On Hardy-Sobolev Inequality and its Application to Certain Singular Problems

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by

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This thesis is dedicated to

My Parents

for their love and affection

Declaration

I hereby declare that the work reported in this thesis is entirely original and has been carried out by me under the supervision of Professor Mrinal Kanti Ghosh, Department of Mathematics, Indian Institute of Science and Professor Mythily Ramaswamy, Tata Institute of Fundamental research, Bangalore centre, in the Department of Mathematics, Indian Institute of Science, Bangalore 560 012. I further declare that this work has not formed the basis for the award of any degree, diploma, fellowship, associateship or similar title of any University or Institution.

Nirmalendu Chaudhuri

Certified

Prof. M. K. Ghosh **Prof. M. Ramaswamy**

Table of Contents

Acknowledgements **ix**

Abstract **xi**

1 Introduction **1**

2 An Improved Hardy-Sobolev Inequality **13**
2.1 Introduction 13
2.2 2-dimensional Hardy inequality 15
2.3 n-dimensional Hardy inequality 17
2.4 Key-Lemma 18
2.5 Proof of Theorem (2.1.1) 19
2.6 An estimate for the best constant 29

3 On the Schröndinger Operator $\Delta + V(x)$ **35**
3.1 Introduction 35
3.2 Preliminary Embeddings 36
3.3 Weighted Eigenvalue Problem 40
3.4 Nonlinear Weighted Eigenvalue Problems 47

4 A Brezis-Nirenberg Type Problem **53**
4.1 Introduction 53
4.2 Proof of Theorem (4.1.1) 54

5 On an Evolution Problem Involving Hardy-Sobolev Operator **63**
5.1 Introduction 63
5.2 Formulation of Weak Solutions 66
5.3 Proof of Theorem (5.1.1) 68
5.4 Proof of Theorem (5.1.3) 75

Bibliography **77**

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Abstract

Inequalities involving integrals of a function and its derivatives appear frequently in various branches of mathematics and represent a useful tool e.g. in the theory and practice of differential equations, in the theory of approximation etc. A general question in this direction can be like:

Let Ω be a open subset of $\mathbb{R}^n$ and w, $w_1, \cdots, w_n$ are given weight functions (measurable and positive almost everywhere). Now the question is, do there exist positive real numbers p and q such that the following inequality

$$\left(\int_\Omega |u(x)|^q \, w(x) \, dx \right)^{1/q} \leq C \left(\int_\Omega \sum_{i=1}^n |\frac{\partial u}{\partial x_i}(x)|^p \, w_i(x) \, dx \right)^{1/p} \tag{0.1}$$

holds for all $u \in C_c^\infty(\Omega)$?

If w, $w_1, \cdots, w_n$ are bounded weights, study of the inequality (0.1) is quite well understood. Here, we are mainly interested in case when w is a singular weight and $w_i = 1$ for all $i = 1, \cdots, n$. In 1920, G. H. Hardy have proved the following inequality: *Let* $1 < p < \infty$, *then the following inequality*

$$\left(\frac{p-1}{p} \right)^p \int_0^\infty \frac{|u(t)|^p}{|t|^p} dt \leq \int_0^\infty |u'(t)|^p \, dt, \tag{0.2}$$

for all $u \in C_c^1(0, \infty)$. *Moreover,* $\left(\frac{p-1}{p} \right)^p$ *is the best constant and the inequality is strict unless* $u \equiv 0$. Analogues to the inequality (0.2) exist in higher dimensions. More precisely, for any p, $1 < p < n$ the following *Hardy-Sobolev inequality*

$$\int_{\mathbb{R}^n} |\nabla u(x)|^p \, dx \geq \left(\frac{n-p}{p} \right)^p \int_{\mathbb{R}^n} \left(\frac{|u(x)|}{|x|} \right)^p \, dx \tag{0.3}$$

holds for all $u \in D^{1,p}(\mathbb{R}^n)$, where $D^{1,p}(\mathbb{R}^n)$ is the completion of $C_c^\infty(\mathbb{R}^n; \mathbb{R})$ in the norm $\|u\|_{D^{1,p}} = \|\nabla u\|_{L^p}$.

The constant $\lambda_{n,p} := \left(\frac{n-p}{p}\right)^p$ is the best in the inequality (0.3) i.e.

$$\lambda_{n,p} = \inf_{\substack{u \in D^{1,p}(\mathbb{R}^n) \\ u \neq 0}} \frac{\int_{\mathbb{R}^n} |\nabla u|^p \, dx}{\int_{\mathbb{R}^n} \frac{|u(x)|^p}{|x|^p} dx}.$$

Moreover, for any bounded domain Ω containing 0, the inequality (0.3) holds for any u in the Sobolev space $W_0^{1,p}(\Omega)$, which is the completion of $C_c^\infty(\Omega)$ in the norm $\|u\|_{1,p,\Omega} := \left(\int_\Omega |u(x)|^p dx + \int_\Omega |\nabla u|^p dx\right)^{1/p}$ and $\lambda_{n,p}$ is independent of the domain. The most interesting fact about the inequality (0.3) is that the constant $\lambda_{n,p}$ is never achieved (in the sense that an equality for some $0 \neq u \in W_0^{1,p}(\Omega)$) in any domain, not even in the whole space and this will be the source of many problems which we deal in this thesis. Some few questions like, is it possible to add a second term (at least when Ω is a bounded domain) in the right hand side of the inequality (0.3)? If yes, what would be the 'best' term? Does this process stop? In Chapter 2 we answer these questions and and we have the following Theorem:

Theorem. *Let $R \geq \sup_\Omega (|x| \; e^{2/p})$ and $1 < p \leq n$, then there exists $C > 0$, depending on n, p and R such that*

$$\int_\Omega |\nabla u|^p dx \geq \left(\frac{n-p}{p}\right)^p \int_\Omega \frac{|u(x)|^p}{|x|^p} \, dx + C \int_\Omega \frac{|u(x)|^p}{|x|^p} \left(\log \frac{R}{|x|}\right)^{-\gamma} dx$$

for any $u \in W_0^{1,p}(\Omega)$ if and only if

(i) $\gamma \geq 2$ *when* $1 < p < n$,

(ii) $\gamma \geq n$ *when* $p = n$.

More generally, for $2 \leq p < n$ and for any $1 < q < p_\beta^* := \frac{p(n-\beta)}{n-p}$, $0 \leq \beta < p$, there exists $C_1 > 0$ depending on n, p, q, R and Ω such that

$$\int_\Omega |\nabla u|^p dx \geq \left(\frac{n-p}{p}\right)^p \int_\Omega \frac{|u(x)|^p}{|x|^p} \, dx + C \int_\Omega \frac{|u(x)|^p}{|x|^p} \left(\log \frac{R}{|x|}\right)^{-2} dx + C_1 \left(\int_\Omega \frac{|u(x)|^q}{|x|^\beta} \, dx\right)^{p/q}$$

for any $u \in W_0^{1,p}(\Omega)$.

Corollary. *Let $1 < p < n$ and $R \geq (\sup_\Omega |x|) \left(e^{e^{e^{\cdot^{\cdot^{\cdot^{e \, (k-times)}}}}}}\right)^{2/p}$. Then there exists $C > 0$ depending on n, p and R such that*

$$\int_\Omega |\nabla u|^p dx \geq \left(\frac{n-p}{p}\right)^p \int_\Omega \frac{|u(x)|^p}{|x|^p} \, dx + C \sum_{j=1}^k \int_\Omega \frac{|u(x)|^p}{|x|^p} \left(\prod_{i=1}^j \log^{(i)} \frac{R}{|x|}\right)^{-2} dx$$

for any $u \in W_0^{1,p}(\Omega)$, where $\log^{(1)}(.) := \log(.)$ and $\log^{(k)}(.) := \log\left(\log^{(k-1)}(.)\right)$ for $k \geq 2$.

Now, let us turn our attention to the differential operator related to the inequality (0.3) in the case $p = 2$, i.e., study of the operator $L_\mu := -\Delta - \dfrac{\mu}{|x|^2}$, $\mu \geq 0$. By using Pohozaev type identity it can be easily shown that the equation $-\Delta u = \dfrac{\mu}{|x|^2}u$, in a *star-shaped* domain Ω, does not have any positive solution in $H_0^1(\Omega)$ for any $\mu \geq 0$. In any bounded domain, the operator L_μ has discrete H_0^1 spectrum if and only if $\mu < \lambda^* := \frac{(n-2)^2}{4}$. Moreover, it is known that the first eigenvalue $\lambda_\mu^1(\Omega)$ i.e.

$$\lambda_\mu^1(\Omega) = \min_{\substack{u \in H_0^1(\Omega) \\ u \neq 0}} \frac{\int\limits_\Omega |\nabla u|^2\, dx - \mu \int\limits_\Omega \frac{|u(x)|^2}{|x|^2}\, dx}{\int\limits_\Omega |u(x)|^2\, dx}$$

goes to positive real number $\lambda(\Omega)$ as μ increases to the critical value λ^*. In Chapter 3, we have the following results.

