabla_{\partial E}\psi\|_{L^2(\partial E)} + \frac{\sigma}{\sqrt{2}}\|\xi_{F,E}\|_{L^2(\partial E)} \\ &\le (1+\sigma K)\|\nabla_{\partial E}\psi\|_{L^2(\partial E)} + \frac{\sigma C_1}{\sqrt{2}}\|\nabla_{\partial E}\xi_{F,E}\|_{L^2(\partial E)} \end{aligned} \tag{4.9}$$

where in the last inequality we used the Poincarè inequality:

$$\|\xi_{F,E}\|_{L^2(\partial E)} \le C_1\|\nabla_{\partial E}\xi_{F,E}\|_{L^2(\partial E)}.$$

Taking σ small enough in (4.9), we obtain

$$\|\nabla_{\partial E}\xi_{F,E}\|_{L^2(\partial E)} \le C\|\nabla_{\partial E}\psi\|_{L^2(\partial E)}. \tag{4.10}$$

From (4.8) and using the Sobolev embedding together with the Hölder's inequality, we get

(4.11)

$$\begin{aligned} \|\nabla_{\partial E}\psi\|_{L^2(\partial E)} &\le \|\nabla_{\partial E}\xi_{F,E}\|_{L^2(\partial E)} + \|\nabla_{\partial E}\psi\|_{L^2(\partial E)}K\|\psi\|_\infty + \|\nabla_{\partial E}\kappa_E\|_{L^2(\partial E)}\|\psi^2\|_{L^2(\partial E)} \\ &\le \|\nabla_{\partial E}\xi_{F,E}\|_{L^2(\partial E)} + \sigma K\|\nabla_{\partial E}\psi\|_{L^2(\partial E)} + 2\sigma K\|\xi_{F,E}\|_{L^2(\partial E)} \\ &\le \|\nabla_{\partial E}\xi_{F,E}\|_{L^2(\partial E)} + \sigma K\|\nabla_{\partial E}\psi\|_{L^2(\partial E)} + 2\sigma K C_1\|\nabla_{\partial E}\xi_{F,E}\|_{L^2(\partial E)} \end{aligned}$$

where the second inequality uses (4.7), and the third uses the Poincaré inequality. Taking σ small enough in (4.11), we get

$$\frac{1}{C}\|\nabla_{\partial E}\psi\|_{L^2(\partial E)} \le \|\nabla_{\partial E}\xi_{F,E}\|_{L^2(\partial E)}. \tag{4.12}$$

Combining (4.10) and (4.12) yields the desired formula, i.e., (4.6). $\square$

Remark 4.5. *Recalling that φ is a regular strictly convex norm, the following inequality holds:*

$$\exists J_\varphi > 0 : D^2\varphi(\nu)\xi\cdot\xi \ge J_\varphi|\xi|^2 \quad \forall \nu \in \mathrm{cl}(\mathcal{I}_{\frac14}(\mathcal{S}^1)),\ \xi \in \mathbb{R}^2 \text{ such that } \nu \perp \xi. \tag{4.13}$$

Lemma 4.6. *Let $E \in \mathfrak{H}^3_{K,\sigma_0}(E_0)$ with $|E| = 1$. Then there exist constants Λ, Λ', σ, depending on K, K_{el}, Ω, such that the following holds: if $F \subset \mathbb{R}^2$ is such that $\partial F = \{x + \psi(x)\nu_E(x) : x \in \partial E\}$, with $\psi \in C^1(\partial F)$ and $\|\psi\|_{C^1(\partial F)} \leq \sigma$, then*

$$\frac{J_\varphi}{4}\|\nabla_\tau \psi\|^2_{L^2(\partial F)} + \mathcal{G}(E) \leq \mathcal{G}(F) + \Lambda d_{H^{-1}}(F,E) \tag{4.14}$$

where J_φ is defined in (4.13)*. Furthermore, if $F \subset \mathbb{R}^2$ is a set of finite perimeter such that $F\Delta E \subset \mathcal{I}_\sigma(\partial E)$, then*

$$\mathcal{G}(E) \leq \mathcal{G}(F) + \Lambda' d_{H^{-1}}(F,E). \tag{4.15}$$

Proof. We fix $\sigma_1 \leq \min\{\sigma_E, \operatorname{dist}(\partial\Omega, \partial E)\}$ and divide the proof into two steps.
Step 1: Proof of (4.14).

By Lemma 3.3 we have $\xi_{F,E} = \psi + \kappa_E \frac{\psi^2}{2}$. If σ_1 is sufficiently small, then

$$\frac{\psi^2}{2} \leq \big(\psi + \frac{\psi^2}{2}\kappa_E\big)^2 = \xi^2_{F,E} \leq 2\psi^2, \qquad 2|\psi\kappa_E| + \psi^2\kappa_E^2 \leq \frac{1}{16}. \tag{4.16}$$

Claim 1: There exists a constant $C(K, K_{el})$ such that for all $\eta > 0$,

$$\mathcal{E}(E(u_E^{K_{el}})) \leq \mathcal{E}(E(u_F^{K_{el}})) + \frac{C(K,K_{el})}{\eta}\|\xi_{F,E}\|_{H^{-1}(\partial E)} + \eta\|\nabla_{\partial E}\psi\|^2_{L^2(\partial E)}. \tag{4.17}$$

Using the definition of $F \to \mathcal{E}(E(u_F^{k_{el}}))$, the minimality of $u_E^{K_{el}}$, and the very definition of $d_{H^{-1}}(F,E)$, we have that

$$\begin{aligned}
\mathcal{E}(E(u_F^{K_{el}})) &= \int_{\Omega\setminus F} Q(E(u_F^{K_{el}}))\,dx = \int_{\mathbb{R}^2} Q(E(u_F^{K_{el}}))(\chi_\Omega - \chi_F)\,dx \\
&= \int_{\Omega\setminus E} Q(E(u_F^{K_{el}}))\,dx + \int_{\mathbb{R}^2} Q(E(u_F^{K_{el}}))(\chi_E - \chi_F)\,dx \\
&\geq \mathcal{E}(E(u_E^{K_{el}})) - \int_{\mathbb{R}^2} Q(E(u_F^{K_{el}}))\circ\pi_{\partial E}(\chi_F - \chi_E)\,dx \\
&\quad + \int_{\mathbb{R}^2}(Q(E(u_F^{K_{el}})) - Q(E(u_F^{K_{el}}))\circ\pi_{\partial E})(\chi_E - \chi_F)\,dx \\
&\geq \mathcal{E}(E(u_E^{K_{el}})) - C(K_{el})d_{H^{-1}}(F,E) \\
&\qquad + \int_{\mathbb{R}^2}(Q(E(u_F^{K_{el}})) - Q(E(u_F^{K_{el}}))\circ\pi_{\partial E})(\chi_E - \chi_F)\,dx.
\end{aligned} \tag{4.18}$$

We observe that

$$|Q(E(u_F^{K_{el}}))(x) - Q(E(u_F^{K_{el}}))\circ\pi_{\partial E}(x)| \leq C(K_{el})|x - \pi_{\partial E}(x)|.$$

By this formula and using the coarea formula and (4.16), we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^2}(Q(E(u_F^{K_{el}})) - Q(E(u_F^{K_{el}}))\circ\pi_{\partial E})(\chi_E - \chi_F)\,dx \\
&\leq C(K_{el})\int_{\mathbb{R}^2}|x - \pi_{\partial E}(x)|\chi_{E\Delta F}(x)\,dx \\
&= C(K,K_{el})\int_{\partial E}|\psi(x)|\int_0^{|\psi(x)|}(1 + t\kappa_E(x))\,dt\,d\mathcal{H}^1_x \\
&\leq C(K,K_{el})\|\xi_{F,E}\|^2_{L^2(\partial E)} \leq C(K,K_{el})\|\xi_{F,E}\|_{H^{-1}(\partial E)}\|\nabla_{\partial E}\xi_{F,E}\|_{L^2(\partial E)} \\
&\leq \frac{C(K,K_{el})}{\eta}\|\xi_{F,E}\|^2_{H^{-1}(\partial E)} + \eta\|\nabla_{\partial E}\xi_{F,E}\|^2_{L^2(\partial E)} \\
&\leq \frac{C(K,K_{el})}{\eta}\|\xi_{F,E}\|^2_{H^{-1}(\partial E)} + \eta C(K)\|\nabla_{\partial E}\psi\|^2_{L^2(\partial E)}
\end{aligned} \tag{4.19}$$

where in the third inequality we have used the interpolation inequality; i.e.,

$$\|f\|_{L^2(\partial E)} \le \|f\|^{\frac{1}{2}}_{H^{-1}(\partial E)} \|\nabla f\|^{\frac{1}{2}}_{L^2(\partial E)} \quad \text{for all } f \text{ such that } \int_{\partial E} f \, d\mathcal{H}^1 = 0,$$

and in the fourth inequality we have used the Young inequality and in the last inequality we have used Lemma 4.4. Combining (4.18) and (4.19), we obtain (4.17).
Claim 2: There exists a constant $C(K)$ such that

$$\frac{3J_\varphi}{8}\|\nabla_{\partial E}\psi\|^2_{L^2(\partial E)} + P_\varphi(E) \le P_\varphi(F) + C(K) d_{H^{-1}}(F, E). \tag{4.20}$$

Define $\Psi : \partial E \to \partial F$ defined as $\Psi(x) := x + \psi(x)\nu_E(x)$. The tangential Jacobian is

$$J_\tau \Psi = \sqrt{(1 + \psi\kappa_E)^2 + |\nabla_{\partial E}\psi|^2}.$$

Using the area formula and the expansion of ν_F, see (2.5), we get:

$$\begin{aligned} P_\varphi(F) = \int_{\partial F} \varphi(\nu_F) d\mathcal{H}^1 &= \int_{\Psi(\partial E)} \varphi(\nu_F \circ \Psi \circ \pi_{\partial E}|_{\partial F}) d\mathcal{H}^1 \\ &= \int_{\partial E} \varphi(\nu_F(\Psi))\sqrt{(1 + \psi\kappa_E)^2 + |\nabla_{\partial E}\psi|^2} d\mathcal{H}^1 \\ &= \int_{\partial E} \varphi(-\nabla_{\partial E}\psi(x) + (1 + \psi(x)\kappa_E(x))\nu_E(x)) d\mathcal{H}^1_x. \end{aligned} \tag{4.21}$$

We observe that

$$\nabla_{\partial E}(\nabla_{\partial E}\varphi(\nu_E)) = \nabla^2_{\partial E}\varphi(\nu_E)\nabla_{\partial E}(\nu_E) = \kappa_E \nabla^2_{\partial E}\varphi(\nu_E)\tau_E \otimes \tau_E,$$

then

$$\mathrm{div}_{\partial E}(\nabla_{\partial E}\varphi(\nu_E)) = \mathrm{Tr}\nabla_{\partial E}(\nabla_{\partial E}\varphi(\nu_E)) = \kappa_E \nabla^2_{\partial E}\varphi(\nu_E)\tau_E \cdot \tau_E. \tag{4.22}$$

By the convexity of φ and using (4.13) (up to take σ_1 small enough) we have

$$\begin{aligned} &\varphi((1 + \psi\kappa_E)\nu_E - \nabla_{\partial E}\psi) \\ &\ge \varphi(\nu_E + \psi\kappa_E\nu_E) - \nabla\varphi(\nu_E) \cdot \nabla_{\partial E}\psi + \frac{J_\varphi}{2}|\nabla_{\partial E}\psi|^2 \\ &\ge \varphi(\nu_E) + \psi\kappa_E\nabla\varphi(\nu_E) \cdot \nu_E - \nabla\varphi(\nu_E) \cdot \nabla_{\partial E}\psi + \frac{J_\varphi}{2}|\nabla_{\partial E}\psi|^2 \\ &= \varphi(\nu_E) + \psi\kappa_E\varphi(\nu_E) - \nabla\varphi(\nu_E) \cdot \nabla_{\partial E}\psi + \frac{J_\varphi}{2}|\nabla_{\partial E}\psi|^2 \end{aligned} \tag{4.23}$$

where in the last equality we have used $\nabla\varphi(x) \cdot x = \varphi(x)$. Let $\varepsilon > 0$ that we choice later, using the divergence theorem and formula (4.22), we get

$$\begin{aligned} \int_{\partial E} \nabla\varphi(\nu_E) \cdot \nabla_{\partial E}\psi &= \int_{\partial E} \nabla_{\partial E}\varphi(\nu_E) \cdot \nabla_{\partial E}\psi = \int_{\partial E} \mathrm{div}_{\partial E}(\nabla_{\partial E}\varphi(\nu_E))\psi \\ &\le \int_{\partial E} \mathrm{div}_{\partial E}(\nabla_{\partial E}\varphi(\nu_E))(\psi + \kappa_E\frac{\psi^2}{2}) + C(K)\int_{\partial E} \psi^2 \\ &\le C(K)\|\psi + \kappa_E\frac{\psi^2}{2}\|_{H^{-1}(\partial E)} + C(K)\int_{\partial E} \xi^2_{F,E} \\ &\le C(K)\|\xi_{F,E}\|_{H^{-1}(\partial E)} + C(K)\|\xi_{F,E}\|_{H^{-1}(\partial E)}\|\nabla_{\partial E}\xi_{F,E}\|_{L^2(\partial E)} \\ &\le \frac{C(K)}{\varepsilon}\|\xi_{F,E}\|_{H^{-1}(\partial E)} + \varepsilon\|\nabla_{\partial E}\xi_{F,E}\|^2_{L^2(\partial E)} \\ &\le \frac{C(K)}{\varepsilon}\|\xi_{F,E}\|_{H^{-1}(\partial E)} + \varepsilon C(K)\|\nabla_{\partial E}\psi\|^2_{L^2(\partial E)} \end{aligned} \tag{4.24}$$

where in the third inequality we have used the interpolation of L^2 between H^{-1} and H^1, in the fourth inequality we have used Young's inequality and in the last inequality we have used Lemma 4.4. Integrating (4.23) over ∂E and using (4.21), (4.24), we get

$$P_\varphi(F) \geq P_\varphi(E) - \frac{C(K)}{\varepsilon}\|\xi_{F,E}\|^2_{H^{-1}(\partial E)} - C(K)\varepsilon\|\nabla_{\partial E}\psi\|^2_{L^2(\partial E)} + \frac{J_\varphi}{2}\|\nabla_{\partial E}\psi\|^2_{L^2(\partial E)}.$$

Choosing $\varepsilon = \frac{J_\varphi}{8C(K)}$ we obtain (4.20).

We choose $\eta = \frac{J_\varphi}{8}$. Combining (4.17), (4.20) and recalling that

$$d_{H^{-1}}(F,E) = \|\xi_{F,E}\|_{H^{-1}(\partial E)},$$

we obtain (4.14).

Step 2: Proof of (4.15).

Let $\widehat{\Lambda} = \widehat{\Lambda}(K, K_{el})$ (to be defined later, see formula (4.38)). By Lemma 4.3 applied with $\Lambda = \widehat{\Lambda}$, we obtain $\delta_0 = \delta_0(K, K_{el})$. Set

$$\mathcal{J}(F) := \mathcal{G}(F) + (\Lambda + 1)d_{H^{-1}}(F,E).$$

Fix $\sigma \leq \min\{\sigma_1, \delta_0\}$. The thesis of the step 2 is equivalent the claim.

Claim 3: The set E is the minimizer of the problem

$$\min\left\{\mathcal{J}(F) \colon F\Delta E \subset \mathrm{cl}(\mathcal{I}_\sigma(\partial E))\right\}. \tag{4.25}$$

Existence of a minimizer follows by the direct method of the calculus of variations. Let F be such a minimizer.

Subclaim: The minimizer of (4.25) is an $\widehat{\Lambda}$-minimizer of the φ-perimeter.

Let $S(K) > 0$ denote the Sobolev embedding constant of $H^1(\partial E)$ into $L^\infty(\partial E)$, which depends only on K. Let $G \subset \mathbb{R}^2$ with $G\Delta E \subset \mathrm{cl}(\mathcal{I}_\sigma(\partial E))$ and $|G| = 1$. Then, by minimality of F, we get:

$$\begin{aligned}
\mathcal{G}(F) - \mathcal{G}(G) &\leq (\Lambda+1)(d_{H^{-1}}(G,E) - d_{H^{-1}}(F,E)) \\
&\leq (\Lambda+1)\int_{\mathbb{R}^2} f_G \circ \pi_{\partial E}(x)(\chi_G(x) - \chi_F(x))\,dx \leq (\Lambda+1)\|f_G \circ \pi_{\partial E}\|_\infty |G\Delta F| \\
&\leq S(K)(\Lambda+1)\|\nabla_{\partial E} f_G\|_{L^2(\partial E)}|G\Delta F| \leq S(K)(\Lambda+1)|G\Delta F|
\end{aligned} \tag{4.26}$$

where f_G is the function that realize the supremum in the definition of $d_{H^{-1}}(G,E)$. We consider now the elastic term:

$$\begin{aligned}
\mathcal{E}(E(u_F^{K_{el}})) - \mathcal{E}(E(u_G^{K_{el}})) &= \int_{\Omega\setminus F} Q(E(u_F^{K_{el}}))dx - \int_{\Omega\setminus G} Q(E(u_G^{K_{el}}))dx \\
&\leq \int_{\Omega\setminus F} Q(E(u_G^{K_{el}}))dx - \int_{\Omega\setminus G} Q(E(u_G^{K_{el}}))dx \\
&\leq K_{el}|G\Delta F|,
\end{aligned} \tag{4.27}$$

here, the first inequality follows from the minimality of $u_F^{K_{el}}$. Combining formulas (4.26), (4.27), and recalling the definition of $\mathcal{G}$, we obtain

$$P_\varphi(F) - P_\varphi(G) \leq (S(K)(\Lambda+1) + K_{el})|F\Delta G| \text{ for all } G\Delta E \subset \mathrm{cl}(\mathcal{I}_\sigma(\partial E)) \text{ and } |G| = 1. \tag{4.28}$$

We now conclude the proof of the subclaim using a standard calibration argument, which we proceed to explain. Let us denote $C_1(K, K_{el}) := (S(K)(\Lambda+1) + K_{el})$, and fix a set $G \subset \mathbb{R}^2$.

Case 1: $|G\Delta F| \geq 1$.

By the minimality of F in (4.25), we have

$$P_\varphi(F) \leq \mathcal{J}(F) \leq \mathcal{J}(E) = \mathcal{G}(E) \leq C_2(K, K_{el}).$$

Therefore,

$$P_\varphi(F) \le P_\varphi(G) + P_\varphi(F) \le P_\varphi(G) + C_2(K, K_{el}) \le P_\varphi(G) + C_2(K, K_{el})|G\Delta F|. \tag{4.29}$$

Case 2: $|G\Delta F| < 1$.
Define $E_s := \{x \in \mathbb{R}^2 \colon d_E(x) < s\}$ for $s \in [-\sigma, \sigma]$, and set

$$\widehat{G} := (G \cap E_\sigma) \cup E_{-\sigma}.$$

Then $\widehat{G}\Delta E \subset \mathrm{cl}\big(\mathcal{I}_\sigma(\partial E)\big)$ and

$$\widehat{G}\Delta G = \big(G \setminus E_\sigma\big) \cup \big(E_{-\sigma} \setminus G\big).$$

Since $F\Delta E \subset \mathrm{cl}\big(\mathcal{I}_\sigma(\partial E)\big)$, we have $F \subset \mathrm{cl}(E_\sigma)$ and $\mathrm{int}(E_{-\sigma}) \subset F$, yielding

$$|G\Delta\widehat{G}| \le |G\Delta F|. \tag{4.30}$$

Subsubclaim:

$$P_\varphi(\widehat{G}) \le P_\varphi(G) + C_3(K)|G\Delta\widehat{G}|. \tag{4.31}$$

We analyze the case $G \cap E_{-\sigma} = E_{-\sigma}$; the other cases are similar. Define the vector field

$$Y : \mathbb{R}^2 \to \mathbb{R}^2, \quad Y := -\nabla\varphi(\nu_{E_\sigma} \circ \pi_{\partial E_\sigma})\xi$$

where $\xi \in C_c^\infty(\mathcal{I}_\sigma(\partial E_\sigma))$ and $\xi(x) = 1$ for all $x \in \mathcal{I}_{\frac{\sigma}{2}}(\partial E_\sigma)$. By the divergence Theorem, we obtain

$$\begin{aligned} \int_{G\setminus\widehat{G}} \mathrm{div}Y\, dx &= \int_{G\setminus E_\sigma} \mathrm{div}Y\, dx \\ &= \int_{\partial E_\sigma \cap G} \nu_{E_\sigma} \cdot \nabla\varphi(\nu_{E_\sigma}) - \int_{\partial^* G \cap E_\sigma^c} \nu_G \cdot \nabla\varphi(\nu_{E_\sigma} \circ \pi_{\partial E_\sigma})\xi. \end{aligned} \tag{4.32}$$

By the convexity of φ and the triangle inequality, we obtain

$$\nabla\varphi(\nu_{E_\sigma} \circ \pi_{\partial E_\sigma}) \cdot \nu_G \le \varphi(\nu_G + \nu_{E_\sigma} \circ \pi_{\partial E_\sigma}) - \varphi(\nu_{E_\sigma} \circ \pi_{\partial E_\sigma}) \le \varphi(\nu_G). \tag{4.33}$$

Furthermore, the one homogeneity of φ gives

$$\nabla\varphi(\nu_{E_\sigma} \circ \pi_{\partial E_\sigma}) \cdot \nu_{E_\sigma} \circ \pi_{\partial E_\sigma} = \varphi(\nu_{E_\sigma} \circ \pi_{\partial E_\sigma}). \tag{4.34}$$

Combining (4.32), (4.33), and (4.34), we obtain

$$P_\varphi(\widehat{G}) \le P_\varphi(G) + \int_{G\setminus\widehat{G}} \mathrm{div}Y\, dx,$$

which implies (4.31).

