

ON THE GEOMETRY OF SYNTHETIC NULL HYPERSURFACES

FABIO CAVALLETTI, DAVIDE MANINI, AND ANDREA MONDINO

ABSTRACT. This paper develops a synthetic framework for the geometric and analytic study of null (lightlike) hypersurfaces in non-smooth spacetimes. Drawing from optimal transport and recent advances in Lorentzian geometry and causality theory, we define a *synthetic null hypersurface* as a triple $(H, G, \mathfrak{m})$: H is a closed achronal set in a topological causal space, G is a gauge function encoding affine parametrizations along null generators, and $\mathfrak{m}$ is a Radon measure serving as a synthetic analog of the rigged measure. This generalizes classical differential geometric structures to potentially singular spacetimes.

A central object is the *synthetic null energy condition* $(\text{NC}^e(N))$, defined via the concavity of an entropy power functional along optimal transport, with parametrization given by the gauge G . This condition is invariant under changes of gauge and measure within natural equivalence classes. It agrees with the classical Null Energy Condition in the smooth setting and it applies to low-regularity spacetimes. A key property of $\text{NC}^e(N)$ is the stability under convergence of synthetic null hypersurfaces, inspired by measured Gromov–Hausdorff convergence.

As a first application, we obtain a synthetic version of Hawking’s area theorem. Moreover, we obtain various sharpenings of the celebrated Penrose’s singularity theorem: for smooth spacetimes we show that the incomplete null geodesic whose existence is guaranteed by Penrose’s argument is actually *maximizing*; we extend Penrose’s singularity theorem to continuous spacetimes; we prove the existence of trapped regions in the general setting of topological causal spaces satisfying the synthetic $\text{NC}^e(N)$.

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1. INTRODUCTION

Null (or lightlike) hypersurfaces are fundamental objects in Lorentzian geometry and mathematical relativity, playing a central role in the causal structure of spacetimes and in the formulation of physically significant results such as Hawking’s area theorem [28] and Penrose’s singularity theorem [57]. Traditionally, the analysis of such hypersurfaces—defined as co-dimension one submanifolds whose normal vector field is null—relies on the smooth differential-geometric setting. However, the increasing interest in low-regularity spacetimes, including those with continuous metrics or metric-measure structures, necessitates the development of a robust synthetic framework capable of encoding the geometric and causal features of null hypersurfaces beyond smooth regimes.

The goal of this paper is to introduce and develop a synthetic theory of null hypersurfaces and the Null Energy Condition (NEC) in non-smooth spacetimes, within the general setting of topological causal spaces [33, 34]. The main outcomes are:

- The *well-posedness* and *covariance/invariance* of the synthetic notions;
- The *compatibility* of the synthetic constructions of the present paper with the classical smooth notions;
- The *stability* of the synthetic NEC under a suitable extension of the measured Gromov–Hausdorff convergence to synthetic null hypersurfaces;
- *Extensions* of certain classical results, namely Hawking’s area theorem and Penrose’s singularity theorem to settings of extremely low regularity (the former to topological causal spaces, the latter to continuous spacetimes), together with *new insights* even for smooth spacetimes.

1.1. Motivations.

Null hypersurfaces. Often, the null hypersurfaces appearing in applications (e.g., as Cauchy horizons) lack smoothness. This motivates the development of a theory for null hypersurfaces under minimal regularity assumptions. Recall that a hypersurface $H \subset (M, g)$, where (M, g) is a Lorentzian manifold, is said to be *null* if the restriction of the Lorentzian metric g to H is degenerate i.e., it has a one-dimensional kernel at every point. Equivalently, the normal vector field to H is also tangent to H . Null hypersurfaces naturally arise in general relativity: they encode the propagation of light-like signals and gravitational radiation, they serve as geometric models for event horizons, Cauchy horizons, and null infinity. A deeper understanding of their geometry may shed new light on fundamental problems in general relativity, for instance see the recent progress on the classification of Cauchy horizons (see for instance [49, 6, 26]). Furthermore, null hypersurfaces feature prominently in theoretical frameworks such as holography and the AdS/CFT correspondence, where boundary structures often possess null character.

Null Energy Condition. The *Null Energy Condition (NEC)*, in its geometric form, requires that the Ricci tensor satisfies

$$(1.1) \quad \text{Ric}(v, v) \geq 0, \quad \text{for all null (i.e., light-like) vectors } v.$$

This condition encapsulates the classical expectation that the effective energy density measured along lightlike directions is nonnegative. The NEC arises as a central assumption in several foundational results in Lorentzian geometry and general relativity. In

the classical (non-quantum) regime, it is expected to hold for all physically reasonable matter models (cf. [9, Ch. 4.6]). Its mathematical significance is underscored by the following applications:

- (1) *Black Hole Area Theorem*: The NEC underpins the proof of Hawking’s area theorem [28], which asserts the non-decreasing nature of the event horizon area of a black hole in classical general relativity. This result serves as a geometric analog of the second law of thermodynamics.
- (2) *Penrose’s Singularity Theorem*: The NEC is a key hypothesis in Penrose’s singularity theorem [57], which guarantees geodesic incompleteness under conditions of gravitational collapse and the existence of trapped surfaces. This theorem represents a foundational result in the mathematical theory of black holes, recognized by the 2020 Nobel Prize in Physics.

Singular spacetimes. As already anticipated in the foundational work of Lichnerowicz in the 1950s [37], many physically realistic models predict spacetimes with matter fields that yield energy-momentum tensors of low regularity. Through the Einstein equations, this leads to Lorentzian metrics of regularity below C^2 . Notable examples include matched spacetimes modeling stellar interiors and exteriors [41], self-gravitating compressible fluids [5], and shock wave solutions. Certain models require even lower regularity, such as impulsive gravitational waves (see Penrose’s original construction [58], and the more recent [24, Ch. 20]), and cyclic cosmologies [36].

Ultimately, a long-term motivation for considering Lorentzian geometries with weak regularity stems from the expectation that at quantum gravitational scales—such as near black-hole singularities or the origin of the universe—the classical manifold model of spacetime may break down. In these regimes, spacetime may exhibit extreme irregularity, potentially eluding approximation by smooth structures.

In the setting of Lorentzian metrics of low regularity, curvature is typically defined in the sense of distributions, leveraging the smooth background manifold. This approach has been developed, for instance, by Geroch and Traschen [22], allowing the definition of distributional curvature tensors for $W_{\text{loc}}^{1,2}$ Lorentzian metrics satisfying an appropriate non-degeneracy condition—automatically satisfied, e.g., for C^1 -metrics (cf. [23]).

A key objective in this paper is to address curvature—not only in the sense of generalized tensors on a smooth underlying manifold, but in settings where *the spacetime itself is singular*, and neither the metric tensor nor the ambient differentiable structure is assumed to be smooth.

Why extending singularity theorems to non-smooth spacetimes. The singularity theorems constitute a cornerstone of Lorentzian geometry and general relativity, rigorously demonstrating that spacetime singularities—understood as causal geodesic incompleteness—arise generically under physically reasonable conditions. Classic results such as those of Penrose [57] and Hawking [27] establish incompleteness in scenarios involving complete gravitational collapse or cosmological expansion, respectively. These foundational theorems, developed in the late 1960s, were originally formulated under the assumption that the spacetime metric g is smooth.

However, already in their seminal monograph [29], Hawking and Ellis raised concerns about the regularity assumptions inherent in these theorems. In particular, they

emphasized that a lack of low-regularity generalizations would limit the physical robustness of the conclusions: if singularity theorems cease to hold for metrics below the C^2 -threshold, then spacetime incompleteness—hence singularity in the standard sense (cf. [18, p. 10], [29, Sec. 8.1])—could potentially be avoided by a mere drop in regularity. For instance, a $C^{1,1}$ -metric would imply, via the Einstein equations, at most a finite discontinuity in the matter fields, which is hardly pathological from a physical standpoint.

While regularity issues surrounding the singularity theorems have been noted for decades (cf. [62, Sec. 6.2]), substantial progress has only recently been achieved. On the one hand, significant developments in Lorentzian causality theory have extended the framework to encompass continuous metrics [17, 61], and even more abstract settings [34, 46]. On the other hand, advances in analytic techniques—especially convolution-based regularization—have led to rigorous proofs of Penrose’s singularity theorem for C^1 metrics [23] and of Hawking’s singularity theorem for $C^{0,1}$ metrics [8].

The present work pushes this frontier decisively further by establishing a version of Penrose’s singularity theorem for continuous metrics, thereby encompassing a broad spectrum of physically relevant, non-smooth spacetimes.

Let us mention that Hawking’s singularity theorem was extended by the first and third author [14] to the setting of Lorentzian length-spaces satisfying timelike Ricci lower bounds in a synthetic sense; see also the more recent extension by Braun–McCann [4] to synthetic variable lower bounds on the timelike Ricci. However, as detailed below, the null setting of Penrose’s singularity theorem poses considerable additional challenges, that are addressed in the present work.

1.2. Previous contributions on NEC and optimal transport. A first synthetic characterization of the NEC for smooth spacetimes was obtained by McCann [43], as a limiting procedure of timelike Ricci lower bounds. This approach builds on top of the optimal transport characterization of timelike Ricci lower bounds obtained for smooth spacetimes in [42, 50] and the synthetic $\text{TCD}(K, N)$ condition studied in [14] (see also [3, 15, 4, 2] for more recent developments) in the framework of Lorentzian length spaces introduced by Kunzinger–Sämman [34]. The advantage of the characterization of the NEC obtained in [43] is that it extends to non-smooth Lorentzian length spaces; the drawback, as already remarked in [43], is the lack of stability under convergence. Moreover, it remains an open question to draw applications from such an overture.

A distinct approach was later proposed by Ketterer [31], who first studied optimal transport inside null hypersurfaces and considered displacement convexity of entropy for *singular* measures supported on co-dimension two spacelike submanifolds embedded in null hypersurfaces.

After [31], the authors [11] established that (1.1) can be equivalently reformulated in terms of entropy convexity for *diffused* probability measures supported on null hypersurfaces. More precisely, [11] provided a systematic study, in smooth spacetimes, of the interplay between Ricci curvature in null directions, rigged measures on null hypersurfaces, and optimal transport of probability measures on null hypersurfaces which are absolutely continuous with respect to a rigged measure.

1.3. A novel theory for synthetic null hypersurfaces. In the present work, we use the smooth characterization of the NEC in terms of optimal transport provided in [11] as a starting point to develop a synthetic theory of null hypersurfaces and of the Null Energy Condition in non-smooth spacetimes.

Synthetic null hypersurfaces. The building blocks of the novel synthetic framework for null hypersurfaces are:

- a closed achronal set H within a topological causal space $(X, \ll, \leq, \mathcal{T})$,
- a gauge function $G : H^\circ \rightarrow \mathbb{R}$ encoding affine parametrization along causal curves (where $H^\circ \subset H$ denotes the subset of intermediate points of the null generators of H , see Definition 3.3),
- a reference measure $\mathbf{m} \in \mathcal{M}_+(H)$ satisfying $\mathbf{m}(\mathfrak{s}(H)) = 0$, where $\mathfrak{s}(H)$ denotes the static (non-causally connected) part of H .

The triple $(H, G, \mathbf{m})$ defines what we call a *synthetic null hypersurface*. This notion is shown to be compatible with classical null hypersurfaces in smooth Lorentzian manifolds, as well as with achronal boundaries arising as $\partial I^+(A)$ for a compact, achronal, space-like, C^2 -submanifold $A \subset M$ of codimension $k \geq 2$ (see Section 3.2).

The lack of a variational characterization for *null* geodesics (contrary to *timelike* geodesics) poses serious challenges in developing a synthetic theory of null hypersurfaces and of the Null Energy Condition in a non-smooth setting.

Indeed, in smooth spacetimes, null geodesics are uniquely determined (up to affine reparametrizations) by the solution of the geodesic equation. In contrast, in the non-smooth setting, causal curves may admit wild reparametrizations—such as those involving non absolutely continuous functions like the Vitali function—that preserve causal behavior without reflecting genuine geodesicity. To overcome this, we define *G-causal curves* as those for which the composition with the gauge function G is affine, a choice that restores the synthetic analog of affine parametrization and allows for a meaningful definition of null geodesics.

A similar ambiguity appears for the reference measure: in the smooth setting, the rigged measure $\mathbf{m}_L$ concentrated on a null hypersurface is not unique and it depends on the choice of a null geodesic vector field L , whose definition requires the differential structure of the manifold. From a more synthetic point of view, different rigged measures can be obtained via multiplication by a transverse function. The synthetic counterpart of the rigged measure $\mathbf{m}_L$ is given by the measure $\mathbf{m} \in \mathcal{M}_+(H)$ which is part of the data defining a synthetic null hypersurface $(H, G, \mathbf{m})$. In order to show well-posedness of the theory, we establish its covariance under multiplication of the reference measure $\mathbf{m}$ by transverse functions (see Section 4).

Synthetic Null Energy Condition. The cornerstone of our theory is the synthetic Null Energy Condition, denoted by $\text{NC}^e(N)$, where $N > 0$ plays the role of a synthetic upper bound on the dimension of the spacetime. This condition is formulated as the concavity of the functional

$$(1.2) \quad \mathbf{U}_{N-1}(\mu|\mathbf{m}) := \exp \left(-\frac{\text{Ent}(\mu|\mathbf{m})}{N-1} \right),$$

where $\text{Ent}(\mu|\mathbf{m})$ denotes the Boltzmann–Shannon entropy of μ relative to $\mathbf{m}$. The functional U_N is well-known in information theory as the *Shannon entropy power* (see, e.g., [20]; for instance, the entropy power on $\mathbb{R}^N$ is the functional $U_{N/2}$), and it was studied in connection to Ricci curvature bounds in [21].

The synthetic $\text{NC}^e(N)$ condition requires that for any pair of probability measures $\mu_0, \mu_1 \in \mathcal{P}(H)$ admitting a causal coupling, there exists a G -causal dynamical plan $\nu \in \text{OptGeo}_G(\mu_0, \mu_1)$ such that $U_{N-1}(\mu_t|\mathbf{m})$ is concave in t , where $\mu_t := (e_t)_\# \nu$ is the Wasserstein geodesic connecting μ_0 to μ_1 via the law ν .

This condition is inspired by the convexity-based characterizations of lower Ricci bounds for Riemannian manifolds obtained in [42, 55, 19, 68] which led to the celebrated $\text{CD}(K, N)$ spaces of Lott–Sturm–Villani [65, 66, 39].

The synthetic framework developed in the present work exhibits important stability and compatibility properties. Notably, we show that the $\text{NC}^e(N)$ condition is independent of the specific choice of gauge and reference measure within appropriate equivalence classes, see Theorem 4.6, thereby extending to the synthetic setting a property previously established in the smooth realm (see [11], in particular Prop. 7.1.). Furthermore, under additional geometric assumptions such as null non-branching or local compactness, we prove that the sets of initial and final points $\mathbf{a}(H)$, $\mathbf{b}(H)$ of the null generators of H are negligible with respect to $\mathbf{m}$, a property crucial for the applications, see Theorem 5.13 and Proposition 3.16, respectively.

In addition to foundational results, we demonstrate that the synthetic theory recovers the classical NEC in smooth settings. Theorems 3.17 and 3.18 establish that if the classical NEC (1.1) holds, then the associated synthetic null hypersurface satisfies the synthetic $\text{NC}^e(N)$ condition. This compatibility is shown both in the presence of global cross-sections and for achronal boundaries of the form $\partial I^+(A)$, even in the absence of smooth global coordinates.

In positive signature, a key property of the Lott–Sturm–Villani $\text{CD}(K, N)$ condition is the stability under measured Gromov–Hausdorff convergence of metric measure spaces [65, 66, 39] (see also [67]). In Section 9 we establish two stability theorems for the Null Energy Condition under a null counterpart of the measured Gromov–Hausdorff convergence for synthetic null hypersurfaces, see Theorem 9.2 and Theorem 9.5 for the precise statements.

Optimal transport along synthetic null hypersurfaces. In Section 6, we establish existence and uniqueness results for the optimal transport problem along a synthetic null hypersurface H . These results enable us to localize the synthetic Ricci curvature lower bounds to the one-dimensional null generators of H (see Theorem 6.8).

Such a localization paradigm has a long history, with roots in convex geometry [56, 25, 30]. It has been later revisited with tools of Optimal Transport in spaces with lower bounds on the Ricci curvature, both smooth [32, 53] and non-smooth [12]. More recently, it has been developed for Lorentzian pre-length spaces satisfying synthetic *timelike* Ricci curvature lower bounds expressed by the $\text{TMCP}(K, N)$ condition [14] or the $\text{TCD}(K, N)$ condition [15], as well as *variable* synthetic timelike Ricci curvature lower bounds [4].

Theorem 6.8 extends such a powerful toolkit to the synthetic *null* setting. These technical tools play a central role in formulating and proving synthetic analogues of Hawking’s area theorem and Penrose’s singularity theorem (see below).

1.4. Applications.

A synthetic version of the Hawking's Area Theorem. Hawking proved the area theorem in 1971 [28]. In its original formulation, it states that in a smooth spacetime satisfying the Null Energy Condition, the area of a smooth black hole horizon can never decrease.

In classical general relativity, this result played a fundamental role in the development of black hole thermodynamics. The area of a black hole horizon is interpreted as a measure of its entropy, and according to the second law of thermodynamics, entropy cannot decrease over time.

The theorem was later revisited by Chruściel, Delay, Galloway, and Howard [16], as well as Minguzzi [44], who extended it by relaxing the regularity assumptions on the null hypersurface while maintaining the assumption of a smooth ambient spacetime.

More recently, Ketterer [31] and—using a different approach more closely aligned with the present work—the authors [11], extended Hawking's Area Theorem to smooth weighted spacetimes satisfying the Null Energy Condition, employing tools from optimal transport theory.

Theorem 7.3 generalizes Hawking's area theorem to the non-smooth setting of a synthetic null hypersurface $(H, G, \mathfrak{m})$ contained in a topological causal space, satisfying the $\text{NC}^e(N)$ condition.

Penrose's Singularity Theorem for continuous spacetimes. In Section 8, we propose a way to extend Penrose's singularity theorem to the setting of a continuous spacetime (i.e., when the Lorentzian metric is continuous). Recall that Penrose's singularity theorem [57] states that, if a spacetime satisfying the Null Energy Condition admits a non-compact Cauchy hypersurface and it contains a trapped surface, then it is null geodesic incomplete; i.e., there exists a null geodesic γ (in the sense that it solves the geodesics-ODE: $\nabla_{\dot{\gamma}}\dot{\gamma}$) whose maximal domain of definition is strictly contained in $\mathbb{R}$.

Besides the proof, a first challenge in generalizing such a result for spacetimes of low regularity (i.e., when the Lorentzian metric is of regularity lower than C^2) is to make sense of the very statement. Penrose's singularity theorem was extended by Kunzinger–Steinbauer–Vickers [35] to $C^{1,1}$ spacetimes and by Graf [23] to C^1 spacetimes (see also [1, 45, 40] for extensions to Finsler spacetimes and [46] to closed cone structures). In such a regularity, the Null Energy Condition is understood in distributional sense and all the other ingredients in the statement (i.e., existence of a compact Cauchy hypersurface, null geodesic completeness, and existence of a trapped surface) can be phrased, with proper care, as in the smooth setting; the proof is then performed via a clever approximation argument by smooth Lorentian metrics.

For continuous spacetimes, Ling [38, Thm. 4.9] was able to extend part of the proof of Penrose's incompleteness theorem; namely, he proved that there are no future trapped sets (recall the classical terminology that a compact achronal set S is future trapped if $\partial I^+(S)$ is compact) in a spacetime admitting a non-compact Cauchy surface (cf., [54, Ch. 14]). Moreover, he raised the question of finding conditions ensuring the existence of a trapped set in continuous spacetimes [38, Rem. p. 34 of 40]. This is where, in Penrose's argument, the NEC, the geodesic completeness, and the future trapped set play a key role.

When the spacetime is merely continuous, the Null Energy Condition cannot be phrased in a distributional sense (this would require the Lorentzian metric g to be in the Geroch–Traschen class, i.e., $g \in L_{loc}^\infty \cap W_{loc}^{1,2}$, $g^{-1} \in L_{loc}^\infty$). We will thus employ the synthetic $\text{NC}^e(N)$ condition discussed above. Also the notions of trapped surface and of geodesic completeness present challenges, as both involve the Christoffel symbols and thus first derivatives of the Lorentzian metric. Thus, ignoring for the moment the technical difficulties involved in the proof, a basic question is how to even phrase the statement of Penrose’s singularity theorem for a C^0 -spacetime. We propose an answer, building on the synthetic optimal transport tools developed in the present work.

- *A synthetic notion of null completeness.* In Section 8.1 we introduce a weaker form of null geodesic completeness, that admits a synthetic formulation. Roughly, (M, g) is *weakly null complete* if for any maximally-defined null geodesic γ , either γ is defined on the whole $\mathbb{R}$, or γ is not maximizing (see Definition 8.1 for the precise notion). A null-geodesically complete manifold trivially satisfies such a property; Proposition 8.2 exhibits an example of a weakly null complete Lorentzian manifold that is not null complete, justifying the terminology. The advantage of the weak null completeness is that it can be rephrased in terms of gauges. Indeed, Proposition 8.3 shows that for a strongly causal Lorentzian manifold (M, g) , the weak null completeness is equivalent to the existence of a (natural) *proper* gauge. Thus, the properness of the gauge shall be seen as a synthetic counterpart of the weak null completeness.
- *A synthetic notion of future convergence.* The classical definition of future convergence for an achronal 2-codimensional submanifold S states that the mean curvature of S on both (incoming and outgoing) sides of $\partial I^+(S)$ to be strictly negative. This notion is meaningful in the smooth setting. Definition 8.4 provides a synthetic formulation, called $(G, \mathbf{m})$ -future convergence. The building blocks of such a definition are the causal relation, the gauge G , and the measure $\mathbf{m}$. The underlying idea is that the mean curvature is the first variation of area, and the area is in turn a first variation of volume.

With the above notions, we prove:

- *A Penrose’s theorem in a synthetic setting*, Theorem 8.5. Roughly, it states that if an achronal set S is $(G, \mathbf{m})$ -future converging and $H = \partial I^+(S)$ is a synthetic null hypersurface satisfying the $\text{NC}^e(N)$ condition for some $N > 2$, then H is compact; i.e., S is future trapped.
- *A Penrose’s singularity theorem in C^0 -spacetimes*, Corollary 8.8. Roughly, it states that if (M, g) is a continuous spacetime admitting a non-compact Cauchy surface and containing a compact achronal set S which is $(G, \mathbf{m})$ -future converging and such that $H = \partial I^+(S)$ satisfies the $\text{NC}^e(N)$ condition, for some $N > 2$, then (M, g) is not weakly null complete. We remark that even in the case of smooth spacetimes, this result yields a stronger conclusion (see Corollary 8.9) than the classical Penrose’s singularity theorem. The new insight is that the incomplete null geodesic in the conclusion must be *maximizing*, thus possessing undiscovered global variational and geometric properties.

