QUANTITATIVE STABILITY CONTROL OF THE FULL SPECTRUM OF THE DIRICHLET LAPLACIAN BY THE SECOND EIGENVALUE

ALEXIS DE VILLEROCHÉ

Abstract. Let $\Omega \subset \mathbb{R}^d$ be an open set of finite measure and let Θ be a disjoint union of two balls of half measure. We study the stability of the full Dirichlet spectrum of Ω when its second eigenvalue is close to the second eigenvalue of Θ. Precisely, for every $k \in \mathbb{N}$, we provide a quantitative control of the difference $|\lambda_k(\Omega) - \lambda_k(\Theta)|$ by the variation of the second eigenvalue $C(d,k)(\lambda_2(\Omega) - \lambda_2(\Theta))^\alpha$, for a suitable exponent α and a positive constant $C(d,k)$ depending only on the dimension of the space and the index k. We are able to find such an estimate for general k and arbitrary Ω with $\alpha = \frac{1}{d+1}$. In the particular case when $\lambda_k(\Omega) \geq \lambda_k(\Theta)$, we can improve the inequality and find an estimate with the sharp exponent $\alpha = \frac{1}{2}$.

1. Introduction

We work in the Euclidean space $\mathbb{R}^d$, for some $d \geq 2$. Let us denote by ω_d the volume of the unit ball of $\mathbb{R}^d$ and set

$$\mathcal{A} = \left\{\Omega \subset \mathbb{R}^d \;\middle|\; \Omega \text{ open and } |\Omega| = \omega_d\right\}.$$

In the following, by $B \in \mathcal{A}$ we denote a ball of radius equal to 1 and by $\Theta \in \mathcal{A}$ a union of two disjoint balls of measure $\omega_d/2$. The position of the balls does not affect the spectrum of the Dirichlet Laplacian, however, in our analysis their position may play a role. This will be specified, when necessary.

For any $\Omega \in \mathcal{A}$, let us consider the k-th eigenvalue of the Dirichlet Laplacian, multiplicity being counted. For $k \geq 1$,

$$\lambda_k(\Omega) = \min_{S_k \subset H_0^1(\Omega)} \sup_{u \in Sk,\, u \neq 0} \frac{\int_\Omega |\nabla u|^2}{\int_\Omega u^2}, \tag{1}$$

where S_k is a subspace of $H_0^1(\Omega)$ of dimension k. The associated eigenfunctions which achieve the minimum are denoted by u_k and solve

$$\begin{cases} -\Delta u_k = \lambda_k(\Omega)\, u_k & \text{in } \Omega, \\ u_k = 0 & \text{in } \partial\Omega. \end{cases} \tag{2}$$

If not otherwise specified, we consider them normalized in $L^2(\Omega)$.

Minimizing $\lambda_k(\Omega)$ for $\Omega \in \mathcal{A}$ is an important question in spectral geometry, as it is related to the celebrated Pólya conjecture stating

$$\forall k \geq 1, \forall\, \Omega \in \mathcal{A}, \;\; \lambda_k(\Omega) \geq 4\pi^2 \Big(\frac{k}{\omega_d}\Big)^{\frac{2}{d}}.$$

One strategy to understand the conjecture is, for given $l \in \mathbb{N}$, to solve the minimization problem

$$\min\{\lambda_l(\Omega) : \Omega \in \mathcal{A}\},$$

precisely to characterize solutions and find their properties. The key element, is that the Pólya conjecture is known to hold for some *particular* class of domains and so, a suitable characterization of the solution may provide useful information.

Assume Ω_l^* is a minimizer of λ_l in $\mathcal{A}$ and $\Omega \in \mathcal{A}$ is a set such that $\lambda_l(\Omega)$ approaches the minimal value $\lambda_l(\Omega_l^*)$. We aim for a sharp control of the variation of the k-th eigenvalue by the variation of the l-th eigenvalue. Precisely, is the following inequality true

$$\forall k \in \mathbb{N}, \forall \Omega \in \mathcal{A}, \ \ |\lambda_k(\Omega) - \lambda_k(\Omega_l^*)| \leq C(d,k)[\lambda_l(\Omega) - \lambda_l(\Omega_l^*)]^\alpha \ ? \tag{3}$$

Above, α is a suitable exponent and $C(d,k)$ is a constant depending only on the dimension and k.

At the moment, the minimizers Ω_l^* are analytically known only for $l = 1$ and $l = 2$ and are respectively the ball B and the union of two disjoint equal balls, that we denote Θ. This is a consequence of the Faber-Krahn and the Krahn-Szegö inequalities (see for example [10] and [11]).

For $l \geq 3$ some qualitative results are known. For instance, Bucur [7] and Mazzoleni and Pratelli [16] proved that a minimizer exists in the larger class of quasi-open sets of measure ω_d. The minimizers are bounded and of finite perimeter. Moreover, the structure of their reduced boundary was analyzed by Kriventsov and Lin in [14, 15], but the qualitative information about their global geometry is missing. Thus, since so little is known in the case $l \geq 3$, we expect to be able to find stability estimates like those in (3) only for the cases $l = 1$ and $l = 2$.

The case $l = 1$ has already been studied in the literature. As we already stated, we know that the ball B is the minimizer to λ_1 as asserted by the Faber-Krahn inequality

$$\forall \Omega \in \mathcal{A}, \qquad \lambda_1(\Omega) \geq \lambda_1(B), \tag{4}$$

with equality if and only if Ω is a ball.