Theorem. *Let $0 \leq \mu < \lambda^*$, $\lambda \in \mathbb{R}^+$ and $f \in \Im_2$, where*

$$\Im_2 := \left\{ f : \Omega \to \mathbb{R}^+ \;\middle|\; \lim_{|x| \to 0} |x|^2 f(x) = 0 \text{ with } f \in L^\infty_{\text{loc}}\left(\Omega \setminus \{0\}\right) \right\}.$$

The eigenvalue problem

$$\left\{ \begin{array}{rcll} -\left(\Delta + \frac{\mu}{|x|^2}\right) u & = & \lambda u f(x) & \text{in} \quad \Omega \\ u & = & 0 & \text{on} \quad \partial\Omega \end{array} \right. \tag{0.4}$$

admits nontrivial weak solutions in $H_0^1(\Omega)$ corresponding to $\lambda \in (\lambda_\mu^k(f))_{k=1}^\infty$, where $0 < \lambda_\mu^1(f) < \lambda_\mu^2(f) \leq \lambda_\mu^3(f) \leq \ldots \to +\infty$. If Ω is $C^{1,1}$, then any weak solution of (1.18) is in $H_0^1(\Omega) \cap W^{2,r}(\Omega)$ for all r, $1 < r < 2n/(n+2)$.
Furthermore, if $f \in \Re_2 := \left\{ f \in \Im_2 \;\middle|\; \limsup_{|x|\to 0} |x|^2 f(x) \left(\log \frac{1}{|x|}\right)^2 < \infty \right\}$, then $\lambda_\mu^1(f) \to \lambda(f) > 0$ as $\mu \to \lambda^*$. If $f \notin \Re_2$ and $|x|^2 f(x) \left(\log \frac{1}{|x|}\right)^2$ tends to ∞ as $|x| \to 0$, then the limit $\lambda(f) = 0$.

For $\mu \geq \lambda^*$, to establish the non-existence of a positive eigenfunction in $H_0^1(\Omega)$, we show more generally the non-existence of any positive $H_0^1(\Omega)$ super-solution of, $-\left(\Delta + \dfrac{\mu}{|x|^2}\right)$, more precisely;

Theorem. *Let $\Omega \subset \mathbb{R}^n$, $n \geq 3$, be a bounded domain with $0 \in \Omega$. Suppose that u is a non-negative function in $H_0^1(\Omega)$ such that*

$$-\Delta u - \frac{(n-2)^2}{4|x|^2} u \geq -\lambda \frac{\phi}{|x|^2} u, \quad \text{in} \quad \Omega$$

where ϕ is a continuous, non-negative function in $\bar{\Omega}$ such that

$$\lim_{|x|\to 0} \phi(x) \left(\log \frac{1}{|x|}\right)^{\alpha} = 0.$$

Then for any $\lambda \in \mathbb{R}$ and $\alpha \geq 2$, $u \equiv 0$.
For $\mu < \lambda^*$, we considered the following nonlinear weighted problem

$$\left\{ \begin{array}{rcll} -\left(\Delta + \frac{\mu}{|x|^2}\right) u &=& u^{q-1} f(x) & \text{in } \Omega \\ u &>& 0 & \text{in } \Omega \\ u &=& 0 & \text{on } \partial\Omega \end{array} \right. \tag{0.5}$$

and proved existence of nontrivial H^1 solution for ceratin $q > 2$, depending on the singularity of f and in Chapter 4 we obtain the following result:

Theorem. *Let $f \in \Re_{2,\beta} := \left\{ f \in \Im_2 \;\middle|\; 0 < \lim_{|x|\to 0} |x|^{\beta} f(x) < \infty \right\}$ for some $0 \leq \beta < 2$. Then for any $2 < q < 2^*_\beta := \dfrac{2(n-\beta)}{n-2}$, Problem (1.19) admits a weak solution for any $0 \leq \mu < \lambda^*$.*

Theorem. *The problem*

$$\left\{ \begin{array}{rcll} -\Delta u &=& \frac{u^{2^*_\alpha - 1}}{|x|^\alpha} + \lambda u^{q-1} f(x) & \text{in } \Omega \\ u &>& 0 & \text{in } \Omega \\ u &=& 0 & \text{on } \partial\Omega \end{array} \right.$$

where $0 \leq \alpha < 2$, $f \in \Re_{2,\beta}$, for some $0 \leq \beta < 2$, $2 < q < 2^*_\beta$ and $\lambda \in \mathbb{R}^+$ admits a weak solution,
(a) for all $n \geq 4$ and $\lambda > 0$
(b) when $n = 3$, in the following different cases:
(i) For all $1 \leq \beta < 2$ and $\lambda > 0$.
(ii) For all $0 \leq \beta < 1$ with $2(2-\beta) < q < 2^*_\beta$ and for all $\lambda > 0$
(iii) For all $0 \leq \beta < 1$ with $2 < q \leq 2(2-\beta)$ and for sufficiently large $\lambda > 0$.

It is well-known that the following parabolic problem

$$u_t = \Delta u + \frac{\lambda^*}{|x|^2} u \qquad \text{in } (0,\infty) \times \Omega \tag{0.6}$$

with $u(x,0) := u_0(x) \in L^2(\Omega)$, $u(x,t) = 0$ for $x \in \partial\Omega$ and $t > 0$, has no $H^1_0(\Omega)$ solution, even locally in time. But by perturbing the stationary operator L_{λ^*}, i.e. by taking $L_{q\lambda^*} := -\left(\Delta + \dfrac{\lambda^* q(x)}{|x|^2}\right)$ for a suitably weight function q, in Chapter 5, we are able to get existence of $H^1_0(\Omega)$ solution for all time $t > 0$ with L^∞ initial data. Perturbation of the operator is heavily depends on the improved Hardy-Sobolev inequality (our first Theorem).

Chapter 1

Introduction

Inequalities involving integrals of a function and its derivatives appear frequently in various branches of mathematics and represent a useful tool e.g. in the theory and practice of differential equations, in the theory of approximation etc. The question about the validity of the inequality in a domain Ω, in the n-dimensional Euclidean space $\mathbb{R}^n$,

$$\left(\int_\Omega |u(x)|^q\, w(x)\, dx\right)^{1/q} \leq C\left(\int_\Omega \sum_{i=1}^{n} |\frac{\partial u}{\partial x_i}(x)|^p\, v_i(x)\, dx\right)^{1/p} \tag{1.1}$$

for p, q positive reals and $w, v_1, \cdots, v_n$ *weight functions* (measurable and positive a.e in Ω), and some of its modifications have become core topics in the study of differential equations. Here, we mention a few inequalities which are particular cases of (1.1) and are well known in the literature, see [1, 28, 45].

For example, in [1, 28], the following inequality has been extensively investigated:

$$\left(\int_\Omega |u(x)|^q\, dx\right)^{1/q} \leq C\left(\int_\Omega |\nabla u(x)|^p\, dx\right)^{1/p} \tag{1.2}$$

where $x := (x_1, \cdots, x_n)$, $\nabla u := \left(\frac{\partial u}{\partial x_1}, \cdots, \frac{\partial u}{\partial x_n}\right)$, $|\nabla u|^p := \sum_{i=1}^n |\frac{\partial u}{\partial x_i}(x)|^p$ and $C > 0$ independent of u. This inequality is known as the *Sobolev inequality* and holds e.g., for $u \in C_c^\infty(\Omega; \mathbb{R})$ and $1 < p < n$, $1 < q \leq \dfrac{np}{n-p}$; provided Ω is a bounded domain with Lipschitzian boundary $\partial\Omega$. For 'worse' domains Ω, the parameter q may change, see e.g., R. A. Adams [1], W. P. Ziemer [45]. For any bounded domain Ω, the inequality (1.2), with $q = p$ holds for all $C_c^\infty(\Omega; \mathbb{R})$ and it is known as Poincaré inequality. Observe that the above inequality is a special case of the inequality (1.1) with $w(x) = v_i(x) = 1,\ i = 1, 2, \cdots, n$.

In 1920, G. H. Hardy proved the following inequality: *Let* $1 < p < \infty$ *and denote*

$$F(t) := \begin{cases} \displaystyle\int_0^t f(x)\, dx, & \text{for } \epsilon < p-1 \\ \displaystyle\int_t^\infty f(x)\, dx, & \text{for } \epsilon > p-1 \end{cases} \tag{1.3}$$

where f is a non-negative measurable function on $(0,\infty)$. Then

$$\int_0^\infty F^p(t)\, t^{\epsilon-p} dt \le C \int_0^\infty f^p(t)\, t^\epsilon \, dt \tag{1.4}$$

with constant $C > 0$ independent of f.

For a proof see, e.g., G. H. Hardy [29] or G. H. Hardy, Littlewood and G. Polya [30]. The exact value of *the best constant* C in the above inequality was given in 1926 by E. Landau [34], which is

$$C = \left(\frac{p}{|\epsilon - p + 1|}\right)^p$$

and the inequality is *strict* unless $f \equiv 0$. Note that, the inequality (1.4) can be written as

$$\int_0^\infty |u(t)|^p\, t^{\epsilon-p} dt \le \left(\frac{p}{|\epsilon - p + 1|}\right)^p \int_0^\infty |u'(t)|^p\, t^\epsilon \, dt \tag{1.5}$$

where $u'(t) := \frac{du}{dt}$. We shall call (1.5), the (*classical*) *Hardy inequality*; which holds, at least for all functions $u \in C_c^\infty((0,\infty);\mathbb{R})$, provided $p > 1$ and $\epsilon \neq p-1$. We observe that the inequality (1.5) is also a particular case of (1.1) with $n = 1$, $\Omega = (0,\infty)$ and $w(x) = x^{\epsilon-p}$, $v(x) = x^\epsilon$. For an excellent reference about the validity of the inequality (1.1), see the book of B. Opic and A. Kufner [38].