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2. BASICS ON CAUSAL SPACES

In this section, we briefly recall some basic notions and results from the theory of causal spaces. We give the definition of causal space, following the convention of [34].

Definition 2.1 (Causal space $(X, \ll, \leq)$ ([34, Def. 2.11])). A *causal space* $(X, \ll, \leq)$ is a set X endowed with a preorder $\leq$ and a transitive relation $\ll$ contained in $\leq$.

Let us point out that this definition differs from the one present in the paper by Kronheimer and Penrose [33], where the notion of causal space was first introduced. For the reader convenience, we compare in the table below the two definitions.

	Causality (antisymmetry of $\leq$)	Push-up property (see Definition 9.1 for the push-up property)
K.–P. [33]	Assumed	Assumed
K.–S. [34]	Not assumed	Not assumed. However, in the setting Lorentzian pre-length spaces (the topic of [34]), the push-up property is always available.

Remark 2.2. The framework of the present paper will be the one of causal spaces, recalled above. Another possible approach could have been to take as starting point the null-distance of Sormani–Vega [63]. It would be interesting to develop such a parallel theory, and investigate the relations with the present work.

We shall write $x < y$ when $x \leq y, x \neq y$. We say that x and y are *timelike* (resp. *causally*) related if $x \ll y$ (resp. $x \leq y$). Let $A \subset X$ be an arbitrary subset of X . We define the *chronological* (resp. *causal*) future of A the set

$$I^+(A) := \{y \in X : \exists x \in A, x \ll y\}$$

$$J^+(A) := \{y \in X : \exists x \in A, x \leq y\}$$

respectively. Analogously, we define $I^-(A)$ (resp. $J^-(A)$) the *chronological* (resp. *causal*) past of A . In case $A = \{x\}$ is a singleton, with a slight abuse of notation, we will write $I^\pm(x)$ (resp. $J^\pm(x)$) instead of $I^\pm(\{x\})$ (resp. $J^\pm(\{x\})$).

Definition 2.3 (Topological causal space). We say that $(X, \ll, \leq, \mathfrak{T})$ is a *topological causal space*, provided:

- $(X, \ll, \leq)$ is a causal space;
- $\mathfrak{T}$ is a Polish and proper topology on X ;
- In the product $X \times X$, the relation $\leq$ is closed and the relation $\ll$ is open.

Recall that a Polish topology is *proper* if there exists a proper and separable metric inducing the topology; recall that a metric is proper if closed balls are compact.

Throughout the paper, $I \subset \mathbb{R}$ will denote an arbitrary interval.

Definition 2.4. [Causal/timelike curves] A non-constant curve $\gamma : I \rightarrow X$ is called (future-directed) *causal* (resp. *timelike*) if γ is continuous and if for all $t_1, t_2 \in I$, with $t_1 < t_2$, it holds $\gamma_{t_1} \leq \gamma_{t_2}$ (resp. $\gamma_{t_1} \ll \gamma_{t_2}$). We say that γ is a *null* curve if, in addition to being causal, no two points on $\gamma(I)$ are related with respect to $\ll$.

We recall the following

Definition 2.5. Let $(X, \ll, \leq, \mathfrak{T})$ be a topological causal space. We say that a set $A \subset X$ is weakly convex, if for all $x, y \in A$, such that $x \leq y$, there exists a causal curve γ contained in A connecting x to y .

Remark 2.6. We point out that Definition 2.4 is more general than the one given in the seminal paper [34] by Kunzinger and Sämman, for we do not require that a causal curve is (locally) Lipschitz. In that paper, the authors proved that if (M, g) is a strongly causal Lorentzian manifold (i.e., any point admits arbitrarily small neighborhoods which are weakly convex) with g continuous, then a causal Lipschitz curve is casual in the classical sense (i.e., its tangent vector is causal almost everywhere).

Next Lemma recovers Kunzinger's and Sämman's result, by proving, in the same setting, that any causal curve admits a Lipschitz reparametrization.

Lemma 2.7. *Let (M, g) be a strongly causal Lorentzian manifold with g continuous. Let $\gamma : I \rightarrow M$ be a causal curve. Then there exists a reparametrization of γ which is locally Lipschitz-continuous.*

Proof. We can assume without loss of generality that γ is injective and that $0 \in I$. Since the statement is local, it suffices to show that γ admits a Lipschitz reparametrization in a neighborhood of 0. Let U a neighborhood of γ_0 . We may assume that U is coordinated, i.e., that $U \subset \mathbb{R}^n$. Up to further restricting U , we can also assume that $g|_U$ is close to the standard Minkowski product, in the sense that, for all $x \in U$, the causal cone in $T_x U = \mathbb{R}^n$ of g_x is contained in the causal cone of $-2dx_n^2 + \frac{1}{2} \sum_{i=1}^{n-1} dx_i^2$, i.e., if X is a g -causal vector, then

$$(2.1) \quad \sum_{i=1}^{n-1} |X_i|^2 \leq 4|X_n|^2.$$

By strong causality, we can assume U to be weakly convex. Let $[a, b]$, such that $\gamma([a, b]) \subset U$, and let $[\alpha, \beta] = \gamma_n([a, b])$.

We claim that, for all $a \leq t_1 \leq t_2 \leq b$, it holds that

$$(2.2) \quad \sum_{i=1}^{n-1} |\gamma_{t_1}^i - \gamma_{t_2}^i|^2 \leq 4|\gamma_{t_2}^n - \gamma_{t_1}^n|^2.$$

Indeed, fix $t_1 \leq t_2$ and let $\delta : [0, 1] \rightarrow U$ be a causal (in the classical, i.e., differential sense) curve connecting γ_{t_1} to γ_{t_2} . A direct computation gives

$$\begin{aligned} \sqrt{\sum_{i=1}^{n-1} |\gamma_{t_1}^i - \gamma_{t_2}^i|^2} &= \sqrt{\sum_{i=1}^{n-1} |\delta_0^i - \delta_1^i|^2} = \sqrt{\sum_{i=1}^{n-1} \left| \int_0^1 \dot{\delta}^i \right|^2} \leq \int_0^1 \sqrt{\sum_{i=1}^{n-1} |\dot{\delta}^i|^2} \\ &\leq 2 \int_0^1 |\dot{\delta}^n| = 2(\delta_1^n - \delta_0^n) = 2|\gamma_t^n - \gamma_r^n|. \end{aligned}$$

It follows that, for all $\tau \in [\alpha, \beta]$, there exists a unique $t \in [a, b]$, such that $\gamma_n(t) = \tau$ (if it were not unique, then (2.2) would contradict the injectivity of γ). We can therefore define the reparametrization $\eta_\tau = \gamma_t$, so that $\tau = \gamma_t^n$. This reparametrization is Lipschitz-continuous, because $|\eta_{\tau_1}^n - \eta_{\tau_2}^n| = |\tau_1 - \tau_2|$, and

$$(2.3) \quad |\eta_{\tau_1}^i - \eta_{\tau_2}^i| = |\gamma_{t_1}^i - \gamma_{t_2}^i| \leq 2|\gamma_{t_1}^n - \gamma_{t_2}^n| = 2|\tau_1 - \tau_2|. \quad \square$$

3. NULL SYNTHETIC SETTING

3.1. Synthetic definitions. In the general setting of topological causal spaces, the role of smooth null-hypersurfaces will be played by closed achronal sets where the causal relation does not trivialise in a sense that will be clarified later.

Among geometrically meaningful closed achronal sets, one can find, e.g., achronal boundaries and Cauchy horizons. We recall below the definition of achronal boundary that is classical in the smooth literature, see for instance [59, 29].

Definition 3.1 (Achronal boundary). Let $(X, \ll, \leq, \Im)$ be a topological causal space. A set $H \subset X$ is an *achronal boundary* if there exists a set $S \subset X$ such that $H = \partial I^+(S)$.

The next lemma shows that achronal boundaries are actually achronal, so the terminology is consistent.

Lemma 3.2. Let $(X, \ll, \leq, \Im)$ be a topological causal space. Then for any subset $S \subset X$, $\partial I^+(S)$ is achronal.

Proof. Assume on the contrary that there exist $x, y \in \partial I^+(S)$, such that $x \in I^-(y)$. Since $I^-(y)$ is open, there exists an open neighborhood $U \subset I^-(y)$ of x . Since $x \in \partial I^+(S)$, then there exists $w \in U \cap I^+(S)$. It follows that $w \ll y$ and thus, by transitivity of $\ll$, we get that $y \in I^+(S)$, a contradiction. $\square$

It is worth underlining that every maximal achronal set is an achronal boundary, see for instance [47, Theorem 2.97]. Since achronal sets are subsets of their own achronal boundary, they provides the natural framework for a synthetic treatment of null geometry that will take into account both the local and the global picture.

We next define the maximal and minimal elements of a closed achronal set H with respect to the pre-order $\leq$, denoted below as $\mathfrak{b}(H)$ and $\mathfrak{a}(H)$ respectively. Such objects are well-known to be useful in the study of achronal boundaries within smooth space-times, see e.g. [29, Section 6.3], and in the analysis of the L^1 optimal transport problem within metric spaces.

Definition 3.3. Given a closed achronal set H , the set of *final* elements of H is given by

$$\mathfrak{b}(H) := \{m \in H : \nexists x \in H, m \leq x, x \neq m\},$$

and the set of *initial* elements of H is given by

$$\mathfrak{a}(H) := \{m \in H : \nexists x \in H, x \leq m, x \neq m\}.$$

Then the achronal set without end-points is defined by

$$H^\circ := H \setminus (\mathfrak{a}(H) \cup \mathfrak{b}(H)).$$

Finally, the set of *static* elements of H is given by

$$\mathfrak{s}(H) := \mathfrak{a}(H) \cap \mathfrak{b}(H).$$

In the sequel, whenever no confusion arises, we shall write $\mathfrak{a} = \mathfrak{a}(H)$, and $\mathfrak{b}$ and $\mathfrak{s}$ with the analogous meaning.

We briefly discuss the measurability of $\mathfrak{a}$ and $\mathfrak{b}$.

Lemma 3.4. *In a topological causal space X , the sets $\mathfrak{a}(H)$, $\mathfrak{b}(H)$, $\mathfrak{s}(H)$, and H° are Borel measurable.*

Proof. Since the topology on X is induced by a proper metric and J^+ is closed, denoting by $\Delta = \{(x, x) : x \in X\}$, the set $J^+ \setminus \Delta \cap H \times H$ is σ -compact, i.e., is countable union of compact sets. As projections of σ -compact set is σ -compact, writing

$$\mathfrak{a} = H \setminus P_2((J^+ \setminus \Delta) \cap H \times H),$$

we obtain that $\mathfrak{a}$ is Borel measurable and the same conclusion is valid for $\mathfrak{b}$ proving the claims. $\square$

Remark 3.5. If $\gamma : I \rightarrow H$ is any causal curve then $\gamma_I \cap \mathfrak{a}$ contains at most a single element: suppose by contradiction there are two of them, then one has to be in the causal future of the other giving a contradiction. The same is true for $\gamma_I \cap \mathfrak{b}$.

Hence $\gamma^{-1}(\mathfrak{a})$ and $\gamma^{-1}(\mathfrak{b})$ are two closed intervals in I on each of which γ is constant. Therefore, without loss of generality we can assume that if γ is a non-constant causal curve and if $I = [0, 1]$, then $\gamma_{(0,1)} \subset H^\circ$.

We introduce the notion of gauge function.

Definition 3.6. Let H be a closed achronal set and let $G : H^\circ \rightarrow \mathbb{R}$ be a Borel map. We say that G is a *gauge* function for H , if it satisfies the following properties:

- (1) for any injective causal curve γ in H° , $G \circ \gamma$ is strictly increasing and continuous;
- (2) for any causal curve $\gamma : [0, 1] \rightarrow H$, such that $\gamma_{(0,1)} \subset H^\circ$, then $\sup_{t \in (0,1)} G(\gamma_t)$ and $\inf_{t \in (0,1)} G(\gamma_t)$ are both finite.

Finally a gauge function G is called *proper* if the pre-image of every compact set is precompact.

Assumption (1) implies that for any causal (possibly non injective) curve γ in H° , $G \circ \gamma$ is non-decreasing and continuous. This fact follows by reparametrizing γ . Assumption (2) above is needed in order to avoid possible blow-ups of the gauge, as the following example illustrates. The concept of properness of a gauge will be used in Section 8 to suitably reformulate the null completeness property.

In the smooth setting, a causal curve $\gamma : [0, 1] \rightarrow H$ solves the geodesics equation if and only if $G \circ \gamma$ is affine for a natural gauge G (see Remark 3.10 below). We take this last observation as the definition of a causal null geodesic in the null synthetic setting.

Definition 3.7 (*G-Causal curves*). Let H be a closed achronal set and let $G : H^\circ \rightarrow \mathbb{R}$ be a gauge. We say that a curve $\gamma : [0, 1] \rightarrow H$ is *G-causal*, if and only if it is causal, $\gamma_{(0,1)} \subset H^\circ$, and $G \circ \gamma|_{(0,1)}$ is an affine function.

The set of *G-causal* curves will be denoted by $\text{Caus}_G \subset C([0, 1]; H)$.

Sometimes, with a slight abuse of notation, we will consider *G-causal* curves whose domain is not the interval $[0, 1]$.

We are now in position to give the definition of synthetic null hypersurface. We denote by $\mathcal{M}^+(H)$ the set of non-negative Radon measures on H .

Definition 3.8 (*Synthetic Null Hypersurface*). Let $(X, \ll, \leq, \mathfrak{T})$ be a topological causal space. We say that the triple $(H, G, \mathbf{m})$ is a *synthetic null hypersurface*, if

- H is a closed achronal set,
- $G : H^\circ \rightarrow \mathbb{R}$ is a gauge for H ,
- $\mathbf{m} \in \mathcal{M}^+(H)$ does not give positive mass to the static elements of H , i.e., $\mathbf{m}(\mathfrak{s}) = 0$.

Remark 3.9. The request of $\mathbf{m}(\mathfrak{s}) = 0$ is necessary to avoid examples of closed achronal sets which do not have the geometry of a null hypersurface. For instance, if H is an acasual set (e.g., a spacelike hypersurface), then the set of static points will coincide with H itself and therefore the only admissible measure to consider over H in order to form a synthetic null hypersurface is $\mathbf{m} \equiv 0$.

3.2. Compatibility with the smooth setting. We next discuss the compatibility of the above synthetic constructions with the smooth theory, by associating to a classical null hypersurface a synthetic one. This is the content of the following remarks.

Remark 3.10 (*Compatibility with the smooth setting: the global case*). We proceed by first discussing the global case, i.e., when a global smooth cross-section is available. Let H be a causal and achronal, smooth null hypersurface, admitting $S \subset H$ a global smooth cross-section for H . Using the cross-section S , we can build a global null geodesic vector field L over H and the corresponding flow map $\Psi_L : S \times \mathbb{R} \rightarrow H$. A natural gauge function $G_{L,S} : H \rightarrow \mathbb{R}$ is then defined as the function such that $z = \Psi_L(p, G_{L,S}(z))$, for some (unique) $p \in S$. To check that $G_{L,S}$ is indeed a gauge function, observe that if γ is a causal curve in H , then it lies in the generator of H passing through γ_0 and, in particular, it must be of the form $\gamma_t = \Psi(\gamma_0, s(t))$, for some non-decreasing function s . Therefore $G_{L,S}(\gamma_t) = G_{L,S}(\gamma_0) + s(t)$, which is non-decreasing. If γ is also injective, then s must be strictly increasing. Moreover, we point out that *a causal curve is $G_{L,S}$ -causal if and only if it is a null geodesic in H* . Finally, as reference measure over H , we choose the rigged measure Vol_L (see [11]). A synthetic null hypersurface associated to H is $(H, G_{L,S}, \text{Vol}_L)$.

Remark 3.11 (*Compatibility with the smooth setting: the local case*). In several situations of interest (e.g., in the presence of recurring geodesics like the case of compact Cauchy horizons [49, 6, 26]) a global cross-section for H does not exist. In these cases, it is possible to proceed via a local argument as follows.

Let (M, g) be a Lorentzian manifold and $U \subset M$ is any open set. For $x, y \in U$, we set by definition $x \ll_U y$, provided that there exists a C^1 curve $\gamma : [0, 1] \rightarrow U$, connecting x to y , such that $\dot{\gamma}$ is a future-directed timelike vector. Notice that $\ll_U$ is (in general) strictly contained in $\ll \cap (U \times U)$; in a similar way we define $\leq_U$ and denote by $\mathcal{T}_U$ the topology on U induced by $\mathcal{T}$. We will write X_U as a shorthand notation for the topological causal space $(U, \leq_U, \ll_U, \mathcal{T}_U)$; in particular X_U is the topological causal space induced by the Lorentzian manifold $(U, g|_U)$.

Now if H is a smooth null hypersurface, then for any $x \in H$, we can take a coordinated neighborhood U for x . Up to restricting U , we have that $H \cap U$ is causal, achronal and closed and that a cross-section S is given by the intersection of H and the 0-level set of the time coordinate. We can thus apply the construction described in Remark 3.10.

Remark 3.12 (Compatibility with the smooth setting: achronal boundaries). Let (M, g) to be a strongly causal, causally closed, C^2 Lorentzian manifold. Let $A \subset M$ be a compact, achronal, space-like, C^2 sub-manifold of codimension $k \geq 2$. Then

$$H := \partial I^+(A)$$

is a locally-Lipschitz topological hypersurface (see, e.g., [59, Lemma 3.17] and [29, Proposition 6.3.1]). If A is a singleton, H is a future cone; if the codimension of A is 2, then H is the central object in the proof of Penrose's singularity theorem. To include achronal boundaries of this type in our setting, we proceed as follows.

Under the above assumptions, it is a well-known fact that

$$(3.1) \quad H = \partial I^+(A) = J^+(A) \setminus I^+(A).$$

Indeed, the inclusion " $\subset$ " is a consequence of the causal closedness of (M, g) and the compactness of A . Conversely, the inclusion " $\supset$ " follows by the C^2 -regularity of g , for a proof see [46, Prop. 2.13].

Firstly, we claim that $\mathfrak{a} \subset A$. Indeed, if $x \in \mathfrak{a} \subset J^+(A)$, there exists $y \in A \cap J^-(x)$; it holds that $y \notin I^+(A)$, otherwise the push-up property would imply that $x \in I^+(A)$, a contradiction. Hence $y \in H$, therefore $y = x$, otherwise x would not be an initial point (i.e., minimal for $\leq$).

We next claim that $J^+(A) = J^+(\mathfrak{a})$. Indeed, fix $x \in J^+(A)$. By compactness, there exists y , a minimal element (w.r.t. $\leq$) of $A \cap J^-(x) \subset J^+(A)$; $y \in H$, otherwise, $y \in I^+(A)$, a contradiction with its minimality. Let $z \in H \cap J^-(y)$; minimality of y implies that $z = y$, therefore $y \in \mathfrak{a}$, thus $x \in J^+(\mathfrak{a})$.

As a consequence, $I^+(A) = I^+(\mathfrak{a})$ and

$$(3.2) \quad H = J^+(A) \setminus I^+(A) = J^+(\mathfrak{a}) \setminus I^+(\mathfrak{a}).$$

Let $M^- : H \setminus \mathfrak{b} \rightarrow A$ be the map defined by

$$M^-(x) := \inf J^-(x) \cap A,$$

where the infimum is taken w.r.t. the causal order $\leq$. Since x is not a final point, then $J^-(x) \cap A$ is totally ordered (i.e., we can always compare elements), hence there is at most one minimal point. Compactness of A guarantees that the infimum is attained. Checking the measurability of M^- is standard. Notice that M^- takes values in $\mathfrak{a}$.

We claim that the set $A \setminus \mathfrak{a}$ is compact. Indeed, let x_n be a sequence in $A \setminus \mathfrak{a}$; by the compactness of A , up to a subsequence, $x_n \rightarrow x \in A$. Let $y_n \in A$, be such that $y_n < x_n$;

up to a subsequence $y_n \rightarrow y \in A$. Causal closedness yields $y \leq x$. If $y \neq x$, then $y < x$, proving that $x \in A \setminus \mathfrak{a}$. If on the contrary $x = y$, then, since x_n and y_n converge to the same point x and are causally related, then $T_x A$ is not space-like, a contradiction. Thus, $A \setminus \mathfrak{a}$ is compact.

It follows that $\mathfrak{a}$ is open in A , thus a sub-manifold. Notice also that $\mathfrak{a}$ is acausal.

Consider $\mathcal{N}\mathfrak{a}$, the normal bundle to $\mathfrak{a}$ and endow it with the restriction of the ambient Lorentzian metric g ; notice that each fiber of $\mathcal{N}\mathfrak{a}$ is a k -dimensional Minkowski space-time. Let $W \in \mathfrak{X}(TM)$ be a vector field inducing the time orientation; since $A \subset M$ is a smooth spacelike hypersurface, we can assume $W|_A \in \mathfrak{X}(\mathcal{N}A)$. Let

$$(3.3) \quad \hat{H} := \{v \in \mathcal{N}\mathfrak{a} : g(v, v) = 0 \text{ and } g(v, W) < 0\}.$$

We claim that $H \setminus \mathfrak{a} \subset \exp(\hat{H})$. Indeed, Equation (3.2) implies that if $x \in H \setminus \mathfrak{a}$, there exists a null curve γ connecting x with $\mathfrak{a}$; by Proposition 5.8, we can assume that γ is a geodesic. It is immediate to see that $\dot{\gamma}_0$ is a future-directed null vector and that it is orthogonal to $\mathfrak{a}$, thus $\dot{\gamma}_0 \in \mathcal{N}\mathfrak{a}$.

We now define a gauge for H : for $x \in H^\circ$, $x = \exp(v)$ for some (unique) $v \in \hat{H} \cap \mathcal{N}_{M^-(x)}\mathfrak{a}$; define

$$(3.4) \quad G(x) := -g(v, W).$$

Notice that G can be continuously extended to 0 on the set $\mathfrak{a}$.