In 2006, Bertrand and Colbois in [3] where able to prove stability estimates near the ball. Precisely, they prove that for any $\Omega \in \mathcal{A}$ such that

$$\lambda_1(\Omega) \leq (1+\varepsilon)\lambda_1(B),$$

it holds, if $\varepsilon < \varepsilon_k$ small enough,

$$|\lambda_k(\Omega) - \lambda_k(B)| \leq C_{d,k}\, \varepsilon^{\frac{1}{80\,d}}, \tag{5}$$

Clearly, the exponent $\frac{1}{80\,d}$ is not optimal, and later in 2019, Mazzoleni and Pratelli [17] improved the result and obtained for $\lambda_1(\Omega) \leq \lambda_1(B) + 1$ and for any $\eta > 0$

$$-C_{d,k,\eta}(\lambda_1(\Omega) - \lambda_1(B))^{\frac{1}{6}-\eta} \leq \lambda_k(\Omega) - \lambda_k(B) \leq C_{d,k,\eta}(\lambda_1(\Omega) - \lambda_1(B))^{\frac{1}{12}-\eta}, \tag{6}$$

and they are able to improve the exponent in dimension 2.
The sharp estimate has been obtained in 2023, by Bucur, Lamboley, Nahon and Prunier in [8], where they prove for any $\Omega \in \mathcal{A}$ and k

$$|\lambda_k(\Omega) - \lambda_k(B)| \leq C_d k^{2+\frac{4}{d}} \lambda_1(\Omega)^{\frac{1}{2}} \left(\lambda_1(\Omega) - \lambda_1(B)\right)^{\frac{1}{2}} \tag{7}$$

A key ingredient of the proof is the quantitative Faber-Krahn inequality proved in 2015 by Brasco, de Philippis and Velichkov in [4],

$$\forall \Omega \in \mathcal{A}, \qquad \lambda_1(\Omega) \geq \lambda_1(B)\,(1 + C_d\, \mathcal{F}_1(\Omega)^2). \tag{8}$$

where $\mathcal{F}_1(\Omega)$ denotes the *Fraenkel asymmetry*

$$\mathcal{F}_1(\Omega) = \min\left\{ \frac{|\Omega \Delta B|}{|\Omega|} \;\middle|\; B \subset \mathbb{R}^d \text{ ball with } |B| = |\Omega| \right\}.$$

We point out that inequality (7) can be improved for some specific values of k. Precisely, if $\lambda_k(B)$ is simple, then the exponent $\frac{1}{2}$ on the right hand side can be replaced by the exponent 1. This result is very fine and relies on the analysis of a degenerate free boundary problem of vectorial type. The key point is that if $\lambda_k(B)$ is simple then the ball is a critical set for λ_k, as for λ_1. Intuitilvely this makes that the variation of λ_k is of the same order as the variation of λ_1.

Our goal is to analyze the case $l = 2$ and to prove an inequality similar to (7) with the variation of the second eigenvalue on the right hand side, variation with respect to its global minimizer which is Θ. Indeed, we know from the Krahn-Szegö inequality that Θ minimizes λ_2 in $\mathcal{A}$,

$$\forall \Omega \in \mathcal{A}, \qquad \lambda_2(\Omega) \geq \lambda_2(\Theta), \tag{9}$$

with equality if and only if $\Omega = \Theta$. Moreover, a quantitative inequality has been proved by Brasco and Pratelli [5] in 2013.

$$\forall \Omega \in \mathcal{A}, \qquad \lambda_2(\Omega) \geq \lambda_2(\Theta)\,(1 + C_d\,\mathcal{F}_2(\Omega)^{d+1}), \tag{10}$$

Here, $\mathcal{F}_2(\Omega)$ is the *Fraenkel 2-asymmetry* defined as follows for any open set $\Omega \subset \mathbb{R}^d$

$$\mathcal{F}_2(\Omega) = \inf\left\{ \frac{|\Omega \Delta (B_1 \cup B_2)|}{|\Omega|} \;\middle|\; B_1, B_2 \subset \mathbb{R}^d \text{ balls, } |B_1 \cap B_2| = 0 \text{ and } |B_1| = |B_2| = \frac{|\Omega|}{2} \right\}.$$

Below we state our main results.

Theorem 1. *There exists a dimensional constant $C_d > 0$ such that for any $\Omega \in \mathcal{A}$, and for any $k \geq 1$, it holds,*

$$|\lambda_k(\Omega) - \lambda_k(\Theta)| \leq C_d\, k^{2+\frac{4}{d}}\, \lambda_2(\Omega)^{\frac{d}{d+1}}\, (\lambda_2(\Omega) - \lambda_2(\Theta))^{\frac{1}{d+1}}.$$

In general, we do not expect the exponent $\frac{1}{d+1}$ to be sharp. We are able to improve the exponent to the sharp one $\frac{1}{2}$ in the case that $\lambda_k(\Omega) \geq \lambda_k(\Theta)$. Our second result reads

Theorem 2. *There exists a dimensional constant $C_d > 0$ such that for any $\Omega \in \mathcal{A}$, and for any $k \geq 1$, it holds,*

$$(\lambda_k(\Omega) - \lambda_k(\Theta))_+ \leq C_d\, k^{2+\frac{4}{d}}\, \lambda_2(\Omega)^{\frac{1}{2}}\, (\lambda_2(\Omega) - \lambda_2(\Theta))^{\frac{1}{2}},$$

where by $(a)_+$ we denote the positive part of the number a.

As an intermediate technical step, we prove the following more general result.

Theorem 3. *There exists a dimensional constant $C_d > 0$ such that for any $\Omega \in \mathcal{A}$ and any pair of disjoint open subset of Ω, Ω^+ and Ω^- verifying $\lambda_2(\Omega) \geq \max(\lambda_1(\Omega^+), \lambda_1(\Omega^-))$, it holds for all $k \geq 1$*

$$|\lambda_k(\Omega^+ \cup \Omega^-) - \lambda_k(\Theta)| \leq C_d\, k^{2+\frac{4}{d}}\, \lambda_2(\Omega)^{\frac{1}{2}} (\lambda_2(\Omega) - \lambda_2(\Theta))^{\frac{1}{2}}.$$

Remark 4. Notice that such couples of subsets Ω^+, Ω^- indeed do exist. For instance, we can choose them to be the two nodal sets of the second eigenfunction of Ω, but other choices could be more relevant in specific situations.