For $\epsilon = 0$, the inequality (1.5) reduces to

$$\int_0^\infty |u'(t)|^p \, dt \ge \left(\frac{p-1}{p}\right)^p \int_0^\infty \left|\frac{u(t)}{t}\right|^p \, dt \tag{1.6}$$

holds for all $u \in C_c^\infty((0,\infty);\mathbb{R})$. Analogues to the inequality (1.6) exist in higher dimensions. More precisely, for any $1 < p < n$ the following *Hardy-Sobolev inequality*

$$\int\limits_{\mathbb{R}^n} |\nabla u(x)|^p \, dx \ge \left(\frac{n-p}{p}\right)^p \int\limits_{\mathbb{R}^n} \left(\frac{|u(x)|}{|x|}\right)^p \, dx \tag{1.7}$$

holds for all $u \in D^{1,p}(\mathbb{R}^n)$, where $D^{1,p}(\mathbb{R}^n)$ is the completion of $C_c^\infty(\mathbb{R}^n;\mathbb{R})$ in the norm $\|u\|_{D^{1,p}} = \|\nabla u\|_{L^p}$. For $p = n \ge 2$, there does not exists any constant $C > 0$ such that

$$\int\limits_{\mathbb{R}^n} |\nabla u(x)|^p \, dx \ge C \int\limits_{\mathbb{R}^n} \left(\frac{|u(x)|}{|x|}\right)^p \, dx\,.$$

This can be seen easily by taking a ball $B(0,r)$ and a smooth function $0 \le \phi \le 1$ such that $\phi = 1$ on $B(0,r/2)$ and 0 out side the ball $B(0,r)$, so that $\int\limits_{\mathbb{R}^n} \frac{|\phi(x)|^n}{|x|^n} dx = \infty$.

The constant $\lambda_{n,p} := \left(\frac{n-p}{p}\right)^p$ is the best in the inequality (1.7) i.e.

$$\lambda_{n,p} = \inf_{\substack{u\in D^{1,p}(\mathbb{R}^n)\\ u\neq 0}} \frac{\int\limits_{\mathbb{R}^n} |\nabla u|^p\, dx}{\int\limits_{\mathbb{R}^n} \frac{|u(x)|^p}{|x|^p} dx}\,. \tag{1.8}$$

Moreover, for any bounded domain Ω containing 0 with $1 < p < n$, the following inequality

$$\int\limits_{\Omega} |\nabla u|^p\, dx \geq \left(\frac{n-p}{p}\right)^p \int\limits_{\Omega} \frac{|u|^p}{|x|^p}\, dx\,, \tag{1.9}$$

holds for any u in the Sobolev space $W_0^{1,p}(\Omega)$, which is the completion of $C_c^\infty(\Omega;\mathbb{R})$ in the norm $\|u\|_{1,p,\Omega} := \left(\int\limits_{\Omega} |u(x)|^p dx + \int\limits_{\Omega} |\nabla u|^p dx\right)^{1/p}$ and $\lambda_{n,p}$ is the best constant, see J. Garcia and I. Peral [27]. Sometimes the Hardy-Sobolev(HS) inequality in the case $p = 2$, is known as *uncertainty principle*, see [23]. We can read the HS inequality saying that the embedding of $W_0^{1,p}(\Omega)$ in $L^p(\Omega)$ with respect to the weight $|x|^{-p}$ is continuous (*Hardy-Sobolev embedding*). The most interesting fact about the inequality (1.9), is that the constant $\lambda_{n,p}$ is never achieved (in the sense that an equality for some $0 \neq u \in W_0^{1,p}(\Omega)$) in any domain, not even in the whole space. This is a major difference with the standard *Sobolev embedding* $W_0^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$, $p^* := \frac{np}{n-p}$. In this case the best constant (*Sobolev constant*) is achieved only when $\Omega = \mathbb{R}^n$, see M. Struwe [40] and references therein. But, we mention that both the Hardy-Sobolev and Sobolev embeddings are non-compact.

Even though, the constant $\lambda_{n,p}$ is not achieved, one can give an explicit minimizing sequence for the best constant. For example, for sufficiently small $\epsilon > 0$, $u_\epsilon := |x|^{-\frac{n-p}{p}+\epsilon}$, for $x \in \mathbb{R}^n$ is in $D^{1,p}(\mathbb{R}^n)$ and it is an approximate minimizer for $\lambda_{n,p}$ in the whole space, i.e.

$$\left(\frac{n-p}{p}\right)^p = \lim_{\epsilon\to 0} \frac{\int\limits_{\mathbb{R}^n} |\nabla u_\epsilon|^p\, dx}{\int\limits_{\mathbb{R}^n} \frac{|u_\epsilon(x)|^p}{|x|^p} dx}\,.$$

The function $u(x) := |x|^{-\frac{n-p}{p}}$, does not belong to $D^{1,p}(\mathbb{R}^n)$ but it satisfies the following equation

$$-\mathrm{div}(|\nabla u|^{p-2}\nabla u) = \lambda_{n,p}|u|^{p-1} \qquad \text{in } \mathbb{R}^n\,,$$

in the sense of distributions, (in short we will write as $D'(\mathbb{R}^n)$) i.e.,

$$\int\limits_{\mathbb{R}^n} |\nabla u|^{p-2}\langle \nabla u, \nabla \phi\rangle\, dx = \lambda_{n,p} \int\limits_{\mathbb{R}^n} |u|^{p-1}\phi\, dx\,,$$

for all $\phi \in C_c^\infty(\mathbb{R}^n;\mathbb{R})$.

Since the inequality (1.9), is strict for any non-zero functions, one can ask a few questions like, is it possible to add a second term in the right hand side of the inequality (1.9)? If yes, what would be the 'best' term? Does this process stop? We answer these questions through the following Theorem:

Theorem. *Let $R \geq \sup_\Omega (|x|\ e^{2/p})$ and $1 < p \leq n$, then there exists $C > 0$, depending on n, p and R such that*

$$\int_\Omega |\nabla u|^p dx \geq \left(\frac{n-p}{p}\right)^p \int_\Omega \frac{|u(x)|^p}{|x|^p}\ dx + C \int_\Omega \frac{|u(x)|^p}{|x|^p} \left(\log \frac{R}{|x|}\right)^{-\gamma} dx \tag{1.10}$$

for any $u \in W_0^{1,p}(\Omega)$, if and only if

(i) $\gamma \geq 2$ *when* $1 < p < n$,

(ii) $\gamma \geq n$ *when* $p = n$.

More generally, for $2 \leq p < n$ and for any $1 < q < p^*_\beta := \dfrac{p(n-\beta)}{n-p}$; $0 \leq \beta < p$, there exists $C_1 > 0$, depending on n, p, q, R and Ω such that for all $u \in W_0^{1,p}(\Omega)$

$$\int_\Omega |\nabla u|^p dx \geq \left(\frac{n-p}{p}\right)^p \int_\Omega \frac{|u(x)|^p}{|x|^p}\, dx + C \int_\Omega \frac{|u(x)|^p}{|x|^p} \left(\log \frac{R}{|x|}\right)^{-2} dx + C_1 \left(\int_\Omega \frac{|u(x)|^q}{|x|^\beta}\, dx\right)^{p/q}.$$

In Chapter 2, we also prove that the series in (1.10) continues till a certain k-th term, which depends only on the choice of R.

Here we give a short review on the recent developments of Hardy-type inequalities involving singularity on the boundary. If Ω be a bounded domain in $\mathbb{R}^n$, with Lipschitz boundary, it is known that the following extension of Hardy-type inequality is valid, see [38]

$$\int_\Omega |\nabla u|^2 \geq \mu \int_\Omega \left(\frac{u(x)}{\delta(x)}\right)^2 dx, \quad \forall\ u \in H_0^1(\Omega) \tag{1.11}$$

where μ is a positive constant and $\delta(x) = \delta_\Omega(x) := \operatorname{dist}(x, \partial\Omega)$. The best constant in (1.11), i.e.

$$\mu(\Omega) := \inf_{\substack{u \in H_0^1(\Omega) \\ u \neq 0}} \frac{\int_\Omega |\nabla u|^2\, dx}{\int_\Omega \left(u(x)/\delta(x)\right)^2 dx} \tag{1.12}$$

depends on Ω. If $\partial\Omega$ possesses a tangent plane at least one point, then $\mu(\Omega) \leq 1/4$, see [18, 36]. For convex domains $\mu(\Omega) = 1/4$, but there are smooth bounded domains such that $\mu(\Omega) < 1/4$ see [36, 37]. Furthermore, it is proved in [36] that, for smooth (at least C^2) bounded domains, the infimum in (1.12) is achieved if and only if $\mu(\Omega) < 1/4$. For

non-smooth domain such type of result is not yet known. For any bounded domain with Lipschitz boundary, define the quantity

$$J_\lambda^\Omega := \inf_{\substack{u\in H_0^1(\Omega)\\ u\neq 0}} \frac{\int_\Omega |\nabla u|^2\,dx - \lambda\int_\Omega u^2\,dx}{\int_\Omega \left(u(x)/\delta(x)\right)^2\,dx}, \quad \forall\ \lambda\in\mathbb{R}. \tag{1.13}$$