Next, we endow the manifold $\hat{H} \subset \mathcal{N}\mathfrak{a}$ with a tangent vector field $\hat{L} \in \mathfrak{X}(T\hat{H})$. In order to do so, notice that at each point $v \in \mathcal{N}_x\mathfrak{a}$, the tangent space splits as the direct sum (in the sense of linear algebra)

$$T_v\mathcal{N}\mathfrak{a} \cong \mathcal{N}_x\mathfrak{a} \oplus T_x\mathfrak{a} = T_xM.$$

We denote by $\Sigma : T\mathcal{N}\mathfrak{a} \rightarrow TM|_{\mathfrak{a}}$, $\Sigma_v : T_v\mathcal{N}\mathfrak{a} \rightarrow T_xM$, such a morphism of fiber bundles. Let $\hat{g} := \Sigma^*g$ be the pull back on $T\mathcal{N}\mathfrak{a}$ of the Lorentzian metric g ; since on each fiber Σ_v is an isomorphism, we can also define $\hat{W} \in \mathfrak{X}(T\mathcal{N}\mathfrak{a})$ as $\hat{W}_v := (\Sigma_v)^{-1}(W)$, giving a time-orientation to $\mathcal{N}\mathfrak{a}$. Observing that $\hat{H}$ is a smooth null hypersurface in the smooth Lorentzian manifold $(\mathcal{N}\mathfrak{a}, \hat{g})$, we can set $\hat{L}$ as the null vector field $\hat{L} \in \mathfrak{X}(\mathcal{N}\hat{H})$, such that $\hat{g}(\hat{L}, \hat{W}) = -1$.

Denote by $U \subset \hat{H}$ the open set where the exponential map is defined and its differential has maximal rank. For any $v \in U$, there exists a neighborhood $U_v \subset U$, such that $\exp|_{U_v}$ is a diffeomorphism on its image. In particular

$$(3.5) \quad H_v := \exp(U_v)$$

is a smooth null hypersurface. We endow H_v with the null geodesic vector field $L^v := d\exp|_{U_v}[\hat{L}]$ and consider the rigged volume Vol_{L^v} . By pulling back the rigged measure Vol_{L^v} associated to L^v , we obtain the measure

$$(3.6) \quad \hat{\mathbf{m}}_v := (\exp|_{U_v})_{\#}^{-1} \text{Vol}_{L^v} \in \mathcal{M}^+(U_v).$$

Given $v_1, v_2 \in U$, it is immediate to see that the measures $\hat{\mathbf{m}}_{v_1}$ and $\hat{\mathbf{m}}_{v_2}$ coincide on $U_{v_1} \cap U_{v_2}$. We can therefore glue these local measures to define a global measure $\hat{\mathbf{m}} \in \mathcal{M}^+(U)$. Let

$$B := \{v \in U : \exists \lambda > 1, \exp(\lambda v) \notin I^+(A)\}.$$

On one hand, it clearly holds that $\exp(B) \subset H^\circ$. On the other hand, if $x \in H^\circ$, then x is not a focal point for the geodesic connecting $\mathfrak{a}$ to x (see [47, Th. 6.16 (a)]); thus it is not a conjugate point, meaning that the differential of the exponential map has maximal rank at $\exp^{-1}(x)$, and therefore $\exp^{-1}(x) \in B$. Thus

$$\exp(B) = H^\circ,$$

and we can endow H° with $\mathfrak{m} := \exp_\#(\hat{\mathfrak{m}}_{\perp B})$. A synthetic null hypersurface associated to H is then $(H, G, \mathfrak{m})$.

3.3. Optimal transport and the synthetic Null Energy Condition $\text{NC}^e(N)$. A concept analogous to the one of optimal dynamical plan can be introduced in the synthetic setting by means of the gauge function and the G -causal curves as follows. Firstly, given any two probability measures μ_0, μ_1 , we introduce

$$\text{OptCaus}^H(\mu_0, \mu_1) := \{\nu \in \mathcal{P}(C([0, 1]; H)) : (e_0)_\# \nu = \mu_0, (e_1)_\# \nu = \mu_1, \nu(\text{Caus}) = 1\},$$

where $\text{Caus} \subset C([0, 1]; H)$ is the closed subset of causal curves and $e_i : C([0, 1]; H) \rightarrow H$ are the evaluation map at $t = i$, with $i = 0, 1$. We also recall the definition of the set of causal couplings

$$\Pi_\leq(\mu_0, \mu_1) = \{\pi \in \mathcal{P}(X \times X) : (e_i)_\# \pi = \mu_i, i = 0, 1\}.$$

In particular, $\text{OptCaus}^H(\mu_0, \mu_1) \neq \emptyset$ only if $\Pi_\leq(\mu_0, \mu_1) \neq \emptyset$. Notice that since H is closed, as topological space H is Polish and therefore $C([0, 1]; H)$ is Polish with the topology of the uniform convergence. In particular, every element of $\mathcal{P}(C([0, 1]; H))$ is inner regular with respect to compact sets of $C([0, 1]; H)$.

Definition 3.13. Let $(H, G, \mathfrak{m})$ be a synthetic null hypersurface and let $\mu_0, \mu_1 \in \mathcal{P}(H)$ be two probability measures. Any $\nu \in \text{OptCaus}^H(\mu_0, \mu_1)$ is said to be G -causal if and only if $\nu(\text{Caus}_G) = 1$. The set of such G -causal plans ν is denoted by $\text{OptGeo}^G(\mu_0, \mu_1)$.

Given a synthetic null hypersurface $(H, G, \mathfrak{m})$, for $\mu \in \mathcal{P}(H)$, the Boltzmann–Shannon entropy $\text{Ent}(\mu|\mathfrak{m})$ of μ with respect to $\mathfrak{m}$ is defined by

$$\text{Ent}(\mu|\mathfrak{m}) := \int_X \rho \log(\rho) \, \mathfrak{m} \in [-\infty, +\infty),$$

if $\mu = \rho \mathfrak{m}$ and $(\rho \log(\rho))_+$ is $\mathfrak{m}$ -integrable; otherwise we set $\text{Ent}(\mu|\mathfrak{m}) = +\infty$. We denote by

$$\text{Dom}(\text{Ent}(\cdot|\mathfrak{m})) := \{\mu \in \mathcal{P}(X) : \text{Ent}(\mu|\mathfrak{m}) \neq +\infty\}.$$

Let us consider the following dimensional variant of the Boltzmann–Shannon entropy: given $N > 0$, for $\mu \in \text{Dom}(\text{Ent}(\cdot|\mathfrak{m}))$, define

$$(3.7) \quad \text{U}_N(\mu|\mathfrak{m}) := \exp\left(-\frac{\text{Ent}(\mu|\mathfrak{m})}{N}\right),$$

and set $\text{U}_N(\mu|\mathfrak{m}) = 0$ otherwise. Such a functional U_N is well-known in information theory as the Shannon entropy power (see, e.g., [20]; for instance, the entropy power on $\mathbb{R}^N$ is the functional $\text{U}_{N/2}$), and it was studied in connection to Ricci curvature bounds in [21].

Definition 3.14 ($\text{NC}^e(N)$ for synthetic null hypersurfaces). Let $N > 0$. Let $(X, \ll, \leq, \mathfrak{T})$ be a topological causal space.

The synthetic null hypersurface $(H, G, \mathbf{m})$ satisfies the Null Energy Condition $\text{NC}^e(N)$ if and only if the following holds: for all measures $\mu_0, \mu_1 \in \mathcal{P}(H)$ such that $\Pi_{\leq}(\mu_0, \mu_1) \neq \emptyset$, there exists $\nu \in \text{OptGeo}^G(\mu_0, \mu_1)$ such that

$$(3.8) \quad U_{N-1}(\mu_t | \mathbf{m}) \geq (1-t) U_{N-1}(\mu_0 | \mathbf{m}) + t U_{N-1}(\mu_1 | \mathbf{m}), \quad \text{for all } t \in [0, 1],$$

where $\mu_t := (e_t)_\# \nu$.

We now collect two geometric consequences of Definition 3.14. The first one is contained in the following

Remark 3.15 (NC^e implies weak convexity). Note that if both μ_0, μ_1 are not contained in $\text{Dom}(\text{Ent}(\cdot | \mathbf{m}))$, then (3.8) is trivially satisfied. On the other hand, if either μ_0 or μ_1 belongs to $\text{Dom}(\text{Ent}(\cdot | \mathbf{m}))$, then (3.8) implies that, for all $t \in (0, 1)$, μ_t is absolutely continuous with respect to $\mathbf{m}$.

Notice also that if $x \leq y$ in H , then taking $\mu_0 = \delta_x$, $\mu_1 = \delta_y$, $\Pi_{\leq}(\mu_0, \mu_1) = \{\delta_{(x,y)}\}$. Definition 3.14 implies that there exists $\nu \in \text{OptGeo}^G(\mu_0, \mu_1)$, producing a G -causal curve from x to y . In other words, H is weakly convex.

We now prove that the sets of initial and final points can be neglected provided the set of G -causal curves satisfy the following compactness property: for $\mathbf{m}$ -a.e. $x \in H$ there exists a neighborhood U , such that $\mathbf{m}(U) > 0$ and the set

$$\{\gamma \in \text{Caus}_G : \gamma_0, \gamma_1 \in K\},$$

is compact, where $K \subset U$ is any compact set. A synthetic null hypersurface $(H, G, \mathbf{m})$ satisfying such assumption will be said *locally compact*.

This assumption is independent from the non-branching condition, that is usually assumed in the literature when that initial and final points have measure zero.

Proposition 3.16. *Let $(H, G, \mathbf{m})$ be a synthetic null hypersurface satisfying the Null Energy Condition $\text{NC}^e(N)$, for some $N > 0$. Assume $(H, G, \mathbf{m})$ to be locally compact. Then*

$$\mathbf{m}(\mathfrak{a}) = \mathbf{m}(\mathfrak{b}) = 0.$$

Proof. It will suffice to prove that $\mathbf{m}(\mathfrak{a} \setminus \mathfrak{b}) = 0$: by assumption $\mathbf{m}(\mathfrak{s}) = 0$ (see Definition 3.8) and therefore it will follow that $\mathbf{m}(\mathfrak{a}) = 0$. The remaining part is symmetric.

Suppose by contradiction

$$(3.9) \quad \mathbf{m}(\mathfrak{a} \setminus \mathfrak{b}) > 0.$$

Define the function $A : H^\circ \times H^\circ \cap J \rightarrow \mathbb{R}$ as $A(x, y) = G(y) - G(x) \geq 0$. Fix $\varepsilon > 0$ and consider the set

$$(3.10) \quad W_\varepsilon := \{(x, z, w) \in \mathfrak{a} \setminus \mathfrak{b} \times H^\circ \times H^\circ : (x, z), (z, w) \in J, A(z, w) > \varepsilon\}.$$

The set W_ε is Borel. It holds that

$$(3.11) \quad \mathfrak{a} \setminus \mathfrak{b} = \bigcup_{\varepsilon > 0} P_1(W_\varepsilon).$$

The (contradiction) assumption gives $\mathbf{m}(P_1(W_\varepsilon)) > 0$, for some $\varepsilon > 0$. By inner regularity of $\mathbf{m}$, there exists a compact set $K \subset P_1(W_\varepsilon)$ with $\mathbf{m}(K) > 0$. We define the following analytic set:

$$\Lambda := P_{1,3}(W_\varepsilon).$$

By Von Neumann's selection Theorem (see for instance [64, Th. 5.5.2]), we deduce that Λ contains the graph of a measurable map T . Define the absolutely continuous measure $\mu_0 := \mathbf{m}_\perp|_K / \mathbf{m}(K) \in \mathcal{P}(H)$ and $\mu_1 := T_\# \mu_0$ so that $(\text{Id}, T)_\# \mu_0 \in \Pi_\leq(\mu_0, \mu_1)$.

Up to neglecting a set of small μ_0 -measure, we can also assume that T is continuous and therefore also μ_1 can be assumed with compact support: hence we have deduced the existence of a compact set $K \subset H$ such that $\mu_0(K) = \mu_1(K) = 1$.

Definition 3.14 gives then a measure $\nu \in \text{OptGeo}^G(\mu_0, \mu_1)$ such that the function $t \mapsto U_N((e_t)_\# \nu)$ is concave. By Remark 3.15 $(e_t)_\# \nu \ll \mathbf{m}$.

Fix any $x \in K$ and an open neighborhood U of x such that $\mu_0(U) > 0$; then we deduce the existence of $\bar{t} > 0$ such that

$$\nu(\{\gamma \in \text{Caus}_G : \gamma_0 \in U, \gamma_t \in U\}) > 0.$$

Indeed by continuity of causal curves,

$$\nu \otimes \mathcal{L}^1(\{(\gamma, t) \in \text{Caus}_G \times (0, 1) : \gamma_0 \in U, \gamma_t \in U\}) > 0,$$

then Fubini's Theorem implies the claim. Then, since ν is inner regular, there exists a compact set $\bar{C}$ such that

$$\bar{C} \subset \{\gamma \in \text{Caus}_G : \gamma_0 \in U, \gamma_{\bar{t}} \in U\}, \quad \text{with} \quad \nu(\bar{C}) > 0.$$

Denote by $\bar{\nu}$ the renormalized restriction of ν to $\bar{C}$ and define the probability measures

$$\bar{\mu}_0 := (e_0)_\# \bar{\nu}, \quad \bar{\mu}_1 := (e_{\bar{t}})_\# \bar{\nu}.$$

By construction, they are causally related and both concentrated on a compact set inside U . Since U was arbitrary, we can take as U the one additional verifying the local compactness hypothesis and deduce the existence of a measure $\eta \in \text{OptGeo}^G(\bar{\mu}_0, \bar{\mu}_1)$ such that the function $t \mapsto U_N((e_t)_\# \bar{\nu})$ is concave and $\eta(C) = 1$, for some compact set C .

For each $t \in [0, 1]$, denote now by $C_t := e_t(C)$. Notice that each C_t is a compact set, C_t converges to C_0 in Hausdorff topology as $t \rightarrow 0$ and $C_0 \subset \mathfrak{a} \setminus \mathfrak{b}$. Finally, by the entropy inequality (3.8) and a double Jensen's inequality we have

$$\mathbf{m}(C_t)^{\frac{1}{N-1}} \geq U_{N-1}(\mu_t | \mathbf{m}) \geq (1-t) U_{N-1}(\mu_0 | \mathbf{m}) = (1-t) \mathbf{m}(C_0)^{\frac{1}{N-1}}.$$

Then we reach a contradiction as follows. Since C_0 is a subset of initial points, necessarily $C_t \cap C_0 = \emptyset$; by Hausdorff convergence for each $\epsilon > 0$ for t sufficiently close to 0 we have

$$\mathbf{m}((C_0)^\epsilon) \geq \mathbf{m}(C_0 \cup C_t) = \mathbf{m}(C_0) + \mathbf{m}(C_t) \geq (1 + (1-t)^N) \mathbf{m}(C_0).$$

Hence letting $t \rightarrow 0$ and then $\epsilon \rightarrow 0$ we reached a contradiction and proved the claim. $\square$

Following the discussion on the compatibility of the notion of synthetic null hypersurfaces with the smooth one (see Remark 3.10), we also report the compatibility of the classical NEC with the synthetic $\text{NC}^e(N)$ condition (see Definition 3.14), established in the authors' previous work [11].

Theorem 3.17 (Compatibility of NC^e with the smooth NEC). *Let (M, g) be a Lorentzian manifold of dimension n . Let $H \subset M$ be a C^2 null hypersurface. Assume that the classical NEC is valid along H , i.e., $\text{Ric}(w, w) \geq 0$, for any normal vector w to H . Then the following two properties hold:*

- (1) *For any open set $U \subset M$, if $H \cap U$ is closed in U and admits a global cross-section S with respect to $\leq_U$, then the synthetic null hypersurface $(H \cap U, \text{Vol}_L, G_{L,S})$ in X_U defined in Remark 3.10 satisfies $\text{NC}^e(n)$.*
- (2) *For any $x \in H$, there exists a neighborhood U of x in M , such that the synthetic null hypersurface $(H \cap U, \text{Vol}_L, G_{L,S})$ in X_U defined in Remark 3.11 satisfies $\text{NC}^e(n)$.*

Proof. Part 1 follows by [11, Theorem 1.6]. The implication $1 \implies 2$: is the content of Remark 3.11. $\square$

We now complement Theorem 3.17 by showing that the classical NEC is also compatible with the synthetic NC^e along achronal boundaries, without any further regularity assumption. Recall that achronal boundaries fall within the class of synthetic null hypersurfaces, see Remark 3.12.

Theorem 3.18. *Let (M, g) be a Lorentzian manifold of dimension n . Assume M to be causally closed and strongly causal. Let $A \subset M$ be a C^2 achronal, space-like sub-manifold of codimension $k \geq 2$. Let $H := \partial I^+(A)$. Assume that the classical NEC is valid on H , i.e., $\text{Ric}(w, w) \geq 0$, for any light-like vector $w \in TH$ at the differentiability points of H .*

Then the synthetic null hypersurface $(H, G, \mathfrak{m})$, as defined in Remark 3.12, satisfies $\text{NC}^e(n)$.

Proof. The claim is a consequence of Theorem 6.8 and the constructions performed in Remark 3.12, to which we refer for the notation. For the sake of brevity, we outline the argument, omitting some straightforward technical details.

Recall the definition (3.3) of $\hat{H}$ as a C^2 -submanifold of TM endowed with the measure $\hat{\mathfrak{m}}$ locally given by (3.6). By its very definition, $\hat{H}$ is a ruled submanifold, generated by the future pointing light-like vectors $v \in \mathcal{N}\mathfrak{a}$. It is convenient to parametrize such generators on a suitable set Q of indices, that can be identified with a suitable measurable subset of $\hat{H}$. We disintegrate the measure $\hat{\mathfrak{m}}$ on $\hat{H}$ along such generators of $\hat{H}$, as follows

$$\hat{\mathfrak{m}} = \int_Q \hat{\mathfrak{m}}_\alpha \mathfrak{q}(d\alpha),$$

where $\mathfrak{q}$ is a probability measure on Q , and each measure $\hat{\mathfrak{m}}_\alpha$ is concentrated on the future pointing half-line generated by v_α , $\alpha \in Q$. Since the exponential map does not mix such generators, we can also disintegrate the measure $\mathfrak{m}$, as follows

$$(3.12) \quad \mathfrak{m} = \int_Q \exp_\#(\hat{\mathfrak{m}}_{\alpha \perp B}) \mathfrak{q}(d\alpha).$$

Fix a vector $v \in \hat{H}$. Consider the C^2 null hypersurface H_v defined in (3.5) and its rigged measure Vol_{L^v} . Since H_v is C^2 , then $\text{Ric}(w, w) \geq 0$ for all light-like $w \in TH_v$ in the differentiability points of H . Moreover H is a locally-Lipschitz sub-manifold, then a.e. differentiable. It follows that

$$\text{Ric}(w, w) \geq 0, \quad \text{for all } w \in TH_v|_H.$$

By disintegrating the rigged measure Vol_{L^v} along the above family of generators, we get

$$\text{Vol}_{L^v} = \int_Q \text{Vol}_{L^v, \alpha} \mathfrak{q}(\mathrm{d}\alpha).$$

For $\mathfrak{q}$ -a.e. $\alpha \in Q$, the conditional measure $\text{Vol}_{L^v, \alpha}$ is absolutely continuous w.r.t. the 1-dimensional Lebesgue measure on the corresponding generator, with density $a_{L^v, \alpha}$. By [11, Theorem 5.11], such densities satisfy

$$(3.13) \quad (a_{L^v, \alpha})''(t) + \frac{((a_{L^v, \alpha})'(t))^2}{n-2} \leq 0, \quad t = -g(W, (\exp|_{U_v})^{-1}(x)), \quad \forall x \in H_v \cap H.$$

Since locally the measure $\mathfrak{m}$ coincides with Vol_{L^v} , the uniqueness of the disintegration implies that the conditional measure $\exp_{\sharp}(\hat{\mathfrak{m}}_{\alpha \perp B})$ has density $a_{L^v, \alpha}$, for $\mathfrak{q}$ -a.e. $\alpha \in Q$. For $v_1, v_2 \in \hat{H}$, one can check that

$$a_{L^{v_1}, \alpha}(t) = a_{L^{v_2}, \alpha}(t), \quad t = -g(W, u), \quad u \in U_{v_1} \cap U_{v_2}.$$

Therefore, the disintegrated measures can be glued, finding that $\exp_{\sharp}(\hat{\mathfrak{m}}_{\alpha \perp B})$ is absolutely continuous, w.r.t. the Lebesgue measure, with density a_{α} satisfying the differential inequality (3.13). The conclusion now follows from Theorem 6.8. $\square$

4. EQUIVALENCE RELATIONS FOR MEASURES AND GAUGES

In the smooth setting, the measure $\mathfrak{m}_L$ and the gauge $G_{L, S}$ depend on the choice of the vector field L . Therefore, a priori Definition 3.14 might depend on these choices as well. One of the goals of our previous work [11] was to show the independence of the NC^e condition from such a choice. The goal of this section is to show such independence also in this more general setting.

In this section, we consider a given topological causal space $(X, \ll, \leq, \mathfrak{T})$ and all the synthetic null hypersurfaces will be subsets of X .

Over H° we can define two equivalence relations, one for measures and one for gauges. Firstly we recall the concept of transverse function that in the smooth setting can be given by asking f to be constant along the generators of the null hypersurface. Since this characterization can be stated without referring to the smooth structure, it is natural to consider it also in our framework.

Definition 4.1 (Transverse function). Let H be a closed achronal set. A function $f : H^\circ \rightarrow \mathbb{R}$ is transverse if, for any causal curve γ in H° , $f \circ \gamma$ is constant.

In the smooth setting, if $L_1, L_2 \in \mathfrak{X}(\mathcal{N}H)$ are two null geodesic vector fields, then $L_1 = \varphi L_2$ for some transverse function φ and the relation between the two rigged measures is $\text{Vol}_{L_1} = \frac{1}{\varphi} \text{Vol}_{L_2}$. Accordingly, we identify measures which are equivalent up to multiplication by a transverse function.

Definition 4.2 (Equivalence relation for measures). Let H be a closed achronal set and $\mathfrak{m}_1, \mathfrak{m}_2 \in \mathcal{M}^+(H)$ be two measures. We set $\mathfrak{m}_1 \sim \mathfrak{m}_2$ if and only if

$$\mathfrak{m}_1 \llcorner_{H \setminus H^\circ} = \mathfrak{m}_2 \llcorner_{H \setminus H^\circ} \quad \text{and} \quad \mathfrak{m}_1 \llcorner_{H^\circ} = f \mathfrak{m}_2 \llcorner_{H^\circ},$$

for some strictly positive transverse function f .

Continuing the analogy with the smooth setting, the definition of $G_{L,S}$ is depending both on L and S but the only relevant aspect of the theory is the set G -causal curves. We therefore introduce an equivalence relation also between gauges.

Definition 4.3 (Equivalence relation for gauges). Let H be a closed achronal set and let $G_1, G_2 : H^\circ \rightarrow \mathbb{R}$ be two gauges. We set $G_1 \sim G_2$ if and only if $\text{Caus}_{G_1} = \text{Caus}_{G_2}$ (see Definition 3.7).

Next we prove that the $\text{NC}^e(N)$ condition is well defined for equivalence classes of gauges and reference measures, i.e., the $\text{NC}^e(N)$ condition does not depend on the choice of the measure and the gauge inside the respective equivalence classes.

