Functions of bounded variation in trace spaces

Domenico Mucci

Abstract. For vector-valued maps in Sobolev-Slobodeckij trace spaces, we prove a necessary and sufficient condition for membership in the class of functions of bounded variation.

Keywords : Sobolev-Slobodeckij trace spaces, functions of bounded variation, approximate differentiability, currents

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1 Introduction

Approximate differentiability a.e. of vector valued non-smooth maps is a fundamental property in order to deal with currents carried by graphs of non-smooth maps, compare Giaquinta-Modica-Souček [9, Sec. 3.1.4]. Calderón-Zygmund theorem [7] implies that the latter property is satisfied by Sobolev functions, see [9, Sec. 3.1.2], and also by functions of bounded variation (BV), see [4, Sec. 3.7]. However, it appears that it is not known whether the same holds true for functions in the Sobolev-Slobodeckij trace spaces $W^{1-1/p,p}$.

More precisely, as communicated to us by G. Crippa in [1], in dimension two there is a function f in $C^{1,\alpha}$ for each $0<\alpha<1$ that does not satisfy the so called *weak Sard property*, see [2]. Correspondingly, the function $b=\nabla^\perp f$ belongs to the fractional Sobolev-Slobodeckij classes $W^{s,p}$ for each $p>1$ and $0<s<1$. Therefore, such a function b does not belong to the class $t^{1,1}$ (functions with first order Taylor expansion in L^1-sense), see [3]. If it were the case, in fact, the corresponding function f had to satisfy the C^2-Lusin property and, definitely, the weak Sard property. Notice that the existence of a first order Taylor expansion in L^1-sense is a slightly stronger property than approximate differentiability a.e.

Therefore, it is reasonable to conjecture the existence of maps in $W^{1-1/p,p}$ that are not a.e. approximately differentiable. Notice that it is known that traces of Sobolev maps may not be functions of bounded variation, but to our knowledge the previous question is an open problem.

In this paper, we find a characterization of the BV property for functions in trace spaces.

For $n,N\geq 2$ and Ω a bounded domain of $\mathbb{R}^n$, we denote by $W^{1-1/p,p}(\Omega,\mathbb{R}^N)$ the Banach space of trace maps in $\Omega\times\{0\}$ of the Sobolev class $W^{1,p}(\Omega\times I,\mathbb{R}^N)$, where $I=(0,1)$ and $p>1$ is a real exponent.

Using the classical extension due to Gagliardo [8], to any map $u\in W^{1-1/p,p}(\Omega,\mathbb{R}^N)$ we associate a Sobolev function $\mathrm{Ext}(u)\in W^{1,p}(\Omega\times I,\mathbb{R}^N)$ given by a minimizer of the infimum problem

$$\inf\left\{\int_{\Omega\times I}|DU(x,t)|^p\,dx\,dt \;\mid\; U\in W^{1,p}(\Omega\times I,\mathbb{R}^N)\,,\ U_{|\Omega\times\{0\}}=u\right\}.$$

The norm of maps u in $W^{1-1/p,p}$ is equivalent to the norm

$$\|u\|_{L^p(\Omega,\mathbb{R}^N)}+\|DU\|_{L^p(\Omega\times I,\mathbb{R}^N)}\,,\quad U=\mathrm{Ext}(u)\,,$$

where

$$\|DU\|^p_{L^p(\Omega\times I,\mathbb{R}^N)}=\int_\Omega\left(\int_0^1|DU(x,t)|^p\,dt\right)dx\,.$$

The main result of this paper is contained in the following

Theorem 1.1 *Let $u\in W^{1-1/p,p}(\Omega,\mathbb{R}^N)$ for some $p\geq 2$. Then, u is a function of bounded variation, say $u\in BV(\Omega,\mathbb{R}^N)$, if and only if we have*

$$\liminf_{\varepsilon\to 0^+}\frac{1}{\varepsilon}\int_\Omega\left(\int_0^\varepsilon\left|\frac{\partial U}{\partial x_i}(x,t)\right|dt\right)dx<\infty\quad\forall\, i=1,\ldots,n\,. \tag{1.1}$$

Notice that property (1.1) does not involve the partial derivative of the extension map U in the direction of the variable t. Moreover, the validity of (1.1) guarantees the a.e. approximate differentiability property of maps that are e.g. traces of Sobolev functions U in $W^{1,2}(\Omega \times I, \mathbb{R}^N)$.

For real valued maps $u \in W^{1-1/p,p}(\Omega, \mathbb{R})$, Theorem 1.1 holds true for any exponent $p > 1$, see Remark 3.1 below. Its proof relies on some ideas contained in [10] and [1]. Roughly speaking, it is based on the analysis of the properties of the lower order strata of the n-current

$$T_u := (-1)^{n-1}(\partial G_U) \llcorner ((\Omega \times \{0\}) \times \mathbb{R}^N)\,, \quad U = \mathrm{Ext}(u)\,,$$

where G_U is the $(n+1)$-current carried by the *graph* of the Sobolev map U, see [9].