2. Preliminary results

To fix the terminology, below we recall some results useful along the proofs.

We begin with the definition and a few properties of the torsional rigidity and of the torsion function. For any $\Omega \in \mathcal{A}$ let us define the torsional rigidity of Ω, $T(\Omega)$ as

$$T(\Omega) = \max_{u \in H^1_0(\Omega)} \int_\Omega 2\,u - \int_\Omega |\nabla u|^2. \tag{11}$$

The function which achieves the maximum is the unique weak solution in $H^1_0(\Omega)$ of

$$\begin{cases} -\Delta u = 1 & \text{in } \Omega, \\ u = 0 & \text{in } \partial\Omega. \end{cases} \tag{12}$$

We denote the solution by w_Ω and call it torsion function. Then

$$T(\Omega) = \int_\Omega 2\,w_\Omega - \int_\Omega |\nabla w_\Omega|^2 = \int_\Omega w_\Omega.$$

We recall the Saint-Venant inequality (see for instance [10] and [11])

$$\forall \Omega \in \mathcal{A}, \qquad T(\Omega) \leq T(B), \tag{13}$$

with equality if and only if Ω is a ball, up to a set of zero capacity.

And we also recall the Talenti inequality (see [18, Theorem 2]) that states that the supremum of the torsion function is also maximized at the ball

$$\forall \Omega \in \mathcal{A}, \qquad \|w_\Omega\|_{L^\infty(\Omega)} \leq \|w_B\|_{L^\infty(B)}. \tag{14}$$

We recall from [7] the following estimate, which shows that one can control the difference of the eigenvalues by the difference between torsional rigidities, between an open set Ω and a subset Ω'.

Lemma 5. *For any $\Omega \in \mathcal{A}$ and $\Omega' \subset \Omega$ open, it holds for all $k \geq 1$,*

$$0 \leq \frac{1}{\lambda_k(\Omega)} - \frac{1}{\lambda_k(\Omega')} \leq exp(1/(4\pi))\, k\, \lambda_k(\Omega)^{d/2}(T(\Omega) - T(\Omega')).$$

We also quote a result by Cheng and Yang (see [9, Theorem 3.1]) which comes as an improvement of an older less general result form Ashbaugh and Benguria [1] that gives a control of the maximal ratio between the k-th and the first eigenvalues of a given open set.

Lemma 6. *For any $\Omega \in \mathcal{A}$, and for all $k \geq 1$,*

$$\lambda_k(\Omega) \leq \left(1 + \frac{4}{d}\right) k^{\frac{2}{d}}\, \lambda_1(\Omega).$$

We also recall the Kohler-Jobin inequality for any open set of finite measure $\Omega \subset \mathbb{R}^d$, see [12] and [13],

$$\lambda_1(\Omega)^{\frac{d+2}{2}} T(\Omega) \geq \lambda_1(B)^{\frac{d+2}{2}} T(B). \tag{15}$$

Notice here that the exponent $\frac{d+2}{2}$ is such that the functional $\Omega \mapsto \lambda_1(\Omega)^{\frac{d+2}{2}} T(\Omega)$ is scale invariant.

We present below some results that are specific to the second eigenvalue λ_2. They play a fundamental role in the proof of Theorem 1. For $l \geq 3$, such results are not available, so that it is not possible to adapt the proof of Theorem 1 to higher eigenvalues λ_l with $l \geq 3$.

We start by recalling a decomposition result for the second eigenvalue proved in [5, Lemma 3.1].

Lemma 7. *Let $\Omega \in \mathcal{A}$. There exists two disjoint subsets $\Omega^+, \Omega^- \subset \Omega$ such that*

$$\lambda_2(\Omega) = \max\{\lambda_1(\Omega^+), \lambda_1(\Omega^-)\}.$$

Note that if Ω is connected, then Ω^+ and Ω^- correspond to the nodal sets of an eigenfunction u_2 associated to $\lambda_2(\Omega)$, namely $\Omega^+ = \{u_2 > 0\}$, $\Omega^- = \{u_2 < 0\}$. In that case $\lambda_2(\Omega) = \lambda_1(\Omega^+) = \lambda_1(\Omega^-)$. If Ω is disconnected, the construction of Ω^+, Ω^- may (or not) use two different connected components, possibly making that the difference $\Omega \setminus (\Omega^+ \cup \Omega^-)$ is a set of positive measure.

At the end this section we recall a Kohler-Jobin type inequality for the second eigenvalue

$$\forall \Omega \subset \mathcal{A}, \qquad \lambda_2(\Omega)^{\frac{d+2}{2}} T(\Omega) \geq \lambda_2(\Theta)^{\frac{d+2}{2}} T(\Theta). \tag{16}$$

This inequality comes as a consequence of [2, Lemma 6], yet we give it here a simple proof for the sake of completeness.

Proof. Take Ω in $\mathcal{A}$, by Lemma 7 there exists two disjoint open subsets of Ω, Ω^+ and Ω^- such that $\lambda_2(\Omega) = \max\{\lambda_1(\Omega^+), \lambda_1(\Omega^-)\}$ and also, by inclusion, $T(\Omega) \geq T(\Omega^+) + T(\Omega^-)$. Then, take $B^{1/2}$ a ball of volume $\omega_d/2$ we have by the Kohler-Jobin inequality (15),

$$\lambda_1(\Omega^+)^{\frac{d+2}{2}} T(\Omega^+) \geq \lambda_1(B^{1/2})^{\frac{d+2}{2}} T(B^{1/2}),$$
$$\lambda_1(\Omega^-)^{\frac{d+2}{2}} T(\Omega^-) \geq \lambda_1(B^{1/2})^{\frac{d+2}{2}} T(B^{1/2}).$$

Then, we can compute

$$\begin{aligned}
\lambda_2(\Omega)^{\frac{d+2}{2}} T(\Omega) &\geq \max\{\lambda_1(\Omega^+), \lambda_1(\Omega^-)\}^{\frac{d+2}{2}} \left(T(\Omega^+) + T(\Omega^-)\right) \\
&\geq \lambda_1(\Omega^+)^{\frac{d+2}{2}} T(\Omega^+) + \lambda_1(\Omega^-)^{\frac{d+2}{2}} T(\Omega^-) \\
&\geq 2\,\lambda_1(B^{1/2})^{\frac{d+2}{2}} T(B^{1/2}) \\
&= \lambda_2(\Theta)^{\frac{d+2}{2}} T(\Theta).
\end{aligned}$$

□

Remark 8. Note that a Kohler-Jobin type inequality for higher order eigenvalues would have its own interest. However, it is very unclear even for $l = 3$ if a minimizer does exist for $\lambda_3(\Omega)^{\frac{d+2}{2}} T(\Omega)$ in the class $\mathcal{A}$.

