RIGIDITY AND FUNCTIONAL PROPERTIES OF $\mathrm{BD}_{\mathrm{dev}}(\Omega)$

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Abstract. We provide a structural analysis of the space of functions of bounded deviatoric deformation, $\mathrm{BD}_{\mathrm{dev}}$, which arises in models of plasticity and fluid mechanics. The main result is the identification of the annihilator and a rigidity theorem for $\mathrm{BD}_{\mathrm{dev}}$-maps with constant polar vector in the wave cone characterizing the structure of singularities for such maps. This result, together with an explicit kernel projection operator, enables an iterative blow-up procedure for relaxation, homogenization, and integral representation problems, allowing for integrands with explicit dependence on u as well as $\mathcal{E}_d u$. Our approach overcomes several difficulties as compared to the BD case, in particular due to the lack of invariance of $\mathcal{E}_d$ under orthogonalization of the polar directions.

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1. Introduction

Rigidity for functions satisfying some differential properties in the sense of measures is encompassed in the general study of the fine properties of these functions, and serves as a powerful tool to tackle several problems in mathematical physics and continuum mechanics. In the field of calculus of variations, rigidity serves in particular to determine blow-ups at some singular points (BV Cantor points for instance), that are a crucial tool for homogenization and relaxation purposes (see [22]). In particular, in phase transition problems, image segmentation, material design, or variational frameworks in solid mechanics involving multi-scale phenomena (like elaso-plasticity), field concentration often occurs on lower-dimensional sets, such as cracks and dislocations. Typical examples of functions with differential constraints are the BV functions (see [6]), whose gradient is a finite Radon measure, or functions of bounded deformations, denoted as BD (see [5, 32, 49]), whose symmetric gradient is constrained to be a finite Radon measure. This latter space is typically of use in linearized elasticity models of fracture mechanics or in infinitesimal elasto-plasticity. However, in the field of plasticity or fluid mechanics (where the shear deformation is the relevant kinematical variable used to define the viscous stresses), shear deformations are favored and the corresponding functional space is the space of bounded deviatoric deformations named as $\mathrm{BD}_{\mathrm{dev}}$. Note that prominent phenomena such as crystal plasticity is also mainly governed by shear efforts due to the formation and motion of dislocations in shear planes, as plasticity is not affected by traction or compression efforts in the material.

Despite their significance, the fine properties and structural results for the space $\mathrm{BD}_{\mathrm{dev}}$ remain much less explored as compared to the classical settings of BV or BD. The aim of the present work is to address this gap by providing a thorough analysis of $\mathrm{BD}_{\mathrm{dev}}$. The main novelty of this manuscript is the characterization of $\mathrm{BD}_{\mathrm{dev}}$ maps with constant polar vector lying in the *wave cone*, which is useful in the theory of relaxation and homogenization via iterated blow-ups, as in [21], [22], [27]. The wave cone is the set of directions along which the *annihilitor* operator looses ellipticity, therefore the importance of its determination and study. For instance, the curl is the annihilator of the gradient, and the incompatibility (or Saint-Venant operator), the annihilator of the symmetric gradient. One of the content of this work is to determine and study the annihilator of the deviatoric symmetric gradient. This will allow us to establish some of the fine properties and to identify the wave-cone, crucial for the rigidity result. The role of rigidity in integral representation results will also be discussed.

A notable advantage of this approach with a view to homogenization, based on iterated blow-ups through rigidity, kernel projection, and integral representation, is that it does not restrict the energy to depend solely on the deviatoric strain $\mathcal{E}_d u$, but also accommodates integrands with explicit dependence on u itself, up to an L^∞- bound.

1.1. The deviatoric operator in continuum mechanics and material science.

The deviatoric operator is of particular importance because it extracts the component of a tensor that is solely responsible for changes in shape, independently of volumetric deformation. This deviatoric part is the most physically relevant contribution in several phenomena in continuum mechanics and materials science (see [40] for a reference on continuum mechanics). For a second-order tensor T (such as stress, strain, velocity gradient, etc.), the deviatoric operator is defined as

$$\operatorname{dev} T = T - \frac{\operatorname{tr} T}{n} \mathrm{Id},$$

where n is the space dimension, and tr stands for the trace operator; in general, in the applications, T is symmetric; in mechanics, $n = 2$ or 3, in general relativity, space-time dimension is $n = 4$.

This operator splits a tensor into a spherical (also called volumetric) part, given by $\frac{\operatorname{tr} T}{n}\mathrm{Id}$, which accounts for volume changes, and a deviatoric part, $\operatorname{dev}\ T$, which accounts for shape changes. This decomposition is not arbitrary: volumetric and deviatoric deformations arise from distinct physical mechanisms, and materials often respond differently to each. In mechanics, a key empirical observation is that plastic deformation and yielding in deformable solids are governed by deviatoric rather than hydrostatic stress. For example, the von Mises yield criterion depends solely on the deviatoric stress, since pure hydrostatic pressure does not induce yielding in ductile metals. Moreover, shear-driven processes such as slip and dislocation motion are intrinsically deviatoric. Indeed, the deviatoric operator isolates precisely the stress component responsible for permanent deformation, making it the appropriate tool for the study of plasticity in ductile materials.

From an energy perspective, the deviatoric operator isolates the part of a tensor that stores or dissipates energy associated with distortion, rather than pure volumetric change. Some examples of applications where the governing equations naturally decouple into deviatoric and spherical parts arise in Navier-Stokes equations: pressure vs shear stress; in linear elasticity: bulk modulus vs shear modulus, and in particular in Cosserat and membrane models; this decomposition also applies to viscoplastic flow rules. For all these models, the dissipation is a functional that depends only on the deviatoric strain.

Another crucial aspect is that mathematicaly it is a projection operator (onto the space of traceless tensors), and in particular it is linear, idempotent ($\operatorname{dev}(\operatorname{dev} T) = \operatorname{dev} T$), and self-adjoint, making this orthogonal operator particularly well-suited for variational formulations, finite element methods, and spectral decompositions.

In the following, we provide a precise mathematical definition of the deviatoric operator (Subsection 1.2), along with selected variational problems in which it appears as the main—or even exclusive—component, in two, three, and four dimensions (Subsection 1.3). So far, these problems have been rarely studied due to the inherent mathematical challenges posed by the deviatoric operator. One objective of this work is to tackle these challenges and clarify important properties of the operator. For example, purely distortional solids do not resist uniform expansion or contraction, so the associated operators are degenerate on volume-changing modes: in the case of the deviatoric operator, these modes correspond to rigid body motions combined with non-uniform dilation and shear/stress modes. We will examine these modes in detail when addressing the characterization of the kernel of the operator (Subsection 1.5).

Rigidity and fine properties are introduced in Subsections 1.4 and 1.6. In Subsection 1.7 we provide the structure of the proof to achieve integral representation results, grounded on the rigidity property, the main result of the present work.

1.2. **The space of bounded deviatoric deformations.** For a general $(n \times n)$-matrix-valued tensor T, two classical orthogonal decompositions apply:

$$T = \underbrace{\frac{T+T^t}{2}}_{\text{sym } T} + \underbrace{\frac{T-T^t}{2}}_{\text{skew } T}, \qquad T = \underbrace{T - \frac{\operatorname{tr} T}{n}\mathrm{Id}}_{\text{dev } T} + \frac{\operatorname{tr} T}{n}\mathrm{Id}.$$

The first decomposition is into the symmetric and skew (or anti-symmetric) part, with T^t denoting the transpose of T. The second decomposition is into the shear part and the volumetric part, with $\operatorname{tr} T$ denoting the trace of the matrix T and Id the $n \times n$ identity matrix. Combining both decompositions, we get the so-called Cartan decomposition of the Lie algebra $\mathfrak{gl}(n)$, i.e., $\mathfrak{gl}(n) = (\mathfrak{sl}(n) \cap \mathrm{Sym}(n)) \oplus \mathfrak{so}(n) \oplus \mathbb{R}\mathrm{Id}$ (see e.g., [13]), namely

$$T = \operatorname{sym} T + \operatorname{skew} T = \operatorname{dev} \operatorname{sym} T + \operatorname{skew} T + \frac{\operatorname{tr} T}{n}\mathrm{Id}.$$

Take now $T = Du$ for $u : \mathbb{R}^n \to \mathbb{R}^n$ a vector-valued function of bounded variation; one obtains

$$Du = \mathcal{E}u + \operatorname{skew} Du = \operatorname{dev} \mathcal{E}u + \operatorname{anticurl} u + \frac{\operatorname{div} u}{n}\mathrm{Id},$$

where $\mathcal{E}u := \operatorname{sym} Du = \left(\frac{Du+Du^t}{2}\right)$, $\operatorname{anticurl} u := \operatorname{skew} Du$, thus showing a decomposition of the deformation gradient Du into an infinitesimal rotation term $\operatorname{anticurl} u$, a volumetric term $\frac{\operatorname{div} u}{n}\mathrm{Id}$, and a pure shear term $\operatorname{dev} \mathcal{E}u$. It turns out that several features in continuum mechanics, such as fluid mechanics or linearized elasticity, but also in general relativity, involve the deviatoric operator

$$\mathcal{E}_d u := \operatorname{dev} \mathcal{E}u = \mathcal{E}u - \frac{\operatorname{div} u}{n}\mathrm{Id}.$$

$\mathcal{E}_d u$ is defined, in principle, on functions $u \in C^1(\Omega;\mathbb{R}^n)$; but, as for the larger spaces BV and BD, it becomes relevant to introduce the space of functions of *bounded deviatoric deformations*:

$$\mathrm{BD}_{\mathrm{dev}}(\Omega) := \left\{ u \in L^1(\Omega;\mathbb{R}^n) \;\middle|\; \mathcal{E}_d u \in \mathcal{M}(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}_0}) \right\} \subset \mathrm{BD}(\Omega).$$

where $\mathcal{M}(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}_0})$ is the space of finite *trace-free, symmetric* $\mathbb{R}^{n\times n}$-valued Radon measures, where $\mathbb{R}^{n\times n}$ are the $(n\times n)$ matrices of $\mathbb{R}^n$ (refer to the beginning of 2 for all the notations).

1.3. **Energy minimization problems on** $\mathrm{BD}_{\mathrm{dev}}$**.** As far as variational problems are concerned, in conventional elasticity or elasto-plasticity, but also soft matter and material engineering, it is classical to consider the stored elastic energy

$$\int_\Omega W(\mathcal{E}u)dx,$$

and to consider the additive decomposition

$$W(\mathcal{E}u) = W_{\mathrm{shear}}(\mathrm{dev}\ \mathcal{E}\mathrm{u}) + \mathrm{W}_{\mathrm{bulk}}(\mathrm{div}\ \ \mathrm{u}),$$

where the kinematical variables u, dev $\mathcal{E}$u and div u obey some regularity properties, and the densities W_{shear}, W_{bulk} some growth properties. Here, the energy density splits into two complementary parts: the energy due to shape changes, or *shear*, writing as $W_{\mathrm{shear}} := GW_0$, where G is the *shear modulus*, and the energy due to volume changes, $W_{\mathrm{bulk}} := KW_{\mathrm{vol}}$, where K is the *bulk modulus*; in the quadratic case, $W_0(E) := E\cdot E$ and $W_{\mathrm{vol}}(v) := \frac{1}{2}v^2$. Let us recall that G is a measure of the elastic shear stiffness of an elastic material (i.e., the resistance to forces acting parallel to a surface), K describes the material's response to uniform hydrostatic pressure, while the *Poisson's ratio*, whose symbol is ν, is a measure of the expansion or contraction of a material in directions perpendicular to the direction of loading; for an isotropic, linear elastic material, Poisson's ratio can be expressed in terms of G and K as $\nu = \frac{3K-2G}{6K+2G}$. For stable isotropic linear elastic materials, Poisson's ratio satisfies $-1 \le \nu \le 0.5$. Most solids exhibit $0.2 \le \nu \le 0.4$ (metals typically $\nu \approx 0.3$). Auxetic materials have $\nu < 0$ and expand laterally under tension. Note that as $\nu \to -1$, we have $K \to 0$.

However, simply setting $W_{\mathrm{bulk}} = 0$ (or $K = 0$) is not a reasonable assumption for any conventional material, as it would imply a vanishing bulk (compressibility) modulus—an unphysical property. Nevertheless, this assumption becomes meaningful as a limit case, and it has attracted growing interest in the context of metamaterial design. In particular, it is relevant to the fabrication of tunable-stiffness materials, whose effective elastic properties—such as bulk modulus, shear modulus, Poisson's ratio, or anisotropy—can be adjusted on demand. These modern artificial solids or fluids, including cellular solids, colloidal crystals, and polymer foams, display a wide range of unusual mechanical behaviors, thereby opening new avenues for advanced industrial applications [23, 31, 41, 14, 51, 36]. Many of these properties are related to materials with negative Poisson ratios [39, 46], like auxetic materials, which, in contrast to classical materials, exhibit a reverse deformation mechanism.

The wide diversity of mechanical properties observed in both natural and engineered materials can be effectively illustrated by plotting the bulk modulus K against the shear modulus G. Materials with a small Poisson's ratio are more easily compressed than sheared, corresponding to a small ratio K/G, whereas materials with a large Poisson's ratio strongly resist compression relative to shear, resulting in a large K/G. In the limiting regime $K/G \ll 1$—which corresponds to Poisson's ratios approaching the lower bound $\nu \to -1$—materials become extremely compressible and exhibit pronounced auxetic behavior. In this case, the material can change volume readily while strongly resisting changes in shape. Representative examples include polymeric re-entrant foams with tailored microstructures [39, 43] and 3D-printed mechanical metamaterials, with applications ranging from soft robotics to biomedical implants.

A prototypical example is the class of so-called Cosserat (micromorphic) continuum ([24, 34]; for recent references [15, 47]), where the energy depends on both the macroscopic displacement u and an independent micro-distortion field $P(x) \in \mathbb{R}^{3\times 3}$ that describe the substructure of the material which can rotate, stretch, shear and shrink (in some simple Cosserat models, the micro-distortion is a pure infinitesimal rotation). Introduce the elastic distortion $e := \nabla u - P$ and $\varepsilon :=$ sym e. Decomposing the relative strain ε into volumetric and deviatoric parts, $\varepsilon = \underbrace{\frac{1}{3}(\operatorname{tr}\varepsilon)I}_{\text{volumetric}} + \underbrace{\varepsilon^{\text{dev}}}_{\text{deviatoric}}$, the elastic energy in the isotropic setting can be expressed as [47, Eq. (2.16)]

$$\int_\Omega \Big[\underbrace{G\,\|\varepsilon^{\text{dev}}\|^2 + \frac{K}{2}(\operatorname{tr}\varepsilon)^2}_{\text{elastic energy}} + \underbrace{G_c\,\|\text{sym }P\|^2}_{\text{microstrain self-energy}} + \underbrace{\ell^2\,\|\text{curl }P\|^2}_{\text{dislocation energy}} \Big]\,dx,$$

where G_c is the Cosserat couple modulus, and ℓ is a characteristic length scale associated with the density of dislocations curl P (if P is a pure rotation, the curl is usually replaced by the rotation gradient). In the auxetic limit where one reaches the (effective) Poisson limit $\nu \to -1$, the bulk modulus $K \to 0$, rendering the volumetric strain energetically unconstrained. Consequently, the energy is dominated by deviatoric distortions and rotational modes, which capture the *mechanism-driven* lateral expansion characteristic of auxetic metamaterials. Crucially, these rotational mechanisms are accompanied by macroscopic volume changes: as the cells rotate, the distance between neighboring cell centers changes, leading to a global expansion or contraction of the structure without energetic cost. Consequently, volumetric (hydrostatic) strain becomes energetically unconstrained, while shape-changing (deviatoric) deformations are still resisted by the stiffness of the structure. At the continuum level, this behavior corresponds to a vanishing bulk modulus with finite shear modulus, and yields an effective Poisson ratio approaching -1. Applications to concrete material are typically granular media, where grains can rotate independently, shear is transmitted along chains, and therefore bulk deformation is dominated by shear and rotation effects.

Summarizing, these three-dimensional examples arise for *mechanism-dominated* material, meaning that the macroscopic deformation is governed primarily by geometric mechanisms of the microstructure, not by the elasticity of the base material. Thus, certain deformation modes cost very little or zero energy (like rotations, and hence volumetric strain), while other modes, typically shear/deviatoric distortions, still require energy. For this class of materials, one is lead to study homogenization problems with energy densities like $W_\varepsilon(\mathcal{E}u) = G(\varepsilon)W_0(\mathcal{E}_d u) + K(\varepsilon)W_{\text{vol}}(\text{div }\ u)$ where $K(\varepsilon)/G(\varepsilon) \to 0$ as $\varepsilon \to 0$. At the limit, no bound on div u exists, and for linear growth W_0 as $C^{-1}|\xi| \le W_0(\xi) \le C|\xi|$, the relevant space is thus BD_{dev}.

Examples can also be found in other dimensions. Let us briefly mention two variational problems appearing in models in dimension two and four. In theoretical physics, in dimension four, the problem of minimizing

$$\int_\Omega (W_{\text{shear}}(\mathcal{E}_d u) - f\cdot u)\,dx$$

is relevant in general relativity (GR–specifically for the study of black holes' momentum and spin), for the so-called "momentum constraint" (see [26] and the references therein; the canonical reference for the deviatoric operator and the functional–analytic structure of the momentum constraint is [52]). Note that in GR the deviatoric operator is called conformal Killing operator, and, by definition, the so-called Killing vectors lie in the kernel of the deviatoric operator; physically, they are related to the symmetries of the black hole.

A membrane is a two-dimensional body embedded in $\mathbb{R}^3$. It cannot sustain transverse normal stress and has negligible bending stiffness. The in-plane (membrane) strain tensor is $\varepsilon_s = \frac{1}{2}\left(\nabla_s u + \nabla_s u^T\right)$, $\varepsilon_s \in \mathbb{R}^{2\times 2}$, and where ∇_s denote the surface gradient. A typical isotropic membrane energy density is

$$W_s = G_s\,\|\varepsilon_s^{\text{dev}}\|^2 + \frac{K_s}{2}\,(\operatorname{tr}\boldsymbol{\varepsilon}_s)^2,$$

where G_s is the surface shear modulus, and K_s is the surface bulk modulus. The latter measures the energetic cost of changing the local surface area of the membrane. In particular, large K_s means that area change is strongly penalized, i.e., the membrane is (almost) inextensible, whereas small K_s implies that area change costs little energy and hence the membrane is easily extensible.

In many physical membranes (e.g. lipid bilayers, soap films), local area change is accommodated by thickness variation or molecular rearrangement. Hence, taking $K_s \to 0$, the energy reduces to

$$W_s = G_s \, \|\varepsilon_s^{\mathrm{dev}}\|^2.$$

In this model, area-preserving distortions cost energy, but pure area change is energetically free: the membrane resists shear but not dilation. This yields a purely deviatoric variational problem that remains well posed because the eliminated volumetric mode corresponds to a physically irrelevant degree of freedom.

In terms of functional spaces, one has $\mathrm{BD}_{\mathrm{dev}} \supset \mathrm{BD} \supset \mathrm{BV}$, and, as we said, the mechanics of the phenomena under analysis do not always allow one to consider the full deformation gradient Du as a kinematical variable on its own in any mathematical model, i.e., one often has no control on all components of the full gradient. Now, suppose for instance we have a functional $\mathcal{F} : \mathrm{S}(\Omega) \times \mathrm{Bor}(\Omega) \to [0, +\infty]$ (here $\mathrm{Bor}(\Omega)$ is the family of Borel sets of Ω) defined on the functional space $\mathrm{S}(\Omega) = \mathrm{BV}(\Omega), \mathrm{BD}(\Omega)$, or $\mathrm{BD}_{\mathrm{dev}}(\Omega)$, lower semicontinuous on S, and satisfying the bound

$$0 \leq \mathcal{F}(u) \leq C|\mu|(\Omega), \tag{1.1}$$

for some $C > 0$, and where $\mu = Du, \mathcal{E}u$, or $\mathcal{E}_d u$, respectively, with $|\mu|$ the total variation measure of μ. Then, according to the physical problem under study, the analysis of $\mathcal{F}$ and in particular the question of the existence of an integral representation for $\mathcal{F}$ arises naturally in each S (see Subsection 1.7 for additional details on this topic). This issue is prominent if, for instance, $\mathcal{F}$ arises from a relaxation process, since then it will be lower semicontinuous, even if its integral expression is not known, or even if it would exist. Also, in homogenization processes, one might wish to pass to the limit with respect to a small scale parameter present in the deviatoric part of the energy $W_{\mathrm{bulk}}(\mathcal{E}_d u)$ and eventually determine the integral form of the limit. For BV, the pioneering work can be found in [18], while in BD it was recently achieved by the authors in [21]. It thus became natural to raise the question of an integral representation for $\mathcal{F}$ as defined in $\mathrm{BD}_{\mathrm{dev}}$. An entire subsection is dedicated to this question, see Subsection 1.7.

The strategy developed in [21], and summarized in Subsection 1.7, goes through an iterative blow-up procedure that is based on rigidity properties of $\mathrm{BD}_{\mathrm{dev}}$-maps with constant polar vectors and a specific projection operator onto $\mathrm{Ker}(\mathcal{E}_d)$ appearing in the Korn-Poincaré inequality. Actually, as shown in [22], where the authors refined a double blow-up technique for this purpose, these two ingredients are enough to tackle homogenization and integral representation problems.

1.4. **Rigidity result.** We here provide a rigidity property for maps with constant polar in $\mathrm{BD}_{\mathrm{dev}}$ and a specific projection operator $\mathcal{R} : L^1 \to \mathrm{Ker}(\mathcal{E}_d)$. With these two results at hand, we will establish the two main ingredients needed to solve integral representation and homogenization problems in $\mathrm{BD}_{\mathrm{dev}}$.

Given a map $u \in \mathrm{BD}_{\mathrm{dev}}(K)$, for a compact convex set K, suppose that

$$\mathcal{E}_d u = M\mu, \quad \mu \in \mathcal{M}(K; \mathbb{R}^+) \tag{1.2}$$

for μ a real-valued non-negative Radon measure and $M \in \mathbb{R}^{n\times n}_{\mathrm{sym}_0}$ a constant matrix. Then what can be said about u? Actually, for the purposes of the present analysis it is not necessary for M to be any matrix, but just a matrix in the *wave cone* $\Lambda_{\mathcal{A}}$: a specific subspace of $\mathbb{R}^{n\times n}_{\mathrm{sym}_0}$ depending on the differential operator $\mathcal{A}$ annihilating $\mathcal{E}_d$ (i.e., such that $\mathcal{A}\mathcal{E}_d = 0$). Indeed, with classical tools (as done in [28]), it is easy to show that if $M \notin \Lambda_{\mathcal{A}}$, then u must be actually C^∞ and thus the relevant case from the technical point of view is the case $M \in \Lambda_{\mathcal{A}}$. Not only this: given $u \in \mathrm{BD}_{\mathrm{dev}}(\Omega)$, we can consider the Radon-Nikodým decomposition

$$\mathcal{E}_d u = \frac{\mathrm{d}\mathcal{E}_d u}{\mathrm{d}\mathcal{L}^n}\mathcal{L}^n + \mathcal{E}_d^s u,$$

where $\mathcal{L}^n$ stands for the n-dimensional Lebesgue measure, and $\mathcal{E}_d^s$ the singular part of the measure with respect to $\mathcal{L}^n$ (cf. Corollary 3.7). Blowing up u at some Lebesgue continuous point x, we have convergence to an affine map given by $y \mapsto e_d(u)(x)y + u(x)$ (here $e_d(u) := e(u) - \frac{\operatorname{div}\, u}{n}\mathrm{Id}$, $e(u) = \frac{\nabla u + \nabla u^t}{2}$, where $\nabla u(x)$ is the *approximate differential of u at x* that exists $\mathcal{L}^n$-a.e.). But for integral representation and homogenization, a characterization of the blow-ups on $\mathrm{spt}(\mathcal{E}^s_d u)$ is also required. A celebrated result (valid for a generic $\mathcal{A}$-free measure μ, with $\mathcal{A}$ a linear constant-coefficient differential operator) due to De Philippis and Rindler [27] implies that

$$\frac{\mathrm{d}\mathcal{E}^s_d u}{\mathrm{d}|\mathcal{E}^s_d u|} \in \Lambda_{\mathcal{A}} \quad \text{for } |\mathcal{E}^s_d u|\text{-a.e. } x \in \Omega\ .$$

If we then consider a blow-up at $x \in \mathrm{spt}(\mathcal{E}^s_d u)$ on a specific compact convex set K:

$$u_{K,\varepsilon,x}(y) = \frac{u(x+\varepsilon y) - \mathcal{R}_K[u](y)}{\frac{|\mathcal{E}_d u|(K_\varepsilon(x))}{|K|\varepsilon^{n-1}}}, \tag{1.3}$$

(the projection operator $\mathcal{R}_K$ and the rescaled set K_ε are defined in Theorem 1.4) we see that $u_{K,\varepsilon,x} \to v$ in L^1 as $\varepsilon \to 0$, where

$$\mathcal{E}_d v = \frac{\mathrm{d}\mathcal{E}^s_d u}{\mathrm{d}|\mathcal{E}^s_d u|}(x)|\mathcal{E}^s_d v|.$$

Thus, in order to characterize the blow-ups, we need to study the structure of solutions to (1.2) in the case when $M \in \Lambda_{\mathcal{A}}$.

Note that the annihilator of a differential operator is not unique, since if $\mathcal{A}$ annihilates $\mathcal{E}_d$ then $\partial_i \mathcal{A}$ still annihilates $\mathcal{E}_d$. But it is true that $\Lambda_{\mathcal{A}} \subset \Lambda_{\partial_i \mathcal{A}}$. In particular, to constrain as much as possible the polar of the singular part, one wants to find the annihilator $\mathcal{A}$ of $\mathcal{E}_d$ of lowest order. We are able to compute such an annihilator, and with this differential operator at hand we can focus our rigidity result only on $M \in \Lambda_{\mathcal{A}}$.

Remark 1.1. A major challenge in the present setting, compared to the classical BD case, arises from the fact that the deviatoric operator $\mathcal{E}_d$ does not behave well under changes of variables that orthogonalize the vectors a and b. Such changes are routinely used in the BD case to simplify the analysis, but are not available here. This fundamental obstacle is one of the main sources of technical complexity in our proof.

We recall the notation:

$$a \odot b := \frac{a \otimes b + b \otimes a}{2}.$$

Theorem 1.2. *Let $n \geq 3$. There exists an annihilator $\mathcal{A}$ for $\mathcal{E}_d$ of order 4 for which it holds that*

$$\Lambda_{\mathcal{A}} = \left\{ a \odot b - \frac{(a \cdot b)}{n}\mathrm{Id} \;\middle|\; a, b \in \mathbb{R}^n \right\}.$$

Moreover, for any $u \in \mathrm{BD}_{\mathrm{dev}}(K)$ satisfying

$$\mathcal{E}_d u = \left(a \odot b - \frac{(a \cdot b)}{n}\mathrm{Id} \right) \mu \tag{1.4}$$

for some $a, b \in \mathbb{R}^n$, $\mu \in \mathcal{M}(K;\mathbb{R}^+)$, one of the following two cases holds:

1) *If a and b are not parallel, then there exist two functions $\psi_1, \psi_2 \in C^\infty(\mathbb{R})$ and $v \in \langle a, b\rangle^\perp$ such that*

$$u(x) = \psi_1(x \cdot a)b + \psi_2(x \cdot b)a + Q(x) + L(x) \tag{1.5}$$

for some $L \in \mathrm{Ker}(\mathcal{E}_d)$ and for some homogeneous third-order degree polynomial Q solving

$$\mathcal{E}_d Q = \left(a \odot b - \frac{(a \cdot b)}{n}\mathrm{Id} \right) \left((v \cdot x) + \eta (a \cdot x)(b \cdot x) - \vartheta \sum_{j=3}^{n} (x \cdot w_j)^2 \right) \tag{1.6}$$

where $\eta, \vartheta \in \mathbb{R}$, $v \in \langle a, b\rangle^\perp$ and $\{w_3, \ldots, w_n\}$ is an orthonormal basis of $\langle a, b\rangle^\perp$;

2) *If $b = \lambda a$ then there exist functions $F \in \mathrm{BV}_{loc}(\mathbb{R})$, $\{P_j\}_{j=2}^n \subset W^{1,1}_{loc}(\mathbb{R})$ with $P'_j \in \mathrm{BV}_{loc}$, $G \in W^{1,1}_{loc}(\mathbb{R})$ with $G' \in \mathrm{BV}_{loc}(\mathbb{R})$ such that*

$$\begin{aligned} u =& F(a\cdot x)a + \left(\sum_{j=2}^n P'_j(a\cdot x)(w_j\cdot x) + \frac{(w_j\cdot x)^2}{2}G'(a\cdot x)\right) a \\ &- \sum_{j=2}^n ((w_j\cdot x)G(a\cdot x) + P_j(a\cdot x))w_j + \varrho Q(x) + L(x) \end{aligned} \tag{1.7}$$

for $\{w_2,\dots,w_n\}$ an orthonormal basis of $a^\perp$, for some $L \in \mathrm{Ker}(\mathcal{E}_d)$, $\varrho \in \mathbb{R}$ and for some homogeneous third-order degree polynomial Q solving

$$\mathcal{E}_d Q = \lambda\left(a\odot a - \frac{|a|^2}{n}\mathrm{Id}\right)\left((w_2\cdot x)^2 - (w_3\cdot x)^2\right). \tag{1.8}$$

Moreover, if $n \geq 4$ then $\varrho = 0$.

The operator $\mathcal{A}$ is explicitly computed in Proposition 3.2.

In both cases, 1) and 2), there is a one-dimensional part with BV regularity, a part orthogonal to a, b with $W^{1,1}$ regularity, and a polynomial part. The main difference between the BD and the $\mathrm{BD}_{\mathrm{dev}}$ case is the polynomial part, which is non-trivial for $\mathrm{BD}_{\mathrm{dev}}$ maps and arises from the fact that (1.6), (1.8) have non-trivial solutions (explicitly computable).

Remark 1.3. The constraint $n \geq 3$ is actually quite important, since for $n = 2$ the operator $\mathcal{E}_d$ lacks a fundamental property called $\mathbb{C}$-ellipticity, required for several structural properties (as, for instance, the existence of traces, as shown in [19]). The difference between $n \geq 4$ and $n = 3$, as made explicit in the condition on ϱ in part 2) of Theorem 1.2, is actually quite common throughout the proofs. Heuristically speaking, the number of differential equations satisfied by a u solving (1.4) depends on the dimension: for $n = 3$ there are simply fewer conditions on u, making the proof of the rigidity structure more challenging. For $n = 3$, when a is parallel to b, we lose an additional equation in (1.4), resulting in the presence of the polynomial part. These considerations seem to strongly indicate that, for $n = 2$, there might be too few equations to constrain the solution of (1.2) for $M \in \Lambda_{\mathcal{A}}$ to have such one-dimensional BV parts.