Note that $J_0^\Omega = \mu(\Omega)$ and the function $\lambda \mapsto J_\lambda^\Omega$ is concave and non-decreasing on $\mathbb{R}$, $J_\lambda^\Omega \to -\infty$ when $\lambda \to +\infty$. Moreover, $J_{\lambda_1}^\Omega = 0$ where λ_1 is the first eigenvalue of $-\Delta$ on $H_0^1(\Omega)$. In [18], E. B. Davies have defined the following quantity,

$$\mu_w(\Omega) := \lim_{\lambda\to-\infty} J_\lambda^\Omega$$

and showed that for C^2-domains, $\mu_w(\Omega) = 1/4$. But for some planar domain with angular points he showed that $\mu_w(\Omega) < 1/4$. He also states the interesting conjecture that,

$$J_\lambda^\Omega \le \frac{1}{4}, \quad \forall\ \lambda\in\mathbb{R},$$

for every domain $\Omega \subset \mathbb{R}^n$, without any smoothness conditions and this question is still remains open. Recently, for smooth bounded domains Ω, Brezis and Marcus [8] have proved that there exists $\lambda^0 = \lambda^0(\Omega)$ such that $J_\lambda^\Omega = 1/4,\ \forall\,\lambda \le \lambda^0,\ J_\lambda^\Omega < 1/4\ ,\forall\,\lambda > \lambda^0$ and the infimum in (1.13) is achieved if and only if $\lambda > \lambda^0$. As a consequence they have for convex domains, an improved Hardy-type inequality,

$$\int_\Omega |\nabla u|^2 \ge \frac{1}{4}\int_\Omega \left(\frac{u(x)}{\delta(x)}\right)^2 dx \ +\ \lambda^0\int_\Omega u^2, \quad \forall\ u\in H_0^1(\Omega) \tag{1.14}$$

with $\lambda^0(\Omega) \ge \dfrac{1}{4\,\mathrm{diam}^2(\Omega)}$, see Theorem I and Theorem II in [8]. Observe that the inequality (1.14) is analogous to the inequality (1.10) when $p = 2$ and $\gamma = 0$. Here we ask a question, whether an inequality of the type (1.10) holds for the above Hardy-type inequality? For more interesting results on Hardy-type inequalities, see [19] and references therein.

Now, let us turn our attention to the differential operator related to the inequality (1.9) in the case $p = 2$, i.e., study of the operator $L_\mu := -\Delta - \dfrac{\mu}{|x|^2}$, $\mu \ge 0$. For $\mu > 0$, the operator L_μ appears in quantum mechanics and it is known as real Schrödinger operator [33]. It also appears in the linearized analysis of standard combustion models which lead to blowup phenomena, see [6, 12, 20, 26, 42]. Let us mention some known facts about the operator L_μ. By Hardy-Sobolev inequality, it follows that $L_\mu u = 0$ has no non-trivial solution in $H_0^1(\Omega)$, for any $\mu \le \lambda^*$, in any domain Ω. From the Theorem (3.3.3) (in Chapter 3), it also follows that $L_\mu u = 0$, has no positive solution in $H_0^1(\Omega)$, for any $\mu > \lambda^*$. In any bounded domain,

the operator L_μ has discrete H_0^1 spectrum if and only if $\mu < \lambda^* := \frac{(n-2)^2}{4}$ and has continuous spectrum for $\mu \geq \lambda^*$, see [13, 31]. Moreover, the first eigenvalue $\lambda_\mu^1(\Omega)$ i.e.

$$\lambda_\mu^1(\Omega) = \min_{\substack{u \in H_0^1(\Omega) \\ u \neq 0}} \frac{\int\limits_\Omega |\nabla u|^2 \, dx - \mu \int\limits_\Omega \frac{|u(x)|^2}{|x|^2} \, dx}{\int\limits_\Omega |u(x)|^2 \, dx} \tag{1.15}$$

goes to positive real number $\lambda(\Omega)$ as μ increases to the critical value (Hardy constant) λ^*. This was proved by E. Jannelli and S. Solomini [31], by using ordinary differential equation techniques. It is worth mentioning here that this asymptotic behaviour implies improvement of the Hardy-Sobolev inequality (1.9) for the case $p = 2$, i.e. an improved Hardy-Sobolev inequality:

$$\int\limits_\Omega |\nabla u|^2 dx \geq \left(\frac{n-2}{2}\right)^2 \int\limits_\Omega \frac{|u(x)|^2}{|x|^2} \, dx + \lambda(\Omega) \int\limits_\Omega |u(x)|^2 \, dx \tag{1.16}$$

holds for all $u \in H_0^1(\Omega)$. H. Brezis and J. L. Vazquez [12] have directly proved this inequality in their study of blowup phenomena of some nonlinear elliptic problems and by using the behaviour of the Bessels function, they have proved that the inequality (1.16) is still strict. Thus there is a two fold connection between the improved Hardy-Sobolev inequality and asymptotic behaviour of the first eigenvalue of the operator L_μ. Motivated by these facts we study the following weighted eigenvalue problem

$$\begin{cases} -\left(\Delta + \frac{\mu}{|x|^2}\right) u &= \lambda u f(x) \quad \text{in} \quad \Omega \\ \qquad\qquad u &= 0 \qquad\quad \text{on} \quad \partial\Omega \,, \end{cases} \tag{1.17}$$

and identified a class of weight functions f (measurable and positive a.e in Ω), so that the operator L_μ admits discrete spectrum for $\mu < \lambda^*$ and our main result is:

Theorem. *Let $0 \leq \mu < \lambda^*$, $\lambda \in \mathbb{R}^+$ and $f \in \Im_2$, where*

$$\Im_2 := \left\{ f : \Omega \to \mathbb{R}^+ \;\middle|\; \lim_{|x| \to 0} |x|^2 f(x) = 0 \text{ with } f \in L^\infty_{\text{loc}}\left(\Omega \setminus \{0\}\right) \right\} .$$

The eigenvalue problem

$$\begin{cases} -\left(\Delta + \frac{\mu}{|x|^2}\right) u &= \lambda u f(x) \quad \text{in} \quad \Omega \\ \qquad\qquad u &= 0 \qquad\quad \text{on} \quad \partial\Omega \,, \end{cases} \tag{1.18}$$

admits nontrivial weak solutions in $H_0^1(\Omega)$ corresponding to $\lambda \in (\lambda_\mu^k(f))_{k=1}^\infty$, where $0 < \lambda_\mu^1(f) < \lambda_\mu^2(f) \leq \lambda_\mu^3(f) \leq \ldots \to +\infty$. If Ω is $C^{1,1}$, then any weak solution of (1.18) is in $H_0^1(\Omega) \cap W^{2,r}(\Omega)$ for all r, $1 < r < 2n/(n+2)$.

Furthermore, if $f \in \Re_2 := \left\{ f \in \Im_2 \;\middle|\; \limsup_{|x| \to 0} |x|^2 f(x) \left(\log \frac{1}{|x|}\right)^2 < \infty \right\}$, then $\lambda_\mu^1(f) \to \lambda(f) > 0$ as $\mu \to \lambda^*$. If $f \notin \Re_2$ and $|x|^2 f(x) \left(\log \frac{1}{|x|}\right)^2$ tends to ∞ as $|x| \to 0$, then the limit $\lambda(f) = 0$.

Here we mention that, the study of asymptotic profile of the first spectrum depends on the improvement of the inequality (1.9). For $\mu \geq \lambda^*$, there exists no positive eigenfunction in $H_0^1(\Omega)$. We show more generally the non-existence of any positive $H_0^1(\Omega)$ *supersolution* of the operator $-\left(\Delta + \dfrac{\mu}{|x|^2}\right)$.

Theorem. *Let $\Omega \subset \mathbb{R}^n$, $n \geq 3$, be a bounded domain with $0 \in \Omega$. Suppose that u is a non-negative function in $H_0^1(\Omega)$ such that*

$$-\Delta u - \frac{(n-2)^2}{4|x|^2}u \geq -\lambda \frac{\phi}{|x|^2}u\,, \quad in \quad \Omega$$

where ϕ is a continuous, non-negative function in $\bar{\Omega}$ such that

$$\lim_{|x|\to 0} \phi(x)\left(\log \frac{1}{|x|}\right)^{\alpha} = 0.$$

Then for any $\lambda \in \mathbb{R}$ and $\alpha \geq 2$, $u \equiv 0$.