Remark 9. Yet the case of the associated maximization problem for the first eigenvalue

$$\max\left\{\lambda_1(\Omega)^p T(\Omega) \;\middle|\; \Omega \in \mathcal{A}\right\},$$

has been studied in [6] by Briani, Buttazzo and Guarino Lo Bianco, where they proved existence of an optimal shape for $p > p_1 > 1$ large enough. In [8], it was proved that the ball is the maximizer for $p > p_2$ for some p_2 larger than p_1.

3. Proof of Theorem 1

The proof of Theorem 1 is based on a sharp approximation of the torsion of a set by the torsion of the best overlapping of the set by two equal balls of half measure. If such an approximation is rough in general, one can reasonably expect that the approximation becomes sharp if the set has a second eigenvalue close to its minimum, on the set Θ. The set Θ is the union of two disjoint balls of half of the measure of Ω, and their positionning has to be done in a optimal manner, we will actually choose them such that they are close to a minimum for the *Fraenkel 2-asymmetry*.

In [8, Lemma 3.2], the authors got a control of the difference of the torsional rigidies between a set and the intersection of the set with a ball. The following lemma is in the same spirit, and gives control of the difference of the torsional rigidities between Ω and $\Omega \cap \Theta$ by the volume of the symmetric difference.

Lemma 10. *Let $\Omega \in \mathcal{A}$. Then*

$$T(\Omega) - T(\Omega \cap \Theta) \leq \left(\frac{1}{d} + \frac{1}{2^{\frac{2}{d}} d^2}\right) |\Omega \setminus \Theta|.$$

Proof. For simplicity, we denote by w the torsion function of Ω and v the torsion function of Θ. Let us denote

$$u = \min(w, v).$$

Notice that $u \in H_0^1(\Omega \cap \Theta)$, then

$$T(\Omega \cap \Theta) \geq \int_{\Omega\cap\Theta} 2u - \int_{\Omega\cap\Theta} |\nabla u|^2.$$

so,

$$\begin{aligned}T(\Omega)-T(\Omega\cap\Theta)+\int_{\Omega\setminus\Theta}|\nabla w|^2 &\le \int_{\Omega\setminus\Theta}2w+\int_{\Omega\cap\Theta}2(w-u)+\int_{\Omega\cap\Theta}(|\nabla u|^2-|\nabla w|^2)\\ &=\int_{\Omega\setminus\Theta}2w+\int_{\Omega\cap\Theta}2(w-v)_+-\int_{\Omega\cap\Theta}\nabla(u+w)\cdot\nabla(w-u).\end{aligned}$$

For any real numbers a,b it holds $(a+b)(a-b)\le 2b(a-b)$, then we deduce

$$T(\Omega)-T(\Omega\cap\Theta)+\int_{\Omega\setminus\Theta}|\nabla w|^2\le\int_{\Omega\setminus\Theta}2w+\int_{\Omega\cap\Theta}2(w-v)_+-\int_{\Omega\cap\Theta}2\nabla u\cdot\nabla(w-v)_+.$$

Notice that on $\Omega\cap\Theta$,

$$\nabla u\cdot\nabla(w-v)_+=\nabla\cdot(w-v)_+\nabla v+(w-v)_+,$$

so by divergence theorem,

$$\begin{aligned}T(\Omega)-T(\Omega\cap\Theta)+\int_{\Omega\setminus\Theta}|\nabla w|^2 &\le \int_{\Omega\setminus\Theta}2w-\int_{\partial(\Omega\cap\Theta)}(w-v)_+\nabla v\cdot\boldsymbol{n}\,\mathrm{d}\mathcal{H}^{d-1}\\ &=\int_{\Omega\setminus\Theta}2w+\int_{\partial\Theta}(w-v)_+|\nabla v|\,\mathrm{d}\mathcal{H}^{d-1}\\ &\le 2\,|\Omega\setminus\Theta|\,\sup_{\Omega}w+\sup_{\partial\Theta}|\nabla v|\int_{\partial\Theta}w\,\mathrm{d}\mathcal{H}^{d-1}.\end{aligned}$$

Next, we use the trace inequality

$$\int_{\partial\Theta}w\,\mathrm{d}\mathcal{H}^{d-1}\le\int_{\Omega\setminus\Theta}|\nabla w|$$

followed by the Jensen inequality to obtain

$$T(\Omega)-T(\Omega\cap\Theta)\le\left(2\sup_{\Omega}w+\sup_{\partial\Theta}|\nabla v|^2\right)|\Omega\setminus\Theta|.$$

It now only remains to express the constant. We know for the ball $B(0,1)$ the expression of its torsion function w_B:

$$w_B(x)=\frac{1-|x|^2}{2d}.$$

Then, since Θ is a disjoint union of two balls of radii $2^{-\frac{1}{d}}$, $\sup_{\partial\Theta}|\nabla v|^2=\frac{1}{2^{\frac{2}{d}}d^2}$ and from the Talenti inequality (14), we get $\sup_\Omega w\le\frac{1}{2d}$. □

We may now tackle the proof of Theorem 1.