We now consider two cases: $|\widehat{G}| \ge |F|$ or $|\widehat{G}| \le |F|$. We analyze the former; the latter is analogous. For all $s \in [-\sigma, \sigma]$, we have

$$|E_{-\sigma}| \le |E_{-\sigma} \cap \widehat{G}| \le |E_s \cap \widehat{G}| \le |E_\sigma \cap \widehat{G}| = |\widehat{G}|.$$

By continuity of $s \to |E_s \cap \widehat{G}|$, there exists $\hat{s} \in [-\sigma, \sigma]$ such that $|E_{\hat{s}} \cap \widehat{G}| = |F|$. Denoting $G_{\hat{s}} := E_{\hat{s}} \cap \widehat{G} \subset \widehat{G}$ and using (4.30), we get

$$\begin{aligned} |\widehat{G}\Delta G_{\hat{s}}| &= |\widehat{G}| - |G_{\hat{s}}| = |\widehat{G}| - |F| \le |\widehat{G}\Delta F| \\ &\le |G\Delta F| + |G\Delta\widehat{G}| \le 2|G\Delta F|. \end{aligned} \tag{4.35}$$

Applying a calibration argument similar to the one in (4.31), we obtain

$$P_\varphi(\widehat{G}_s) \le P_\varphi(\widehat{G}) + C_4(K)|\widehat{G}_s\Delta\widehat{G}|.$$

Combining this with (4.30), (4.31), and (4.35) we conclude

$$P_\varphi(\widehat{G}_s) \le P_\varphi(G) + C_5(K)|G\Delta F|. \tag{4.36}$$

From (4.29), (4.28), and (4.36), we obtain

$$P_\varphi(F) \le P_\varphi(G) + \widehat{\Lambda}(K, K_{el})|G\Delta F|, \tag{4.37}$$

where

$$\widehat{\Lambda}(K, K_{el}) := C_1(K, K_{el}) + C_2(K, K_{el}) + C_5(K). \tag{4.38}$$

Thus, from (4.37) we deduce that F is an $\widehat{\Lambda}$-minimizer of the φ-perimeter. Therefore subclaim has been proven.

Applying Lemma 4.3, we obtain that ∂F coincide with a normal graph of a C^1 function over ∂E, i.e.,

$$\partial F = \{x + \psi(x)\nu_E(x) \colon x \in \partial E\}.$$

Applying Step 1, we get

$$\begin{aligned} \frac{J_\varphi}{4}\|\nabla_\tau \psi\|^2_{L^2(\partial F)} + \mathcal{G}(E) &\le \mathcal{G}(F) + \Lambda d_{H^{-1}}(F, E) \\ &\le \mathcal{J}(F) \le \mathcal{J}(E) = \mathcal{G}(E) \end{aligned} \tag{4.39}$$

where in the last inequality we have used the minimality of F. Hence, from (4.39), we conclude that ψ must be constant. Since ψ is constant and $|F| = |E|$, it follows that $\psi = 0$, and therefore $F = E$. □

In the next lemma, we show that every minimizer F of the problem (4.1) is an Λ-minimizer of the φ-perimeter, where Λ depends only on K, K_{el}. Moreover, we establish that the discrete velocity in H^{-1} is bounded

$$\frac{d_{H^{-1}}(E, F)}{h} \le C.$$

Lemma 4.7. *Let $E \in \mathfrak{H}^3_{K,\sigma_0}(E_0)$ with $|E| = 1$ and let σ be the constant from 4.6. Then, for every minimizer F of the problem* (4.1) *with $\eta < \sigma$, the following properties hold:*

1) *Let Λ' be the constant from Lemma 4.6. Then*

$$d_{H^{-1}}(F, E) \le 2\Lambda' h. \tag{4.40}$$

2) *There exists a constant $\lambda = \lambda(K, K_{el})$ such that*

$$P_\varphi(F) \le P_\varphi(G) + \lambda|G\Delta F| \quad \textit{for all } G \subset \mathbb{R}^2. \tag{4.41}$$

Proof. We divide the proof into two steps.
Step 1 In this step, we prove (4.40).

Using formula (4.15) and the minimality of F, we obtain

$$\mathcal{G}(F) + \frac{1}{2h} d^2_{H^{-1}}(F, E) \le \mathcal{G}(E) \le \mathcal{G}(F) + \Lambda' d_{H^{-1}}(F, E).$$

Hence, inequality (4.40) follows.
Step 2 In this step, we prove (4.41).
Claim: For every set $G \subset \mathbb{R}^2$ such that $G\Delta E \subset \mathrm{cl}(\mathcal{I}_\eta(\partial E))$ and $|G| = 1$, the following inequality holds:

$$\mathcal{G}(F) \le \mathcal{G}(G) + 3\Lambda'(d_{H^{-1}}(G, E) - d_{H^{-1}}(F, E)).$$

Case 1) $d_{H^{-1}}(G, E) > 4\Lambda' h$.
By (4.40), we have

$$2d_{H^{-1}}(F, E) \le 4\Lambda' h < d_{H^{-1}}(G, E)$$

and hence,

$$d_{H^{-1}}(F, E) \le d_{H^{-1}}(G, E) - d_{H^{-1}}(F, E). \tag{4.42}$$

Using the minimality of F along with formulas (4.15) and (4.42), we deduce

$$\begin{aligned}\mathcal{G}(F) &\leq \mathcal{G}(F)+\frac{1}{2h}d^2_{H^{-1}}(F,E)\leq \mathcal{G}(E)\\ &\leq \mathcal{G}(G)+\Lambda' d_{H^{-1}}(F,E)\leq \mathcal{G}(G)+\Lambda'(d_{H^{-1}}(G,E)-d_{H^{-1}}(F,E)).\end{aligned}\tag{4.43}$$

Case 2) $d_{H^{-1}}(G,E)\leq 4\Lambda' h$.
Using the minimality of F and inequality (4.40), we obtain

$$\begin{aligned}\mathcal{G}(F)-\mathcal{G}(G) &\leq \frac{1}{2h}(d_{H^{-1}}(G,E)+d_{H^{-1}}(F,E))(d_{H^{-1}}(G,E)-d_{H^{-1}}(F,E))\\ &\leq 3\Lambda'(d_{H^{-1}}(G,E)-d_{H^{-1}}(F,E)).\end{aligned}\tag{4.44}$$

The conclusion of the claim follows from inequalities (4.43) and (4.44).
Claim: There exits $\lambda_1=\lambda_1(K,K_{el})$ such that for every set $G\subset\mathbb{R}^2$ with $G\Delta E\subset \mathrm{cl}(\mathcal{I}_\sigma(\partial E))$ and $|G|=1$, the following holds:

$$P_\varphi(F)\leq P_\varphi(G)+\lambda_1|F\Delta G|.\tag{4.45}$$

Using the definition of $\mathcal{G}$ (see formula (3.22)), and the previous claim, we have

$$P_\varphi(F)-P_\varphi(G)\leq \mathcal{E}(E(u_G^{K_{el}}))-\mathcal{E}(E(u_F^{K_{el}}))+3\Lambda'(d_{H^{-1}}(G,E)-d_{H^{-1}}(F,E))$$

The claim then follows by applying the same reasoning used in formulas (4.26) (to estimate the difference between $d_{H^{-1}}(G,E)$ and $d_{H^{-1}}(F,E)$) and (4.27) (to estimate the difference between $\mathcal{E}(E(u_G^{K_{el}}))$ and $\mathcal{E}(E(u_F^{K_{el}}))$).
Claim: There exits $\lambda=\lambda(K,K_{el})$ such that for every $G\subset\mathbb{R}^2$, the inequality (4.41).

This claim follows by (4.45) and adapting the same argument used in the subclaim of Step 2 in Lemma 4.6. □

4.2. **Estimate for the heightfunction.** In this subsection, we prove that ∂F coincides with the graph of a smooth function $\psi:\partial E\to\mathbb{R}$, where F is a minimizer of (4.1). Moreover, we establish regularity estimates for ψ.

We begin with a remark that provides an analogue of formula (2.6) for the anisotropic curvature defined in (2.12).

Remark 4.8. *Let $E\in\mathfrak{C}^3_{K,\sigma_0}(E_0)$ and let $A\subset\mathbb{R}^2$ be a set of class C^2 such that ∂A is a normal graph over ∂E, i.e.*

$$\partial A=\{x+\psi(x)\nu_E(x)\colon x\in\partial E\}.$$

Let $g\in C^\infty(\mathbb{R}^2\setminus\{0\})$ be the function defined in (2.12)*. Then, using formula* (2.5) *and the Taylor expansion of g, we obtain for all $x\in\partial E$*

$$g(\nu_A(x+\psi(x)\nu_E(x)))=g(\nu_E(x))+R_1(\psi(x)\kappa_E(x),\partial_\tau\psi(x),\nu_E(x)),\tag{4.46}$$

where $R_1\in C^\infty$. Using formulas (2.6), (2.7), (2.12) *and* (4.46)*, we obtain*

$$\kappa^\varphi_A(x+\psi(x)\nu_E(x))=-g(\nu_E(x))\partial^2_\tau\psi(x)+\kappa^\varphi_E(x)+R(x)\ x\in\partial E,\tag{4.47}$$

where

$$R=r_0(\psi,\partial_\tau\psi,\kappa_E,\nu_E)+r_1(\psi\kappa_E,\partial_\tau\psi,\nu_E)\partial^2_\tau\psi+r_2(\psi\kappa_E,\partial_\tau\psi,\nu_E)\partial_\tau(\psi\kappa_E)\tag{4.48}$$

and r_0,r_1,r_2 are smooth functions satisfying

$$r_0(0,0,\cdot,\cdot)=r_1(0,0,\cdot)=r_2(0,0,\cdot)=0.$$

We now state and prove three propositions that will enable us to prove the Theorem 4.1.

Proposition 4.9. *Let $E \in \mathfrak{H}^4_{K,\sigma_0}(E_0)$. Then there exist constants $\eta_0 = \eta_0(K, K_{el})$, $h_0 = h_0(K, K_{el})$ and $C = C(K, K_{el})$ such that, if $0 < h \le h_0$ then any minimizer of the problem (4.1) (with $\eta = \eta_0$) $F \subset \mathbb{R}^2$ has the property that ∂F coincides with the graph of a smooth function $\psi : \partial E \to \mathbb{R}$ satisfying*

$$\|\psi\|_{L^2(\partial E)} \le C h^{\frac{3}{4}}, \quad \|\nabla_{\partial E}\psi\|_{L^2(\partial E)} \le C\sqrt{h}, \quad \|\kappa^\varphi_F\|_{H^1(\partial F)} \le C. \tag{4.49}$$

Proof. Let σ be the constant from Lemma 4.6, and let δ_0 be the constant obtained in Lemma 4.1 for $\Lambda = \lambda(K, K_{el})$, where $\lambda(K, K_{el})$ is the constant defined in Lemma 4.7 (see formula (4.41)). We set $\eta_0 := \min\{\sigma, \delta_0\}$. Let F be a minimizer of (4.1) for $\eta = \eta_0$. By Lemma 4.7, we have that F is a $\lambda(K, K_{el})$-minimizer of the φ-perimeter. Applying Lemma 4.3, we obtain

$$\partial F = \{x + \psi(x)\nu_E(x) \colon x \in \partial E\},$$

where $\psi \in C^1(\partial E)$ and $\|\psi\|_{C^1(\partial E)} \le C(K, K_{el})$. Using formula (4.14), the minimality of F, and formula (4.40), we get

$$\|\nabla_{\partial E}\psi\|_{L^2(\partial E)} \le C(K)\sqrt{h}. \tag{4.50}$$

Using formulas (4.6), (4.7), and the Poincaré inequality, we have

$$\|\psi\|_{L^2(\partial E)} \le C(K)\|\xi_{F,E}\|_{L^2(\partial E)} \le C(K)\|\nabla_{\partial E}\xi_{F,E}\|_{L^2(\partial E)} \le C(K)\|\nabla_{\partial E}\psi\|_{L^2(\partial E)}, \tag{4.51}$$

where $\xi_{F,E} = \psi + \frac{\psi\kappa_E^2}{2}$. Therefore by the Sobolev embedding, (4.50) and (4.51), we obtain

$$\|\psi\|_{L^\infty(\partial E)} \le C(K)\|\psi\|_{H^1(\partial E)} \le C(K)\sqrt{h}.$$

Hence, for h_0 small enough, we have $\partial F \Subset \mathcal{I}_\eta(\partial E)$. We are now in a position to compute the Euler–Lagrange equation for the functional $F \to \mathcal{F}_h(F, E)$. Applying formula (2.11), Proposition 3.4, and Proposition 3.6 we obtain

$$\kappa^\varphi_F(y) - Q(E(u_F^{K_{el}}))(y) + \frac{d_{H^{-1}}(F,E)}{h} f(\pi_{\partial E}(y)) = L \quad \text{for all } y \in \partial F \tag{4.52}$$

where $f \in H^1(\partial E)$ is the function that attains the supremum in (3.1), and L is the Lagrange multiplier. Integrating equation (4.52) over ∂F, we get

$$\begin{aligned} L &\le \frac{1}{P(F)}\Big[\int_{\partial F}\kappa^\varphi_F\, d\mathcal{H}^1 + C(K, K_{el}) + \frac{d_{H^{-1}}(F,E)}{h}\int_{\partial F} f\circ\pi_{\partial E}\, d\mathcal{H}^1\Big] \\ &= \frac{1}{P(F)}\Big[\int_{\partial F}\kappa^\varphi_F\, d\mathcal{H}^1 + C(K, K_{el}) \\ &\qquad + \frac{d_{H^{-1}}(F,E)}{h}\int_{\partial E} f(x)\sqrt{(1+\psi(x)\kappa_E(x))^2 + |\nabla_{\partial E}\psi(x)|^2}\, d\mathcal{H}^1_x\Big] \\ &\le \frac{1}{\sqrt{4\pi}}\Big[C_\varphi + 2\Lambda' C(K)\Big] = C(K, K_{el}) \end{aligned} \tag{4.53}$$

where we have used: a change of variable $x \in \partial E \to x + \psi(x)\nu_E(x) \in \partial F$, whose tangential Jacobian is $x \to \sqrt{(1+\psi(x)\kappa_E(x))^2 + |\nabla_{\partial E}\psi(x)|^2}$, the isoperimeteric inequality $1 = |F| \le \frac{1}{4\pi}P(F)^2$, Lemma 2.3, formula (4.40), and the bound $\|f\|_{L^2(\partial E)} \le C(K)$. By the Euler–Lagrange equation and (4.53), we can now estimate the L^2 norm of κ^φ_F, and we get

$$\|\kappa^\varphi_F\|_{L^2(\partial F)} \le C(K, K_{el}). \tag{4.54}$$

Differentiating the equation (4.52), we obtain

$$\nabla_{\partial F}\kappa^\varphi_F(y) - \nabla_{\partial F}Q(E(u_F^{K_{el}}))(y) + \frac{d_{H^{-1}}(F,E)}{h}\nabla_{\partial F} f(\pi_{\partial E}(y)) = 0 \quad \text{for all } y \in \partial F.$$

Using this equation, formula (2.8), and $\|\nabla_{\partial E} f\|_{L^2(\partial E)} \leq 1$, we get

$$\|\nabla_{\partial F}\kappa_F^{\varphi}\|^2_{L^2(\partial F)} \leq C(K, K_{el}) + \int_{\partial E} \frac{|\nabla_{\partial E} f|^2}{\sqrt{(1+\psi\kappa_E) + |\nabla_{\partial E}\psi|^2}}\, d\mathcal{H}^1 \leq C(K, K_{el}). \tag{4.55}$$

From formulas (4.4) and (4.50), we deduce

$$\|\nabla_{\partial E}\xi_{F,E}\|_{L^2(\partial E)} \leq C\sqrt{h}.$$

Now using formula (4.16) and the interpolation of $L^2(\partial E)$ between $H^1(\partial E)$ and $H^{-1}(\partial E)$, we obtain

$$\frac{1}{\sqrt{2}}\|\psi\|_{L^2(\partial E)} \leq \|\xi_{F,E}\|_{L^2(\partial E)} \leq \|\nabla_{\partial E}\xi_{F,E}\|^{\frac{1}{2}}_{L^2(\partial E)}\|\xi_{F,E}\|^{\frac{1}{2}}_{H^{-1}(\partial E)} \leq C(K, K_{el})h^{\frac{3}{4}} \tag{4.56}$$

where we have used formula (4.40), i.e., $d_{H^{-1}}(F,E) = \|\xi_{F,E}\|_{H^{-1}(\partial E)} \leq 2\Lambda' h$. Finally, the estimate (4.49) follows from (4.50), (4.54), (4.55), and (4.56). □

An important consequence of Proposition 4.9 is that the boundary of any minimizer of problem (4.1) does not intersect the boundary of the constraint $\partial\mathcal{I}_{\eta_0}(\partial E)$ for $h \leq h_0$. This allows us to write the Euler-Lagrange equation for F, and by applying (3.8) and (4.52), we obtain:

$$\begin{cases} \kappa_F^{\varphi} - Q(E(u_F^{K_{el}})) + \dfrac{d_{H^{-1}}(F,E)}{h} f \circ \pi_{\partial E} = L & \text{on } \partial F, \\ -\Delta_{\partial E} f = \dfrac{\xi_{F,E}}{d_{H^{-1}}(F,E)} & \text{on } \partial E \end{cases} \tag{4.57}$$

where $\xi_{F,E}$ is defined in (3.2)(see also formula (3.3)) and L is the Lagrange multiplier. We remark that if E is $C^{5,\gamma}$-regular for some $\gamma \in (0,1)$, then by the elliptic regularity theory implies that F is also C^5-regular. Using formula (4.47), we can combine the two equations above into the following single equation:

$$\begin{aligned} &\frac{1}{h}\big(\psi(x) + \kappa_E(x)\frac{\psi(x)^2}{2}\big) \\ &\quad = \partial^2_\tau(-g(\nu_E(x))\partial^2_\tau\psi(x) + \kappa_E^{\varphi}(x)) - \partial^2_\tau\big(Q(E(u_F^{K_{el}}))(x + \psi(x)\nu_E(x))\big) + \partial^2_\tau R(x) \end{aligned} \tag{4.58}$$

for all $x \in \partial E$, where R is define in (4.48).

Let us recall a lemma that will be useful in the upcoming proofs; see [18, Lemma 2.3], [38, Lemma 2.5 & Proposition 2.6].

Lemma 4.10. *Let $A \subset \mathbb{R}^2$ be a set of class C^5 and such that $A \in \mathfrak{C}^2_M(E_0)$. For all $f \in C^4(\partial A)$ it holds*

$$\begin{aligned} \|f\|_{H^1(\partial A)} &\leq C\big(\|\partial_\tau f\|_{L^2(\partial A)} + \|f\|_{L^\infty(\partial A)}(1 + \|\kappa_E\|_{L^2(\partial A)})\big), \\ \|f\|_{H^2(\partial A)} &\leq C\big(\|\partial^2_\tau f\|_{L^2(\partial A)} + \|f\|_{L^\infty(\partial A)}(1 + \|\partial_\tau\kappa_A\|_{L^2(\partial A)})\big), \\ \|f\|_{H^3(\partial A)} &\leq C\big(\|\partial^3_\tau f\|_{L^2(\partial A)} + \|f\|_{L^\infty(\partial A)}(1 + \|\partial^2_\tau\kappa_A\|_{L^2(\partial A)})\big), \\ \|f\|_{H^4(\partial A)} &\leq C\big(\|\partial^4_\tau f\|_{L^2(\partial A)} + \|f\|_{L^\infty(\partial A)}(1 + \|\partial^3_\tau\kappa_A\|_{L^2(\partial A)})\big), \end{aligned}$$

where C is a universal constant.

In the next proposition, we will prove a sharp estimate for the L^2-norm of the heightfunction in (4.49), namely:

$$\|\psi\|_{L^2(\partial E)} \lesssim h.$$

Proposition 4.11. *Let E be a set of class C^5 such that $E \in \mathfrak{H}^4_{K,\sigma_0}(E_0)$ and $\|\partial^3_\tau\kappa_E^{\varphi}\|_{L^2(\partial E)} \leq \frac{K}{h^{\frac{1}{4}}}$. Let $F \subset \mathbb{R}^2$ be a minimizer of (4.1) for $\eta = \eta_0$, where η_0 is given in Proposition 4.9. Then, for the heightfunction in (4.49), we have*

$$\|\psi\|_{L^2(\partial E)} \leq C_1 h, \quad \|\psi\|_{H^4(\partial E)} \leq C_1 \tag{4.59}$$

for all $h \leq h_0$ where h_0 is the constant from Proposition 4.9. The constant C_1 depends on K and K_{el}.