2 Notation and preliminary results

We deal with mappings $u : \mathcal{X} \to \mathbb{R}^N$ defined in a smooth, connected, compact Riemannian manifold $\mathcal{X}$ without boundary, of dimension $n \geq 2$. Actually, we let $\mathcal{X} = \partial\mathcal{M}$ for some smooth and compact $(n+1)$-manifold $\mathcal{M}$, the model case being $\mathcal{X} = \mathbb{S}^n$, the unit sphere in $\mathbb{R}^{n+1}$. By Nash-Moser theorem, the manifold $\mathcal{M}$ is isometrically embedded into some Euclidean space $\mathbb{R}^\ell$. We shall equip $\mathcal{M}$ and $\mathcal{X}$ with the metric induced by the Euclidean norm on the ambient space.

For $x \in \mathcal{X}$ and $0 < h < r_0$, where $r_0 > 0$ is the injectivity radius of $\mathcal{X}$, denote by $B(x,h)$ the geodesic n-ball of radius h centered at $x \in \mathcal{X}$. For $0 < \delta < r_0$ small, let

$$\mathcal{M}_\delta := \{z \in \overline{\mathcal{M}} \mid \mathrm{dist}(z, \mathcal{X}) \leq \delta\}\,, \qquad \mathcal{X} = \partial\mathcal{M}\,.$$

There exists $0 < d < r_0$ such that the nearest point projection $\Pi_{\mathcal{M}}$ from $\mathcal{M}_d$ onto $\mathcal{X}$ is well-defined, and hence we may consider the fibration

$$\Phi^{-1} : \mathcal{X} \times [0,d] \to \mathcal{M}_d\,, \tag{2.1}$$

where $\Phi(z) := (\Pi_{\mathcal{M}}(z), \mathrm{dist}(z, \mathcal{X}))$ for any $z \in \mathcal{M}_d$.

2.1 Trace spaces

The fractional Sobolev-Slobodeckij space $W^{1-1/p,p}(\mathcal{X})$, where $p > 1$ is a real exponent, is the Banach space of L^p-functions $u : \mathcal{X} \to \mathbb{R}$ which have finite $W^{1-1/p,p}$-seminorm

$$|u|^p_{1-1/p,\mathcal{X}} := \int_{\mathcal{X}}\int_{\mathcal{X}} \frac{|u(x)-u(y)|^p}{|x-y|^{n+p-1}}\, d\mathcal{H}^n(x)\, d\mathcal{H}^n(y)\,,$$

where $\mathcal{H}^k$ is the k-dimensional Hausdorff measure in $\mathbb{R}^\ell$, endowed with the norm

$$\|u\|^p_{1-1/p,\mathcal{X}} := \|u\|^p_{L^p(\mathcal{X})} + |u|^p_{1-1/p,\mathcal{X}}\,. \tag{2.2}$$

We denote by $W^{1-1/p,p}(\mathcal{X}, \mathbb{R}^N)$ the space of vector valued maps $u : \mathcal{X} \to \mathbb{R}^N$ with components $u^j \in W^{1-1/p,p}(\mathcal{X})$ for every $j = 1, \ldots, N$. Since $\mathcal{X} = \partial\mathcal{M}$ for some smooth manifold $\mathcal{M}$, then $W^{1-1/p,p}(\partial\mathcal{M}, \mathbb{R}^N)$ can be characterized as the space of functions u that are *traces* of functions U in the Sobolev space $W^{1,p}(\mathcal{M}, \mathbb{R}^N)$.

Following Bethuel-Demengel [5], to each map $u \in W^{1-1/p,p}(\mathcal{X}, \mathbb{R}^N)$ we associate a function $\widetilde{U}$ in $W^{1,p}(\mathcal{M}_d, \mathbb{R}^N)$ given by $\widetilde{U} = v \circ \Phi$, where

$$v(x,h) := \fint_{B(x,h)} u\, d\mathcal{H}^n\,, \qquad (x,h) \in \mathcal{X} \times]0,d]\,. \tag{2.3}$$

It turns out that $\widetilde{U} \in W^{1,p}(\mathcal{M}_d, \mathbb{R}^N)$ and $\widetilde{U}$ is smooth outside $\mathcal{X}$, with $\widetilde{U}_{|\mathcal{X}} = u$ in the sense of the traces, compare [8]. Moreover, setting $u_h(x) := v(x,h)$, we have (cf. [1, Prop. 1.1])

Proposition 2.1 *$u_h \to u$ strongly in $W^{1-1/p,p}$ as $h \to 0^+$.*

2.2 Semi-currents carried by graphs

We refer to [9] for the basic notation on the theory of currents. According to [10], every compactly supported smooth differential k-form $\omega \in \mathcal{D}^k(\mathcal{X} \times \mathbb{R}^N)$, where $k \leq n$, splits as a sum

$$\omega = \sum_{j=0}^{\underline{k}} \omega^{(j)}, \quad \underline{k} := \min(k, N).$$