Proof. (of Theorem 1) Along the proof we will denote by C_d a purely dimensional constant which may increase from line to line.

Take Ω in $\mathcal{A}$ and Θ a disjoint union of two balls of volume $\omega_d/2$ and fix $k\ge 1$.

First, if $\lambda_2(\Omega)\ge 2\,\lambda_2(\Theta)$, we know from the spectral inequality given by Lemma 6 that

$$|\lambda_k(\Omega)-\lambda_k(\Theta)|\le k^{\frac{2}{d}}\left(1+\frac{4}{d}\right)(\lambda_2(\Omega)+\lambda_2(\Theta)),$$

and then we can compute

$$\begin{aligned}|\lambda_k(\Omega)-\lambda_k(\Theta)| &\le 2\left(1+\frac{4}{d}\right) k^{\frac{2}{d}}\,\lambda_2(\Omega)\\ &\le 2^{\frac{d+2}{d+1}}\left(1+\frac{4}{d}\right) k^{\frac{2}{d}}\,\lambda_2(\Omega)^{1-\frac{1}{d+1}}\left(\lambda_2(\Omega)-\lambda_2(\Theta)\right)^{\frac{1}{d+1}}\\ &\le C\,k^{2+\frac{4}{d}}\,\lambda_2(\Omega)^{1-\frac{1}{d+1}}\left(\lambda_2(\Omega)-\lambda_2(\Theta)\right)^{\frac{1}{d+1}}.\end{aligned}$$

Assume now $\lambda_2(\Omega)<2\,\lambda_2(\Theta)$, by Lemma 5, since $\Omega\cap\Theta\subset\Omega$ and $\Omega\cap\Theta\subset\Theta$, we obtain

$$\begin{aligned}\frac{1}{\lambda_k(\Omega)}-\frac{1}{\lambda_k(\Omega\cap\Theta)} &\le exp(1/(4\pi))\,k\,\lambda_k(\Omega)^{d/2}(T(\Omega)-T(\Omega\cap\Theta)),\\ \frac{1}{\lambda_k(\Theta)}-\frac{1}{\lambda_k(\Omega\cap\Theta)} &\le exp(1/(4\pi))\,k\,\lambda_k(\Theta)^{d/2}(T(\Theta)-T(\Omega\cap\Theta)).\end{aligned}$$

We combine this with Lemma 6 and the minimality of Θ for λ_2 to get

$$\begin{aligned}\frac{1}{\lambda_k(\Omega)}-\frac{1}{\lambda_k(\Omega\cap\Theta)} &\le exp(1/(4\pi))\,k^2\left(1+\frac{4}{d}\right)^{\frac{d}{2}}\lambda_2(\Omega)^{d/2}\big(T(\Omega)-T(\Omega\cap\Theta)\big),\\ \frac{1}{\lambda_k(\Theta)}-\frac{1}{\lambda_k(\Omega\cap\Theta)} &\le exp(1/(4\pi))\,k^2\left(1+\frac{4}{d}\right)^{\frac{d}{2}}\lambda_2(\Omega)^{d/2}\big(T(\Theta)-T(\Omega\cap\Theta)\big).\end{aligned}$$

And we deduce, adding these two inequalities

$$\left|\frac{1}{\lambda_k(\Omega)}-\frac{1}{\lambda_k(\Theta)}\right| \le exp(1/(4\pi))\,k^2\left(1+\frac{4}{d}\right)^{\frac{d}{2}}\lambda_2(\Omega)^{d/2}\big(T(\Omega)+T(\Theta)-2\,T(\Omega\cap\Theta)\big),$$

which rewrites, using Lemma 6 again, as

$$|\lambda_k(\Omega)-\lambda_k(\Theta)|\le C_d k^{2+\frac{4}{d}}\,\lambda_2(\Omega)^{2+\frac{d}{2}}\big(T(\Omega)+T(\Theta)-2\,T(\Omega\cap\Theta)\big).$$

It remains to estimate the factor $T(\Omega)+T(\Theta)-2\,T(\Omega\cap\Theta)$, to do so, we rewrite it

$$T(\Theta)-T(\Omega)+2\big(T(\Omega)-T(\Omega\cap\Theta)\big).$$

In Lemma 10 we have already computed

$$T(\Omega)-T(\Omega\cap\Theta)\le\left(\frac{1}{d}+\frac{1}{2^{\frac{2}{d}}\,d^2}\right)|\Omega\setminus\Theta|.$$

Choosing rightfully the two balls that compose Θ we get

$$|\Omega\setminus\Theta|\le\omega_d\,\mathcal{F}_2(\Omega).$$

Then we have by the quantitative Krahn-Szegö inequality (10),

$$|\Omega\setminus\Theta|\le C_d\,\lambda_2(\Theta)^{-\frac{1}{d+1}}\left(\lambda_2(\Omega)-\lambda_2(\Theta)\right)^{\frac{1}{d+1}}.$$

Combining the two we get that

$$2\left(T(\Omega)-T(\Omega\cap\Theta)\right)\le C_d\,\lambda_2(\Theta)^{-\frac{1}{d+1}}\left(\lambda_2(\Omega)-\lambda_2(\Theta)\right)^{\frac{1}{d+1}}.$$

Now, from the Kohler-Jobin inequality for the second eigenvalue (16) and since $\lambda_2(\Omega)<2\,\lambda_2(\Theta)$, we obtain

$$\begin{aligned}T(\Theta)-T(\Omega) &\le T(\Omega)\left[\left(\frac{\lambda_2(\Omega)}{\lambda_2(\Theta)}\right)^{\frac{d+2}{2}}-1\right]\\ &\le 2^{\frac{d}{2}}\,\frac{d+2}{2}\,T(\Omega)\,\lambda_2(\Theta)^{-\frac{1}{d+1}}\left[\lambda_2(\Omega)-\lambda_2(\Theta)\right]^{\frac{1}{d+1}}.\end{aligned}$$