Proof. By the assumption on E, we deduce

$$\nu_E \in H^3(\partial E) \text{ and } \|\nu_E\|_{H^3(\partial E)} \leq C(K). \tag{4.60}$$

We multiply the Euler equation (4.58) by $\partial^4_\tau \psi$ and integrate over ∂E, obtaining

$$\begin{aligned} \frac{1}{h}\int_{\partial E} |\partial^2_\tau \psi|^2 + \int_{\partial E} g(\nu_E)|\partial^4_\tau \psi|^2 = \frac{1}{h}\int_{\partial E} \kappa_E \frac{\psi^2}{2}\partial^4_\tau \psi - \int_{\partial E} \partial^2_\tau g(\nu_E)\partial^2_\tau \psi \partial^4_\tau \psi \\ - \int_{\partial E} \partial_\tau g(\nu_E)\partial^3_\tau \psi \partial^4_\tau \psi + \int_{\partial E} \partial^2_\tau \kappa^\varphi_E \partial^4_\tau \psi \\ - \int_{\partial E} \partial^2_\tau (Q(E(u^{K_{el}}_F)))(\cdot + \psi(\cdot)\nu_E(\cdot))\partial^4_\tau \psi + \int_{\partial E} \partial^2_\tau R \partial^4_\tau \psi. \end{aligned} \tag{4.61}$$

We now proceed to estimate the right-hand side of the above equation. Let us fix $\varepsilon > 0$ to be chosen later.

Estimate of $\frac{1}{h}\int_{\partial E} \kappa_E \frac{\psi^2}{2}\partial^4_\tau \psi$.

Using the Cauchy–Schwarz and Young inequalities, together with formula (4.49) and the Sobolev embedding, we obtain

$$\begin{aligned} \frac{1}{h}\int_{\partial E} \kappa_E \psi^2 \partial^4_\tau \psi &\leq \frac{C(K)}{h}\|\psi\|_{L^\infty(\partial E)}\|\psi\|_{L^2(\partial E)}\|\partial^4_\tau \psi\|_{L^2(\partial E)} \\ &\leq \frac{C(K,\varepsilon)}{h^2}\|\psi\|^2_{L^\infty(\partial E)}\|\psi\|^2_{L^2(\partial E)} + \varepsilon\|\partial^4_\tau \psi\|^2_{L^2(\partial E)} \\ &\leq \frac{C(K,\varepsilon)}{h^2} h^{2(\frac{1}{2}+\frac{3}{4})} + \varepsilon\|\partial^4_\tau \psi\|^2_{L^2(\partial E)} \\ &\leq C(K,\varepsilon) + \varepsilon\|\partial^4_\tau \psi\|^2_{L^2(\partial E)}. \end{aligned} \tag{4.62}$$

Estimate of $\int_{\partial E} \partial^2_\tau g(\nu_E)\partial^2_\tau \psi \partial^4_\tau \psi$.

Using the Cauchy–Schwarz and Young inequalities, we obtain

$$\begin{aligned} \int_{\partial E} \partial^2_\tau g(\nu_E)\partial^2_\tau \psi \partial^4_\tau \psi &\leq \|\partial^2_\tau g(\nu_E)\|_{L^\infty(\partial E)}\|\partial^2_\tau \psi\|_{L^2(\partial E)}\|\partial^4_\tau \psi\|_{L^2(\partial E)} \\ &\leq C(K,\varepsilon)\|g(\nu_E)\|^2_{H^3(\partial E)}\|\partial^2_\tau \psi\|^2_{L^2(\partial E)} + \varepsilon\|\partial^4_\tau \psi\|^2_{L^2(\partial E)} \\ &\leq C(K,\varepsilon)\|\partial^2_\tau \psi\|^2_{L^2(\partial E)} + \varepsilon\|\partial^4_\tau \psi\|^2_{L^2(\partial E)} \end{aligned} \tag{4.63}$$

where in the last inequality we have used the smoothness of g is smooth and formula (4.60).

Estimate of $\int_{\partial E} \partial_\tau g(\nu_E)\partial^3_\tau \psi \partial^4_\tau \psi$.

Recalling the interpolation inequality (see Proposition 2.4),

$$\|\partial^3_\tau \psi\|_{L^2(\partial E)} \leq C\|\psi\|^{\frac{3}{4}}_{H^4(\partial E)}\|\psi\|^{\frac{1}{4}}_{L^2(\partial E)}$$

and using the Cauchy–Schwarz and Young inequalities, we obtain

$$\begin{aligned} &\int_{\partial E} \partial_\tau g(\nu_E)\partial^3_\tau \psi \partial^4_\tau \psi \leq C(K)\|\partial^3_\tau \psi\|_{L^2(\partial E)}\|\partial^4_\tau \psi\|_{L^2(\partial E)} \\ &\leq C(K,\varepsilon)\|\partial^3_\tau \psi\|^2_{L^2(\partial E)} + \frac{\varepsilon}{2}\|\partial^4_\tau \psi\|^2_{L^2(\partial E)} \\ &\leq C(K,\varepsilon)\|\psi\|^2_{L^2(\partial E)} + \varepsilon\|\psi\|^2_{H^4(\partial E)} \\ &\leq C(K,\varepsilon)\|\psi\|^2_{L^2(\partial E)} + \varepsilon C\big(\|\partial^4_\tau \psi\|_{L^2(\partial E)} + \|\psi\|_{L^\infty(\partial E)}(1 + \|\partial^3_\tau \kappa_E\|_{L^2(\partial E)})\big)^2 \\ &\leq C(K,\varepsilon)\|\psi\|^2_{L^2(\partial E)} + \varepsilon C\|\partial^4_\tau \psi\|^2_{L^2(\partial E)} + C(K), \end{aligned} \tag{4.64}$$

where in the fourth inequality we have used Lemma 4.10 and (4.49) that gives

$$\|\psi\|_{L^\infty(\partial A)}(1+\|\partial_\tau^3\kappa_E\|_{L^2(\partial E)})^2 \le C(K)h^{\frac{1}{2}}\frac{1}{h^{\frac{1}{2}}} \le C(K).$$

Estimate of $\int_{\partial E}\partial_\tau^2\kappa_E^\varphi\partial_\tau^4\psi$.

Using the Cauchy–Schwarz and Young inequalities and the bound $\|\partial_\tau^2\kappa_E^\varphi\|_{L^2(\partial E)} \le C(K)$, we obtain

$$\int_{\partial E}\partial_\tau^2\kappa_E^\varphi\partial_\tau^4\psi \le \|\partial_\tau^2\kappa_E^\varphi\|_{L^2(\partial E)}\|\partial_\tau^4\psi\|_{L^2(\partial E)} \le C(K,\varepsilon)+\varepsilon\|\partial_\tau^4\psi\|^2_{L^2(\partial E)}. \tag{4.65}$$

Estimate of $\int_{\partial E}\partial_\tau^2(Q(E(u_F^{K_{el}})))(\cdot+\psi(\cdot)\nu_E(\cdot))\partial_\tau^4\psi$.
Claim: It is holds

$$\|\partial_\tau^2(Q(E(u_F^{K_{el}})))(\cdot+\psi(\cdot)\nu_E(\cdot))\|_{L^2(\partial E)} \le C(K,K_{el})\big(\|\psi\|_{L^2(\partial E)}+\|\partial_\tau^2\psi\|_{L^2(\partial E)}\big). \tag{4.66}$$

Set $\hat F(x) := Q(E(u_F^{K_{el}}))(x)$ for all $x\in\Omega$. Then, for all $x\in\partial E$, we have

$$\partial_\tau\big(\hat F(x+\psi(x)\nu_E(x))\big) = \nabla\hat F(x+\psi(x)\nu_E(x))\cdot(\partial_\tau\psi(x)\nu_E(x)+(1+\psi(x)\kappa_E(x))\tau_E(x)).$$

Moreover,

$$\partial_\tau\nabla\hat F(x+\psi(x)\nu_E(x)) = \nabla^2\hat F(x+\psi(x)\nu_E(x))[(1+\psi(x)\kappa_E(x))\tau_E(x)+\partial_\tau\psi(x)\nu_E(x)]$$

and

$$\partial_\tau[(1+\psi\kappa_E)\tau_E+\partial_\tau\psi\nu_E] = \tau_E[2\kappa_E\partial_\tau\psi+\psi\partial_\tau\kappa_E]+\nu_E[\kappa_E+\psi\kappa_E^2+\partial_\tau^2\psi].$$

Therefore, by the Leibniz rule, we obtain

$$\begin{aligned}&\partial_\tau^2\big(\hat F(x+\psi(x)\nu_E(x))\big)\\ &\quad = \nabla^2\hat F(x+\psi(x)\nu_E(x))G(x,\psi(x)\kappa_E(x),\partial_\tau\psi(x))\cdot G(x,\psi(x)\kappa_E(x),\partial_\tau\psi(x))\\ &\qquad + \nabla\hat F(x+\psi(x)\nu_E(x))\cdot\bar G(x,\partial_\tau\psi(x)\kappa_E(x),\psi(x)\partial_\tau\kappa_E(x),\psi(x)\kappa_E^2(x),\kappa_E^2(x),\partial_\tau^2\psi(x)),\end{aligned} \tag{4.67}$$

where $G\in C^\infty(\mathbb{R}^3)$ and $\bar G\in C^\infty(\mathbb{R}^6,\mathbb{R}^2)$ satisfy $G(\cdot,0,0)=0$ and $\bar G(\cdot,0,0,0,0,0)=0$. Using formula (4.67) and recalling the very definition of $\hat F$ and that $u_F^{K_{el}}\in\mathfrak{C}^{3,\frac14}_{K_{el}}(\Omega,\mathbb{R}^2)$, see (3.15) and using the Soblev embedding we can estimate the $L^2(\partial E)$-norm of $x\to\partial_\tau^2\big(Q(E(u_F^{K_{el}}))(x+\psi(x)\nu_E(x))\big)$ and we obtain the claim.

Using the claim, along with the Cauchy–Schwarz and Young inequalities and Lemma 4.10, we finally obtain

$$\int_{\partial E}\partial_\tau^2(Q(E(u_F^{K_{el}})))(\cdot+\psi(\cdot)\nu_E(\cdot))\partial_\tau^4\psi \le C(K,K_{el},\varepsilon)\|\partial_\tau^2\psi\|^2_{L^2(\partial E)}+\varepsilon\|\partial_\tau^4\psi\|^2_{L^2(\partial E)}. \tag{4.68}$$

Estimate of $\int_{\partial E}\partial_\tau^2R\partial_\tau^4\psi$.

Using the Cauchy–Schwarz and Young inequalities, we obtain

$$\int_{\partial E}\partial_\tau^2R\partial_\tau^4\psi \le \|\partial_\tau^2R\|_{L^2(\partial E)}\|\partial_\tau^4\psi\|_{L^2(\partial E)} \le C(\varepsilon)\|\partial_\tau^2R\|^2_{L^2(\partial E)}+\varepsilon\|\partial_\tau^4\psi\|^2_{L^2(\partial E)}.$$

Hence we need to estimate $\|\partial_\tau^2R\|_{L^2(\partial E)}$. To estimate this term, we recall the form of R, see (4.48). Applying the Leibniz rule, we obtain

$$\begin{aligned}\|\partial_\tau^2R\|_{L^2(\partial E)} \le& C\sum_{j+k=2}\|\partial_\tau^jr_1(\psi\kappa_E,\partial_\tau\psi,\nu_E)\partial_\tau^{2+k}\psi\|_{L^2(\partial E)}\\ &+C\sum_{j+k=2}\||\partial_\tau^jr_2(\psi\kappa_E,\partial_\tau\psi,\nu_E)||\partial_\tau^{1+k}(\psi\kappa_E)|\|_{L^2(\partial E)}\\ &+\|r_0(\psi,\partial_\tau\psi,\kappa_E,\nu_E)\|_{H^2(\partial E)}.\end{aligned} \tag{4.69}$$

Let $j,k\in\mathbb{N}$ be such that $j+k=2$, we apply Lemma 2.5 with $f_1=r_1(\psi\kappa_E,\partial_\tau\psi,\nu_E)$ and $f_2=\partial_\tau\psi$ to estimate

$$\begin{aligned}\|\partial_\tau^j(r_1(\psi\kappa_e,\partial_\tau\psi,\nu_E))\partial_\tau^{2+k}\psi\|_{L^2(\partial E)} \leq & C\|r_1(\psi\kappa_E,\partial_\tau\psi,\nu_E)\|_{L^\infty(\partial E)}\|\psi\|_{H^4(\partial E)}\\ &+C\|\psi\|_{C^1(\partial E)}\|r_1(\psi\kappa_E,\partial_\tau\psi,\nu_E)\|_{H^3(\partial E)}.\end{aligned}\tag{4.70}$$

Similarly, with $f_1=r_2(\psi\kappa_E,\partial_\tau\psi,\nu_E)$ and $f_2=\psi\kappa_E$

$$\begin{aligned}\|\partial_\tau^j(r_2(\psi\kappa_E,\partial_\tau\psi,\nu_E))\partial_\tau^{1+k}(\psi\kappa_E)\|_{L^2(\partial E)} \leq & C\|r_2(\psi\kappa_E,\partial_\tau\psi,\nu_E)\|_{L^\infty(\partial E)}\|\psi\kappa_E\|_{H^3(\partial E)}\\ &+C\|\psi\kappa_E\|_{L^\infty(\partial E)}\|r_2(\psi\kappa_E,\partial_\tau\psi,\nu_E)\|_{H^3(\partial E)}.\end{aligned}\tag{4.71}$$

Since r_i is smooth and satisfies $r_i(0,0,\cdot)=0$, we have that

$$\|r_i(\psi\kappa_E,\partial_\tau\psi,\nu_E)\|_{L^\infty(\partial E)}\leq C\|\psi\|_{C^1(\partial E)}.$$

Furthermore, by the smoothness of r_i and the chain rule, we obtain the following pointwise estimate

$$\begin{aligned}|\partial_\tau r_i(\psi\kappa_E,\partial_\tau\psi,\nu_E)|&\leq C\big(1+|\partial_\tau^2\psi|+|\partial_\tau(\psi\kappa_E)|\big),\\ |\partial_\tau^2 r_i(\psi\kappa_E,\partial_\tau\psi,\nu_E)|&\leq C\sum_{\alpha\in\mathbb{N}^6,|\alpha|\leq 2}\prod_{k=1}^{2}(1+|\partial_\tau^{\alpha_k}(\psi\kappa_E)|)(1+|\partial_\tau^{1+\alpha_{2+k}}\psi|),\\ |\partial_\tau^3 r_i(\psi\kappa_E,\partial_\tau\psi,\nu_E)|&\leq C\sum_{\alpha\in\mathbb{N}^6,|\alpha|\leq 3}\prod_{k=1}^{3}(1+|\partial_\tau^{\alpha_k}(\psi\kappa_E)|)(1+|\partial_\tau^{1+\alpha_{3+k}}\psi|).\end{aligned}$$

Therefore, using Lemma 2.5 with $f_1=f_2=f_3=\psi\kappa_E$ and $f_4=f_5=f_6=\partial_\tau\psi$, we get

$$\begin{aligned}&\|r_i(\psi\kappa_E,\partial_\tau\psi,\nu_E)\|_{H^3(\partial E)}\\ &\quad\leq C(1+\|\psi\kappa_E\|_{L^\infty(\partial E)})(1+\|\psi\|_{H^4(\partial E)})+(1+\|\psi\|_{C^1(\partial E)})(1+\|\psi\kappa_E\|_{H^3(\partial E)})\\ &\quad\leq C(1+\|\psi\|_{H^4(\partial E)}+\|\psi\kappa_E\|_{H^3(\partial E)})\\ &\quad\leq C(1+\|\psi\|_{H^4(\partial E)}+\|\psi\|_{L^\infty(\partial E)}\|\kappa_E\|_{H^3(\partial E)})\\ &\quad\leq C(1+\|\psi\|_{H^4(\partial E)}),\end{aligned}\tag{4.72}$$

where we used the assumption $\|\partial_\tau^3\kappa_E^\varphi\|_{L^2(\partial E)}\leq\frac{K}{h^{\frac14}}$ and formula (4.49). To estimate the third term, we again apply the chain rule, the regularity of r_0, Lemma 2.5 and Proposition 2.4:

$$\begin{aligned}\|r_0(\psi,\partial_\tau\psi,\kappa_E,\nu_E)\|_{H^2(\partial E)}&\leq C(1+\|\psi\|_{H^3(\partial E)}+\|\kappa_E\|_{H^2(\partial E)})\leq C(1+\|\psi\|_{H^3(\partial E)})\\ &\leq\varepsilon\|\psi\|_{H^4(\partial E)}+C(K,\varepsilon)\|\psi\|_{H^2(\partial E)}+C(K).\end{aligned}\tag{4.73}$$

Combining estimates (4.69), (4.70), (4.71), (4.72), (4.73) and using Lemma 4.10 we obtain

$$\|\partial_\tau^2 R\|_{L^2(\partial E)}\leq C(K,\varepsilon)\|\psi\|^2_{H^2(\partial E)}+\varepsilon\|\partial_\tau^4\psi\|^2_{L^2(\partial E)}.\tag{4.74}$$

Using (4.61),(4.62), (4.63), (4.64),(4.65),(4.68), (4.74) and recalling (2.13) and for ε,h sufficiently small we deduce

$$\frac{1}{h}\|\partial_\tau^2\psi\|^2_{L^2(\partial E)}+\frac{C_g}{2}\|\partial_\tau^4\psi\|^2_{L^2(\partial E)}\leq C(K,K_{el}).$$

Therefore, we obtain the bound

$$\|\psi\|_{H^4(\partial E)}\leq C(K,K_{el}).\tag{4.75}$$

This implies that the right-hand side of equation (4.58) is bounded in $L^2(\partial E)$, i.e.,

$$\|\partial_\tau^2(-g(\nu_E)\partial_\tau^2\psi+\kappa_E^\varphi)-\partial_\tau^2(Q(E(u_F^{K_{el}}))(\cdot+\psi(\cdot)\nu_E(\cdot)))+\partial_\tau^2 R\|_{L^2(\partial E)}\leq C(K,K_{el}).\tag{4.76}$$

To prove $\|\psi\|_{L^2(\partial E)} \le C(K, K_{el})h$, we multiply the Euler–Lagrange equation (4.58) by $\psi + \frac{\psi^2}{2}\kappa_E$, integrate over ∂E, and apply the Cauchy–Schwarz inequality along with (4.76):

$$
\begin{aligned}
&\frac{1}{h}\|\psi + \frac{\psi^2}{2}\kappa_E\|^2_{L^2(\partial E)} \\
&\quad \le \|(\partial^2_\tau(-g(\nu_E)\partial^2_\tau\psi + \kappa^\varphi_E) - \partial^2_\tau(Q(E(u_F^{K_{el}}))(\cdot + \psi(\cdot)\nu_E(\cdot))) + \partial^2_\tau R)(\psi + \frac{\psi^2}{2}\kappa_E)\|_{L^2(\partial E)} \\
&\quad \le C(K, K_{el})\|\psi + \frac{\psi^2}{2}\kappa_E\|_{L^2(\partial E)}.
\end{aligned}
$$

Therefore we obtain $\|\psi + \frac{\psi^2}{2}\kappa_E\|_{L^2(\partial E)} \le C(K, K_{el})h$. Recalling that $\frac{\psi^2}{2} \le \big(\psi + \frac{\psi^2}{2}\kappa_E\big)^2$ (see formula (4.16)), we obtain

$$
\|\psi\|_{L^2(\partial E)} \le C(K, K_{el})h. \tag{4.77}
$$

Combining (4.75) and (4.77), we conclude the proof of (4.59). □

We need the following technical lemma; see [38, Lemma 5.3] for the proof. We state the lemma in $\mathbb{R}^2$, as this is the setting relevant to our context.

Lemma 4.12. *Let $E \subset \mathbb{R}^2$ of class C^5 be such that $E \in \mathfrak{H}^4_{K,\sigma_0}(E_0)$. Then, for all $u \in C^3(\partial E)$, the following estimates hold:*

$$
\begin{aligned}
&|\nabla(u \circ \pi_{\partial E})(x)| \le C(1 + |\kappa_E \circ \pi_{\partial E}(x)|)|\partial_{\partial E} u \circ \pi_{\partial E}(x)|, \\
&|\nabla^2(u \circ \pi_{\partial E})(x)| \le C \sum_{i=0,1}(1 + |\partial^i_{\partial E}\kappa_E \circ \pi_{\partial E}(x)|)|\partial^{2-i}_{\partial E} u \circ \pi_{\partial E}(x)|, \\
&|\nabla^3(u \circ \pi_{\partial E})(x)| \le C \sum_{i=0,1,2}(1 + |\partial^i_{\partial E}\kappa_E \circ \pi_{\partial E}(x)|)|\partial^{3-i}_{\partial E} u \circ \pi_{\partial E}(x)|,
\end{aligned} \tag{4.78}
$$

for all $x \in \mathcal{I}_{\sigma_E}(\partial E)$.