Proposition 4.4. *Let H be a closed achronal set, let $\nu \in \text{OptCaus}^H(\mu_0, \mu_1)$ for a pair of probability measures $\mu_0, \mu_1 \in \mathcal{P}(H)$ which are concentrated on H° .*

Set $\mu_t := (e_t)_\# \nu$. Then, for every transverse function $f : H^\circ \rightarrow \mathbb{R}$, it holds that

$$(4.1) \quad \int_H f \, d\mu_t = \int_H f \, d\mu_s, \quad \text{for all } t, s \in [0, 1].$$

If moreover f is non-negative and the integrals above are equal to 1, defining $\tilde{\mu}_t := f\mu_t$, then $\tilde{\nu} := (f \circ e_0)\nu \in \text{OptCaus}^H(\tilde{\mu}_0, \tilde{\mu}_1)$ and $\tilde{\mu}_t = (e_t)_\# \tilde{\nu}$.

Proof. Since f is transverse, then $f \circ \gamma$ is constant for all $\gamma \in \text{Caus}_H$, such that $\gamma_{[0,1]} \subset H^\circ$, therefore, since $\nu(\text{Caus}_H) = 1$, we can compute

$$\int_H f \, d\mu_t - \int_H f \, d\mu_s = \int_{C([0,1];H)} (f \circ e_t - f \circ e_s) \, d\nu = 0.$$

For the second part, a direct computation gives

$$\tilde{\mu}_t = f\mu_t = f(e_t)_\#(\nu) = (e_t)_\#(f \circ e_t\nu) = (e_t)_\#(f \circ e_0\nu) = (e_t)_\#(\tilde{\nu}). \quad \square$$

Proposition 4.5. *Let H be a closed achronal set, $\nu \in \text{OptCaus}^H(\mu_0, \mu_1)$ for a pair of probability measures $\mu_0, \mu_1 \in \mathcal{P}(H)$ concentrated on H° and let $\mathbf{m}_1, \mathbf{m}_2 \in \mathcal{M}^+(H)$ such that $\mathbf{m}_1 \sim \mathbf{m}_2$.*

Then the function $t \mapsto \text{Ent}(\mu_t|\mathbf{m}_1) - \text{Ent}(\mu_t|\mathbf{m}_2)$ is constant.

Proof. Let f be any transverse function such that $\mathbf{m}_1 = f\mathbf{m}_2$ and $\rho_{k,t}$ be the density of μ_t w.r.t. the measure $\mathbf{m}_k$, $k = 1, 2$. Since $\rho_{2,t} = \rho_{1,t}f$, a direct computation gives

$$\begin{aligned} \text{Ent}(\mu_t|\mathbf{m}_2) &= \int_H \log(\rho_{2,t}) \, d\mu_t = \int_H \log(\rho_{1,t}) \, d\mu_t + \int_H \log f \, d\mu_t \\ &= \text{Ent}(\mu_t|\mathbf{m}_1) + \int_H \log f \, d\mu_t \stackrel{(4.1)}{=} \text{Ent}(\mu_t|\mathbf{m}_1) + \int_H \log f \, d\mu_0. \end{aligned} \quad \square$$

The next result follows straightforwardly from the propositions above.

Theorem 4.6. *Let $N > 0$ and $(X, \ll, \leq, \mathfrak{T})$ be a topological causal space. Consider a closed achronal set H , two measures $\mathbf{m}_1, \mathbf{m}_2 \in \mathcal{M}^+(H)$ and two gauges G_1, G_2 forming two locally compact synthetic null hypersurfaces $(H, G_1, \mathbf{m}_1)$ and $(H, G_2, \mathbf{m}_2)$.*

If $\mathbf{m}_1 \sim \mathbf{m}_2$ and $G_1 \sim G_2$, then $(H, G_1, \mathbf{m}_1)$ satisfies the $\text{NC}^e(N)$ condition if and only if $(H, G_2, \mathbf{m}_2)$ does.

Proof. Assume $(H, G_1, \mathbf{m}_1)$ to be $\text{NC}^e(N)$. By Proposition 3.16, $\mathbf{m}_1(H \setminus H^\circ) = 0$, thus $\mathbf{m}_2(H \setminus H^\circ) = 0$. This means that the only relevant probability measures for the $\text{NC}^e(N)$ condition are the ones concentrated on H° . Then Proposition 4.5 implies that also $(H, G_2, \mathbf{m}_2)$ satisfies $\text{NC}^e(N)$. $\square$

5. THE GEOMETRY OF SYNTHETIC NULL HYPERSURFACES

This section investigates more in details the geometry of synthetic null hypersurfaces, starting with relations between gauges and causal curves.

In this section we will often assume the closed achronal set to be weakly convex; this assumption is justified by Remark 3.15.

Proposition 5.1. *Let H be a closed achronal set, G be a gauge and $\gamma : [0, 1] \rightarrow H$ be a causal curve. Then there exists a reparametrization of γ which is G -causal.*

Proof. By Remark 3.5 we can assume without loss of generality that $[0, 1] \setminus \gamma^{-1}(H^\circ) \subset \{0, 1\}$. Also assuming $\gamma|_{(0,1)}$ injective is not restrictive. Define $l(t) := G(\gamma_t)$ and notice that by construction l is strictly increasing. Let $(a, b) := l((0, 1))$ and $m : (a, b) \rightarrow [0, 1]$ be the inverse of $l|_{(0,1)}$. Define $\eta := \gamma \circ m : (a, b) \rightarrow H$. This function is continuous. At this point extend η by declaring that $\eta_a := \gamma_0$ and $\eta_b = \gamma_1$. To check that this function satisfies the desired properties is trivial. $\square$

Example 5.2. Consider the Minkowski space $\mathbb{R}^2$ (t is the time variable, x is the space variable). Let H be the future light cone. Let $G : H \rightarrow \mathbb{R}$ given by $G(x, t) = \log t$. This function fails hypothesis (2) in the definition of gauge.

Let $\gamma : [0, 1] \rightarrow H$ be given by $\gamma_r = (r, r)$. There is no reparametrization of γ such that its post-composition with G is affine.

It is then immediate to obtain the following

Corollary 5.3. *Let H be a weakly convex closed achronal set, G be a gauge and $x, y \in H$. If $x \leq y$, $x \neq y$, then there exists a G -causal curve connecting x to y .*

Proposition 5.4. *Let H be a weakly convex closed achronal set and G be a gauge for H . Let $x, y \in H^\circ$ be such that $G(x) = G(y)$.*

Then either $x = y$ or x and y are not causally related.

Proof. Assume that $x \neq y$ and, without loss of generality, that $x \leq y$. Then by weakly convexity of H there exists γ , a causal curve connecting x to y , that, up to reparametrization, we can assume to be injective. Then $G \circ \gamma$ is strictly increasing giving a contradiction. $\square$

Proposition 5.5. *Let H be a weakly convex closed achronal set admitting a gauge G . Then H is causal, that is $\leq$ is a partial order relation over H (or, equivalently, H does not contain causal loops).*

Proof. Assume by contradiction the existence of a non-constant causal curve $\gamma : [0, 1] \rightarrow H$ with $\gamma_0 = \gamma_1$. Without loss of generality $\gamma_0 \neq \gamma_{1/2}$ and $\gamma_{1/2} \neq \gamma_1$. By definition of end points, $\gamma([0, 1]) \subset H^\circ$ and by Proposition 5.1 γ can be assumed to be G -causal.

Since $\gamma_0 = \gamma_1$ and $G \circ \gamma$ is monotone, then $G \circ \gamma$ is constant. Thus γ itself is constant by Proposition 5.4, giving a contradiction. $\square$

5.1. Null non-branching and first properties. Assuming that branching phenomena do not occur within H permits to deduce much more on the structure of the achronal set. We start giving the definition of null non-branching.

Definition 5.6. Let $H \subset X$ be a closed achronal set. We say that H is *forward null non-branching* if the following holds. For every pair of injective causal curves $\gamma^1, \gamma^2 : [0, 1] \rightarrow H$, such that $\gamma_0^1 = \gamma_0^2$ and $\gamma_{1/2}^1 = \gamma_{1/2}^2$, then either $\gamma_{[0,1]}^1 \subset \gamma_{[0,1]}^2$ or $\gamma_{[0,1]}^2 \subset \gamma_{[0,1]}^1$.

We say that H is *backward null non-branching* if H is forward null non-branching in the causally reversed space.

We say that H is *null non-branching* if it is both forward and backward null non-branching.

The next two propositions guarantees that the null non-branching condition is available for a broad class of Lorentzian manifolds.

Proposition 5.7. *Let (M, g) be a strongly causal Lipschitz Lorentzian manifold. Let S be an achronal set and $H = \partial I^+(S)$ its achronal boundary. Assume H° to be $C_{loc}^{1,1}$. Then H is null non-branching.*

Proof. Since H° is $C^{1,1}$, there exists a Lipschitz-continuous, null vector field $L \in \mathfrak{X}(TH^\circ)$. First, we claim that if $\gamma : I \rightarrow H^\circ$ is a causal curve, then there exists a reparametrization $\eta : J \rightarrow H^\circ$ of γ , such that η is C^1 and $\dot{\eta} = L$. Indeed, by Lemma 2.7 we can assume γ to be Lipschitz-continuous. Then in the differentiability points, $\dot{\eta} \in TH^\circ$ and $\dot{\eta}$ is a future-directed causal vector. It thus holds that $\dot{\eta}_r$ is parallel to L_{η_r} . Therefore, we can define the L^∞ function $\varphi : I \rightarrow (0, \infty)$ by the property

$$\dot{\gamma}_r = \varphi(r) L_{\gamma_r}.$$

If we define

$$T(s) := \left(\int_0^s \varphi(r) dr \right)^{-1} \quad \text{and} \quad \eta_s := \gamma_{T(s)},$$

it is then immediate to see that η satisfies the desired property. Consider now a pair of injective causal curves $\gamma^1, \gamma^2 : [0, 1] \rightarrow U$, such that $\gamma_0^1 = \gamma_0^2$ and $\gamma_{1/2}^1 = \gamma_{1/2}^2$. By the first part, we know that there exist $\eta^i : J^i \rightarrow H^\circ$, which are reparametrizations of $\gamma^i|_{(0,1)}$, satisfying $\dot{\eta}^i = L$, for $i = 0, 1$. We can assume $0 \in J^i$ and $\eta_0^i = \gamma_{1/2}^i$. By continuity, they can be extended to the closure $\overline{J^i}$. Since these two curves satisfy the same ODE, then they coincide in the intersection $\overline{J^1} \cap \overline{J^2}$. The thesis follows. $\square$

Proposition 5.8. *Let (M, g) be a strongly causal C^2 Lorentzian manifold. Let $\gamma : I \rightarrow M$ be a null curve (i.e., a causal curve such that its points are not chronologically related).*

Then γ is the reparametrization of a geodesic.

Proof. By Lemma 2.7, we may assume γ to be locally-Lipschitz and that $0 \in I$. Since the statement is local, it suffices to show that γ can be reparametrized into a geodesic in a neighborhood of 0. Let $p = \gamma_0$. Let H be the light-cone in M , emanating from p , and let $\hat{H}$ be the light-cone in the Minkowski spacetime. Denote by $\hat{L}$ the radial vector field along $\hat{H}$. Since g is C^2 , the exponential is a map of class C^1 , and therefore a diffeomorphism between a neighborhood of 0 in $T_p M$ and a neighborhood of p . As

a consequence, $H \setminus \{p\} = \exp(\hat{H} \setminus \{0\})$ is a C^1 sub-manifold of M . It is standard fact that $L := \exp_* \hat{L}$ is a null geodesic vector field along H . We define the gauge $G(x) := |\exp^{-1}(x)|$, where $|\cdot|$ is the Euclidean norm in $T_p M$; the map G is of class C^1 . By strong causality, γ takes values in H , therefore $\dot{\gamma}_t \in TH$, for a.e. t . Since by assumption $\dot{\gamma}$ is causal, it follows that $\dot{\gamma}$ is parallel to L . Denoting $\hat{\gamma} := \exp^{-1} \circ \gamma$, we infer that $\hat{\gamma}$ is parallel to $\hat{L}$, a.e.. As a consequence there exists $\hat{\eta}$, a reparametrization of $\hat{\gamma}$, such that $\hat{\eta}_t = \hat{L}_{\hat{\eta}_t}$, for a.e. t . Since $\hat{L}$ is a geodesic vector field, then $\hat{\eta}$ is a geodesic in $T_p M$, therefore, $\hat{\eta}$ is a linear ray starting from 0. We conclude that $\eta := \exp \circ \hat{\eta}$ is a geodesic in (M, g) . $\square$

Corollary 5.9. *Let (M, g) be a strongly causal C^2 Lorentzian manifold.*

Then every closed achronal set is null non-branching.

Proof. Let $H \subset M$ be a closed achronal subset and fix γ^0, γ^1 two causal curves in H , such that $\gamma_s^0 = \gamma_s^1$, $s = 0, \frac{1}{2}$. By the previous proposition, we can assume that γ^i is a geodesic, $i = 0, 1$, with $\gamma_0^0 = \gamma_0^1$ and $\gamma_{s_0}^0 = \gamma_{s_1}^1$, with $s_0, s_1 \in (0, 1)$. The vectors $\dot{\gamma}_{s_0}^0$ and $\dot{\gamma}_{s_1}^1$ are parallel; if not, then $\gamma_{s_0}^0$ is a focal point for γ_0^0 , thus γ_0 stops to be a null curve after s_0 , a contradiction. We deduce that γ^0 and γ^1 are part of the same maximal geodesic and coincide at 0. The thesis follows by the Cauchy–Lipschitz theorem, ensuring (existence and) uniqueness of solutions of the geodesic equation in a Lorentian metric g of class C^2 . $\square$

We next draw some consequences from the null non-branching condition.

Proposition 5.10. *Let H be a weakly convex, null non-branching, closed achronal set, G a gauge for H , and $x, y \in H$, with $x \leq y$ and $x \neq y$. Assume that either $x \in H^\circ$ or $y \in H^\circ$.*

Then there exists a unique G -causal curve connecting x to y .

Proof. The existence is given by Proposition 5.1.

We assume $x \in H^\circ$, the other case being symmetric. Let γ and η be two G -causal curves from x to y . Let $z \in J^-(x) \cap H$ with $z \neq x$, and let δ be a curve connecting z to x .

Consider first the case $G \circ \gamma = G \circ \eta$. If by contradiction $\gamma_t \neq \eta_t$, thus, $\gamma([0, 1]) \not\subset \eta([0, 1])$, nor $\eta([0, 1]) \subset \gamma([0, 1])$. By concatenating δ with γ and δ with η , we obtain a contradiction with the non-branching hypothesis.

Consider now the case $G \circ \gamma \neq G \circ \eta$. In this case, by affinity, since $\gamma_0 = \eta_0 = x \in H^\circ$, then $G \circ \gamma$ and $G \circ \eta$ have different derivatives. If $G \circ \eta$ is steeper, take $\tilde{\eta}$ as a restriction and affine reparametrization of η . By following the previous part we deduce $\gamma = \tilde{\eta}$, which is a contradiction. $\square$

Remark 5.11. Proposition 5.10 states that there exists a map

$$\Gamma : H \times H^\circ \cup H^\circ \times H \rightarrow C_G, \quad \text{such that } \Gamma(x_0, x_1)_i = x_i, \quad i = 0, 1.$$

Note that Γ is the right inverse of (e_0, e_1) , therefore the graph of Γ is closed. It follows that Γ is Borel.

Proposition 5.12. *Let H be a null non-branching weakly convex closed achronal set and let G be a gauge. Assume that γ is a non-constant curve in H , such that $\gamma_s \leq \gamma_t, \forall s \leq t$,*

continuous at the end-points, such that $\gamma_{(0,1)} \subset H^\circ$ and $G \circ \gamma|_{(0,1)}$ is affine and non-decreasing.

Then γ is continuous and therefore a causal curve.

Proof. Fix $\varepsilon > 0$. Let $\eta^\varepsilon : [\varepsilon, 1 - \varepsilon] \rightarrow H$ be the G -causal curve given by Proposition 5.10 connecting γ_ε to $\gamma_{1-\varepsilon}$. Fix $t \in (\varepsilon, 1 - \varepsilon)$. Arguing as in the proof of Proposition 5.10, one can check that $\gamma_t = \eta_t^\varepsilon$. By taking the limit as $\varepsilon \rightarrow 0$, we obtain that any restriction of $\gamma|_{(0,1)}$ to a closed interval is causal. Since γ is continuous at the endpoints, we conclude. $\square$

We prove that the sets of initial and final points are negligible without the local compactness assumption (see Proposition 3.16) but assuming instead the non-branching condition.

Theorem 5.13. *Let $(H, G, \mathbf{m})$ be a null non-branching synthetic null hypersurface satisfying the Null Energy Condition $\text{NC}^e(N)$, for some $N > 0$. Then*

$$\mathbf{m}(\mathfrak{a}) = \mathbf{m}(\mathfrak{b}) = 0.$$

Proof. The first part of the proof repeats verbatim the one of Proposition 3.16. Hence, we will argue by contradiction that $\mathbf{m}(\mathfrak{a} \setminus \mathfrak{b}) > 0$ and deduce the existence of continuous map $T : K \rightarrow H$, with $K \subset \mathfrak{a} \setminus \mathfrak{b}$ of positive $\mathbf{m}$ -measure. Define the absolutely continuous measure $\mu_0 := \mathbf{m}_\perp K / \mathbf{m}(K) \in \mathcal{P}(H)$ and $\mu_1 := T_\# \mu_0$ so that $(\text{Id}, T)_\# \mu_0 \in \Pi_\leq(\mu_0, \mu_1)$.

Definition 3.14 gives then a measure $\nu \in \text{OptGeo}^G(\mu_0, \mu_1)$ such that the function $t \mapsto \text{U}_N((e_t)_\# \nu)$ is concave. By Remark 3.15 $(e_t)_\# \nu \ll \mathbf{m}$. However, since μ_0 is concentrated on the set of initial points $\mathfrak{a}$, necessarily $(e_t)_\# \nu \perp (e_s)_\# \nu$ for all $s \neq t$, giving a contradiction with the σ -finiteness of $\mathbf{m}$: indeed, suppose by contradiction that there exist two causal curves γ^1, γ^2 , such that $\gamma_t^1 = \gamma_s^2$, with $1 > t > s$ but $\gamma_0^1, \gamma_0^2 \in K$. Then by the null non-branching assumption (γ_t^1 is not a final point), necessarily $\gamma_{(0,1)}^2 \subset \gamma_{(0,1)}^1$. Hence, γ_0^2 cannot be an initial point, giving a contradiction. $\square$

Remark 5.14. As a consequence of Theorem 5.13, we can prove that, under the null non-branching assumption, the $\text{NC}^e(N)$ condition is independent of the representative in the equivalence class of gauge and measure. That is, one can prove the same conclusion of Theorem 4.6, under the null non-branching assumption, in place of local compactness of the gauge.

5.2. Ray decomposition of achronal sets and disintegration. We fix the following definition: the set $\mathcal{R} := (J \cup J^{-1}) \cap (H \times H)$ will be called the *transport relation*.

Proposition 5.15. *Let H be a null non-branching, weakly convex closed achronal set. Then $\mathcal{R}$ is an equivalence relation on H° .*

Proof. The symmetric and reflexive properties are trivially verified. We are therefore only left with checking the transitive property.

Suppose then $(x, y), (y, z) \in \mathcal{R}$. Without any loss of generality we can assume that all these points are different and that $x \leq y$ and $z \leq y$ (the other cases are either trivial or equivalent to this one). Assume on the contrary that neither $x \leq z$, nor $z \leq x$. Let $p \in J^+(y) \cap H$, such that $p \neq y$. Let γ and δ be two curves connecting x and z (respectively) to p , passing through y . Up to reparametrizing we can assume that $\gamma_{1/2} = \delta_{1/2} = y$, a contradiction with the backward non-branching-ness. $\square$

If $\alpha \in H^\circ$, we will denote by H_α° the equivalence class containing α , also called ray:

$$(5.1) \quad H_\alpha^\circ := \{x \in H^\circ : (x, \alpha) \in \mathcal{R}\}.$$

We will denote by H_α the ray with end-points:

$$(5.2) \quad H_\alpha := \{x \in H : (x, \alpha) \in \mathcal{R}\}.$$

The set H_α is closed, because $\mathcal{R}$ is closed. We will denote by $\overline{H_\alpha^\circ}$ the topological closure of H_α° in H . By definition of closure $\overline{H_\alpha^\circ} \subset H_\alpha$.

Remark 5.16. As a consequence of non-branching, each of the sets $H_\alpha \cap \mathfrak{a}$ and $H_\alpha \cap \mathfrak{b}$ contains at most one element. Therefore, $\overline{H_\alpha^\circ} = H_\alpha$. Indeed, by contradiction assume $\{x\} = H_\alpha \cap \mathfrak{a} \setminus \overline{H_\alpha^\circ}$. Since $x \leq \alpha$, by weakly convexity of H , there exists a G -causal curve γ connecting x to α . Since $\gamma_t \in H_\alpha^\circ$, $t > 0$, then $x = \gamma_0$ can be approximated by points in H_α° .

Since the causal relation $\mathcal{R}$ is induced by $\leq$, we obtain the following consequence of Proposition 5.15.

Proposition 5.17. *Let H be a null non-branching, weakly convex closed achronal set, admitting a gauge G . Then each equivalence class with end-points of $\mathcal{R}$ inside H is contained in the image of a causal curve.*

Proof. Fix $\alpha \in H^\circ$. Consider the set

$$(5.3) \quad \zeta := \{(s, z) \in \mathbb{R} \times H_\alpha^\circ : G(z) = s + G(\alpha)\}$$

Let $I := P_1(\zeta) \subset \mathbb{R}$. We want to see that the set ζ is the graph of a G -causal curve covering all H_α° . By Proposition 5.4, for any s there exists at most one $z \in H_\alpha^\circ$, such that $G(z) = s$. We need to see that I is an interval, i.e., that I is connected. Given $s, t \in I$, there exist $x, y \in H_\alpha^\circ$, such that $(s, x), (t, y) \in \zeta$. Thus either $x \leq y$ or $y \leq x$. Take γ a causal curve connecting x to y , contained in H_α° . By continuity of $G \circ \gamma$, I contains all the elements between s and t , hence it is an interval. Therefore, ζ is the graph of a function $\eta : I \rightarrow H_\alpha^\circ$ and η satisfies $G(\eta_t) = G(\alpha) + t$.

It holds that $\eta_s \leq \eta_t$, for all $t \leq s$, therefore we can apply Proposition 5.12 and deduce that η is G -causal. Finally, for any point $w \in H_\alpha^\circ$, $(G(w) - G(\alpha), w) \in \zeta$, therefore $\eta_I = H_\alpha^\circ$.