1.5. **Kernel projection.** While the rigidity result presented above constitutes the main analytical cornerstone of our approach, the development of a comprehensive blow-up strategy in the space $\mathrm{BD}_{\mathrm{dev}}$ requires a further structural ingredient. In particular, although the following result has a somewhat less pronounced impact compared to the rigidity theorem, it remains an essential tool in iterative blow-up procedure as clarified in Subsection 1.7. Here and in the sequel we denote by $P(K)$ the *distributional perimeter of* K (i.e. $P(K) = \mathcal{H}^{n-1}(\partial K)$ for regular sets) and we intend K to be a *set of finite perimeter* (i.e. $P(K) < +\infty$). We refer to [44] for specific properties of such an object.

Theorem 1.4. *Let $n \geq 3$ and K be a center-symmetric convex body*[1]*. Set*

$$\mathrm{s}_K[u] := -\frac{1}{(n-1)|K|}\int_{\partial K} u^t \,\mathrm{d}D^{\mathrm{Tan}}\nu_K(y) \tag{1.9}$$

$$\mathrm{A}_K[u] := \frac{1}{2|K|}\int_{\partial K} (u\otimes\nu_K - \nu_K\otimes u)\,\mathrm{d}\mathcal{H}^{n-1}(x) \tag{1.10}$$

$$\gamma_K[u] := \frac{1}{n|K|}\int_{\partial K}(u\cdot\nu_K)\,\mathrm{d}\mathcal{H}^{n-1}(y) \tag{1.11}$$

$$\mathrm{b}_K[u] := \frac{1}{P(K)}\int_{\partial K} u\,\mathrm{d}\mathcal{H}^{n-1}(y) + \tau_K \mathrm{s}_K[u], \tag{1.12}$$

[1] Meaning that K is invariant under reflection through some point $c \in \mathbb{R}^n$. In other words, there exists a center of symmetry of K.

where

$$\tau_K := \frac{1}{P(K)}\int_{\partial K}\left(\frac{|y-\mathrm{bar}(K)|^2}{2}\mathrm{Id} \,-(y-\mathrm{bar}(K))\otimes(y-\mathrm{bar}(K))\right)\,\mathrm{d}\mathcal{H}^{n-1}(y)$$

and $\mathrm{bar}(K)=\fint_K x\,\mathrm{d}x$ *Let us define the map* $\mathcal{R}_K:\mathrm{BD}_{\mathrm{dev}}(\Omega)\to\mathrm{Ker}(\mathcal{E}_d)$ *as*

$$\begin{aligned}\mathcal{R}_K[u](y) &:= (\mathrm{A}_K[u]+\gamma_K[u]\mathrm{Id}\,)\,(y-\mathrm{bar}(K))\\ &\qquad +(\mathrm{s}_K[u]\cdot(y-\mathrm{bar}(K)))(y-\mathrm{bar}(K))-\mathrm{s}_K[u]\frac{|y-\mathrm{bar}(K)|^2}{2}+\mathrm{b}_K[u]\end{aligned}\tag{1.13}$$

is a linear, bounded operator satisfying $\mathcal{R}_K(L)=L$ *for all* $L\in\mathrm{Ker}(\mathcal{E}_d)$*. As a consequence,*

$$\|u-\mathcal{R}_K[u]\|_{L^1(\Omega)}\le c|\mathcal{E}_d u|(\Omega)\quad \textit{for all } u\in\mathrm{BD}_{\mathrm{dev}}(\Omega).\tag{1.14}$$

Note that the Poincaré inequality is actually well-known for a whole series of operators (see [30, Theorem 3.7]), holding for any linear, bounded kernel projection operator. Here we simply restrict ourselves to computing a specific kernel operator.

1.6. **Fine properties.** Once the annihilator is computed, as a corollary we can derive a structure theorem for $\mathcal{E}_d u$ in the spirit of the one holding for BV. In particular, (see Section 4) we can prove that $|\mathcal{E}_d u|\ll\mathcal{H}^{n-1}$, and obtain the specific structure of the jump part (see also [19]):

$$\mathcal{E}_d u = e_d(u)\mathcal{L}^n+\left([u]\odot\nu_u-\frac{([u]\cdot\nu_u)}{n}\mathrm{Id}\right)\mathcal{H}^{n-1}\llcorner J_u+\left(a(x)\odot b(x)-\frac{(a(x)\cdot b(x))}{n}\mathrm{Id}\right)|\mathcal{E}_d^c u|$$

where $|\mathcal{E}^c_d u|$ is the Cantor part and $a,b:\mathbb{R}^n\to\mathbb{R}^n$ are Borel-measurable vector fields. We recall that for $u\in L^1_{loc}$ the so-called jump set J_u is defined as the set of points x for which there exists a triplets $(u^+(x),u^-(x),\nu_u(x))$ such that $u^+\neq u^-$, $\nu_u\in\mathbb{S}^{n-1}$ and

$$0=\lim_{r\to0+}\fint_{B_r^{\pm}(x)}|u(y)-u^{\pm}(x)|\,\mathrm{d}y$$

where

$$B_r^-(x):=\{y\in B_r(x)\mid y\cdot\nu_u(x)\le0\},\qquad B_r^+(x):=\{y\in B_r(x)\mid y\cdot\nu_u(x)\ge0\}.$$

Let us spend a few words about a major difference that we actually encounter when looking at the decomposition of $\mathcal{E}_d u$ compared to the decomposition of $\mathcal{E}u$ or $\mathcal{D}u$. Calling S_u the set of points where u is not *approximately continuous* (i.e., x is not a Lebesgue point for u), it is known that BD and BV-maps charge this set almost all on J_u, namely $|\mathcal{E}u|(S_u\setminus J_u)=0$ for BD, and $\mathcal{H}^{n-1}(S_u\setminus J_u)=0$ for BV. It is actually an open problem whether the same property holds for $\mathrm{BD}_{\mathrm{dev}}$ maps: $|\mathcal{E}_d u|(S_u\setminus J_u)=0$? The slicing technique developed in [7], as a generalization of [5], seems not to work for $\mathcal{E}_d u$ due to a specific missing property of the operator: $\mathcal{E}_d u$ does not satisfies a specific *mixing property* ([7, Property $(\mathfrak{m})$ in Theorem 1]). This property is crucial in ensuring that the operator can be decomposed into one-dimensional slicing. Due to its structure, it seems not true in general that $\mathcal{E}_d$ can be recovered as the integration of one-dimensional slices, actually the only tool available to prove approximate continuity at Cantor points.

As a corollary of Theorem 1.4, by applying an argument similar to the one developed in [5, 42], we can rewrite the integral operators defining $\mathrm{s}_K,\mathrm{A}_K,\gamma_K$ and b_K (in (6.3)-(6.6)) as nonlocal interaction integrals against $\mathcal{E}_d u$. This allows us to control the infinitesimal behavior of these quantities, providing quasi-continuity $|\mathcal{E}_d u|$-a.e. for $\mathrm{BD}_{\mathrm{dev}}$ functions, a weaker notion than approximate continuity.

We therefore report this very natural conjecture about the fine properties of $\mathrm{BD}_{\mathrm{dev}}$ functions.

Conjecture 1.5. *For all* $u\in\mathrm{BD}_{\mathrm{dev}}(\Omega)$ *it holds that* $|\mathcal{E}_d u|(S_u\setminus J_u)=0$.

The conjecture might be true only for $n\ge3$, since in $n=2$ the operator fails to be $\mathbb{C}$-elliptic, although it is not clear how important this property is for the $|\mathcal{E}_d u|$-a.e. approximate continuity.

1.7. **The role of Theorems 1.2 and 1.4 for integral representation results.** In recent decades, following the pioneering work of [18], the quest for an integral representation of local functionals, over variationally and mechanically relevant functional spaces, has become one of the most powerful tools in the modeling process (here a - definetely partial - list of some topic-related contributions [2, 3, 4, 12, 11, 18, 17, 20, 25, 45]. In particular, with regard to the case of $\mathrm{BD}_{\mathrm{dev}}$ (see also [21] and [33] for the BD case), in this framework we consider an abstract functional $\mathcal{F} : \mathrm{BD}_{\mathrm{dev}}(\Omega) \times \mathcal{O}(\Omega) \to \mathbb{R}$ satisfying the following assumptions.

(H1) Lower semicontinuity. For every $A \in \mathcal{O}(\Omega)$ the map

$$u \mapsto \mathcal{F}(u, A)$$

is sequentially lower semicontinuous with respect to $L^1(A; \mathbb{R}^n)$ convergence.

(H2) Linear growth and coercivity in $\mathcal{E}_d u$. There exists a constant $C \geq 1$ such that for every $A \in \mathcal{O}(\Omega)$ and every $u \in \mathrm{BD}_{\mathrm{dev}}(\Omega)$ one has

$$\frac{1}{C} |\mathcal{E}_d u| (A) \leq \mathcal{F}(u, A) \leq C \left(\mathcal{L}^n(A) + |\mathcal{E}_d u|(A) \right).$$

(H3) Measure property and locality. For every $u \in \mathrm{BD}_{\mathrm{dev}}(\Omega)$, the set function $A \mapsto \mathcal{F}(u, A)$ is the restriction to $\mathcal{O}(\Omega)$ of a Radon measure on Ω (denoted again by $\mathcal{F}(u, \cdot)$), and $\mathcal{F}$ is local in the sense that if $u = v$ $\mathcal{L}^n$-a.e. on A, then

$$\mathcal{F}(u, A) = \mathcal{F}(v, A).$$

(H4) Kernel invariance. For every Lipschitz set $A \subset\subset \Omega$, every $u \in \mathrm{BD}_{\mathrm{dev}}(\Omega)$, and every $R \in \ker(\mathcal{E}_d) \cap W^{1,\infty}(A; \mathbb{R}^n)$ one has

$$\mathcal{F}(u + R; A) = \mathcal{F}(u; A).$$

We then ask whether the functional $\mathcal{F}$ admits an integral representation of the form

$$\mathcal{F}(u; A) = \int_A f(x, e_d(u))\, \mathrm{d}x + \int_A g\left(x, \frac{\mathrm{d}\mathcal{E}_d^s u}{|\mathcal{E}_d^s u|}(x)\right) \mathrm{d}|\mathcal{E}_d^s u|(x). \tag{1.15}$$

The consequences of such a characterization are quite broad: the identification of Γ-limits over the considered space of functions, the identification of the homogenized functionals and the identification of relaxed local energies.

As a matter of fact, if one combines the ideas emerging from the work [18] with the *iterative blow-up technique* (see [28]), one can develop a specific technique (that the author started to implement in [21] in the BD setting) to derive an integral representation as in (1.15), a technique that relies only on the following two ingredients: a) a well-defined functional space, with the sought fine properties (and this is typically ensured by the $\mathbb{C}$-ellipticity of the underlying differential operator—in our case $\mathcal{E}_d$ is $\mathbb{C}$-elliptic for $n \geq 3$); b) a strong rigidity structure for functions with constant polar, i.e. $\mathcal{E}_d u = P\nu$ implies some one-dimensionality of u (and this is precisely the role of our main contribution).

We briefly indicate how an integral representation can be obtained in our setting. First, a *cell problem* is defined:

$$\mathrm{m}(u; A) := \inf \left\{ \mathcal{F}(v; A) \mid v \in \mathrm{BD}_{\mathrm{dev}}(A),\ v = u \text{ on } \partial A \right\}.$$

Afterwards, by abstract (general) measure theory, it is shown that

$$\lim_{\varepsilon \to 0} \frac{\mathrm{m}(u; Q_\varepsilon(x))}{\mu(Q_\varepsilon(x))} = \lim_{\varepsilon \to 0} \frac{\mathcal{F}(u; Q_\varepsilon(x))}{\mu(Q_\varepsilon(x))}$$

for $\mu = \mathcal{L}^n + |\mathcal{E}_d^s u|$. The argument in this part is not very sensitive to the differential operator $\mathcal{E}_d$ used, and the tools already developed in [18], [33] are quite enough to secure this step for all operators satisfying H2. Also, m turns out to be continuous with respect to strict convergence in $\mathrm{BD}_{\mathrm{dev}}$.

After this, by blowing up at points x which are Lebesgue points of $e_d(u)$, and exploiting approximate differentiability, we can characterize

$$f(x, e_d(u)(x)) = \lim_{\varepsilon\to 0} \frac{\mathcal{F}(u; Q_\varepsilon(x))}{\mathcal{L}^n(Q_\varepsilon(x))} \tag{1.16}$$

by means of the cell function, since f is indeed defined as

$$f(x, M) := \limsup_{\varepsilon\to 0} \frac{\mathrm{m}(M(\cdot - x), Q_\varepsilon(x))}{\varepsilon^n}, \tag{1.17}$$

where M is a generic symmetric matrix. This provides a very useful characterization, since it means that the behavior of the energy around regular points can be almost captured by its behavior over the affine functions $M(\cdot - x)$. Due to the kernel invariance of the energy, of course the function f turns out to be also invariant with respect to sums of matrices of the type $A + \beta\mathrm{Id}$ (skew-symmetric plus constant diagonal), thus depending only on the deviatoric part of M, $\mathrm{dev}M$.

To identify the contribution of $\mathcal{F}$ around singular points, we then make use of our refined iterative blow-up techniques [22]. By picking a point $x \in \mathrm{spt}(\mathcal{E}_d^s u)$ and blowing up around x at specific scales, we see that (up to a subsequence)

$$u_{Q,x,\varepsilon}(y) := \frac{|Q|\,\varepsilon^{n-1}}{|\mathcal{E}_d u|(Q_\varepsilon(x))}\Big(u(x+\varepsilon y) - \mathcal{R}_Q[u(x+\varepsilon\cdot)](y)\Big) \;\to\; v(y), \qquad y \in Q,$$

in L^1 as $\varepsilon \to 0$. The blow-up v now enjoys a constant polar property

$$\frac{\mathrm{d}\mathcal{E}_d v}{\mathrm{d}|\mathcal{E}_d v|} = a \otimes_{\mathcal{E}_d} b,$$

thus (by means of our rigidity result) yielding a structure of the kind

$$v = \psi_1(a\cdot y)\, b + \psi_2(b \cdot y)\, a + W^{1,1}\text{-regular terms}$$

(we also consider the parallel case $a = b$, with a unique description). A further blow-up of v at a specific point y in Q yields (by means of an additional application of our rigidity theorem) that (again up to a subsequence)

$$v_{Q,y,\delta}(z) := \frac{|Q|\,\delta^{n-1}}{|\mathcal{E}_d v|(Q_\delta(y))}\Big(v(y+\delta z) - \mathcal{R}_Q[v(y+\delta\cdot)](z)\Big) \;\to\; g(z) = \psi(a\cdot z)b + Mz. \quad z \in Q$$

The role of a, b might turn out to be eventually swapped. Crucially, the profile g consists of a single one-dimensional oscillation $\psi(a\cdot z)b$ plus an affine part Mz. This is the "good" blow-up selected by the iterative procedure: it is canonical enough to be compared with the affine boundary datum in the cell formula, and it is the only place where the constant-polar rigidity enters. Now three interesting phenomena take place:

1) since blow-ups of blow-ups are blow-ups, we can find a specific sequence ε_i such that
$$u_{Q,x,\varepsilon_i}(y) \to \quad g(y) = \psi(a\cdot y)b + My, \qquad y \in Q,$$
in L^1 as $\varepsilon \to 0$.
2) Since $\mathcal{R}_Q(u_{Q,x,\varepsilon_i}) = 0$, we have $\mathcal{R}_Q(g) = 0$, giving a specific coupling relation between (ψ, M);
3) The strict continuity of the cell problem with respect to $\mathrm{BD}_{\mathrm{dev}}$–strict convergence, the coupling relation on (ψ, M) coming from $\mathcal{R}_Q(g) = 0$, and the invariance of m with respect to kernel elements imply that, at the scale of $|\mathcal{E}_d^s u|$, the one-dimensional BV oscillation ψ becomes energetically negligible. More precisely, one is led to
$$\frac{\mathrm{d}\mathcal{F}(u,\cdot)}{\mathrm{d}|\mathcal{E}_d^s u|}(x) = \liminf_{\varepsilon_i\downarrow 0} \frac{\mathrm{m}\left(\frac{|\mathcal{E}_d^s u|(Q_{\varepsilon_i}(x))}{\varepsilon_i^n}\, a \otimes_{\mathcal{E}_d} b(\cdot - x), Q_{\varepsilon_i}(x)\right)}{|\mathcal{E}_d^s u|(Q_{\varepsilon_i}(x))},$$
so that only the affine datum survives, with the diverging amplitude $t_i := \frac{|\mathcal{E}_d^s u|(Q_{\varepsilon_i}(x))}{\varepsilon_i^n} \uparrow +\infty$. In particular, this is exactly the regime in which the cell formula probes the recession of f.

It is now immediate to link this quantity to our f. By means of (1.17) and a specific *one-dimensional* geometric construction (that we will comment on later; see also [18, Lemma 3.11] or [21, Lemma 5.4]) we can show that

$$\frac{f\left(x,\lambda a\otimes_{\mathcal{E}_d} b\right)-f(x,0)}{\lambda}\leq \liminf_{\varepsilon_i\downarrow 0}\frac{\mathrm{m}\left(\frac{|\mathcal{E}_d^s u|(Q_{\varepsilon_i}(x))}{\varepsilon_i^n}a\otimes_{\mathcal{E}_d} b(\cdot-x),Q_{\varepsilon_i}(x)\right)}{|\mathcal{E}_d^s u|(Q_{\varepsilon_i}(x))} \tag{1.18}$$

for all $\lambda\geq 0$. By taking the limit as $\lambda\to+\infty$ we thus get

$$f_\infty(x,a\otimes_{\mathcal{E}_d} b)\leq \frac{\mathrm{d}\mathcal{F}(u,\cdot)}{\mathrm{d}|\mathcal{E}_d^s u|}(x),$$

where f_∞ is the so-called *recession function of* f, defined as

$$f_\infty(x,M):=\limsup_{\lambda\to+\infty}\frac{f(x,\lambda M)}{\lambda}.$$

The other inequality is simply obtained from

$$\begin{aligned}\liminf_{\varepsilon_i\downarrow 0}\frac{\mathrm{m}\left(\frac{|\mathcal{E}_d^s u|(Q_{\varepsilon_i}(x))}{\varepsilon_i^n}a\otimes_{\mathcal{E}_d} b(\cdot-x),Q_{\varepsilon_i}(x)\right)}{|\mathcal{E}_d^s u|(Q_{\varepsilon_i}(x))}&=\liminf_{\varepsilon_i\downarrow 0}\frac{\mathrm{m}\left(t_i a\otimes_{\mathcal{E}_d} b(\cdot-x),Q_{\varepsilon_i}(x)\right)}{t_i\varepsilon_i^n}\\&\leq\liminf_{t_i\uparrow+\infty}\frac{f(x,t_i a\otimes_{\mathcal{E}_d} b)}{t_i}+o(1)\\&=f_\infty(x,a\otimes_{\mathcal{E}_d} b),\end{aligned}$$

(We are cutting corners a bit in explaining the whole machinery: to correctly obtain the result, a double-limit procedure is required, accounting for a shrinking of $Q_{\varepsilon_i}^{\varrho_j}\approx\varepsilon_i\left([-\varrho_j,\varrho_j]\times[-1,1]^{n-1}\right)$ along the direction a (or b), which we neglect here in order to keep the general picture clear, without noisy technical details). So, in the end, we also get

$$\frac{\mathrm{d}\mathcal{F}(u;\cdot)}{\mathrm{d}|\mathcal{E}_d^s u|}(x)=f_\infty\left(x,\frac{\mathrm{d}\mathcal{E}_d^s u}{\mathrm{d}|\mathcal{E}_d^s u|}(x)\right),$$

which, combined with (1.16), gives

$$\mathcal{F}(u;A)=\int_A f(x,e_d(u)(x))\,\mathrm{d}x+\int_A f_\infty\left(x,\frac{\mathrm{d}\mathcal{E}_d^s u}{\mathrm{d}|\mathcal{E}_d^s u|}(x)\right)\,\mathrm{d}|\mathcal{E}_d^s u|(x).$$

All ingredients except the constant-polar rigidity Theorem 1.2, are available in the existing BV/BD integral-representation literature; in a forthcoming paper we will combine these tools with 1.2 to obtain the full representation under (H1)–(H4).

From the technical point of view, the iterative blow-up is not merely a convenient way of extracting a limit: it is the mechanism that *selects* a blow-up profile which is simple enough to be inserted into the cell formula and, at the same time, rich enough to encode the polar. Concretely, after the second blow-up we reduce to profiles of the form

$$g(z)=\psi(a\cdot z)b+Mz,$$

with a *single* one-dimensional oscillation ψ. This is the decisive simplification: because of kernel invariance and the coupling constraint $\mathcal{R}_Q(g)=0$, the cell problem cannot distinguish ψ at the scale relevant for $|\mathcal{E}_d^s u|$, and it "collapses" onto the affine datum $t_i(a\otimes_{\mathcal{E}_d} b)$ with $t_i\uparrow\infty$. This is exactly why the singular density is identified with the recession function f_∞.

There is also a more structural reason why such one-dimensional blow-ups are *necessary* in the comparison step (1.18). To prove (1.18) one must compare the energy of the blow-up sequence with that of an explicit competitor imposing the affine boundary condition $v(x)=t_i(a\otimes_{\mathcal{E}_d} b)x$ on ∂Q, while generating a purely Cantor-type contribution in the interior. All currently available geometric constructions producing Cantor measures with prescribed constant polar are intrinsically one-dimensional

(Cantor staircase, slicing/stratification along $(n-1)$-planes, and variants thereof). Therefore, the reduction to a one-dimensional oscillation in the blow-up is what makes it possible to implement the required "Cantor competitor" and to close the link between the singular density and the recession f_∞.

The present work, which will be combined with the preparatory paper [22] exploring and refining the iterative blow-up procedure over abstract functional spaces, represents therefore the key and most important step in the development of such an integral representation for local functionals over $\mathrm{BD}_{\mathrm{dev}}$. Moreover, by setting up, such an approach, we aim to establish a roadmap for running a solid machinery also for further applications to more general function spaces $\mathrm{BV}^{\mathcal{A}}$—with $\mathcal{A}$ a $\mathbb{C}$-elliptic operator—that relies indeed on rigidity arguments.

Finally, let us underline that the global method approach, by working on blow-ups at the level of u and not at the level of $\mathcal{E}_d u$, allows one to potentially account also for functionals with a mild dependence on u. Note indeed that what cuts out functionals with explicit dependence on u is hypothesis H4. By weakening H4 a bit (as done in [21]), is it possible to preserve the general picture and allow functionals depending also on u. This however requires a stronger knowledge of the fine properties of $\mathrm{BD}_{\mathrm{dev}}$ functions and, in particular: 1) the validity of Conjecture 1.5 for u and 2) a deeper knowledge of the kernel operator $\mathcal{R}_K$ appearing in the blow-up sequence. Whereas the first ingredient is still an open problem, the second ingredient is precisely Theorem 1.4 placing ourselves already in the position to extend the method as soon as fine properties are available. The main difference that this would imply is that in obtaining the function f we would have

$$f(x,u(x),e_d(u)(x)) = \lim_{\varepsilon\to 0}\frac{\mathcal{F}(u;Q_\varepsilon(x))}{\mathcal{L}^n(Q_\varepsilon(x))} = \lim_{\varepsilon\to 0}\frac{\mathrm{m}(u(x)+e_d(u)(x)(\cdot-x);Q_\varepsilon(x))}{\mathcal{L}^n(Q_\varepsilon(x))}$$

with

$$f(x,v,M) := \limsup_{\varepsilon\to 0}\frac{\mathrm{m}(v+M(\cdot-x),Q_\varepsilon(x))}{\varepsilon^n}.$$

Clearly then, when double blowing-up we would need to trace also the point-wise value of $u(x)$, and this is where a finer knowledge of the quantities involved in $\mathcal{R}_Q$ is required, together with the validity of Conjecture 1.5 for the function u. However, let us stress that this is only a technical improvement within the roadmap of the global method, which can be here easily understood in the u-independent case.

So, to summarize, with this work at hand we are in the position to provide two integral representation results on $\mathrm{BD}_{\mathrm{dev}}$: 1) a complete one for functionals independent of u (or satisfying H4, in the abstract setting) and 2) a partial one for u-dependent functionals, holding on all functions u satisfying $|\mathcal{E}_d u|(S_u\setminus J_u)=0$.

1.8. **Strategy of the proof for the rigidity.** The most important result, where the highest nontriviality lies, is the proof of the rigidity part in Theorem 1.2. The proof is quite computational, so we spend a few words to explain the underlying strategy, which is fully developed in Section 5.

We first consider a general function $u\in C^\infty(K;\mathbb{R}^n)$ satisfying

$$\mathcal{E}_d u = \left(a\odot b - \frac{(a\cdot b)}{n}\mathrm{Id}\right) g \tag{1.19}$$

for some $g\in C^\infty(\mathbb{R}^n;\mathbb{R})$. We exploit this structure of $\mathcal{E}_d u$ and apply Schwarz's theorem to the differential relation

$$\nabla\left(\frac{\nabla u-\nabla u^t}{2}\right)_{ij} = \partial_i((\mathcal{E}_d u)e_j) - \partial_j((\mathcal{E}_d u)e_i) + \partial_i\left(\frac{\mathrm{div}\ u}{n}\right)e_j - \partial_j\left(\frac{\mathrm{div}\ u}{n}\right)e_i$$

(which is a variant of the differential relation exploited for the rigidity of BD maps with constant polar vector). This leads to some general considerations and to a set of PDEs (listed in Lemma 5.1) that u must solve. In particular, we give these equations in terms of u, g, and $f := \frac{\operatorname{div}\ u-(a\cdot b)g}{n}$, which quantifies how much u fails to satisfy BD rigidity, since

$$\mathcal{E}u = (a \odot b)g + f\mathrm{Id}$$

($f = 0$ implies that u has the BD rigidity [28, Theorem 2.10]). To perform this computation, we use specific coordinates. Having established this set of PDEs in Lemma 5.1, for $u \in C^\infty$ solving (1.19), we proceed to treat separately the cases $a \nparallel b$ and $a \parallel b$.

The first case, treated in Subsection 5.1, which is the most technical one, is handled by showing that the set of PDEs in Lemma 5.1—after suitable manipulation—actually leads to wave equations for $\partial_a g$ and $\partial_b g$. The D'Alembert formula then implies the one-dimensionality of $\partial_a g$ and $\partial_b g$ up to a polynomial remainder. This characterizes g as being the sum of two waves plus a polynomial part, and this gives the precise structure (1.5) for $u \in C^\infty$. This is done in Subsection 5.1.1. Now, for a general $u \in \mathrm{BD}_{\mathrm{dev}}(\Omega)$ solving (1.4), we consider a mollification $u_\varepsilon := u \star \varrho_\varepsilon \in C^\infty$. Since mollification preserves the structure of (1.4), u_ε will solve (1.19) (with $g_\varepsilon = \mu \star \varrho_\varepsilon$). Thus the mollified u_ε must have the claimed structure — being C^∞ — and the main point now consists in showing that such a structure is stable when taking the limit as $\varepsilon \to 0$. The most difficult part here is handling the polynomial part Q_ε. Indeed, the coefficient of Q_ε will involve the quantity $\tau_\varepsilon = \partial_{12}\left(\frac{\operatorname{div}\ u_\varepsilon - g_\varepsilon}{n}\right)$, and such a quantity might in principle have no limit as $\varepsilon \to 0$, since $\operatorname{div}\ u_\varepsilon$ might not converge for $\mathrm{BD}_{\mathrm{dev}}$ maps ($\mathcal{E}_d u$ does not provide any control on $\operatorname{div}\ u$). However, the constant polar structure of $\mathcal{E}_d u$ suggests that a u solving (1.5) is somewhat more than just $\mathrm{BD}_{\mathrm{dev}}$. Thus, in Subsection 5.1.2, with an explicit computation of (part of) the polynomial remainder Q, we gain control on τ_ε and are able to pass to the limit in Subsection 5.1.3.

The second case, when $a \parallel b$, is treated in 5.2, where we again argue first for $u \in C^\infty$, and we see that the equations given by Lemma 5.1 are easily integrable, yielding a g which has a one-dimensional part plus a purely polynomial part plus a mixed term, which is a polynomial in x_j for $j \geq 2$ with coefficients depending on x_1. Again, once g is identified, the shape of u follows. For a general function, we again need to pass to the limit, and we need to handle the floating constants that might diverge. We obtain the required control by testing (1.5) against specific test functions. In this part, we find it convenient to treat separately the case $n = 3$ and $n \geq 4$ due to a slight difference between the two cases.

In the end, being able to pass to the limit somehow amounts to showing that u is actually more regular than $\mathrm{BD}_{\mathrm{dev}}$ (at least BD). This fact should not be a surprise, since $\mathcal{E}_d u$ has constant polar vector, and a posteriori, as Theorem 1.2 clarifies, such a u is actually in BV.

1.9. **Organization of the paper.** In Section 2, we introduce the main ingredients required to fully treat the topics contained herein. Section 3 is devoted to computing $\mathrm{Ker}(\mathcal{E}_d)$ and the annihilator $\mathbb{A}$. Section 4 exploits the annihilator to derive some fine properties of $\mathcal{E}_d u$, in the spirit of [28], and lays the basis for the general analysis in Section 5, where the proof of the rigidity Theorem 1.2 is completed. Section 6 in turn provides the explicit computations leading to the identification of the kernel operator $\mathcal{R}_K$ in Theorem 1.4. Finally, in Section 7 and the Appendix 8, we provide some well-known computations based on the non-local approach from Kohn [42], giving infinitesimal information on the quantities defining $\mathcal{R}_K$ and also implying *quasi-continuity* $|\mathcal{E}u|$-a.e. on Ω.

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2. Preliminaries and main result

2.1. **General notations.** The letter n will always denote the ambient Euclidean space dimension. We will denote by $B_r(x)$ the ball of radius r and centered at x. Whenever $x = 0$ we just write B_r, as well as in the case $r = 1$ when we simply write $B(x)$. More in general, given a convex body K we denote by $K_r(x) := x + rK$. We denote by $\mathbb{R}^{m\times n}$ the set of $n \times n$ matrices. The notation e_i stands for the i-th vector of the canonical basis of $\mathbb{R}^n$, Id denotes the $n \times n$ identity matrix. With $\mathbb{R}^{n\times n}_{sym}$, $\mathbb{R}^{n\times n}_{sym_0}$, $\mathbb{R}^{n\times n}_{skew}$ we denote the subsets of $\mathbb{R}^{n\times n}$ made respectively by all symmetric matrices, all trace free symmetric matrices and skew-symmetric matrices. The space $\mathrm{Lin}(X; Y)$ denotes the family of all linear maps between the two vector spaces X and Y. We will use the notation $\langle a, b\rangle$ to denote the linear space spanned by the vectors a and b, i.e. $\langle a, b\rangle := \mathrm{span}\{a, b\}$. Given $v, w \in \mathbb{R}^n$ we will often consider the rank-one matrix $v \otimes w$ acting as $(v \otimes w)z = v(w \cdot z)$ for all $z \in \mathbb{R}^n$ and the matrix

$$v \odot w := \frac{v \otimes w + w \otimes v}{2}.$$

The notation $\mathcal{L}^n$, $\mathcal{H}^{n-1}$ stand for the n-dimensional Lebesgue measure and the $(n-1)$-dimensional Hausdorff measure on $\mathbb{R}^n$ while $\mathcal{M}(\Omega; V)$ is the space of all finite V-valued Radon measures on Ω and all V.