In the next proposition, we prove that if $F \subset \mathbb{R}^2$ is a minimizer of (4.1), then the following estimates hold:

$$
\|\kappa^\varphi_F\|_{H^2(\partial F)} \le C_2, \quad \|\kappa^\varphi_F\|_{H^3(\partial F)} \le \frac{C_2}{h^{\frac14}},
$$

where $C_2 := C_2(K, K_{el})$.

Proposition 4.13. *Let $E \subset \mathbb{R}^2$ of class C^5 be such that $E \in \mathfrak{H}^4_{K,\sigma_0}(E_0)$ and $\|\partial^3_{\partial E}\kappa^\varphi_E\|_{L^2(\partial E)} \le \frac{K}{h^{\frac14}}$. Let $F \subset \mathbb{R}^2$ be a minimizer of (4.1) for $\eta = \eta_0$, where η_0 is given by the Proposition 4.9. Let ψ be the heightfunction in (4.49) satisfying (4.59), that is,*

$$
\|\psi\|_{L^2(\partial E)} \le C_1 h, \qquad \|\partial^4_{\partial E}\psi\|_{L^2(\partial E)} \le C_1.
$$

Then there exists a constant C_2, depending only on K, K_{el}, such that

$$
\|\partial^2_{\partial F}\kappa^\varphi_F\|_{L^2(\partial F)} \le C_2, \qquad \|\partial^3_{\partial F}\kappa^\varphi_F\|_{L^2(\partial F)} \le \frac{C_2}{h^{\frac14}}. \tag{4.79}
$$

Proof. In what follows, we denote by C a generic constant that depends on K, K_{el}, C_1. We derive the estimates from the Euler–Lagrange equations (4.57). We define

$$
\tilde f := \frac{d_{H^{-1}}(F, E)}{h} f
$$

where f is the function that realizes the supremum in (3.1). The Euler-Lagrange equation then becomes

$$\begin{cases} \kappa_F^\varphi - Q(E(u_F^{K_{el}})) + \tilde{f} \circ \pi_{\partial E} = L & \text{on } \partial F, \\ -\Delta_{\partial E} \tilde{f} = \dfrac{\xi_{F,E}}{h} & \text{on } \partial E, \end{cases} \tag{4.80}$$

where $\xi_{F,E}$ is defined in (3.3) and L is the Lagrange multiplier. From (4.59) and interpolation of $H^1(\partial E)$ between $L^2(\partial E)$ and $H^4(\partial E)$, see Proposition 2.4, we obtain

$$\|\psi\|_{H^1}(\partial E) \leq Ch^{\frac{3}{4}}.$$

Using this estimate together with (4.6) and (4.59), we deduce

$$\|\xi_{F,E}\|_{L^2(\partial E)} \leq Ch \qquad \|\xi_{F,E}\|_{H^1(\partial E)} \leq Ch^{\frac{3}{4}}. \tag{4.81}$$

Therefore, by (4.81) and the second equation in (4.80), we conclude

$$\begin{aligned} &\|\tilde{f}\|_{H^2(\partial E)} \leq C\frac{\|\xi_{F,E}\|_{L^2(\partial E)}}{h} \leq C, \\ &\|\tilde{f}\|_{H^3(\partial E)} \leq C(1 + \|\kappa_E\|_{H^2(\partial E)} + \|\partial_{\partial E}\tilde{f}\|_{H^1(\partial E)}) \leq C(1 + \frac{\|\xi_{F,E}\|_{H^1(\partial E)}}{h}) \leq Ch^{-\frac{1}{4}}. \end{aligned} \tag{4.82}$$

We now need to estimate the derivatives of $\tilde{f} \circ \pi_{\partial E}$ on ∂F. From formula (4.78), we find that for all $x \in \mathcal{I}_{\eta_0}(\partial E)$,

$$\begin{aligned} &|\nabla(\tilde{f} \circ \pi_{\partial E})(x)| \leq C(1 + |\kappa_E \circ \pi_{\partial E}(x)|)|\partial_{\partial E}\tilde{f} \circ \pi_{\partial E}(x)|, \\ &|\nabla^2(\tilde{f} \circ \pi_{\partial E})(x)| \leq C\sum_{i=0,1}(1 + |\partial^i_{\partial E}\kappa_E \circ \pi_{\partial E}(x)|)|\partial^{2-i}_{\partial E}\tilde{f} \circ \pi_{\partial E}(x)|, \\ &|\nabla^3(\tilde{f} \circ \pi_{\partial E})(x)| \leq C\sum_{i=0,1,2}(1 + |\partial^i_{\partial E}\kappa_E \circ \pi_{\partial E}(x)|)|\partial^{3-i}_{\partial E}\tilde{f} \circ \pi_{\partial E}(x)|. \end{aligned} \tag{4.83}$$

To obtain (4.79), we need to estimate $\|\partial^2_{\partial F}\kappa_F^\varphi\|_{L^2(\partial F)}$ and $\|\partial^3_{\partial F}\kappa_F^\varphi\|_{L^2(\partial F)}$.

Estimate of $\|\partial^2_{\partial F}\kappa_F^\varphi\|_{L^2(\partial F)}$.

Recalling the first equation of (4.80), we need to estimate

$$\|\partial^2_{\partial F}(\tilde{f} \circ \pi_{\partial E})\|_{L^2(\partial F)} \text{ and } \|\partial^2_{\partial F}Q(E(u_F^{K_{el}}))\|_{L^2(\partial F)}.$$

We begin with the estimate of $\|\partial^2_{\partial F}(\tilde{f} \circ \pi_{\partial E})\|_{L^2(\partial F)}$. Using formula (4.83), we obtain

$$|\nabla^2(\tilde{f} \circ \pi_{\partial E})(x)| \leq C|\partial^2_{\partial E}\tilde{f} \circ \pi_{\partial E}(x)| + C(1 + |\partial_{\partial E}\kappa_E \circ \pi_{\partial E}(x)|)|\partial_{\partial E}\tilde{f} \circ \pi_{\partial E}(x)|$$

for all $x \in \partial F$. Therefore, by Sobolev embedding and (4.82), we get

$$\|\nabla^2(\tilde{f} \circ \pi_{\partial E})\|_{L^2(\partial F)} \leq C\big(\|\tilde{f}\|_{H^2(\partial E)} + (1 + \|\partial_{\partial E}\kappa_E\|_\infty\|\partial_{\partial E}\tilde{f}\|_{L^2(\partial E)})\big) \leq C. \tag{4.84}$$

Next, we recall that the Laplacian of $\tilde{f} \circ \pi_{\partial E}$ on ∂F can be written as

$$\partial^2_{\partial F}(\tilde{f} \circ \pi_{\partial E}) = \Delta_{\mathbb{R}^2}(\tilde{f} \circ \pi_{\partial E}) - \nabla^2(\tilde{f} \circ \pi_{\partial E})\nu_F \cdot \nu_F - \kappa_F\nabla(\tilde{f} \circ \pi_{\partial E}) \cdot \nu_F. \tag{4.85}$$

Thus, using this formula together with (4.84), we obtain

$$\|\partial^2_{\partial F}(\tilde{f} \circ \pi_{\partial E})\|_{L^2(\partial F)} \leq C. \tag{4.86}$$

It remains to estimate

$$\|\partial^2_{\partial F}Q(E(u_F^{K_{el}}))\|_{L^2(\partial F)}.$$

A direct computation yields

$$\begin{aligned} \partial^2_{\partial F}Q(E(u_F^{K_{el}})) &= \partial_{\partial F}[\partial_{\partial F}Q(E(u_F^{K_{el}}))] = \partial_{\partial F}[\nabla Q(E(u_F^{K_{el}})) \cdot \tau_F] \\ &= \nabla^2 Q(E(u_F^{K_{el}}))\tau_F \cdot \tau_F + \kappa_F\nabla Q(E(u_F^{K_{el}})) \cdot \nu_F \end{aligned} \tag{4.87}$$

where we have used $\partial_{\partial F}\tau_F = \kappa_F \nu_F$. Applying this formula, we deduce

$$\|\partial^2_{\partial F} Q(E(u_F^{K_{el}}))\|_{L^2(\partial F)} \le C. \tag{4.88}$$

Combining estimates (4.86) and (4.88), we conclude that

$$\|\partial^2_{\partial F} \kappa_F^{\varphi}\|_{L^2(\partial F)} \le C. \tag{4.89}$$

Estimate of $\|\partial^3_{\partial F}\kappa_F^{\varphi}\|_{L^2(\partial F)}$.

Recalling the first equation in (4.80), we need to estimate

$$\|\partial^3_{\partial F}(\tilde f \circ \pi_{\partial E})\|_{L^2(\partial F)} \text{ and } \|\partial^3_{\partial F} Q(E(u_F^{K_{el}}))\|_{L^2(\partial F)}.$$

We begin by estimating $\|\partial^3_{\partial F}(\tilde f \circ \pi_{\partial E})\|_{L^2(\partial F)}$.

Using formula (4.83), we obtain

$$\begin{aligned}|\nabla^3(\tilde f \circ \pi_{\partial E})(x)| \le{}& C|\partial^3_{\partial E}\tilde f \circ \pi_{\partial E}(x)| + C(1 + |\partial_{\partial E}\kappa_E \circ \pi_{\partial E}(x)|)|\partial^2_{\partial E}\tilde f \circ \pi_{\partial E}(x)| \\ & C\big(1 + |\partial^2_{\partial E}\kappa_E \circ \pi_{\partial E}(x)| + |\partial_{\partial E}\kappa_E \circ \pi_{\partial E}(x)|^2\big)|\partial_{\partial E}\tilde f \circ \pi_{\partial E}(x)|\end{aligned} \tag{4.90}$$

for all $x \in \partial F$. Using formula (4.85), we compute $\partial^3_{\partial F}(\tilde f \circ \pi_{\partial E})$

$$\begin{aligned}\partial^3_{\partial F}(\tilde f \circ \pi_{\partial E}) &= \nabla \Delta_{\partial F}(\tilde f \circ \pi_{\partial E}) \cdot \tau_F \\ &= \nabla\big[\Delta_{\mathbb{R}^2}(\tilde f \circ \pi_{\partial E}) - \nabla^2(\tilde f \circ \pi_{\partial E})\nu_F \cdot \nu_F - \kappa_F \nabla(\tilde f \circ \pi_{\partial E}) \cdot \nu_F\big] \cdot \tau_F \\ &= T(\nabla^3(\tilde f \circ \pi_{\partial E}), \nabla^2(\tilde f \circ \pi_{\partial E}), \partial_{\partial F}\kappa_F \nabla(\tilde f \circ \pi_{\partial E})),\end{aligned}$$

where $T \in C^\infty$ such that $T(0,0,0) = 0$. Hence, using the regularity of T, we deduce the pointwise estimate: for all $x \in \partial F$,

$$|\partial^3_{\partial F}(\tilde f \circ \pi_{\partial E})|(x) \le C\big(|\nabla^3(\tilde f \circ \pi_{\partial E})(x)| + |\nabla^2(\tilde f \circ \pi_{\partial E})(x)| + |\nabla(\tilde f \circ \pi_{\partial E})(x)|\big). \tag{4.91}$$

Therefore, combining (4.91), (4.90), (4.84), and (4.82), we obtain

$$\|\partial^3_{\partial F}(\tilde f \circ \pi_{\partial E})\|_{L^2(\partial F)} \le C\big(\|\tilde f\|_{H^3(\partial E)} + \|\kappa_E\|_{H^2(\partial E)}\big) \le Ch^{-\frac14}. \tag{4.92}$$

It remains to estimate $\|\partial^3_{\partial F} Q(E(u_F^{K_{el}}))\|_{L^2(\partial F)}$.

Differentiating formula (4.87), we get

$$\begin{aligned}\partial^3_{\partial F} Q(E(u_F^{K_{el}})) &= \partial_{\partial F}\big[\nabla^2 Q(E(u_F^{K_{el}}))\tau_F \cdot \tau_F\big] + \partial_{\partial F}[\kappa_F \nabla Q(E(u_F^{K_{el}})) \cdot \nu_F] \\ &= 2\kappa_F \nabla^2 Q(E(u_F^{K_{el}}))\tau_F \cdot \nu_F + M(\nabla^3 Q(E(u_F^{K_{el}})), \tau_F)\tau_F \cdot \tau_F \\ &\quad - \kappa_F^2 \tau_F \cdot \nabla Q(E(u_F^{K_{el}})) + \partial_{\partial F}\kappa_F \nabla Q(E(u_F^{K_{el}})) \cdot \nu_F \\ &\quad + \kappa_F \nabla^2 Q(E(u_F^{K_{el}}))\tau_F \cdot \nu_F\end{aligned} \tag{4.93}$$

where $M(\nabla^3 Q(E(u_F^{K_{el}})), \tau_F)$ is a matrix 2×2 matrix whose coefficients depend on $\nabla^3 Q(E(u_F^{K_{el}}))$ and τ_F, and satisfy

$$|M(\nabla^3 Q(E(u_F^{K_{el}})), \tau_F)| \le C|\nabla^3 Q(E(u_F^{K_{el}}))|.$$

Therefore, using (4.82), (4.93), and recalling that

$$\|\nabla^3 Q(E(u_F^{K_{el}}))\|_{L^\infty(\Omega)} \le \frac{K_{el}}{h^{\frac14}},$$

(see formula (3.13)), we obtain

$$\|\partial^3_{\partial F} Q(E(u_F^{K_{el}}))\|_{L^2(\partial F)} \le C\big(\|\nabla^3 Q(E(u_F^{K_{el}}))\|_{L^2(\partial F)} + C\|\partial_{\partial F}\kappa_F\|_{L^2(\partial F)}\big) \le \frac{C}{h^{\frac14}}. \tag{4.94}$$

Therefore, combining (4.92) and (4.94), we conclude that

$$\|\partial^3_{\partial F}\kappa_F^{\varphi}\|_{L^2(\partial F)} \le \frac{C}{h^{\frac14}}. \tag{4.95}$$

Combining (4.89) and (4.95), we finally obtain(4.79). □

We are now in position to prove Theorem 4.1.

Proof of Theorem 4.1. The existence of constants η_0 and h_0 is guaranteed by Proposition 4.9. Using this proposition, it is also established that $\partial F \Subset \mathcal{I}_{\eta_0}(\partial E)$ and that

$$\partial F = \{x + \psi(x)\nu_E(x) : x \in \partial E\}.$$

Proposition 4.11 establishes the existence of a constant C_1 and the validity of formula (4.2). Similarly, Proposition 4.13 proves the existence of a constant C_2 and formula (4.3).
Claim There exists $\hat{\sigma}$ such that $F \in \mathfrak{H}^4_{K_1,\hat{\sigma}}(E_0)$, for some $K_1 = K_1(K, K_{el})$.

By Lemma 4.7, the set F is a λ-minimizer of the φ-perimeter, with $\lambda = \lambda(K, K_{el})$. Applying Lemma 4.3 with $E = E_0$ and $\Lambda = \lambda$ we obtain the existence of the constant $\delta_0 = \delta_0(\lambda)$. Now we take σ_0, η_0 such that $\sigma_0 + \eta_0 \leq \frac{\delta_0}{2}$. Then we have:

$$\partial F \Subset \mathcal{I}_{\eta_0}(\partial E),\ \partial E \Subset \mathcal{I}_{\sigma_0}(\partial E_0) \text{ and } \sigma_0 + \eta_0 \leq \frac{\delta_0}{2} \implies \partial F \Subset \mathcal{I}_{\delta_0}(\partial E_0). \tag{4.96}$$

Applying Lemma 4.3 once again, we obtain the existence of a function $u : \partial E_0 \to \mathbb{R}$ such that

$$\partial F = \{x + u(x)\nu_{E_0}(x) : x \in \partial E_0\},$$

with $u \in C^{1,\gamma}(\partial E_0)$. Therefore, using (4.3) we conclude that

$$u \in H^4(\partial E_0), \quad \|u\|_{H^4(\partial E_0)} \leq K_1 \quad \text{for some } K_1 = K_1(K, K_{el}). \tag{4.97}$$

Combining (4.96) and (4.97), we obtain $F \in \mathfrak{H}^4_{K_1,\hat{\sigma}}$ as claimed. □

5. Iteration

In this section, we prove a crucial iteration formula. To this end, we fix a set $E \subset \mathbb{R}^2$ of class C^5 such that $E \in \mathfrak{H}^4_{K,\sigma_0}(E_0)$ and $\|\partial^3_{\partial E}\kappa^\varphi_E\|_{L^2(\partial E)} \leq \frac{K}{h^{\frac{1}{4}}}$. We recall that $E_0 \Subset \Omega$ be open and connected set of class C^5. We consider two sets $F, G \subset \mathbb{R}^2$ constructed as follows. By the Theorem 4.1, there exist constants $h_0, \eta_0, C_1, C_2, K_1, \hat{\sigma}$, depending only on K and K_{el}, such that if $0 < h \leq h_0$ and

$$F \in \operatorname{argmin}\{\mathcal{F}_h(A, E) \colon A \Delta E \subset \operatorname{cl}(\mathcal{I}_{\eta_0}(\partial E))\}$$

then F is of class C^5 and $F \in \mathfrak{H}^4_{K_1,\hat{\sigma}}(E_0)$. Again, by Proposition 2.4 and Theorem 4.1, we have $\partial F \Subset \mathcal{I}_{\eta_0}(\partial E)$, and $\partial F = \{x + \psi_{F,E}(x)\nu_E(x) : x \in \partial E\}$, with the following estimates for $\psi_{F,E}$:

(5.1)

$$\begin{aligned}
&\|\psi_{F,E}\|_{L^2(\partial E)} \leq C_1 h,\ \|\psi_{F,E}\|_{H^4(\partial E)} \leq C_1,\ \|\kappa^\varphi_F\|_{H^2(\partial F)} \leq C_2,\ \|\partial^3_{\partial F}\kappa^\varphi_F\|_{L^2(\partial F)} \leq \frac{C_2}{h^{\frac{1}{4}}}\\
&\|\partial_{\partial E}\psi_{F,E}\|_{L^2(\partial E)} \leq C_1 h^{\frac{3}{4}},\ \|\partial^2_{\partial E}\psi_{F,E}\|_{L^2(\partial E)} \leq C_1 h^{\frac{1}{2}},\ \|\partial^3_{\partial E}\psi_{F,E}\|_{L^2(\partial E)} \leq C_1 h^{\frac{1}{4}},
\end{aligned}$$

where the second line follows from the first and an application of Proposition 2.4. Applying Theorem 4.1 again this time with $F, K_1, \hat{\sigma}$ in place of E, K, σ_0, we get new constants $\eta_1, h_1, C_3, C_4, K_2, \tilde{\sigma}$, depending only on K_1 and K_{el} (and hence ultimately on K and K_{el}). If $\eta \leq \eta_1$ and $0 < h \leq h_2 := \min\{h_0, h_1\}$ the set G is given by

$$G \in \operatorname{argmin}\{\mathcal{F}_h(A, F) \colon A \Delta E \subset \operatorname{cl}(\mathcal{I}_\eta(\partial F))\}$$

and G is of class C^5, $G \in \mathfrak{H}^4_{K_2,\tilde{\sigma}}(E_0)$. Again, by Proposition 2.4 and Theorem 4.1, we have $\partial G \Subset \mathcal{I}_{\eta_1}(\partial F)$, and $\partial G = \{x + \psi_{G,F}(x)\nu_F(x) : x \in \partial F\}$, with the following estimates for

$\psi_{G,F}$:

(5.2)
$$\|\psi_{G,F}\|_{L^2(\partial F)} \le C_3 h,\ \|\psi_{G,F}\|_{H^4(\partial F)} \le C_3,\ \|\kappa_G^\varphi\|_{H^2(\partial G)} \le C_4,\ \|\partial^3_{\partial G}\kappa_G^\varphi\|_{L^2(\partial G)} \le \frac{C_4}{h^{\frac14}},$$
$$\|\partial_{\partial F}\psi_{G,F}\|_{L^2(\partial F)} \le C_3 h^{\frac34},\ \|\partial^2_{\partial F}\psi_{G,F}\|_{L^2(\partial F)} \le C_3 h^{\frac12},\ \|\partial^3_{\partial F}\psi_{G,F}\|_{L^2(\partial F)} \le C_3 h^{\frac14},$$

where the second line again follows from the first using Proposition 2.4. Throughout this section, we will use the notation just introduced. We now state a lemma that will be essential for proving the main result of this section.

Lemma 5.1. *Let $\eta < \eta_1$, where η_1 is as defined above. Let E, F and G as above, and we set $\xi_{G,F} = \psi_{G,F} + \kappa_G \frac{\psi^2_{G,F}}{2}$. There exists a constant $h_3 > 0$, depending only on K and K_{el}, such that the following inequality holds:*

$$\int_{\partial F} (1 - Ch)\xi^2_{G,F} + \frac{3h}{4} g(\nu_F)|\partial^2_{\partial F}\psi_{G,F}|^2 \, d\mathcal{H}^1 \le h \int_{\partial F} \kappa_F^\varphi \partial^2_{\partial F}\xi_{G,F} \, d\mathcal{H}^1. \tag{5.3}$$

for $0 < h \le h_3$, where $C = C(K, K_{el})$.