For $\mu < \lambda^*$, we consider the following nonlinear weighted problem

$$\left\{\begin{array}{rcll} -\left(\Delta + \frac{\mu}{|x|^2}\right)u &=& u^{q-1}f(x) & \text{in } \Omega \\ u &>& 0 & \text{in } \Omega \\ u &=& 0 & \text{on } \partial\Omega\,, \end{array}\right. \tag{1.19}$$

and prove existence of nontrivial H^1 solution for ceratin $q > 2$, depending on the singularity of f and we obtain the following result:

Theorem. *Let $f \in \Re_{2,\beta} := \left\{ f \in \Im_2 \;\middle|\; 0 < \lim_{|x|\to 0} |x|^{\beta} f(x) < \infty \right\}$ for some $0 \leq \beta < 2$. Then for any $2 < q < 2^*_\beta := \dfrac{2(n-\beta)}{n-2}$, problem (1.19) admits a weak solution for any $0 \leq \mu < \lambda^*$.*

For any $0 < \alpha < 2$, it is well known that the following problem

$$\left\{\begin{array}{rcll} -\Delta u &=& \frac{u}{|x|^{\alpha}}|u|^{2^*_\alpha - 2} & \text{in } \Omega \\ u &>& 0 & \text{in } \Omega \\ u &=& 0 & \text{on } \partial\Omega\,, \end{array}\right. \tag{1.20}$$

where $2^*_\alpha := \dfrac{2(n-\alpha)}{n-2}$, admits solution in $H^1(\Omega)$ only when Ω is invariant under scaling centered at $x = 0$, for example $\Omega = \mathbb{R}^n$ or $\mathbb{R}^n_+$, see Theorem 2, in [31]. Let us note that when $\alpha = 0$, this problem becomes the standard problem in critical exponent theory and the literature is huge, see [40] and references therein. It was Pohozaev [39], who first proved the non-existence of any solution for the problem

$$\left\{\begin{array}{rcll} -\Delta u &=& |u|^{2^*-2}u & \text{in } \Omega \\ u &>& 0 & \text{in } \Omega \\ u &=& 0 & \text{on } \partial\Omega\,, \end{array}\right.$$

if Ω is star-shaped. However, motivated by some variational problems in geometry and physics, e.g. *Yamabe problem* [44, 35], H. Brezis and L. Nirenberg considered the above problem in their fundamental paper [11] and showed that certain lower order term can reverse the situation. More precisely they have proved the following:

Theorem (Brezis-Nirenberg): *Let $\Omega \subset \mathbb{R}^n$, $n \geq 3$ be a bounded domain, $\lambda \in \mathbb{R}$ and λ_1 denote the first eigenvalue of $-\Delta$ with zero Dirichlet boundary value. If $n \geq 4$ then the problem*

$$\left\{ \begin{array}{rcll} -\Delta u & = & |u|^{2^*-2}u + \lambda u & \text{in } \Omega \\ u & > & 0 & \text{in } \Omega \\ u & = & 0 & \text{on } \partial\Omega\,, \end{array} \right. \tag{1.21}$$

admits a solution for every $\lambda \in (0, \lambda_1)$, and if $n = 3$ then there is a $\lambda^0 \in (0, \lambda_1)$ such that (1.21) admit solutions if and only if $\lambda \in (\lambda^0, \lambda_1)$.

One obvious question that comes to our mind, is whether a similar Berzis-Nirenberg phenomenon occurs for the problem (1.20)? The answer is yes. By following the proof of Brezis-Nirenberg [11], it is easy to prove that, a similar result holds for the problem

$$\left\{ \begin{array}{rcll} -\Delta u & = & \frac{u}{|x|^\alpha}|u|^{2^*_\alpha-2} + \lambda\, u & \text{in } \Omega \\ u & > & 0 & \text{in } \Omega \\ u & = & 0 & \text{on } \partial\Omega\,. \end{array} \right.$$

In a bounded domain Ω, we study a similar problem with a superlinear lower order term $g(u)$ having a singular coefficient, where an interesting balance between the singularity and nonlinearity is essential for the existence of a solution. Here also a similar Brezis-Nirenberg type of result occurs but there is beautiful interplay between the singularity and the superlinear growth of g.

Theorem. *The problem*

$$\left\{ \begin{array}{rcll} -\Delta u & = & \frac{u^{2^*_\alpha-1}}{|x|^\alpha} + \lambda u^{q-1} f(x) & \text{in } \Omega \\ u & > & 0 & \text{in } \Omega \\ u & = & 0 & \text{on } \partial\Omega\,, \end{array} \right.$$

where $0 \leq \alpha < 2$, $f \in \Re_{2,\beta}$; for some $0 \leq \beta < 2$, $2 < q < 2^*_\beta$ and $\lambda \in \mathbb{R}^+$, admits a weak solution :

(a) For all $n \geq 4$ and $\lambda > 0$.

(b) When $n = 3$, in the following different cases:

(i) For all $1 \leq \beta < 2$ and $\lambda > 0$.

(ii) For all $0 \leq \beta < 1$ with $2(2-\beta) < q < 2^*_\beta$ and for all $\lambda > 0$.

(iii) For all $0 \leq \beta < 1$ with $2 < q \leq 2(2-\beta)$ and for sufficiently large $\lambda > 0$.

The main difference between the cases $\alpha = 2$ and $\alpha < 2$, is that the functional corresponding to the PDE satisfies *Palais-Smale* $(P.S)$, condition (see Definition 3 at the end of this

Chapter) at any energy level when $\alpha = 2$, but for the case $\alpha < 2$ it satisfies $P.S$ condition only below a certain energy level.

Finally, we apply the improved Hardy-Sobolev inequality to a evolution problem, which complements the work of Adimurthi and Sandeep [3] on the stationary problem. In 1984, Baras and Goldstein in [7], have considered the following heat equation

$$\left\{\begin{array}{rcll} u_t - \Delta u &=& V(x)u & \text{in} \quad \Omega \times (0,\infty) \\ u(x,0) &=& u_0(x) & \text{in} \quad \Omega \\ u(x,t) &=& 0 & \text{on} \quad \partial\Omega \times (0,\infty)\,, \end{array}\right. \tag{1.22}$$

in a domain $\Omega \subset \mathbb{R}^n$. For the critical potential $V(x) = \dfrac{\mu}{|x|^2}$, they have proved the following result:

Theorem (Baras-Goldstein): *The Cauchy-Dirichlet problem (1.22) has a global solution (in the sense of distributions) if* $\mu \leq \lambda^* = \dfrac{(n-2)^2}{4}$ *and no solution, even locally in time, if* $\mu > \lambda^*$.

By the Hardy-Sobolev inequality, it is easy to show that the problem (1.22) admits solution in $C\left([0,\infty); L^2(\Omega)\right) \cap L^2\left(0,\infty; H_0^1(\Omega)\right)$, (this spaces are defined in Chapter 5) if $\mu < \lambda^*$. But a strong change of behaviour takes place at the transition value $\mu = \lambda^*$. In this case the behaviour of the solution is intimately related to the improved form of the classical Hardy inequality (1.9). For the case $\mu = \lambda^*$, Vazquez and Zuazua [43] proved that the solution $u(t)$ of the problem (1.1) with $u_0 \in L^2(\Omega)$ and zero Dirichlet data is in the Hilbert space $H(\Omega)$, for all time $t > 0$ but not in $H_0^1(\Omega)$, even locally in time, where $H(\Omega)$ is the completion of $H_0^1(\Omega)$ with respect to the *energy norm* $\|u\|_H := \left(\int_\Omega \{|\nabla u|^2 - \lambda^* \frac{u^2}{|x|^2}\} dx\right)^{1/2}$. It is interesting to notice that $H_0^1(\Omega) \subset H(\Omega) \subset \cap_{q<2} W_0^{1,q}(\Omega)$ and the inclusions are proper. So naturally a question arises: Is it possible to perturb the *Hardy-Sobolev operator* $L_{\lambda^*} := -\Delta - \dfrac{\lambda^*}{|x|^2}$, in such a way that the heat equation admits a non-negative solution $u(t) \in H_0^1(\Omega)$ for all $t > 0$ with the initial data $u_0 \geq 0$? Or what should be the perturbation of the stationary operator L_{λ^*} which determines the border line between existence and non-existence of H_0^1 solutions? Here we are only able to answer the existence part.

Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$ be a bounded domain, $0 \in \Omega$. Take q and η to be weight functions (measurable and positive a.e) satisfying the following hypothesis:

(H_1) $\qquad 0 \leq q \leq 1, \quad \eta \in L^\infty_{\text{loc}}(\Omega \setminus \{0\}),$

(H_2)

$$\begin{cases} \lim\limits_{x\to 0} |x|^2 \left(\log \frac{1}{|x|}\right)^2 \eta(x) &= 0, \quad n \geq 3\,, \\ \\ \lim\limits_{x\to 0} |x|^2 \left(\log \frac{1}{|x|}\right)^2 \left(\log\log \frac{1}{|x|}\right)^2 \eta(x) &= 0, \qquad n = 2. \end{cases}$$

Now, let us define the following constant,

$$C(q,\eta) := \begin{cases} \inf\limits_{H_0^1(\Omega)} \left\{ \int\limits_\Omega \left[|\nabla u|^2 - \lambda^* \frac{|u|^2 q(x)}{|x|^2} \right] dx \;\Big|\; \int\limits_\Omega |u|^2 \eta(x)\, dx = 1 \right\}, & \text{for } n \geq 3\,, \\ \\ \inf\limits_{H_0^1(\Omega)} \left\{ \int\limits_\Omega \left[|\nabla u|^2 - \frac{1}{4} \frac{|u|^2 q(x)}{(|x| \log\, R/|x|)^2} \right] dx \;\Big|\; \int\limits_\Omega |u|^2 \eta(x)\, dx = 1 \right\} & \text{for } n = 2, \end{cases}$$

and by improved Hardy-Sobolev inequality (1.10), it follows that $C(q,\eta) > 0$. Here, we have the following Theorems;