For $u : \mathbb{R}^n \to \mathbb{R}^m$ we specify that

$$\partial_j u := \begin{pmatrix} \partial_j u_1 \\ \vdots \\ \partial_j u_m \end{pmatrix} \in \mathbb{R}^m, \quad \nabla u := (\partial_1 u, \dots, \partial_n u) \in \mathbb{R}^{m\times n}. \tag{2.1}$$

For the matrix-valued function $F : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ we define its gradient as

$$(\nabla F)_{lmj} = (\partial_j F)_{lm} = \partial_j F_{lm},$$

while its divergence is define as the row-divergence operator, namely the vector field

$$\mathrm{div}\ F \cdot e_i = \sum_{j=1}^{n} \partial_j F_{ij}.$$

2.2. **Maps of bounded deformation.** For $u \in C^\infty(\mathbb{R}^n; \mathbb{R}^n)$ the *symmetric gradient* is defined as

$$\mathcal{E}u = \frac{\nabla u + \nabla^t u}{2}$$

The adjoint operator on $F \in C^\infty(\mathbb{R}^n; \mathbb{R}^{n\times n}_{sym})$, reads

$$\mathcal{E}^* F = \mathrm{div}\ F.$$

We also define, for $\xi \in \mathbb{R}^n$, the symbols $\mathbb{E}[\xi] : \mathbb{R}^n \to \mathbb{R}^{n\times n}_{sym}$ as

$$\mathbb{E}[\xi]w = \xi \odot w \quad \text{for all } \xi, w \in \mathbb{R}^n. \tag{2.2}$$

The symmetric gradient measure $\mathcal{E}u \in \mathcal{M}(\Omega; \mathbb{R}^{n\times n}_{sym})$ is defined as

$$\int_\Omega \varphi(x) \cdot \mathrm{d}\mathcal{E}u(x) := \int_\Omega \mathrm{div}\ (\varphi(x)) \cdot u(x)\, \mathrm{d}x \ \ \forall\, \varphi \in C^\infty_c(\Omega; \mathbb{R}^{n\times n}_{sym}).$$

and the space of function with bounded deformation is

$$\mathrm{BD}(\Omega) := \left\{ u \in L^1(\Omega; \mathbb{R}^n) \;\middle|\; \mathcal{E}u \in \mathcal{M}(\Omega; \mathbb{R}^{n\times n}_{sym}) \right\} \subset \mathrm{BV}(\Omega).$$

Several properties are already well-studied for this operator. In particular see [5] or [28] for a more recent approach, closer to the one in this paper. It is well-known that the kernel of $\mathcal{E}$ is made by anti-symmetric affine transformations:

$$\mathrm{Ker}(\mathcal{E}) := \{Ax + b \mid A \in \mathbb{R}^{n\times n}_{skew}, b \in \mathbb{R}^n\}. \tag{2.3}$$

Moreover, for $u \in \mathrm{BD}(\Omega)$ we have that u is $\mathcal{L}^n$-a.e. *approximately differentiable* and the gradient decomposition is in force

$$\mathcal{E}u = e(u)\mathcal{L}^n + ([u] \odot \nu)\mathcal{H}^{n-1}\llcorner_{J_u} + \mathcal{E}^c u$$

where $e(u) = \frac{\nabla u + \nabla u^t}{2}$, J_u is the *jump set*, $[u]$ is the *jump of* u and $\mathcal{E}^c u$ is the *Cantor part*[2]. Recent development on this topic [27] allows also to say that the Cantor part has a very rigid polar vector field

$$\frac{\mathrm{d}\mathcal{E}^c u}{\mathrm{d}|\mathcal{E}^c u|}(x) = a(x) \odot b(x),$$

for some $a, b : \Omega \to \mathbb{R}^n$ $|\mathcal{E}^c u|$ measurable Borel vector-fields. The *Saint-Venant* compatibility condition are also a well-established fact: setting

$$(\mathcal{SV}(M))_{jk} := \sum_{i=1}^{n} \partial_{ik} M_{ij} + \partial_{ij} M_{ik} - \partial_{jk} M_{ii} - \partial_{ii} M_{jk}, \quad \text{for all } j, k = 1, \ldots, n, \tag{2.4}$$

for $M \in C^\infty(\Omega; \mathbb{R}^{n\times n}_{sym})$, it holds that

$$\mathcal{SV}(\mathcal{E}u) = 0, \quad \text{for all } u \in C^\infty(\Omega; \mathbb{R}^n).$$

Note that its symbol is given by

$$\mathbb{SV}[\xi]M := [(M\xi) \otimes \xi + \xi \otimes (M\xi)] - \mathrm{tr}\,(M)(\xi \otimes \xi) - |\xi|^2 M.$$

2.3. **Maps of bounded deviatoric deformation.** Now we define the set of functions with bounded $\mathcal{E}_d$-variation. For $u \in C^\infty(\mathbb{R}^n; \mathbb{R}^n)$ we consider the differential operator

$$\mathcal{E}_d u = \mathcal{E}u - \frac{\mathrm{tr}\ \mathcal{E}u}{n}\mathrm{Id} = \mathcal{E}u - \frac{\mathrm{div}\ \ u}{n}\mathrm{Id}.$$

The adjoint operator on $F \in C^\infty(\mathbb{R}^n; \mathbb{R}^{n\times n})$, reads

$$\mathcal{E}_d^* F = \mathrm{div}\ (\mathrm{sym}\ F) - \frac{\nabla(\mathrm{tr}\ \ F)}{n}.$$

We also define, for $\xi \in \mathbb{R}^n$, the symbols $\mathbb{E}_d[\xi] : \mathbb{R}^n \to \mathbb{R}^{n\times n}_{sym_0}$ as

$$\mathbb{E}_d[\xi]w = \xi \odot w - \frac{(\xi \cdot w)}{n}\mathrm{Id}, \quad \text{for all } \xi, w \in \mathbb{R}^n. \tag{2.5}$$

We will often use the notation introduced in [19]

$$v \otimes_{\mathcal{E}_d} w := \mathbb{E}_d[v]w = \mathbb{E}_d[w]v.$$

We can define then the measure $\mathcal{E}_d u \in \mathcal{M}(\Omega; \mathbb{R}^{n\times n}_{sym_0})$ as

$$\int_\Omega \Phi(x) \cdot \mathrm{d}\mathcal{E}_d u(x) := -\int_\Omega \left(\mathrm{div}\ (\mathrm{sym}\ \Phi(x)) - \frac{\nabla(\mathrm{tr}\ \ \Phi)}{n}\right) \cdot u(x)\,\mathrm{d}x \ \ \forall\ \Phi \in C_c^\infty(\Omega; \mathbb{R}^{n\times n}).$$

Remark 2.1. Since $\mathcal{E}_d u$ takes values in $\mathbb{R}^{n\times n}_{sym_0}$, it only acts on the deviatoric symmetric component of a matrix field: for every $\Phi \in C_c^\infty(\Omega; \mathbb{R}^{n\times n})$ one has

$$\begin{aligned}\int_\Omega \Phi(x) \cdot \mathrm{d}\mathcal{E}_d u(x) &= -\int_\Omega \left(\mathrm{div}\,(\mathrm{sym}\ \Phi(x)) - \frac{\nabla(\mathrm{tr}\ \ \Phi)}{n}\right) \cdot u(x)\,\mathrm{d}x \\ &= -\int_\Omega \mathrm{div}\ (\mathrm{dev\ sym}\ \Phi) \cdot u(x)\,\mathrm{d}x = \int_\Omega \mathrm{dev\ sym}\ \Phi(x) \cdot \mathrm{d}\mathcal{E}_d u(x).\end{aligned}$$

In particular, to extract the required property (as we will do in Section 6) it is often enough to test $\mathcal{E}_d u$ against fields $\varphi \in C_c^\infty(\Omega; \mathbb{R}^{n\times n}_{sym_0})$.

[2]By definition, the Cantor part $\mathcal{E}^c u$ is concentrated on points where u is approximately continuous, but the measure $\mathcal{E}^c u$ is singular and not representable by a density with respect to $\mathcal{L}^n$. In particular, it gives no mass to the jump set.

Given this notation we consider the space

$$\mathrm{BD}_{\mathrm{dev}}(\Omega) := \left\{ u \in L^1(\Omega;\mathbb{R}^n) \;\middle|\; \mathcal{E}_d u \in \mathcal{M}(\Omega;\mathbb{R}^{n\times n}_{symo_0}) \right\} \subset \mathrm{BD}(\Omega).$$

Let us report the following results, proved in [19, Theorem, 4.20] for general differential operators, and clarifying the relation between $\mathcal{E}_d$ and $\mathbb{E}_d$.

Proposition 2.2. *Let $n \geq 3$. Let $u \in \mathrm{BD}_{\mathrm{dev}}(\Omega)$. Then for any $\mathcal{H}^{n-1}$-rectifiable set $R \subset \Omega$ there exists a trace $u\big|_R$. Moreover, for any $u \in \mathrm{BD}_{\mathrm{dev}}(\Omega), F \in C^\infty(\mathbb{R}^n;\mathbb{R}^{n\times n})$ it holds that*

$$\begin{aligned}\int_\Omega F \cdot \mathrm{d}\mathcal{E}_d u(x) &= -\int_\Omega u \cdot \mathcal{E}_d^* F(x)\,\mathrm{d}x + \int_{\partial\Omega} (\mathbb{E}_d[\nu_\Omega](u) \cdot F \,\mathrm{d}\mathcal{H}^{n-1}(x) \\ &= -\int_\Omega u \cdot \mathcal{E}_d^* F(x)\,\mathrm{d}x + \int_{\partial\Omega} (\nu_\Omega \otimes_{\mathcal{E}_d} u) \cdot F \,\mathrm{d}\mathcal{H}^{n-1}(x). \end{aligned} \tag{2.6}$$

Remark 2.3. In [19] the result is proven for any $\mathbb{C}$-elliptic operator $\mathcal{A} : C^\infty(\Omega;\mathbb{R}^n) \to C^\infty(\Omega; V)$ for some vector space V. The notion of $\mathbb{C}$-ellipticity can be stated as the *injectivity of the symbol* $\mathbb{A}[\xi] : \mathbb{C}^n \to V + iV$ *as a linear map from* $\mathbb{C}^n$ *into* $V + iV$*, for all* $\xi \neq 0$. Actually, $\mathbb{C}$-ellipticity is a very important property in order to ensure structural properties to the operator and its functional spaces. In particular for instance, the existence of the trace cannot be guaranteed for non $\mathbb{C}$-elliptic operator. We refer the interested reader to [19] and the literature therein for more details on $\mathbb{C}$-ellipticity and functional spaces.

Remark 2.4. It is a simple computation to show that $\mathbb{E}_d[\xi] : \mathbb{C}^n \to \mathbb{R}^{n\times n}_{symo_0} + i\mathbb{R}^{n\times n}_{symo_0}$ is $\mathbb{C}$-elliptic for all $n \geq 3$ while it is not $\mathbb{C}$-elliptic for $n \geq 3$. This is probably one of the main reasons for the failing of a lot of properties in $n = 2$, such as for instance the existence of trace (see [19]). Also our rigidity Theorems 5.3 and 5.11 are proven for $n \geq 3$. It is not clear though whether $n = 2$ still allows for a rigidity structure. Our computation - and the full proofs - seem to strongly suggest that probably there are not enough equation in dimension 2, to derive a strong rigidity structure on a u with constant polar vector in the wave cone.

A simple algebraic computation yields the following Leibniz rule, together with a useful property of the adjoint operator:

$$\mathcal{E}_d(\varphi u) = \varphi\mathcal{E}_d u + \mathbb{E}_d[\nabla\varphi]u, \qquad \varphi \in C^\infty(\mathbb{R}^n), u \in C^\infty(\mathbb{R}^n;\mathbb{R}^n); \tag{2.7}$$

$$(\mathbb{E}_d[\xi]z) \cdot M = z \cdot \mathbb{E}_d^*[\xi]M, \qquad z \in \mathbb{R}^n, M \in \mathbb{R}^{n\times n}_{symo_0}, \xi \in \mathbb{R}^n. \tag{2.8}$$

2.4. **Poincaré-Sobolev and compactness.** Finally we underline that as a consequence of several general results in the literature on $\mathbb{C}$-elliptic operator we have also the following Poincaré-Sobolev inequality. In the following, $\Pi^U : L^1(U;\mathbb{R}^m) \to \mathrm{Ker}(\mathcal{E}_d)$ stands for a bounded linear projection operator onto the kernel of $\mathcal{E}_d$. We refer to [38, Proposition 2.5], [22, Proposition 2.5, Remark 2.6]

Proposition 2.5 (Poincaré-Sobolev inequality)**.** *Let $n \geq 3$ and K be a center-symmetric convex set. Then there exists a constant c depending on n and K only such that*

$$\|u - \Pi^{K_r(x)}u\|_{L^{\frac{n}{n-1}}(K_r(x);\mathbb{R}^n)} \leq c|\mathcal{E}_d u|(\overline{K_r(x)}) \tag{2.9}$$

for all $x \in \mathbb{R}^n$, $r > 0$ and $u \in \mathrm{BD}_{\mathrm{dev}}(\mathbb{R}^n)$.

The space $\mathrm{BD}_{\mathrm{dev}}(\Omega)$, endowed with the norm $\|u\|_{\mathrm{BD}_{\mathrm{dev}}} := |\mathcal{E}_d u|(\Omega) + \|u\|_{L^1}$, is a Banach space. The Poincaré-Sobolev inequality in 2.5 provide a standard argument, by following for instance the ideas in [42] (combined with the extension argument in [37] to prove the following compactness theorem.

Theorem 2.6 (Compactness Theorem)**.** *Let $\Omega \subset \mathbb{R}^n$ be an open bounded set with Lipschitz boundary and $n \geq 3$ be a first order linear operator. Let $\{u_k\}_{k\in\mathbb{N}} \subset \mathrm{BD}_{\mathrm{dev}}(\Omega)$. Suppose that*

$$\sup_{k\in\mathbb{N}}\{\|u_k\|_{\mathrm{BD}_{\mathrm{dev}}}\} < +\infty.$$

Then there exists $u \in \mathrm{BD}_{\mathrm{dev}}(\Omega)$ and a subsequence $h(k)$ such that $u_{h(k)} \to u$ in L^1 and $\mathcal{E}_d u_{h(k)} \overset{*}{\rightharpoonup} \mathcal{E}_d u$.

The notation $\mathcal{E}_d u_{h(k)} \rightharpoonup^* \mathcal{E}_d u$ stands for the standard *weak* convergence* of Radon measures (see [35] or [44]).

As a consequence of [30, Theorem 3.7] or [22, Proposition 2.8] we have the following Poincaré inequality.

Proposition 2.7 (Poincaré inquality)**.** *Let $n \geq 3$, $K \subset \mathbb{R}^n$ be a fixed convex set. Let $\mathcal{R} : L^1(K;\mathbb{R}^n) \to \mathrm{Ker}(\mathcal{E}_d)$ be a linear, bounded operator such that $\mathcal{R}(L) = L$ for all $L \in \mathrm{Ker}(\mathcal{E}_d)$. Then there exists a uniform constant $c = c(\mathcal{R}, K, n)$ depending on $n, \mathcal{R}$ and K such that*

$$\|u - \mathcal{R}[u]\|_{L^1(K;\mathbb{R}^n)} \leq c\,\mathrm{diam}(K)|\mathcal{E}_d u|(K). \tag{2.10}$$

3. Kernel and annihilator

In this section we present the kernel and the Annihilator of $\mathcal{E}_d$.

3.1. **Kernel.** It is a well-known fact that the kernel of $\mathcal{E}_d$ is made by Killing vector fields

$$\mathrm{Ker}(\mathcal{E}_d) := \left\{ L(y) = (A + \gamma \mathrm{Id})y + s\frac{|y|^2}{2} - (s\cdot y)y + b \;\middle|\, A \in \mathbb{R}^{n\times n}_{skew},\ s, b \in \mathbb{R}^n,\ \gamma \in \mathbb{R} \right\}.$$

Since the proof of the kernel structure and the ingredients required are quite enlightening, in order to deepen the approach used to prove rigidity we present here the proof of this result.

Set, for $u \in C^\infty(A;\mathbb{R}^n)$ the quantity $Wu := \frac{\nabla u - \nabla u^t}{2}$ and observe that

$$\nabla(Wu)_{ij} = \partial_i((\mathcal{E}_d u)e_j) - \partial_j((\mathcal{E}_d u)e_i) + \partial_i\left(\frac{\mathrm{div}\ u}{n}\right)e_j - \partial_j\left(\frac{\mathrm{div}\ u}{n}\right)e_i. \tag{3.1}$$

Proposition 3.1 (Kernel structure)**.** *Let $n \geq 3$, $u \in C^\infty(\Omega;\mathbb{R}^n)$ with $\Omega \subseteq \mathbb{R}^n$ a connected set. Suppose that $\mathcal{E}_d u = 0$. Then there exists $s, b \in \mathbb{R}^n$, $A \in \mathbb{R}^{n\times n}_{skew}$, $\gamma \in \mathbb{R}$ such that*

$$u(y) = (A + \gamma \mathrm{Id})y + s\frac{|y|^2}{2} - (s\cdot y)y + b.$$

Proof. Because of (3.1) and $\mathcal{E}_d u = 0$ we have for all $i \neq j$

$$\nabla(Wu)_{ij} = \partial_i\left(\frac{\mathrm{div}\ u}{n}\right)e_j - \partial_j\left(\frac{\mathrm{div}\ u}{n}\right)e_i.$$

In particular, by taking the curl we get

$$\partial_{ii}\mathrm{div}\ u + \partial_{jj}\mathrm{div}\ u = 0 \quad \text{for all } j \neq i \tag{3.2}$$

and

$$\partial_{ki}\mathrm{div}\ u = 0 \quad \text{for all } k \neq i. \tag{3.3}$$

Since we have $n \geq 3$ we have at least another index $m \neq i \neq j$ for which

$$\partial_{mm}\mathrm{div}\ u + \partial_{ii}\mathrm{div}\ u = 0, \quad \partial_{mm}\mathrm{div}\ u + \partial_{jj}\mathrm{div}\ u = 0, \quad \partial_{ii}\mathrm{div}\ u + \partial_{jj}\mathrm{div}\ u = 0.$$

But then

$$\partial_{ii}\mathrm{div}\ u = -\partial_{mm}\mathrm{div}\ u = \partial_{jj}\mathrm{div}\ u,$$

from which it follows $\partial_{ii}\mathrm{div}\ u = \partial_{jj}\mathrm{div}\ u = \partial_{mm}\mathrm{div}\ u = 0$. By combining this with (3.3) we have

$$\nabla\left(\partial_i \mathrm{div}\ u\right) = 0,$$

this being true for every fixed $i = 1, \dots, n$. Then, since Ω is connected, for some $s \in \mathbb{R}^n$

$$\partial_i \frac{\mathrm{div}\ u}{n} = s_i, \quad \nabla(\mathrm{div}\ u) = ns,$$

which again implies that div $u = n(s\cdot y + \gamma)$ for $s \in \mathbb{R}^n$, $\gamma \in \mathbb{R}$. This implies

$$0 = \mathcal{E}u - \frac{\mathrm{div}\ u}{n}\mathrm{Id} \;\Rightarrow\; \mathcal{E}u = (s\cdot y + \gamma)\mathrm{Id}.$$

Observe that, setting $p(y) := (s\cdot y)y - s\frac{|y|^2}{2} + \gamma y$ we have

$$\mathcal{E}p(y) = (s\cdot y+\gamma)\mathrm{Id},$$

and thus $\mathcal{E}(u-p)=0$. By now invoking the characterization of the kernel of $\mathcal{E}$ we conclude that, for some $A\in\mathbb{R}^{n\times n}_{skew}$,

$$u(y)-p(y) = Ay+b \;\Rightarrow\; u(y) = Ay+p(y)+b,$$

as claimed. $\square$

3.2. **Annihilator.** Given a function $F\in C^\infty(\Omega;\mathbb{R}^{n\times n}_{symo_0})$ we seek for an operator $\mathcal{A}$ such that $\mathcal{A}F=0$ whenever $F=\mathcal{E}_d u$ for some potential u. In particular the existence of such an object for $\mathbb{C}$-elliptic operators is always guaranteed by a result of Van Schaftingen [50, Remark 4.1, Lemma 4.4] (see also [9, Proposition 17] for an extension of Van Schaftingen's construction). The results in the papers are abstract and not constructive.

The annihilator, together with the powerful result in [27], will allow us to determine the structure of the singular measure $\frac{\mathrm{d}\mathcal{E}_d^s u}{\mathrm{d}|\mathcal{E}_d^s u|}$. Indeed, setting

$$\Lambda_{\mathcal{A}} := \bigcup_{|\xi|=1}\mathrm{Ker}(\mathbb{A}[\xi]), \tag{3.4}$$

then (cf. with [27, Theorem 1.1])

$$\frac{\mathrm{d}\mathcal{E}_d u}{\mathrm{d}|\mathcal{E}_d u|}(x)\in\Lambda_{\mathcal{A}}\quad\text{for } |\mathcal{E}_d^s u|\text{-a.e. } x\in\Omega. \tag{3.5}$$

Since the annihilator is not unique (think about curl and ∇(curl) both annihilates ∇u) we need to seek for the operator with the lowest possible order so as to have the kernel of its symbol (and thus its wave cone) the smallest possible in order to find the sharpest constraint on the polar vector of the singular part.

Proposition 3.2 (Annihilator)**.** *Define for $F\in C^\infty(\Omega;\mathbb{R}^{n\times n}_{symo_0})$ the fourth-order operator $\mathcal{A}: C^\infty(\Omega;\mathbb{R}^{n\times n}_{symo_0})\to C^\infty(\Omega;\mathbb{R}^{n\times n}_{symo_0})$*

$$\begin{aligned}(\mathcal{A}(F))_{jk} := &\sum_{i,\ell=1}^n \partial_{iij\ell}F_{k\ell}+\partial_{iik\ell}F_{\ell j} - \sum_{i,\ell=1}^n\partial_{ii\ell\ell}F_{jk}\\ &-\frac{n-2}{n-1}\sum_{i,\ell=1}^n\partial_{i\ell jk}F_{i\ell} - \frac{\delta_{jk}}{n-1}\sum_{i,\ell,m=1}^n\partial_{ii\ell m}F_{\ell m}.\end{aligned} \tag{3.6}$$

Then, if $u\in C^\infty(\mathbb{R}^n;\mathbb{R}^n)$ it holds that

$$\mathcal{A}(\mathcal{E}_d u) = 0.$$

Proof. It is verifiable via a direct computation or by arguing on the symbol as explained in Remark 3.3. We report, for the sake of completeness, the main step of the computations where we make repeated use of Schwarz's theorem.

$$\begin{aligned}\sum_{i,\ell=1}^n \partial_{iij\ell}(\mathcal{E}_d u)_{k\ell}+\partial_{iik\ell}(\mathcal{E}_d u)_{\ell j} = &\frac12\sum_{i,\ell=1}^n\partial_{iij\ell}\left(\partial_\ell u_k+\partial_k u_\ell\right) - \frac1n\sum_{i=1}^n\partial_{iijk}\mathrm{div}\;u\\ &+\frac12\sum_{i,\ell=1}^n\partial_{iik\ell}\left(\partial_\ell u_j+\partial_j u_\ell\right)-\frac1n\sum_{i=1}^n\partial_{iikj}\mathrm{div}\;u\\ =&\frac12\sum_{i,\ell=1}^n\partial_{ii\ell\ell}(\partial_j u_k+\partial_k u_j)+\frac{n-2}{n}\sum_{i=1}^n\partial_{iijk}\mathrm{div}\;u\,,\end{aligned}$$

$$
\begin{aligned}
-\sum_{i,\ell=1}^{n}\partial_{i\ell\ell}(\mathcal{E}_d u)_{jk} &= -\frac{1}{2}\sum_{i,\ell=1}^{n}\partial_{i\ell\ell}\left(\partial_j u_k+\partial_k u_j\right)+\frac{\delta_{jk}}{n}\sum_{i,\ell=1}^{n}\partial_{ii}\partial_{\ell\ell}\mathrm{div}\ \ u\,,\\
-\frac{n-2}{n-1}\sum_{i,\ell=1}^{n}\partial_{i\ell jk}(\mathcal{E}_d u)_{i\ell} &= -\frac{n-2}{2(n-1)}\sum_{i,\ell=1}^{n}\partial_{i\ell jk}(\partial_i u_\ell+\partial_\ell u_i)+\frac{n-2}{n(n-1)}\sum_{i=1}^{n}\partial_{iijk}\mathrm{div}\ \ u\\
&= -\frac{n-2}{n}\sum_{i=1}^{n}\partial_{iijk}\mathrm{div}\ \ u\,,\\
-\frac{\delta_{jk}}{n-1}\sum_{i,\ell,m=1}^{n}\partial_{ii\ell m}(\mathcal{E}_d u)_{\ell m} &= -\frac{\delta_{jk}}{2(n-1)}\sum_{i,\ell,m=1}^{n}\partial_{ii\ell m}(\partial_\ell u_m+\partial_m u_\ell)+\frac{\delta_{jk}}{n(n-1)}\sum_{i,\ell=1}^{n}\partial_{ii\ell\ell}\mathrm{div}\ \ u\\
&= -\frac{\delta_{jk}}{n}\sum_{i,\ell=1}^{n}\partial_{ii\ell\ell}\mathrm{div}\ \ u\,.
\end{aligned}
$$

Now by simply adding up the above relations we obtain $\mathcal{A}(\mathcal{E}_d u)=0$. □

Remark 3.3. Notice that the symbol of $\mathcal{A}$ is given by $\mathbb{A}[\xi]:\mathbb{R}^{n\times n}_{sym_0}\to\mathbb{R}^{n\times n}_{sym_0}$

$$
\mathbb{A}[\xi]M:=|\xi|^2(M\xi\otimes\xi+\xi\otimes M\xi)-|\xi|^4M-\frac{(\xi^tM\xi)}{n-1}\left[(n-2)\xi\otimes\xi+|\xi|^2\mathrm{Id}\right]. \tag{3.7}
$$

Notice also that (3.6) is a very natural choice since, starting from

$$
\mathbb{E}_d[\xi]u=u\odot\xi-\frac{(u\cdot\xi)}{n}\mathrm{Id},
$$

we have

$$
\xi^t(\mathbb{E}_d[\xi]u)\xi=|\xi|^2(u\cdot\xi)\frac{(n-1)}{n}\quad\Rightarrow\quad\frac{(u\cdot\xi)}{n}=\frac{\xi^t(\mathbb{E}_d[\xi]u)\xi}{|\xi|^2(n-1)}.
$$

Thus

$$
\begin{aligned}
(\mathbb{E}_d[\xi]u)\xi &= \frac{1}{2}\left((u\cdot\xi)\xi+u|\xi|^2\right)-\frac{\xi^t(\mathbb{E}_d[\xi]u)\xi}{|\xi|^2(n-1)}\xi\\
&= \frac{(n-2)}{2(n-1)}\frac{\xi^t(\mathbb{E}_d[\xi]u)\xi}{|\xi|^2}\xi+\frac{u|\xi|^2}{2},
\end{aligned}
$$

yielding also

$$
u=\frac{2}{|\xi|^2}\left[(\mathbb{E}_d[\xi]u)\xi-\frac{(n-2)}{2(n-1)}\frac{\xi^t(\mathbb{E}_d[\xi]u)\xi}{|\xi|^2}\xi\right]
$$

and

$$
\mathbb{E}_d[\xi]u=\frac{2}{|\xi|^2}\left[((\mathbb{E}_d[\xi]u)\xi)\odot\xi-\frac{(n-2)}{2(n-1)}\frac{\xi^t(\mathbb{E}_d[\xi]u)\xi}{|\xi|^2}\xi\odot\xi\right]-\frac{\xi^t(\mathbb{E}_d[\xi]u)\xi}{|\xi|^2(n-1)}\mathrm{Id}.
$$

Multiplying by $|\xi|^4$ we get

$$
\begin{aligned}
0 &= 2|\xi|^2((\mathbb{E}_d[\xi]u)\xi)\odot\xi-\frac{(n-2)}{(n-1)}(\xi^t(\mathbb{E}_d[\xi]u)\xi)\xi\odot\xi-|\xi|^2\frac{\xi^t(\mathbb{E}_d[\xi]u)\xi}{(n-1)}\mathrm{Id}-|\xi|^4(\mathbb{E}_d[\xi]u)\\
&= \mathbb{A}[\xi](\mathbb{E}_d[\xi]u).
\end{aligned}
$$

So somehow the fourth order seems the minimum required in order to find a linear function $\mathbb{A}[\xi]$ for which $\mathbb{E}_d[\xi]u\in\mathrm{Ker}(\mathbb{A}[\xi])$. In some sense, from the wave-cone point of view, as clarified by Proposition 3.6, the operator $\mathcal{A}$ is minimal indeed, i.e. provides a wave cone that is indeed minimal (recall that the jump part has surely the claimed shape, by means of the rectifiability result [19]).

Remark 3.4. Note that $\mathcal{A}(F)$ in (3.6), for $F\in C^\infty(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}_0})$ can be expressed also as

$$
\mathcal{A}(F)_{jk}=\Delta\left(\partial_j(\mathrm{div}\ \ F)_k+\partial_k(\mathrm{div}\ \ F)_j\right)-\Delta^2F_{jk}-\frac{n-2}{n-1}\partial_{jk}\left(\mathrm{div}\ \ (\mathrm{div}\ \ F\right)-\frac{\delta_{jk}}{n-1}\Delta\left(\mathrm{div}\ \ (\mathrm{div}\ \ F)\right).
$$

Remark 3.5. The computations in Remark 3.3 are consistent with the Saint-Venant condition (2.4) annhilating $\mathcal{E}u$. Indeed

$$\mathbb{E}[\xi]u = u \odot \xi, \quad \mathrm{tr}\,(\mathbb{E}[\xi]u) = (u \cdot \xi),$$

and

$$2(\mathbb{E}[\xi]u)\xi = (u \cdot \xi)\xi + u|\xi|^2, \quad \Rightarrow \quad u = \frac{2}{|\xi|^2}(\mathbb{E}[\xi]u)\xi - \frac{(u \cdot \xi)\xi}{|\xi|^2},$$

yielding

$$(\mathbb{E}[\xi]u) = \frac{2}{|\xi|^2}((\mathbb{E}[\xi]u)\xi) \odot \xi - \frac{(u \cdot \xi)}{|\xi|^2}(\xi \odot \xi),$$

and

$$0 = 2((\mathbb{E}[\xi]u)\xi) \odot \xi - \mathrm{tr}\,(\mathbb{E}[\xi]u)(\xi \odot \xi) - \mathbb{E}[\xi]u|\xi|^2 = \mathbb{SV}[\xi](\mathbb{E}[\xi]u).$$

So the second order is the minimum required to find a linear function $\mathbb{SV}[\xi]$ for which $\mathbb{E}[\xi]u \in \mathrm{Ker}(\mathbb{SV}[\xi])$ In the deviatoric operator the control on $(u \cdot \xi)$ requires an additional $|\xi|^2$, differently from the symmetric case.