Here the $\omega^{(j)}$'s are the k-forms that contain exactly j differentials in the vertical $\mathbb{R}^N$ variables. For fixed $r = 1, \ldots, \underline{k}$ we denote by $\mathcal{D}^{k,r}(\mathcal{X} \times \mathbb{R}^N)$ the subspace of $\mathcal{D}^k(\mathcal{X} \times \mathbb{R}^N)$ of k-forms of the type $\omega = \sum_{j=0}^{r} \omega^{(j)}$. The dual space of "semi-currents" is denoted by $\mathcal{D}_{k,r}(\mathcal{X} \times \mathbb{R}^N)$. Of course we have $\mathcal{D}_{k,k} = \mathcal{D}_k$, the space of all k-currents. A similar notation holds by replacing $\mathcal{X}$ with $\mathcal{M}$ or $\mathcal{M}_d$.

Example 2.2 If $U \in W^{1,p}(\mathcal{M}, \mathbb{R}^N)$, then the graph current G_U is a well defined $(n+1, \mathbf{p})$-current in $\mathcal{D}_{n+1,\mathbf{p}}(\mathcal{M} \times \mathbb{R}^N)$, where we have set

$$\mathbf{p} := \min\{[p], N\}, \quad \text{with } [p] \text{ the integer part of } p.$$

Denoting by $f \bowtie g$ the join map $(f \bowtie g)(x) := (f(x), g(x))$, in an approximate sense we have

$$G_U := (\mathrm{Id}_{\mathcal{M}} \bowtie U)_{\#}[\![\mathcal{M}]\!].$$

For example, if $\omega = \gamma \wedge \eta \in \mathcal{D}^{n+1}(\mathcal{M} \times \mathbb{R}^N)$, where $\gamma \in \mathcal{D}^{n+1-h}(\mathcal{M})$, $\eta \in \mathcal{D}^h(\mathbb{R}^N)$, and $0 \leq h \leq \min\{n+1, \mathbf{p}\}$, by the area formula we have

$$\langle G_U, \gamma \wedge \eta \rangle = \langle [\![\mathcal{M}]\!], (\mathrm{Id}_{\mathcal{M}} \bowtie U)^{\#}(\gamma \wedge \eta) \rangle = \langle [\![\mathcal{M}]\!], \gamma \wedge U^{\#}\eta \rangle = \int_{\mathcal{M}} \gamma \wedge U^{\#}\eta.$$

Setting moreover

$$\|G_U\| := \sup\left\{ \langle G_U, \omega \rangle \mid \omega \in \mathcal{D}^{n+1,\mathbf{p}}(\mathcal{M} \times \mathbb{R}^N), \ \|\omega\| \leq 1 \right\},$$

where $\|\omega\|$ is the *comass* norm of ω, by using the parallelogram inequality we infer that

$$\|G_U\| \leq C \int_{\mathcal{M}} (1 + |DU|^p)\, d\mathcal{H}^{n+1} < \infty$$

for some absolute constant $C = C(n, p, \mathcal{M}) > 0$, not depending on U. As a consequence, if $p \geq N$ it turns out that G_U is an integer multiplicity rectifiable $(n+1)$-current in $\mathcal{M}_d \times \mathbb{R}^N$ with finite mass, $\mathbf{M}(G_U) = \|G_U\| < \infty$, compare [9].

Definition 2.3 To any map $u \in W^{1-1/p,p}(\mathcal{X}, \mathbb{R}^N)$ we associate a Sobolev map $\mathrm{Ext}(u) \in W^{1,p}(\mathcal{M}, \mathbb{R}^N)$ given by a minimizer of the infimum problem

$$\inf \left\{ \int_{\mathcal{M}} |DU|^p \, d\mathcal{H}^{n+1} \mid U \in W^{1,p}(\mathcal{M}, \mathbb{R}^N), \ U_{|\mathcal{X}} = u \right\}.$$

The $(n, \mathbf{p}-1)$-current T_u in $\mathcal{D}_{n,\mathbf{p}-1}(\mathcal{X} \times \mathbb{R}^N)$ carried by the graph of u is given by

$$T_u := (-1)^{n-1} (\partial G_U) \llcorner (\mathcal{X} \times \mathbb{R}^N) \qquad \text{on} \quad \mathcal{D}^{n,\mathbf{p}-1}(\mathcal{X} \times \mathbb{R}^N), \tag{2.4}$$

where $U := \mathrm{Ext}(u)$ and $G_U \in \mathcal{D}_{n+1,\mathbf{p}}(\mathcal{M} \times \mathbb{R}^N)$ is defined as in Example 2.2.