Theorem. *Assume $n \geq 3$ and let q, η satisfies (H_1) to (H_2). Then the problem*

$$\begin{cases} u_t - \Delta u - \lambda^* \frac{q(x)}{|x|^2}\, u &= \lambda \eta u \quad \text{in} \quad \Omega \times (0,\infty) \\ \\ u(x,0) &= u_0(x) \quad \text{in} \quad \Omega \\ \\ u(x,t) &= 0 \qquad \text{on} \quad \partial\Omega \times (0,\infty), \end{cases}$$

where $u_0 \in L^\infty(\Omega)$, $0 \leq u_0 \not\equiv 0$, admits an unique, non-negative global solution $u(t) \in H_0^1(\Omega)$ for all λ, $0 \leq \lambda \leq C(q,\eta)$, provided q satisfies

$$\liminf_{x\to 0} \left(\log \frac{1}{|x|}\right)^2 (1 - q(x)) > \frac{3}{(n-2)^2}. \tag{1.23}$$

Theorem. *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with $0 \in \Omega$ and q, η as in the hypotheses. Then the problem*

$$\begin{cases} u_t - \Delta u - \frac{1}{4} \frac{q(x)}{(|x| \log\, R/|x|)^2}\, u &= \lambda \eta u \quad \text{in} \quad \Omega \times (0,\infty) \\ \\ u(x,0) &= u_0(x) \quad \text{in} \quad \Omega \\ \\ u(x,t) &= 0 \qquad \text{on} \quad \partial\Omega \times (0,\infty), \end{cases}$$

where $u_0 \in L^\infty(\Omega)$, $0 \leq u_0 \not\equiv 0$, admits an unique, non-negative global solution $u(t) \in H_0^1(\Omega)$ for all λ, $0 \leq \lambda \leq C(q,\eta)$, provided q satisfies

$$\liminf_{x\to 0}\left(\log\log\frac{1}{|x|}\right)^2(1-q(x))>3. \tag{1.24}$$

Remark. *We are expecting that, if the conditions in (1.23) and (1.24) get reversed, there will be non-existence of H_0^1 solutions, even locally in time.*

Most of the results in Chapter 2 are taken from [2], Chapter 3- Chapter 4 are from [15] and Chapter 5 is from [16]. Before going to next chapters, we introduce some of the basic notations and definitions which we follow throughout in this thesis we follow.

Definitions and Notations.

- $\mathbb{R}$, the set of real numbers.
- $\mathbb{R}^+$, the set of non-negative real numbers.
- $\mathbb{R}^n$, the n-dimensional Euclidean space.
- $x=(x_1,\cdots,x_n)$, a points in $\mathbb{R}^n$.
- $|x|:=(x_1^2+\cdots+x_n^2)^{1/2}$, the Euclidean norm.
- For x, $y\in\mathbb{R}^n$, $\langle x,y\rangle:=x_1y_1+\cdots+x_ny_n$, the inner product.
- $S^{n-1}:=\{x\in\mathbb{R}^n \mid |x|=1\}$, the unit sphere in $\mathbb{R}^n$.
- Ω, an open bounded set (domain) in $\mathbb{R}^n$.
- $\partial\Omega$, the boundary of Ω.
- $|\Omega|_n$, the n-dimensional Lebesgue measure.
- $\omega_n:=|S^{n-1}|_{n-1}$, volume of the unit sphere.
- For $0\le k\le\infty$, $C_c^k(\Omega;\mathbb{R})$ $(C_0^k(\Omega;\mathbb{R}))$, k-times differentiable real valued functions with compact support (vanishing on $\partial\Omega$). In short, we simply denote by $C_c^k(\Omega)$ $(C_0^k(\Omega))$.
- For $k\ge 1$ and $u\in C^k(\Omega)$, $\nabla u:=\left(\dfrac{\partial u}{\partial x_1},\cdots,\dfrac{\partial u}{\partial x_n}\right)$, the gradient of u.
- $\Delta:=\dfrac{\partial^2}{\partial x_1^2}+\cdots+\dfrac{\partial^2}{\partial x_n^2}$, the Laplacian operator.
- For $1\le p<\infty$, $L^p(\Omega):=\left\{u:\Omega\to\mathbb{R}\;\middle|\;\text{measurable } s.t\int\limits_\Omega|u(x)|^p\,dx<\infty\right\}$, the Lebesgue space and $\|u\|_{L^p(\Omega)}:=\left(\int\limits_\Omega|u(x)|^p\,dx\right)^{1/p}$, its norm.

- For $1 \leq p < \infty$, $L^p(\Omega, f) := \left\{ u : \Omega \to \mathbb{R} \;\middle|\; \text{measurable } s.t \int\limits_{\Omega} |u(x)|^p f(x)\, dx < \infty \right\}$, the weighted Lebesgue space and $\|u\|_{L^p(\Omega,f)} := \left(\int\limits_{\Omega} |u(x)|^p f(x)\, dx \right)^{1/p}$, its norm, where $f : \Omega \to \mathbb{R}^+$ measurable and positive almost everywhere (a.e.).
- For $1 \leq p < \infty$, $W_0^{1,p}(\Omega)$ is the completion of $C_c^\infty(\Omega)$ with respect to the Sobolev norm $\|u\|_{1,p} := \|\nabla u\|_{L^p(\Omega)} + \|u\|_{L^p(\Omega)}$.
- $W_0^{1,2}(\Omega) := H_0^1(\Omega)$.
- For $p > 1$, $W^{-1,p'}(\Omega) := \left\{ T : W_0^{1,p}(\Omega) \to \mathbb{R} \;\middle|\; T \text{ is linear and continuous} \right\}$, the dual of $W_0^{1,p}(\Omega)$, where $p' = p/(p-1)$.
- $W_0^{-1,2}(\Omega) := H^{-1}(\Omega)$.
- $D'(\Omega) := \left\{ T : C_c^\infty(\Omega) \to \mathbb{R} \;\middle|\; T \text{ is linear and continuous} \right\}$, space of distributions.

Here we give few definitions, which we use in this thesis.

Definition 1. A function $u \in H_0^1(\Omega)$, is said to a weak solution for the equation $L_\mu u := -\left(\Delta + \dfrac{\mu}{|x|^2} \right) u = g(u) f(x)$ in Ω, if the following integral identity

$$\int\limits_{\Omega} \langle \nabla u, \nabla \phi \rangle \, dx - \mu \int\limits_{\Omega} \frac{u}{|x|^2} \phi \, dx = \int\limits_{\Omega} g(u) \phi f(x) \, dx$$

holds for all $\phi \in H_0^1(\Omega)$, whenever the right hand side is finite.

Definition 2. A function $u \in H_0^1(\Omega)$ is said to satisfy, $L_\mu u \geq g(u) f(x)$ in Ω, if the following inequality

$$\int\limits_{\Omega} \langle \nabla u, \nabla \phi \rangle \, dx - \mu \int\limits_{\Omega} \frac{u}{|x|^2} \phi \, dx \geq \int\limits_{\Omega} g(u) \phi f(x) \, dx,$$

holds for all $\phi \in H_0^1(\Omega)$, $\phi \geq 0$. The function u is called *supersolution*.

Definition 3. Let H be a Hilbert space, $I \in C^1(H; \mathbb{R})$. Then any sequence $(u_k) \subset H$, is said to be a *Palais-Smale P.S*, sequence if the sequence $(I(u_k))$ is bounded and $DI(u_k) \to 0$ strongly in H^* as $k \to \infty$, where DI is the Fréchet derivative of I and H^* is dual of H. The functional I is said to satisfy *Palais-Smale* compactness condition, if every $P.S$ sequence has strongly convergence subsequence in H, (relatively compact).

Definition 4. We say that I satisfies $(P.S)_c$, if any sequence (u_k) in H such that $I(u_k) \to c$ while $DI(u_k) \to 0$ strongly in H^* as $k \to \infty$ is relatively compact. Such a sequel will be referred to as $(P.S)_c$-sequence.