Then, from Saint-Venant inequality (13), for B a ball of radius 1, we obtain

$$T(\Theta) - T(\Omega) \leq 2^{\frac{d}{2}} \frac{d+2}{2} T(B)\, \lambda_2(\Theta)^{-\frac{1}{d+1}} \left[\lambda_2(\Omega) - \lambda_2(\Theta)\right]^{\frac{1}{d+1}}$$
$$\leq C_d\, \lambda_2(\Theta)^{-\frac{1}{d+1}} \left[\lambda_2(\Omega) - \lambda_2(\Theta)\right]^{\frac{1}{d+1}}.$$

Finally, we obtain using the upper bound on $\lambda_2(\Omega)$

$$|\lambda_k(\Omega) - \lambda_k(\Theta)| \leq C_d\, k^{2+\frac{4}{d}}\, \lambda_2(\Omega)^{2+\frac{d}{2}}\, \lambda_2(\Theta)^{-\frac{1}{d+1}} \left(\lambda_2(\Omega) - \lambda_2(\Theta)\right)^{\frac{1}{d+1}}$$
$$\leq C_d\, k^{2+\frac{4}{d}}\, \lambda_2(\Omega)^{1-\frac{1}{d+1}} \left(\lambda_2(\Omega) - \lambda_2(\Theta)\right)^{\frac{1}{d+1}}.$$

□

4. Proof of Theorems 2 and 3

The statement of Theorem 2 is a consequence of the more technical Theorem 3. We will prove first Theorem 3 and get, as a consequence, Theorem 2.

The idea is that using the two subsets Ω^+ and Ω^- we are able to benefit from the sharp estimation (7) that was proved for the first eigenvalue and thus obtain the sharp exponent $1/2$.

Proof. (of Theorem 3) Take Ω in $\mathcal{A}$ and Θ a disjoint union of two balls of volume $\omega_d/2$ and fix $k \geq 1$. Now take Ω^+ and Ω^- two disjoint open subsets of Ω such that $\lambda_2(\Omega) \geq \max(\lambda_1(\Omega^+), \lambda_1(\Omega^-))$. We know the existence of such sets by the decomposition lemma 7. Finally consider two disjoint balls B^+ and B^- of respective volumes $|\Omega^+|$ and $|\Omega^-|$, we will specify their position later in the proof.

First, as in the proof Theorem 1, if $\lambda_2(\Omega) \geq 2\,\lambda_2(\Theta)$, we know from Lemma 6 that

$$|\lambda_k(\Omega^+ \cup \Omega^-) - \lambda_k(\Theta)| \leq k^{\frac{2}{d}} \left(1 + \frac{4}{d}\right) (\lambda_2(\Omega^+ \cup \Omega^-) + \lambda_2(\Theta)),$$
$$\leq k^{\frac{2}{d}} \left(1 + \frac{4}{d}\right) (\lambda_2(\Omega) + \lambda_2(\Theta)),$$

and then, by a similar computation,

$$|\lambda_k(\Omega^+ \cup \Omega^-) - \lambda_k(\Theta)| \leq C_d\, k^{2+\frac{4}{d}}\, \lambda_2(\Omega)^{\frac{1}{2}} \left(\lambda_2(\Omega) - \lambda_2(\Theta)\right)^{\frac{1}{2}}.$$

Now, consider $\lambda_2(\Omega) < 2\,\lambda_2(\Theta)$, then

$$|\lambda_k(\Omega^+ \cup \Omega^-) - \lambda_k(\Theta)| \leq |\lambda_k(\Omega^+ \cup \Omega^-) - \lambda_k(B^+ \cup B^-)| + |\lambda_k(B^+ \cup B^-) - \lambda_k(\Theta)|.$$

For each term, as in the previous proof, by Lemma 5 we obtain the two estimates

$$|\lambda_k(\Omega^+ \cup \Omega^-) - \lambda_k(B^+ \cup B^-)$$
$$\leq C\, k^{2+\frac{4}{d}}\, \lambda_2(\Omega)^{2+\frac{d}{2}} \left(T(\Omega^+ \cup \Omega^-) + T(B^+ \cup B^-) - 2\,T((\Omega^+ \cup \Omega^-) \cap (B^+ \cup B^-))\right),$$

and,

$$|\lambda_k(B^+ \cup B^-) - \lambda_k(\Theta)| \leq C\, k^{2+\frac{4}{d}}\, \lambda_2(\Omega)^{2+\frac{d}{2}} \big(T(B^+ \cup B^-) + T(\Theta) - 2\,T((B^+ \cup B^-) \cap \Theta)\big).$$

We start by working with the first term, since the torsion is an increasing functional for the inclusion of sets,

$$T((\Omega^+ \cup \Omega^-) \cap (B^+ \cup B^-)) \geq T(\Omega^+ \cap B^+) + T(\Omega^- \cap B^-),$$

and we only have to estimate separately the two terms

$$T(\Omega^+) + T(B^+) - 2\,T(\Omega^+ \cap B^+),$$
$$T(\Omega^-) + T(B^-) - 2\,T(\Omega^- \cap B^-).$$

We know, from [8, Theorem 1.1], applied to each set Ω^+ and Ω^- that we can choose the two balls B^+ and B^- such that

$$T(\Omega^+) + T(B^+) - 2\,T(\Omega^+ \cap B^-) \le C(\lambda_1(\Omega^+) - \lambda_1(B^+))^{\frac{1}{2}},$$
$$T(\Omega^-) + T(B^-) - 2\,T(\Omega^- \cap B^-) \le C(\lambda_1(\Omega^-) - \lambda_1(B^-))^{\frac{1}{2}},$$

then, since $\lambda_2(\Omega) = \min(\lambda_1(\Omega^+), \lambda_1(\Omega^-))$, we deduce that

$$T(\Omega^+) + T(B^+) - 2\,T(\Omega^+ \cap B^+) \le C(\lambda_2(\Omega) - \lambda_1(B^+))^{\frac{1}{2}},$$
$$T(\Omega^-) + T(B^-) - 2\,T(\Omega^- \cap B^-) \le C(\lambda_2(\Omega) - \lambda_1(B^-))^{\frac{1}{2}}.$$