Proof. In what follows, we denote by C a generic constant depending on K and K_{el}. We recall that, as stated in formula (4.16)

$$\frac{1}{\sqrt{2}}\psi^2_{G,F} \le \xi^2_{G,F} \le \sqrt{2}\psi^2_{G,F}. \tag{5.4}$$

From the discussion at the beginning of the section, the Euler–Lagrange equation (4.58) for the set G can be written as

$$\begin{aligned}\frac{1}{h}\xi_{G,F}(x) =& \partial^2_{\partial F}(-g(\nu_F(x))\partial^2_{\partial F}\psi_{G,F}(x) + \kappa_F^\varphi(x)) \\ &- \partial^2_{\partial F}\big(Q(E(u_F^{K_{el}}))(x + \psi_{G,F}(x)\nu_F(x))\big) + \partial^2_{\partial F}R(x) \text{ for } x \in \partial F\end{aligned}$$

where R is define in (4.48). Multiplying the equation above by $\xi_{G,F}$ and integrating by parts yields

$$\begin{aligned}\int_{\partial F} \frac{\xi^2_{G,F}}{h} \, d\mathcal{H}^1 =& \int_{\partial F} -g(\nu_F)\partial^2_{\partial F}\psi_{G,F}\partial^2_{\partial F}\xi_{G,F} \, d\mathcal{H}^1 + \int_{\partial F} \kappa_F^\varphi \partial^2_{\partial F}\xi_{G,F} \, d\mathcal{H}^1 \\ &- \int_{\partial F} \xi_{G,F}\partial^2_{\partial F}\big(Q(E(u_F^{K_{el}}))(\cdot + \psi_{G,F}(\cdot)\nu_F(\cdot))\big) \, d\mathcal{H}^1 + \int_{\partial F} R\partial^2_{\partial F}\xi_{G,F} \, d\mathcal{H}^1.\end{aligned}$$

By the very definition of $\xi_{G,F}$, we obtain

$$\begin{aligned}&\int_{\partial F} \frac{\xi^2_{G,F}}{h} \, d\mathcal{H}^1 + \int_{\partial F} g(\nu_F)|\partial^2_{\partial F}\psi_{G,F}|^2 \, d\mathcal{H}^1 \\ &= \int_{\partial F} -g(\nu_F)\partial^2_{\partial F}\psi_{G,F}\partial^2_{\partial F}\left(\frac{\kappa_F\psi^2_{G,F}}{2}\right) d\mathcal{H}^1 + \int_{\partial F} \kappa_F^\varphi \partial^2_{\partial F}\xi_{G,F} \, d\mathcal{H}^1 \\ &\quad - \int_{\partial F} \xi_{G,F}\partial^2_{\partial F}\big(Q(E(u_F^{K_{el}}))(\cdot + \psi_{G,F}(\cdot)\nu_F(\cdot))\big) \, d\mathcal{H}^1 + \int_{\partial F} R\partial^2_{\partial F}\xi_{G,F} \, d\mathcal{H}^1.\end{aligned} \tag{5.5}$$

We now need to estimate the following integrals:

$$\int_{\partial F} g(\nu_F)\partial^2_{\partial F}\psi_{G,F}\partial^2_{\partial F}\left(\frac{\kappa_F\psi^2_{G,F}}{2}\right) d\mathcal{H}^1; \tag{5.6}$$

$$\int_{\partial F} \xi_{G,F}\partial^2_{\partial F}\big(Q(E(u_F^{K_{el}}))(\cdot + \psi_{G,F}(\cdot)\nu_F(\cdot))\big) \, d\mathcal{H}^1; \tag{5.7}$$

$$\int_{\partial F} R\partial^2_{\partial F}\xi_{G,F} \, d\mathcal{H}^1. \tag{5.8}$$

Let $\varepsilon > 0$ be fixed, to be chosen later.
Estimate of (5.6).

A straightforward computation yields

$$\begin{aligned}\partial^2_{\partial F}\left(\kappa_F\frac{\psi^2_{G,F}}{2}\right) =&\partial^2_{\partial F}\kappa_F\frac{\psi^2_{G,F}}{2} + 2\partial_{\partial F}\kappa_F\psi_{G,F}\partial_{\partial F}\psi_{G,F}\\ &+ \kappa_F(\partial_F\psi_{G,F})^2 + \kappa_F\psi_{G,F}\partial^2_{\partial F}\psi_{G,F}.\end{aligned} \tag{5.9}$$

Using formulas (5.2) and (5.4), a together with the Sobolev embedding and the Hölder inequality, we obtain

(5.10)
$$\begin{aligned}&\|\partial^2_{\partial F}\kappa_F\frac{\psi^2_{G,F}}{2} + 2\partial_{\partial F}\kappa_F\psi_{G,F}\partial_{\partial F}\psi_{G,F}\|_{L^2(\partial F)}\\ &\quad\le \big(\frac{\|\psi_{G,F}\|_{L^\infty(\partial F)}}{2}\|\kappa_F\|_{H^2(\partial F)} + 2\|\partial_{\partial F}\kappa_F\|_{L^2(\partial F)}\|\partial_{\partial F}\psi_{G,F}\|_{L^2(\partial F)}\big)\|\psi_{G,F}\|_{L^2(\partial F)}\\ &\quad\le C\big(\frac{C_2h^{\frac12}}{2} + 2C_2h^{\frac34}\big)\|\xi_{G,F}\|_{L^2(\partial E)} = Ch^{\frac12}\|\xi_{G,F}\|_{L^2(\partial E)}.\end{aligned}$$

Using again formula (5.2), the Sobolev embedding, and the Hölder inequality, we get

$$\begin{aligned}\int_{\partial F} g(\nu_F)\partial^2_{\partial F}\psi_{G,F}\kappa_F(\partial_F\psi_{G,F})^2\, d\mathcal{H}^1 &\le C\|\partial_{\partial F}\psi_{G,F}\|^2_{L^\infty(\partial F)}\|\partial^2_{\partial F}\psi_{G,F}\|_{L^2(\partial F)}\\ &\le Ch^{\frac12}\|\partial^2_{\partial F}\psi_{G,F}\|^2_{L^2(\partial F)}.\end{aligned} \tag{5.11}$$

Still using (5.2), the Sobolev embedding, and the Hölder inequality, we deduce

$$\begin{aligned}\int_{\partial F} g(\nu_F)\big(\partial^2_{\partial F}\psi_{G,F}\big)^2\kappa_F\psi_{G,F}\, d\mathcal{H}^1 &\le C\|\psi_{G,F}\|_{L^\infty(\partial F)}\|\partial^2_{\partial F}\psi_{G,F}\|^2_{L^2(\partial F)}\\ &\le Ch^{\frac34}\|\partial^2_{\partial F}\psi_{G,F}\|^2_{L^2(\partial F)}.\end{aligned} \tag{5.12}$$

Combining (5.6), (5.9), (5.10), (5.11), (5.12), and using the Hölder inequality, we obtain that

$$\int_{\partial F} g(\nu_F)\partial^2_{\partial F}\psi_{G,F}\partial^2_{\partial F}\left(\frac{\kappa_F\psi^2_{G,F}}{2}\right) d\mathcal{H}^1 \le \|\xi_{G,F}\|^2_{L^2(\partial F)} + h^{\frac12}C\|\partial^2_{\partial F}\psi_{G,F}\|^2_{L^2(\partial F)}. \tag{5.13}$$

Estimate of (5.7).

Recalling formula (4.66) and applying the Cauchy–Schwarz and Young inequalities, we get

$$\begin{aligned}&\int_{\partial F}\xi_{G,F}\partial^2_{\partial F}\big(Q(E(u_F^{K_{el}}))(\cdot + \psi_{G,F}(\cdot)\nu_F(\cdot))\big)\, d\mathcal{H}^1\\ &\qquad\qquad \le C(K,K_{el},\varepsilon)\|\xi_{G,F}\|^2_{L^2(\partial F)} + \varepsilon\|\partial^2_{\partial F}\psi_{G,F}\|^2_{L^2(\partial F)}.\end{aligned} \tag{5.14}$$

Estimate of (5.8).
Claim:

$$\|\partial_{\partial F}\psi_{G,F}\|_{L^2(\partial F)} \le \varepsilon C\|\partial^2_{\partial F}\psi_{G,F}\|_{L^2(\partial F)} + C(\varepsilon)\|\psi_{G,F}\|_{L^2(\partial F)}. \tag{5.15}$$

From formulas (4.6), (5.2), and (5.9), and using the Sobolev embedding and Proposition 2.4, we deduce

$$
\begin{aligned}
\|\partial_{\partial F}\psi_{G,F}\|_{L^2(\partial F)} &\le C\|\partial_{\partial F}\xi_{G,F}\|_{L^2(\partial F)} \le C\|\partial^2_{\partial F}\xi_{G,F}\|^{\frac12}_{L^2(\partial F)}\|\xi_{G,F}\|^{\frac12}_{L^2(\partial F)} \\
&= \varepsilon\|\partial^2_{\partial F}\psi_{G,F} + \partial^2_{\partial F}\big(\kappa_F\frac{\psi^2_{G,F}}{2}\big)\|_{L^2(\partial F)} + C(\varepsilon)\|\xi_{G,F}\|_{L^2(\partial F)} \\
&\le \varepsilon\|\partial^2_{\partial F}\psi_{G,F}\|_{L^2(\partial F)} + C\|\partial^2_{\partial F}\kappa_F\|_{L^2(\partial F)}\|\psi_{G,F}\|_\infty\|\psi_{G,F}\|_{L^2(\partial F)} \\
&\qquad + C\|\partial_{\partial F}\psi_{G,F}\|^2_{L^2(\partial F)} + C(\varepsilon)\|\xi_{G,F}\|_{L^2(\partial F)} \\
&\le \varepsilon\|\partial^2_{\partial F}\psi_{G,F}\|_{L^2(\partial F)} + Ch^{\frac34}\|\partial_{\partial F}\psi_{G,F}\|_{L^2(\partial F)} + C(\varepsilon)\|\xi_{G,F}\|_{L^2(\partial F)}.
\end{aligned}
\tag{5.16}
$$

Thus, for sufficiently small h, estimate (5.15) follows.
Claim:

$$
\|R\|_{L^2(\partial F)} \le \varepsilon C\|\partial^2_{\partial F}\psi_{G,F}\|_{L^2(\partial F)} + C(\varepsilon)\|\psi_{G,F}\|_{L^2(\partial F)}.
$$

To this end, we first require a pointwise estimate for R on ∂F. By the very definition of R, and using formula (4.48) along with the smallness of $\|\psi_{G,F}\|_{C^1(\partial F)} \le Ch^{\frac12}$ (this estimate follows from formula (5.2) and the Sobolev embedding), we obtain the pointwise estimate

$$
|R| \le C\big(|\psi_{G,F}| + |\partial_{\partial F}\psi_{G,F}|\big)\big(1 + |\partial^2_{\partial F}\psi_{G,F}| + |\partial_{\partial F}(\psi_{G,F}\kappa_F)|\big) \quad \text{on } \partial F.
\tag{5.17}
$$

From formula (5.2) and the Sobolev embedding, we derive the following estimates:

$$
\begin{aligned}
&\|\partial_{\partial F}(\psi_{G,F}\kappa_F)\|_{L^\infty(\partial F)} \le \|\psi_{G,F}\|_\infty\|\partial_{\partial F}\kappa_F\|_\infty + \|\partial_{\partial F}\psi_{G,F}\|_\infty\|\kappa_F\|_\infty \le Ch^{\frac12}, \\
&\|\partial^2_{\partial F}\psi_{G,F}\|_{L^\infty(\partial F)} \le Ch^{\frac14}.
\end{aligned}
\tag{5.18}
$$

Combining (5.17), (5.18), and using (4.16), (5.15), we obtain

$$
\begin{aligned}
\|R\|_{L^2(\partial F)} &\le C\big(\|\partial_{\partial F}\psi_{G,F}\|_{L^2(\partial F)} + \|\psi_{G,F}\|_{L^2(\partial F)}\big) \\
&\le \varepsilon C\|\partial^2_{\partial F}\psi_{G,F}\|_{L^2(\partial F)} + C(\varepsilon)\|\xi_{G,F}\|_{L^2(\partial F)}.
\end{aligned}
\tag{5.19}
$$

We are now in a position to estimate (5.8). Using formula (5.19), the definition of $\xi_{G,F}$, the Cauchy–Schwarz inequality, and the Young's inequality, we obtain

$$
\begin{aligned}
\int_{\partial F} R\partial^2_{\partial F}\xi_{G,F}\,d\mathcal{H}^1 &= \int_{\partial F} R\partial^2_{\partial F}\psi_{G,F}\,d\mathcal{H}^1 + \int_{\partial F} R\partial^2_{\partial F}\left(\kappa_F\frac{\psi^2_{G,F}}{2}\right)d\mathcal{H}^1 \\
&\le \|R\|_{L^2(\partial F)}\|\partial^2_{\partial F}\psi_{G,F}\|_{L^2(\partial F)} + \int_{\partial F} R\partial^2_{\partial F}\left(\kappa_F\frac{\psi^2_{G,F}}{2}\right)d\mathcal{H}^1 \\
&\le \varepsilon C\|\partial^2_{\partial F}\psi_{G,F}\|^2_{L^2(\partial F)} + C(\varepsilon)\|\psi_{G,F}\|^2_{L^2(\partial F)} + \int_{\partial F} R\partial^2_{\partial F}\left(\kappa_F\frac{\xi^2_{G,F}}{2}\right)d\mathcal{H}^1.
\end{aligned}
$$

It remains to estimate the term $\int_{\partial F} R\partial^2_{\partial F}\left(\kappa_F\frac{\psi^2_{G,F}}{2}\right)d\mathcal{H}^1$. Using (5.2), (5.9), (5.10), (5.15), the Sobolev embedding, we get

$$
\begin{aligned}
\|\partial^2_{\partial F}\left(\kappa_F\frac{\psi^2_{G,F}}{2}\right)\|_{L^2(\partial F)} &\le C\big(h^{\frac12}(\|\xi_{G,F}\|_{L^2(\partial F)} + \|\partial_{\partial F}\psi_{G,F}\|_{L^2(\partial F)}) + h\|\partial^2_{\partial F}\psi_{G,F}\|_{L^2(\partial F)}\big) \\
&\le Ch^{\frac12}\big(\|\xi_{G,F}\|_{L^2(\partial F)} + \|\partial^2_{\partial F}\psi_{G,F}\|_{L^2(\partial F)}\big).
\end{aligned}
\tag{5.20}
$$

Therefore, using the Cauchy–Schwarz inequality, estimates (5.19), (5.20), and Young's inequality, we obtain

$$\begin{aligned}\int_{\partial F} R\partial^2_{\partial F}\big(\kappa_F\frac{\psi^2_{G,F}}{2}\big)\,d\mathcal{H}^1 &\le C\|R\|_{L^2(\partial F)}h^{\frac12}\big(\|\xi_{G,F}\|_{L^2(\partial F)}+\|\partial^2_{\partial F}\psi_{G,F}\|_{L^2(\partial F)}\big)\\ &\le C(\varepsilon)\|\xi_{G,F}\|^2_{L^2(\partial F)}+\varepsilon\|\partial^2_{\partial F}\psi_{G,F}\|^2_{L^2(\partial F)}.\end{aligned} \tag{5.21}$$

Finally, inserting the estimates (5.13) for (5.6), (5.14) for (5.7), and (5.21) for (5.8), into formula (5.5), we obtain

$$\Big(\frac{1}{h}-C(\varepsilon)\Big)\int_{\partial F}\xi^2_{G,F}\,d\mathcal{H}^1+\int_{\partial F}(g(\nu_F)-\varepsilon)|\partial^2_{\partial F}\psi_{G,F}|\,d\mathcal{H}^1\le\int_{\partial F}\kappa^\varphi_F\partial^2_{\partial F}\xi_{G,F}\,d\mathcal{H}^1.$$

Recalling that $g\ge m_\varphi>0$ (see formula (2.1)), the inequality above implies formula (5.3) for sufficiently small h and ε. ☐

In the proof of the next proposition, we will use the following well-known inequality, whose proof follows from a classical homogenization argument and the Sobolev embedding of H^1 in L^∞. Let $A\in\mathfrak{C}^1_M(E_0)$, for some $M>0$. If f is a smooth function on ∂A, then there exists a constant $C(M)$ such that for every $\varepsilon\in(0,1)$,

$$\|f\|^2_{L^\infty(\partial A)}\le C(M)\big(\frac{1}{\varepsilon}\|f\|^2_{L^2(\partial A)}+\varepsilon\|\partial_A f\|^2_{L^2(\partial A)}\big). \tag{5.22}$$

We are now in a position to prove the main result of this section.

Proposition 5.2 (Iteration)**.** *Let E,F,G be as in Lemma 5.1 and, we set*

$$\xi_{F,E}=\psi_{F,E}+\kappa_E\frac{\psi^2_{F,E}}{2},\ \xi_{G,F}=\psi_{G,F}+\kappa_G\frac{\psi^2_{G,F}}{2}.$$

There exist M,h_4, depending only on K and K_{el}, such that

$$\int_{\partial F}\big(\xi^2_{G,F}+\frac{h}{2}g(\nu_F)|\Delta_{\partial F}\psi_{G,F}|^2\big)\,d\mathcal{H}^1\le(1+Mh)\int_{\partial E}\big(\xi^2_{F,E}+\frac{h}{8}g(\nu_F)|\Delta_{\partial E}\psi_{F,E}|^2\big)\,d\mathcal{H}^1 \tag{5.23}$$

for $0<h\le h_4$.

Proof. In what follows, we denote by C a generic constant depending on K and K_{el}. To prove the thesis, we need to estimate the term on the right-hand side of inequality (5.3), namely

$$h\int_{\partial F}\kappa^\varphi_F\partial^2_{\partial F}\xi_{G,F}\,d\mathcal{H}^1. \tag{5.24}$$

To this end, we consider the diffeomorphism

$$\Psi_{F,E}:\partial E\to\partial F\quad \Psi_{F,E}(x)=x+\psi_{F,E}(x)\nu_E(x)$$

and we define

$$\hat\kappa^\varphi_F(x):=\kappa^\varphi_F(\Psi_{F,E}(x)),\quad \hat\xi_{G,F}(x):=\xi_{G,F}(\Psi_{F,E}(x)),\quad \forall x\in\partial E.$$

We fix $\varepsilon>0$, to be chosen later. We set

$$J_{F,E}:=\sqrt{(1+\kappa_E\psi_{F,E})^2+|\partial_{\partial E}\psi_{F,E}|^2}.$$

By integrating by parts in (5.24), and using formula (2.8), the Young inequality and the Taylor expansion of the function $t \to \frac{1}{\sqrt{1+t}}$, we obtain

(5.25)
$$
\begin{aligned}
h\int_{\partial F} \kappa_F^\varphi \partial^2_{\partial F}\xi_{G,F}\, d\mathcal{H}^1 &= -\int_{\partial F} \nabla_{\partial F}\kappa_F^\varphi \cdot \nabla_{\partial F}\xi_{G,F}\, d\mathcal{H}^1 = -h\int_{\partial E} \frac{\nabla_{\partial E}\hat{\kappa}_F^\varphi \cdot \nabla_{\partial E}\hat{\xi}_{G,F}}{J_{F,E}}\, d\mathcal{H}^1 \\
&= -h\int_{\partial E} \nabla_{\partial E}\hat{\kappa}_F^\varphi \cdot \nabla_{\partial E}\hat{\xi}_{G,F}\, d\mathcal{H}^1 + h\int_{\partial E} \nabla_{\partial E}\hat{\kappa}_F^\varphi \cdot \nabla_{\partial E}\hat{\xi}_{G,F}\Big(1-\frac{1}{J_{F,E}}\Big)\, d\mathcal{H}^1 \\
&\le -h\int_{\partial E} \partial_{\partial E}\hat{\kappa}_F^\varphi \partial_{\partial E}\hat{\xi}_{G,F}\, d\mathcal{H}^1 + \varepsilon h C\int_{\partial F} |\partial_{\partial F}\xi_{G,F}|^2\, d\mathcal{H}^1 \\
&\qquad + \frac{C}{\varepsilon}h\int_{\partial E} |\partial_{\partial E}\hat{\kappa}_F^\varphi|^2\big(\psi_{F,E}^2+\psi_{F,E}^4+|\partial_{\partial E}\psi_{F,E}|^4\big)\, d\mathcal{H}^1.
\end{aligned}
$$

Using (5.22), (5.1), and the Sobolev embedding, and assuming h is sufficiently small with respect to ε, we estimate the last integral:

(5.26)
$$
\begin{aligned}
&\frac{Ch}{\varepsilon}\int_{\partial E} |\partial_{\partial E}\hat{\kappa}_F^\varphi|^2(\psi_{F,E}^2+\psi_{F,E}^4+|\partial_{\partial E}\psi_{F,E}|^4)\, d\mathcal{H}^1 \\
&\qquad \le \frac{Ch}{\varepsilon}\big(\|\psi_{F,E}\|^2_{L^\infty(\partial E)} + \|\psi_{F,E}\|^4_{L^\infty(\partial E)} + \|\partial_{\partial E}\psi_{F,E}\|^4_{L^\infty(\partial E)}\big) \\
&\qquad \le \frac{Ch}{\varepsilon}\Big[\frac{1}{\varepsilon^2}\|\psi_{F,E}\|^2_{L^2(\partial E)} + \varepsilon^2\|\partial_{\partial E}\psi_{F,E}\|^2_{L^2(\partial E)} \\
&\qquad\qquad + h^{\frac{3}{4}}\Big(\frac{1}{\varepsilon^2}\|\partial_{\partial E}\psi_{F,E}\|^2_{L^2(\partial E)} + \varepsilon^2\|\partial^2_{\partial E}\psi_{F,E}\|^2_{L^2(\partial E)}\Big)\Big] \\
&\qquad \le C(\varepsilon)h\|\psi_{F,E}\|^2_{L^2(\partial E)} + Ch\varepsilon(\|\partial_{\partial E}\psi_{F,E}\|^2_{L^2(\partial E)} + \|\partial^2_{\partial E}\psi_{F,E}\|^2_{L^2(\partial E)}) \\
&\qquad \le C(\varepsilon)h\|\xi_{F,E}\|^2_{L^2(\partial E)} + Ch\varepsilon\|\partial^2_{\partial E}\psi_{F,E}\|^2_{L^2(\partial E)},
\end{aligned}
$$

where in the last inequality we have used (5.4) and (5.15). Using the same reasoning as in (5.16), and recalling (5.4), we obtain

$$
\|\partial_{\partial F}\xi_{G,F}\|^2_{L^2(\partial F)} \le C\big(\|\xi_{G,F}\|^2_{L^2(\partial F)} + \|\partial^2_{\partial F}\psi_{G,F}\|^2_{L^2(\partial F)}\big). \tag{5.27}
$$

Plugging (5.26) and (5.27) into formula (5.25) and performing integration by parts yields

(5.28)
$$
\begin{aligned}
h\int_{\partial F} \kappa_F^\varphi \partial^2_{\partial F}\xi_{G,F}\, d\mathcal{H}^1 \le{}& h\int_{\partial E} \partial^2_{\partial E}\hat{\kappa}_F^\varphi \hat{\xi}_{G,F}\, d\mathcal{H}^1 + \varepsilon h C\big(\|\xi_{G,F}\|^2_{L^2(\partial F)} + \|\partial^2_{\partial F}\psi_{G,F}\|^2_{L^2(\partial F)}\big) \\
&+ C(\varepsilon)h\|\xi_{F,E}\|^2_{L^2(\partial E)} + Ch\varepsilon\|\partial^2_{\partial E}\psi_{F,E}\|^2_{L^2(\partial E)}.
\end{aligned}
$$

Recalling that F satisfies the Euler–Lagrange equation (4.58), we have

$$
\partial^2_{\partial E}\hat{\kappa}_F^\varphi(x) = \frac{1}{h}\xi_{F,E}(x) + \partial^2_E\big(Q(E(u_F^{K_{el}}))(x+\psi_{F,E}(x)\nu_E(x))\big), \quad x \in \partial E. \tag{5.29}
$$

We need to estimate

$$
h\int_{\partial E} \partial^2_{\partial E}\hat{\kappa}_F^\varphi \hat{\xi}_{G,F}\, d\mathcal{H}^1.
$$

Using (5.29), the Cauchy–Schwarz and Young inequalities, and (4.66), we obtain

$$
\begin{aligned}
&h\int_{\partial E}\partial^2_{\partial E}\hat\kappa^\varphi_F\hat\xi_{G,F}\,d\mathcal{H}^1=\int_{\partial E}\xi_{F,E}\hat\xi_{G,F}\,d\mathcal{H}^1\\
&\qquad\qquad+h\int_{\partial E}\partial^2_E\big(Q(E(u^{K_{el}}_F))(x+\psi_{F,E}(x)\nu_E(x))\big)\hat\xi_{G,F}(x)\,d\mathcal{H}^1_x\\
&\quad\le\frac12\int_{\partial E}\xi^2_{F,E}\,d\mathcal{H}^1+\frac12\int_{\partial E}\hat\xi^2_{G,F}\,d\mathcal{H}^1\\
&\qquad+h\Big(\int_{\partial E}(\partial^2_E\big(Q(E(u^{K_{el}}_F))(x+\psi_{F,E}(x)\nu_E(x))\big))^2\,d\mathcal{H}^1_x\Big)^{\frac12}\Big(\int_{\partial E}\hat\xi^2_{G,F}\,d\mathcal{H}^1\Big)^{\frac12}\\
&\quad\le\frac12\int_{\partial E}\xi^2_{F,E}\,d\mathcal{H}^1+\frac12\int_{\partial E}\frac{\hat\xi^2_{G,F}}{J_{F,E}}\,d\mathcal{H}^1+\frac12\int_{\partial E}\hat\xi^2_{G,F}\Big(1-\frac{1}{J_{F,E}}\Big)\,d\mathcal{H}^1\\
&\qquad+Ch\varepsilon\|\partial^2_{\partial E}\psi_{F,E}\|^2_{L^2(\partial E)}+hC(\varepsilon)\int_{\partial E}\frac{\hat\xi^2_{G,F}}{J_{F,E}}\,d\mathcal{H}^1\\
&\quad\le\frac12\int_{\partial E}\xi^2_{F,E}\,d\mathcal{H}^1+\frac12\int_{\partial F}\xi^2_{G,F}\,d\mathcal{H}^1+\frac12\int_{\partial E}\hat\xi^2_{G,F}\kappa_E\psi_{F,E}\,d\mathcal{H}^1\\
&\qquad+Ch\varepsilon\|\partial^2_{\partial E}\psi_{F,E}\|^2_{L^2(\partial E)}+hC(\varepsilon)\int_{\partial F}\xi^2_{G,F}\,d\mathcal{H}^1+Ch\|\psi_{F,E}\|^2_{L^2(\partial E)}\\
&\quad\le\frac12\int_{\partial E}\xi^2_{F,E}\,d\mathcal{H}^1+\frac12\int_{\partial F}\xi^2_{G,F}\,d\mathcal{H}^1+\frac12\int_{\partial E}\hat\xi^2_{G,F}\kappa_E\psi_{F,E}\,d\mathcal{H}^1\\
&\qquad+Ch\varepsilon\|\partial^2_{\partial E}\psi_{F,E}\|^2_{L^2(\partial E)}+hC(\varepsilon)\int_{\partial F}\xi^2_{G,F}\,d\mathcal{H}^1+Ch\|\psi_{F,E}\|^2_{L^2(\partial E)}.
\end{aligned}
\tag{5.30}
$$

We need to estimate $\frac12\int_{\partial E}\hat\xi^2_{G,F}\kappa_E\psi_{F,E}\,d\mathcal{H}^1$. To this aim we recall that

$$
\begin{cases}
-\Delta_{\partial E}v_{F,E}=\xi_{F,E}=\psi_{F,E}+\dfrac{\psi^2_{F,E}}{2}\kappa_E & \text{on } \partial E,\\
\displaystyle\int_{\partial E}v_{F,E}\,d\mathcal{H}^1=0
\end{cases}
$$

and $\|\nabla_{\partial E}v_{F,E}\|_{L^2(\partial E)}=d_{H^{-1}}(F,E)\le Ch$, see formulas (3.7), (3.8) and (4.40). Therefore we have that

$$
\int_{\partial E}\hat\xi^2_{G,F}\kappa_E\psi_{F,E}\,d\mathcal{H}^1=-\int_{\partial E}\hat\xi^2_{G,F}\kappa_E\frac{\psi^2_{F,E}}{2}\,d\mathcal{H}^1-\int_{\partial E}\hat\xi^2_{G,F}\kappa_E\Delta_{\partial E}v_{F,E}\,d\mathcal{H}^1. \tag{5.31}
$$

Using formula (5.1) we obtain

$$
\int_{\partial E}\hat\xi^2_{G,F}\kappa_E\frac{\psi^2_{F,E}}{2}\le h^{\frac32}C\int_{\partial E}\hat\xi^2_{G,F}\le hC\int_{\partial F}\xi^2_{G,F}\,d\mathcal{H}^1. \tag{5.32}
$$

Now by the divergence theorem and using (4.40), (5.1) and the Poincarè inequality and the Sobolev embedding we have that

$$
\begin{aligned}
&\int_{\partial E} -\hat{\xi}^2_{G,F}\kappa_E \Delta_{\partial E} v_{F,E}\, d\mathcal{H}^1 \\
&\qquad = \int_{\partial E} \nabla_{\partial E}\hat{\xi}_{G,F}\cdot\nabla_{\partial E} v_{F,E}\kappa_E\hat{\xi}_{G,F}\, d\mathcal{H}^1 + \int_{\partial E}\hat{\xi}^2_{G,F}\nabla_{\partial E} v_{F,E}\cdot\nabla_{\partial E}\kappa_E\, d\mathcal{H}^1 \\
&\qquad \le C\|\nabla_{\partial E} v_{F,E}\|_{L^2(\partial E)}\big(\|\hat{\xi}^2_{G,F}\|_{L^2(\partial E)} + \|\nabla_{\partial E}\hat{\xi}_{G,F}\|^2_{L^2(\partial E)}\big) \\
&\qquad \le Ch\big(\|\hat{\xi}_{G,F}\|_{L^\infty(\partial E)}\|\hat{\xi}_{G,F}\|_{L^2(\partial E)} + \|\nabla_{\partial F}\xi_{G,F}\|^2_{L^2(\partial F)}\big) \\
&\qquad \le Ch\big(\|\nabla_{\partial E}\hat{\xi}_{G,F}\|_{L^2(\partial E)}\|\hat{\xi}_{G,F}\|_{L^2(\partial E)} + \|\xi_{G,F}\|^2_{L^2(\partial F)} + \varepsilon\|\Delta_{\partial F}\psi_{G,F}\|^2_{L^2(\partial F)}\big) \\
&\qquad \le Ch\big(\|\nabla_{\partial E}\hat{\xi}^2_{G,F}\|^2_{L^2(\partial E)} + \|\xi_{G,F}\|^2_{L^2(\partial F)} + \varepsilon\|\Delta_{\partial F}\psi_{G,F}\|^2_{L^2(\partial F)}\big) \\
&\qquad \le Ch\big(\|\xi_{G,F}\|^2_{L^2(\partial F)} + \varepsilon\|\Delta_{\partial F}\psi_{G,F}\|^2_{L^2(\partial F)}\big)
\end{aligned} \tag{5.33}
$$

where we have used Lemma 2.2 to get

$$
\|\nabla_{\partial E}\hat{\xi}_{G,F}\|^2_{L^2(\partial E)} \le C(K)\int_{\partial E}\frac{|\nabla_{\partial E}\hat{\xi}_{G,F}|^2}{J_{F,E}}\, d\mathcal{H}^1 = C\|\nabla_{\partial F}\xi_{G,F}\|^2_{L^2(\partial F)}.
$$

Using formulas (5.30), (5.31), (5.32), and (5.33), we get

$$
\begin{aligned}
h\int_{\partial E}\partial^2_{\partial E}\hat{\kappa}^{\varphi}_F\hat{\xi}_{G,F}\, d\mathcal{H}^1 \le{}& \frac{1}{2}\int_{\partial E}\xi^2_{F,E}\, d\mathcal{H}^1 + \frac{1}{2}\int_{\partial F}\xi^2_{G,F}\, d\mathcal{H}^1 + Ch\varepsilon\|\partial^2_{\partial E}\psi_{F,E}\|^2_{L^2(\partial E)} \\
&+ hC(\varepsilon)\int_{\partial F}\xi^2_{G,F}\, d\mathcal{H}^1 + Ch\|\psi_{F,E}\|^2_{L^2(\partial E)} + \varepsilon h\|\partial^2_{\partial F}\psi_{G,F}\|^2_{L^2(\partial F)}.
\end{aligned} \tag{5.34}
$$

Therefore, combining (5.3), (5.28), and (5.34), and recalling that $M_\varphi \ge g \ge m_\varphi > 0$ (see formula (2.1)), we conclude:

$$
\begin{aligned}
&\big(\frac{1}{2} - hC\big)\int_{\partial F}\xi^2_{G,F}\, d\mathcal{H}^1 + h\big(\frac{3}{4} - \varepsilon C\big)\int_{\partial F} g(\nu_F)|\partial^2_{\partial F}\psi_{G,F}|^2\, d\mathcal{H}^1 \\
&\qquad\qquad \le \big(\frac{1}{2} + hC\big)\int_{\partial E}\xi^2_{F,E}\, d\mathcal{H}^1 + hC\varepsilon\int_{\partial E} g(\nu_E)|\partial^2_{\partial E}\psi_{F,E}|^2\, d\mathcal{H}^1.
\end{aligned}
$$

Choosing ε and h sufficiently small concludes the proof of (5.23). □

6. Proof of the main theorems

In this section, we use the iteration estimates proved in the previous section to show that the constrained discrete flat flow, defined in 3.7, converge, as $h \to 0$, to the classical solution of the equation (1.5), provided K_{el} is sufficiently large. We recall that $E_0 \Subset \Omega$ be open and connected set of class C^5.

Here and in the following, we reuse the notation introduced in formula (3.13) and (3.14). Specifically, we denote by $u_F^{K_{el},h}$ a solution to the minimization problem

$$
\min\left\{\int_{\Omega\setminus F} Q(E(u))\, dx \colon u \in \mathfrak{C}^{3,\frac{1}{4}}_{K_{el}}(\Omega,\mathbb{R}^2),\ \|\nabla^4 u\|_{C^{0,\frac{1}{4}}(\Omega)} \le \frac{K_{el}}{h^{\frac{1}{4}}}, u|_{\partial\Omega} = w_0\right\} \tag{6.1}
$$

where $w_0 \in C^{3,\frac{1}{4}}(\partial\Omega)$ is the prescribed boundary displacement. We denote by $u_F^{K_{el},0}$ a solution to the minimization problem

$$
\min\left\{\int_{\Omega\setminus F} Q(E(u))\, dx \colon u \in \mathfrak{C}^{3,\frac{1}{4}}_{K_{el}}(\Omega,\mathbb{R}^2),\ u|_{\partial\Omega} = w_0\right\} \tag{6.2}
$$

where w_0 is as above.

Before proving the first theorem of this section, we establish a lemma that ensures, under suitable assumptions, that the minimizers of problem (6.1) converge to those of problem (6.2) as $h \to 0^+$.

Lemma 6.1. *Let $F_h \Subset \Omega$ be such that $\chi_{F_h} \to \chi_F$ in $L^1(\Omega)$ with $F \Subset \Omega$. Let $u_{F_h}^{K_{el},h}$ be a minimizer of* (6.1) *for $h > 0$. Then $u_{F_h}^{K_{el},h} \to u_F^{K_{el},0}$ as $h \to 0^+$ in $\mathfrak{C}^{3,\frac{1}{4}}_{K_{el}}(\Omega, \mathbb{R}^2)$ where $u_F^{K_{el},0}$ is a minimizer of* (6.2).

Proof. In the proof of the lemma, we will omit explicitly mentioning 'up to subsequences' for the sake of brevity. By the Ascoli-Arzelà Theorem we have that $u_{F_h}^{K_{el},h} \to u$ in $C^{3,\frac{1}{4}}(\Omega)$. Let $v \in C^4(\Omega)$ with $\|v\|_{C^{3,\frac{1}{4}}(\Omega)} \leq K_{el}$ then for h sufficiently small we get

$$\begin{aligned}\int_{\Omega\setminus F} Q(E(u))\,dx &= \lim_{h\to 0^+}\int_{\Omega\setminus F_h} Q(E(u_{F_h}^{K_{el},h}))\,dx\\ &\leq \lim_{h\to 0^+}\int_{\Omega\setminus F_h} Q(E(v))\,dx = \int_{\Omega\setminus F} Q(E(v))\,dx.\end{aligned}$$

Therefore by the above formula and using a standard density argument we get the thesis. □

Theorem 6.2. *Let $K_{el} > 0$ be fixed. There exist $T_0, C_0, \beta_0, \sigma_1$ with the following property: for every $\beta < \beta_0$ there exists $\tilde{h}$ such that $E_t^{h,\beta} \in \mathfrak{H}^4_{C_0,\sigma_1}(E_0)$, i.e.,*

$$\partial E_t^{h,\beta} = \{x + f^{h,\beta}(t,x)\nu_{E_0}(x) \colon x \in \partial E_0\},\ \|f^{h,\beta}\|_{H^4(\partial E_0)} \leq C_0,\ \|f^{h,\beta}\|_{L^\infty(\partial E_0)} \leq \sigma_1,$$

for all $t \in [0, T_0]$ and $0 < h \leq \tilde{h}$, where $\{E_t^{h,\beta}\}_{t\geq 0}$ is a discrete constrained flat flow starting from E_0.

The function $f^{h,\beta}$ converge in $L^\infty([0,T_0], H^4(\partial E_0))$ to a function f^β such that the family $\{E_t^\beta\}_{t\in[0,T_0]}$ with

$$\partial E_t^\beta = \{x + f^\beta(t,x)\nu_{E_0}(x) : x \in \partial E_0\}$$

is a distributional solution of the problem

$$\begin{cases} V_t = \partial^2_{\partial E_t^\beta}\big(\kappa^\varphi_{E_t^\beta} - Q(E(u_{E_t^\beta}^{K_{el},0}))\big), \text{ on } \partial E_t^\beta\\ E_0^\beta = E_0,\\ u_{E_t^\beta}^{K_{el},0} \in \operatorname{argmin}\left\{\displaystyle\int_{\Omega\setminus E_t^\beta} Q(E(u))\,dx \colon u \in \mathfrak{C}^{3,\frac{1}{4}}_{K_{el}}(\Omega,\mathbb{R}^2),\ u|_{\partial\Omega} = w_0\right\}. \end{cases} \tag{6.3}$$

Moreover $f^\beta \in \mathrm{Lip}([0,T_0], L^2(\partial E_0))$ and

$$\|f^\beta(t,\cdot)\|_{C^{3,\frac{1}{4}}(\partial E_0)} \leq Ct^{\frac{1}{21}} \tag{6.4}$$

where $C = C(K_{el})$.

Proof. In the proof of the theorem, we will omit explicit mention of 'up to subsequences' for the sake of brevity, unless it is strictly necessary for clarity. We fix a large constant $K_0 = K_0(K_{el}, \sigma_0)$, which will be chosen later. Let $\beta_0 < \eta_1$ where η_1 is the constant from Lemma 5.1. We fix $\beta \leq \beta_0$. Let $\{E_{hk}^{h,\beta}\}_{k\in\mathbb{N}}$ be a constrained discrete flat flow starting from E_0; see definition 3.7. To simplify notation, we write $E_k = E_{hk}^{h,\beta}$ for $k \geq 0$. We are now in a position to apply Theorem 4.1, which yields

$$
\begin{aligned}
&\partial E_1 = \{x + \psi_1(x)\nu_{E_0}(x) \colon x \in \partial E^0\},\\
&\|\psi_1\|_{L^2(\partial E^0)} \le L_0 h, \quad \|\psi_1\|_{H^4(\partial E^0)} \le L_0,\\
&\|\kappa^{\varphi}_{E_1}\|_{H^2(\partial E_1)} \le K_0, \quad \|\partial^3_{\partial E_1}\kappa^{\varphi}_{E_1}\|_{L^2(\partial E_1)} \le \frac{K_0}{h^{\frac14}},
\end{aligned} \tag{6.5}
$$

where $L_0 = L_0(K_{el})$. Moreover, using Proposition 2.4, we have

$$
\|\partial_{\partial E_0}\psi_1\|_{L^2(\partial E_0)} \le L_0 h^{\frac34}, \quad \|\partial^2_{\partial E_0}\psi_1\|_{L^2(\partial E_0)} \le L_0 h^{\frac12}, \quad \|\partial^3_{\partial E_0}\psi_1\|_{L^2(\partial E_0)} \le L_0 h^{\frac14}. \tag{6.6}
$$

We denote by $k_0 \in \mathbb{N}$ the largest index such that it holds

$$
\partial E_k \subset \mathcal{I}_\beta(\partial E_0) \quad \forall k \le k_0.
$$

We set $T_0 := k_0 h$.