We can now compute the the wave cone of $\mathcal{A}$.

Proposition 3.6 (Wave cone of $\mathcal{A}$)**.** *If $M \in \mathbb{R}^{n\times n}_{sym_0}$ and $|\xi| = 1$ then*

$$\mathbb{A}[\xi]M = 0 \quad \Leftrightarrow \quad M = v \odot \xi - \frac{(v \cdot \xi)}{n}\mathrm{Id} \text{ for some } v \in \mathbb{R}^n.$$

In particular

$$\Lambda_{\mathcal{A}} = \left\{ v \odot \xi - \frac{(v \cdot \xi)}{n}\mathrm{Id} \;\middle|\; v \in \mathbb{R}^n,\ \xi \neq 0 \right\}.$$

Proof. If $M = v \odot \xi - \frac{(v\cdot\xi)}{n}\mathrm{Id}$ for some v then, computing as in Remark (3.3), it follows that $\mathbb{A}[\xi]M = 0$. So we prove the other implication. Up to a rotation we can assume without loss of generality that $\xi = e_1$. Then $\mathbb{A}[e_1]M = 0$ implies, from (3.7)

$$M = (Me_1 \otimes e_1 + e_1 \otimes Me_1) - \frac{(e_1^t M e_1)}{n-1}\left[(n-2)e_1 \otimes e_1 + \mathrm{Id}\right].$$

This gives us

$$M_{11} = (e_1^t M e_1) =: \varrho, \quad M_{ij} = 0 \ \text{ for } i \neq j,\ i,j > 1, \quad M_{ii} = -\frac{\varrho}{(n-1)},\ i > 1.$$

Thus, setting $w_j := M_{1j} = M_{j1}$ for $j > 1$ we have that

$$M = \begin{pmatrix} \varrho & w_2 & \dots & \dots & w_n \\ w_2 & -\frac{\varrho}{n-1} & \dots & \dots & 0 \\ \vdots & \vdots & -\frac{\varrho}{n-1} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ w_n & 0 & \dots & 0 & -\frac{\varrho}{n-1} \end{pmatrix}$$

By now choosing $v_j := 2w_j$ for $j > 1$ and $v_1 = \frac{n\varrho}{n-1}$ we get also $\frac{\varrho}{n-1} = \frac{v_1}{n}$ and thus

$$M = \begin{pmatrix} v_1\left(1 - \frac{1}{n}\right) & \frac{v_2}{2} & \dots & \dots & \frac{v_n}{2} \\ \frac{v_2}{2} & -\frac{v_1}{n} & \dots & \dots & 0 \\ \vdots & \vdots & -\frac{v_1}{n} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ \frac{v_n}{2} & 0 & \dots & 0 & -\frac{v_1}{n} \end{pmatrix} = v \odot e_1 - \frac{(v \cdot e_1)}{n}\mathrm{Id}$$

and the claim follows. □

Corollary 3.7 (Polar vector of $\mathcal{E}_d u$)**.** *Let $u \in \mathrm{BD}_{\mathrm{dev}}(\Omega)$. Then there exists two Borel vector fields $a, b : \Omega \to \mathbb{R}^n$ such that, for $|\mathcal{E}_d^s u|$-a.e. $x \in \Omega$*

$$\frac{\mathrm{d}\mathcal{E}_d^s u}{\mathrm{d}|\mathcal{E}_d^s u|}(x) = a(x) \odot b(x) - \frac{(a(x) \cdot b(x))}{n}\mathrm{Id} = a(x) \otimes_{\mathcal{E}_d} b(x),$$

where $\mathcal{E}_d^s u$ is the singular part in the Radon-Nikodým derivative of $\mathcal{E}_d u$, with respect to $\mathcal{L}^n$.

Proof. Due to (3.5) we must have

$$\frac{\mathrm{d}\mathcal{E}_d u}{\mathrm{d}|\mathcal{E}_d u|}(x) \in \Lambda_{\mathcal{A}} \quad \text{for } |\mathcal{E}_d^s u|\text{-a.e. } x \in \Omega.$$

Thanks to Proposition 3.6 we conclude. □

The above Corollary gives a precise structure to blow-ups around singular points and motivates Theorem 1.2 (see Section 5).

4. Fine properties from the annihilator

4.1. **Structure of the gradient.** For any $u \in L^1(\Omega; \mathbb{R}^n)$ the Lebesgue point theorem ensures that for $\mathcal{L}^n$-a.e. $x \in$ there exists a precise representative $u(x)$ such that

$$\lim_{r \to 0} \fint_{B_r(x)} |u(y) - u(x)| \,\mathrm{d}x = 0.$$

The set of points where this property fails is denoted as S_u and is the *discontinuity set of* u. For $x \in \Omega \setminus S_u$ the value $u(x)$ is also called the *approximate limit* of u.

We recall that $u \in L^1(\Omega; \mathbb{R}^n)$ is said to be *approximately differentiable at* $x \in \Omega \setminus S_u$ if there exists a matrix $L \in \mathbb{R}^{n \times n}$ such that

$$\lim_{r \to 0} \fint_{B_r(x)} \frac{|u(x) - u(y) - L(x - y)|}{r} \,\mathrm{d}y = 0.$$

In this case L is also called the approximate gradient and the notation $\nabla u(x) = M$ is adopted. Thanks to [48, Theorem 1.1] (see also [1, Theorem 3.4], that requires a non-local representation of the involved quantity in the spirit of Proposition 8.2 in the Appendix) we can deduce that $u \in \mathrm{BD}_{\mathrm{dev}}(\Omega)$ is approximately differentiable at $\mathcal{L}^n$-a.e. $x \in \Omega$. Moreover the same result ensures that

$$\frac{\mathrm{d}\mathcal{E}_d u}{\mathrm{d}\mathcal{L}^n}(x) = e_d(u)(x), \quad e_d(u)(x) := e(u)(x) - \frac{\mathrm{tr}\,(e(u)(x))}{n}\mathrm{Id}$$

where we recall that $e(u)(x) = \frac{\nabla u(x) + (\nabla u(x))^t}{2}$.

Thus, by Corollary 3.7 we have

$$\mathcal{E}_d u = e_d(u)(x)\mathcal{L}^n + a(x) \otimes_{\mathcal{E}_d} b(x) |\mathcal{E}_d^s u|$$

for two measurable vector fields $a, b : \Omega \to \mathbb{R}^n$.

We recall that for $u \in L^1_{loc}$ the set J_u is defined as the set of points x for which there exists a triplet $(u^+(x), u^-(x), \nu_u(x))$ such that $u^+ \neq u^-$, $\nu_u \in \mathbb{S}^{n-1}$ and

$$0 = \lim_{r \to 0+} \fint_{B_r^\pm(x)} |u(y) - u^\pm(x)| \,\mathrm{d}y$$

where

$$B_r^-(x) := \{y \in B_r(x) \mid y \cdot \nu_u(x) \leq 0\}, \qquad B_r^+(x) := \{y \in B_r(x) \mid y \cdot \nu_u(x) \geq 0\}.$$

Clearly $J_u \subseteq S_u$. A recent result [29] shows that J_u is always $n-1$ rectifiable and that ν_u is the unitary vector field orienting J_u (nameley $\nu_u(x)^\perp = \mathrm{Tan}(J_u, x)$, the approximate tangent plane to J_u at x, for $\mathcal{H}^{n-1}$-a.e. $x \in J_u$, cf. [6]).

4.2. **Structure of the jump part.** Define

$$\Theta_u := \left\{ x \in \Omega \;\middle|\; \limsup_{r \to 0} \frac{|\mathcal{E}_d u|(B_r(x))}{r^{n-1}} > 0 \right\}. \tag{4.1}$$

We make use of the results in [8] and [19] to prove that $|\mathcal{E}_d u| \ll \mathcal{H}^{n-1}$ and a reasonable structure result for the gradient on the jump set. Set

$$\Lambda_{\mathcal{A}}^{n-1} := \bigcap_{v \in \mathbb{R}^n} \bigcup_{\xi \in v^\perp \setminus \{0\}} \ker(\mathbb{A}(\xi)).$$

Proposition 4.1. *Let $n \geq 3$, $u \in \mathrm{BD}_{\mathrm{dev}}(\Omega)$ and $\mathcal{A}$ be the annihilator given by Proposition 3.2. Then*

$$\Lambda_{\mathcal{A}}^{n-1} = \{0\}. \tag{4.2}$$

As a consequence we have that:

1) $|\mathcal{E}_d u| \ll \mathcal{H}^{n-1}$;
2) $|\mathcal{E}_d u| \left(\left\{ x \in \Omega \;\middle|\; \limsup_{r \to 0} \frac{|\mathcal{E}_d u|(B_r(x))}{r^{n-1}} = +\infty \right\} \right) = 0$;
3) $\mathcal{H}^{n-1}(\Theta_u \Delta J_u) = 0$ *and*

$$\mathcal{E}_d u \llcorner \Theta_u = \mathcal{E}_d u \llcorner J_u = [u] \otimes_{\mathcal{E}_d} \nu_u \mathcal{H}^{n-1} \llcorner J_u.$$

We underline that, thanks to the deep result in [8], properties 1)-3) follows immediately from (4.2). Before proceeding to the proof we first provide a simple Lemma from linear algebra that will simplify our argument in computing $\Lambda_{\mathcal{A}}^{n-1}$.

Lemma 4.2. *Let $a, \xi \in \mathbb{R}^n \setminus \{0\}$ with $|\xi| = 1$. Then*

1) *If a and ξ are not parallel then $a \odot \xi$ has two distinct eigenvalues*

$$\mu_1 = \frac{(a \cdot \xi) + |a|}{2}, \quad \mu_2 = \frac{(a \cdot \xi) - |a|}{2};$$

2) *If a and ξ are parallel then $a \odot \xi$ has one eigenvalue*

$$\mu_1 = (a \cdot \xi) = |a|.$$

Proof. We treat the two cases separately.

Case 1). Without loss of generality we can suppose that $\xi = e_1$ and that $a \in \operatorname{span}\{e_1, e_2\}$. Note that

$$a \odot e_1 = (a \cdot e_1) e_1 \odot e_1 + (a \cdot e_2) e_1 \odot e_2 = \begin{pmatrix} (a \cdot e_1) & \frac{(a \cdot e_2)}{2} \\ \frac{(a \cdot e_2)}{2} & 0 \end{pmatrix}. \tag{4.3}$$

To find the eigenvalues we need to solve

$$0 = \det(a \odot e_1 - \mu \mathrm{Id}) = -((a \cdot e_1) - \mu)\mu - \frac{(a \cdot e_2)^2}{4} = \mu^2 - \mu(a \cdot e_1) - \frac{(a \cdot e_2)^2}{4},$$

whose solutions are precisely

$$\mu_{1,2} = \frac{(a \cdot e_1) \pm \sqrt{(a \cdot e_1)^2 + (a \cdot e_2)^2}}{2} = \frac{(a \cdot e_1) \pm |a|}{2},$$

which are the claimed values.

Case 2). Without loss of generality we can suppose that $\xi = |\xi| e_1$, $a = |a| e_1$. Then, the only eigenvector is e_1 itself with eigenvalue

$$\mu_1 = |a||\xi| = (a \cdot \xi).$$

□

Proof of Proposition 4.1. It is enough to prove (4.2). From this relation indeed, by [8, Corollary 1.4, Theorem 1.5] and a simple application of the theory in [19], we will conclude properties 1)-3).

Proof of (4.2). Let $A \in \Lambda_{\mathcal{A}}^{n-1}$. Then, by Definition of $\Lambda_{\mathcal{A}}^{n-1}$ and by Proposition 3.6 in particular

$$A \in \bigcap_{v \in \mathbb{R}^n} \bigcup_{\xi \in v^\perp \setminus \{0\}} \left\{ a \odot \xi - \frac{(a \cdot \xi)}{n} \mathrm{Id} \;\middle|\; a \in \mathbb{R}^n \right\}.$$

Fix any $v \in \mathbb{R}^n$ and let $\xi \in v^\perp \setminus \{0\}$, $a \in \mathbb{R}^n$ be such that

$$A = a \odot \xi - \frac{(a \cdot \xi)}{n} \mathrm{Id}\,.$$

Without loss of generality, up to redefining a, we can assume that $|\xi| = 1$. Denote by

$$\mathrm{eig}(A) = \{\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n\}$$

the family of eigenvalues of A. Note that the eigenvectors of A are given by the eigenvectors of $M := a \odot \xi$ and by a base of $\mathrm{Ker}(M)$. All the eigenvectors $v \in \mathrm{Ker}(M)$ have the same eigenvalue $-\frac{(a \cdot \xi)}{n}$. Note that $\mathrm{Ker}(M)$ has dimension either $n-2$ (for $a \nparallel \xi$) or $n-1$ (for $a \parallel \xi$), (as shown in [27]). In any case A has at least $n-2$ coincident eigenvalues $\lambda_3 = \ldots = \lambda_n = -\frac{(a \cdot \xi)}{n}$. Let v_k be eigenvectors relative to λ_k. For $i = 1, 2, 3$ let now $\xi_i \in v_i^\perp \setminus \{0\}$ and $a_i \in \mathbb{R}^n$ be such that

$$A = a_i \odot \xi_i - \frac{(a_i \cdot \xi_i)}{n} \mathrm{Id}\,.$$

By fixing $z \in a^\perp \cap a_i^\perp$ we get

$$-\frac{(a \cdot \xi)}{n} |z|^2 = z^t A z = -\frac{(a_i \cdot \xi_i)}{n} |z|^2 \quad \Rightarrow \quad -\frac{(a_i \cdot \xi_i)}{n} = \lambda_3 \;\text{ for all } i = 1, 2, 3.$$

Now note that (since $\xi_i \cdot v_i = 0$)

$$\lambda_2 v_2 = A v_2 = \frac{\xi_2 (a_2 \cdot v_2)}{2} + \lambda_3 v_2 \quad \Rightarrow \quad (a_2 \cdot v_2) = 0,\; \lambda_2 = \lambda_3,$$

and

$$\lambda_1 v_1 = A v_1 = \frac{\xi_1 (a_1 \cdot v_1)}{2} + \lambda_3 v_1 \quad \Rightarrow \quad (a_1 \cdot v_1) = 0,\; \lambda_1 = \lambda_3.$$

In particular $\lambda_1 = \lambda_2 = \ldots = \lambda_n$. Now, by Lemma 4.2, if $a \nparallel \xi$ we just observe that

$$\lambda_1 = \frac{n-2}{2n}(a \cdot \xi) - \frac{|a|}{2}, \quad \lambda_2 = \frac{n-2}{2n}(a \cdot \xi) + \frac{|a|}{2}$$

while for $a \parallel \xi$

$$\lambda_1 = \frac{(n-1)}{n} |a|.$$

In the first case, from $\lambda_1 = \lambda_2$, we immediately have $|a| = 0$. In the second case, from $\lambda_1 = \lambda_3$ we have $\frac{(n-1)}{n}|a| = -\frac{|a|}{n}$ which again implies $a = 0$. In particular $A = 0$ is the only possibility and (4.2) holds true. ∎

So, by means of Proposition 3.2 and [27] we conclude that the measure $\mathcal{E}_d u$ can be split in a singular part with a specific structure of the polar. By means of Proposition 4.1 we have a further characterization of the singular part and we also conclude that, for $u \in \mathrm{BD}_{\mathrm{dev}}(\Omega)$, we have the splitting in three mutually singular measures

$$\mathcal{E}_d u = e_d(u)\mathcal{L}^n + [u] \otimes_{\mathcal{E}_d} \nu_u \mathcal{H}^{n-1} \llcorner J_u + a(x) \otimes_{\mathcal{E}_d} b(x) |\mathcal{E}_d^c u|$$

where $|\mathcal{E}_d^c u|$ is the Cantor part. Note that also by invoking [19] we could have derived the structure of the Jump part.

While for BD it is known the further important property $|\mathcal{E}u|(S_u \setminus J_u) = 0$ (for BVit holds that in the stronger form $\mathcal{H}^{n-1}(S_u \setminus J_u) = 0$) this is actually not known in the $\mathrm{BD}_{\mathrm{dev}}$ context. At the current state it seems technically difficult to be established and the available technology, such as [5], [7] does not seem to apply to the deviatoric operator since it does not satisfy a one-dimensional slicing property. We refer the reader to [10] for a partial result in this sense.

5. Proof of rigidity Theorem 1.2

In this Section we prove the rigidity structure for maps with constant polar vector field, i.e., that satisfies

$$\mathcal{E}_d u = (a \otimes_{\mathcal{E}_d} b)\mu, \quad \mu \in \mathcal{M}(\Omega; \mathbb{R}^+), \ a, b \in \mathbb{R}^n.$$

In homogenization problem this scenario is the only one that occurs when dealing with Cantor points where the characterization of the blow-up is required. It is also the most challenging from the technical point of view.

A very important difference between rigidity in BD (cf. [28]) and rigidity in $\mathrm{BD}_{\mathrm{dev}}(\Omega)$ is that we cannot, in the proof, perform a change of variable that will make $a \perp b$. Indeed while for $\tilde{u}(x) = Au(A^t x)$it holds that

$$e(\tilde{u})(x) = Ae(u)(A^t x)A^t$$

we cannot express $\mathcal{E}_d \tilde{u}$ as a linear transformation of $\mathcal{E}_d(u)$. So somehow the operator $\mathcal{E}_d$ does not behave well under change of variables. However, if the matrix A is a rotation then we can infer

$$\mathcal{E}_d(\tilde{u})(x) = A\mathcal{E}_d(u)(A^t x)A^t.$$

This property allows us, without loss of generality, to rotate the coordinates in order to have a more explicit relation between a and b. In particular without loss of generality we can assume that $a = e_1$, $b = \alpha e_1 + \beta e_2$. In this way, by selectively chosing $\alpha = 0$, or $\beta = 0$, we can deal with the case of perpendicular vectors, parallel vectors or general position vectors, respectively.

We find convenient to introduce a function f, quantifying how much u is far from being a BD rigid.

Lemma 5.1. *Let $n \geq 3$ and $u \in C^\infty(\mathbb{R}^n; \mathbb{R}^n)$ be such that*

$$\mathcal{E}_d u = \left[e_1 \odot (\alpha e_1 + \beta e_2) - \frac{\alpha}{n}\mathrm{Id}\right] g$$

for some $g \in C^\infty(\mathbb{R}^n)$. Setting $f = \frac{\mathrm{div}\ u - \alpha g}{n}$, then the following set of equation hold

$$\beta\partial_{21}g - \alpha\partial_{22}g - \partial_{22}f - \partial_{11}f = 0 \tag{5.1}$$

$$\partial_{jj}g\frac{\beta}{2} + \partial_{21}f = 0 \qquad \text{for all } j \geq 3, \tag{5.2}$$

$$\partial_{11}f + \partial_{jj}f + \alpha\partial_{jj}g = 0 \qquad \text{for all } j \geq 3, \tag{5.3}$$

$$\frac{\beta}{2}\partial_{1j}g - \alpha\partial_{2j}g - \partial_{2j}f = 0 \qquad \text{for all } j \geq 3, \tag{5.4}$$

$$\partial_{22}f + \partial_{jj}f = 0 \qquad \text{for all } j \geq 3, \tag{5.5}$$

$$\partial_{2j}g\frac{\beta}{2} - \partial_{1j}f = 0 \qquad \text{for all } j \geq 3, \tag{5.6}$$

$$\partial_{kj}g = 0 \qquad \text{for all } k, j \geq 3,\ k \neq j, \tag{5.7}$$

$$\partial_{kj}f = 0 \qquad \text{for all } k \neq j \text{ and } j \geq 3, \tag{5.8}$$

$$\partial_{ii}f + \partial_{jj}f = 0 \qquad \text{for all } i, j \geq 3,\ i \neq j. \tag{5.9}$$

If $\beta \neq 0$ we further get

$$\partial_{2j}g = \partial_{1j}g = 0 \qquad \text{for all } j \geq 3, \tag{5.10}$$

$$\partial_j g = -\frac{2\partial_{12}f}{\beta}x_j + v_j \qquad \text{for all } j \geq 3, \tag{5.11}$$

for some $v_j \in \mathbb{R}$ *and* $\mathrm{Hess}(f)$ *is constant.*

Proof. Observe that

$$(\mathcal{E}_d u)e_k = \left(e_1\alpha\delta_{1k} + \frac{\beta}{2}[e_1\delta_{2k} + e_2\delta_{1k}] - \frac{\alpha}{n}e_k\right)g = \left(e_1\left[\alpha\delta_{1k} + \frac{\beta}{2}\delta_{2k}\right] + e_2\frac{\beta}{2}\delta_{1k} - \frac{\alpha}{n}e_k\right)g.$$

Then, by recalling that $Wu := \frac{\nabla u - \nabla u^t}{2}$, combining the above with (3.1) we have

$$\begin{aligned}\nabla(Wu)_{i,j} = &\left(e_1\left[\alpha\delta_{1j} + \frac{\beta}{2}\delta_{2j}\right] + e_2\frac{\beta}{2}\delta_{1j} - \frac{\alpha}{n}e_j\right)\partial_i g\\ &- \left(e_1\left[\alpha\delta_{1i} + \frac{\beta}{2}\delta_{2i}\right] + e_2\frac{\beta}{2}\delta_{1i} - \frac{\alpha}{n}e_i\right)\partial_j g + \partial_i\left(\frac{\mathrm{div}\ u}{n}\right)e_j - \partial_j\left(\frac{\mathrm{div}\ u}{n}\right)e_i.\end{aligned}$$

We can rewrite the above as

$$\begin{aligned}\nabla(Wu)_{ij} =& e_1\left[\left(\alpha\delta_{1j} + \frac{\beta}{2}\delta_{2j}\right)\partial_i g - \left(\alpha\delta_{1i} + \frac{\beta}{2}\delta_{2i}\right)\partial_j g\right] + e_2\frac{\beta}{2}\left[\delta_{1j}\partial_i g - \delta_{1i}\partial_j g\right]\\ &+ e_j\partial_i\left(\frac{\mathrm{div}\ u - \alpha g}{n}\right) - e_i\partial_j\left(\frac{\mathrm{div}\ u - \alpha g}{n}\right), \qquad (5.12)\end{aligned}$$

this being valid for all $i, j = 1, \ldots, n$. Thus

$$\nabla(Wu)_{12} = e_1\left(\frac{\beta}{2}\partial_1 g - \alpha\partial_2 g - \partial_2 f\right) - e_2\left(\frac{\beta}{2}\partial_2 g - \partial_1 f\right), \qquad (5.13)$$

$$\nabla(Wu)_{1j} = -e_1(\alpha\partial_j g + \partial_j f) - e_2\frac{\beta}{2}\partial_j g + e_j\partial_1 f, \qquad \text{for all } j \geq 3, \qquad (5.14)$$

$$\nabla(Wu)_{2j} = -e_1\frac{\beta}{2}\partial_j g + e_j\partial_2 f - e_2\partial_j f, \qquad \text{for all } j \geq 3, \qquad (5.15)$$

$$\nabla(Wu)_{ij} = e_j\partial_i f - e_i\partial_j f, \qquad \text{for all } i, j \geq 3. \qquad (5.16)$$

By considering the curl of (5.13) we get (5.1). By taking the curl of (5.14) we get (5.2),(5.3) and (5.4). By considering the curl of (5.15) we get (5.5), (5.6) and (5.7). By considering the curl of (5.16) we get (5.8) and (5.9).

If thence $\beta \neq 0$, by (5.8) and (5.6) we obtain

$$\partial_{2j} g = 0 \quad \text{for all } j \geq 3,$$

and this combined with (5.4) (still for $\beta \neq 0$) yields (5.10). Finally (5.10), (5.7) and (5.2) implies

$$\nabla(\partial_j g) = -\frac{2}{\beta}(\partial_{12} f)e_j \quad \text{for all } j \geq 3.$$

From this we immediately get $\partial_{k12} f = 0$ for all k (it is immediate if $n \geq 4$ while in dimension $n = 3$ we obtain it by deriving in ∂_2 the relation $\partial_{13} f = 0$ given by (5.6)). This yields (5.11). By deriving in ∂_1 (or ∂_2) (5.3) for $j = 3$ we get $\partial_{111} f = 0$ (or $\partial_{211} f = 0$) that yields $\nabla(f_{11}) = 0$, thanks to (5.7) and (5.8). By doing the same on (5.5) we get also $\nabla(f_{ii}) = 0$ for all $i \geq 2$. As a consequence we also have $\nabla(f_{ij}) = 0$ for all i, j and thus $\mathrm{Hess}(f)$ must finally be a constant matrix. □

We now treat two separate cases, depending on β being zero or different from zero. Before proceeding let us recall some well established facts in the next Remark.

Remark 5.2. Observe that it is immediate to verify that if w solves

$$\mathcal{E}w = (a \odot b)g,$$

then it will solve also $\mathrm{div}\ w = \mathrm{tr}\,(\mathcal{E}w) = (a \cdot b)g$ and thence

$$\mathcal{E}_d w = (a \otimes_{\mathcal{E}_d} b)g.$$

Since (see for instance [28, Theorem 2.10, Assertion (i)]) the function

$$w = a\psi_1(b \cdot x) + b\psi_2(a \cdot x) + (v \cdot x)[a(b \cdot x) + b(a \cdot x)] - v(a \cdot x)(b \cdot x), \tag{5.17}$$

where $\psi_{1,2}$ are scalar functions, and v is a fixed vector, solves– for a, b not parallel–

$$\mathcal{E}w = (a \odot b)(\psi_1'(b \cdot x) + \psi_2'(a \cdot x) + 2(v \cdot x)),$$

and thus

$$\mathcal{E}_d w = (a \otimes_{\mathcal{E}_d} b)(\psi_1'(b \cdot x) + \psi_2'(a \cdot x) + 2(v \cdot x)). \tag{5.18}$$

With this established we can now state and proceed.

5.1. **Rigidity for non parallel vectors ($\beta \neq 0$).** In this Section we provide the proof to the following Theorem.

Theorem 5.3. *Let $u \in \mathrm{BD}_{\mathrm{dev}}(A)$ for a connected open set $A \subset \mathbb{R}^n$. Suppose that*

$$\mathcal{E}_d u = (a \otimes_{\mathcal{E}_d} b)\nu, \tag{5.19}$$

for some $a, b \in \mathbb{R}^n$, $a \nparallel b$ and some positive Radon measure $\nu \in \mathcal{M}(A; \mathbb{R}^+)$. Then there exists two functions $\psi_1, \psi_2 \in \mathrm{BV}_{loc}(\mathbb{R})$ and $v \in \langle a, b\rangle^\perp$ such that

$$u(x) = \psi_1(x \cdot a)b + \psi_2(x \cdot b)a + Q(x) + L(x) \tag{5.20}$$

for some $L \in \mathrm{Ker}(\mathcal{E}_d)$ and for some third order degree polynomial Q solving

$$\mathcal{E}_d Q = (a \otimes_{\mathcal{E}_d} b)\left((v \cdot x) + \eta(a \cdot x)(b \cdot x) - \vartheta \sum_{j=3}^n (x \cdot w_j)^2\right), \tag{5.21}$$

where $\eta, \vartheta \in \mathbb{R}$, $v \in \langle a, b\rangle^\perp$ and $\{w_3, \dots, w_n\}$ is an orthonormal basis of $\langle a, b\rangle^\perp$.

The proof of Theorem 5.3 is achieved by arguing first on regular functions and then by a density argument. In order to correctly pass to the limit we need to gather control on η, θ. To do this some features on the general integral of the polynomial equation (5.21) must be found.

5.1.1. *Rigidity for regular functions.*

Proposition 5.4. *Let $A \subset \mathbb{R}^n$ be a connected set and $u \in C^\infty(A; \mathbb{R}^n)$ be such that*

$$\mathcal{E}_d u = (a \otimes_{\mathcal{E}_d} b)g, \tag{5.22}$$

for some $a, b \in \mathbb{R}^n$, $a \nparallel b$, $g \in C^\infty(A; \mathbb{R}^+)$. Then there exists two functions $\psi_1, \psi_2 \in C^\infty(\mathbb{R})$ and $v \in \langle a, b\rangle^\perp$ such that

$$u(x) = \psi_1(x \cdot a)b + \psi_2(x \cdot b)a + Q(x) + L(x), \tag{5.23}$$

for some $L \in \mathrm{Ker}(\mathcal{E}_d)$ and for some Q solving

$$\mathcal{E}_d Q = (a \otimes_{\mathcal{E}_d} b)\left((v \cdot x) + \eta(a \cdot x)(b \cdot x) - \vartheta \sum_{j=3}^n (x \cdot w_j)^2\right), \tag{5.24}$$

where $\eta, \vartheta \in \mathbb{R}$, $v \in \langle a, b\rangle^\perp$ and $\{w_3, \dots, w_n\}$ is an orthonormal basis of $\langle a, b\rangle^\perp$.

Proof of Proposition 5.4. We place ourselves in the coordinate $a = e_1$, $b = \alpha e_1 + \beta e_2$ for $\beta \neq 0$. Recall that, by Lemma 5.1 we have $\mathrm{Hess}(f)$ is a constant. By (5.1):

$$\zeta = \beta \partial_{12} g - \alpha \partial_{22} g, \tag{5.25}$$

where $\zeta \in \mathbb{R}$ is a constant. Also (5.10), (5.7) and (5.11) imply that

$$\partial_{ki} g = 0 \ \text{ for all } k \neq i,\ i \geq 3, \quad \partial_j g = -2\tau x_j + v_j \quad j \geq 3,$$

for some $\tau = \partial_{12} f \in \mathbb{R}$. We need just to identify $\partial_1 g$ and $\partial_2 g$. Let us now consider separately the case $\alpha = 0$ and $\alpha \neq 0$.