More precisely, for each $\delta > 0$ we choose a cut-off function $\eta = \eta_\delta \in C^\infty([0, \delta], [0, 1])$ such that $\eta(t) = 1$ for $0 \leq t \leq \delta/4$, $\eta(t) = 0$ for $3\delta/4 \leq t \leq \delta$, and $\|\eta'\| \leq 4/\delta$. Then, on account of (2.1), to each smooth n-form $\omega \in \mathcal{D}^n(\mathcal{X} \times \mathbb{R}^N)$ we associate the smooth n-form $\widetilde{\omega}$ in $\mathcal{M}_\delta \times \mathbb{R}^N$ given by

$$\widetilde{\omega} := (\Phi \bowtie \mathrm{Id}_{\mathbb{R}}^N)^{\#} \omega \wedge \eta, \qquad (\Phi \bowtie \mathrm{Id}_{\mathbb{R}}^N)(z, y) := (\Phi(z), y). \tag{2.5}$$

Now, since U is smooth out of $\mathcal{X}$, the above formula (2.4) reads as

$$\langle T_u, \omega \rangle = \langle T_u, \widetilde{\omega} \rangle := (-1)^{n-1} \langle G_U, d\widetilde{\omega} \rangle \qquad \forall\, \omega \in \mathcal{D}^{n,\mathbf{p}-1}(\mathcal{X} \times \mathbb{R}^N), \tag{2.6}$$

where we can choose $\eta = \eta(\delta)$ independently of $0 < \delta < d$.

Remark 2.4 The above definition, introduced in [10], does not depend on the choice of the Sobolev extension. In fact, in [9, Sec. 3.2.5] it is shown that *two Sobolev maps* $U_1, U_2 \in W^{1,p}(\mathcal{M}, \mathbb{R}^N)$ *have the same traces on* $\partial\mathcal{M}$*, i.e.,* $U_{1|\mathcal{X}} = U_{2|\mathcal{X}}$*, if and only if*

$$(\partial G_{U_1}) \llcorner (\mathcal{X} \times \mathbb{R}^N) = (\partial G_{U_2}) \llcorner (\mathcal{X} \times \mathbb{R}^N) \qquad \text{on} \quad \mathcal{D}^{n,\mathbf{p}-1}(\mathcal{X} \times \mathbb{R}^N)\,.$$

Moreover, the following null-boundary condition holds true (cf. [1, Prop. 2.4]).

Proposition 2.5 *If* $\mathbf{p} \geq 2$*, for every* $u \in W^{1-1/p,p}(\mathcal{X}, \mathbb{R}^N)$ *we have*

$$\langle \partial T_u, \xi \rangle := \langle T_u, d\xi \rangle = 0 \qquad \forall\, \xi \in \mathcal{D}^{n-1,\mathbf{p}-2}(\mathcal{X} \times \mathbb{R}^N)\,. \tag{2.7}$$

Notice that if $\mathcal{X} = \mathbb{S}^2$ and $N = 2$, the function

$$u(x_1, x_2, x_3) := \frac{(x_1, x_2)}{|(x_1, x_2)|}$$

belongs to $u \in W^{1-1/p,p}(\mathbb{S}^2, \mathbb{R}^2)$ for each exponent $p < 3$, whence $\mathbf{p} = 2$, but one has

$$\partial T_u = (\delta_{P_-} - \delta_{P_+}) \times [\![\, \mathbb{S}^1 \,]\!] \qquad \text{on} \quad \mathcal{D}^1(\mathbb{S}^2 \times \mathbb{R}^2)\,,$$

where $[\![\, \mathbb{S}^1 \,]\!]$ is the 1-current corresponding to integration on the naturally oriented unit circle $\mathbb{S}^1$, and $\delta_{P_\pm}$ denotes the unit Dirac mass at the point $P_\pm := (0, 0, \pm 1)$. Therefore, u is not a Cartesian map in the sense of [9].

2.3 Functions of bounded variation

If $\Omega \subset \mathbb{R}^n$ is a bounded domain, a summable function $u \in L^1(\Omega, \mathbb{R}^N)$ is said to be of bounded variation, $u \in BV(\Omega, \mathbb{R}^N)$, if the distributional derivative Du is an $\mathbb{R}^{N \times n}$-valued Borel measure in Ω with finite total variation, $|Du|(\Omega) < \infty$. In that case, u is approximately differentiable a.e. in Ω and the approximate gradient ∇u agrees with the Radon-Nikodym derivative of Du with respect to the Lebesgue measure in $\mathbb{R}^n$. We refer to the treatise [4] for further details.

3 Currents carried by graphs in $W^{1-1/p,p}$

Let T_u be the semi-current given by Definition 2.3 for some map $u \in W^{1-1/p,p}(\mathcal{X}, \mathbb{R}^N)$, where $p \geq 2$. Following [10], in this section we write explicitly the action of the "lower" components of T_u in terms of the $W^{1,p}$ extension map U.

We assume for simplicity $\mathcal{X} = \Omega$, a bounded domain in $\mathbb{R}^n$, and $\mathcal{M}_\delta = \Omega \times [0, \delta]$. In fact, the general case of mappings $u : \mathcal{X} \to \mathbb{R}^N$ is recovered by means of local coordinates and a partition of unity argument.