Now, if $|\Omega^+|, |\Omega^-| \le \omega_d/2$ we have since the eigenvalue is decreasing for the inclusion of sets that $\lambda_1(B^+), \lambda_1(B^-) \ge \lambda_2(\Theta)$ and we can conclude. So it only remains to consider the case $|\Omega^-| < \omega_d/2 < |\Omega^+|$, we still have the same argument working for $|\Omega^-|$ and we claim that

$$\lambda_2(\Omega) - \lambda_1(B^+) \le 2\,(\lambda_2(\Omega) - \lambda_2(\Theta)), \tag{17}$$

then we have showed that

$$|\lambda_k(\Omega^+ \cup \Omega^-) - \lambda_k(B^+ \cup B^-)| \le Ck^{2+\frac{4}{d}}\,\lambda_2(\Omega)^{2+\frac{d}{2}}\,(\lambda_2(\Omega) - \lambda_2(\Theta))^{\frac{1}{2}},$$

and using the upper bound $\lambda_2(\Omega) < 2\,\lambda_2(\Theta)$, we deduce

$$|\lambda_k(\Omega^+ \cup \Omega^-) - \lambda_k(B^+ \cup B^-)| \le Ck^{2+\frac{4}{d}}\,\lambda_2(\Omega)^{\frac{1}{2}}\,(\lambda_2(\Omega) - \lambda_2(\Theta))^{\frac{1}{2}}.$$

To prove this claim we just compute, for B any ball of radius 1, and any $0 < t < 1$,

$$\lambda_2(\Theta) = \left(\frac{1}{2}\right)^{-\frac{2}{d}} \lambda_1(B) \le \frac{1}{2}\,(t^{-\frac{2}{d}} + (1-t)^{-\frac{2}{d}})\,\lambda_1(B),$$

choosing $t = \frac{|B^+|}{|B|}$ we obtain $t^{-\frac{2}{d}}\,\lambda_1(B) = \lambda_1(B^+)$ and $(1-t)^{-\frac{2}{d}}\,\lambda_1(B) \le \lambda_1(B^-) \le \lambda_2(\Omega)$. And then, we get

$$\lambda_2(\Theta) \le \lambda_2(\Omega) + \frac{1}{2}(\lambda_1(B^+) - \lambda_2(\Omega)),$$

which proves the claim.

We now consider the second term. Note that since the torsion functional and the eigenvalues are invariant by translation of each connected components, we can consider that B^+ and B^- are disjoint. Once again we need to consider two cases depending on the volume. If $|B^+| \le \omega_d/2$ and $|B^-| \le \omega_d/2$, we choose the two balls B^1 and B^2 composing Θ such that $B^+ \subset B^1$ and $B^- \subset B^2$ and then we are left with estimating the term

$$T(B^1) - T(B^+) + T(B^2) - T(B^-).$$

Then, by Kohler Jobin inequality (15)

$$\begin{aligned} T(B^1) - T(B^+) + T(B^2) - T(B^-) &\le C_d(\lambda_1(B^+) - \lambda_2(\Theta) + \lambda_1(B^-) - \lambda_2(\Theta)) \\ &\le 2\,C_d(\lambda_2(\Omega) - \lambda_2(\Theta)). \end{aligned}$$

If we have $|B^-| < \omega_d/2 < |B^+|$ by choosing $B^1 \subset B^+$ and $B^2 \supset B^-$, we are left with estimating $T(B^+) - T(B^-)$ and once again, using the Kohler Jobin inequality and inequality (17) we obtain,

$$T(B^+) - T(B^-) \le 2\,C(\lambda_2(\Omega) - \lambda_2(\Theta)),$$

and then we have showed

$$|\lambda_k(B^+ \cup B^-) - \lambda_k(\Theta)| \le C_d\,k^{2+\frac{4}{d}}\,\lambda_2(\Omega)^{2+\frac{d}{2}}\,(\lambda_2(\Omega) - \lambda_2(\Theta)),$$

and once again using the upper bound $\lambda_2(\Omega) < 2\,\lambda_2(\Theta)$, we deduce

$$|\lambda_k(B^+ \cup B^-) - \lambda_k(\Theta)| \le C_d\,k^{2+\frac{4}{d}}\,2^{2+\frac{d-1}{2}}\,\lambda_2(\Theta)^{2+\frac{d}{2}}\lambda_2(\Omega)^{\frac{1}{2}}\,(\lambda_2(\Omega) - \lambda_2(\Theta))^{\frac{1}{2}}.$$

Wich we rewritte

$$|\lambda_k(B^+ \cup B^-) - \lambda_k(\Theta)| \leq C_d\, k^{2+\frac{4}{d}}\, \lambda_2(\Omega)^{\frac{1}{2}}\, (\lambda_2(\Omega) - \lambda_2(\Theta))^{\frac{1}{2}},$$

and this concludes the proof □

From Theorem 3 we can now deduce Theorem 2 wich comes as a direct corollary.

Proof. (of Theorem 2) Consider $\Omega \in \mathcal{A}$ and choose two disjoints subsets Ω^+, Ω^- of Ω satisfying the eigenvalue condition $\lambda_2(\Omega) \geq \max(\lambda_1(\Omega^+), \lambda_1(\Omega^-))$. Since the eigenvalues are decreasing for the inclusion of sets, it holds for all k in $\mathbb{N}$,

$$\lambda_k(\Omega) \leq \lambda_k(\Omega^+ \cup \Omega^-).$$

Then we can deduce

$$(\lambda_k(\Omega) - \lambda_k(\Theta))_+ \leq \big(\lambda_k(\Omega^+ \cup \Omega^-) - \lambda_k(\Theta)\big)_+ \leq \big|\lambda_k(\Omega^+ \cup \Omega^-) - \lambda_k(\Theta)\big|.$$

Finally, we can apply Theorem 3 and obtain

$$(\lambda_k(\Omega) - \lambda_k(\Theta))_+ \leq C_d\, k^{2+\frac{4}{d}}\, \lambda_2(\Omega)^{\frac{1}{2}}\, (\lambda_2(\Omega) - \lambda_2(\Theta))^{\frac{1}{2}}.$$

□

5. Further remarks and open questions

As a conclusion, a few remarks are in order.