The case $\alpha = 0$ ($a \perp b$). Up to replace g with βg (keeping the same notation) there is no loss of generality in assuming $\beta = 1$. From (5.25) we get $\partial_{12} g = \eta$. From this and (5.10) we derive that

$$\partial_1 g = h_1(x_1) + \eta x_2, \quad \partial_2 g = h_2(x_2) + \eta x_1,$$

for some functions $h_1, h_2 \in C^\infty(\mathbb{R})$. Thus

$$\nabla g = e_1(h(x_1) + \eta x_2) + e_2(h(x_2) + \eta x_1) + \sum_{j=3}^{n}(v_j - 2\tau x_j)e_j, \tag{5.26}$$

and thus

$$g = H_1(x_1) + H_2(x_2) + \eta x_1 x_2 + \sum_{j=3}^{n} v_j x_j - \tau x_j^2,$$

with H_1, H_2 such that $H_1' = h_1$, $H_2' = h_2$. In particular according to Remark 5.2 we have that, setting $\psi_1' = H_1$, $\psi_2' = H_2$, u as in (5.23), with Q solving

$$\mathcal{E}_d Q = (e_1 \otimes_{\mathcal{E}_d} e_2)\left((v \cdot x) + \eta x_1 x_2 - \sum_{j=3}^{n} \tau x_j^2\right),$$

must solve

$$\mathcal{E}_d u = (e_1 \otimes_{\mathcal{E}_d} e_2) g.$$

The case $\alpha \neq 0$. By (5.25) we now show that the following wave equations are in force.

$$\beta^2 \partial_{11}(\partial_2 g) - \alpha^2 \partial_{22}(\partial_2 g) = 0 \tag{5.27}$$

$$\beta^2 \partial_{11}(\partial_1 g) - \alpha^2 \partial_{22}(\partial_1 g) = w(x_1). \tag{5.28}$$

for some $w : \mathbb{R} \to \mathbb{R}$. Indeed, (5.27) comes from

$$\beta^2 \partial_{11}(\partial_2 g) = \beta \partial_1(\beta \partial_{12} g) = \alpha\beta \partial_1(\partial_{22} g) = \alpha \partial_2(\beta \partial_{12} g) = \alpha^2 \partial_{22}(\partial_2 g).$$

Equation (5.28) just comes from the fact that (by (5.10))

$$\partial_j(\beta^2 \partial_{11}(\partial_1 g) - \alpha^2 \partial_{22}(\partial_1 g)) = 0 \quad \text{for all } j \geq 3$$

and

$$\partial_2 \left(\beta^2 \partial_{11}(\partial_1 g) - \alpha^2 \partial_{22}(\partial_1 g)\right) = \partial_1 \left(\beta^2 \partial_{11}(\partial_2 g) - \alpha^2 \partial_{22}(\partial_2 g)\right) =_{(5.27)} 0. \tag{5.29}$$

Notice that $\partial_1 g, \partial_2 g$ are functions depending only on x_1, x_2 (by (5.10)). By the well-known D'Alambert formula for the general solutions of the planar wave equation we thus conclude

$$\partial_1 g = f_0(x_1) + f_1^1(\alpha x_1 - \beta x_2) + f_1^2(\alpha x_1 + \beta x_2) \tag{5.30}$$

$$\partial_2 g = f_2^1(\alpha x_1 - \beta x_2) + f_2^2(\alpha x_1 + \beta x_2). \tag{5.31}$$

We now integrate ∇g from (5.30), (5.31). We first observe the relation $\partial_{21} g = \partial_{12} g$ implies

$$-\beta (f_1^1)'(\alpha x_1 - \beta x_2) + \beta (f_1^2)'(\alpha x_1 + \beta x_2) = \alpha (f_2^1)'(\alpha x_1 - \beta x_2) + \alpha (f_2^2)'(\alpha x_1 + \beta x_2)$$

which yields by computing on $\alpha x_1 = \beta x_2$ and $\alpha x_1 = -\beta x_2$:

$$-\beta f_1^1(t) = \alpha f_2^1(t) + c_1 \quad \beta f_1^2(t) = \alpha f_2^2(t) + c_2,$$

Then, accounting for (5.11) and the above we have

$$g(x) = \int_0^1 \nabla g(tx) \cdot x \, \mathrm{d}t = -\frac{\tau}{\beta} \sum_{j=3}^{n} x_j^2 + v_j x_j + \gamma_j$$

$$+ \int_0^1 [f_0^1(tx_1)x_1 + f_1^1(t(\alpha x_1 - \beta x_2))x_1 + f_1^2(t(\alpha x_1 + \beta x_2))x_1] \, \mathrm{d}t$$

$$
\begin{aligned}
&+\int_0^1 [f_2^1(t(\alpha x_1-\beta x_2))x_2+f_2^2(t(\alpha x_1+\beta x_2))x_2]\,\mathrm{d}t\\
=&\int_0^1 \nabla g(tx)\cdot x\,\mathrm{d}t=-\frac{\tau}{\beta}\sum_{j=3}^n x_j^2+v_jx_j+\gamma_j\\
&+\int_0^1\left(f_0^1(tx_1)x_1-\left(\frac{c_1}{\beta}-\frac{c_2}{\beta}\right)x_1\right)\,\mathrm{d}t\\
&-\frac{1}{\beta}\int_0^1 f_2^1(t(\alpha x_1-\beta x_2))\,(\alpha x_1-\beta x_2)\,\mathrm{d}t\\
&+\frac{1}{\beta}\int_0^1 f_2^2(t(\alpha x_1+\beta x_2))\,(\alpha x_1+\beta x_2)\,\mathrm{d}t
\end{aligned}
$$

and thus

$$
g(x)=h_1(x_1)+h_2(\alpha x_1+\beta x_2)+h_3(\alpha x_1-\beta x_2)-\sum_{j=3}^n\frac{\tau}{\beta}|x_j|^2+v\cdot x+\gamma,
$$

where $h_{1,2,3}$ are functions obtained from the above antiderivatives of f_k^l, and $\gamma:=\sum_{j=3}^n\gamma_j$, and for $v\in\operatorname{span}\{e_1,e_2\}^\perp$. By exploiting (5.25) again we also derive that

$$
-2\alpha\beta^2h_3''(\alpha x_1-\beta x_2)=\zeta.
$$

Since $\alpha\neq 0$:

$$
h_3(t)=-\frac{\zeta}{4\alpha\beta^2}t^2+\sigma t+\tilde\gamma
$$

By observing that

$$
\begin{aligned}
(\alpha x_1-\beta x_2)^2=&(\alpha x_1+\beta x_2)^2-4\alpha\beta x_1x_2\\
\alpha x_1-\beta x_2=&2\alpha x_1-(\alpha x_1+\beta x_2)
\end{aligned}
$$

and up to redefining h_1, we can rewrite g as

$$
\begin{aligned}
g(x)=&h_1(x_1)+h_2(\alpha x_1+\beta x_2)+\frac{\zeta}{\beta}x_1x_2-\frac{\tau}{\beta}\sum_{j=3}^n|x_j|^2+v\cdot x\\
=&h_1(x_1)-\frac{\zeta\alpha}{\beta^2}x_1^2+h_2(\alpha x_1+\beta x_2)+\frac{\zeta}{\beta^2}x_1(\alpha x_1+\beta x_2)-\frac{\tau}{\beta}\sum_{j=3}^n|x_j|^2+v\cdot x\\
=&h_1(x_1)+h_2(\alpha x_1+\beta x_2)+\frac{\zeta}{\beta^2}x_1(\alpha x_1+\beta x_2)-\frac{\tau}{\beta}\sum_{j=3}^n|x_j|^2+v\cdot x
\end{aligned}\tag{5.32}
$$

Setting

$$
\bar u:=\psi_1(x\cdot a)b+\psi_2(x\cdot b)a
$$

with $\psi_1'(t)=h_1(t)$, $\psi_2'(t)=h_2(t)$ we have

$$
\mathcal{E}_d\bar u=\left[e_1\odot(\alpha e_1+\beta e_2)-\frac{\alpha}{n}\mathrm{Id}\right](h_1(x_1)+h_2(\alpha x_1+\beta x_2)).
$$

Thus $Q:=u-\bar u$ satisfies

$$
\mathcal{E}_dQ=\left[e_1\odot(\alpha e_1+\beta e_2)-\frac{\alpha}{n}\mathrm{Id}\right]\left((x\cdot v)+\eta x_1(\alpha x_1+\beta x_2)-\vartheta\sum_{j=3}^n x_j^2\right).
$$

and hence $u=\bar u+Q+L$ as claimed with $\eta=\frac{\zeta}{\beta^2}$, $\vartheta=\frac{\tau}{\beta}$. □

5.1.2. *Features of the polynomial solutions.* In this section we derive some specific features of the polynomial Q satisfying (5.24). This is required in order to pass to the limit in our density argument. To do this we observe that, since

$$w = (v \cdot x)[a(b \cdot x) + b(a \cdot x)] - v(a \cdot x)(b \cdot x),$$

due to Remark 5.2 solves

$$\mathcal{E}_d w = (a \otimes_{\mathcal{E}_d} b)(2v \cdot x),$$

we are just left to compute the solution to

$$\mathcal{E}_d P = (a \otimes_{\mathcal{E}_d} b) \left[\eta(a \cdot x)(b \cdot x) - \vartheta \sum_{j=3}^{n} (w_j \cdot x)^2 \right] \tag{5.33}$$

for $\{w_j\}_{j=3}^n$ orthonormal basis of $\langle a, b\rangle^\perp$. We will not need the explicit solutions but just the features required to pass to the limit from C^∞ solutions of (5.22) to $\mathrm{BD_{dev}}$ solutions of (5.19).

Proposition 5.5. *Set $a = e_1, b = \alpha e_1 + \beta e_2$, $\beta \neq 0$ and consider the equation*

$$\mathcal{E}_d P = e_1 \otimes_{\mathcal{E}_d} (\alpha e_1 + \beta e_2) \left[\eta x_1(\alpha x_1 + \beta x_2) - \vartheta \sum_{j=3}^{n} x_j^2 \right]. \tag{5.34}$$

Then any particular solution $P \in C^2$ of (5.34) is a third order degree polynomial and satisfies

$$\partial_{123} P_3(x) = \vartheta\beta, \quad \partial_{223} P_3(x) = -\frac{2\alpha\vartheta - \eta\beta^2}{2}. \tag{5.35}$$

Remark 5.6. We underline that, for $n \geq 4$ a slightly stronger result - not needed in the purpose of computing the limit - holds: for $n \geq 4$ (5.34) has a solution if and only if $\eta = \frac{2\alpha\vartheta}{\beta^2}$. Indeed for $n \geq 4$ from (5.5), (5.9) we get $\partial_{ii} f = 0$ for all $i \geq 2$. Suppose now that we have a solution $P \in C^2$ for $n \geq 4$ and set as in previous computation

$$g := \eta x_1(\alpha x_1 + \beta x_2) - \vartheta \sum_{j=3}^{n} x_j^2, \quad f := \frac{\mathrm{div}\ P - \alpha g}{n}$$

Then (5.3) gives $\partial_{11} f = 2\alpha\vartheta$. This, plugged in (5.1) and combined with the fact that $\partial_{22} f = 0$, gives

$$2\alpha\vartheta = \beta\partial_{12} g - \alpha\partial_{22} g = \beta^2\eta \quad \Rightarrow \quad \eta = \frac{2\alpha\vartheta}{\beta^2}.$$

This is not the case in $n = 3$ where a particular solution to (5.34) can be provided even for independent η, ϑ.

Proof. Let P be a particular solution of (5.34). We start by observing that, having set $M := (a \otimes_{\mathcal{E}_d} b)$

$$\mathcal{E}_d(\partial_1 P) = M\eta[\alpha x_1 + (\alpha x_1 + \beta x_2)] \tag{5.36}$$

$$\mathcal{E}_d(\partial_2 P) = M\beta\eta x_1 \tag{5.37}$$

$$\mathcal{E}_d(\partial_j P) = -M 2\vartheta x_j \quad j \geq 3 \tag{5.38}$$

All these equations have the structure of (5.18). Being $\mathcal{E}_d$ a differential linear operator we have that the space of solution of $\mathcal{E}_d u = f$ is described by $\{u_p + R \mid R \in \mathrm{Ker}(\mathcal{E}_d)\}$ where u_p is any *particular* solution for $\mathcal{E}_d u = f$. Thus all the solutions must be given, up to some element of $\mathrm{Ker}(\mathcal{E}_d)$, by Formula (5.17). In particular this tells us that $\partial_k P$ is a second order degree polynomial. Once integrated we get that P is a polynomial with degree less or equal to 3. Note that, since the right hand side of (5.34) is a second order degree polynomial and $\mathcal{E}_d$ is a first order differential operator, P needs to have at least one term of third degree: so P is a third degree polynomial. We now focus on the proof of (5.35).

Formula (5.17) gives the exact structure:

$$\begin{aligned}
\partial_1 P =&\eta\left[\frac{(\alpha x_1+\beta x_2)^2}{2}e_1+\frac{\alpha x_1^2}{2}(\alpha e_1+\beta e_2)\right]+L_1(x)\\
\partial_2 P =&\frac{\beta\eta}{2}x_1^2(\alpha e_1+\beta e_2)+L_2(x)\\
\partial_j P =&-\vartheta[e_1(\alpha x_1+\beta x_2)x_j+(\alpha e_1+\beta e_2)x_1x_j-e_jx_1(\alpha x_1+\beta x_2)]+L_j(x)\quad j\geq 3
\end{aligned}$$

for $L_i\in\operatorname{Ker}(\mathcal{E}_d)$. Since $L_i(x)=A_ix+(s_i\cdot x)x-s_i\frac{|x|^2}{2}+b_i$ for some $s_i,b_i\in\mathbb{R}^n$, $A=R_i+\gamma_i\mathrm{Id}$ with $R_i\in\mathbb{R}^{n\times n}_{sym_0}$, $\gamma_i\in\mathbb{R}$ then

$$\partial_k L_i(x)=A_ie_k+(s_i\cdot e_k)x+(s_i\cdot x)e_k-s_ix_k.$$

Then

$$\partial_{123}P=\vartheta\beta e_3+\partial_{12}L_3(x),\quad \partial_{123}P_3=\vartheta\beta+\partial_{12}L_3(x)\cdot e_3.$$

Since

$$\partial_{12}L_3(x)=(s_3\cdot e_2)e_1+(s_3\cdot e_1)e_2$$

Then we have immediately

$$\partial_{123}P_3=\vartheta\beta.$$

Also

$$\partial_{223}P_3=\partial_{22}(L_3(x)\cdot e_3)=-(s_3\cdot e_3). \tag{5.39}$$

To compute this value we now take advantage of Schwarz's theorem to derive information on s_1,s_2,s_3. In particular by $\partial_{13}P=\partial_{31}P$ we obtain

$$-\vartheta[e_1\alpha x_3+(\alpha e_1+\beta e_2)x_3-e_3(2\alpha x_1+\beta x_2)]+\partial_1L_3(x)=\partial_3L_1(x).$$

By computing in $x=0$ we get rid of the affine part $A_3e_1=A_1e_3$ and

$$\begin{aligned}
-\vartheta[e_1\alpha x_3+(\alpha e_1+\beta e_2)x_3-e_3(2\alpha x_1+\beta x_2)]+(s_3\cdot e_1)x+(s_3\cdot x)e_1-s_3x_1&\\
=(s_1\cdot e_3)x+(s_1\cdot x)e_3-s_1x_3.&
\end{aligned}$$

and computing in $x=e_1$, $x=e_2$ and $x=e_3$ gives

$$\begin{aligned}
2\vartheta\alpha e_3+2(s_3\cdot e_1)e_1-s_3&=(s_1\cdot e_3)e_1+(s_1\cdot e_1)e_3\\
\vartheta\beta e_3+(s_3\cdot e_1)e_2+(s_3\cdot e_2)e_1&=(s_1\cdot e_3)e_2+(s_1\cdot e_2)e_3\\
-2\alpha\vartheta e_1-\vartheta\beta e_2+(s_3\cdot e_1)e_3+(s_3\cdot e_3)e_1&=2(s_1\cdot e_3)e_3-s_1.
\end{aligned}$$

This gives

$$s_3\cdot e_1=s_1\cdot e_3,\quad s_3\cdot e_2=0,\quad (s_1\cdot e_2)=\vartheta\beta \tag{5.40}$$

and

$$s_1=(2\alpha\vartheta-(s_3\cdot e_3))e_1+\vartheta\beta e_2+(s_1\cdot e_3)e_3 \tag{5.41}$$

$$s_3=(s_1\cdot e_3)e_1+(2\alpha\vartheta-(s_1\cdot e_1))e_3 \tag{5.42}$$

From $\partial_{23}P=\partial_{32}P$ we get

$$-\vartheta[\beta x_3e_1-\beta e_3x_1]+\partial_2L_3(x)=\partial_3L_2(x).$$

Again computing at $x=0$ allows to ignore the part $A_3e_2=A_2e_3$ and obtain

$$\begin{aligned}
-\vartheta[\beta x_3e_1-\beta e_3x_1]+(s_3\cdot e_2)x+(s_3\cdot x)e_2-s_3x_2&\\
=(s_2\cdot e_3)x+(s_2\cdot x)e_3-s_2x_3.&
\end{aligned}$$

Computing at $x=e_1$, $x=e_2$ and $x=e_3$ yields

$$\begin{aligned}
\vartheta\beta e_3+(s_3\cdot e_2)e_1+(s_3\cdot e_1)e_2&=(s_2\cdot e_3)e_1+(s_2\cdot e_1)e_3\\
(s_3\cdot e_2)e_2+(s_3\cdot e_2)e_2-s_3&=(s_2\cdot e_3)e_2+(s_2\cdot e_2)e_3\\
-\vartheta\beta e_1+(s_3\cdot e_2)e_3+(s_3\cdot e_3)e_2&=(s_2\cdot e_3)e_3+(s_2\cdot e_3)e_3-s_2
\end{aligned}$$

that results in

$$(s_3 \cdot e_1) = 0, \quad (s_3 \cdot e_2) = (s_2 \cdot e_3) = 0$$

which combined with (5.40), (5.41) and (5.42) gives

$$\begin{aligned} s_2 &= \vartheta\beta e_1 - (s_3 \cdot e_3)e_2 \\ s_3 &= -(s_2 \cdot e_2)e_3. \end{aligned} \tag{5.43}$$

Finally by $\partial_{21}P = \partial_{12}P$ we get, still after neglecting the affine part $A_1e_2 = A_2e_1$

$$\begin{aligned} &\eta\beta(\alpha x_1 + \beta x_2)e_1 + (s_1 \cdot e_2)x + (s_1 \cdot x)e_2 - s_1x_2 \\ &\qquad\qquad = \eta\beta x_1(\alpha e_1 + \beta e_2) + (s_2 \cdot e_1)x + (s_2 \cdot x)e_1 - s_2x_1. \end{aligned}$$

computing in $x = e_1$ yields finally

$$\eta\beta\alpha e_1 + (s_1 \cdot e_2)e_1 + (s_1 \cdot e_1)e_2 = \eta\beta(\alpha e_1 + \beta e_2) + 2(s_2 \cdot e_1)e_1 - s_2$$

which implies by means of (5.40), (5.41) and (5.43), that

$$s_2 = \vartheta\beta e_1 + (\eta\beta^2 - (s_1 \cdot e_1))e_2.$$

The above combined with (5.43) and (5.42) gives

$$(s_2 \cdot e_2) = -(s_3 \cdot e_3) \;\Rightarrow\; (\eta\beta^2 - (s_1 \cdot e_1)) = -(s_3 \cdot e_3) = -(2\alpha\vartheta - (s_1 \cdot e_1))$$

that gives

$$(s_1 \cdot e_1) = \frac{2\alpha\vartheta + \eta\beta^2}{2}, \quad (s_3 \cdot e_3) = \frac{2\alpha\vartheta - \eta\beta^2}{2}.$$

By plugging $(s_3 \cdot e_3)$ into (5.39) we conclude. □

Remark 5.7. Observe that, with (a discrete amount of) patience the approach proposed in the proof of Proposition 5.5 allows one to build a complete particular polynomial solution of the PDE (5.34). Indeed the choice of s_1, s_2, s_j are forced by the gradient structure of ∇P and thus, once identified it is possible to choose $\{A_i\}_{i=}^n \subset \mathbb{R}^{n\times n}_{skew}$, $\{b_i\}_{i=1}^n \subset \mathbb{R}^n$ so that we can integrate ∇P to get P. By doing so it is possible to observe that there is a choice ($A_i = 0$, $b_i = 0$) so that a particular solution P is also a *homogeneous* polynomial of degree 3.

5.1.3. *Approximation argument.* The following is a standard approximation argument.

Lemma 5.8 (Approximation Lemma)**.** *Let $u \in \mathrm{BD}_{\mathrm{dev}}(\mathbb{R}^n)$. Let $\varrho_\varepsilon \in C_c^\infty(B_\varepsilon(0))$ be a family of mollifying kernels. Then it holds that $u_\varepsilon := u \star \varrho_\varepsilon \in C^\infty$ and*

(1) $\mathcal{E}_d(u_\varepsilon)(x) = (\mathcal{E}_d u \star \varrho_\varepsilon)(x)$;
(2) $u_\varepsilon \to u$ *in* $L^1_{\mathrm{loc}}(\mathbb{R}^n)$.

Lemma 5.9. *Let $s_\varepsilon(x) = \psi_\varepsilon(x \cdot \ell)$ for some $\ell \in \mathbb{R}^n$ and for some sequence of measurable functions $\psi_\varepsilon \in \mathrm{BV}_{\mathrm{loc}}(\mathbb{R})$. Suppose that $s_\varepsilon \to s$ in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$. Then $s(x) = \psi(x \cdot \ell)$ for some measurable function $\psi \in L^1_{\mathrm{loc}}(\mathbb{R})$.*

Proof. Suppose without loss of generality that $\ell = e_1$. Since $\psi_\varepsilon \in \mathrm{BV}_{\mathrm{loc}}(\mathbb{R})$ then $s_\varepsilon \in \mathrm{BV}_{\mathrm{loc}}(\mathbb{R}^n)$ with $Ds_\varepsilon = e_1 D\psi_\varepsilon(\mathrm{d}x_1) \otimes \mathcal{L}^{n-1}(\mathrm{d}x_2 \ldots \mathrm{d}x_n)$. Here we recall the notation

$$\mu(\mathrm{d}x_1) \otimes \nu(\mathrm{d}x_2 \ldots \mathrm{d}x_n)(\varphi) := \int_{\mathbb{R}^n} \varphi \,\mathrm{d}\mu(x_1)\,\mathrm{d}\nu(x_2) \ldots \mathrm{d}\nu(x_n).$$

For $\varphi \in C_c^\infty(\mathbb{R}^n)$ we also have for $k \neq 1$:

$$\int_{\mathbb{R}^n} s\partial_k\varphi \,\mathrm{d}x = \lim_\varepsilon \int_{\mathbb{R}^n} s_\varepsilon \partial_k \varphi \,\mathrm{d}x = -\lim_\varepsilon \int_{\mathbb{R}^n} (e_1 \cdot e_k)\varphi \,\mathrm{d}D\psi_\varepsilon(\mathrm{d}x_1)\,\mathrm{d}x_2 \ldots \mathrm{d}x_d = 0.$$

Thence, distributionally $\partial_k s = 0$ for all $k \neq 1$. This implies that $s(x) = \psi(x \cdot e_1)$ for a measurable function $\psi : \mathbb{R} \to \mathbb{R}$. □

Finally, to handle the polynomial part we need to employ the following Lemma.

Lemma 5.10. *Let $Q_\varepsilon : \mathbb{R}^n \to \mathbb{R}$ be a family of polynomials in $x_1, \dots, x_n$ of degree k. Suppose that $Q_\varepsilon \to Q$ in L^1_{loc} for some $Q \in L^1_{loc}$. Then Q is a polynomial of degree at most k and the coefficients of Q_ε converge to the coefficients of Q.*

Proof. Fix an open bounded set $A \subset \mathbb{R}^n$ and note that the space

$$S := \{Q : A \to \mathbb{R} \mid Q \text{ is a polynomial of degree at most } k\}$$

is a closed finite-dimensional vector space. Let $\underline{i} \in \{0, 1, \dots, k\}^n$ be a multindex and, for $Q \in S$, $c^Q_{\underline{i}}$ be denoting the coefficient of Q in front of a monomial $x^{\underline{i}} := x_1^{i_1} \dots x_n^{i_n}$, $|\underline{i}| = i_1 + \dots + i_n \leq k$. Then the application

$$T : S \to \mathbb{R}^N, \quad T(Q) := (c^Q_{\underline{i}})_{\underline{i} \in \{0,1,\dots,k\}^n}$$

is a linear application between finite-dimensional vector spaces and hence is continuous. Thus, since Q_ε is converging in L^1_{loc}, we have $Q_\varepsilon \to Q$ in $L^1(A)$ for some $Q \in S$. Thence

$$|T(Q_\varepsilon) - T(Q)| \leq C_T \|Q_\varepsilon - Q\|_{L^1} \to 0.$$

By the very definition of T we have that the coefficient of Q_ε converges to those of Q. □

We are now in the position for proving Theorem 5.3.

Proof of Theorem 5.3. Let $u \in \mathrm{BD}_{\mathrm{dev}}(\mathbb{R}^n)$ satisfy

$$\mathcal{E}_d u = (a \otimes_{\mathcal{E}_d} b)\nu.$$

Since a, b are not parallel, without loss of generality, we can assume that $a = e_1, b = \alpha e_1 + \beta e_2$ for $\beta \neq 0$ and $\alpha \in \mathbb{R}$. Consider u_ε the approximation as in Lemma 5.8 and note that

$$\mathcal{E}_d u_\varepsilon = \mathcal{E}_d u \star \varrho_\varepsilon = (a \otimes_{\mathcal{E}_d} b)(\nu \star \varrho_\varepsilon)(x).$$

We invoke Proposition 5.4 with $g_\varepsilon = (\nu \star \varrho_\varepsilon) \in C^\infty$ and we conclude that

$$u_\varepsilon(x) = a\psi_1^\varepsilon(x \cdot b) + b\psi_2^\varepsilon(x \cdot a) + Q_\varepsilon(x) + L_\varepsilon(x).$$

with Q_ε solving (5.24). We just need to show that the claimed structure is stable under the limit as $\varepsilon \to 0$, yielding the thesis also on u. Notice that $u_\varepsilon \to u$ in L^1_{loc} and since

$$\sup_\varepsilon \{|\mathcal{E}_d(u_\varepsilon - L_\varepsilon)|(A)\} < +\infty,$$

we have $u_\varepsilon - L_\varepsilon \to \bar{u}$. Since

$$\mathcal{E}_d u = \operatorname*{w^*\text{-}lim}_{\varepsilon \to 0} \mathcal{E}_d u_\varepsilon = \operatorname*{w^*\text{-}lim}_{\varepsilon \to 0} \mathcal{E}_d(u_\varepsilon - L_\varepsilon) = \mathcal{E}_d \bar{u},$$

where w*-lim denote the *weak star limit*, we have also that $L := u - \bar{u} \in \mathrm{Ker}(\mathcal{E}_d)$ and that

$$L_\varepsilon = u_\varepsilon - (u_\varepsilon - L_\varepsilon) \to u - \bar{u} = L.$$

We now complete the proof by separately analyzing the polynomial part and the one dimensional part.

Step one: *limit of the polynomial part.* Suppose that $a = e_1$, $b = \alpha e_1 + \beta e_2$. We know that (up to an element of $\mathrm{Ker}(\mathcal{E}_d)$) it must hold (cf. Remark 5.2)

$$Q_\varepsilon(x) = (v_\varepsilon \cdot x)[e_1(\alpha x_1 + \beta x_2) + (\alpha e_1 + \beta e_2)x_1] - v_\varepsilon x_1(\alpha x_1 + \beta x_2) + P_\varepsilon(x),$$

for some $v_\varepsilon \in \langle e_1, e_2 \rangle^\perp$ and for some P_ε solving (5.34)

$$\mathcal{E}_d P_\varepsilon = e_1 \otimes_{\mathcal{E}_d} (\alpha e_1 + \beta e_2) \left[\eta_\varepsilon x_1(\alpha x_1 + \beta x_2) - \vartheta_\varepsilon \sum_{j=3}^n x_j^2 \right].$$

Since $(u_\varepsilon - L_\varepsilon) \to \bar{u}$ in L^1 for any $j \geq 3$ we have

$$Q_\varepsilon(x) \cdot e_j = (u_\varepsilon - L_\varepsilon) \cdot e_j \to (\bar{u} \cdot e_j) \ \text{ in } L^1.$$

But

$$Q_\varepsilon(x) \cdot e_j = -(v_\varepsilon \cdot e_j)x_1(\alpha x_1 + \beta x_2) + P_\varepsilon \cdot e_j$$

We use Proposition 5.5 to infer that a solution to (5.34) must be of the form

$$(P_\varepsilon(x)\cdot e_3) := \vartheta_\varepsilon \beta x_1 x_2 x_3 - (2\alpha\vartheta_\varepsilon - \eta_\varepsilon\beta^2)\frac{x_2^2 x_3}{4} + \sum_{\underline{i}\in\{0,\dots,3\}^n\setminus\{(1,1,1,0\dots,0),(0,2,1,0,\dots,0)\}} (c_{\underline{i}}(\eta_\varepsilon,\vartheta_\varepsilon)\cdot e_3)\, x^{\underline{i}}$$

and hence

$$\begin{aligned} Q_\varepsilon \cdot e_3 = &-(v_\varepsilon\cdot e_3)x_1(\alpha x_1 + \beta x_2) + \vartheta_\varepsilon\beta x_1 x_2 x_3 - (2\alpha\vartheta_\varepsilon - \eta_\varepsilon\beta^2)\frac{x_2^2 x_3}{4} \\ &+ \sum_{\underline{i}\in\{0,\dots,3\}^n\setminus\{(1,1,1,0\dots,0),(0,2,1,0,\dots,0)\}} (c_{\underline{i}}(\eta_\varepsilon,\vartheta_\varepsilon)\cdot e_3)\, x^{\underline{i}}. \end{aligned}$$

By means of Lemma 5.10 we have that the coefficients of $Q_\varepsilon \cdot e_3$ must converge to something and thus $\eta_\varepsilon, \vartheta_\varepsilon$ and $(v_\varepsilon\cdot e_3)$ converges to something. In particular we have that $v_\varepsilon \to v$, $\vartheta_\varepsilon \to \vartheta$ and $\eta_\varepsilon \to \eta$. Since the coefficients of Q_ε are all expressed as polynomial functions of $\eta_\varepsilon, \vartheta_\varepsilon$ and v_ε we have that $Q_\varepsilon \to Q$ for Q solving (5.24).