Notice that T_u agrees with the n-current G_u carried by the rectifiable graph $\mathcal{G}_u$ if u is a Sobolev map in $W^{1,q}(\Omega, \mathbb{R}^N)$, where $q \geq \min\{n, N\}$, or, more generally, if $u \in BV(\Omega, \mathbb{R}^N)$ and the determinant of any minor of the approximate gradient matrix $\nabla u \in \mathbb{R}^{N \times n}$ is summable in Ω.

According to (2.4), we decompose $T_u = \sum_{j=0}^{\mathbf{p}-1} T_{u(j)}$, where $T_{u(j)}$ is the component of T_u acting on forms in $\mathcal{D}^n(\Omega \times \mathbb{R}^N)$ with exactly j vertical differentials:

$$\langle T_{u(j)}, \omega \rangle := \langle T_u, \omega^{(j)} \rangle\,, \qquad \omega \in \mathcal{D}^n(\Omega \times \mathbb{R}^N)\,.$$

For $0 < \varepsilon \ll \delta$, set $\eta_\varepsilon(t) := 1 - t/\varepsilon$ for $0 \leq t \leq \varepsilon$ and $\eta_\varepsilon(t) \equiv 0$ for $t \geq \varepsilon$. For each $\omega \in \mathcal{D}^{n,\mathbf{p}-1}(\Omega \times \mathbb{R}^N)$ we have

$$(-1)^{n-1} \langle T_u, \omega \rangle = \langle G_U, \eta'_\varepsilon(t)\, \omega \wedge dt + \eta_\varepsilon(t) \wedge d\omega \rangle\,. \tag{3.1}$$

Setting $U = (U^1, \ldots, U^N)$, for $j = 1, \ldots, N$ we denote

$$D_t U^j(x,t) := \frac{\partial U^j}{\partial t}(x,t)\,, \qquad D_i U^j(x,t) := \frac{\partial U^j}{\partial x_i}(x,t)\,, \quad i = 1, \ldots, n\,.$$

3.1 The component $T_{u(0)}$

If $\omega = \phi(x)\,\psi(y)\,dx$, where $\phi \in C_c^\infty(\Omega)$ and $\psi \in C_c^\infty(\mathbb{R}^N)$, formula (3.1) gives

$$\begin{aligned}\langle T_u, \phi(x)\,\psi(y)\,dx\rangle \quad &= \quad \int_\Omega \phi(x) \fint_{[0,\varepsilon]} \psi(U(x,t))\,dt\,dx \\ &\quad -\sum_{j=1}^N \int_\Omega \phi(x) \int_0^\varepsilon \eta_\varepsilon(t) \frac{\partial \psi}{\partial y_j}(U(x,t))\,D_t U^j(x,t)\,dt\,dx\,.\end{aligned}$$

Since $U \in W^{1,1}(\Omega \times (0,\delta), \mathbb{R}^N)$, passing to the limit as $\varepsilon \to 0$ we get

$$\langle T_u, \phi(x)\psi(y)\,dx\rangle = \int_\Omega \phi(x)\psi(u(x))\,dx\,.$$

By a density argument, this yields that

$$\langle T_u, \phi(x,y)\,dx\rangle = \int_\Omega \phi(x,u(x))\,dx \qquad \forall\, \phi \in C_c^\infty(\Omega \times \mathbb{R}^N)\,. \tag{3.2}$$

In particular, we have: $\mathbf{M}(T_{u(0)}) < \infty$.

3.2 The component $T_{u(1)}$

If $\omega = \phi(x)\,\psi(y)\,\widehat{dx^i} \wedge dy^j$, where $\widehat{dx^i} := dx^1 \wedge \cdots \wedge dx^{i-1} \wedge dx^{i+1} \wedge \cdots \wedge dx^n$, by (3.1) we get

$$\begin{aligned}\langle T_u, \phi(x)\,\psi(y)\,\widehat{dx^i} \wedge dy^j\rangle &= (-1)^{i-1} \int_\Omega \phi(x) \fint_{[0,\varepsilon]} \psi(U(x,t)) D_i U^j(x,t)\,dt\,dx \\ +(-1)^{i-1} &\int_\Omega \frac{\partial \phi}{\partial x_i}(x) \int_0^\varepsilon \eta_\varepsilon(t)\,\psi(U(x,t)) D_t U^j(x,t)\,dt\,dx \\ +(-1)^{n+i-1} &\sum_{k\neq j} \int_\Omega \phi(x) \int_0^\varepsilon \eta_\varepsilon(t) \frac{\partial \psi}{\partial y_k}(U(x,t))\,\frac{\partial(U^j,U^k)}{\partial(x_i,t)}(x,t)\,dt\,dx\,.\end{aligned} \tag{3.3}$$

Since $U \in W^{1,2}(\Omega \times (0,\delta))$, both the terms $\frac{\partial(U^j,U^k)}{\partial(x_i,t)}$ and $D_t U$ are summable functions in $\Omega \times (0,\delta)$, and hence the last two integrals in (3.3) go to zero as $\varepsilon \to 0$.

Remark 3.1 We thus deduce that $T_{u(1)}$ *has finite mass provided that property* (1.1) *holds true.* In addition, in case $N = 1$, the same conclusion holds true for any exponent $p > 1$.