We now extend η in order to cover the endpoints. If $\mathfrak{a} \cap H_\alpha$ is empty there is nothing to do, thus we consider $x \in \mathfrak{a} \cap H_\alpha$. Let $\delta : [0, 1] \rightarrow H$ be the G -causal curve connecting x to α . Let $r := \inf_{t>0} G(\delta_t)$. We claim that $\inf I = r$. If $\inf I < r$, then one can take the curve starting from $\eta_{r-\varepsilon}$, for $r - \varepsilon \in I$, passing through $\delta_{1/2}$, and reaching δ_1 , obtaining a contradiction; the other inequality is trivial. We can therefore define $\eta_r := x$. Both $G \circ \eta$ and $G \circ \delta$ are affine and increasing functions, thus $T := (G \circ \delta)^{-1} \circ (G \circ \eta)$ is affine and increasing. Since $G(\eta_t) = G(\delta_{T(t)})$, for $\eta > r$, by definition of ζ and η , we deduce $\eta_t = \delta_{T(t)}$, $t > r$. As a consequence $\lim_{t \rightarrow r+} \eta_t = \lim_{s \rightarrow 0+} \delta_s = x$, proving that η is continuous in r , and therefore causal. $\square$

Notice that in Proposition 5.17 we have constructed a parametrization of each equivalence class with G -speed equal to one. We define the map $\Psi_G : \text{Dom}(\Psi_G) \subset H^\circ \times \mathbb{R} \rightarrow H^\circ$, as the map whose graph is given by

$$(5.4) \quad \text{Graph}(\Psi_G) = \{(x, t, y) : G(y) - G(x) = t\}.$$

Following Proposition 5.17, we can extend Ψ_G to cover also the possible end-points.

The next proposition characterizes the equivalence of gauges as an affine transformation with transverse coefficients.

Proposition 5.18. *Let H be a null non-branching, weakly convex, closed achronal set, admitting two gauges G_1 and G_2 . Then $G_1 \sim G_2$, if and only if $G_1 = f + hG_2$, for some transverse functions f, h , with $h > 0$.*

Proof. The “if” part is trivial, so we consider the other implication. Fix $G_1 \sim G_2$ two equivalent gauges, and let

$$h(z) := (G_1 \circ \Psi_{G_2}(z, \cdot))'(0).$$

Since $\Psi_{G_2}(z, \cdot)$ is G_2 -causal, hence G_1 -causal, therefore $G_1 \circ \Psi_{G_2}(z, \cdot)$ is affine; it follows that the derivative in the equation above exists and h is transverse. Since

$$G_2(\Psi_{G_1}(z, h(z)t)) = G_2(z) + t,$$

then $\Psi_{G_1}(z, h(z)t) = \Psi_{G_2}(z, t)$. We can thus compute

$$\begin{aligned} ((G_1 - hG_2) \circ \Psi_{G_1}(z, t))' &= 1 - h(z)(G_2 \circ \Psi_{G_1}(z, t))' \\ &= 1 - h(z)\left(G_2 \circ \Psi_{G_2}\left(z, \frac{t}{h(z)}\right)\right)' = 0, \end{aligned}$$

therefore $G_1 - hG_2 =: f$ is transverse. $\square$

The next proposition guarantees the existence of a $\mathcal{A}$ -measurable cross-section for the equivalence relation $\mathcal{R}$. Here $\mathcal{A}$ is the σ -algebra generated by the set of analytic sets, i.e., projections of Borel sets inside a Polish space.

Recall also that we use the notation $x < y$ to denote $x \leq y$, $x \neq y$.

Proposition 5.19 (Measurable selection). *Let H be a null non-branching, causal, weakly convex closed achronal set. Then there exists a $\mathcal{A}$ -measurable quotient map $\Omega : H^\circ \rightarrow H^\circ$ for the equivalence relation $\mathcal{R}$, i.e., for all $x \in H$, $(x, \Omega(x)) \in \mathcal{R}$ and*

$$\Omega(x) = \Omega(y) \iff (x, y) \in \mathcal{R}(x).$$

Proof. We construct a cross-section $Q \subset H^\circ$, as follows. Let $(U_n)_n$ be a countable base for the topology of X , such that U_n are precompact. Consider the compact sets $C_n := \overline{U_n \cap H^\circ} \subset H$. Define

$$(5.5) \quad A_n := P_1((J^{-1} \setminus \Delta) \cap (C_n \times C_n)) = \{x \in C_n : \exists y \in C_n, x < y\},$$

The sets A_n are σ -compact, as they are projections of intersections of σ -compact sets (Δ is the diagonal, whose complementary is open, thus σ -compact, by the properness of the topology). Define

$$(5.6) \quad L_n := C_n \cap H^\circ \setminus A_n = \{x \in C_n \cap H^\circ : \nexists y \in C_n, x < y\},$$

which is a Borel set. Define

$$B_n := P_1(\mathcal{R} \cap (H^\circ \times L_n)) \cap H^\circ = \{x \in H^\circ : \text{the set } H_x^\circ \cap L_n \text{ is non-empty}\},$$

which is an analytic set. Finally, define

$$(5.7) \quad E_n := B_n \setminus \left(\bigcup_{i=1}^{n-1} B_i \right), \quad Q_n := L_n \cap E_n,$$

which belong to the σ -algebra generated by analytic sets. Define

$$(5.8) \quad Q := \bigcup_{n=1}^{\infty} Q_n \subset V.$$

Next, we show that the set Q is indeed a cross-sections.

Step 1. For all $x \in H^\circ$, $n \in \mathbb{N}$, $L_n \cap H_x^\circ$ contains at most one element. Fix $x \in H^\circ \cap \mathcal{R}(U)$. Assume on the contrary that there exists $v, w \in L_n \cap H_x^\circ$, $w \neq v$. Then either $v \leq w$ or $w \leq v$. If $v \leq w$, then $v \notin L_n$, because (5.6) would be contradicted by $y = w$. Analogously, $w \leq v$ leads to a contradiction.

Step 2. For all $x \in H^\circ$, there exists $n \in \mathbb{N}$, such that $L_n \cap H_x^\circ$ contain one element.

Fix $x \in H^\circ$. Then there exist two points $y, z \in H_x^\circ$, such that $y < x < z$. Since H is causal, $x \notin J^-(y) \cup J^+(z)$, therefore there exist n and W , such that U_n is a neighborhood of x , and W a neighborhood of $J^-(y) \cup J^+(z)$, such that $W \cap U_n = \emptyset$. Thus, $C_n \cap W = \emptyset$. Let $\gamma : [0, 1] \rightarrow H^\circ$ be a causal curve connecting x to z . Let $\bar{t} := \sup\{t : \gamma_t \in C_n\} < 1$ and let $v = \gamma_{\bar{t}} \in C_n \cap H_x^\circ$.

We claim that $v \in L_n$; if not, there exists $w \in C_n$, such that $v < w$, thus $w \in H_x^\circ$. If $w \in J^+(z) \subset W$, then we obtain a contradiction with $C_n \cap W = \emptyset$, thus $w \leq z$; by forward non-branching, there exists $s \in (\bar{t}, 1)$, such that $w = \gamma_s$, a contradiction with the definition of $\bar{t}$.

We notice that $v \neq x$, otherwise $x \in \partial U_n$, a contradiction with the fact that U_n is neighborhood of x .

An immediate consequence of the last step is that for all $x \in H^\circ$ there exists n , such that $x \in B_n$. Therefore, it holds that $Q_n \cap H_x^\circ = L_n \cap H_x^\circ$, for the smallest n , such that $x \in B_n$. As a consequence, for all x , $Q \cap H_x^\circ$ is a singleton.

Step 3. Construction and measurability of the quotient map $\mathfrak{Q}$.

Define the map $\mathfrak{Q}$ as $\mathfrak{Q}(x) = \alpha$, if $(x, \alpha) \in \mathcal{R}$ and $\alpha \in Q$. In other words, the graph of the map $\mathfrak{Q}$ is given by

$$(5.9) \quad \text{Graph } \mathfrak{Q} = \mathcal{R} \cap (H^\circ \times Q).$$

We next show that the map $\mathfrak{Q}$ is measurable. We adopt the convention that $\times$ is computed before $\cap$. Fix an open set $W \subset X$ and compute

$$\begin{aligned} \mathfrak{Q}^{-1}(W) &= P_1(\mathcal{R} \cap H^\circ \times (Q \cap W)) \stackrel{(5.8)}{=} P_1\left(\mathcal{R} \cap H^\circ \times \bigcup_{n=1}^{\infty} (L_n \cap W \cap E_n)\right) \\ &= \bigcup_{n=1}^{\infty} P_1(\mathcal{R} \cap H^\circ \times (L_n \cap W \cap E_n)) \\ &\stackrel{(5.10)}{=} \bigcup_{n=1}^{\infty} P_1(\mathcal{R} \cap H^\circ \times (L_n \cap W) \cap E_n \times E_n) \\ &= \bigcup_{n=1}^{\infty} P_1(\mathcal{R} \cap H^\circ \times (L_n \cap W)) \cap E_n, \end{aligned}$$

which belongs to the σ -algebra generated by analytic sets; we have used the fact

$$(5.10) \quad \mathcal{R} \cap H^\circ \times (E_n \cap Y) = \mathcal{R} \cap E_n \times E_n \cap H^\circ \times Y, \quad \forall Y \subset X. \quad \square$$

Once a measurable quotient map is available, one can invoke Disintegration theorem and obtain a strongly consistent disintegration formula for any element of the class $[\mathfrak{m}]$ with respect to $\mathcal{R}$. We refer to [14, Section 4.2] for this implication in the synthetic Lorentzian setting; for metric spaces we point to the references the cited in [14, Section 4.2] (see in particular [13, Th. 3.4]). We only recall that we will denote by Q the quotient set associated to $\mathfrak{Q}$, i.e. $Q = \mathfrak{Q}(H)$.

Theorem 5.20. *Let $(H, G, \mathfrak{m})$ be a weakly convex, null non-branching synthetic null hypersurface. Then the following disintegration formula holds:*

$$\mathfrak{m}_{\perp H^\circ} = \int_Q \mathfrak{m}_\alpha \mathfrak{q}(\mathrm{d}\alpha),$$

where $\mathfrak{q}$ is a Borel probability measure over $Q \subset H^\circ$ such that $\mathfrak{Q}_\# \mathfrak{m}_{\perp H^\circ} \ll \mathfrak{q}$ and the map $Q \ni \alpha \mapsto \mathfrak{m}_\alpha \in \mathcal{M}_+(H^\circ)$ satisfies the following properties:

- (1) for any $\mathfrak{m}$ -measurable set B , the map $\alpha \mapsto \mathfrak{m}_\alpha(B)$ is $\mathfrak{q}$ -measurable;
- (2) for $\mathfrak{q}$ -a.e. $\alpha \in Q$, $\mathfrak{m}_\alpha$ is concentrated on $\mathfrak{Q}^{-1}(\alpha) = H_\alpha$ (strong consistency);
- (3) for any $\mathfrak{m}$ -measurable set $B \subset H^\circ$ and $\mathfrak{q}$ -measurable set C , the following disintegration formula holds:

$$\mathfrak{m}(B \cap \mathfrak{Q}^{-1}(C)) = \int_C \mathfrak{m}_\alpha(B) \mathfrak{q}(\mathrm{d}\alpha);$$

- (4) For every compact subset $\mathcal{K} \subset X$ there exists a constant $C_{\mathcal{K}} \in (0, \infty)$ such that

$$\mathfrak{m}_\alpha(\mathcal{K}) \leq C_{\mathcal{K}}, \quad \text{for } \mathfrak{q}\text{-a.e. } \alpha \in Q.$$

Moreover, fixed any $\mathfrak{q}$ as above such that $\mathfrak{Q}_\# \mathfrak{m} \ll \mathfrak{q}$, the disintegration is $\mathfrak{q}$ -essentially unique.

A few comments are in order. If $\mathfrak{m}_1 \sim \mathfrak{m}_2$ and $\mathfrak{q} \in \mathcal{P}(Q)$ is such that $\mathfrak{q} \ll \mathfrak{Q}_\# \mathfrak{m}_{1 \perp H^\circ}$, then also $\mathfrak{q} \ll \mathfrak{Q}_\# \mathfrak{m}_{2 \perp H^\circ}$. Moreover, if $\mathfrak{m}_{2 \perp H^\circ} = f \mathfrak{m}_{1 \perp H^\circ}$ for some transverse function, then $\mathfrak{m}_{2, \alpha} = f(\alpha) \mathfrak{m}_{1, \alpha}$, for $\mathfrak{q}$ -a.e. $\alpha \in Q$.

6. LIGHT-LIKE OPTIMAL TRANSPORT AND LOCALIZATION

In this section, we will establish existence and uniqueness results for the optimal transport problem along a synthetic null hypersurface H , and use these to localize the synthetic Ricci curvature lower bounds to the measures $\mathbf{m}_\alpha$ concentrated on the null generators H_α of H , as in the disintegration Theorem 5.20.

6.1. Existence and uniqueness of a monotone light-like Optimal Transport.

Definition 6.1 (Monotone set). Given $(Y, \leq)$ a partially ordered set, $A \subset Y \times Y$ is said to be *monotone* if

$$x_1 \leq x_2, x_1 \neq x_2 \implies y_1 \leq y_2, \quad \text{for all } (x_1, y_1), (x_2, y_2) \in A.$$

Definition 6.2 (Monotone plans). Given a topological causal space $(X, \ll, \leq, \mathfrak{T})$ and two probability measures $\mu_0, \mu_1 \in \mathcal{P}(X)$, a coupling $\pi \in \Pi_\leq(\mu_0, \mu_1)$ is called *monotone* if it is concentrated on a Borel set $A \subset X \times X$ that is monotone in the sense of Definition 6.1 in $(X, \leq)$.

Lemma 6.3. *Let H be a null non-branching, weakly convex, closed achronal set admitting a gauge function G . Let $A \subset J \cap (H^\circ \times H^\circ)$. The following are equivalent.*

- (1) *The set A is monotone in $(H, \leq)$.*
- (2) *for all $\alpha \in Q$, there exists a monotone set $B_\alpha \subset \mathbb{R} \times \mathbb{R}$, such that*

$$(6.1) \quad A \cap (H_\alpha^\circ \times H_\alpha^\circ) \subset \Psi_G(\alpha, \cdot) \otimes \Psi_G(\alpha, \cdot)(B_\alpha)$$

Proof. (1) $\implies$ (2). Take as $B_\alpha := (\Psi_G(\alpha, \cdot) \otimes \Psi_G(\alpha, \cdot))^{-1}(A)$. The fact that B_α is monotone is trivial.

(2) $\implies$ (1). Fix $(x_1, y_1), (x_2, y_2) \in A$. If $x_1 < x_2$, then all these four points belong to the same class H_α . Then $x_i = \Psi_G(\alpha, t_i)$ and $s_i = \Psi_G(\alpha, s_i)$, for some $(t_i, s_i) \in B_\alpha$, $i = 1, 2$. The assumption $x_1 \leq x_2$ gives $t_1 \leq t_2$ and the monotonicity of B_α gives $s_1 \leq s_2$, concluding the proof. $\square$

Proposition 6.4. *Let H be a non-branching, weakly convex, closed achronal set, admitting a gauge function G . Let $\mu_0, \mu_1 \in \mathcal{P}(H^\circ)$ such that $\Pi_\leq(\mu_0, \mu_1) \neq \emptyset$.*

Then there exists a unique monotone coupling $\pi \in \Pi_\leq(\mu_0, \mu_1)$.

Proof. Let $Q \subset H^\circ$ be a measurable cross-section of $\mathcal{R}$ and let $\mathfrak{Q} : H^\circ \rightarrow Q$ be the quotient map.

Step 1. Existence. Let $\bar{\pi} \in \Pi_\leq(\mu_0, \mu_1)$. Consider the map

$$(6.2) \quad \mathfrak{Q}^1 := \mathfrak{Q} \circ P_1 : H^\circ \times H^\circ \rightarrow Q,$$

where P_1 is the projection on the first variable. By definition of coupling, we get

$$(6.3) \quad \mathfrak{q} := \mathfrak{Q}_\#(\mu_1) = \mathfrak{Q}_\#^1(\bar{\pi}).$$

We are therefore in position to apply the disintegration theorem, deducing that

$$(6.4) \quad \bar{\pi} = \int_Q \bar{\pi}_\alpha \mathfrak{q}(\mathrm{d}\alpha),$$

where the probability measures $\bar{\pi}_\alpha$ are concentrated on the fibers of $\mathfrak{Q}^1$. Since $\pi(J) = 1$, then the measure $\bar{\pi}_\alpha$ are concentrated on

$$(6.5) \quad (\mathfrak{Q}^1)^{-1}(\alpha) \cap J = \{(x, y) \in H^\circ : (x, \alpha) \in \mathcal{R}\} \cap J \subset H_\alpha^\circ \times H_\alpha^\circ.$$

At this point we let

$$\tilde{\pi}_\alpha := (\Psi(\alpha, \cdot)^{-1} \otimes \Psi(\alpha, \cdot)^{-1})_\# \bar{\pi}_\alpha \in \mathcal{P}(\mathbb{R}^2) \quad \text{and} \quad \tilde{\mu}_i := (P_i)_\# \tilde{\pi}_\alpha \in \mathcal{P}(\mathbb{R}), \quad i = 0, 1.$$

Let $\tilde{\pi}_\alpha \in \mathcal{P}(\mathbb{R}^2)$ be the monotone rearrangement of $\tilde{\mu}_0$ into $\tilde{\mu}_1$, and let $B_\alpha := \text{supp } \tilde{\pi}_\alpha$. By [11, Lemma 6.4], we have that

$$B_\alpha \subset \{(x, y) \in \mathbb{R}^2 : x \leq y\}.$$

Define

$$(6.6) \quad \pi := \int_Q (\Psi_G(\alpha, \cdot) \otimes \Psi_G(\alpha, \cdot))_\# \tilde{\pi}_\alpha d\alpha.$$

Since $\Psi_G(\alpha, \cdot)$ is order-preserving and $\tilde{\pi}_\alpha$ is concentrated on $B_\alpha \subset \{(x, y) : y \geq x\}$, π is concentrated on J . Finally, it is a routine calculation to see that π is coupling between μ_0 and μ_1 .

Step 2. Uniqueness. Let π^1 and π^2 be two monotone transport plans. It holds that

$$(6.7) \quad \mathfrak{Q}_\#^1 \pi^1 = \mathfrak{Q}_\#^1 \pi^2 =: \mathfrak{q}.$$

We can therefore apply the disintegration theorem, finding that

$$(6.8) \quad \pi^i = \int_Q \pi_\alpha^i \mathfrak{q}(d\alpha).$$

Let

$$\tilde{\pi}_\alpha^k := ((\Psi_G(\alpha, \cdot))^{-1} \otimes (\Psi_G(\alpha, \cdot))^{-1})_\# \pi_\alpha^k \in \mathcal{P}(\mathbb{R}^2).$$

Since $\tilde{\pi}_\alpha^1$ and $\tilde{\pi}_\alpha^2$ have the same marginals, by the uniqueness of the monotone rearrangement, we deduce $\tilde{\pi}_\alpha^1 = \tilde{\pi}_\alpha^2$ giving $\pi^1 = \pi^2$. $\square$

Proposition 6.5. *Let H be a null non-branching, weakly convex, closed achronal set, admitting a gauge function G . Let $\mu_0, \mu_1 \in \mathcal{P}(H^\circ)$ be two probability measures, and assume there exists a causal coupling $\pi \in \Pi_{\leq}(\mu_0, \mu_1)$.*

Then there exists a unique G -causal dynamical transport plan $\nu \in \text{OptGeo}_G(\mu_0, \mu_1)$, such that $(e_0 \otimes e_1)_\# \nu = \pi$.

Proof. The plan π gives full measure to $B := J \cap (H^\circ \times H^\circ)$. Consider the map $\Gamma : B \rightarrow C_G$ given by Remark 5.11, such that, for each $(x_0, x_1) \in B$, $\Gamma((x_0, x_1))_i = x_i$, $i = 0, 1$. Then $\nu := \Gamma_\# \pi$, satisfies the claim. $\square$

Corollary 6.6. *Let H be a null non-branching, weakly convex, closed achronal set, admitting a gauge function G . Let $\mu_0, \mu_1 \in \mathcal{P}(H^\circ)$ be two probability measures, such that $\Pi_{\leq}(\mu_0, \mu_1) \neq \emptyset$.*

Then there exists $\nu \in \text{OptGeo}_G(\mu_0, \mu_1)$, a unique G -causal, dynamical optimal transport plan, such that $(e_0, e_1)_\# \nu$ is a monotone coupling.

Proof. Combine Proposition 6.4 with Proposition 6.5. $\square$

As the following example shows, the requirement that the marginal measures give no mass to the endpoints is necessary for the existence of a monotone coupling and it cannot be weakened by the requirement that only one measure is concentrated on H° .

Example 6.7. Consider the 2-dimensional Minkowski space $\mathbb{R}^2$ and let $H = \{(x, |x|)\}$. Consider the measures

$$\mu_0 = \frac{1}{2}(\delta_{(0,0)} + \delta_{(1,1)}), \quad \mu_1 = \frac{1}{2}(\delta_{(-2,2)} + \delta_{(2,2)}).$$

The only transport plan $\pi \in \Pi_{\leq}(\mu_0, \mu_1)$ is induced by a map T given by $T(0,0) = (-2,2)$ and $T(1,1) = (2,2)$. Notice that

$$\text{supp } \pi = \{((0,0), (-2,2)), ((1,1), (2,2))\}.$$

For this set, it holds that $(1,1) \in J^+(0,0)$, but $(2,2) \notin J^+(-2,2)$, thus the plan is not monotone.

6.2. Localization of the Null Energy Condition.

Theorem 6.8. *Let $(H, G, \mathbf{m})$ be a null-non-branching synthetic null hypersurface. Then the following are equivalent.*

- $(H, G, \mathbf{m})$ satisfies the Null Energy Condition $\text{NC}^e(N)$.
- Considering the disintegration given by Theorem 5.20, for $\mathbf{q}$ -a.e. $\alpha \in Q$, the metric measure space $(H_\alpha, |G(\cdot) - G(\cdot)|, \mathbf{m}_\alpha)$ satisfies the $\text{CD}(0, N-1)$ condition. That is, using the identification $\Psi_G(\alpha, \cdot)$ between H_α and a real interval $I_\alpha \subset \mathbb{R}$, the measure $\mathbf{m}_\alpha$ is absolutely continuous and, denoting by $h_\alpha : I_\alpha \rightarrow [0, \infty)$ its density, $t \mapsto h_\alpha^{1/(N-2)}(t)$ is concave.

Proof. First notice that by Remark 3.15, the $\text{NC}^e(N)$ condition implies that H is weakly convex. Hence Theorem 5.20 applies.