Step two: *limit of the one-dimensional part.* Set $w_\varepsilon := u_\varepsilon - Q_\varepsilon - L_\varepsilon$. Thanks to Step one, two and a standard compactness argument we know that $w_\varepsilon \to w := \bar{u} - Q$ in L^1 for some Q solving (5.21). Since

$$w_\varepsilon(x) = a\psi_1^\varepsilon(x\cdot b) + b\psi_2^\varepsilon(x\cdot a),$$

pick now $z\in b^\perp$ for which $a\cdot z\neq 0$ and note that

$$s_\varepsilon^z(x) := \frac{z}{a\cdot z}\cdot w_\varepsilon = \psi_1^\varepsilon(x\cdot b)$$

where ψ_1^ε is actually smooth since w_ε is smooth. Since $w_\varepsilon \to w$ in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$ hence $s_\varepsilon^z \to \frac{z}{a\cdot z}\cdot w$. From the other side, by applying Lemma 5.9 we conclude $s_\varepsilon^z \to s(x) = \psi_1(x\cdot b)$ for a measurable function $\psi_1\in L^1_{\mathrm{loc}}(\mathbb{R})$ independent of z (since $s_\varepsilon^z(x) = \psi_1^\varepsilon(x\cdot b)$). Thence

$$z\cdot w = (a\cdot z)\psi_1(x\cdot b) \quad \text{for all } z\in b^\perp. \tag{5.44}$$

Analogously, starting from $h\in a^\perp$ we conclude

$$h\cdot w = (b\cdot h)\psi_2(x\cdot a) \quad \text{for all } h\in a^\perp, \tag{5.45}$$

for some measurable function $\psi_2\in L^1_{\mathrm{loc}}(\mathbb{R})$. We now choose $z_1,\dots,z_{n-1}$ orthonormal basis of $b^\perp$ and $h_1,\dots,h_{n-1}$ orthonormal basis of $a^\perp$. Observe - by (5.44) - that

$$w(x) = \sum_{i=1}^{n-1} z_i(z_i\cdot w(x)) + \left(\frac{b}{|b|}\cdot w(x)\right)\frac{b}{|b|} = \sum_{i=1}^{n-1} z_i\,(a\cdot z_i)\,\psi_1(x\cdot b) + \left(\frac{b}{|b|}\cdot w(x)\right)\frac{b}{|b|}.$$

By also expressing

$$\frac{b}{|b|} = \sum_{i=1}^{n-1}\left(\frac{b}{|b|}\cdot h_i\right)h_i + \left(\frac{b}{|b|}\cdot\frac{a}{|a|}\right)\frac{a}{|a|}$$

and using (5.45), we can further write

$$\begin{aligned} w(x) =& \sum_{i=1}^{n-1} z_i\,(a\cdot z_i)\,\psi_1(x\cdot b) + \sum_{i=1}^{n-1}\left(\frac{b}{|b|}\cdot h_i\right)(h_i\cdot w(x))\frac{b}{|b|} + \left(\frac{b}{|b|}\cdot\frac{a}{|a|}\right)\left(\frac{a}{|a|}\cdot w(x)\right)\frac{b}{|b|} \\ =& a\psi_1(x\cdot b) + \psi_2(x\cdot a)\frac{b}{|b|^2}\sum_{i=1}^{n-1}(b\cdot h_i)^2 + \psi_2(x\cdot a)\frac{b}{|b|^2}\left(b\cdot\frac{a}{|a|}\right)^2 \\ &+ \left(\frac{b}{|b|}\cdot\frac{a}{|a|}\right)\left(\frac{a}{|a|}\cdot w(x)\right)\frac{b}{|b|} - \psi_1(x\cdot b)\left(a\cdot\frac{b}{|b|}\right)\frac{b}{|b|} - \psi_2(x\cdot a)\frac{b}{|b|^2}\left(b\cdot\frac{a}{|a|}\right)^2 \\ =& a\psi_1(x\cdot b) + b\psi_2(x\cdot a) \end{aligned}$$

$$+\left(\frac{b}{|b|}\cdot\frac{a}{|a|}\right)\frac{b}{|b|}\left[\left(\frac{a}{|a|}\cdot w(x)\right)-|a|\psi_1(x\cdot b)-\left(b\cdot\frac{a}{|a|}\right)\psi_2(x\cdot a)\right]. \tag{5.46}$$

We immediately conclude if $b\cdot a=0$. Otherwise we multiply (5.46) by $h\in a^\perp$ and, using (5.45), we have

$$\begin{aligned}(b\cdot h)\psi_2(x\cdot a)=&(b\cdot h)\psi_2(x\cdot a)\\&+\left(\frac{b}{|b|}\cdot\frac{a}{|a|}\right)\frac{b\cdot h}{|b|}\left[\left(\frac{a}{|a|}\cdot w(x)\right)-|a|\psi_1(x\cdot b)-\left(b\cdot\frac{a}{|a|}\right)\psi_2(x\cdot a)\right]\end{aligned}$$

yielding

$$\left(\frac{b}{|b|}\cdot\frac{a}{|a|}\right)\frac{b\cdot h}{|b|}\left[\left(\frac{a}{|a|}\cdot w(x)\right)-|a|\psi_1(x\cdot b)-\left(b\cdot\frac{a}{|a|}\right)\psi_2(x\cdot a)\right]=0\ \text{ for all } h\in a^\perp.$$

By selecting an h for which $(b\cdot h)\neq 0$ (which exists since $a\nparallel b$) we conclude

$$\left[\left(\frac{a}{|a|}\cdot w(x)\right)-|a|\psi_1(x\cdot b)-\left(b\cdot\frac{a}{|a|}\right)\psi_2(x\cdot a)\right]=0,$$

which implies, by (5.46)

$$w(x)=a\psi_1(x\cdot b)+b\psi_2(x\cdot a).$$

Step three: *bounded variation of the one-dimensional part.* We are just left to show that the ψ's are $\mathrm{BV}_{\mathrm{loc}}$. We thus express (for the sake of simplicity) $a=e_1$, $h=e_2$ and $b=\alpha e_1+\beta e_2$. Then

$$u(x)=e_1[\psi_1(\alpha x_1+\beta x_2)+\alpha\psi_2(x_1)]+\beta e_2\psi_2(x_1).$$

We choose $\varphi\in C_c^\infty([-1,1])$ with $\|\varphi\|_\infty\leq 1$ and $\eta,\eta_3,\ldots,\eta_n\in C_c^\infty(\mathbb{R})$ with $\int\eta_i\,\mathrm{d}t=1$. We consider thus

$$\Phi(x):=\varphi(x_1)\eta(x_2)\eta_3(x_3)\ldots\eta_n(x_n)(e_1\odot e_2).$$

Note that

$$2\mathrm{div}\ \ (\Phi(x))=\varphi(x_1)\eta'(x_2)\eta_3(x_3)\ldots\eta_n(x_n))e_1+\varphi'(x_1)\eta_2(x_2)\eta_3(x_3)\ldots\eta_n(x_n))e_2.$$

We thus apply (2.6) and the fact that $\int\eta_i=\int\eta=1$ to infer

$$\begin{aligned}2\int_{\mathbb{R}^n}&\Phi\cdot\,\mathrm{d}\mathcal{E}_d u(x)=2\int_{\mathbb{R}^n}(\mathrm{div}\ \ \Phi\cdot u)\,\mathrm{d}x\\&=\int_{\mathbb{R}^n}\varphi(x_1)\eta'(x_2)\eta_3(x_3)\ldots\eta_n(x_n))u_1\,\mathrm{d}x+\int_{\mathbb{R}^n}\varphi'(x_1)\eta(x_2)\eta_3(x_3)\ldots\eta_n(x_n))u_2\,\mathrm{d}x\\&=\int_{\mathbb{R}^2}\varphi(x_1)\eta'(x_2)[\psi_1(\alpha x_1+\beta x_2)+\alpha\psi_2(x_1)]\,\mathrm{d}x_1\,\mathrm{d}x_2+\beta\int_{\mathbb{R}}\varphi'(x_1)\psi_2(x_1)\,\mathrm{d}x_1.\end{aligned}$$

By rearranging the above relation, setting $K=\mathrm{spt}(\Phi)$, $K_2:=\mathrm{spt}(\varphi\eta)\subset\mathbb{R}^2$, we obtain

$$\left|\int_{\mathbb{R}}\varphi'(x_1)\psi_2(x_1)\,\mathrm{d}x_1\right|\leq C\left[|\mathcal{E}_d u|(K)+\int_{K_2}[|\psi_1(\alpha x_1+\beta x_2)|+|\psi_2(x_1)|]\,\mathrm{d}x_1\,\mathrm{d}x_2\right],$$

with the constant C possibly depending on η,η_i' but not on φ. For any fixed $T>0$, the right hand side is independent of $\varphi\in C_c^\infty([-T,T])$ with $\|\varphi\|_\infty\leq 1$ - being $K\subset[-T,T]\times\mathrm{spt}(\eta)\times\mathrm{spt}(\eta_3)\times\ldots\times\mathrm{spt}(\eta_d)$ - yielding that $\psi_2\in\mathrm{BV}_{\mathrm{loc}}(\mathbb{R})$.

Writing $a=(a\cdot\frac{b}{|b|})\frac{b}{|b|}+(a\cdot z)z$ for some $z\in b^\perp$) (setting again without loss of generality $b=|b|e_1$, $z=e_2$ and $a=\alpha_1e_1+\alpha_2e_2$ for some α_1,α_2) and replicating the above argument (since we have - formally - inverted the role of x_1, x_2 we can test with the same $\Phi(x)$) we obtain also $\psi_1\in\mathrm{BV}_{\mathrm{loc}}(\mathbb{R})$. $\square$

5.2. **Rigidity for parallel vectors ($\beta = 0$).** In this Section we instead provide the proof to the following

Theorem 5.11. *Let $u \in \mathrm{BD}_{\mathrm{dev}}(A)$ for a connected open set $A \subset \mathbb{R}^n$. Suppose that*

$$\mathcal{E}_d u = (a \otimes_{\mathcal{E}_d} a)\nu,$$

for some $a \in \mathbb{R}^n$, and some Radon measure $\nu \in \mathcal{M}(A;\mathbb{R})$. Then there exists functions $F \in \mathrm{BV}_{loc}(\mathbb{R}))$, $\{P_j\}_{j=2}^n \subset W^{1,1}_{loc}(\mathbb{R})$ with $P_j' \in \mathrm{BV}_{loc}$, $G \in W^{1,1}_{loc}(\mathbb{R})$ with $G' \in \mathrm{BV}_{loc}(\mathbb{R})$ such that

$$\begin{aligned} u =& F(a\cdot x)a + \left(\sum_{j=2}^n P_j'(a\cdot x)(w_j\cdot x) + \frac{(w_j\cdot x)^2}{2}G'(a\cdot x)\right)a \\ &- \sum_{j=2}^n ((w_j\cdot x)G(a\cdot x) + P_j(a\cdot x))w_j + \varrho Q(x) + L(x), \end{aligned} \tag{5.47}$$

for $\{w_2,\dots,w_n\}$ orthonormal basis of $a^\perp$, for some $L \in \mathrm{Ker}(\mathcal{E}_d)$, $\varrho \in \mathbb{R}$ and for some third order degree polynomial Q solving

$$\mathcal{E}_d Q = (a \otimes_{\mathcal{E}_d} a)\left((w_2\cdot x)^2 - (w_3\cdot x)^2\right). \tag{5.48}$$

Moreover if $n \geq 4$ then $\varrho = 0$.

Remark 5.12. Note that since having $\beta = 0$ results in fewer controls on the relevant quantities (cf. Lemma 5.1) there is no surprise that we have a more complex structure on the u, compared to the case $\beta \neq 0$. However still the lowest regularity informations is on the one directional part along a which is only BV and no more.

As before, we place ourselves in coordinates so that $a = e_1$ and we focus first on the C^∞ case.

Lemma 5.13. *Let $u \in C^\infty(\mathbb{R}^n;\mathbb{R}^n)$, $g \in C^\infty(\mathbb{R}^n)$ be such that*

$$\mathcal{E}_d u = \left[e_1 \odot e_1 - \frac{1}{n}\mathrm{Id}\right] g. \tag{5.49}$$

Then necessarily, for some $\psi, h, p_j : \mathbb{R} \to \mathbb{R}$it holds that

$$g = h(x_1) + \sum_{j=2}^n p_j(x_1)x_j + \psi(x_1)\sum_{j=2}^n \frac{x_j^2}{2} + \varrho(x_2^2 - x_3^2), \tag{5.50}$$

and consequently

$$\begin{aligned} u =& (H(x_1) - \Psi(x_1))e_1 + \left(\sum_{j=2}^n P_j'(x_1)x_j + \frac{x_j^2}{2}\Psi''(x_1)\right)e_1 \\ &- \sum_{j=2}^n (x_j\Psi'(x_1) + P_j(x_1))e_j + \varrho Q(x) + L(x), \end{aligned} \tag{5.51}$$

for $L \in \mathrm{Ker}(\mathcal{E}_d)$, $\Psi, H, P_j : \mathbb{R} \to \mathbb{R}$ such that $\Psi''' = \psi$, $H' = h$, $P_j'' = p_j$ and for a polynomial Q solving

$$\mathcal{E}_d Q = (e_1 \otimes_{\mathcal{E}_d} e_1)\left(x_2^2 - x_3^2\right). \tag{5.52}$$

Moreover, if $n \geq 4$, $\varrho = 0$.

Proof. We now split the proof in three steps: the first two to compute g depending on whether $n = 3$ or $n \geq 4$, and the last one to conclude. Just restricted to this proof will be useful the notation: for $w : \mathbb{R}^n \to \mathbb{R}^n$ write $w(\hat{x}_{i_1},\dots,\hat{x}_{i_k})$ to emphasize that the function w does not depend on the variable $x_{i_1},\dots x_{i_k}$ (but possibly depends on all the others).

Step one: *computing g for $n \geq 4$.* In this case, as already observed in Remark 5.6, we have (due to (5.5), (5.9)) $\partial_{jj} f = 0$ for all $j \geq 2$. Also, by invoking Lemma 5.1 (for $\alpha = 1, \beta = 0$) we obtain the relevant set of equations

$$
\begin{aligned}
\partial_{11} f + \partial_{22} f =& - \partial_{22} g, && && (5.53)\\
\partial_{21} f =& 0, && && (5.54)\\
-\partial_{2j} g - \partial_{2j} f =& 0, && \text{for all } j \geq 3, && (5.55)\\
\partial_{22} f + \partial_{jj} f =& 0, && \text{for all } j \geq 3, && (5.56)\\
-\partial_{1j} f =& 0, && \text{for all } j \geq 3, && (5.57)\\
\partial_{kj} g =& 0, && \text{for all } k, j \geq 3,\ k \neq j, && (5.58)\\
\partial_{kj} f =& 0, && \text{for all } k \neq j \text{ and } j \geq 3, && (5.59)
\end{aligned}
$$

By (5.59) and (5.55):

$$\partial_{2j} f = 0 \quad j \geq 3, \quad \partial_{2j} g = 0 \quad j \geq 3. \tag{5.60}$$

The above and (5.54) implies

$$\nabla(\partial_2 f) = 0, \quad \nabla(\partial_1 f) = (-\partial_{jj} g) e_1 \quad j \geq 3, \quad \nabla(\partial_j f) = 0 \quad j \geq 3. \tag{5.61}$$

and (5.53) gives additionally

$$-\partial_{22} g = \partial_{11} f.$$

Since (5.54), (5.57) yields $\partial_k(\partial_{11} f) = 0$ for $k \neq 1$ it follows that

$$\partial_{22} g = \partial_{jj} g = \psi(x_1) \quad j \geq 3. \tag{5.62}$$

for some $\psi : \mathbb{R} \to \mathbb{R}$. Thence, for some p_2, it must hold

$$\partial_2 g = p_2(\hat{x}_2) + \psi(x_1) x_2.$$

By (5.60) we now must have $p_2(\hat{x}_2) = p_2(x_1)$ and (for some G),

$$g = G(\hat{x}_2) + p_2(x_1) x_2 + \frac{x_2^2}{2} \psi(x_1).$$

By (5.58) and (5.62) we have $\partial_{jj} G(\hat{x}_2) = \psi(x_1)$ for $j \geq 2$ and $\partial_{kj} G(\hat{x}_2) = 0$. Thus, for some p_j,

$$\partial_j G(\hat{x}_2) = p_j(x_1) + \psi(x_1) x_j, \ j \geq 3, \ \partial_2 G = 0.$$

In particular

$$G(\hat{x}_2) = h_j(\hat{x_j}, \hat{x}_2) + p_j(x_1) x_j + \psi(x_1) \frac{x_j^2}{2} \quad j \geq 3,$$

for some h_j, which means

$$G(\hat{x}_2) = h(\hat{x}_2) + \frac{1}{n-2} \sum_{j=3}^{n} p_j(x_1) x_j + \frac{\psi(x_1)}{n-2} \sum_{j=3}^{n} \frac{x_j^2}{2},$$

where $h(\hat{x}_2) = \frac{1}{n-2} \sum_{j=3}^{n} h_j(\hat{x}_2, \hat{x}_j)$. Since $\partial_j G(\hat{x}_2) = \psi(x_1) x_j + p_j(x_1)$ we immediately have $\partial_j h(\hat{x}_2) = 0$ for all $j \geq 3$ and thus $h(\hat{x}_2) = h(x_1)$. Hence, up to rename ψ, p_j to include the factor $\frac{1}{n-2}$):

$$g = h(x_1) + \sum_{j=2}^{n} p_j(x_1) x_j + \psi(x_1) \sum_{j=2}^{n} \frac{x_j^2}{2}.$$

Step two: *computing the g for $n = 3$.* In this case the relevant set of equation from Lemma 5.1 reduces to

$$
\begin{aligned}
\partial_{11} f + \partial_{22} f =& - \partial_{22} g, && (5.63)\\
\partial_{21} f =& 0, && (5.64)\\
\partial_{11} f + \partial_{33} f + \partial_{33} g =& 0, && (5.65)
\end{aligned}
$$

$$-\partial_{23}g - \partial_{23}f = 0, \tag{5.66}$$
$$\partial_{22}f + \partial_{33}f = 0, \tag{5.67}$$
$$-\partial_{13}f = 0, \tag{5.68}$$
$$\partial_{23}f = 0. \tag{5.69}$$

From (5.64) and (5.68) we get

$$\partial_{211}f = \partial_{311}f = 0 \quad \Rightarrow \quad \partial_{11}f = S_1(x_1).$$

From (5.64) and (5.69) we get

$$\partial_{122}f = \partial_{322}f = 0 \quad \Rightarrow \quad \partial_{22}f = S_2(x_2).$$

From (5.68) and (5.69) we get

$$\partial_{133}f = \partial_{233}f = 0 \quad \Rightarrow \quad \partial_{33}f = S_3(x_3).$$

But because of (5.67) clearly $S_2(x_2) = \varrho$, $S_3(x_3) = -\varrho$ for some constant $\varrho \in \mathbb{R}$. Thus setting $\psi = S_1$ we have from (5.63) and (5.65):

$$-\partial_{22}g = \psi(x_1) + \varrho, \tag{5.70}$$
$$-\partial_{33}g = \psi(x_1) - \varrho. \tag{5.71}$$

From (5.70) and $\partial_{23}g = 0$ (obtained by combining (5.69) and (5.66)) we get

$$-\partial_2 g = p_2(x_1) + (\psi(x_1) + \varrho)x_2, \tag{5.72}$$

and thus

$$-g = h_2(x_1, x_3) + p_2(x_1)x_2 + (\psi(x_1) + \varrho)\,\frac{x_2^2}{2}.$$

From (5.71) and again $\partial_{23}g = 0$ we in turn get

$$-\partial_3 g = p_3(x_1) + (\psi(x_1) - \varrho)x_3, \tag{5.73}$$

and

$$-g = h_3(x_1, x_2) + p_3(x_1)x_3 + (\psi(x_1) - \varrho)\,\frac{x_3^2}{2}.$$

Thus, we have (setting $h(x_1, x_2, x_3) = h_2(x_1, x_3) + h_3(x_1, x_2)$),

$$-2g = (\psi(x_1) + \varrho)\,\frac{x_2^2}{2} + (\psi(x_1) - \varrho)\,\frac{x_3^2}{2} + p_2(x_1)x_2 + p_3(x_1)x_3 + h(x_1, x_2, x_3),$$

but the relations (5.72), (5.73) also implies $\partial_2 h = \partial_3 h = 0$ and thence (up to a renaming of ψ, p_j, ϱ and h)

$$g = \psi(x_1)\sum_{j=2}^{3}\frac{x_j^2}{2} + \varrho(x_2^2 - x_3^2) + h(x_1) + \sum_{j=2}^{3} p_j(x_1)x_j.$$

Step three: *computing* u. It is now a simple computation to check that a special solution to (5.49) with a g as in (5.50) is given by

$$w = (H(x_1) - \Psi(x_1))e_1 + e_1\left(\sum_{j=2}^{n} P_j'(x_1)x_j + \frac{x_j^2}{2}\Psi''(x_1)\right) - \sum_{j=2}^{n} e_j\left(P_j(x_1) + x_j\Psi'(x_1)\right) + \varrho Q,$$

with $H' = h$, $\Psi''' = \psi$ e $P_j'' = p_j$, and where Q solves

$$\mathcal{E}_d Q = (e_1 \otimes_{\mathcal{E}_d} e_1)(x_2^2 - x_3^2). \tag{5.74}$$

Indeed

$$\nabla w = (H'(x_1) - \Psi'(x_1))e_1 \otimes e_1 + e_1 \otimes e_1 \sum_{j=2}^{n}\left(P_j''(x_1)x_j + \frac{x_j^2}{2}\Psi'''(x_1)\right)$$

$$
\begin{aligned}
&+\sum_{j=2}^{n} e_1 \otimes e_j \left(P_j'(x_1) + x_j \Psi''(x_1)\right) - \sum_{j=2}^{n} e_j \otimes e_1 \left(P_j'(x_1) + x_j \Psi''(x_1)\right)) \\
&-\sum_{j=2}^{n} e_j \otimes e_j \Psi'(x_1) + \varrho \nabla Q,
\end{aligned}
$$

and thus

$$
\begin{aligned}
\mathcal{E}w =& \left((H'(x_1) - \Psi'(x_1)) + \sum_{j=2}^{n} P_j''(x_1) x_j + \frac{x_j^2}{2} \Psi'''(x_1) \right) e_1 \otimes e_1 - \sum_{j=2}^{n} e_j \otimes e_j \Psi'(x_1) + \varrho \mathcal{E} Q \\
=& (g(x) - \varrho(x_2^2 - x_3^2) - \Psi'(x_1)) e_1 \otimes e_1 - \sum_{j=2}^{n} e_j \otimes e_j \Psi'(x_1) + \varrho \mathcal{E} Q \\
=& (g(x) - \varrho(x_2^2 - x_3^2)) e_1 \otimes e_1 - \Psi'(x_1) \mathrm{Id} + \varrho \mathcal{E} Q.
\end{aligned}
$$

Since

$$
\mathrm{tr}\,(\mathcal{E}w) = g(x) - \varrho(x_2^2 - x_3^2) - n\Psi'(x_1) + \varrho \mathrm{tr}\,(\mathcal{E}Q),
$$

we conclude by (5.74) that

$$
\begin{aligned}
\mathcal{E}_d w =& (g(x) - \varrho(x_2^2 - x_3^2)) e_1 \otimes e_1 - \Psi'(x_1) \mathrm{Id} - \frac{g(x) - \varrho(x_2^2 - x_3^2) - n\Psi'(x_1)}{n} \mathrm{Id} + \varrho \mathcal{E}_d Q \\
=& (g(x) - \varrho(x_2^2 - x_3^2)) e_1 \otimes_{\mathcal{E}_d} e_1 + \varrho \mathcal{E}_d Q = g(x) e_1 \otimes_{\mathcal{E}_d} e_1.
\end{aligned}
$$

Thus u must be equal to w up to a kernel element $L \in \mathrm{Ker}(\mathcal{E}_d)$ achieving the structure as in (5.51) and concluding the proof. □

We can now prove the rigidity structure for parallel vectors.

Proof of Theorem 5.11. Without loss of generality we can assume that $a = e_1$. Thus by mollifying $u_\varepsilon := u \star \varrho_\varepsilon$ and applying Lemma 5.13 we get that

$$
g_\varepsilon = h_\varepsilon(x_1) + \sum_{j=2}^{n} p_{\varepsilon,j}(x_1) x_j + \psi_\varepsilon(x_1) \sum_{j=2}^{n} \frac{x_j^2}{2} + \varrho_\varepsilon(x_2^2 - x_3^2),
$$

and that

$$
\begin{aligned}
u_\varepsilon =& (H_\varepsilon(x_1) - \Psi_\varepsilon(x_1)) e_1 + \left(\sum_{j=2}^{n} P_{\varepsilon,j}'(x_1) x_j + \frac{x_j^2}{2} \Psi_\varepsilon''(x_1) \right) e_1 \\
&- \sum_{j=2}^{n} (x_j \Psi_\varepsilon'(x_1) + P_{\varepsilon,j}(x_1)) e_j + \varrho_\varepsilon Q(x) + L_\varepsilon(x),
\end{aligned} \tag{5.75}
$$

with $P_{\varepsilon,j}'' = p_{\varepsilon,j}$, $\Psi_\varepsilon''' = \psi_\varepsilon$, $H_\varepsilon' = h_\varepsilon$.

All is left to do is to compute the limit. Note that as before we conclude immediately that $L_\varepsilon \to L \in \mathrm{Ker}(\mathcal{E}_d)$ (as in the proof of Theorem (5.3)) so without loss of generality we can neglect the term in L_ε in computing the limit and assume right away that $u_\varepsilon \to u$ in L^1. Recall also that $g_\varepsilon \mathcal{L}^n \rightharpoonup^* \nu$ and thus

$$
\sup_{\varepsilon > 0} \{ \|g_\varepsilon\|_{L^1} \} < +\infty.
$$

Up to redefining the functions $H_\varepsilon, \Psi_\varepsilon, P_{\varepsilon,j}$ and operating a translation we can restrict ourselves to consider the convergence on the compact set $[-1, 1]^n$.

We now split the proof in two steps, depending on n being 3 or bigger than 3.

Step one: *the case* $n \geq 4$. In this case we do not have to handle the polynomial term Q since $\varrho_\varepsilon = 0$. Let $\eta_i \in C_c^\infty([-1,1])$ be such that

$$\eta_i \geq 0 \quad \text{for all } i = 1, \dots, n,$$
$$\int_{[-1,1]} \eta_i(t)\,\mathrm{d}t = 1 \quad \text{for all } i = 2, \dots, n.$$

Set also

$$r_i := \int_{[-1,1]} t\eta_i(t)\,\mathrm{d}t, \quad s_i := \int_{[-1,1]} \frac{t^2}{2}\eta_i(t)\,\mathrm{d}t \quad \text{for } i = 2, \dots, n.$$

Let us denote by $\eta_{\hat{j}}(\hat{x}_j) := \eta_1(x_1)\dots\eta_{j-1}(x_{j-1})\eta_{j+1}(x_{j+1})\dots\eta_n(x_n)$. Since for a fixed $j \geq 2$ we have $(\mathcal{E}_d u_\varepsilon)_{1j} = 0$ we get $-\partial_j(u_\varepsilon \cdot e_1) = \partial_1(u_\varepsilon \cdot e_j)$:

$$\begin{aligned}
-\int_{[-1,1]^n} (u_\varepsilon \cdot e_1)\eta_j'(x_j)\eta_{\hat{j}}(\hat{x}_j)\,\mathrm{d}x &= \int_{[-1,1]^n} (u_\varepsilon \cdot e_j)\eta_1'(x_1)\eta_{\hat{1}}(\hat{x}_1)\,\mathrm{d}x \\
&= -\int_{[-1,1]^2} (x_j\Psi_\varepsilon'(x_1) + P_{\varepsilon,j}(x_1))\eta_1'(x_1)\eta_j(x_j)\,\mathrm{d}x_1\,\mathrm{d}x_j \\
&= -r_j\int_{[-1,1]} \Psi_\varepsilon'(x_1)\eta_1'(x_1)\,\mathrm{d}x_1 - \int_{[-1,1]} P_{\varepsilon,j}(x_1))\eta_1'(x_1)\,\mathrm{d}x_1.
\end{aligned}$$

By selecting an even function η_j we get $r_j = 0$ and therefore

$$\left|\int_{[-1,1]} P_{\varepsilon,j}(x_1))\eta_1'(x_1)\,\mathrm{d}x_1\right| \leq \|\eta_j'\|_\infty \int_{[-1,1]^n} |(u_\varepsilon \cdot e_1)|\,\mathrm{d}x.$$

Since $u_\varepsilon \cdot e_1 \to u \cdot e_1$ it follows that

$$\sup_\varepsilon \left\{\|P_{\varepsilon,j}'\|_{L^1}\right\} \leq C. \tag{5.76}$$

By integrating (5.75) against $e_j\varphi\eta_2\dots\eta_n$ for $j \geq 3$, $\varphi \in L^\infty([-1,1])$ and even $\eta_i \in C^1$, $i \geq 2$ we get

$$\int_Q (u_\varepsilon(x) \cdot e_j)\varphi(x_1)\eta_{\hat{1}}(\hat{x}_1)\,\mathrm{d}x = -\int_{[-1,1]} P_{\varepsilon,j}(x_1)\varphi(x_1)\,\mathrm{d}x,$$

and thus, for all $\varepsilon > 0$, $j = 2, \dots, n$, by duality

$$\|P_{\varepsilon,j}\|_{L^1} = \sup_{\|\varphi\|_{L^\infty} \leq 1}\left\{\left|\int_{[-1,1]} P_{\varepsilon,j}(x_1)\varphi(x_1)\,\mathrm{d}x_1\right|\right\} \leq C\|u_\varepsilon\|_{L^1} \leq C.$$

Thence, the above combined with (5.76), we have $P_{\varepsilon,j} \to P_j$ (up to a subsequence) in L^1 and with $P_j \in \mathrm{BV}([-1,1])$. By observing that $u_\varepsilon \cdot e_j \to u \cdot e_j$ in L^1 we deduce also that $x_j\Psi_\varepsilon'(x_1) \to (u \cdot e_j) - P_j$ in L^1. By integrating over a rectangle for which $x_j \neq 0$ we get $\Psi_\varepsilon' \to G$ for some $G \in L^1$ and thence

$$u_\varepsilon \cdot e_j \to x_jG(x_1) + P_j(x_1) = u \cdot e_j,$$

and

$$\sup_\varepsilon \left\{\|\Psi_\varepsilon'\|_{L^1}\right\} < +\infty. \tag{5.77}$$

Fix $0 < s \leq 1$. We now choose $\eta_j = \eta_2$ for all $j \geq 2$ with η_2 even so that $s_j = s_2 = s$ and $r_j = 0$ for all $j \geq 2$ and this yields

$$\begin{aligned}
&\int_Q g_\varepsilon\eta_1(x_1)\dots\eta_n(x_n)\,\mathrm{d}x = \\
&\quad = \int_Q \left(h_\varepsilon(x_1) + \sum_{j=2}^n p_{\varepsilon,j}(x_1)x_j + \psi_\varepsilon(x_1)\sum_{j=2}^n \frac{x_j^2}{2} + \varrho_\varepsilon(x_2^2 - x_3^2)\right)\eta_1(x_1)\dots\eta_n(x_n)\,\mathrm{d}x
\end{aligned}$$

$$= \int_{[-1,1]} h_\varepsilon(x_1)\eta_1(x_1)\,\mathrm{d}x_1 + \sum_{j=2}^n \int_{[-1,1]} \psi_\varepsilon(x_1)\eta_1(x_1)\sum_{j=2}^n s\,\mathrm{d}x_1$$
$$= \int_{[-1,1]} H_\varepsilon'(x_1)\eta_1(x_1)\,\mathrm{d}x_1 + s(n-1)\int_{[-1,1]} \Psi_\varepsilon'''(x_1)\eta_1(x_1)\,\mathrm{d}x_1.$$

Note that this means that, for all $0 < s \le 1$, there is $C = C(s)$ such that

$$\sup_\varepsilon \left\{ \|s(n-1)\Psi_\varepsilon''' + H_\varepsilon'\|_{L^1} \right\} < C(s).$$

By selecting two different values for s we can decouple the uniform bound as

$$\sup_\varepsilon \left\{ \|H_\varepsilon'\|_{L^1} \right\} < +\infty, \quad \sup_\varepsilon \left\{ \|\Psi_\varepsilon'''\|_{L^1} \right\} < +\infty. \tag{5.78}$$

Fix now any $j \ge 2$. By now selecting η_i's for $i \ge 2$ to be all equal and even except for $i = j$ (so that $s_i = s$, $r_i = 0$ for $i \ne j$, $i \ge 2$). Then

$$\int_Q g_\varepsilon \eta_1(x_1)\dots\eta_n(x_n)\,\mathrm{d}x = s(n-2)\int_{[-1,1]} \Psi_\varepsilon'''(x_1)\eta_1(x_1)\,\mathrm{d}x_1 + s_j \int_{[-1,1]} \Psi_\varepsilon'''(x_1)\eta_1(x_1)\,\mathrm{d}x_1$$
$$+ r_j \int_{[-1,1]} P_{\varepsilon,j}''(x_1)\eta_1(x_1)\,\mathrm{d}x_1 + \int_{[-1,1]} H_\varepsilon'(x_1)\eta_1(x_1)\,\mathrm{d}x_1,$$

and thus

$$\|P_{\varepsilon,j}''\|_{L^1} \le C(\|\Psi_\varepsilon'''\|_{L^1} + \|H_\varepsilon'\|_{L^1} + \|g_\varepsilon\|_{L^1}),$$

yielding

$$\sup_\varepsilon \{\|P_{\varepsilon,j}''\|_{L^1}\} < +\infty \quad \text{for all } j \ge 2. \tag{5.79}$$

Analogously considering $\varphi \in L^\infty$ and even $\eta_j = \eta_2$ for all $j \ge 2$ we get

$$\int_Q (u_\varepsilon \cdot e_1)\varphi(x_1)\eta_2(x_2)\dots\eta_n(x_n) = \int_{[-1,1]} (H_\varepsilon(x_1) - \Psi_\varepsilon(x_1))\varphi(x_1)\,\mathrm{d}x_1$$
$$+ s(n-1)\int_{[-1,1]} \Psi_\varepsilon''(x_1)\varphi(x_1)\,\mathrm{d}x_1,$$

meaning

$$\sup_{\varepsilon>0}\{\|H_\varepsilon - \Psi_\varepsilon + s(n-1)\Psi_\varepsilon''\|_{L^1}\} \le C(s).$$

By selecting two different values for s we can decouple the uniform bound as

$$\sup_{\varepsilon>0}\{\|H_\varepsilon - \Psi_\varepsilon\|_{L^1}\} < +\infty, \quad \sup_{\varepsilon>0}\{\|\Psi_\varepsilon''\|_{L^1}\} < +\infty. \tag{5.80}$$

By combining (5.80), (5.78) and (5.77) we get that

$$H_\varepsilon - \Psi_\varepsilon \to F, \quad \Psi_\varepsilon'' \to G' \ \text{ in } L^1,$$

for some $F \in \mathrm{BV}([-1,1])$, $G \in W^{1,1}([-1,1])$, $G' \in \mathrm{BV}([-1,1])$. By combining (5.76) and (5.79) we get also that

$$P_{\varepsilon,j}' \to P_j' \ \text{ in } L^1,$$

with $P_j' \in \mathrm{BV}([-1,1])$ and the final limit u has the claimed structure.