4 Proof of the Main Result

We proof Theorem 1.1 in two steps. We then give an example of $W^{1/2,2}$ maps that are not functions of bounded variation, and collect some final remarks.

4.1 Step 1

We show that if (1.1) holds, then $u \in BV(\Omega, \mathbb{R}^N)$. We follow the lines in [1, Prop. 3.4], where we used arguments taken from Thm. 3 in [9, Sec. 4.2.3].

Since $u \in W^{1-1/p,p}(\Omega, \mathbb{R}^N)$ for some $p \geq 2$, we have seen in Remark 3.1 that if property (1.1) holds, the component $T_{u(1)}$ has finite mass. We now choose the form $\xi = y^j\,\varphi(x)\,\widehat{dx^i}$, where $\varphi \in C^1(\Omega)$ and $|D\varphi| \in L^\infty$, so that

$$d\xi = (-1)^{i-1} D_i\varphi\, y^j\,dx + \varphi(x)\,dy^j \wedge \widehat{dx^i}\,.$$

Since the coefficient of $(d\xi)^{(0)}$ grows linearly in y and the coefficient of $(d\xi)^{(1)}$ is bounded, using that $\mathbf{M}(T_{u(0)}) + \mathbf{M}(T_{u(1)}) < \infty$, the action of T_u on $d\xi$ can be computed, by approximation, as limit of $\langle T_u, d\alpha_h\rangle$, the α_h being smooth $(n-1)$-forms in $\Omega\times\mathbb{R}^N$ with compact support and such that $\alpha_h=\alpha_h^{(0)}$. Since $\mathbf{p}\geq 2$, property (2.7) gives $\langle T_u, d\alpha_h\rangle = 0$, and passing to the limit

$$0=\langle T_u, d\xi\rangle = \langle T_u, (-1)^{i-1}D_i\varphi\, y^j\, dx\rangle + \langle T_u, \varphi(x)\, dy^j\wedge\widehat{dx^i}\rangle\,,$$

whence by the formula (3.2) we have

$$\int_\Omega D_i\varphi\, u^j\, dx = (-1)^i\langle T_u, \varphi(x)\, dy^j\wedge\widehat{dx^i}\rangle\,.$$

Setting for every $\phi=(\phi^1,\dots,\phi^N)\in C_c^\infty(\Omega,\mathbb{R}^{N\times n})$

$$\omega_\phi := \sum_{j=1}^N\sum_{i=1}^n(-1)^i\phi_i^j\,\widehat{dx^i}\wedge dy^j\,,\qquad \phi^j=(\phi_1^j,\dots,\phi_n^j)\,,$$

by linearity this gives

$$\sum_{j=1}^N\int_\Omega \operatorname{div}\phi^j\, u^j\, dx = \langle T_u,\omega_\phi\rangle\,,$$

and hence the estimate

$$|Du|(\Omega)\leq \mathbf{M}(T_{u(1)}) \tag{4.1}$$

follows from the definition of variation, see [4]. We thus conclude that $u\in BV(\Omega,\mathbb{R}^N)$.

4.2 Step 2

We show that if u is a function of bounded variation, then property (1.1) is satisfied. To this purpose, we recall from [1] the following approximation property:

Proposition 4.1 *For $n\geq 2$, let $\rho:\mathbb{R}^n\to\mathbb{R}$ be the summable symmetric convolution kernel given by*

$$\rho(z):=\begin{cases}\dfrac{1}{(n-1)\,\alpha_n}\,(|z|^{1-n}-1) & \text{if}\quad 0<|z|<1\\ 0 & \text{elsewhere}\end{cases}\qquad \alpha_n:=|B^n|$$

so that $\rho\in L^1(\mathbb{R}^n)$, $\operatorname{spt}\rho=\overline{B}^n$, $\rho\geq 0$, and $\int\rho(z)\,dz=1$. Let $u\in L^1(B^n,\mathbb{R}^N)$ and $U(x,t)=⨍_{B_t(x)}u(y)\,dy$. Then for each $\varepsilon>0$ and $x\in B^n_{1-\varepsilon}$ we have

$$(u*\rho_\varepsilon)(x) = ⨍_{[0,\varepsilon]}U(x,t)\,dt\,,\qquad \rho_\varepsilon(z):=\varepsilon^{-n}\rho(z/\varepsilon)\,.$$

Therefore, if $u\in \mathrm{BV}(B^n,\mathbb{R}^N)$, for every $i=1,\dots,n$ we have that

$$\lim_{\varepsilon\to 0}\int_{B^n}\Big|⨍_{[0,\varepsilon]}D_iU(x,t),dt\Big|\,dx = \lim_{\varepsilon\to0}\int_{B^n}|D_i(u*\rho_\varepsilon)(x)|\,dx = |D_iu|(B^n)<\infty \tag{4.2}$$

and if $u\in W^{1,q}(B^n,\mathbb{R}^N)$, the map $x\mapsto ⨍_{[0,\varepsilon]}U(x,t)\,dt$ converges to u strongly in $W^{1,q}$, as $\varepsilon\to 0$.