One can check that $(H_\alpha, |G(\cdot) - G(\cdot)|)$ is a metric space isometric to the metric space $(I_\alpha, |\cdot - \cdot|)$, via the map $\Psi_G(\alpha, \cdot)$. Therefore, it is sufficient to check that $(I_\alpha, |\cdot - \cdot|, \tilde{\mathbf{m}}_\alpha)$ satisfies the $\text{CD}(0, N-1)$ condition, where $\tilde{\mathbf{m}}_\alpha := ((\Psi_G(\alpha, \cdot))^{-1})_\# \mathbf{m}_\alpha$.

We first show that for $\mathbf{q}$ -a.e. α , the measure $\tilde{\mathbf{m}}_\alpha$ satisfies the $\text{MCP}(0, N-1)$ condition. Suppose the contrary. Then there exists $\varepsilon > 0$, such that the set

$$B := \{\alpha \in Q : \exists \mu_{0,\alpha} \in \mathcal{P}(\mathbb{R}), \exists r_\alpha \in \text{supp } \tilde{\mathbf{m}}_\alpha, \exists t_\alpha \in (0, 1) : \\ U_{N-1}(\mu_{t_\alpha, \alpha} | \tilde{\mathbf{m}}_\alpha) \leq (1 - t_\alpha) U_{N-1}(\mu_{0,\alpha} | \tilde{\mathbf{m}}_\alpha) - \varepsilon\}$$

has positive $\mathbf{q}$ -measure, where μ_{t_α} is the Wasserstein interpolation at time t_α between $\mu_{0,\alpha}$ and δ_{r_α} . Then the set

$$C = \left\{ (\alpha, t) \in Q \times [0, 1] : U_{N-1}(\mu_{t,\alpha} | \tilde{\mathbf{m}}_\alpha) \leq (1 - t) U_{N-1}(\mu_{0,\alpha} | \tilde{\mathbf{m}}_\alpha) - \frac{\varepsilon}{2} \right\}$$

has positive $\mathbf{q} \otimes \mathcal{L}^1$ -measure. Indeed, by upper semicontinuity of $t \mapsto U_{N-1}(\mu_{t,\alpha} | \tilde{\mathbf{m}}_\alpha)$, for any $\alpha \in B$, the slice $C(\alpha)$ is a non-empty open set. By Fubini's Theorem, there exists $\bar{t} \in (0, 1)$ such that the inequality in the definition of C is verified on a subset of B , still denoted by B , of positive $\mathbf{q}$ -measure. Integrating the said inequality at $\bar{t}$ over B contradicts the $\text{NC}^e(N)$ inequality between

$$\mu_0 := \frac{1}{\mathbf{q}(B)} \int_B \mu_{0,\alpha} \mathbf{q}(d\alpha) \quad \text{and} \quad \mu_1 := \frac{1}{\mathbf{q}(B)} \int_B \delta_{r_\alpha} \mathbf{q}(d\alpha).$$

Moreover $\text{OptGeo}^G(\mu_0, \mu_1)$ has only one element: indeed, since $\Pi_{\leq}(\mu_0, \mu_1)$ is a singleton, then Proposition 6.5 ensures that there exists a unique G -causal dynamical plan from μ_0 to μ_1 .

A consequence of the $\text{MCP}(0, N-1)$ condition is that the measures $\tilde{\mathbf{m}}_\alpha$ are $\mathcal{L}^1$ -absolutely continuous. Therefore arguing as in [11, Sec. 8.2], one can prove that $\mathbf{m}_\alpha$ satisfies the $\text{CD}(0, N-1)$ condition.

The proof of the converse implication follows the arguments of [11, Sec. 7]. $\square$

Remark 6.9. As a consequence of the proof, if $\mathbf{m}$ has full support in H then, for $\mathbf{q}$ -a.e. $\alpha \in Q$, $\mathbf{m}_\alpha$ has full support in H_α .

Remark 6.10 (A Brenier-like result). As a consequence of Theorem 6.8, one can prove a Brenier-like theorem; i.e, if $(H, G, \mathbf{m})$ is a null non-branching synthetic null hypersurface satisfying $\text{NC}^e(N)$, if $\mu_0 \ll \mathbf{m}$ and $\Pi_{\leq}(\mu_0, \mu_1) \neq \emptyset$, then the unique monotone coupling given by Proposition 6.4 is induced by a map. The proof follows standard arguments of L^1 -Optimal Transport (see, e.g., [10, Sec. 3.6.1]), once proved that the disintegration of $\mathbf{m}$ along the transport set satisfies the property stated in the second item of Theorem 6.8.

Remark 6.11. The second item in Theorem 6.8 shall be seen as a synthetic counterpart of the $\text{NC}^1(N)$ condition introduced in [11, Def. 5.13] in the smooth setting. Thus Theorem 6.8 shall be seen as a synthetic counterpart of the equivalence between the $\text{NC}^e(N)$ and $\text{NC}^1(N)$ conditions, proved in the smooth framework in [11, Sec. 8].

7. SYNTHETIC HAWKING'S AREA THEOREM

In this section, as an application of the theory developed above, we provide a new formulation of Hawking's Area Theorem, where the assumption that the Ricci curvature is non-negative on null vectors is replaced by the NC^e condition.

Firstly, we give a suitable definition for the area of a cross-section that is compatible with the smooth one. It will be convenient to consider also its local version; for this purpose, recall the notation

$$\mathcal{R}(A) = (J^-(A) \cup J^+(A)) \cap H,$$

for any subset $A \subset H$ of the null non-branching synthetic null hypersurface $(H, G, \mathbf{m})$.

Definition 7.1. Let $(H, G, \mathbf{m})$ be a null non-branching synthetic null hypersurface. Let $S \subset H$ be a set. We define the (future) ε -enlargement of S by:

$$(7.1) \quad S_\varepsilon^+ := \{x \in H^\circ : \Psi_G(x, -r) \in S, r \in (0, \varepsilon)\}.$$

The (future) Minkowski content of S is defined by

$$(7.2) \quad \mathbf{m}_G^+(S) := \limsup_{\varepsilon \rightarrow 0^+} \frac{\mathbf{m}(S_\varepsilon^+)}{\varepsilon}.$$

Let $A \subset H$. The relative (future) Minkowski content of S with respect to A is defined by

$$(7.3) \quad \mathbf{m}_G^+(S; A) := \limsup_{\varepsilon \rightarrow 0^+} \frac{\mathbf{m}(S_\varepsilon^+ \cap \mathcal{R}(A))}{\varepsilon}.$$

The definition below shall be read as a synthetic counterpart of the smooth notion of a null hypersurface being future geodesically complete.

Definition 7.2. Let H be a closed achronal set admitting a gauge function G . We say that H is *future complete for G* , if for all $z \in H^\circ$

$$(7.4) \quad \sup_{y \in J^+(z) \cap H^\circ} G(y) = \infty.$$

Notice that if H is null non-branching, then future completeness is equivalent to requiring that for all $z \in H^\circ$

$$(7.5) \quad \sup_{y \in H_z^\circ} G(y) = \infty;$$

or, equivalently, if for all $z \in H^\circ$ and all $t > 0$ it holds that $(z, t) \in \text{Dom}(\Psi_G)$.

We are now in position to state and prove Hawking's area theorem.

Theorem 7.3. *Let $(H, G, \mathbf{m})$ be a null non-branching, synthetic null hypersurface, satisfying the $\text{NC}^e(N)$ condition. Assume that H is future complete for G and that $\text{supp } \mathbf{m} = H$. Let $S_1, S_2 \subset H^\circ$ be two acausal sets with $S_1 \subset J^-(S_2)$. Then*

$$(7.6) \quad \mathbf{m}_G^+(S_1) \leq \mathbf{m}_G^+(S_2),$$

where the value $+\infty$ is admitted.

Proof. We use Theorem 5.20 and we adopt its notation. Since $H = \text{supp } \mathbf{m}$ and H is future complete for G , then the density h_α of $\mathbf{m}_\alpha$ is positive on a unbounded interval. Since by Theorem 6.8 $h_\alpha^{1/(N-2)}$ is concave, we infer that h_α is non-decreasing. The assumption $S_1 \subset J^-(S_2)$ implies that $\Omega((S_1)_\varepsilon^+) \subset \Omega((S_2)_\varepsilon^+)$. Using that $(S_i)_\varepsilon^+ \subset H^\circ$

$$\begin{aligned} \mathbf{m}((S_2)_\varepsilon^+) &= \int_{\Omega((S_2)_\varepsilon^+)} \mathbf{m}_\alpha((S_2)_\varepsilon^+) \mathbf{q}(d\alpha) \geq \int_{\Omega((S_1)_\varepsilon^+)} \mathbf{m}_\alpha((S_2)_\varepsilon^+) \mathbf{q}(d\alpha) \\ &\geq \int_{\Omega((S_1)_\varepsilon^+)} \mathbf{m}_\alpha((S_1)_\varepsilon^+) \mathbf{q}(d\alpha) = \mathbf{m}((S_1)_\varepsilon^+), \end{aligned}$$

proving the theorem. $\square$

Remark 7.4. Inspecting the proof of Theorem 7.3, one can check that the assumption (7.5) can be replaced by the following weaker (but slightly more technical) condition: for $\mathbf{q}$ -a.e. α , the ray H_α (see Theorem 6.8) is isometric to a real interval unbounded from above.

A priori, the Minkowski content of S could depend on the choice of $\mathbf{m}$ and G . In the next proposition we show that, under suitable covariance relations linking G with $\mathbf{m}$, the Minkowski content is independent from such choices.

Proposition 7.5. *Let $(X, \ll, \leq, \mathfrak{T})$ be a topological causal space and let $H \subset X$ be a null non-branching closed achronal set. Let $\mathbf{m}_i \in \mathcal{M}^+(H)$ be a measure and G_i be a gauge on H , $i = 1, 2$. Let $S \subset H$ be an acausal set. Assume that $G_1 = f + hG_2$, for some transverse functions h, f (cf. Proposition 5.18) and that $\mathbf{m}_1 = \frac{1}{h}\mathbf{m}_2$. Assume that $(H, G_1, \mathbf{m}_1)$ satisfies the $\text{NC}^e(N)$ condition (hence $(H, G_2, \mathbf{m}_2)$ does, as well, see Remark 5.14). Assume that H is future complete for G_1 (hence for G_2).*

If

$$(\mathbf{m}_1)_{G_1}^+(S) < \infty \quad \text{and} \quad (\mathbf{m}_2)_{G_2}^+(S) < \infty,$$

then

$$(7.7) \quad (\mathbf{m}_1)_{G_1}^+(S) = (\mathbf{m}_2)_{G_2}^+(S).$$

Proof. Let $\mathbf{m}_{i,\alpha}$ be the conditional measures given by the disintegration of $\mathbf{m}_i$ and let $h_{i,\alpha}$ be the density of $\mathbf{m}_i$, $i = 1, 2$, see Theorem 5.20. Since by assumption $\mathbf{m}_2 = h\mathbf{m}_1$, and $G_1 = f + hG_2$, it is immediate to see that

$$(7.8) \quad h_{1,\alpha}(t) = h_{2,\alpha}(f(\alpha) + h(\alpha)t)$$

Denote by $S_{i,\varepsilon}^+$ the ε enlargement w.r.t. the gauge G_i , $i = 1, 2$. Let $A := \Omega(S_{1,\varepsilon}^+) = \Omega(S_{2,\varepsilon}^+)$. For every $\alpha \in A$, let $a_{i,\alpha} \in \mathbb{R}$ be such that $(a_{i,\alpha}, a_{i,\alpha} + \varepsilon)$ is the maximal interval satisfying $\Psi_{G_i}((a_{i,\alpha}, a_{i,\alpha} + \varepsilon)) \in S_{i,\varepsilon}^+$, $i = 1, 2$. It holds that

$$(7.9) \quad a_{1,\alpha} = f(\alpha) + h(\alpha)a_{2,\alpha}.$$

We compute

$$\begin{aligned} \mathbf{m}_{G_i}^+(S) &= \limsup_{\varepsilon \rightarrow 0^+} \frac{\mathbf{m}(S_{i,\varepsilon}^+)}{\varepsilon} = \limsup_{\varepsilon \rightarrow 0^+} \int_A \frac{\mathbf{m}_\alpha(S_{i,\varepsilon}^+)}{\varepsilon} \mathbf{q}(\mathrm{d}\alpha) \\ &= \limsup_{\varepsilon \rightarrow 0^+} \int_A \int_{a_{i,\alpha}}^{a_{i,\alpha} + \varepsilon} \frac{h_{i,\alpha}(t)}{\varepsilon} \mathrm{d}t \mathbf{q}(\mathrm{d}\alpha). \end{aligned}$$

Denote by $v_{i,\alpha}$ the function

$$(7.10) \quad v_{i,\alpha}(t) := \int_{a_{i,\alpha}}^t h_{i,\alpha}(s) \mathrm{d}s.$$

The $\mathrm{NC}^e(N)$ condition implies that $h_{i,\alpha}^{\frac{1}{N-2}}$ is concave, thus the future completeness w.r.t. G_i guarantees that $h_{i,\alpha}$ is non-decreasing, therefore $v_{i,\alpha}$ is convex. It follows that the limit $\lim_{\varepsilon \rightarrow 0^+} \frac{v_{i,\alpha}(\varepsilon)}{\varepsilon}$ exists and it is monotone decreasing. The hypothesis $\mathbf{m}_{G_i}^+(S) < \infty$, guarantees that for $\varepsilon > 0$ small enough, $\frac{v_{i,\alpha}(\varepsilon)}{\varepsilon} \in L^1(\mathbf{q})$. We are in position to apply the dominated convergence theorem, deducing

$$(7.11) \quad \mathbf{m}_{G_i}^+(S) = \int_A h_{i,\alpha}(a_{i,\alpha}) \mathbf{q}(\mathrm{d}\alpha).$$

We conclude by combining (7.8) and (7.9). $\square$

8. PENROSE'S SINGULARITY THEOREM FOR SYNTHETIC AND CONTINUOUS SPACETIMES

In this section, we propose a way to extend Penrose's singularity theorem to the setting of a continuous spacetime (i.e., when the Lorentzian metric is continuous). Recall that Penrose's singularity theorem [57] states that if a spacetime satisfying the Null Energy Condition admits a non-compact Cauchy hypersurface and it contains a trapped surface, then it is null geodesic incomplete; i.e., there exists a null geodesic γ (in the sense that it solves the geodesics-ODE: $\nabla_{\dot{\gamma}}\dot{\gamma}$) whose maximal domain of definition is strictly contained in $\mathbb{R}$. Besides the proof, a first challenge in generalizing such a result for a continuous

spacetime is to make sense of the very statement. We first start with the concept of null completeness.

8.1. Weak null completeness. In the smooth setting, (geodesic) null completeness is formulated as the global existence of the solutions of the geodesic equation, which is not available in the synthetic setting.

We overcome this issue by first considering in the smooth setting a weaker form of null completeness that we will then show to have a surprising equivalent formulation that makes sense also in the synthetic setting.

Definition 8.1 (Weak null completeness). Let (M, g) be a strongly causal Lorentzian manifold. We say that (M, g) is *weakly future null complete* if for any maximally-defined (on the right) null geodesic (in the differential sense) $\gamma : [0, b) \rightarrow M$, either $b = \infty$, or $\gamma_0 \ll \gamma_t$, for some $t \in [0, b)$. The notion of *weakly past null complete* is analogous, by considering maximally defined (on the left) null geodesics. We say that (M, g) is *weakly null complete* if it is both weakly future and past null complete.

It is clear that a null complete manifold is weakly null complete as the case of $b < \infty$ never occur (and therefore being weakly null incomplete will be a stronger property of being null incomplete). On the other hand, we now exhibit an example of a weakly null complete Lorentzian manifold that is not null complete justifying the terminology.

In particular, we now provide a sufficient condition for a warped product to be weakly null complete but not null complete. This amounts to show that all the incomplete inextendible null geodesics are not globally maximizing. The geometric idea is, roughly, of an upside down cone-like shape where all the causal geodesic end in the tip and all the null geodesics move in spiral, more and more densely as they approach the tip.

Proposition 8.2. *Let (F, g_F) be a compact Riemannian manifold and consider the warped product $M := (-\infty, 0) \times_f F$, endowed with the metric*

$$(8.1) \quad g = dt^2 - f^2(t) g_F.$$

Assume that $f(t) \leq O(\sqrt{-t})$ as $t \rightarrow 0^-$.

Then M is a weakly null complete, but both null and time-like incomplete. More precisely, every future directed geodesic (either timelike or null) is future incomplete, and every future directed null geodesic is not globally maximizing.

Proof. In this proof, we denote by $\gamma : [0, b) \rightarrow M$ an arbitrary maximally-defined future-directed geodesic in M , with $\gamma_s = (t_s, x_s)$. We will write $(\dot{t}, \dot{x}) = \dot{\gamma} := \frac{d}{ds}\gamma$ and $\nabla_{\dot{x}_s} \dot{x}_s$ for the covariant derivative in (F, g_F) of $\dot{x}$ with respect to itself, i.e., the acceleration of the curve $s \mapsto x_s \in F$. Also, we denote the derivative of the warping function $f : (-\infty, 0) \rightarrow \mathbb{R}$ by $f' := \frac{d}{dt}f$. The equations governing the evolution of γ are [54, Prop. 7.38, p. 208]

$$(8.2) \quad \ddot{t}_s = -g_F(\dot{x}_s, \dot{x}_s) f(t_s) f'(t_s),$$

$$(8.3) \quad \nabla_{\dot{x}_s} \dot{x}_s = -2 \frac{f'(t_s)}{f(t_s)} \dot{t}_s \dot{x}_s = -2 \frac{d \log(f \circ t_s)}{ds} \dot{x}_s.$$

Notice that $|\dot{\gamma}|^2 = |\dot{t}_s|^2 - f^2(t_s)g_F(\dot{x}_s, \dot{x}_s)$ is constant in s , since γ is a geodesic. We can rewrite (8.2) as

$$(8.4) \quad \ddot{t}_s = -(|\dot{t}_s|^2 - |\dot{\gamma}|^2) \frac{f'(t_s)}{f(t_s)}.$$

If $\dot{x}_s \equiv 0$ then, from (8.2), we get that t is affine and thus b must be finite.

If instead $\dot{x}_s \not\equiv 0$, then

$$\frac{\dot{t}_s \ddot{t}_s}{|\dot{t}_s|^2 - |\dot{\gamma}|^2} = \frac{1}{2} \frac{d \log(|\dot{t}_s|^2 - |\dot{\gamma}|^2)}{ds} = -\frac{f'(t_s) \dot{t}_s}{f(t_s)} = -\frac{d \log(f(t_s))}{ds}.$$

By integrating, we deduce that

$$\frac{|\dot{t}_{s_2}|^2 - |\dot{\gamma}|^2}{|\dot{t}_{s_1}|^2 - |\dot{\gamma}|^2} = \frac{f^2(t_{s_1})}{f^2(t_{s_2})}, \quad \forall s_1, s_2 \in [0, b), s_1 \leq s_2,$$

therefore

$$(8.5) \quad \dot{t}_{s_2} = \sqrt{|\dot{\gamma}|^2 + (|\dot{t}_{s_1}|^2 - |\dot{\gamma}|^2) \frac{f^2(t_{s_1})}{f^2(t_{s_2})}}.$$

We can now take the limit on the r.h.s. as $s_2 \rightarrow b^-$, deducing that $\lim_{s \rightarrow b^-} \dot{t}_s = \infty$, therefore b must be finite. This proves future incompleteness.

A consequence of the limits $\lim_{s \rightarrow b^-} t_s = 0$ and $\lim_{s \rightarrow b^-} \dot{t}_s = \infty$ is that

$$t_s = \int_b^s \dot{t}_w dw \geq s - b, \quad \text{for } s \text{ large enough,}$$

therefore

$$(8.6) \quad f(t_s) \leq O(\sqrt{b - s}).$$

We now show that any null geodesic starting at a given point $p = (t_0, x_0)$ will definitely enter in $I^+(p)$. We claim that for any $z \in M$,

$$(8.7) \quad J^+(p) \cup (-\infty, 0) \times \{z\} \neq \emptyset.$$

Suppose on the contrary that there exists z contradicting (8.7). Let $L := (\exp_{x_0}^{g_F})^{-1}(z)$ and $\dot{t}_0 := f(t_0)\sqrt{g_F(L, L)}$, and consider the null geodesic γ starting from p , with initial velocity $\dot{\gamma}_0 = (\dot{t}_0, L)$. We will obtain a contradiction by showing that the projection of γ on F passes through z . Equation (8.3) states that the geodesic acceleration of x is parallel to its velocity, thus x is a reparametrization of a geodesic y in F , i.e., $x_s = y_{r(s)}$. Since F is compact, then Hopf–Rinow theorem guarantees it is complete. Then y_r is defined for all positive r and y accumulates at some point in F . A simple computation gives

$$\nabla_{\dot{x}_s} \dot{x}_s = \nabla_{\dot{r}(s)\dot{y}_{r(s)}} (\dot{r}(s)\dot{y}_{r(s)}) = \ddot{r}(s)\dot{y}_{r(s)} = \frac{\ddot{r}(s)}{\dot{r}(s)} \dot{x}_s,$$

which, combined with (8.3), yields

$$\frac{d \log \dot{r}}{ds} = \frac{\ddot{r}}{\dot{r}} = -2 \frac{d \log(f \circ t)}{ds},$$

or, equivalently:

$$\dot{r}(s) = \frac{C}{f^2(t_s)},$$

for some constant C . Integrating, we get that

$$r(s) = \int_0^s \frac{C}{f^2(t_w)} dw.$$

Plugging the expansion (8.6) in the equation above, we deduce that $r(s) \rightarrow +\infty$, as $s \rightarrow b^-$. This means that x runs on the whole trajectory of y and therefore the curve x passes through the point z .

Having proved (8.7), we deduce that $\partial I^+(p)$ intersect each line $(-\infty, 0) \times \{z\}$ in only one point, for each fixed $z \in F$. Moreover, it is well-known (see [54, Cor. 14.27, p. 415]) that $\partial I^+(p)$ is a C^0 manifold, thus $\partial I^+(p)$ can be seen as a graph hypersurface over F (in particular, it is a Cauchy hypersurface). Since F is compact and $\partial I^+(p)$ is a graph surface, then there exists $t_1 \in (t_0, 0)$ such that

$$\partial I^+(p) \subset (t_0, t_1) \times F.$$

It follows that any maximal null geodesic starting from p will definitely enter $I^+(p)$, thus failing to be maximizing. $\square$

The weak null completeness condition can be rephrased using gauges. We recall that if M is a smooth Lorentzian manifold and $H = \partial I^+(A)$, with A a compact acausal submanifold, a *natural* gauge function on H is given by pushing via the exponential a fiber-wise affine function in the normal bundle $\mathcal{N}\mathfrak{a}$ (see Remark 3.12). In particular G can be extended to 0 along $\mathfrak{a}$, continuously on each generator. Recall also that G is called *proper* if the pre-image of every compact set is precompact.