Claim 1: For every $k \le k_0$, the following holds:

$$
\|\kappa^{\varphi}_{E_k}\|_{H^2(\partial E_k)} \le K_0, \quad \|\partial^3_{\partial E_k}\kappa^{\varphi}_{E_k}\|_{L^2(\partial E_k)} \le \frac{K_0}{h^{\frac14}}. \tag{6.7}
$$

We prove (6.7) by induction. The base case is verified since the claim holds for $k = 1$; see formula (6.5). Assume that the claim holds for all integers up to $k-1$. Then, by applying Theorem (4.1), we obtain

$$
\begin{aligned}
&\partial E_k = \{x + \psi_k(x)\nu_{E_{k-1}}(x) \colon x \in \partial E_{k-1}\},\\
&\|\psi_k\|_{L^2(\partial E_{k-1})} \le L_1 h, \quad \|\psi_k\|_{H^4(\partial E_{k-1})} \le L_1,
\end{aligned}
$$

where $L_1 = L_1(K, K_{el})$. For every $j \ge 1$, we set $\xi_j = \xi_{E^j, E^{j-1}}$; see (3.3). Using Proposition 5.2, we get

$$
\begin{aligned}
&\int_{\partial E_{j-1}} \big(\xi_j^2 + \frac{h}{2} g(\nu_{E_{j-1}})|\Delta_{\partial E_{j-1}}\psi_j|^2\big)\, d\mathcal{H}^1\\
&\qquad\qquad \le (1 + Mh)\int_{\partial E_{j-2}} \big(\xi_{j-1}^2 + \frac{h}{8} g(\nu_{E_{j-2}})|\Delta_{\partial E_{j-2}}\psi_{j-1}|^2\big)\, d\mathcal{H}^1
\end{aligned}
$$

for every $1 \le j \le k$. We recall (see formula (4.16)) that

$$
\frac{1}{\sqrt{2}}\psi_j^2 \le \xi_j^2 \le \sqrt{2}\psi_j^2 \text{ for every } j. \tag{6.8}
$$

By iterating the estimate above and using (6.5) and (6.6), we obtain

$$
\begin{aligned}
\int_{\partial E_{k-1}} \big(\xi_k^2 + \frac{h}{4}\sum_{j=1}^{k} g(\nu_{E_{j-1}})|\Delta_{\partial E_{j-1}}\psi_j|^2\big) &\le (1 + Mh)^{k-1}\int_{\partial E^0} \big(\xi_1^2 + \frac{h}{2} g(\nu_{E_0})|\Delta_{\partial E_0}\psi_1|^2\big)\\
&\le e^{2Mhk} L_0^2 h^2\\
&\le e^{2Mhk_0} L_0^2 h^2 \le 2L_0^2 h^2,
\end{aligned}
$$

where we have used $hk_0 = T_0$ and that T_0 is small. Possibly increasing the value of L_0, and using the above inequality along with (6.8), we obtain

$$
\|\psi_k\|^2_{L^2(\partial E_{k-1})} + h\sum_{j=1}^{k} \|\Delta_{\partial E_{k-1}}\psi_k\|^2_{L^2(\partial E_{k-1})} \le L_0^2 h^2. \tag{6.9}
$$

Therefore, we can apply Proposition 4.13 and conclude the proof of the claim (6.7), possibly after increasing the constant K_0.

Claim 2: $T_0 > 0$.

We can assume that for the set E_{k_0}, there exists a point $x_0 \in \partial E_{k_0}$ such that $\operatorname{dist}(x_0, E_0) \ge \frac{\beta}{2}$. The set E_{k_0} satisfies the assumptions of Lemma 4.7 and therefore it is a Λ-minimizer

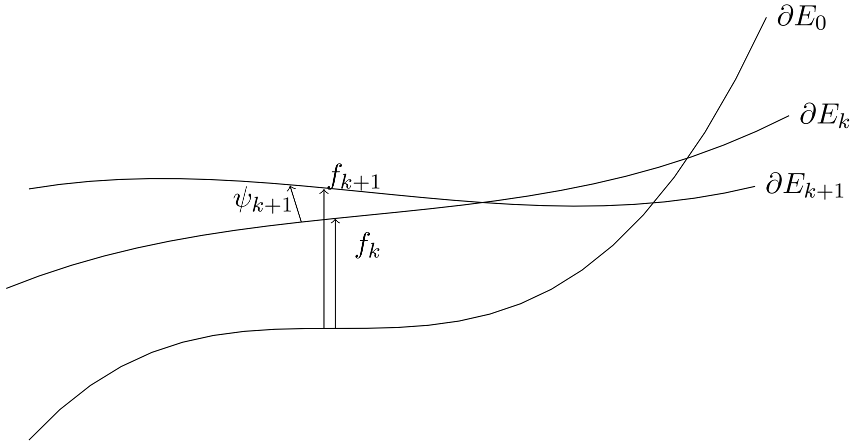

Figure 1. Boundary of E_k, E_{k+1} and functions f_k, f_{k+1}, ψ_{k+1}

of the φ-perimeter for a Λ that is independent of h. Consequently, for E_{k_0} the density estimates are satisfied, both for the perimeter and for the volume; see [8]. Using these density estimates together with the inequality $\mathrm{dist}(x_0, E_0) \geq \frac{\beta}{2}$, we obtain

$$|E_{k_0} \Delta E_0| \geq c\beta^2,$$

where c depends on Λ. Now, using the inequality above together with (6.8), (6.9) and the triangular inequality, we derive

$$\begin{aligned}
c\beta^2 \leq |E_{k_0}\Delta E_0| &\leq \sum_{j=1}^{k_0} |E_j \Delta E_{j-1}| = \sum_{j=1}^{k_0} \|\xi_j\|_{L^1(\partial E_{j-1})} \leq P(E_{j-1})^{\frac{1}{2}} \sum_{j=1}^{k_0} \|\xi_j\|_{L^2(\partial E_{j-1})} \\
&\leq C_\varphi P_\varphi(E_{j-1})^{\frac{1}{2}} \sum_{j=1}^{k_0} \|\xi_j\|_{L^2(\partial E_{j-1})} \leq C_\varphi (P_\varphi(E_0) + K_{el}^2|\Omega|)^{\frac{1}{2}} \sum_{j=1}^{k_0} \|\psi_j\|_{L^2(\partial E_{j-1})} \\
&\leq C_\varphi (P_\varphi(E_0) + K_{el}^2|\Omega|)^{\frac{1}{2}} L_0 k_0 h = C_\varphi (P_\varphi(E_0) + K_{el}^2|\Omega|)^{\frac{1}{2}} L_0 T_0
\end{aligned}$$

where we have used that $P_\varphi(E_j) \leq K_{el}^2|\Omega| + P_\varphi(E_0)$, which follows from the minimizing movements scheme.

Claim 3: There exist constants $C_0, \sigma_1 > 0$ such that

$$E_j \in \mathfrak{H}^4_{C_0,\sigma_1}(E_0) \text{ for all } 0 \leq j \leq k_0. \tag{6.10}$$

This claim follows by adapting the arguments from the proof of Theorem 4.1. We provide here a sketch of the proof. As in the previous claim, we may apply Lemma 4.7, which implies that each E_j is a Λ-minimizer of the φ-perimeter for some Λ independent of h. Then, using Lemma 4.3, we deduce that each E_j is a normal graph over ∂E_0, with

$$\mathrm{dist}(\partial E_j, \partial E_0) \leq \beta \leq \sigma_1 \text{ for all } j.$$

If σ_1 is small enough, we can again apply Lemma 4.3 to conclude that each E_j is a normal graph over ∂E_0. Therefore, there exist functions $f_j : \partial E_0 \to \mathbb{R}$ such that

$$\partial E_j = \{x + f_j(x)\nu_{E_0}(x) \colon x \in \partial E_0\}.$$

Moreover, by Lemma 4.3, we have that $\|f_j\|_{C^{1,\gamma}}(\partial E_0) \leq C$ for some $C > 0$. Using formula (6.7) (the bound $\|\kappa^\varphi_{E_j}\|_{H^2(\partial E^j)} \leq K_0$) we deduce that $\|f_j\|_{H^4(\partial E_0)} \leq C_0$.

Claim 4: There exists a constant $L_{lip} > 0$ such that for all $0 \leq i, k \leq k_0$,

$$\|f_i - f_k\|_{L^1(\partial E_0)} \leq L_{lip} h| - 1 + i - k|. \tag{6.11}$$

Without loss of generality, suppose $i < k$. The claim follows from the following estimate:

$$\begin{aligned}\|f_i - f_k\|_{L^1(\partial E_0)} &\leq |E_k \Delta E_i| \leq \sum_{j=i+1}^{k} |E_j \Delta E_{j-1}| = \sum_{j=i+1}^{k} \|\xi_j\|_{L^1(\partial E_{j-1})}\\ &\leq P(E_{j-1})^{\frac{1}{2}} \sum_{j=i+1}^{k} \|\xi_j\|_{L^2(\partial E_{j-1})}\\ &\leq \sqrt{2}(K_{el}^2|\Omega| + P(E_0))^{\frac{1}{2}} \sum_{j=i+1}^{k} \|\psi_j\|_{L^2(\partial E_{j-1})}\\ &\leq \sqrt{2}(K_{el}^2|\Omega| + P(E_0))^{\frac{1}{2}} L_0(k-i-1)h,\end{aligned}$$

where we have used (6.9).

Hence, combining (6.10) and (6.11) and by a standard application of the Ascoli Arzelà Theorem, commonly used in the analysis of minimizing movements, we conclude that there exists a subsequence $\{h_m\}_{m\in\mathbb{N}}$ such that $f_{h_m}(t) \to f^\beta(t)$ in $L^1(\partial E_0)$ for a.e. $t \in [0, T_0]$ as $m \to +\infty$, where

$$f^\beta \in \mathrm{Lip}([0, T_0], L^1(\partial E_0)), \quad f^\beta \in L^\infty([0, T_0], H^4(\partial E_0)). \tag{6.12}$$

in what follows we omit the dependence on m for this subsequence. Therefore by the Sobolev embedding, we get $f^\beta \in L^\infty([0, T_0], C^{3,\frac{1}{2}}(\partial E_0))$. We define the family $\{E_t^\beta\}_{t\in[0,T_0]}$ by

$$E_t^\beta \Delta E_0 \subset \mathcal{I}_{\sigma_1}(\partial E_0) \text{ and } \partial E_t^\beta := \{x + f^\beta(t,x)\nu_{E_0}(x)\colon\ x \in \partial E_0\}. \tag{6.13}$$

Recall that $u_{E_j}^{K_{el},h}$ is a minimizer of the problem (6.1), with $u_{E_j}^{K_{el},h} \in \mathfrak{C}_{K_{el}}^{3,\frac{1}{4}}(\Omega, \mathbb{R}^2)$. Then, up to subsequence, by Lemma 6.1, we obtain

$$u_{E_{j_h}}^{K_{el},h} \to u_{E_t^\beta}^{K_{el},0} \text{ in } \mathfrak{C}_{K_{el}}^{3,\frac{1}{4}}(\Omega, \mathbb{R}^2) \text{ as } h \to 0^+,$$

where $u_{E_t^\beta}^{K_{el},0}$ is a minimizer of (6.2). Thus, we obtain that

$$u_{E_t^\beta}^{K_{el},0} \in \operatorname{argmin}\left\{\int_{\Omega\setminus E^\beta(t)} Q(E(u))\,dx\colon u \in \mathfrak{C}_{K_{el}}^{3,\frac{1}{4}}(\Omega, \mathbb{R}^2),\ u|_{\partial\Omega} = w_0\right\}.$$

Claim 5: E_t^β is a solution of equation (6.3).

We define the discrete normal velocity on ∂E_j as

$$V_j : \partial E_j \to \mathbb{R}, \quad V_j := \frac{\psi_{j+1}}{h}.$$

Let $\Psi_j : \partial E_0 \to \partial E_j$ be defined by

$$\Psi_j(x) := x + f_j(x)\nu_{E_0}(x).$$

We recall that

$$J_\tau\Psi_j(x) = \sqrt{(1 + f_j(x)\kappa_{E_0}(x))^2 + |\partial_{\partial E_0} f_j(x)|^2} \quad x \in \partial E_0.$$

We also define $N_j : \partial E_0 \to \mathbb{R}^2$ by

$$N_j(x) := \frac{-\partial_{\partial E_0} f_j(x)}{1 + \kappa_{E_0}(x) f_j(x)}\tau_{E_0}(x) + \nu_{E_0}(x).$$

We observe that

$$|N_j| = \frac{J_\tau\Psi_j}{1 + \kappa_{E_0} f_j}.$$

Subclaim: The following holds:

$$\lim_{h\to 0^+} \|V_j \circ \Psi_j - \frac{f_{j+1}-f_j}{|N_j|h}\|_{L^2(\partial E_0)} = 0. \tag{6.14}$$

Using the estimate in (6.9) we obtain

$$\|\psi_{j+1}\circ\Psi_j\|_{C^1(\partial E_0)} \le Ch^{\frac{1}{2}} \text{ and } \|\psi_{j+1}\circ\Psi_j\|_{L^2(\partial E_0)} \le Ch.$$

From the bounds $\|f_j\|_{C^{1,\gamma}}(\partial E_0) \le \varepsilon$ and the previous estimate, we deduce

$$|f_{j+1}(x) - f_j(x)| \le C|\psi_{j+1}\circ\Psi_j(x)| \ \forall x \in \partial E_0 \text{ and } \|f_{j+1}-f_j\|_{C^1(\partial E_0)} \le Ch^{\frac{1}{2}}.$$

Let $G : \partial E_0 \to \mathbb{R}$ be a function such that $\|G\|_{C^1(\partial E_0)} \le Ch^\gamma$ for some γ. We define

$$\Psi_t : \partial E_0 \to \mathbb{R}^2, \ \Psi_t(x) := x + t\nu_{E_0}(x),$$

and we recall that $J_\tau\Psi_t = 1 + t\kappa_{E_0}$.Applying the coarea formula, we get:

$$\begin{aligned}
&\int_{\mathbb{R}^2} G\circ\pi_{\partial E_0}(x)\big(\chi_{E_{j+1}}(x) - \chi_{E_j}(x)\big)\,dx\\
&= \int_{\partial E_0} G(x)\int_{-\sigma_1}^{\sigma_1}\big(\chi_{E_{j+1}}(\Psi_t(x)) - \chi_{E_j}(\Psi_t(x))\big)(1+t\kappa_{E_0}(x)\,dt\,d\mathcal{H}^1_x\\
&= \int_{\partial E_0} G(x)\int_{f_j(x)}^{f_{j+1}(x)}(1+t\kappa_{E_0}(x))\,dt\,d\mathcal{H}^1_x\\
&= \int_{\partial E_0} G(x)(f_{j+1}(x)-f_j(x))(1+f_j(x)\kappa_{E_0}(x))\,d\mathcal{H}^1_x + o(h^2)\\
&= \int_{\partial E_0} G(x)J_\tau\Psi_j(x)\frac{f_{j+1}(x)-f_j(x)}{|N_j(x)|}\,d\mathcal{H}^1_x + o(h^2).
\end{aligned} \tag{6.15}$$

We define

$$\Phi_{j,t} : \partial E^j \to \mathbb{R}^2, \ \Phi_{j,t}(x) := x + t\nu_{E_j}(x),$$

and we recall $J_\tau\Phi_{j,t} = 1 + t\kappa_{E_j}$. We compute the integral $\int_{\mathbb{R}^2} G\circ\pi_{\partial E_0}(x)\big(\chi_{E_{j+1}}(x) - \chi_{E_j}(x)\big)\,dx$ in a different way from (6.15):

(6.16)

$$\begin{aligned}
&\int_{\mathbb{R}^2} G\circ\pi_{\partial E_0}(x)\big(\chi_{E_{j+1}}(x) - \chi_{E_j}(x)\big)\,dx\\
&= \int_{\partial E_j}\int_{-\beta}^{\beta} G\circ\pi_{\partial E_0}(\Phi_{j,t}(x))\big(\chi_{E_{j+1}}(\Phi_{j,t}(x)) - \chi_{E^j}(\Phi_{j,t}(x))\big)(1+t\kappa_{E^j}(x))\,dt\,d\mathcal{H}^1_x\\
&= \int_{\partial E_j}\int_0^{\psi_{j+1}(x)} G\circ\pi_{\partial E_0}(\Psi_{j,t}(x))(1+t\kappa_{E_j}(x))\,dt\,d\mathcal{H}^1_x\\
&= \int_{\partial E_j}\int_0^{\psi_{j+1}(x)}\big(G\circ\pi_{\partial E_0}(\Psi_{j,t}(x)) - G\circ\pi_{\partial E_0}(x) + G\circ\pi_{\partial E_0}(x)\big)(1+t\kappa_{E_j}(x))\,dt\,d\mathcal{H}^1_x\\
&= \int_{\partial E_j}\psi_{j+1}(x)G\circ\pi_{\partial E_0}(x)\,d\mathcal{H}^1_x + o(h^2)\\
&= \int_{\partial E_0}\psi_{j+1}\circ\Psi_j(x)G(x)J_\tau\Psi_j(x)\,d\mathcal{H}^1_x + o(h^2).
\end{aligned}$$

Comparing (6.15) and (6.16) we find that for all $G : \partial E_0 \to \mathbb{R}$ with $\|G\|_{C^1(\partial E_0)} \le Ch^\gamma$, it holds true

$$\int_{\partial E_0} G(x)J_\tau\Psi_j(x)\Big[\psi_{j+1}\circ\Psi_j(x) - \frac{f_{j+1}(x)-f_j(x)}{|N_j(x)|}\Big]\,d\mathcal{H}^1_x = o(h^2). \tag{6.17}$$

We define

$$G(x) := \frac{1}{J_\tau \Psi_j(x)} \left[\psi_{j+1} \circ \Psi_j(x) - \frac{f_{j+1}(x) - f_j(x)}{|N_j(x)|} \right]. \tag{6.18}$$

A straightforward computation yields $\|G\|_{C^1(\partial E_0)} \le Ch^\gamma$ for some $\gamma \in (0,1)$. Plugging (6.18) into (6.17) gives (6.14).

We now return to the main claim: "$E^\beta(t)$ is a solution of (6.3)". Up to now, we have established:

$$\|f_j\|_{H^4(\partial E_0)} \le C_0,\ \|f_j\|_{L^\infty(\partial E_0)} \le \sigma_1,\ \left\| \frac{f_{j+1} - f_j}{|N_j| h} \right\|_{L^2(\partial E_0)} \le C \ \text{ for all } jh \le T_0. \tag{6.19}$$

Therefore, using (6.19), along with (6.14) and (6.12) we conclude:

$$\exists\, L^2(\partial E_0) - \lim_{h \to 0^+} V_j \circ \Psi_j(\cdot) = \frac{\partial_t f^\beta(t,\cdot)}{|N(t,\cdot)|}, \quad \text{for } t \in [0, T_0], \tag{6.20}$$

where $|N(t,x)| = \frac{J_\tau \Psi_t(x)}{1 + f^\beta(t,x)\kappa_{E_0}(x)}$ and $\Psi_t(x) := x + f^\beta(t,x)\nu_{E_0}(x)$ for $x \in \partial E_0$. Let $l \in C^2_c(\mathbb{R}^2)$. Multiplying the Euler–Lagrange equation (4.58) by l and integrating by parts yields:

$$\begin{aligned} &\int_{\partial E_j} \frac{\psi_{j+1}(x)}{h} l(x)\, d\mathcal{H}^1_x + \int_{\partial E_j} \frac{\psi^2_{j+1}(x)}{2h} \kappa_{E_j}(x) l(x)\, d\mathcal{H}^1_x \\ &\qquad = \int_{\partial E_j} -g(\nu_{E_j}(x)) \partial^2_{\partial E_j} \psi(x) \partial^2_{\partial E_j} l(x)\, d\mathcal{H}^1_x \\ &\qquad\quad - \int_{\partial E_j} Q(E(u^{K_{el}}_{E_{j+1}}))(x + \psi_{j+1}(x)\nu_{E_j}(x)) \partial^2_{\partial E_j} l(x)\, d\mathcal{H}^1_x \\ &\qquad\quad + \int_{\partial E_j} R(x) \partial^2_{\partial E_j} l(x)\, d\mathcal{H}^1_x. \end{aligned} \tag{6.21}$$

We observe that

$$\begin{aligned} &\lim_{h \to 0^+} \int_{\partial E_j} \frac{\psi^2_{j+1}(x)}{2h} \kappa_{E_j}(x) l(x)\, d\mathcal{H}^1_x \\ &\quad \le \lim_{h \to 0^+} \|l\|_{L^\infty(\mathbb{R}^2)} \frac{\|\psi_{j+1}\|_{L^2(\partial E_j)}}{2h} \|\psi_{j+1}\|_{L^\infty(\partial E_j)} \|\kappa_{E_j}\|_{L^2(\partial E_j)} = 0, \end{aligned} \tag{6.22}$$

where we have used (6.9). Thank the result of the previous claim we also have that

$$\|\psi_{j+1}\|_{H^2(\partial E_j)} \le Ch^\gamma.$$

Recalling the definition of R (see formula (4.48)), we have:

$$\|R\|_{L^2(\partial E^j)} \le Ch^\gamma. \tag{6.23}$$

Therefore, we can pass to the limit as $h \to 0^+$, in the equation (6.21), and using (6.22) and (6.23), we conclude that $E^\beta(t)$ is a distributional solution of (6.3) in $[0, T_0]$. Moreover from (6.19) and (6.13). we get $f^\beta \in \mathrm{Lip}([0,T_0], L^2(\partial E_0))$.

Claim 6: It is holds true (6.4).

Using formula (6.12) and Proposition 2.4, we get

$$\begin{aligned} \|\partial_{\partial E_0} f^\beta(t,\cdot)\|_{L^2(\partial E_0)} &\le C \|f^\beta(t,\cdot)\|^{\frac{2}{3}}_{L^1(\partial E_0)} \sup_{t \in [0,T_0]} \|f^\beta(t,\cdot)\|^{\frac{1}{3}}_{H^4(\partial E_0)} \\ &\le Ct^{\frac{2}{3}}. \end{aligned}$$

Using the estimate above, together with (6.12), (2.14) and the Sobolev embedding, we deduce

$$\begin{aligned}\|f^\beta(t,\cdot)\|_{C^{3,\frac14}(\partial E_0)} &\le C\big(\sup_{t\in[0,T_0]}\|f^\beta(t,\cdot)\|_{H^4(\partial E_0)}^{\frac{13}{14}}\big)\|f^\beta(t,\cdot)\|_{C^{0,\frac12}(\partial E_0)}^{\frac{1}{14}}\\ &\le Ct^{\frac{1}{21}}.\end{aligned}$$

□

We now recall the statement of Lemma 3.2 from [30], which we will use in the next theorem.