PROOF: Denote by χ_A the characteristic function of a set $A\subset\mathbb{R}^n$. We have

$$\begin{aligned}⨍_{[0,\varepsilon]}U(x,\lambda)\,d\lambda &= \frac{1}{\alpha_n\,\varepsilon^n}\int_0^1\frac{1}{t^n}\int u(y)\,\chi_{B_{\varepsilon\, t}(x)}(y)\,dy\,dt\\ &= \frac{1}{\alpha_n\,\varepsilon^n}\int_{B_\varepsilon(x)}u(y)\int_0^1\frac{1}{t^n}\chi_{B_{\varepsilon\,t}(x)}(y)\,dt\,dy\\ &= \frac{1}{\alpha_n\,\varepsilon^n}\int_{B_\varepsilon(x)}u(y)\int_{|x-y|/\varepsilon}^1\frac{1}{t^n}\,dt\,dy\\ &= \frac{1}{\varepsilon^n}\int_{B_\varepsilon(x)}u(y)\,\frac{1}{(n-1)\,\alpha_n}\Big(\Big|\frac{x-y}{\varepsilon}\Big|^{1-n}-1\Big)\,dy\\ &= \int u(y)\rho_\varepsilon(x-y)\,dy =: (u*\rho_\varepsilon)(x)\,.\end{aligned}$$

The other assertions follow from standard arguments, compare [4]. □

Now, since for every $u \in W^{1-1/p,p}(\Omega,\mathbb{R}^N)$ and every $\varepsilon > 0$ small we have

$$\frac{1}{\varepsilon}\int_\Omega \int_0^\varepsilon |D_i U(x,t)|\,dt\,dx \le \int_{B^n} \Big| ⨍_{[0,\varepsilon]} D_i U(x,t), dt \Big|\,dx$$

for $i = 1,\dots,n$, by the limit in (4.2) we infer that if $u \in BV(\Omega,\mathbb{R}^N)$, then property (1.1) holds, and the proof is complete.

4.3 A counterexample

Denoting for simplicity $H^{1/2} = W^{1/2,2}$, we give an example taken from [1] of maps $\overline{u} \in H^{1/2}(B^2,\mathbb{R}^2)$ that do not have bounded variation, $\overline{u} \notin \mathrm{BV}(B^2,\mathbb{R}^2)$.

Following [6, Ex. 5], let $f(x) = |\log|x||^\alpha$, where $x \in \mathbb{R}^n$ and $0 < \alpha < 1$. Then $f \in W^{1,n}_{\mathrm{loc}}(\mathbb{R}^n)$ provided that $n > 1/(1-\alpha)$. As a consequence, setting $\Omega = (-1,1)^2$ and $x = (x_1,x_2) \in \Omega$, the function $v : \Omega \to \mathbb{R}^2$ given by

$$v(x_1,x_2) := (|\log|x_1||^\alpha, |\log|x_2||^\alpha)\,, \qquad 0 < \alpha < \frac{1}{2}$$

belongs to the class $H^{1/2}(\Omega,\mathbb{R}^2)$. Furthermore, denoting $v = (v^1,v^2)$, we have

$$|Dv^i| = \frac{\alpha}{|x_i|}\,|\log|x_1||^{\alpha-1}\,, \qquad |\det Dv| = |Dv^1|\cdot|Dv^2|\,,$$

whence $|Dv| \notin L^1(\Omega)$ and $\det Dv \notin L^1(\Omega)$. In particular, $v \notin \mathrm{BV}(\Omega,\mathbb{R}^2)$.

We now modify the function v to obtain a function $u = (u^1,u^2) \in H^{1/2}(\Omega,\mathbb{R}^2)$ such that $0 \le u^i(x) \le 1$ for each i, so that u takes values into the unit square $[0,1]^2$. To this purpose, define $t_0 = 1$ and $t_n := \mathbf{e}^{-n^{1/\alpha}}$, so that $0 < t_n < t_{n-1}$ and $|\log|t_n||^\alpha = n$ for each $n \in \mathbb{N}^+$, and set, for $i = 1,2$,

$$u^i(x) := \begin{cases} |\log|x_i||^\alpha - n & \text{if} \quad t_{n+1} \le |x_i| \le t_n \quad \text{and} \quad n \in \mathbb{N} \quad \text{is even} \\ n - |\log|x_i||^\alpha & \text{if} \quad t_{n+1} \le |x_i| \le t_n \quad \text{and} \quad n \in \mathbb{N} \quad \text{is odd}\,. \end{cases}$$

The function u can be easily extended to a function u from $(-2,2)^2$ onto $[0,1]^2$ that belongs to the class $H^{1/2}$ and such that $u \equiv (0,0)$ at the boundary of $(-2,2)^2$.