We will now show that if M is a strongly causal Lorentzian manifold, the weak null completeness of M is equivalent to the existence of natural proper gauges, with the latter having the great advantage of being formulated without any need of an underlying manifold structure.

Thus, *the properness of the gauge shall be seen as a synthetic counterpart of the weak null completeness*. The statement of the next proposition is given for weak future null completeness; one can state a similar theorem for past completeness.

Proposition 8.3. *Let (M, g) be a strongly causal, causally closed Lorentzian manifold. Let $H = \partial I^+(A)$, for some C^2 compact space-like achronal submanifold A .*

- (1) *Let G be any natural gauge for H which is continuously extendible to 0 at A (see Remark 3.12). If (M, g) is weakly future null complete, then G is proper.*
- (2) *Conversely, if H admits a proper natural gauge G , then for every maximally (on the right) defined null geodesic $\gamma : [0, b) \rightarrow M$, the following dichotomy holds: either $b = \infty$ or γ will leave H definitely.*
- (3) *Finally, if every future light-cone of M admits a proper natural gauge, then (M, g) is weakly future null future complete.*

Proof. Proof of 1. We recall from Remark 3.12 that

$$(8.8) \quad H^\circ \subset \exp(\{v \in \mathcal{N}A : g(v, v) = 0 \text{ and } g(v, W) < 0\}),$$

where W is a time-like vector field given by the time-orientation of M (notice that here, differently from Remark 3.12, we are considering the larger set $A \supset \mathfrak{a}$). If we define

$$C := \{v \in \mathcal{N}A \cap \text{Dom}(\exp) : g(v, v) = 0, g(v, W) \leq 0, \text{ and } \exp(v) \notin I^+(A)\},$$

it is then clear that $H = \exp(C)$. By definition of natural gauge, it holds that

$$G^{-1}([a, b]) \subset \exp(D), \quad \text{where} \quad D := \{v \in C : -g(v, W) \in [a, b]\} \subset \mathcal{N}A.$$

We check that D is compact; by continuity of the exponential the thesis will follow. Boundedness of D is immediate. Regarding closedness, let v_n be a sequence in D , converging to $v_\infty \in \mathcal{N}A$. Assume by contradiction that $v_\infty \notin D$. Then $v_\infty \notin \text{Dom}(\exp)$, since the other conditions defining D are closed. It follows that the curve $\gamma_t := \exp(tv_\infty)$ is defined on $[0, b)$ with $b \leq 1$. We exploit the weak null completeness to infer that, for some $t < b$, $\gamma_0 \ll \gamma_t \in I^+(\gamma_0) \subset I^+(A)$. Since $\exp(tv_n) \xrightarrow{n \rightarrow \infty} \gamma_t$, then, for n large enough, $\exp(tv_n) \in I^+(A)$. Therefore, $\exp(v_n) \in I^+(A)$, yielding a contradiction with the definition of D .

Proof of 2. Fix $\gamma : [0, b) \rightarrow M$ an inextendible geodesic. Up to a reparametrization, we can assume $G(\gamma_t) = t + G(\gamma_0)$. Suppose by contradiction that $b < \infty$ and that γ is contained in H . Let $F := \overline{G^{-1}([0, b + G(\gamma_0)])} \subset H$. By the properness assumption on G , we have that F is compact. Let $t_n \uparrow b$. Up to a subsequence, $\gamma_{t_n} \in F$ is converging in F . By standard properties of the geodesic flow, it follows that $\exp_{\gamma_0}(bL)$ is well-defined, a contradiction with the inextendability of γ .

Proof of 3. Fix $\gamma : [0, b) \rightarrow M$ a future inextendable null geodesic. Consider $H = \partial I^+(\gamma_0)$ the future light-cone emanating from γ_0 . Applying part 2, yields that either $b = \infty$ or γ leaves H . In the latter case, there exists $t \in (0, b)$ such that $\gamma_0 \ll \gamma_t$. $\square$

8.2. Penrose's singularity theorem. Having thoroughly discussed the completeness issue, a possible way to state the desired result on the existence of trapped sets is the following: if S is a compact achronal surface, such that $(\partial I^+(S), G, \mathfrak{m})$ satisfies the $\text{NC}^e(N)$ with a proper gauge and S is $(G, \mathfrak{m})$ -future converging, then $\partial I^+(H)$ is compact.

The $\text{NC}^e(N)$ has already been defined. It remains to discuss the “ $(G, \mathfrak{m})$ -future converging” condition. This is a delicate point, firstly because of the non-smooth synthetic setting, secondly because of the presence of initial points of H inside S , where the parametrization given by the gauge is not available.

We next define the “ $(G, \mathfrak{m})$ -future converging” condition, followed by a discussion of its compatibility with the smooth notions.

Definition 8.4. Let $(H, G, \mathfrak{m})$ be a synthetic null hypersurface satisfying the $\text{NC}^e(N)$ condition, with $H = \partial I^+(S)$, for some achronal set S .

We say that S is $(G, \mathfrak{m})$ -future converging, if there exists $\theta < 0$, such that

$$(8.9) \quad \limsup_{\varepsilon \rightarrow 0^+} \frac{\mathfrak{m}(S_\varepsilon^+ \cap \mathcal{R}(A)) - \varepsilon \mathfrak{m}_G^+(S; A)}{\varepsilon^2/2} \leq \theta \mathfrak{m}_G^+(S; A),$$

for all $A \subset H^\circ$.

Recall that the classical future convergence requires the mean curvature of S on both (incoming and outgoing) sides of $\partial I^+(S)$ to be strictly negative. If S is smooth and compact, this is equivalent to assume that its mean curvature (in both the incoming and

outgoing directions) is bounded above by some $\theta < 0$. Equation (8.9) is then an integral characterization of this curvature bound. The basic idea is that the mean curvature is given by first variation of area, and area is in turn given as first variation of volume. Notice indeed that (8.9) corresponds to a second order Taylor expansion of the volume (cf. [14, Remark 5.4], for a similar construction in the timelike framework).

A careful reader may wonder about the necessity of the set $A \subset H^\circ$ in Definition 8.4 and whether one can simplify the definition by stating everything just in terms of S . The reason for including $A \subset H^\circ$ in Definition 8.4 is that, in general, there is more than one generator for H leaving at each point of S (in the classical setting, exactly two generators) and it is desirable to have a criterion to select exactly one. Restricting the analysis to $\mathcal{R}(A)$, for any $A \subset H^\circ$, allows to overcome this issue.

In the statement of the synthetic Penrose's theorem, we will assume the gauge to satisfy the additional condition

$$(8.10) \quad \inf_{x \in H_\alpha} G(x) = 0, \quad \text{for } \mathbf{q}\text{-a.e. } \alpha \in Q,$$

where Q is the quotient set of Theorem 5.20. In the smooth setting, one can always find a gauge satisfying (8.10), as discussed in Remark 3.12.

Moreover we recall the classical terminology that a compact achronal set S is *future trapped* if $\partial I^+(S)$ is compact.

Theorem 8.5 (Penrose's theorem in a synthetic setting). *Let $(H, G, \mathbf{m})$ be a null non-branching, synthetic null hypersurface satisfying the $\text{NC}^e(N)$ condition for some $N > 2$, where $H = \partial I^+(S)$ for some compact achronal set S , and $\text{supp } \mathbf{m} = H$. Assume moreover (8.10).*

If S is $(G, \mathbf{m})$ -future converging, then $\|G\|_{L^\infty(\mathbf{m})} < \infty$. In particular, if the gauge G is proper, then H is compact; i.e., S is future trapped.

Proof. We divide the proof in two parts.

Step 1. By the Localization Theorem 6.8, there exists a disintegration of $\mathbf{m}$, given by the measures $\mathbf{m}_\alpha$ satisfying the $\text{CD}(0, N-1)$ condition. Let $h_\alpha : [0, b_\alpha] \rightarrow [0, \infty)$ be the density of $\mathbf{m}_\alpha$. For the class of synthetic null hypersurfaces fulfilling (8.10) the relative Minkowski content can be more easily described. In particular, it will be convenient to look for a class of sets $A \subset H^\circ$ for which the following identity is valid:

$$(8.11) \quad \int_B h_\alpha(0) \mathbf{q}(d\alpha) = \mathbf{m}_G^+(S; A),$$

where $B = \mathcal{R}(A) \cap Q$. To establish (8.11), notice that

$$\limsup_{\varepsilon \rightarrow 0^+} \frac{\mathbf{m}(S_\varepsilon^+ \cap \mathcal{R}(A))}{\varepsilon} = \limsup_{\varepsilon \rightarrow 0^+} \frac{1}{\varepsilon} \int_B \int_0^\varepsilon h_\alpha(t) dt.$$

Since $t \mapsto h_\alpha(t)$ is continuous, we can consider the real-valued measurable map $Q \ni \alpha \mapsto \|h_\alpha(\cdot)\|_{L^\infty(0,1)}$. For each $\delta > 0$ there exists a compact set $K \subset Q$ such that $\mathbf{q}(K) > 1 - \delta$ and, for all $\alpha \in K$, it holds that $\|h_\alpha(\cdot)\|_{L^\infty(0,1)} \leq C$, for some positive constant. A direct application of dominated convergence theorem implies that for all $A \subset \mathcal{R}(K) \cap H^\circ$, the identity in Equation (8.11) is valid.

In the next step, we will use Equation (8.11) in order to obtain an upper bound on b_α independent on $\delta > 0$; this will imply that $\|G\|_{L^\infty} < \infty$.

Step 2. The concavity of the function $t \mapsto h_\alpha(t)^{\frac{1}{N-2}}$ implies that

$$(8.12) \quad h_\alpha(t) \geq h_\alpha(0) \left(\frac{b_\alpha - t}{b_\alpha} \right)^{N-2}, \quad \forall t \in [0, b_\alpha].$$

If $N \geq 3$, Bernoulli's inequality (i.e., $x^{N-2} \geq 1 + (N-2)(x-1)$, for all $x \geq 0$) yields

$$(8.13) \quad \int_0^\varepsilon \left(\left(\frac{b_\alpha - t}{b_\alpha} \right)^{N-2} - 1 \right) dt \geq \frac{2-N}{b_\alpha} \int_0^\varepsilon t dt = \varepsilon^2 \frac{2-N}{2b_\alpha}, \quad \forall \varepsilon > 0.$$

If $2 < N < 3$, the trivial inequality $x^{N-2} \geq x$, holding for all $x \in [0, 1]$, implies that

$$(8.14) \quad \int_0^\varepsilon \left(\left(\frac{b_\alpha - t}{b_\alpha} \right)^{N-2} - 1 \right) dt \geq b_\alpha \int_0^\varepsilon t dt = \frac{\varepsilon^2}{2b_\alpha}, \quad \forall \varepsilon > 0.$$

In the rest of the proof, we will assume that $N \geq 3$ and we will use (8.13); in case $2 < N < 3$, the argument is completely analogous, using (8.14).

Let $\theta < 0$ be given by Definition 8.4. Consider any $\delta > 0$ and let $K \subset Q$ be given by the step 1 of the proof. Let $A \subset \mathcal{R}(K) \cap H^\circ$ be any measurable set where (8.11) holds, with $B = \mathcal{R}(A) \cap Q$. Then (8.9) can be written as

$$\begin{aligned} \theta \int_B h_\alpha(0) \mathbf{q}(d\alpha) &\stackrel{(8.9)}{\geq} \limsup_{\varepsilon \rightarrow 0^+} \frac{2}{\varepsilon^2} \int_B \int_0^\varepsilon (h_\alpha(t) - h_\alpha(0)) dt \mathbf{q}(d\alpha) \\ &\stackrel{(8.12)}{\geq} \limsup_{\varepsilon \rightarrow 0^+} \frac{2}{\varepsilon^2} \int_B h_\alpha(0) \int_0^\varepsilon \left(\left(\frac{b_\alpha - t}{b_\alpha} \right)^{N-2} - 1 \right) dt \mathbf{q}(d\alpha) \\ &\stackrel{(8.13)}{\geq} \int_B h_\alpha(0) \frac{2-N}{b_\alpha} \mathbf{q}(d\alpha). \end{aligned}$$

The arbitrariness of $A \subset \mathcal{R}(K) \cap H^\circ$ implies the arbitrariness of $B \subset K$, and thus

$$(8.15) \quad b_\alpha \leq -\frac{N-2}{\theta}, \quad \text{for } \mathbf{q}\text{-a.e. } \alpha \in K.$$

Since the estimate is independent of $\delta > 0$, taking the limit as $\delta \rightarrow 0$ yields that the inequality in (8.15) is valid for $\mathbf{q}$ -a.e. $\alpha \in Q$. Combining this with (8.10), we deduce that

$$G(x) \in \left[0, \frac{2-N}{\theta} \right], \quad \text{for } \mathbf{m}\text{-a.e. } x \in H^\circ.$$

The properness of G implies that $G^{-1}([0, \frac{2-N}{\theta}])$ is precompact. Therefore $\mathbf{m}$ is concentrated on a precompact set, yielding that $H = \text{supp } \mathbf{m}$ is compact. $\square$

Remark 8.6. Inspecting the proof of Theorem 8.5, one can check that the assumption (8.10) can be replaced by the following weaker (but slightly more technical) condition: the function $\alpha \mapsto \mathcal{G}(\alpha) := \inf_{x \in H_\alpha} G(x)$ belongs to $L^\infty(\mathbf{q})$. In this case, the proof gives that $G \leq \frac{2-N}{\theta} + \|\mathcal{G}\|_{L^\infty(\mathbf{q})}$, $\mathbf{m}$ -a.e.. The same argument as above yields that, if G is proper, then H is compact.

Remark 8.7. In the spirit of Proposition 7.5, we discuss how Theorem 8.5 behaves under a different choice of gauge and reference measure. Namely, let $H = \partial I^+(S)$ and

let G_i be a gauge for H and $\mathbf{m}_i$ be a measure, $i = 1, 2$. Assume that G_1 and G_2 (resp. $\mathbf{m}_1$ and $\mathbf{m}_2$) are related by the covariance relation (cf. Proposition 5.18)

$$G_1 = f + hG_2, \quad \mathbf{m}_1 = \frac{1}{h}\mathbf{m}_2,$$

for some transverse functions h and f ,

$$(8.16) \quad -C \leq f \leq C, \quad 1/C \leq h \leq C,$$

for some constant $C \geq 1$. Let us discuss the invariance of the assumptions of Theorem 8.5, under such transformations.

- The assumption that the reference measure has full support is trivially preserved (i.e., $\text{supp } \mathbf{m}_1 = H \iff \text{supp } \mathbf{m}_2 = H$).
- Eq. (8.10) is not preserved exactly; however, one can apply Remark 8.6, noting that the condition $\|\mathcal{G}\|_{L^\infty(q)} < \infty$ is preserved.
- Regarding the future converging assumption, following the argument of Proposition 7.5 one can prove that S is $(G_1, \mathbf{m}_1)$ -future converging if and only if it is $(G_2, \mathbf{m}_2)$ -future converging. In this case, the constant θ is replaced by θ/C .

Notice that also the thesis of Proposition 7.5 is covariant, i.e., properness of the gauge does not depend on the choice of the gauge but only on its equivalence class; indeed, $G_2^{-1}([-a, a]) \subset G_1^{-1}([-C(a+1), C(a+1)])$.

Finally, let us point out that the bound (8.16) is necessary, otherwise both the hypothesis and the thesis of Proposition 7.5 would cease to be invariant.

The following corollary follows by the combination of Theorem 8.5 and the C^0 -extension obtained by Ling [38, Thm. 4.9] of the classical result by Penrose, stating that there are no future trapped sets in a spacetime admitting a non-compact Cauchy surface (see, for instance [54, Sec. 14]). We refer to [17, 61, 38] for the basics of causality theory and useful notions for continuous spacetimes.

Corollary 8.8 (Penrose's singularity theorem in C^0 -spacetimes). *Let (M, g) be a space-time endowed with a continuous Lorentzian metric. Let $S \subset M$ be a compact achronal set. Let $H = \partial I^+(S)$ and endow H with a gauge function G satisfying (8.10) and a positive Radon measure $\mathbf{m} \in \mathcal{M}^+(H)$ with $\text{supp } \mathbf{m} = H$. Assume that H is null non-branching¹ and that:*

- (1) *The synthetic null hypersurface $(H, G, \mathbf{m})$ satisfies the $\text{NC}^e(N)$ condition, for some $N > 2$.*
- (2) *(M, g) admits a non-compact Cauchy surface.*
- (3) *S is $(G, \mathbf{m})$ -future converging.*

Then the gauge G cannot be proper.

Proof. Assume by contradiction that (M, g) contains a compact achronal set $S \subset M$ such that $H = \partial I^+(S)$ satisfies the conditions (1), (2), (3) and moreover the gauge G is

¹The assumption that H is null non-branching is always satisfied if

- H° is $C^{1,1}$ and g is locally Lipschitz (see Proposition 5.7),
- g is C^2 , without any further regularity assumption on H (see Corollary 5.9).

proper. Then, Theorem 8.5 implies that S is future trapped. Since by assumption (M, g) admits a non-compact Cauchy surface, we obtain a contradiction with [38, Thm. 4.9]. $\square$

Finally, we obtain the following geometrically sharpened conclusion in the celebrated Penrose's incompleteness theorem, where the sharpening consists in the fact that the incomplete null geodesic is *maximizing*; i.e., it is also a geodesic in the synthetic sense. We denote by $\tau_g(\cdot, \cdot)$ the time-separation function on (M, g) .

Corollary 8.9 (Sharpened Penrose's singularity theorem). *Let (M, g) be a spacetime endowed with a C^2 Lorentzian metric. Assume that:*

- (1) *The Null Energy Condition holds.*
- (2) *M admits a non-compact Cauchy hypersurface.*
- (3) *There exists a future-converging compact $(n - 2)$ -dimensional achronal space-like C^2 submanifold S .*

Then there exists an inextendible (on the right), null geodesic $\gamma : [0, b) \rightarrow \mathbb{R}$, $b < \infty$. Moreover, γ is maximizing, i.e., $\tau_g(\gamma_s, \gamma_t) = 0$ for all $s \leq t \in [0, b)$.

Proof. The existence of a Cauchy hypersurface implies that (M, g) is strongly causal. We can therefore apply Remark 3.12 and endow $H := \partial I^+(S)$ with a natural gauge G and the associated rigged measure $\mathfrak{m}$, making $(H, G, \mathfrak{m})$ a synthetic null hypersurface, which, by Theorem 3.17, satisfies $\text{NC}^e(n)$. As observed after (3.4), G can be continuously extended to 0 on the set $\mathfrak{a}$ of initial points of the null generators of H , thus it satisfies (8.10). We next claim that S is $(G, \mathfrak{m})$ -future-converging. Indeed, one can check (8.9) by performing a second-order Taylor expansion in ε of $\mathfrak{m}(A_\varepsilon^+)$ and then apply the future-converging hypothesis (cf. [14, Remark 5.4]). Applying Corollary 8.8, yields that G is not proper. By the first point of Proposition 8.3, it follows that (M, g) is not weakly future null complete; i.e., there exists an inextendible null geodesic like in the claim which is a global maximizer. $\square$

9. STABILITY OF THE $\text{NC}^e(N)$ CONDITION

In this section we state and prove two stability theorems for the Null Energy Condition. The first one is a null counterpart of Sturm's stability of the CD condition [65, 66].

We recall that, in a metric space (X, d) given a sequence of closed sets B_n , $n \in \mathbb{N}$ the Kuratowski-lim-sup of B_n is

$$(9.1) \quad \text{K-}\limsup_{n \rightarrow \infty} B_n = \{x \in X : \exists \text{ a subsequence } x_{n_k} \in B_{n_k} \text{ such that } x_{n_k} \rightarrow x\}.$$

We recall the definition of push-up property.

Definition 9.1. Let $(X, \leq, \ll)$ be a causal space. We say that it has the *push-up property*, if $\forall x, y, z \in X$

$$(9.2) \quad x \leq y \ll z \text{ or } x \ll y \leq z \quad \implies \quad x \ll z.$$

The push-up property is satisfied in locally-Lipschitz Lorentzian manifolds [17] and, more generally, in Lorentzian pre-length spaces [34] (see also [38] for the case of C^0 -Lorentzian metrics).

Theorem 9.2. *Let $(X, \ll, \leq, \mathfrak{T})$ be a topological causal space satisfying the push-up property and let $N > 1$. Let $(H_n, G_n, \mathbf{m}_n)$, $n \in \mathbb{N} \cup \{\infty\}$, be a sequence of synthetic null hypersurfaces, with $\mathbf{m}_n \in \mathcal{P}(H_n)$. Assume the following.*

(1) *There exist monotone (recall Definition 6.2) transport plans $\Lambda_n \in \Pi(\mathbf{m}_\infty, \mathbf{m}_n)$, such that*

$$(9.3) \quad \Lambda_n \rightharpoonup (\text{Id}, \text{Id})_\# \mathbf{m}_\infty, \quad \text{in duality with continuous and bounded functions.}$$

(2) *For every compact set $K \subset X$, the set of curves*

$$(9.4) \quad \{\gamma \in C([0, 1]; X) : \gamma \in C_{G_n} \text{ for some } n \in \mathbb{N} \text{ and } \gamma_i \in K, i = 0, 1\} \text{ is precompact.}$$

(3) *Denoted by C_{G_n} the set of G_n -causal curves, it holds that*

$$(9.5) \quad \text{K-} \limsup_{n \rightarrow \infty} C_{G_n} \subset C_{G_\infty}.$$

If the synthetic null hypersurfaces $(H_n, G_n, \mathbf{m}_n)$ satisfy the $\text{NC}^e(N)$ condition, for all $n \in \mathbb{N}$, then also $(H_\infty, G_\infty, \mathbf{m}_\infty)$ satisfies the $\text{NC}^e(N)$ condition, as well.

Proof. Let $\mu_i^\infty \in \mathcal{P}(H_\infty)$, $i = 0, 1$, be two probability measures, transported one in the other by a plan $\hat{\pi}_\infty \in \Pi_\leq(\mu_0^\infty, \mu_1^\infty)$. Notice that, if μ_i^∞ is not absolutely continuous w.r.t. $\mathbf{m}_\infty$, then the concavity inequality of $\text{NC}^e(N)$ trivializes; therefore we can assume that $\mu_i^\infty \ll \mathbf{m}_\infty$.

Step 1. Definition of the approximations $\mu_i^n \in \mathcal{P}(H_n)$, $i = 1, 2$.