Remark 11 (Sharpness of the exponent $\frac{1}{2}$)**.** We have claimed that $\frac{1}{2}$ should be the sharp exponent in our inequality. By this we mean that one cannot find an exponent $\alpha^* > \frac{1}{2}$ such that for all $\Omega \in \mathcal{A}$

$$|\lambda_k(\Omega) - \lambda_k(\Theta)| \leq C\, k^{2+\frac{4}{d}}\, \lambda_2(\Omega)^{1-\alpha^*}\, (\lambda_2(\Omega) - \lambda_2(\Theta))^{\alpha^*}, \tag{18}$$

and that the inequality should be true for $\alpha^* = \frac{1}{2}$.

Even though we are not able to prove that the inequality is true for the exponent $\frac{1}{2}$ Theorem 2 gives good hope that it should be, and we can actually show that we cannot expect a better exponent.

Suppose that (18) is true for some $\alpha^* > \frac{1}{2}$, for any Ω and k. We can take $\Omega \in \mathcal{A}$. Denoting by $\Omega^{1/2} \sqcup \Omega^{1/2}$ the disjoint union of two copies of Ω rescaled to have measure $\omega_d/2$ we have for any k,

$$|\lambda_k(\Omega) - \lambda_k(B)| = 2^{-\frac{1}{d}}\, \left|\lambda_{2k}(\Omega^{1/2} \sqcup \Omega^{1/2}) - \lambda_{2k}(\Theta)\right|.$$

We then deduce from inequality (18)

$$|\lambda_k(\Omega) - \lambda_k(\Theta)| \leq 2^{-\frac{1}{d}} C\, (2k)^{2+\frac{4}{d}}\, \lambda_2(\Omega^{1/2} \sqcup \Omega^{1/2})^{1-\alpha^*}\, (\lambda_2(\Omega^{1/2} \sqcup \Omega^{1/2}) - \lambda_2(\Theta))^{\alpha^*}.$$

Which implies that

$$|\lambda_k(\Omega) - \lambda_k(\Theta)| \leq 2^{2+\frac{4}{d}}\, C\, k^{2+\frac{4}{d}}\, \lambda_1(\Omega)^{1-\alpha^*}\, (\lambda_1(\Omega) - \lambda_1(B))^{\alpha^*}.$$

Since this inequality would then be true for any $\Omega \in \mathcal{A}$ and $k \geq 1$, it is in contradiction with the fact that $\frac{1}{2}$ is the sharp exponent for the stability inequality by the first eigenvalue near the ball. In the setting of the first eigenvalue, the sharpness of the exponent $\frac{1}{2}$ can actually be easyly observed. Since the second eignevalue is not critical on the ball B, one can find a smooth volume preserving deformation of the ball Ω_t such that $\lambda_2(\Omega_t) - \lambda_2(B) = c_2\, t + o(t)$ and by minimality of the ball for the first eigenvalue, $\lambda_1(\Omega_t) - \lambda_1(B) = c_1\, t^2 + o(t^2)$, where c_1 and c_2 are two positve constants. Then having

$$|\lambda_2(\Omega_t) - \lambda_2(B)| \leq C(\lambda_1(\Omega_t) - \lambda_1(B))^{\beta},$$

for t small implies $\beta \leq \frac{1}{2}$.

Remark 12 (General estimate with exponent $\frac{1}{2}$)**.** We proved in Theorem 2, for any $\Omega \in \mathcal{A}$

$$\big(\lambda_k(\Omega) - \lambda_k(\Theta)\big)_+ \leq C\, k^{2+\frac{4}{d}}\, \lambda_2(\Omega)^{\frac{1}{2}}\, (\lambda_2(\Omega) - \lambda_2(\Theta))^{\frac{1}{2}}. \tag{19}$$

which is of course non trivial only when $\lambda_k(\Omega) - \lambda_k(\Theta) \geq 0$. To obtain the full inequality with exponent $\frac{1}{2}$, it remains to prove an inequality of the form

$$\lambda_k(\Theta) - \lambda_k(\Omega) \leq C_d\, k^{2+\frac{4}{d}}\, \lambda_2(\Omega)^{\frac{1}{2}}\, (\lambda_2(\Omega) - \lambda_2(\Theta))^{\frac{1}{2}}. \tag{20}$$

Taking into account Theorem 3 and Lemma 5 this reduces to the following question:
Is there a dimensionnal constant C_d such that for any $\Omega \in \mathcal{A}$, statisfying $\lambda_2(\Omega) \leq 2\,\lambda_2(\Theta)$, the following quantitative Krahn-Szegö inequality holds

$$\frac{\lambda_2(\Omega) - \lambda_2(\Theta)}{\lambda_2(\Theta)} \geq C_d \left(\frac{T(\Omega) - T(\Omega_1 \cup \Omega_2)}{T(\Omega)}\right)^2, \tag{21}$$

for some Ω_1 and Ω_2 disjoint subsets of Ω satisfying $max\{\lambda_1(\Omega_1), \lambda_1(\Omega_2)\} \leq \lambda_2(\Omega)$?

Remark 13 (About the restriction on the sets Ω)**.** In order to simplify the notations in the proofs, we fixed the volume of Ω to ω_d but all the inequalities we proved are actually scale invariant and remain true for any Ω of finite measure as long as Θ is chosen with the same volume. As for the restriction to open sets, by continuity for the γ-convergence of the eigenvalues, the theorems remain true for any quasi open set of finite measure.

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