Chapter 2

An Improved Hardy-Sobolev Inequality

§2.1 Introduction

Let Ω be a bounded domain in $\mathbb{R}^n, n \geq 2$, with $0 \in \Omega$. For any p, $1 < p < n$ the well known Hardy-Sobolev inequality

$$\int_\Omega |\nabla u|^p dx \geq \left(\frac{n-p}{p}\right)^p \int_\Omega \frac{|u(x)|^p}{|x|^p}\, dx \tag{2.1}$$

holds for $u \in W_0^{1,p}(\Omega)$, where $W_0^{1,p}(\Omega)$ is the completion of $C_c^\infty(\Omega)$ in the norm $\|u\|_{1,p,\Omega} := \left(\int_\Omega |u(x)|^p dx + \int_\Omega |\nabla u|^p dx\right)^{1/p}$. It is known that there is no function $u \in W_0^{1,p}(\Omega)$, for which the best constant $\lambda_{n,p} := \left(\frac{n-p}{p}\right)^p$ is achieved. So, one anticipates to have an estimate of the error term on the right hand side of the inequality (2.1). For the case $p = 2$, such improved Hardy-Sobolev inequalities are known. For example, Brezis-Vazquez [12] have proved that there exists a constant $C > 0$ depending only on $n,\ q$ and Ω such that

$$\int_\Omega |\nabla u|^2 dx \geq \left(\frac{n-2}{2}\right)^2 \int_\Omega \frac{|u(x)|^2}{|x|^2}\, dx + C\left(\int_\Omega |u(x)|^q\, dx\right)^{2/q} \tag{2.2}$$

holds for any $u \in H_0^1(\Omega)$ and $1 < q < \dfrac{2n}{n-2}$. We mention here that for the one dimensional Hardy inequality (1.6), Brezis and Marcus [10] have obtained the following improved inequality:

$$\int_0^1 |u'(t)|^2\, dt \geq \frac{1}{4}\int_0^1 \frac{|u(t)|^2}{t^2}\left(1 + (\log e/t)^{-2}\right) dt$$

holds for every $u \in H_0^1(0,1)$. Moreover this inequality is in some sense optimal: the weight function $X(t) := (\log e/t)^{-2}$ cannot be replaced by a lower power of X and the constant on the right hand side is sharp. Recently, Cabre and Martel have considered critical potentials in their study of existence versus instantaneous explosion for the heat equation with singular

potentials [14], where they have an improved inequality similar to (2.3) below, for the case $n = 2 = p$.

Our aim of this Chapter, is to achieve an"optimal" improvement of the inequality (2.1), by adding a second term involving the singular weight $\left(\frac{1}{\log 1/|x|}\right)^2$, in the sense that the improved inequality holds for this weight but fails for any weight more singular than this one. The improvement of the inequality (2.1) for the case $1 < p < 2$, is more delicate than for the case $p \geq 2$. As far our understanding goes, it is because of the fact that for $1 < p < 2$; $(1+x)^p$, $x \geq -1$, does not have a global estimate in terms of either x^2 or $|x|^p$. Before stating our main results, let us introduce the following notations: $\log^{(1)}(.) := \log(.)$, and $\log^{(k)}(.) := \log\left(\log^{k-1}(.)\right)$, for $k \geq 2$. Our main results are

(2.1.1) Theorem. *Let $R \geq \sup_\Omega\ (|x|\ e^{2/p})$ and $1 < p \leq n$, then there exists $C > 0$, depending on n, p and R such that*

$$\int\limits_\Omega |\nabla u|^p dx \geq \left(\frac{n-p}{p}\right)^p \int\limits_\Omega \frac{|u(x)|^p}{|x|^p}\ dx + C \int\limits_\Omega \frac{|u(x)|^p}{|x|^p} \left(\log \frac{R}{|x|}\right)^{-\gamma} dx \tag{2.3}$$

for any $u \in W_0^{1,p}(\Omega)$, if and only if

(i) $\gamma \geq 2$ *when* $1 < p < n$,

(ii) $\gamma \geq n$ *when* $p = n$.

More generally, for $2 \leq p < n$ and for any $1 < q < p^*_\beta := \frac{p(n-\beta)}{n-p}$; $0 \leq \beta < p$, there exists $C_1 > 0$ depending on n, p, q, R and Ω such that

$$\int\limits_\Omega |\nabla u|^p dx \geq \left(\frac{n-p}{p}\right)^p \int\limits_\Omega \frac{|u(x)|^p}{|x|^p}\ dx + C \int\limits_\Omega \frac{|u(x)|^p}{|x|^p} \left(\log \frac{R}{|x|}\right)^{-2} dx + C_1 \left(\int\limits_\Omega \frac{|u(x)|^q}{|x|^\beta}\ dx\right)^{p/q} \tag{2.4}$$

for any $u \in W_0^{1,p}(\Omega)$.

(2.1.2) Corollary. *Let $1 < p < n$ and $R = (\sup_\Omega |x|) \left(e^{e^{e^{\cdot^{\cdot^{\cdot^{e\,(k-times)}}}}}}\right)^{2/p}$. Then there exists $C > 0$ depending on n, p and R such that*

$$\int\limits_\Omega |\nabla u|^p dx \geq \left(\frac{n-p}{p}\right)^p \int\limits_\Omega \frac{|u(x)|^p}{|x|^p}\ dx + C \sum_{j=1}^{k} \int\limits_\Omega \frac{|u(x)|^p}{|x|^p} \left(\prod_{i=1}^{j} \log^{(i)} \frac{R}{|x|}\right)^{-2} dx$$

holds for any $u \in W_0^{1,p}(\Omega)$.

(2.1.3) Remark. *One of the open problem mentioned in [12] (Problem 2), is whether the two terms on the RHS of the the inequality (2.2) for $q = 2$, are just the first two terms of a series or not. The Corollary (2.1.2) for the particular case $p = 2$, shows that, indeed the series continues till a certain k-th term, which depends only on the choice of R.*

(2.1.4) Corollary. *Let $1 < p < n$ and define,*

$$\Re_p := \left\{ f : \Omega \to \mathbb{R}^+ \;\middle|\; f \in L^\infty_{\rm loc}(\Omega \setminus \{0\}) \text{ with } \limsup_{|x|\to 0} |x|^p f(x) \left(\log \frac{1}{|x|}\right)^2 < \infty \right\}.$$

If $f \in \Re_p$, then there exists $\lambda(f) > 0$ such that for all $u \in W^{1,p}_0(\Omega)$,

$$\int_\Omega |\nabla u|^p dx \geq \left(\frac{n-p}{p}\right)^p \int_\Omega \frac{|u(x)|^p}{|x|^p}\, dx + \lambda(f) \int_\Omega |u(x)|^p\, f(x)\, dx\,. \tag{2.5}$$

If $f \notin \Re_p$ and if $|x|^p f(x) \left(\log \frac{1}{|x|}\right)^2$ tends to $+\infty$ as $|x| \to 0$, then no inequality of the type (2.5) can hold.

§2.2 2-dimensional Hardy inequality

One of the main ingredient for the proof of the Theorem (2.1.1) is the following 2-dimensional Hardy inequality and a similar one for the case $k = 1$ appears in [10] (inequality (A.4)). Here, again we recall the following notations: $\log^{(1)}(.) := \log(.)$, and $\log^{(k)}(.) := \log\left(\log^{k-1}(.)\right)$, for $k \geq 2$.

(2.2.1) Lemma. *For any $h \in C^1[a,T]$, $0 \leq a < T$ and for $R > T$, the following inequality holds for $k = k(R)$:*

$$\begin{aligned}\int_a^T |h'(r)|^2 r\, dr \;\geq\; & \frac{1}{4}\sum_{j=1}^k \int_a^T \frac{|h(r)|^2}{\left(r\, \prod_{i=1}^j \log^{(i)} R/r\right)^2} r\, dr \\ & + \frac{1}{2}\sum_{j=1}^k \left\{ h^2(a) \left(\prod_{i=1}^j \log^{(i)} R/a\right)^{-1} - h^2(T) \left(\prod_{i=1}^j \log^{(i)} R/T\right)^{-1} \right\} \end{aligned} \tag{2.6}$$

where $k(R)$ is the first positive integer for which $0 < \log^{(k)} R/T \leq 1$.

Proof. Let h be in $C^1[a,T]$. For $r \in [a,T]$, define

$$\psi_1(r) := (\log R/r)^{-1/2} h(r)\,.$$

Then

$$\begin{aligned}\int_a^T |h'(r)|^2 r\ dr &= \frac{1}{4}\int_a^T \frac{|\psi_1(r)|^2}{(r\ \log R/r)}dr + \int_a^T |\psi_1'(r)|^2(\log R/r)\, r\, dr \\ &\quad - \int_a^T \psi_1(r)\,\psi_1'(r)dr \\ &= \frac{1}{4}\int_a^T \frac{|h(r)|^2}{(r\ \log R/r)^2}\, r\, dr + \int_a^T |\psi_1'(r)|^2(\log R/r)\, r\, dr \\ &\quad + \frac{1}{2}\left\{\psi_1^2(a) - \psi_1^2(T)\right\}.\end{aligned}$$

To estimate the middle term on the RHS, we introduce the iterative function $\psi_2(r) := \psi_1(r)\left(\log^{(2)} R/r\right)^{-1/2}$. A simple calculations shows that

$$\begin{aligned}\int_a^T |h'(r)|^2 r\ dr &= \frac{1}{4}\sum_{j=1}^{2}\int_a^T \frac{|h(r)|^2}{\left(r\ \prod_{i=1}^{j}\log^{(i)} R/r\right)^2}\, r\, dr \\ &\quad + \int_a^T |\psi_2'(r)|^2(\log R/r)\,(\log\log R/r)\, r\, dr \\ &\quad + \frac{1}{2}\sum_{j=1}^{2}\left\{h^2(a)\left(\prod_{i=1}^{j}\log^{(i)} R/a\right)^{-1} - h^2(T)\left(\prod_{i=1}^{j}\log^{(i)} R/T\right)^{-1}\right\}.\end{aligned}$$

We can again estimate the middle term on the RHS, by introducing $\psi_3(r) := \psi_2(r)\left(\log^{(3)} R/r\right)^{-1/2}$. This process can be continued till we reach $k = k(R)$ for which $0 < \log^{(k)} R/T \le 1$ and by defining, $\psi_k(r) := \psi_{k-1}(r)\left(\log^{(k)} R/r\right)^{-1/2}$, ignoring the integral involving the term $|\psi_k'(r)|^2$ we obtain the inequality (2.6). ∎

By the standard approximation argument we have the following corollary.