Lemma 6.3. *Let $0<\alpha<\beta\le 1$, $M>0$ and $k\in\mathbb{N}$. Then there exists $C>0$ such that for any $F,\tilde F\in\mathfrak{C}^{k,\beta}_M(E_0)$, the following estimate holds:*

$$\|u_F(\cdot+\varphi_F(\cdot)\nu_{E_0}(\cdot))-u_{\tilde F}(\cdot+\varphi_{\tilde F}(\cdot)\nu_{E_0}(\cdot))\|_{C^{k,\alpha}(\partial E_0)}\le C\|\varphi_F-\varphi_{\tilde F}\|_{C^{k,\alpha}(\partial E_0)} \tag{6.24}$$

where

$$\partial F=\{x+\varphi_F(x)\nu_{E_0}(x):\,x\in\partial E_0\},\ \partial\tilde F=\{x+\varphi_{\tilde F}(x)\nu_{E_0}(x):\,x\in\partial E_0\}.$$

We are now in a position to prove that the minimizing movement scheme converges to the classical solution of (1.5), provided that the initial data E is sufficiently smooth and K_{el} is sufficiently large.

Theorem 6.4. *There exist constants K_{el} and T_s such that any family $\{E^\beta_t\}_{t\in[0,T_0]}$ obtained in Theorem 6.2 is a solution of the problem* (1.5) *in* $[0,T_s]$.

Proof. Thanks the regularity of ∂E_0 and using classical elliptic regularity theory, see [2], [32, Proposition 8.9], we know that the function u_{E_0}, which minimizes problem (3.15) for $F=E_0$, is of class $C^{3,\frac14}(\Omega\setminus E_0)$ and satisfies equation (3.16). We define the function

$$\tilde u_{E_0}(x)=\begin{cases}u_{E_0}(x) & \text{if } x\in\Omega\setminus E_0,\\ \eta\Big(\dfrac{d_{E_0}(x)}{\sigma_{E_0}}\Big)u_{E_0}\big(\pi_{\partial E_0}(x)\big) & \text{if } x\in E_0\cap\mathcal{I}_{\sigma_{E_0}}(\partial E_0),\\ 0 & \text{if } x\in E_0\setminus\mathcal{I}_{\sigma_{E_0}}(\partial E_0),\end{cases}$$

where $\eta\in C^\infty_c(-2,2)$, and $\eta\ge0$, $\eta=1$ in $(-1,1)$. We observe that

$$\|\tilde u_{E_0}\|_{C^{3,\frac14}(\Omega)}\le C_1\|u_{E_0}\|_{C^{3,\frac14}(\Omega\setminus E_0)}, \tag{6.25}$$

where $C_1=C(\|\kappa_{E_0}\|_{C^2(\partial E_0)})$. Moreover, since u_{E_0} solves the equation (3.16), we get

$$\|u_{E_0}\|_{C^{3,\frac14}(\Omega\setminus E_0)}\le L(\|w_0\|_{C^{3,\frac14}(\partial\Omega)}+\|u_{E_0}\|_{C^{3,\frac14}(\partial E_0)}), \tag{6.26}$$

where L is a universal constant. We define the constant K_{el} as

$$K_{el}:=2\max\{1,C_1L(\|w_0\|_{C^{3,\frac14}(\partial\Omega)}+\|u_{E_0}\|_{C^{3,\frac14}(\partial E_0)})\}.$$

Therefore, $\tilde u_{E_0}$ is a minimizer of the problem (6.2) with this choice of K_{el}, and satisfies

$$\|\tilde u_{E_0}\|_{C^{3,\frac14}(\Omega)}<K_{el}.$$

Let f^β be the function obtained in Theorem 6.2, and let T_0 be the corresponding time from the same theorem. Recall that $f^\beta(0,\cdot)=0$. By combining formulas (6.4) and (6.24) (with $F=E_0$ and $\tilde F=E^\beta_t$), we get, for all $t\in[0,T_0]$,

$$\|u_{E_0}-u_{E^\beta_t}(\cdot+f^\beta(t,\cdot)\nu_{E_0}(\cdot))\|_{C^{3,\frac14}(\partial E_0)}\le Ct^{\frac{1}{21}}, \tag{6.27}$$

where $u_{E_t^\beta}$ is a minimizer of (3.15) for $F = E_t^\beta$.
Claim: There exits T_s such that for all $0 \le t \le T_s$, the function

$$\tilde{u}_{E_t^\beta}(x) = \begin{cases} u_{E_t^\beta}(x) \text{ if } x \in \Omega \setminus E_t^\beta, \\ \eta\Big(\dfrac{d_{E_0}(x)}{\sigma_{E_0}}\Big) u_{E_t^\beta}\Big(\pi_{\partial E_0}(x) + f^\beta(t,x)\nu_{E_0}(\pi_{E_0}(x))\Big) \text{ if } x \in E^{E^\beta(t)} \end{cases}$$

is a minimizer of (6.2) for $F = E_t^\beta$ and satisfies $\|\tilde{u}_{E_t^\beta}\|_{C^{3,\frac14}(\Omega)} < K_{el}$.

As in formulas (6.25) and (6.26), we obtain

$$\|\tilde{u}_{E_t^\beta}\|_{C^{3,\frac14}(\Omega)} \le C_1 L(\|w_0\|_{C^{3,\frac14}(\partial\Omega)} + \|u_{E_t^\beta}(\cdot + f^\beta(t,\cdot)\nu_{E_0}(\cdot))\|_{C^{3,\frac14}(\partial E_0)}).$$

Using this and (6.27), we conclude that

$$\begin{aligned} \|\tilde{u}_{E_t^\beta}\|_{C^{3,\frac14}(\Omega)} &\le C_1 L(\|w_0\|_{C^{3,\frac14}(\partial\Omega)} + \|u_{E_0}\|_{C^{3,\frac14}(\partial E_0)}) + Ct^{\frac{1}{21}} \\ &< K_{el}, \end{aligned}$$

for t sufficiently small. By the minimality of $u_{E_t^\beta}$ in (3.15), we get that $\tilde{u}_{E_t^\beta}$ is a minimizer of (6.2).

Therefore, there exist constants K_{el}, T_s such that the family $\{E_t^\beta\}_{t\in[0,T_s]}$ satisfies (6.3). Moreover, the minimizer $\tilde{u}_{E_t^\beta}$ of (6.2) satisfies $\|\tilde{u}_{E_t^\beta}\|_{C^{3,\frac14}(\Omega)} < K_{el}$ and

$$\tilde{u}_{E_t^\beta}(x) = u_{E_t^\beta}(x) \text{ for all } x \in \Omega \setminus E_t^\beta,$$

where $u_{E_t^\beta}$ is a minimizer of (3.15). Hence, we conclude that the family $\{E_t^\beta\}_{t\in[0,T_s]}$, parametrized by the diffeomorphisms $\Phi_t(x) = x + f^\beta(t,x)\nu_{E_0}$, constitutes a strong solution to the anisotropic surface diffusion equation with elasticity. More precisely, using the expansion of the curvature from (4.47) and the expansion of the Laplace–Beltrami operator as in [30], we find that the function $f^\beta : [0,T_s] \times \partial E_0 \to \mathbb{R}$ is a strong solution to the equation (see formulas (3.6), (3.32), and (3.38) in [30])

$$\begin{cases} \partial_t f^\beta = \dfrac{1}{1 + f^\beta \kappa_{E_0}} \partial_\tau \Big(\dfrac{\partial_\tau\big(\big(g(\nu_{E_t^\beta})\kappa_{E_t^\beta} - Q(E(u_{E_t^\beta}))\big) \circ \pi^{-1}_{\partial E_t^\beta}\big)}{\sqrt{(1 + f^\beta\kappa_{E_0})^2 + |\partial_\tau f^\beta|^2}} \Big), \text{ on } \partial E_0 \\ f^\beta(0,\cdot) = 0 \text{ on } \partial E_0. \end{cases} \tag{6.28}$$

By a strong solution, we mean that $f^\beta \in \mathrm{Lip}([0,T_s], L^2(\partial E_0)) \cap L^\infty([0,T_s], H^4(\partial E_0))$ and that it satisfies equation (6.28) almost everywhere. Using Grönwall's lemma, one can deduce that the strong solution to (6.28) with zero initial data is unique. This implies that the limiting flat flow coincides with the classical solution of (1.5) on the interval $[0,T_s]$. $\square$

We recall the definition of uniform ball condition.

Definition 6.5. *We say that a set $E \subset \mathbb{R}^2$ satisfies the uniform ball condition (UBC) with a given radius $r > 0$ if for every $x \in \partial E$ there are balls $B_r(x_+)$ and $B_r(x_-)$ such that*

$$B_r(x_+) \subset \mathbb{R}^2 \setminus E, \quad B_r(x_-) \subset E \text{ and } x \in \partial B_r(x_+) \cap \partial B_r(x_-).$$

Remark 6.6. *We may quantify the statement of Theorem 6.4 as follows : Let $E_0 \Subset \Omega$ be a open connected set of class C^5 and that satisfies the UBC with radius $2r_0$, and the heightfunction ψ_1, see Theorem 6.2, satisfies*

$$\|\psi_1\|_{L^2(\partial E_0)} \le L_0 h \quad \text{and} \quad \|\Delta_{\partial E_0}\psi_1\|_{L^2(\Sigma_0)} \le L_0\sqrt{h}.$$

Then there is $K_0 = K_0(r_0, L_0)$ and $\beta_0 = \beta_0(r_0, K_0)$ such that if

$$\|\kappa^\varphi_{E_0}\|_{H^2(\partial E_0)} \le K_0 \quad \text{and} \quad \|\nabla_{\partial E_0}\Delta_{\partial E_0}\kappa^\varphi_{\partial E_0}\|_{L^2(\partial E_0)} \le K_0 h^{-\frac14}$$

then the discrete constrained flat flow $\{E_t^{h,\beta}\}_{t\geq 0}$*, where* $\beta \leq \beta_0$*, also satisfies the UBC with radius* r_0*, and*

$$\|\kappa^{\varphi}_{E_t^{h,\beta}}\|_{H^2(\partial E_t^{h,\beta})} \leq K_0 \quad \text{and} \quad \|\nabla_{\partial E_t^{h,\beta}} \Delta_{\partial E_t^{h,\beta}} \kappa^{\varphi}_{E_t^{h,\beta}}\|_{L^2(\partial E_t^{h,\beta})} \leq K_0 h^{-\frac{1}{4}}$$

for all $t \in [0, T_s]$*, where* $T_s = T_s(r_0, K_0)$*.*

Remark 6.7. *The arguments in the proofs of Theorems 6.4 and 6.2 imply that if a constrained discrete flat flow* $\{E_t^{h,\beta}\}_{t\geq 0}$*, starting from* E_0*, satisfies*

$$\|\kappa^{\varphi}_{E_t^{h,\beta}}\|_{H^2(\partial E_t^{h,\beta})} \leq K_0 \quad \text{and} \quad \|\nabla_{\partial E_t^{h,\beta}} \Delta_{\partial E_t^{h,\beta}} \kappa^{\varphi}_{E_t^{h,\beta}}\|_{L^2(\partial E_t^{h,\beta})} \leq K_0 h^{-\frac{1}{4}}$$

for all $t \in [0, T_s]$*, then the limiting flat flow coincides with the classical solution on the time interval* $[0, T_s]$*.*

7. Convergence to the global solution

We recall that the classical solution of (1.5) with initial datum E_0 exists on the interval $[0, T_e)$, where $T_e$denotes the maximal existence time. In this subsection, we prove that for every $T < T_e$, there exist $\beta(T)$ and $K_{el}(T)$ such that the constrained discrete flat flow with initial datum E_0 converges to the classical solution of (1.5) on $[0, T]$ as $h \to 0^+$. The proof of the next theorem is similar to the one presented in [18][Theorem 1.2], but we include it here for the reader's convenience.

Theorem 7.1. *Let* $\{E_t\}_{t\in[0,T_e)}$ *be a classical solution of* (1.5) *with initial datum* E_0*. Then for every* $T < T_e$ *there exist* $\beta(T)$ *and* $K_{el}(T)$ *such that for all* $\beta \in (0, \beta(T)]$ *the constrained flat flow* E_t^{β}*, starting from* E_0*, coincide with* E_t *in* $[0, T]$*.*

Proof. Let $\{E_t\}_{t\in[0,T_e)}$ be the classical solution of (1.5), and let $T < T_e$ be fixed. Since the classical solution is regular on $[0, T]$, there exist constants K_2, σ_2, and $\tilde{K}_{el}$ such that

$$E_t \in \mathfrak{H}^4_{K_2,\sigma_2}(E_0),\ \|\tilde{u}_{E_t}\|_{C^3(\Omega)} < \tilde{K}_{el} \ \text{ for all } t \in [0, T],$$

where $\tilde{u}_{E_t}$ is the function defined by (6.2), where we replaced E_t^{β} with E_t. It is easy to check that the condition $E_t \in \mathfrak{H}^4_{K_2,\sigma_2}(E_0)$ implies that there exists $r_0 > 0$ such that E_t satisfies the UBC with r_0. Let β_0, T_s be the constants obtained in Theorems 6.2, 6.4 and Remark 6.6 for

$$K_{el} = 4\tilde{K}_{el}.$$

We fix $\beta \leq \beta_0$. Let $k_0 \in \mathbb{N}$ be such that $T_0 \in [hk_0, h(k_0 + 1))$, and let $(E_{hk}^{h,\beta})_{k\in\mathbb{N}}$ be a discrete constrained flat flow starting from E_0. As in Theorem 6.2, we have

$$\begin{aligned}
&\partial E_h^{h,\beta} = \{x + \psi_1(x)\nu_{E^0}(x) \colon x \in \partial E^0\}, \\
&\|\psi_1\|_{L^2(\partial E^0)} \leq L_0 h, \quad \|\psi_1\|_{H^4(\partial E^0)} \leq L_0, \\
&\|\kappa^{\varphi}_{E_h^{h,\beta}}\|_{H^2(\partial E_h^{h,\beta})} \leq K_0, \quad \|\partial^3_{\partial E^{h,\beta}} \kappa^{\varphi}_{E_h^{h,\beta}}\|_{L^2(\partial E_h^{h,\beta})} \leq \frac{K_0}{h^{\frac{1}{4}}},
\end{aligned}$$

where L_0 and K_0 are as in (6.5), with $K_0 > K_{el}$. We adopt the following notation: $E_t^h := E^{h,\beta}(t)$, $E_k := E_{hk}^{h,\beta}$, and recall (3.23) for the definition of $E_t^{h,\beta}$. The conclusion of the theorem follows from the next claim, together with Remarks 6.6 and 6.7.
Claim: For evert $t \in [0, T]$

$$\|\kappa^{\varphi}_{E_t^h}\|_{H^2(\partial E_t^h)} \leq K_0, \quad \|\partial^3_{\partial E_t^h} \kappa^{\varphi}_{E_t^h}\|_{L^2(\partial E_t^h)} \leq \frac{K_0}{h^{\frac{1}{4}}}. \tag{7.1}$$

By Theorem 6.2, estimate (7.1) holds for all $t \in [0, T_s]$. We define

$$\tilde{t} := \inf\{t \in [T_s, T] : \ \text{formula (7.1) is true for all } t \in [0, \tilde{t}]\}. \tag{7.2}$$

We will show that (7.1) continues to hold for all $t \le \tilde{t} + \frac{T_s}{2}$, which implies the claim. To this end, let $\tilde{k} \in \mathbb{N}$ be such that $\tilde{t} - \frac{T_s}{2} \in [h\tilde{k}, (\tilde{k}+1)h)$ satisfies (7.1), we apply Theorems 6.2 and 6.4 with $E_0 = E_{\tilde{k}}$ to obtain there exist $k_1 \in \mathbb{N}$ and $c > 0$ (we recall that $c = c(K_{el}, K_0)$) such that $0 < c \le hk_1 = T_1 \le T_s$, and for all $k \in \{\tilde{k}, \dots, \tilde{k} + k_1\}$

$$\partial E_k = \{x + \psi_k(x)\nu_{E_{k-1}}(x) \colon x \in \partial E_{k-1}\}.$$

Using formula (6.9), we obtain

$$\|\psi_k\|^2_{L^2(\partial E_{k-1})} + h \sum_{j=\tilde{k}}^{\tilde{k}+k_1} \|\Delta_{\partial E_{k-1}} \psi_k\|^2_{L^2(\partial E_{k-1})} \le Ch^2,$$

for some constant C. Since $0 < c \le hk_1 = T_1$, there exists $\hat{k} \in \{\tilde{k}, \dots, \tilde{k} + \lfloor \frac{k_1}{4} \rfloor\}$ such that

$$\|\psi_{\hat{k}}\|_{L^2(\partial E_{\hat{k}-1})} + \|\Delta_{\partial E_{\hat{k}-1}} \psi_{\hat{k}}\|_{L^2(\partial E_{\hat{k}-1})} \le Ch. \tag{7.3}$$

From the above and using the very definition of $\tilde{t}$, see(7.2), we get

$$h\hat{k} \le h(\tilde{k} + \lfloor \frac{k_1}{4} \rfloor) \le \tilde{t}.$$

Recall that in the minimizing movement scheme, each set E_j is of class C^5, since it solves the Euler–Lagrange equation (4.57). Moreover, $E_{\hat{k}}$ is uniformly $C^{3,\frac{1}{2}}$-regular. Let $t_h = \hat{k}h$ and we set

$$v^h(t_h, x) = \frac{\psi_{\hat{k}}(x)}{h}.$$

By (7.3) and the Sobolev embedding theorem, we obtain

$$\|v^h(t_h, \cdot)\|_{C^{1,\frac{1}{2}}(\partial E_{\hat{k}-1})} \le C. \tag{7.4}$$

Since $t_h = \hat{k}h \in [\tilde{t} - \frac{T_s}{2}, \tilde{t}]$, by passing to a subsequence if necessary, we can assume

$$\exists \lim_{h \to 0^+} t_h = \bar{t}. \tag{7.5}$$

From (7.4) and (7.5), we conclude

$$v^h(t_h, \cdot) \to v(\bar{t}, \cdot) \text{ in } C^{1,\frac{1}{2}} \text{ as } h \to 0^+,\ \|v(\bar{t}, \cdot)\|_{C^{1,\frac{1}{2}}(\partial E^\beta(\bar{t}))} \le C.$$

Hence,

$$\lim_{h \to 0^+} \|v^h(t_h, \cdot)\|_{L^2(\partial E_{\hat{k}-1})} = \|v(\bar{t}, \cdot)\|_{L^2(\partial E^\beta_{\bar{t}})}.$$

Since we assumed that (7.1) holds for all $t \le \tilde{t}$, and $\bar{t} \le \tilde{t}$, Remark 6.7 implies that the flat flow agrees with the classical solution up to time $\tilde{t}$. Using (6.20) and (6.14) with $E(\tilde{t} - \frac{T_0}{2})$ in place of E_0, we find that $v(\bar{t}, \cdot)$ coincides with the normal velocity $V_{\bar{t}}$ of the classical solution $\{E_t\}_{t \ge 0}$, and

$$\|V_{\bar{t}}\|_{L^2(\partial E_{\bar{t}})} = \|v(\bar{t}, \cdot)\|_{L^2(\partial E^\beta_{\bar{t}})}.$$

By the definition of K_{el} ($K_{el} = 4\max\{K_2, \tilde{K}_{el}\}$) and using equation (1.5), we get

$$\|V_{\hat{t}}\|_{L^2(\partial E_{\hat{t}})} = \|\Delta_{\partial E_t}\big(\kappa^\varphi_{E_{\hat{t}}} - Q(E(u_{E_{\hat{t}}}))\big)\|_{L^2(\partial E_{\hat{t}})} \le K_2 + \tilde{K}_{el} \le \frac{K_{el}}{2}.$$

Hence, we conclude

$$\|\psi_k\|_{L^2(\partial E_{k-1})} \le \frac{K_{el}}{2} h.$$

Using (7.3), we also have

$$\|\Delta_{\partial E_{k-1}} \psi_k\|_{L^2(\partial E_{k-1})} \le Ch \le K_{el}\sqrt{h}$$

for h small enough. Finally, since $E_{\bar{t}} = E^{\beta}_{\bar{t}}$ is uniformly $C^{3,\frac{1}{2}}$ regular with bound C, the same holds for $E_{\hat{k}}$ is uniformly $C^{3,\frac{1}{2}}$, with a bound $2C$. Then, applying Remarks 6.6 and 6.7 with $E_{\hat{k}}$ instead of E_0, we deduce that (7.1) holds on $[\hat{k}h, \hat{k}h + T_s]$,. Since $\hat{k}h \in [\tilde{t} - \frac{T_s}{2}, \tilde{t}]$, this implies that E^h_t satisfies (7.1) on $[0, \tilde{t} + \frac{T_s}{2}]$. Repeating this argument a finite number of times yields the claim. □

Aknowledgments

I was supported by the Academy of Finland grant 314227.

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