Step two: *the case* $n = 3$. The main issue here is in deducing the convergence of ϱ_ε and then everything follows exactly as in Step one. By choosing $\eta_2 = \eta_3$ even, integrating $g_\varepsilon\eta_1\eta_2\eta_3$ over Q we can obtain again (5.78) since the polynomial part cancels out. By now considering $\varphi \in L^\infty$ and only η_2 to be even (and still integrating $g_\varepsilon\eta_1\eta_2\eta_3$ over Q) we conclude that

$$\left| \int_{[-1,1]} (P_{\varepsilon,3}''(x_1) + \varrho_\varepsilon(s_2 - s_3))\varphi(x_1)\,\mathrm{d}x_1 \right| \le C(s_2, s_3)(\|H_\varepsilon'\|_{L^1} + \|\Psi_\varepsilon'''\|_{L^1} + \|g\|_{L^1}).$$

By selecting specific s_2, s_3 we can again decouple the bound as

$$\sup_{\varepsilon}\{\|P''_{\varepsilon,3}\|_{L^1}\} \leq C, \quad \sup_{\varepsilon}\{\varrho_\varepsilon\} \leq C.$$

In particular This implies that $\varrho_\varepsilon \to \varrho$ and thus $\varrho_\varepsilon Q \to \varrho Q$. Now, by considering $w_\varepsilon := u_\varepsilon - \varrho_\varepsilon Q$ and by applying the exact arguments of Step one to w_ε we conclude that $u - \varrho Q$ has the claimed structure. The proof is complete. □

Proof of Theorem 1.2. It comes as a consequence of Proposition 3.2, Theorem 5.3 and Theorem 5.11. □

6. Proof of kernel projection Theorem 1.4

We now provide a proof of Theorem 1.4, yielding an explicit map $\mathcal{R}_K$ which is particularly useful in homogenization and integral representation problems.

For any given convex set K we recall the intrinsic quantity defined in Theorem 1.4:

$$\tau_K := \frac{1}{P(K)} \int_{\partial K} \left(\frac{|y - \mathrm{bar}(K)|^2}{2} \mathrm{Id} - (y - \mathrm{bar}(K)) \otimes (y - \mathrm{bar}(K)) \right) \mathrm{d}\mathcal{H}^{n-1}(y).$$

where $P(K)$ is the distributional perimeter of K (see [44]),

$$\mathrm{bar}(K) := \frac{1}{|K|} \int_K y \, \mathrm{d}y.$$

and we denote by ν_K the outer unit normal (defined $\mathcal{H}^{n-1}$-a.e. on ∂K). For such sets the distributional tangential derivative of ν_K is a matrix-valued Radon measure [35, Theorem 2, Section 6.3]. We will denote it as $D^{\mathrm{Tan}}\nu_K \in \mathcal{M}(\partial K; \mathbb{R}^{n\times n})$, we recall that it is supported on ∂K and

$$\int_{\partial K} F^t(y) \, \mathrm{d}D^{\mathrm{Tan}}\nu_K(y) := -\int_{\partial K} \nu_K^t(y)(\nabla^{\mathrm{Tan}} F)(y) d\mathcal{H}^{n-1}(y) \quad \text{for all } F \in C^\infty(\mathbb{R}^n; \mathbb{R}^n), \tag{6.1}$$

where we have defined

$$\nabla^{\mathrm{Tan}} F(y) := \nabla F(y) - \nabla F(y)\nu_K(y) \otimes \nu_K(y).$$

Remark 6.1. If K has C^2 boundary the $D^{\tan}\nu_K$ can be made explicit in terms of K as follows: let $H_K(y)$ be the scalar mean curvature of ∂K at y and define the vector $\mathbf{H}_K(y) := (n-1)H_K(y)\nu_K(y)$. Then we recall that (see [44, Theorem 11.8]) for such regular sets we have the Gauss-Green formula on surfaces

$$\int_{\partial K} \nabla^{\mathrm{Tan}} (v \cdot \nu_K) \, d\mathcal{H}^{n-1} = \int_{\partial K} \mathbf{H}_K v \cdot \nu_K d\mathcal{H}^{n-1},$$

in force for all $v \in C^\infty(\mathbb{R}^n; \mathbb{R}^n)$. In particular since

$$\nabla^{\mathrm{Tan}} (v \cdot \nu_K) = v^t \nabla^{\mathrm{Tan}} \nu_K + \nu_K^t \nabla^{\mathrm{Tan}} v,$$

we have

$$\int_{\partial K} \nu_K^t \left(\nabla^{\mathrm{Tan}} v\right) \, \mathrm{d}\mathcal{H}^{n-1} = \int_{\partial K} \mathbf{H}_K v \cdot \nu_K - v^t (\nabla^{\mathrm{Tan}} \nu_K) \, \mathrm{d}\mathcal{H}^{n-1}. \tag{6.2}$$

Thus

$$D^{\mathrm{Tan}}\nu_K = \left(\nabla^{\mathrm{Tan}}\nu_K - \mathbf{H}_K \otimes \nu_K\right) \mathcal{H}^{n-1}_{\llcorner \partial K}.$$

According to our convention, the mean curvature H_K is always positive for convex sets K. As a title of example we recall that, for $K = B_r$ we know that $H_{\partial B_r} = \frac{n-1}{r}$.

Remark 6.2. If K has only piece-wise C^2 boundary then $D^{\mathrm{Tan}}\nu_K$ might contains parts orthogonal to $\mathcal{H}^{n-1}_{\llcorner \partial K}$. Consider, as a title of example, $K = [-1,1]^2$ in $\mathbb{R}^2$. Then, still as a consequence of the Gauss-Green formula on surfaces (with the boundary terms accounted with the conormal as in [44, Theorem 11.8]):

$$D^{\mathrm{Tan}}\nu_K = \left[-(e_1+e_2)\mathcal{H}^0_{\llcorner (1,1)} - (e_1 - e_2)\mathcal{H}^0_{\llcorner (1,-1)} + (e_1+e_2)\mathcal{H}^0_{\llcorner (-1,-1)} - (-e_1+e_2)\mathcal{H}^0_{\llcorner (-1,1)} \right] \otimes \nu_K,$$

which is nothing but the mean curvature measure (supported only on the vertexes of Q) tensorized with ν_K.

Given these intrinsic quantities, now for $u \in \mathrm{BD}_{\mathrm{dev}}(\Omega)$, $x \in \mathbb{R}^n$ and for K a convex set we recall the Definition of the relevant quantities

$$\mathrm{s}_K[u] := -\frac{1}{(n-1)|K|}\int_{\partial K} u^t \,\mathrm{d}D^{\mathrm{Tan}}\nu_K(y), \tag{6.3}$$

$$\mathrm{A}_K[u] := \frac{1}{2|K|}\int_{\partial K} (u \otimes \nu_K - \nu_K \otimes u)\,\mathrm{d}\mathcal{H}^{n-1}(x), \tag{6.4}$$

$$\gamma_K[u] := \frac{1}{n|K|}\int_{\partial K} (u \cdot \nu_K)\,\mathrm{d}\mathcal{H}^{n-1}(y), \tag{6.5}$$

$$\mathrm{b}_K[u] := \frac{1}{P(K)}\int_{\partial K} u\,\mathrm{d}\mathcal{H}^{n-1}(y) + \tau_K \mathrm{s}_K[u], \tag{6.6}$$

and we set

$$\begin{aligned}\mathcal{R}_K[u](y) :=& (\mathrm{A}_K[u] + \gamma_K[u]\mathrm{Id}\,)\,(y - \mathrm{bar}(K)) \\ &+ (\mathrm{s}_K[u]\cdot(y-\mathrm{bar}(K)))(y-\mathrm{bar}(K)) - \mathrm{s}_K[u]\frac{|y-\mathrm{bar}(K)|^2}{2} + \mathrm{b}_K[u].\end{aligned} \tag{6.7}$$

This correction with $\mathrm{bar}(K)$ makes the quantity $\mathcal{R}_K$ depend only on the shape and the size of K and not on the position. We now prove Theorem 1.4.

Proof of Theorem 1.4. The Poincarè inequality (1.14) comes from Proposition 2.7 as soon as we show that $\mathcal{R}_K$ is linear, bounded and fixes $\mathrm{Ker}(\mathcal{E}_d)$ (see [22, 19]). Thus we focus on the first part of the statement.

Clearly $\mathcal{R}_K$ is linear. Since the convergence in $\mathrm{BD}_{\mathrm{dev}}$ implies the convergence of the traces, and since the elements identifying $\mathcal{R}_K$ are defined on the traces, the continuity (and thus the boundedness) follows at once. We focus on showing that $\mathcal{R}_K$ fixes the elements of $\mathcal{E}_{\mathrm{d}}$. For $L \in \mathcal{E}_d$ we have

$$L(y) = (A + \gamma\mathrm{Id}\,)y + (s\cdot y)y - s\frac{|y|^2}{2} + b,$$

for some $R \in \mathbb{R}^{n\times n}_{skew}, \gamma \in \mathbb{R}$,$s$, $b \in \mathbb{R}^n$. Let us split

$$L(y) := M(y) + B(y), \quad M(y) := (A + \gamma\mathrm{Id}\,)y + b, \quad B(y) := (s\cdot y)y - s\frac{|y|^2}{2}.$$

It is straightforward to see that, for $u \in BV(\Omega)$it holds that

$$\mathrm{A}_K[u] = \frac{1}{2|K|}(Du(K) - Du^t(K)), \tag{6.8}$$

$$\gamma_K[u] = \frac{1}{n|K|}\mathrm{tr}\,(Du(K)), \tag{6.9}$$

implying

$$\mathrm{A}_K[M] = A, \;\; \gamma_K[M] = \gamma.$$

Due to the very definition of s_K we also have, for regular maps $v \in C^1(\mathbb{R}^n;\mathbb{R}^n)$ that

$$\mathrm{s}_K[v] = -\frac{1}{(n-1)|K|}\int_{\partial K} \nu_K^t(y)\nabla^{\mathrm{Tan}} v(y)\,\mathrm{d}\mathcal{H}^{n-1}(y).$$

In particular we have

$$\nu_K^t\nabla^{\mathrm{Tan}}M(y) = (\nu_K^t A + \gamma\nu_K) - (\nu_K^t A\nu_K)\nu_K(y) - \gamma\nu_K(y) = \nu_K^t A - (\nu_K^t A \nu_K)\nu_K(y).$$

Since K is center-symmetric, this facts immediately implies

$$\mathrm{s}_K[M] = 0, \quad \mathrm{b}_K[M] = b.$$

Moreover, since

$$DB = (s\cdot y)\mathrm{Id} + y\otimes s - s\otimes y,$$

then, again by center-symmetry, $DB(K)=0$, and thence we have (recalling (6.8), (6.9)):

$$\mathrm{A}_K[B]=0, \quad \gamma_K[B]=0.$$

Furthermore (setting to alleviate the notation $\nu=\nu_K$)

$$\nabla^{\mathrm{Tan}}B = (s\cdot y)\mathrm{Id} + y\otimes s - s\otimes y - \left[(s\cdot y)(\nu\otimes\nu) + (s\cdot\nu)y\otimes\nu - (y\cdot\nu)s\otimes\nu\right],$$

and thus

$$\begin{aligned}\nu^t\nabla^{\mathrm{Tan}}B &= (s\cdot y)\nu + (\nu\cdot y)s - (\nu\cdot s)y - [(s\cdot y)\nu + (s\cdot\nu)(\nu\cdot y)\nu - (y\cdot\nu)(\nu\cdot s)\nu]\\ &= (\nu\cdot y)s - (\nu\cdot s)y.\end{aligned}$$

Therefore

$$\mathrm{s}_K[B] = \frac{1}{(n-1)|K|}\int_{\partial K}\nu^t\nabla^{\mathrm{Tan}}B\,\mathrm{d}\mathcal{H}^{n-1}(y) = \frac{1}{(n-1)|K|}\int_{\partial K}((\nu\cdot y)s - (\nu\cdot s)y)\,\mathrm{d}\mathcal{H}^{n-1}(y).$$

Since, clearly

$$\begin{aligned}\int_{\partial K} s(y\cdot\nu)\,\mathrm{d}y &= \sum_{i=1}^n e_i\int_K \mathrm{div}\ \ \psi(s_i y)\,\mathrm{d}y = n|K|s,\\ \int_{\partial K} y(s\cdot\nu)\,\mathrm{d}y &= \sum_{i=1}^n e_i\int_K \mathrm{div}\ \ \psi(y_i s)\,\mathrm{d}y = s|K|,\end{aligned}$$

then

$$\mathrm{s}_K[B] = \frac{1}{(n-1)|K|}\left(n|K|s - s|K|\right)s = s.$$

Finally, this also shows that

$$\mathrm{b}_K[B] = -\tau_K s + \tau_K\mathrm{s}_K[B] = 0.$$

Summarizing

$$\begin{aligned}\mathrm{s}_K[L] &= \mathrm{s}_K[B] + \mathrm{s}_K[M] = s,\\ \mathrm{A}_K[L] &= \mathrm{A}_K[B] + \mathrm{A}_K[M] = A,\\ \gamma_K[L] &= \gamma_K[B] + \gamma_K[M] = \gamma,\\ \mathrm{b}_K[L] &= \mathrm{b}_K[B] + \mathrm{b}_K[M] = b.\end{aligned}$$

yielding $\mathcal{R}_K[L]=L$. □

Before concluding the Section we state a Lemma which turns out to be quite useful in the blow-up theory. We recall the notion of *pushforward measure* (see also [16] for additional details). Let $\mu\in\mathcal{M}(A_1;\mathbb{R}^{n\times n})$ and $f: A_1\to A_2$ be a μ-measurable function. The pushforward measure $f_{\#}\mu\in\mathcal{M}(A_2;\mathbb{R}^{n\times n})$ is defined as

$$f_{\#}\mu(B) := \mu(f^{-1}(B)) \quad \text{for all } \mu\text{-measurable } B\subset A_2.$$

We recall that the following integral formula holds

$$\int_B g(x)\,\mathrm{d}(f_{\#}\mu)(x) = \int_{f^{-1}(B)} g(f(x))\,\mathrm{d}\mu(x). \tag{6.10}$$

Lemma 6.3. *Let K be a center symmetric convex body. If $u\in\mathrm{BD}_{\mathrm{dev}}(K_\varrho(x))$ then $v_\varrho(y) := \frac{u(x+\varrho y)}{\varrho}\in\mathrm{BD}_{\mathrm{dev}}(K)$ and*

$$\begin{aligned}|\mathcal{E}_d v_\varrho|(K) &= \varrho^{-n}|\mathcal{E}_d u|(K_\varrho(x)),\\ \mathcal{R}_K[v_\varrho](y) &= \frac{\mathcal{R}_{K_\varrho(x)}[u](x+\varrho y)}{\varrho}.\end{aligned}$$

Proof. We can suppose without loss of generality that $\mathrm{bar}(K) = 0$. If $u \in C^\infty(K_\varrho(x))$ we have

$$\mathcal{E}_d v_\varrho(y) = \mathcal{E}_d \left(\frac{u(x+\varrho y)}{\varrho} \right) = (\mathcal{E}_d u)(x+\varrho y)$$

and thence,

$$\begin{aligned} |\mathcal{E}_d v_\varrho|(K) &= \int_K |(\mathcal{E}_d u)(x+\varrho y)| \,\mathrm{d}y \\ &= \varrho^{-n} \int_{K_\varrho(x)} |\mathcal{E}_d u(z)| \,\mathrm{d}z = \varrho^{-n} |\mathcal{E}_d u|(K_\varrho(x)). \end{aligned}$$

By approximation in the strict convergence (see [19, Theorem 2.8]) we now pass to the whole $\mathrm{BD}_{\mathrm{dev}}$. We now check the scaling properties of $\mathcal{R}_K$. We consider

$$\begin{aligned} \mathrm{A}_K[v_\varrho] &= \frac{1}{2\varrho|K|} \int_{\partial K} [u(x+\varrho y) \otimes \nu_K - \nu_K \otimes u(x+\varrho y)] \,\mathrm{d}\mathcal{H}^{n-1}(y) \\ &= \frac{1}{2|K_\varrho(x)|} \int_{\partial K_\varrho(x)} \left[u(z) \otimes \nu_K \left(\frac{z-x}{\varrho} \right) - \nu_K \left(\frac{z-x}{\varrho} \right) \otimes u(z) \right] \,\mathrm{d}\mathcal{H}^{n-1}(y) \\ &= \frac{1}{2|K_\varrho(x)|} \int_{\partial K_\varrho(x)} \left[u(z) \otimes \nu_{K_\varrho(x)}(z) - \nu_{K_\varrho(x)}(z) \otimes u(z) \right] \,\mathrm{d}\mathcal{H}^{n-1}(y) \\ &= \mathrm{A}_{K_\varrho(x)}[u], \end{aligned} \tag{6.11}$$

since

$$\nu_{K_\varrho(x)}(z) = \nu_K \left(\frac{z-x}{\varrho} \right) \quad \text{for all } z \in \partial K_\varrho(x).$$

The exact same computation also yields

$$\gamma_K[v_\varrho] = \gamma_{K_\varrho(x)}[u]. \tag{6.12}$$

Moreover, setting $g^{x,\varrho}(y) := x + \varrho y$, $g^{x,\varrho}(\partial K) = \partial K_\varrho(x)$ we have from (6.10)

$$\int_{\partial K_\varrho(x)} f(z) \,\mathrm{d}(g_\#(D^{\mathrm{Tan}}\nu_K)(z) = \int_{\partial K} f(g(y)) \,\mathrm{d}(D^{\mathrm{Tan}}\nu_K)(y). \tag{6.13}$$

Therefore

$$\begin{aligned} \mathrm{s}_K[v_\varrho] &= -\frac{1}{(n-1)\varrho|K|} \int_{\partial K} v(x+\varrho y)^t \,\mathrm{d}(D^{\mathrm{Tan}}\nu_K)(y) \\ &= -\frac{1}{(n-1)\varrho|K|} \int_{\partial K_\varrho(x)} v(z)^t \,\mathrm{d}(g_\#^{x,\varrho} D^{\mathrm{Tan}}\nu_K)(z). \end{aligned} \tag{6.14}$$

Since

$$\nabla^{\mathrm{Tan}} \left(F(x+\varrho y) \right) = \varrho \left(\nabla^{\mathrm{Tan}} F \right)(x+\varrho y)$$

we have

$$\begin{aligned} \int_{\partial K} \nu_K^t(y) \left(\nabla^{\mathrm{Tan}} F \right)(x+\varrho y) \,\mathrm{d}\mathcal{H}^{n-1}(y) &= \frac{1}{\varrho} \int_{\partial K} \nu_K^t(y) \nabla^{\mathrm{Tan}} \left(F(x+\varrho y) \right) \,\mathrm{d}\mathcal{H}^{n-1}(y) \\ &= -\frac{1}{\varrho} \int_{\partial K} F(x+\varrho y)^t \,\mathrm{d}(D^{\mathrm{Tan}}\nu_K)(y) = -\frac{1}{\varrho} \int_{\partial K_\varrho(x)} F(y)^t \,\mathrm{d}(g_\#^{x,\varrho} D^{\mathrm{Tan}}\nu_K)(y). \end{aligned}$$

where the last equality follows from (6.13). Also

$$\begin{aligned} \int_{\partial K} \nu_K^t(y) \left(\nabla^{\mathrm{Tan}} F \right)(x+\varrho y) \,\mathrm{d}\mathcal{H}^{n-1}(y) &= \frac{1}{\varrho^{n-1}} \int_{\partial K_\varrho(x)} \nu_K \left(\frac{z-x}{\varrho} \right)^t \left(\nabla^{\mathrm{Tan}} F \right)(z) \,\mathrm{d}\mathcal{H}^{n-1}(z) \\ &= -\frac{1}{\varrho^{n-1}} \int_{\partial K_\varrho(x)} F(z)^t \,\mathrm{d}(D^{\mathrm{Tan}}\nu_{K_\varrho(x)})(z). \end{aligned}$$

in particular we deduce that

$$\int_{\partial K_\varrho(x)} F(z)^t\, \mathrm{d}(D^{\mathrm{Tan}}\nu_{K_\varrho(x)})(z) = \varrho^{n-2}\int_{\partial K_\varrho(x)} F(y)^t\, \mathrm{d}(g_{\#}^{x,\varrho}D^{\mathrm{Tan}}\nu_K)(y) \quad \text{for all } F \in C^\infty(\mathbb{R}^n;\mathbb{R}^n).$$

Thence $g_{\#}^{x,\varrho}D^{\mathrm{Tan}}\nu_K = \varrho^{2-n}D^{\mathrm{Tan}}\nu_{K_\varrho(x)}$. Thus, from (6.14):

$$\begin{aligned} \mathrm{s}_K[v_\varrho] &= -\frac{1}{(n-1)\varrho|K|}\int_{\partial K_\varrho(x)} v(z)^t\, \mathrm{d}(g_{\#}^{x,\varrho}D^{\mathrm{Tan}}\nu_K)(z) \\ &= -\frac{\varrho}{(n-1)|K_\varrho(x)|}\int_{\partial K_\varrho(x)} v(z)^t\, \mathrm{d}(D^{\mathrm{Tan}}\nu_{K_\varrho(x)})(z) = \varrho\mathrm{s}_{K_\varrho(x)}[v]. \end{aligned} \tag{6.15}$$

Finally we note that

$$\tau_{K_\varrho(x)} = \frac{\varrho^2}{P(K)}\int_{\partial K}\left(\frac{|y|^2}{2}\mathrm{Id} - y\otimes y\right)\, \mathrm{d}\mathcal{H}^{n-1}(y) = \varrho^2\tau_K,$$

and

$$\frac{1}{P(K)}\int_{\partial K}\frac{u(x+\varrho y)}{\varrho}\, \mathrm{d}\mathcal{H}^{n-1}(y) = \frac{1}{\varrho P(K_\varrho(x))}\int_{\partial K_\varrho(x)} u(z)\, \mathrm{d}\mathcal{H}^{n-1}(z).$$

Thus

$$\begin{aligned} \mathrm{b}_K[v_\varrho] &= \frac{1}{\varrho P(K_\varrho(x))}\int_{\partial K_\varrho(x)} u(z)\, \mathrm{d}\mathcal{H}^{n-1}(z) + \frac{\tau_{K_\varrho(x)}}{\varrho^2}\mathrm{s}_K[v_\varrho] \\ &= \frac{1}{\varrho P(K_\varrho(x))}\int_{\partial K_\varrho(x)} u(z)\, \mathrm{d}\mathcal{H}^{n-1}(z) + \frac{\tau_{K_\varrho(x)}}{\varrho}\mathrm{s}_{K_\varrho(x)}[u] \\ &= \frac{1}{\varrho}\mathrm{b}_{K_\varrho(x)}[u]. \end{aligned} \tag{6.16}$$

In particular, by collecting (6.11),(6.12),(6.15), (6.16) and the very definition of $\mathcal{R}_K, \mathcal{R}_{K_\varrho(x)}$ we conclude

$$\mathcal{R}_K[v_\varrho](y) = \frac{\mathcal{R}_{K_\varrho(x)}[u](x+\varrho y)}{\varrho}.$$

□

7. Vanishing properties of the projection

The final Proposition of this Section states a key property, usually required in the application, for the map $\mathcal{R}_K$.

Proposition 7.1. *Let K be a center-symmetric convex body and let $u \in \mathrm{BD}_{\mathrm{dev}}(\Omega)$. Then, for any $x \notin \Theta_u$ (as in Definition (4.1))it holds that*

$$\lim_{\varrho\to 0}\varrho^2|\mathrm{s}_{K_\varrho(x)}[u]| = \lim_{\varrho\to 0}\varrho|\mathrm{A}_{K_\varrho(x)}[u]| = \lim_{\varrho\to 0}\varrho|\gamma_{K_\varrho(x)}[u]| = 0. \tag{7.1}$$

and

$$\lim_{\varrho\to 0}⨍_{K_\varrho(x)}|u(y) - \mathrm{b}_{K_\varrho(x)}[u]|\, \mathrm{d}y = 0.$$

In particular the above slightly extend a result for quasi-continuous points for $\mathrm{BD}_{\mathrm{dev}}$ maps, proved firstly in [10] in the general context of elliptic operators. We recall that a map u is said to be *approximately quasi-continuous* at $x_0 \in \Omega$ if

$$\lim_{\varrho\to 0}\min_{b\in\mathbb{R}^n}\left\{⨍_{K_\varrho(x_0)}|u(y) - b|\, \mathrm{d}y\right\} = 0.$$

With the following Proposition we can ensure that $\mathrm{BD}_{\mathrm{dev}}$ maps are approximately quasi-continuous at $\mathcal{H}^{n-1}$-a.e. $x \in \Omega$.

Let us underline that this notion of continuity is weaker than the notion of approximate continuity, for which fewer things are known. To obtain this fact we will exploit the specific case $K = B$, in particular the following Proposition.

Proposition 7.2. *Let $\varrho > 0$, $u \in \mathrm{BD}_{\mathrm{dev}}(\Omega)$. Then for all $x \notin \Theta_u$ (as in Definition (4.1))it holds that*

$$\lim_{\varrho\to 0}\varrho^2|\mathrm{s}_{B_\varrho(x)}[u]| = \lim_{\varrho\to 0}\varrho|\mathrm{A}_{B_\varrho(x)}[u]| = \lim_{\varrho\to 0}\varrho|\gamma_{B_\varrho(x)}[u]| = 0 \tag{7.2}$$

and

$$\lim_{\varrho\to 0}\fint_{B_\varrho(x)}|u(y) - \mathrm{b}_{B_\varrho(x)}[u]|\,\mathrm{d}y = 0.$$

In particular u is approximately quasi-continuous at $\mathcal{H}^{n-1}$-a.e. point $x \in \Omega$.

The proof of the above Proposition is based on an useful characterization of $\mathrm{A}_{B_\varrho(x)}$, $\gamma_{B_\varrho(x)}$, $\mathrm{s}_{B_\varrho(x)}$ developed in the spirit of [5], [42]. The proof is quite technical and the techniques are well-known therefore we postpone it to the Appendix. We instead report here the proof of Proposition 7.1.