4.4 Final remarks

We finally point out that if $u \in W^{1-1/p,p}(\Omega,\mathbb{R}^N) \cap BV$, where $p \ge 2$, we have

$$\mathbf{M}(T_{u(1)}) = |Du|(\Omega)\,. \tag{4.3}$$

In addition, for every $\varphi \in C^\infty_c(\Omega \times \mathbb{R}^N)$ we have

$$\langle T_u, \phi(x,y)\,\widehat{dx^i} \wedge dy^j \rangle = (-1)^{n-i}\int_\Omega \phi(x,u(x))\,dD_i u^j(x) \tag{4.4}$$

In fact, the averaged integral $⨍_{B_\varepsilon(x)} u(y)\,dy$ agrees (up to an absolute constant) with the convolution product $(u * \rho_\varepsilon)(x)$, where $\rho_\varepsilon(z) := \varepsilon^{-n}\rho(z/\varepsilon)$ for some symmetric kernel $\rho \in L^1(\mathbb{R}^n)$, with $\operatorname{spt}\rho = \overline{B}^n$, $\rho \ge 0$, and $\int \rho(z)\,dz = 1$.

Let $\Omega_\varepsilon := \{x \in \Omega \mid \operatorname{dist}(x,\partial\Omega) < \varepsilon\}$. By [4, Prop. 3.2], we have $\nabla(u * \rho_\varepsilon) = Du * \rho_\varepsilon$ in Ω_ε, and $\int_U |\nabla(u * \rho_\varepsilon)|\,dx \to |Du|(U)$ as $\varepsilon \to 0^+$, for every $U \subset\subset \Omega$ such that $|Du|(\partial U) = 0$, see [4, Prop. 3.7]. We thus deduce that the graph currents G_{u_ε} weakly converge (along a sequence $\{\varepsilon_j\}$ with $\varepsilon_j \searrow 0$) in $\mathcal{D}_{n,1}(\Omega \times \mathbb{R}^N)$ to the current T_u.

By lower semicontinuity of the mass we have

$$\mathbf{M}(T_{u(1)}) \le \liminf_{j\to\infty} \mathbf{M}(G_{u_{\varepsilon_j}(1)})\,,$$

where $\mathbf{M}(G_{u_{\varepsilon(1)}}) \leq \int_\Omega |\nabla(u * \rho_\varepsilon)|\, dx$. Moreover, by the weak-* BV-convergence with the total variation convergence, we have

$$\mathbf{M}(G_{u_{\varepsilon(1)}}) = \int_\Omega |\nabla(u * \rho_\varepsilon)|\, dx \to |Du|(U) \quad \text{as } \varepsilon \to 0^+ .$$

Therefore, the inequality $\mathbf{M}(T_{u(1)}) \leq |Du|(\Omega)$ holds, and hence eq. (4.3) follows from the inequality (4.1). In addition, the structure property (4.4) readily follows.

In conclusion, the maps $\overline{u}$ in the previous counterexample are such that $\mathbf{M}(T_{\overline{u}(1)}) = \infty$.

References

[1] Acerbi, E., Mucci, D.: Graphs of maps between manifolds in trace spaces and with vanishing mean oscillation *Ann. Inst. H. Poincaré C Anal. Non Linéaire* **31** (2014) no. 5, 1015–1034.

[2] Alberti, G., Bianchini, S., Crippa, G.: Structure of level sets and Sard-type properties of Lipschitz maps. *Ann. Sc. Norm. Sup. Pisa Cl. Sci. (5)* **12** (2012), 863–902.

[3] Alberti, G., Bianchini, S., Crippa, G.: On the L^p differentiability of certain classes of functions. *Rev. Mat. Iberoam.* **30** (2012), 349–367.

[4] Ambrosio, L., Fusco, N., Pallara, D.: *Functions of bounded variation and free discontinuity problems,* Oxford Math. Monographs, Oxford, 2000.

[5] Bethuel, F., Demengel, F.: Extensions for Sobolev mappings between manifolds. *Calc. Var.* **3** (1995), 475–491.

[6] Brezis, H., Nirenberg, L.: Degree theory and BMO; Part I: compact manifolds without boundaries. *Selecta Math. N. S.* **1** (1995), 197–263.

[7] Calderón, A.-P., Zygmund, A.: On the differentiability of functions which are of bounded variation in Tonelli's sense. *Rev. Un. Mat. Argentina* **20** (1962), 102–121.

[8] Gagliardo, E.: Caratterizzazione delle tracce sulla frontiera relative ad alcune classi di funzioni in n variabili. *Rend. Sem. Mat. Univ. Padova* **27** (1957), 284–305.

[9] Giaquinta, M., Modica, G., Souček, J.: *Cartesian currents in the calculus of variations, Vol I, "Cartesian Currents"*. Ergebnisse Math. Grenzgebiete (III Ser), 37, Springer, Berlin, 1998.

[10] Mucci, D.: The homological singularities of maps in trace spaces between manifolds. *Math. Z.* **266** (2010), 817–849.

Dipartimento di Scienze Matematiche, Fisiche e Informatiche, Università di Parma.
Parco Area delle Scienze 53/A, I-43124 Parma, Italy
E-mail: domenico.mucci@unipr.it