We disintegrate Λ_n , in the following way

$$(9.6) \quad \Lambda_n = \int_{H_\infty} \Lambda_{n,x} \mathbf{m}_\infty(dx),$$

where $\Lambda_{n,x}$ is a probability measure concentrated on $\{x\} \times H_n$. Notice that, by monotonicity of Λ_n , if $x \leq y$, then

$$(9.7) \quad (P_2)_\# \Lambda_{n,x} \otimes (P_2)_\# \Lambda_{n,y}(J) = 1.$$

We push the measures μ_i^∞ and $\hat{\pi}_\infty$, as follows

$$(9.8) \quad \mu_i^n := \int_{H_\infty} (P_2)_\# \Lambda_{n,x} \mu_i^\infty(dx), \quad \hat{\pi}_n := \int_{H_\infty \times H_\infty} ((P_2)_\# \Lambda_{n,x}) \otimes ((P_2)_\# \Lambda_{n,y}) \hat{\pi}_\infty(dx dy),$$

where P_2 is the projection on the second variable. We now check that $\hat{\pi}_n$ is a transport plan between μ_0^n and μ_1^n :

$$\begin{aligned} (P_1)_\# \hat{\pi}_n &= (P_1)_\# \int_{H_\infty \times H_\infty} ((P_2)_\# \Lambda_{n,x}) \otimes ((P_2)_\# \Lambda_{n,y}) \hat{\pi}_\infty(dx dy) \\ &= \int_{H_\infty \times H_\infty} (P_1)_\# (((P_2)_\# \Lambda_{n,x}) \otimes ((P_2)_\# \Lambda_{n,y})) \hat{\pi}_\infty(dx dy) \\ &= \int_{H_\infty \times H_\infty} (P_2)_\# \Lambda_{n,x} \hat{\pi}_\infty(dx dy) = \int_{H_\infty \times H_\infty} (P_2)_\# \Lambda_{n,x} \mu_0^\infty(dx) = \mu_0^n, \end{aligned}$$

and analogously one can prove the same for the other projection. The next step is to prove that $\hat{\pi}_n(J) = 1$:

$$\begin{aligned}\hat{\pi}_n(J) &= \int_{H_\infty \times H_\infty} ((P_2)_\# \Lambda_{n,x}) \otimes ((P_2)_\# \Lambda_{n,y})(J) \hat{\pi}_\infty(dx dy) \\ &= \int_{H_\infty \times H_\infty \cap J} ((P_2)_\# \Lambda_{n,x}) \otimes ((P_2)_\# \Lambda_{n,y})(J) \hat{\pi}_\infty(dx dy) \stackrel{(9.7)}{=} 1.\end{aligned}$$

Step 2. Proof that $\mu_i^n \rightharpoonup \mu_i^\infty$, $i = 0, 1$.

We first claim that, up to a subsequence,

$$(9.9) \quad (P_2)_\#(\Lambda_{n,x}) \rightharpoonup \delta_x, \quad \text{for } \mathbf{m}_\infty\text{-a.e. } x.$$

Indeed, using assumption (9.3), we compute

$$\begin{aligned}0 &= \int_{X \times X} d \wedge 1 \, d((\text{Id}, \text{Id})_\# \mathbf{m}_\infty) = \lim_{n \rightarrow \infty} \int_{X \times X} d \wedge 1 \, d\Lambda_n \\ &= \lim_{n \rightarrow \infty} \int_{H^\circ} \int_{X \times X} d \wedge 1 \, d\Lambda_{n,x} \, \mathbf{m}_\infty(dx).\end{aligned}$$

Therefore, the function

$$(9.10) \quad x \mapsto \int_{X \times X} d \wedge 1 \, d\Lambda_{n,x} = \int_X d(x, y) \wedge 1 \, ((P_2)_\# \Lambda_{n,x})(dy)$$

converges in $L^1(\mathbf{m}_\infty)$ to 0 (in the last equality we used the fact that $\Lambda_{n,x}$ is concentrated on $\{x\} \times H_n$). Up to a subsequence, it also converges $\mathbf{m}_\infty$ -a.e. to 0, proving the claim (9.9).

Using the claim (9.9), we next show that $\mu_i^n \rightharpoonup \mu_i^\infty$, $i = 0, 1$. Since the weak convergence is metrizable, it is enough to show that any subsequence of μ_i^n converges to μ_i^∞ . To this aim, fix $\varphi \in C_b(X)$ and compute, using (9.9) and dominated convergence theorem:

$$\begin{aligned}\lim_{n \rightarrow \infty} \int_X \varphi d\mu_i^n &= \lim_{n \rightarrow \infty} \int_{H^\circ} \int_X \varphi(y) (P_2)_\#(\Lambda_{n,x})(dy) \mu_i^\infty(dx) \\ &= \int_{H^\circ} \left(\lim_{n \rightarrow \infty} \int_X \varphi(y) (P_2)_\#(\Lambda_{n,x})(dy) \right) \mu_i^\infty(dx) \stackrel{(9.9)}{=} \int_{H^\circ} \varphi(x) \mu_i^\infty(dx),\end{aligned}$$

yielding that $\mu_i^n \rightharpoonup \mu_i^\infty$, $i = 0, 1$.

Arguing as in of [65, Lemma 4.19], since $\mu_i^n = (\Lambda_n)_\# \mu_i^\infty$ and $\mathbf{m}_n = (\Lambda_n)_\# \mathbf{m}_\infty$, we deduce that

$$(9.11) \quad \text{Ent}(\mu_i^n | \mathbf{m}_n) \leq \text{Ent}(\mu_i^\infty | \mathbf{m}_\infty).$$

Step 3. Weak convergence of the G_n -causal dynamical optimal transport plans.

Let now $\nu_n \in \text{OptGeo}^{G_n}(\mu_0^n, \mu_1^n)$ be the G_n -causal dynamical optimal transport plan given by the definition of $\text{NC}^e(N)$, i.e., (recall (3.7), for the definition of U_N)

$$(9.12) \quad U_{N-1}((e_t)_\# \nu_n | \mathbf{m}_n) \geq (1-t) U_{N-1}((e_0)_\# \nu_n | \mathbf{m}_n) + t U_{N-1}((e_1)_\# \nu_n | \mathbf{m}_n).$$

We show that the sequence ν_n is tight. Fix $\varepsilon > 0$. By tightness of $(\mu_i^n)_n$, there exists a compact set $K \subset X$ such that $\mu_i^\infty(K) \geq 1 - \varepsilon$, $i = 0, 1$, for all $n \in \mathbb{N} \cap \{\infty\}$. We deduce that $\nu_n(C_K) \geq 1 - 2\varepsilon$, where $C_K \subset C([0, 1]; X)$ is the family set of G_n -causal curves

with endpoints in K . By (9.4), we have that C_K is compact, therefore the family $(\nu_n)_n$ is tight. By Prokhorov's Theorem, up to taking a subsequence, it holds that $\nu_n \rightarrow \nu_\infty$, for some probability measure ν_∞ .

We claim that ν_∞ is concentrated on G_∞ -causal curves in H_∞ . Since $(e_0, e_1)_\# \nu_\infty \in \Pi_{\leq}(\mu_0^\infty, \mu_1^\infty)$, then ν_∞ is concentrated on causal curves whose end-points belong to H_∞ . Since H_∞ is a closed achronal set, by the push-up property, any causal curve with endpoints in H_∞ lays in H_∞ , therefore ν_∞ is concentrated on causal curves in H_∞ . Making use of Lemma 9.4 below, we can thus compute

$$1 = \limsup_{n \rightarrow \infty} \nu_n(C_{G_n}) \stackrel{(9.15)}{\leq} \nu_\infty \left(\text{K-lim sup}_{n \rightarrow \infty} C_{G_n} \right) \stackrel{(9.5)}{\leq} \nu_\infty(C_{G_\infty}) \leq 1.$$

Using the joint lower semicontinuity of the entropy under weak convergence of probability measures [67, Theorem 29.20], we deduce that

$$(9.13) \quad \liminf_{n \rightarrow \infty} \text{Ent}((e_t)_\#(\nu_n)|\mathbf{m}_n) \geq \text{Ent}((e_t)_\#(\nu_\infty)|\mathbf{m}_\infty).$$

We can thus compute

$$\begin{aligned} U_{N-1}((e_t)_\# \nu_\infty | \mathbf{m}_\infty) &\stackrel{(9.13)}{\geq} \limsup_{n \rightarrow \infty} U_{N-1}((e_t)_\# \nu_n | \mathbf{m}_\infty) \\ &\stackrel{(9.12)}{\geq} \limsup_{n \rightarrow \infty} ((1-t) U_{N-1}((e_0)_\# \nu_n | \mathbf{m}_n) + t U_{N-1}((e_1)_\# \nu_n | \mathbf{m}_n)) \\ &\stackrel{(9.11)}{\geq} (1-t) U_{N-1}((e_0)_\# \nu_\infty | \mathbf{m}_\infty) + t U_{N-1}((e_1)_\# \nu_\infty | \mathbf{m}_\infty). \end{aligned}$$

deducing that ν_∞ enjoys concavity for the entropy, thus $(H_\infty, G_\infty, \mathbf{m}_\infty)$ is $\text{NC}^e(N)$. $\square$

The next two technical lemmas (included for the reader's convenience) recall two elementary facts linking the Kuratowski convergence of sets and the weak convergence of probability measures.

Lemma 9.3. *Let (X, d) be a metric space. Let $(C_n)_n$ be a sequence of subsets of X . Assume that $\text{K-lim sup}_{n \rightarrow \infty} C_n \subset C$, for some subset $C \subset X$.*

Then, for any compact subset $K \subset X$, the following holds: for every $\varepsilon > 0$ there exists $k \in \mathbb{N}$ such that

$$(9.14) \quad C_n \cap K \subset C^\varepsilon, \quad \text{for all } n > k.$$

Proof. Assume on the contrary that there exists $\varepsilon_0 > 0$ such that, up to passing to a unlabeled subsequence, there exists a sequence $x_n \in C_n \cap K$, such that $\text{dist}(x_n, C) > \varepsilon_0$. Up to a further subsequence, using the compactness of K , we have that $x_n \rightarrow x$, for some $x \in K$. Therefore, by definition of Kuratowski convergence (9.1), we infer that $x \in \text{K-lim sup}_{n \rightarrow \infty} C_n \subset C$. This contradicts that $\text{dist}(x, C) \geq \varepsilon_0$. $\square$

Lemma 9.4. *Let (X, d) be a Polish space. Let $(\mu_n)_n$ be a sequence of probability measures, weakly converging to μ . Let $(C_n)_n$ be a sequence of subsets of X . Then*

$$(9.15) \quad \limsup_{n \rightarrow \infty} \mu_n(C_n) \leq \mu \left(\text{K-lim sup}_{n \rightarrow \infty} C_n \right).$$

Proof. Define $C := \text{K-lim sup}_{n \rightarrow \infty} C_n$. Fix $\varepsilon > 0$. By tightness, there exists a compact set $K \subset X$, such that $\mu_n(K) \geq 1 - \varepsilon$, for all n . From Lemma 9.3, we deduce that $C_n \cap K \subset C^\varepsilon$, for all n large enough. It follows that

$$\begin{aligned} \limsup_{n \rightarrow \infty} \mu_n(C_n) &\leq \limsup_{n \rightarrow \infty} \mu_n(C_n \cap K) + \varepsilon \leq \limsup_{n \rightarrow \infty} \mu_n(C^\varepsilon) + \varepsilon \\ &\leq \mu(\overline{C^\varepsilon}) + \varepsilon \leq \mu(C^{2\varepsilon}) + \varepsilon, \end{aligned}$$

having used the upper semicontinuity of measures of a closed set under weak convergence. By arbitrariness of $\varepsilon > 0$, since C is closed, we conclude. $\square$

On a closed achronal set H endowed with a gauge G , we define the set of (G, ε) -causal curves as

$$\begin{aligned} C_{G,\varepsilon} := \Big\{ \gamma : [0, 1] \rightarrow H \text{ causal: } \gamma_r \in H^\circ, \forall r \in (0, 1) \text{ and} \\ \left| \frac{G(\gamma_t) - G(\gamma_s)}{t - s} - \lim_{u \rightarrow 0^+} (G(\gamma_{1-u}) - G(\gamma_u)) \right| \leq \varepsilon \lim_{u \rightarrow 0^+} (G(\gamma_{1-u}) - G(\gamma_u)), \\ \forall t, s \in (0, 1) : t < s \Big\}. \end{aligned}$$

We stress out that the limit in the definition above exists by monotonicity. Moreover, the definition is invariant by an affine transformation of G (cf. Proposition 5.18).

The next result shall be seen as a null counterpart of Lott and Villani's [39, 67] stability of the $\text{CD}(K, N)$ condition under pointed measured Gromov–Hausdorff convergence of pointed metric measure spaces. More precisely, the next theorem establishes a stability result for the $\text{NC}^e(N)$ condition under a suitable pointed measured convergence of (possibly) non-compact synthetic null hypersurfaces endowed with a σ -finite measure (note that, in the previous stability result, Theorem 9.2, the reference measures $\mathbf{m}_n$ were probabilities). In order to handle convergence in such a higher generality, it is convenient to consider *pointed* synthetic null hypersurfaces $(H, G, \mathbf{m}, \star)$, where $\star \in H$ is a marked point.

Theorem 9.5. *Let $N > 1$. Let $(X_n, \leq_n, \ll_n, \mathfrak{T}_n)$, $n \in \mathbb{N} \cup \{\infty\}$ be a sequence of topological causal spaces. Let $(H_n, G_n, \mathbf{m}_n, \star_n)$, $n \in \mathbb{N} \cup \{\infty\}$ be a pointed synthetic null hypersurface in $(X_n, \leq_n, \ll_n, \mathfrak{T}_n)$, $n \in \mathbb{N} \cup \{\infty\}$. Assume the following.*

- (1) *There exist Borel and monotone maps $h_n : H_n \rightarrow H_\infty$ and $g_n : H_\infty \rightarrow H_n$ and an infinitesimal sequence $\varepsilon_n \downarrow 0$, such that $h_n(\star_n) = \star_\infty$ and $g_n(\star_\infty) = \star_n$,*

$$(9.16) \quad (h_n)_\# \mathbf{m}_n \rightharpoonup \mathbf{m}_\infty,$$

$$(9.17) \quad (g_n)_\# (\mathbf{m}_\infty) = \sigma_n \mathbf{m}_n, \quad \text{with} \quad \sigma_n \leq 1 + \varepsilon_n,$$

$$(9.18) \quad h_n \circ g_n(x) \rightarrow x, \quad \text{for } \mathbf{m}_\infty\text{-a.e. } x \in H_\infty.$$

Moreover h_n transforms, by post-composition, G_n -causal curves into $(G_\infty, \varepsilon_n)$ -causal curves; i.e.,

$$h_n \circ \gamma \in C_{G_\infty, \varepsilon_n}, \quad \text{for all } \gamma \in C_{G_n}.$$

- (2) *For every precompact set $K \subset H_\infty^\circ$ and for every $\varepsilon > 0$, it holds that the set*

$$(9.19) \quad \{\gamma \in C_{G_\infty, \varepsilon} : \gamma_i \in K, i = 0, 1\} \text{ is precompact.}$$

(3) It holds that²

$$(9.20) \quad \text{K-}\limsup_{\varepsilon \rightarrow 0} C_{G_\infty, \varepsilon} \subset C_{G_\infty}.$$

If the synthetic null hypersurfaces $(H_n, G_n, \mathbf{m}_n)$ satisfy the $\text{NC}^e(N)$ condition, for all $n \in \mathbb{N}$, then also $(H_\infty, G_\infty, \mathbf{m}_\infty)$ satisfies the $\text{NC}^e(N)$ condition, as well.

Remark 9.6. Recently, inspired by the measured Gromov–Hausdorff convergence in positive signature, several notions of convergence for smooth and non-smooth space-times appeared in the literature, see e.g., [14, 48, 52, 7, 60, 51]. It is an interesting problem, that we do not address here due to length constraint, to compare these with the convergence used in the stability results.

Proof. Let $\mu_i^\infty \in \mathcal{P}(H_\infty)$, $i = 0, 1$, be such that $\Pi_\leq(\mu_0^\infty, \mu_1^\infty) \neq \emptyset$. Notice that, if μ_i^∞ is not absolutely continuous w.r.t. $\mathbf{m}_\infty$, then the concavity inequality of $\text{NC}^e(N)$ trivializes; therefore we can assume that

$$(9.21) \quad \mu_i^\infty = \rho_{i, \infty} \mathbf{m}_\infty \ll \mathbf{m}_\infty.$$

Let

$$(9.22) \quad \mu_i^n := (g_n)_\# \mu_i^\infty = \rho_{i, n}(g_n)_\# \mathbf{m}_\infty = \rho_{i, n} \sigma_n \mathbf{m}_n,$$

for some function $\rho_{i, n}$, $n \in \mathbb{N}$.

Let $\nu_n \in \text{OptGeo}^{G_n}(\mu_0^n, \mu_1^n)$ be a G_n -causal dynamical transport, given by the definition of $\text{NC}^e(N)$, i.e., (recall (3.7), for the definition of U_N)

$$(9.23) \quad U_{N-1}((e_t)_\# \nu_n | \mathbf{m}_n) \geq (1-t) U_{N-1}((e_0)_\# \nu_n | \mathbf{m}_n) + t U_{N-1}((e_1)_\# \nu_n | \mathbf{m}_n).$$

Step 1. Convergence at $t = 0, 1$.

Using that $\text{Ent}(\cdot | \cdot)$ decreases after push-forward on both entries (see for instance [67, Th. 29.20 (ii)]), we can estimate the entropy at the extremals of the interval, as follows (assumption (9.17) has been taken into account)

$$\begin{aligned} \text{Ent}(\mu_i^\infty | \mathbf{m}_\infty) &\geq \text{Ent}(\mu_i^n | (g_n)_\# \mathbf{m}_\infty) = \int_{H_n} \log(\rho_{i, n}) d\mu_i^n \\ &= \int_{H_n} \log(\rho_{i, n} \sigma_n) d\mu_i^n - \int_{H_n} \log \sigma_n d\mu_i^n \\ &\stackrel{(9.17)}{\geq} \int_{H_n} \log(\rho_{i, n} \sigma_n) d\mu_i^n - \int_{H_n} \log(1 + \varepsilon_n) d\mu_i^n \\ &\stackrel{(9.22)}{=} \text{Ent}(\mu_i^n | \mathbf{m}_n) - \int_{H_n} \log(1 + \varepsilon_n) d\mu_i^n, \quad i = 0, 1. \end{aligned}$$

Since by assumption $\varepsilon_n \downarrow 0$, we deduce that

$$(9.24) \quad \liminf_{n \rightarrow \infty} U_{N-1}((e_i)_\# \nu_n | \mathbf{m}_n) \geq U_{N-1}((e_i)_\# \nu_n | \mathbf{m}_n), \quad i = 0, 1.$$

²Since $C_{G_\infty, \varepsilon}$ is a decreasing sequence of sets, the assumption (9.20) is equivalent to require that $C_{G_\infty} = \bigcap_{\varepsilon > 0} C_{G_\infty, \varepsilon} = \text{K-}\lim_{\varepsilon \rightarrow 0} C_{G_\infty, \varepsilon}$.

We next prove that $(h_n)_\# \mu_i^n \rightharpoonup \mu_i^\infty$. Fix $\varphi \in C_b(H_\infty)$ and compute, using dominated convergence theorem

$$\begin{aligned} \int_{H_n} \varphi d((h_n)_\# \mu_i^n) &= \int_{H_n} \varphi d((h_n \circ g_n)_\# \mu_i^\infty) = \int_{H_n} \varphi(h_n \circ g_n(x)) \mu_i^\infty(dx) \\ &= \int_{H_n} \varphi(h_n \circ g_n(x)) \rho_{i,\infty}(x) \mathbf{m}_\infty(dx) \xrightarrow{(9.18)} \int_{H_n} \varphi(x) \rho_{i,\infty}(x) \mathbf{m}_\infty(dx). \end{aligned}$$

Step 2. Construction of the dynamical transport plan.

Let $\hat{h}_n : C_{G_n} \rightarrow C_{G_\infty, \varepsilon_n}$ denote the post-composition by h_n , (i.e., $\hat{h}_n(\gamma) = h_n \circ \gamma$) and define $\eta_n := (\hat{h}_n)_\# \nu_n$. We claim that the sequence $(\eta_n)_n$ is tight. Up to taking a subsequence, we can assume ε_n to be decreasing, therefore, by hypothesis, η_n is concentrated on $C_{G_\infty, \varepsilon_m}$, if $n \geq m$. Since $(h_n)_\# \mu_i^n \rightharpoonup \mu_i^\infty$, then the family $\{(h_n)_\# \mu_i^n\}_{i=0,1, n \in \mathbb{N}}$ is tight. For every $\delta > 0$, let $K_\delta \subset H_\infty$ be a compact set, such that

$$(e_i)_\# \eta_n(K_\delta) = (h_n)_\# \mu_i^n(K) \geq 1 - \delta, \quad i = 0, 1, \quad \text{for all } n \in \mathbb{N}.$$

Define

$$C_{G_\infty, \varepsilon_m}^{K_\delta} := \{\gamma \in C_{G_\infty, \varepsilon_m} : \gamma_i \in K_\delta, i = 0, 1\}.$$

By hypothesis (9.19), $C_{G_\infty, \varepsilon_m}^{K_\delta}$ is precompact and it holds that $\eta_n(C_{G_\infty, \varepsilon_m}^{K_\delta}) \geq 1 - 2\delta$. Therefore the family $(\eta_n)_n$ is tight. By Prokhorov's Theorem, up to taking a subsequence, $\eta_n \rightharpoonup \eta_\infty$, for some measure η_∞ . Since η_n is concentrated on $C_{G_\infty, \varepsilon_m}$, $n > m$, then η_∞ is concentrated on $\overline{C_{G_\infty, \varepsilon_m}}$, for all m . Therefore, by (9.20) we have that η_∞ is concentrated on C_{G_∞} .

Step 3. Convergence at $t \in (0, 1)$.

Using again [67, Thm. 29.20 (ii)], we infer that

$$\begin{aligned} \text{Ent}((e_t)_\# \nu_n | \mathbf{m}_n) &\geq \text{Ent}((h_n)_\# (e_t)_\# \nu_n | (h_n)_\# \mathbf{m}_n) \\ &= \text{Ent}((e_t)_\# (\hat{h}_n)_\# \nu_n | (h_n)_\# \mathbf{m}_n) = \text{Ent}((e_t)_\# \eta_n | (h_n)_\# \mathbf{m}_n). \end{aligned}$$

Using the joint lower-semicontinuity of the entropy under weak convergence and assumption (9.16), we deduce that

$$(9.25) \quad \liminf_{n \rightarrow \infty} \text{Ent}((e_t)_\# \nu_n | \mathbf{m}_n) \geq \liminf_{n \rightarrow \infty} \text{Ent}((e_t)_\# \eta_n | (h_n)_\# \mathbf{m}_n) \geq \text{Ent}((e_t)_\# \eta_\infty | \mathbf{m}_\infty).$$

Combining, (9.24) and (9.25), we can pass to the limit (9.23), concluding the proof (see the last steps of the proof of Theorem 9.2 for the details). $\square$

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