Proof. Let $x_0 \notin \Theta_u$. The strategy is to start from $\mathcal{R}_{B_\varrho(x_0)}$ and replace each quantity with the one under analysis. For example, to deduce the correct decay rate on $\mathrm{s}_{K_\varrho(x_0)}$ we consider the linear application

$$\begin{aligned}\bar{\mathcal{R}}_{\varrho,x_0}[u] :=&(\mathrm{s}_{K_\varrho(x_0)}[u]\cdot(y-x_0))(y-x_0) - \frac{|y-x_0|^2}{2}\mathrm{s}_{K_\varrho(x_0)}[u]\\ &+ (\mathrm{A}_{B_\varrho(x_0)}[u] + \gamma_{B_\varrho(x_0)}[u]\mathrm{Id}\,)(y-x_0) + \mathrm{b}_{B_\varrho(x_0)}[u]\end{aligned}$$

which is basically $\mathcal{R}_{B_\varrho(x_0)}$ where we replaced $s_{B_\varrho(x_0)}[u]$ with $s_{K_\varrho(x_0)}[u]$. The same computation as exploited in the proof of Lemma 6.3 tells us that

$$\bar{\mathcal{R}}_{1,0}[u(x_0+\varrho\cdot)](y) = \bar{\mathcal{R}}_{\varrho,x_0}[u](x_0+\varrho y),$$

while the same arguments in the Proof of Theorem 1.4 in Section 6 yields $\bar{\mathcal{R}}_{\varrho,x_0}[L] = L$ for all $L \in \mathfrak{K}$. In particular Proposition 2.7 combined with these two facts, as in Proposition 2.7, leads us to say that

$$\|u - \bar{\mathcal{R}}_{\varrho,x_0}[u]\|_{L^1(B_\varrho(x_0))} \le c\varrho|\mathcal{E}_d|(B_\varrho(x_0)) \quad \text{for all } u \in \mathrm{BD}_{\mathrm{dev}}(\Omega)$$

for a constant $c > 0$ depending on $\bar{\mathcal{R}}_{1,0}$ only. Hence

$$\begin{aligned}\|u - \bar{\mathcal{R}}_{\varrho,x_0}[u]\|_{L^1(B_\varrho(x_0))} \ge &\int_{B_\varrho(x_0)}\left|(s_{K_\varrho(x_0)}[u]\cdot(y-x_0))(y-x_0) - \frac{|y-x_0|^2}{2}s_{K_\varrho(x_0)}[u]\right|\,\mathrm{d}y\\ &-\int_{B_\varrho(x_0)}\varrho\left|\mathrm{A}_{B_\varrho(x_0)}[u] + \gamma_{B_\varrho(x_0)}[u]\mathrm{Id}\,\right|\,\mathrm{d}y\\ &-\int_{B_\varrho(x_0)}\left|u(y) - \mathrm{b}_{B_\varrho(x_0)}[u]\right|\,\mathrm{d}y.\end{aligned}$$

Moreover, for $v \in \mathbb{R}^n$it holds that

$$\begin{aligned}\int_{B_\varrho(x_0)}&\left|(v\cdot(y-x_0))(y-x_0) - \frac{|y-x_0|^2}{2}v\right|\,\mathrm{d}y = \varrho^n\int_B\varrho^2\left|(v\cdot z)z - \frac{|z|^2}{2}v\right|\,\mathrm{d}z\\ &\ge \varrho^{n+2}\left|\int_B\left[(v\cdot z)z - \frac{|z|^2}{2}v\right]\,\mathrm{d}z\right| = \varrho^{n+2}\left|\int_0^1 t^{n+1}\int_{\partial B}\left[(v\cdot z)z - \frac{v}{2}\right]\,\mathrm{d}\mathcal{H}^{n-1}(z)\right|\\ &= \varrho^{n+2}C(n)|v|.\end{aligned}$$

Hence

$$\begin{aligned}\varrho^2|s_{K_\varrho(x_0)}[u]| \le &c\frac{|\mathcal{E}_d u|(B_\varrho(x_0))}{\varrho^{n-1}}\\ &+ \kappa\left(\fint_{B_\varrho(x_0)}\left|u(y) - \mathrm{b}_{B_\varrho(x_0)}[u]\right|\,\mathrm{d}y + \varrho\left|\mathrm{A}_{B_\varrho(x_0)}[u] + \gamma_{B_\varrho(x_0)}[u]\mathrm{Id}\,\right|\right).\end{aligned}$$

Thanks to Proposition 7.2 and the fact that $x_0 \notin \Theta_u$ we get

$$\lim_{\varrho\to 0} \varrho^2 |s_{K_\varrho(x_0)}[u]| = 0.$$

The other quantities can be treated similarly by applying the previous argument to the linear application $\mathcal{R}_{B_\varrho(x_0)}$, where we have replaced $\mathrm{A}_{B_\varrho(x_0)}$ by $\mathrm{A}_{K_\varrho(x_0)}$,

$$\begin{aligned}\bar{\mathcal{R}}_{\varrho,x_0}[u] :=&(\mathrm{s}_{B_\varrho(x_0)}[u]\cdot(y-x_0))(y-x_0) - \frac{|y-x_0|^2}{2}\mathrm{s}_{B_\varrho(x_0)}[u]\\ &+ (\mathrm{A}_{K_\varrho(x_0)}[u] + \gamma_{B_\varrho(x_0)}[u]\mathrm{Id}\,)(y-x_0) + \mathrm{b}_{B_\varrho(x_0)}[u],\end{aligned}$$

and to the linear application $\mathcal{R}_{B_\varrho(x_0)}$, where we have replaced $\gamma_{B_\varrho(x_0)}$ by $\gamma_{K_\varrho(x_0)}$,

$$\begin{aligned}\bar{\mathcal{R}}_{\varrho,x_0}[u] :=&(\mathrm{s}_{B_\varrho(x_0)}[u]\cdot(y-x_0))(y-x_0) - \frac{|y-x_0|^2}{2}s_{B_\varrho(x_0)}[u]\\ &+ (\mathrm{A}_{B_\varrho(x_0)}[u] + \gamma_{K_\varrho(x_0)}[u]\mathrm{Id}\,)(y-x_0) + \mathrm{b}_{B_\varrho(x_0)}[u].\end{aligned}$$

□

8. Appendix

We here provide a proof for Proposition 7.2. In order to do this we set up the lighter notation

$$\mathrm{A}_{\varrho,x} := \mathrm{A}_{B_\varrho(x)}, \quad \gamma_{\varrho,x} := \gamma_{B_\varrho(x)},$$
$$\mathrm{s}_{\varrho,x} := \mathrm{s}_{B_\varrho(x)}, \quad \mathrm{b}_{\varrho,x} := \mathrm{b}_{B_\varrho(x)}.$$

Moreover, in this case we can explicitely compute the above quantities.

Lemma 8.1. *It holds*

$$\begin{aligned}\mathrm{s}_{\varrho,x}[u] &= \frac{1}{(n-1)\omega_n\varrho^{n+1}}\int_{\partial B_\varrho(x)}\left[u(y) - n\left(\frac{y-x}{|y-x|}\cdot u(y)\right)\frac{y-x}{|y-x|}\right]\mathrm{d}\mathcal{H}^{n-1}(y),\\ \mathrm{A}_{\varrho,x}[u] &= \frac{1}{2\omega_n\varrho^n}\int_{\partial B_\varrho(x)}\left[u(y)\otimes\frac{y-x}{|y-x|} - \frac{y-x}{|y-x|}\otimes u(y)\right]\mathrm{d}\mathcal{H}^{n-1}(y),\\ \gamma_{\varrho,x}[u] &= \frac{1}{n\omega_n\varrho^n}\int_{\partial B_\varrho(x)}\left(u\cdot\frac{y-x}{|y-x|}\right)\mathrm{d}\mathcal{H}^{n-1}(y),\\ \mathrm{b}_{\varrho,x}[u] &= \frac{1}{n\omega_n\varrho^{n-1}}\int_{\partial B_\varrho(x)} u(y)\,\mathrm{d}\mathcal{H}^{n-1}(y) + \frac{n-2}{2n}\mathrm{s}_{\varrho,x}[u]\varrho^2.\end{aligned}$$

Proof. For $\mathrm{A}_{\varrho,x}$, $\gamma_{\varrho,x}$ it is enough to translate the definition given in (6.4), (6.5). Also $\mathrm{b}_{\varrho,x}$ can be obtained by confronting it with (6.6) and by observing that

$$\begin{aligned}\tau_{B_\varrho(x)} =& \frac{1}{n\omega_n\varrho^{n-1}}\int_{\partial B_\varrho(x)}\left[\frac{|y-x|^2}{2}\mathrm{Id} - (y-x)\otimes(y-x)\right]\mathrm{d}\mathcal{H}^{n-1}(y)\\ =&\frac{\varrho^2}{n\omega_n\varrho^{n-1}}\int_{\partial B_\varrho(x)}\left[\frac{1}{2}\mathrm{Id} - \frac{(y-x)}{|y-x|}\otimes\frac{(y-x)}{|y-x|}\right]\mathrm{d}\mathcal{H}^{n-1}(y)\\ =&\frac{\varrho^2}{2}\mathrm{Id} - \frac{\varrho^2}{n\omega_n}\int_{\partial B} z\otimes z\,\mathrm{d}\mathcal{H}^{n-1}(z) = \frac{\varrho^2}{2}\mathrm{Id} - \frac{\varrho^2}{n}\mathrm{Id} = \varrho^2\frac{(n-2)}{2n}\mathrm{Id}\end{aligned}$$

since

$$\int_{\partial B} z\otimes z\,\mathrm{d}\mathcal{H}^{n-1}(z) = |B|\mathrm{Id}\,.$$

We also compute $\mathrm{s}_{\varrho,x}$ starting from (6.3) and by recalling also Remark 6.1 :

$$\mathrm{s}_{\varrho,x} = \frac{1}{(n-1)\omega_n\varrho^n}\int_{\partial B_\varrho(x)} u^t(y)\nabla^{\mathrm{Tan}}\left(\frac{y-x}{|y-x|}\right)\mathrm{d}\mathcal{H}^{n-1}(y)$$

$$
\begin{aligned}
&-\frac{1}{(n-1)\omega_n\varrho^n}\int_{\partial B_\varrho(x)}\frac{(n-1)}{\varrho}\left(\frac{y-x}{|y-x|}\cdot u(y)\right)\frac{y-x}{|y-x|}\,\mathrm{d}\mathcal{H}^{n-1}(y)\\
=&\frac{1}{(n-1)\omega_n\varrho^{n+1}}\int_{\partial B_\varrho(x)} u)^t(y)\left[\mathrm{Id}-\frac{y-x}{|y-x|}\otimes\frac{y-x}{|y-x|}\right]\mathrm{d}\mathcal{H}^{n-1}(y)\\
&-\frac{1}{(n-1)\omega_n\varrho^{n+1}}\int_{\partial B_\varrho(x)}(n-1)\left(\frac{y-x}{|y-x|}\cdot u(y)\right)\frac{y-x}{|y-x|}\,\mathrm{d}\mathcal{H}^{n-1}(y)\\
=&\frac{1}{(n-1)\omega_n\varrho^{n+1}}\int_{\partial B_\varrho(x)}\left[u(y)-\left(u(y)\cdot\frac{y-x}{|y-x|}\right)\frac{y-x}{|y-x|}\right]\mathrm{d}\mathcal{H}^{n-1}(y)\\
&-\frac{1}{(n-1)\omega_n\varrho^{n+1}}\int_{\partial B_\varrho(x)}(n-1)\left(\frac{y-x}{|y-x|}\cdot u(y)\right)\frac{y-x}{|y-x|}\,\mathrm{d}\mathcal{H}^{n-1}(y)\\
=&\frac{1}{(n-1)\omega_n\varrho^{n+1}}\int_{\partial B_\varrho(x)}\left[u(y)-n\left(\frac{y-x}{|y-x|}\cdot u(y)\right)\frac{y-x}{|y-x|}\right]\mathrm{d}\mathcal{H}^{n-1}(y).
\end{aligned}
$$

□

The next Proposition allows us to re-write $\mathrm{A}_{\varrho,x}$, $\gamma_{\varrho,x}$, $\mathrm{s}_{\varrho,x}$ in terms of a non-local interaction. These computations have been done by mimicking the approach in [5], [42].

Proposition 8.2. *Let $u\in \mathrm{BD}_{\mathrm{dev}}(\mathbb{R}^n)$. Then, for any $\tau>0$, and $0<\varrho<\tau$ it holds that*

$$
\mathrm{A}_{\varrho,x}[u]=-\frac{1}{\omega_n}\int_{B_\tau(x)\setminus B_\varrho(x)}\frac{\Gamma(y-x)}{|y-x|^n}\,\mathrm{d}\mathcal{E}_d u(y)+\mathrm{A}_{\tau,x}[u], \tag{8.1}
$$

$$
\gamma_{\varrho,x}[u]\mathrm{Id}=-\frac{1}{(n-1)\omega_n}\int_{B_\tau(x)\setminus B_\varrho(x)}\frac{\Xi(y-x)}{|y-x|^n}\,\mathrm{d}\mathcal{E}_d u(y)+\gamma_{\tau,x}[u]\mathrm{Id}\,, \tag{8.2}
$$

$$
\mathrm{s}_{\varrho,x}[u]=-\frac{1}{(n-1)\omega_n}\int_{B_\tau(x)\setminus B_\varrho(x)}\frac{\Upsilon(y-x)}{|y-x|^{n+1}}\,\mathrm{d}\mathcal{E}_d u(y)+\mathrm{s}_{\tau,x}[u], \tag{8.3}
$$

where $\Gamma(z),\Xi(z):\mathbb{R}^{n\times n}_{\mathrm{sym}_0}\to\mathbb{R}^{n\times n}$ are the zero homogeneus tensors acting as

$$
\Gamma(z)M:=\frac{Mz\otimes z}{|z|^2}-\frac{z\otimes Mz}{|z|^2},
$$

$$
\Xi(z)M:=\left(\frac{(z^tMz)}{|z|^2}-\frac{\mathrm{tr}\,(M)}{n}\right)\mathrm{Id}\,,
$$

and $\Upsilon(z):\mathbb{R}^{n\times n}_{\mathrm{sym}_0}\to\mathbb{R}^n$ is the zero homogeneus tensor acting as

$$
\Upsilon(z)M:=2\frac{Mz}{|z|}-(n+2)\frac{(z^tMz)}{|z|^2}\frac{z}{|z|}+\mathrm{tr}\,(M)\frac{z}{|z|}.
$$

Proof. Set, without loss of generality, $x=0$. Recall that for $f:\mathbb{R}^n\to\mathbb{R}$, $M:\mathbb{R}^n\to\mathbb{R}^{n\times n}$ the following chain rule formula holds:

$$
\mathrm{div}\ \psi(f(y)M(y))=f(y)\mathrm{div}\ \psi(M(y))+M(y)\nabla f(y).
$$

We now split the proof in three steps.

Step one: *Proof of* (8.1). Define the trace free symmetric matrix-valued map

$$
\psi_{ij}(y):=\frac{y_j(e_i\odot y)-y_i(e_j\odot y)}{\omega_n|y|^{n+2}}\in C^\infty(\mathbb{R}^n\setminus\{0\};\mathbb{R}^{n\times n}_{sym_0}).
$$

Then we claim that

$$
\begin{cases}
(\mathbb{E}_d\left[\frac{y}{|y|}\right]u)\cdot\psi_{ij}(y) & =\frac{(u\otimes\frac{y}{|y|}-\frac{y}{|y|}\otimes u)_{ij}}{2\omega_n|y|^n}\\
\mathcal{E}_d^*\psi_{ij} & =0\quad \text{on } \mathbb{R}^n\setminus\{0\}
\end{cases}. \tag{8.4}
$$

Indeed, since ψ_{ij} is symmetric and trace free we can compute the adjoint operator

$$\mathcal{E}_d^*\psi(y) = \mathrm{div}\ \ (\psi_{ij}(y))\,,$$

where recall that div M is meant to be the row divergence. Hence, since

$$\begin{aligned} \mathrm{div}\ \ (y_h(e_r \otimes y)) &= y_h(n+1)e_r,\\ \mathrm{div}\ \ (y_h(y \otimes e_r)) &= y\delta_{hr} + y_h e_r,\\ \mathrm{div}\ \ (y_h(e_r \odot y)) &= y_h(n+2)e_r + y\delta_{hr}, \end{aligned}$$

we have

$$\mathrm{div}\ \ \left(\frac{y_j}{|y|^{n+2}}(e_i \odot y)\right) = \frac{1}{|y|^{n+2}}[y_j(n+2)e_i + y\delta_{ij}] - \frac{(n+2)}{|y|^{n+4}}(y_j y_i y + e_i|y|^2).$$

The above expression is symmetric in i,j and thus

$$\mathrm{div}\ \ (\psi_{ij}(y)) = \mathrm{div}\ \ \left(\frac{y_j}{\omega_n|y|^{n+2}}(e_i \odot y)\right) - \mathrm{div}\ \ \left(\frac{y_i}{\omega_n|y|^{n+2}}(e_j \odot y)\right) = 0.$$

Moreover

$$\begin{aligned} \left(\mathbb{E}_d\left[\frac{y}{|y|}\right]u\right)\cdot\psi_{i,j}(y) &= \frac{1}{\omega_n|y|^{n+2}}\left[u \odot \frac{y}{|y|} - \frac{1}{n}\left(u\cdot\frac{y}{|y|}\right)\mathrm{Id}\right]\cdot(y_j(e_i\odot y) - y_i(e_j\odot y))\\ &= \frac{1}{\omega_n|y|^{n+2}}\left(u\odot\frac{y}{|y|}\right)\cdot(y_j(e_i\odot y) - y_i(e_j\odot y)) \end{aligned}$$

and

$$\begin{aligned} \left(u\odot\frac{y}{|y|}\right)\cdot(y_j(e_i\odot y) - y_i(e_j\odot y)) &= y_j\left(u\odot\frac{y}{|y|}\right)\cdot(e_i\odot y) - y_i\,(u\odot y)\cdot\left(e_j\odot\frac{y}{|y|}\right)\\ &= \frac{y_j}{2|y|}(u_i|y|^2 + (u\cdot y)y_i) - \frac{y_i}{2|y|}(u_j|y|^2 + (u\cdot y)y_j)\\ &= \frac{|y|^2}{2}\left(u\otimes\frac{y}{|y|} - \frac{y}{|y|}\otimes u\right)_{ij} \end{aligned}$$

since

$$\begin{aligned} y_j\left(u\odot\frac{y}{|y|}\right)\cdot(e_i\odot y) &= \frac{y_j}{4|y|}\,(u\otimes y + y\otimes u)\cdot(e_i\otimes y + y\otimes e_i)\\ &= \frac{y_j}{2|y|}(u_i|y|^2 + (u\cdot y)y_i) \end{aligned}$$

which finally prov(8.4). Finally note that

$$\psi_{ij}(y)\cdot M = (e_i\otimes e_j)\cdot\frac{\Gamma(y)M}{\omega_n|y|^n}, \quad \text{for all } M\in\mathbb{R}^{n\times n}_{\mathrm{sym}_0}. \tag{8.5}$$

With such a ψ_{ij} at hand we observe that, for $0<\varrho<\tau$ we have thanks to (8.5),(2.6) and (8.4) that

$$\begin{aligned} (e_i\otimes e_j)\cdot\int_{B_\tau(0)\setminus B_\varrho(0)}\frac{\Gamma(y)}{\omega_n|y|^{n+2}}\,\mathrm{d}\mathcal{E}_d u(y) &= \int_{B_\tau(0)\setminus B_\varrho(0)}\psi_{ij}(y)\cdot\mathrm{d}\mathcal{E}_d u(y)\\ &= \int_{\partial(B_\tau(0)\setminus B_\varrho(0))}(\mathbb{E}_d[\nu]u)\cdot\psi_{ij}(y)\,\mathrm{d}\mathcal{H}^{n-1}(y)\\ &= \int_{\partial B_\tau(0)}\left(\mathbb{E}_d\left[\frac{y}{|y|}\right]u\right)\cdot\psi_{ij}(y)\,\mathrm{d}\mathcal{H}^{n-1}(y)\\ &\quad - \int_{\partial B_\varrho(0))}\left(\mathbb{E}_d\left[\frac{y}{|y|}\right]u\right)\cdot\psi_{ij}(y)\,\mathrm{d}\mathcal{H}^{n-1}(y)\\ &= \frac{1}{2\omega_n\tau^n}\int_{\partial B_\tau(0)}(u\otimes\nu(y) - \nu(y)\otimes u)_{ij}\,\mathrm{d}\mathcal{H}^{n-1}(y) \end{aligned}$$

$$
\begin{aligned}
&- \frac{1}{2\omega_n \varrho^n} \int_{\partial B_\varrho(0)} (u \otimes \nu(y) - \nu(y) \otimes u)_{ij} \, \mathrm{d}\mathcal{H}^{n-1}(y) \\
=&(e_i \otimes e_j) \cdot (\mathrm{A}_{\tau,x}[u] - \mathrm{A}_{\varrho,x}[u]).
\end{aligned}
$$

Step two: *Proof of* (8.2). Define the trace free symmetric-matrix valued map

$$
\psi_{ij}(y) := \frac{\delta_{ij}}{(n-1)\omega_n |y|^n} \left[\frac{y}{|y|} \otimes \frac{y}{|y|} - \frac{\mathrm{Id}}{n} \right],
$$

and again notice that

$$
\begin{cases} \left(\mathbb{E}_d \left[\frac{y}{|y|} \right] u \right) \cdot \psi_{ij}(y) & = \delta_{ij} \frac{\left(u \cdot \frac{y}{|y|} \right)}{n\omega_n |y|^n} \\ \mathcal{E}_d^* \psi_{ij} & = 0 \quad \text{on } \mathbb{R}^n \setminus \{0\} \end{cases} . \tag{8.6}
$$

Indeed

$$
\begin{aligned}
\operatorname{div}\ \psi \left(\frac{y \otimes y}{|y|^{n+2}} \right) &= -(n+2) \frac{y}{|y|^{n+2}} + (n+1) \frac{y}{|y|^{n+2}} = \frac{y}{|y|^{n+2}} \\
\operatorname{div}\ \psi \left(\frac{1}{n|y|^n} \mathrm{Id} \right) &= - \frac{y}{|y|^{n+2}}
\end{aligned}
$$

giving for $y \neq 0$

$$
\mathcal{E}_d^* \psi_{ij}(y) = 0.
$$

Moreover

$$
\begin{aligned}
\left(\mathbb{E}_d \left[\frac{y}{|y|} \right] u \right) \cdot \psi_{ij}(y) =& \frac{\delta_{ij}}{(n-1)\omega_n |y|^n} \left(u \odot \frac{y}{|y|} \right) \cdot \left(\frac{y}{|y|} \otimes \frac{y}{|y|} - \frac{1}{n} \mathrm{Id} \right) \\
=& \frac{\delta_{ij}}{(n-1)\omega_n |y|^n} \left[\left(u \odot \frac{y}{|y|} \right) \cdot \frac{y}{|y|} \otimes \frac{y}{|y|} - \frac{1}{n} \left(u \cdot \frac{y}{|y|} \right) \right] \\
=& \frac{\delta_{ij}}{n\omega_n |y|^n} \left(u \cdot \frac{y}{|y|} \right).
\end{aligned}
$$

which finally proves (8.6). Notice that

$$
(\psi_{ij}(y) \cdot M) = (e_i \otimes e_j) \cdot \frac{\Xi(y) M}{(n-1)\omega_n |y|^n}, \quad \text{for any } M \in \mathbb{R}^{n \times n}_{\mathrm{sym}_0}, \tag{8.7}
$$

and hence, again due to (2.6), (8.6) and (8.7) we get

$$
\begin{aligned}
(e_i \otimes e_j) \cdot \frac{1}{(n-1)\omega_n} \int_{B_\tau(0) \setminus B_\varrho(0)} \frac{\Xi(y)}{|y|^n} \, \mathrm{d}\mathcal{E}_d u(y) =& \int_{B_\tau(0) \setminus B_\varrho(0)} \psi_{ij}(y) \cdot \, \mathrm{d}\mathcal{E}_d u(y) \\
=& \int_{\partial B_\tau(0)} \left(\mathbb{E}_d \left[\frac{y}{|y|} \right] u \right) \cdot \psi_{ij}(y) \, \mathrm{d}\mathcal{H}^{n-1} \\
&- \int_{\partial B_\varrho(0)} \left(\mathbb{E}_d \left[\frac{y}{|y|} \right] u \right) \cdot \psi_{ij}(y) \, \mathrm{d}\mathcal{H}^{n-1} \\
=& (\gamma_{\tau,x}[u] - \gamma_{\varrho,x}[u]) \delta_{ij}.
\end{aligned}
$$

Step Three: *Proof of* (8.3). Given the trace free symmetric matrix defined as

$$
\psi_k(y) := \frac{1}{(n-1)\omega_n |y|^{n+2}} \left(2e_k \odot y - (n+2) y_k \frac{y}{|y|} \otimes \frac{y}{|y|} + y_k \mathrm{Id} \right),
$$

we have

$$
2\operatorname{div}\ \psi \left(\frac{e_k \odot y}{|y|^{n+2}} \right) = (n+1) \frac{e_k}{|y|^{n+2}} - \frac{(n+2)}{|y|^{n+4}} (e_k |y|^2 + y_k y)
$$

$$-(n+2)\mathrm{div}\ \psi\left(\frac{y_k}{|y|^{n+2}}\frac{y}{|y|}\otimes\frac{y}{|y|}\right)=-n(n+2)y_k\frac{y}{|y|^{n+4}}+\frac{(n+2)^2}{|y|^{n+4}}y_k y$$
$$\mathrm{div}\ \psi\left(\frac{y_k}{|y|^{n+2}}\mathrm{Id}\right)=\frac{e_k}{|y|^{n+2}}-\frac{(n+2)}{|y|^{n+4}}y_k y.$$

Hence we immediately see that

$$\mathcal{E}_d^*\psi_k(y)=\mathrm{div}\ \psi_k=0.$$

Also

$$\begin{aligned}\psi_k(y)\cdot\mathbb{E}_d u\left[\frac{y}{|y|}\right]=&\frac{1}{(n-1)\omega_n|y|^{n+2}}\left(2e_k\odot y-(n+2)y_k\frac{y}{|y|}\otimes\frac{y}{|y|}+y_k\mathrm{Id}\right)\cdot\left(u\odot\frac{y}{|y|}\right)\\ =&\frac{1}{(n-1)\omega_n|y|^{n+2}}\left(u_k|y|+\frac{y_k}{|y|}(u\cdot y)-(n+2)\frac{y_k}{|y|}(u\cdot y)+\frac{y_k}{|y|}(u\cdot y)\right)\\ =&\frac{1}{(n-1)\omega_n|y|^{n+1}}\left(u_k-n\frac{y_k}{|y|}\left(u\cdot\frac{y}{|y|}\right)\right)\\ =&\frac{1}{(n-1)\omega_n|y|^{n+1}}\left(\left[\mathrm{Id}-n\frac{y}{|y|}\otimes\frac{y}{|y|}\right]u\right)_k.\end{aligned}$$

Since we see that

$$\frac{e_k\cdot\Upsilon(y)M}{(n-1)\omega_n|y|^{n+1}}=\psi_k(y)\cdot M\quad\text{for any } M\in\mathbb{R}^{n\times n}_{\mathrm{sym}_0},$$

we conclude again as in Step one and two. ☐

Proposition 8.3. *If $x\notin\Theta_u$ then*

$$\lim_{\varrho\to 0}\varrho^2|\mathrm{s}_{\varrho,x}[u]|=\lim_{\varrho\to 0}\varrho|\mathrm{A}_{\varrho,x}[u]|=\lim_{\varrho\to 0}\varrho|\gamma_{\varrho,x}[u]|=0.$$

Proof. We assume $x=0$ without loss of generality and we also write A_ϱ, γ_ϱ, s_ϱ in place of $\mathrm{A}_{\varrho,0}$, $\gamma_{\varrho,0}$, $\mathrm{s}_{\varrho,0}$. We notice that, the condition $0\notin\Theta_u$ can be translated into

$$\lim_{\tau\to 0^+}\sup_{t\in(0,\tau)}\frac{|\mathcal{E}_d u|(B_t(0))}{t^{n-1}}=0.$$

We start by showing the key fact

$$\lim_{\tau\to 0^+}\limsup_{\varrho\to 0^+}\varrho\int_{B_\tau(0)\setminus B_\varrho(0)}\frac{1}{|y|^n}\,\mathrm{d}|\mathcal{E}_d u|(y)=0. \tag{8.8}$$

This comes as a consequence of the layer-cake representation for the measure

$$\mu:=|\mathcal{E}_d u|\llcorner_{B_\tau(0)\setminus B_\varrho(0)},$$

since

$$\begin{aligned}\int_{B_\tau(0)\setminus B_\varrho(0)}\frac{1}{|y|^n}\,\mathrm{d}|\mathcal{E}_d u|(y)=\int_{\mathbb{R}^n}\frac{1}{|y|^n}\,\mathrm{d}\mu(y)&=\int_0^{+\infty}\mu\left(\left\{\frac{1}{|y|^n}>t\right\}\right)\mathrm{d}t\\ &=n\int_0^{+\infty}\frac{\mu(\{|y|\le s\}}{s^{n+1}}\,\mathrm{d}s\\ &=n\int_\varrho^\tau\frac{|\mathcal{E}_d u|(B_s(0)\setminus B_\varrho(0))}{s^{n+1}}\,\mathrm{d}s+\frac{|\mathcal{E}_d u|(B_\tau(0)\setminus B_\varrho(0))}{\tau^n}.\end{aligned}$$

Moreover

$$n\int_\varrho^\tau\frac{|\mathcal{E}_d u|(B_s(0)\setminus B_\varrho(0))}{s^{n+1}}\,\mathrm{d}s\le n\sup_{t\in(0,\tau)}\frac{|\mathcal{E}_d u|(B_t(0))}{t^{n-1}}\int_\varrho^\tau s^{-2}\,\mathrm{d}s$$

$$= n \sup_{t\in(0,\tau)} \frac{|\mathcal{E}_d u|(B_t(0))}{t^{n-1}} \left(\frac{1}{\varrho} - \frac{1}{\tau}\right).$$

Therefore

$$\limsup_{\varrho\to 0^+} \varrho \int_{B_\tau(0)\setminus B_\varrho(0)} \frac{1}{|y|^n} \,\mathrm{d}|\mathcal{E}_d u|(y) \le n \sup_{t\in(0,\tau)} \frac{|\mathcal{E}_d u|(B_t(0))}{t^{n-1}},$$

yielding (8.8), since $0 \notin \Theta_u$. By now using the representation (8.1), (8.2), (8.3) we see that (by definition)

$$|\Gamma(y)| + |\Xi(y)| + |\Upsilon(y)| \le C,$$

giving that

$$\varrho^2|\mathrm{s}_\varrho[u]| + \varrho|\mathrm{A}_\varrho[u]| + \varrho|\gamma_\varrho[u]| \le C\varrho \int_{B_\tau(0)\setminus B_\varrho(0)} \frac{1}{|y|^n} \,\mathrm{d}|\mathcal{E}_d u|(y) + \varrho\kappa(\tau),$$

where $\kappa(\tau)$ is a constant depending on τ only. We now first take the limit in ϱ and then in τ, which by exploiting (8.8), achieves the proof. □

We are now ready to prove Proposition 7.2.

Proof of Proposition 7.2. Without loss of generality set $x = 0$. Relation (7.2) comes from Proposition 8.3. We now set $\mathcal{R}_\varrho = \mathcal{R}_{B_\varrho(0)}$ and we first invoke Poincaré ineqality 2.7 to see that

$$\frac{1}{\omega_n \varrho^n} \int_{B_\varrho(0)} |u(y) - \mathcal{R}_\varrho[u](y)| \,\mathrm{d}y \le c \frac{|\mathcal{E}_d u|(B_\varrho(0))}{\varrho^{n-1}},$$

for a universal constant c independent of ϱ. Moreover

$$\begin{aligned}
\fint_{B_\varrho(0)} |u(y) - \mathcal{R}_\varrho[u](y)| \,\mathrm{d}y \ge& \fint_{B_\varrho(0)} |u(y) - d_\varrho[u]| \,\mathrm{d}y \\
&- \kappa \left[\fint_{B_\varrho(0)} |R_\varrho[u]y| \,\mathrm{d}y + \fint_{B_\varrho(0)} |\gamma_\varrho[u]y| \,\mathrm{d}y\right] \\
&- \kappa \left|\fint_{B_\varrho(0)} [(s_\varrho[u]\cdot y)y - s_\varrho[u]|y|^2] \,\mathrm{d}y\right| \\
\ge& \fint_{B_\varrho(0)} |u(y) - d_\varrho[u]| \,\mathrm{d}y - \kappa\left[\varrho|R_\varrho[u]| + \varrho|\gamma_\varrho[u]|\right].
\end{aligned}$$

for a universal constant $\kappa > 0$. Hence

$$\begin{aligned}
\fint_{B_\varrho(0)} |u(y) - d_\varrho[u]| \,\mathrm{d}y \le& \fint_{B_\varrho(0)} |u(y) - \mathcal{R}_\varrho[u](y)| \,\mathrm{d}y + \kappa\left[\varrho|R_\varrho[u]| + \varrho|\gamma_\varrho[u]| + \varrho^2|s_\varrho[u]|\right] \\
\le& \kappa' \left[\frac{|\mathcal{E}_d u|(B_\varrho(0))}{\varrho^{n-1}} + \varrho|R_\varrho[u]| + \varrho|\gamma_\varrho[u]| + \varrho^2|s_\varrho[u]|\right].
\end{aligned}$$

By taking the limit as $\varrho \to 0$ and by exploiting $x \notin \Theta_u$, together with Proposition 8.3 we achieve the proof. □

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