(2.2.2) Corollary. *Given any $T > 0$ and for $R > T$, the inequality*

$$\int_0^T |h'(r)|^2 r\ dr \ \ge\ \frac{1}{4}\sum_{j=1}^{k}\int_0^T \frac{|h(r)|^2}{\left(r\ \prod_{i=i}^{j}\log^{(i)} R/r\right)^2}\, r\, dr \tag{2.7}$$

holds for any radial function h in $H_0^1(B)$, where B is the ball of radius T centered at origin, where $k = k(R)$ is the first integer for which $0 < \log^{(k)} R/T \le 1$.

(2.2.3) Remark. *In fact (2.7) holds for any h in $H_0^1(B)$ with $R = e^{e^{e^{\cdot^{\cdot^{\cdot^{e\,(k-times)}}}}}} T$, by using a symmetrization argument as in the proof of Theorem (2.1.1), below. This choice of R enables one to conclude that the singular weights $\left(r\prod_{i=1}^{j}\log^{(i)} R/r\right)^{-2}$ are decreasing functions of r for $1 \le j \le k$, which is essential for symmetrization.*

§2.3 n-dimensional Hardy inequality

We have seen in our previous section a Hardy-type inequality for $n = p = 2$ in the most general form. The following Lemma is generalization of the Lemma (2.2.1) but only with the first term.

(2.3.1) Lemma. *Let $n \geq 2$, $\Omega = B(0,T) \subset \mathbb{R}^n$, be the open ball of radius T, centered at origin and $R > T$. Then*

$$\int_\Omega |\nabla h|^n dx \geq \left(\frac{n-1}{n}\right)^n \int_\Omega \frac{|h|^n}{|x|^n} \left(\log R/|x|\right)^{-n} dx \tag{2.8}$$

for all radial $h \in W_0^{1,n}(\Omega)$.

Proof. For any $h \in C_0^1(\Omega)$, $h > 0$, radially nonincreasing, we define $\psi(r) := (\log R/r)^{-(n-1)/n} h(r)$ for $r \in [0,T]$. Then

$$\begin{aligned} h'(r) &= \psi'(r)(\log R/r)^{(n-1)/n} - \frac{n-1}{n\,r}\psi(r)(\log R/r)^{-1/n} \\ &= -\frac{n-1}{n\,r}\psi(r)(\log R/r)^{-1/n}\left(1 - \frac{n\,r\,\psi'(r)}{(n-1)\,\psi(r)}\log R/r\right). \end{aligned}$$

Since $h'(r) \leq 0$, we have $\left(1 - \dfrac{n\,r\,\psi'(r)}{(n-1)\,\psi(r)}\right) \geq 0$, and hence

$$\begin{aligned} |h'(r)|^n &= \left(\frac{n-1}{n\,r}\right)^n \psi^n(r)(\log R/r)^{-1}\left(1 - \frac{n\,r\,\psi'(r)}{(n-1)\,\psi(r)}\log R/r\right)^n \\ &\geq \left(\frac{n-1}{n\,r}\right)^n \psi^n(r)(\log R/r)^{-1}\left(1 - n\,\frac{n\,r\,\psi'(r)}{(n-1)\,\psi(r)}\log R/r\right) \\ &= \left(\frac{n-1}{n\,r}\right)^n \psi^n(r)(\log R/r)^{-1} - n\left(\frac{n-1}{n\,r}\right)^{n-1}\psi'(r)\psi^{n-1}(r). \end{aligned}$$

Since $\psi(r) \to 0$ as $r \to 0$ and $\psi(T) = 0$, we obtain

$$\begin{aligned} \int_\Omega |\nabla h|^n dx &= \omega_n \int_0^T |h'(r)|^n r^{n-1} dr \\ &\geq \omega_n \left(\frac{n-1}{n}\right)^n \int_0^T |h(r)|^n \left(\log R/r\right)^{-n} \frac{dr}{r} \\ &\quad - \omega_n \left(\frac{n-1}{n}\right)^{n-1} \int_0^T \left(\frac{d}{dr}\psi^n(r)\right) dr \\ &= \left(\frac{n-1}{n}\right)^n \int_\Omega \frac{|h(x)|^n}{|x|^n} \left(\log R/r\right)^{-n} dx. \end{aligned}$$

For any radial function h in $W_0^{1,n}(\Omega)$, approximate h by smooth radial h_m. Then using strong convergence of h_m to h in the gradient norm and also Fatou's lemma, the above inequality holds for all radial h in $W_0^{1,n}(\Omega)$. ∎

(2.3.2) Remark. *Recently, Adimurthi and Sandeep in [3], have shown that the best constant for the n-dimensional Hardy inequality (inequality (2.8), for the case $p = n$) is $\left(\frac{n-1}{n}\right)^n$.*

§2.4 Key-Lemma

In this section, we prove a lemma just by using elementary calculus but it is very important for proving the Theorem (2.1.1). The case $1 < p < 2$, is really crucial and is very interesting.

(2.4.1) Lemma. *If $p \geq 2$, then there exist positive constants B and C such that*

$$(1+x)^p \geq 1 + px + Cx^2 + B|x|^p \quad \forall\ x \geq -1. \tag{2.9}$$

If $1 < p < 2$ and $M \geq 1$

$$(1+x)^p \geq \begin{cases} 1 \ + \ px \ + \ 2^{p-4}p(p-1)M^{p-2}\,x^2, & \text{for } -1 \leq x \leq M\,, \\ 1 \ + \ px \ + \ 2^{p-3}p(p-1)\,x^p, & \text{for } x \geq M\,. \end{cases} \tag{2.10}$$

Proof. First let us assume $p \geq 2$. For $x \geq -1$, define $h_x(t) := (1+tx)^p$, for all $0 \leq t \leq 1$. Then $h \in C^2[0,1]$ and hence by Taylors expansion we have

$$h_x(t) = h_x(0) + t\,h_x'(0) + t^2 \int_0^1 h_x''(\theta t)(1-\theta)d\theta$$

therefore

$$\begin{aligned} h_x(1) &= 1 \ + \ px \ + \ p(p-1)\,x^2 \int_0^1 (1+tx)^{p-2}(1-t)dt \\ &\geq 1 \ + \ px \ + \ p(p-1)\,x^2 \int_0^1 (1-t)^{p-1}dt \\ &= 1 \ + \ px \ + \ (p-1)\,x^2\,. \end{aligned} \tag{2.11}$$

Let $0 < \alpha < 1$, and choose $M > 1$ such that for $x \geq M$, $\alpha x^p \geq 1 + px + (\alpha/2)x^2$. Then for $-1 \leq x \leq M$

$$\begin{aligned} (1+x)^p &\geq 1 \ + \ px \ + \ (p-1)\,x^2 \\ &= 1 \ + \ px \ + \ (\alpha/2)x^2 \ + \ (p-1-\alpha/2)\ x^2 \\ &\geq 1 \ + \ px \ + \ (\alpha/2)x^2 \ + \ M^{2-p}\,(p-1-\alpha/2)\ |x|^p \end{aligned}$$

and for $x \geq M$, $(1+x)^p \geq x^p = \alpha x^p + (1-\alpha)x^p \geq 1 + px + (\alpha/2)x^2 + (1-\alpha)x^p$. Combining these inequalities by replacing $C = \alpha/2$ and $B = \min\left\{1-\alpha, M^{2-p}(p-1-\alpha)\right\}$, to we obtain (2.9).

Let us consider the case $1 < p < 2$, and take the function h_x for $x > -1$, as above. Notice that $1 + tx \neq 0$ and hence h_x'' exists and we have

$$(1+x)^p = 1 + px + p(p-1)\,x^2 \int_0^1 (1+tx)^{p-2}(1-t)dt\,. \tag{2.12}$$

Now for the case $-1 < x \leq M$, from 2.12 we get

$$\begin{aligned}
(1+x)^p &\geq 1 + px + p(p-1)\,x^2 \int_0^{1/2} (1+tx)^{p-2}(1-t)dt \\
&\geq 1 + px + p(p-1)\,x^2 \int_0^{1/2} (2M)^{p-2}(1-t)dt \\
&= 1 + px + \frac{p(p-1)}{4}\,(2M)^{p-2}x^2 \\
&= 1 + px + 2^{p-4}p(p-1)M^{p-2}\,x^2. \qquad (2.13)
\end{aligned}$$

Since $M \geq 1$, from (2.12), we have all for $x \geq M$

$$\begin{aligned}
(1+x)^p &\geq 1 + px + p(p-1)\,x^2 \int_0^1 (2x)^{p-2}(1-t)dt \\
&= 1 + px + 2^{p-3}p(p-1)\,x^p. \qquad (2.14)
\end{aligned}$$

Observe that for $x = -1$, the inequalities (2.13) and (2.14) are trivially true and hence the lemma. ∎

§2.5 Proof of Theorem (2.1.1)

Now we are in a position to give a proof of Theorem (2.1.1) and we organize the proof in the following manner: First we prove the validity of the inequality (2.4) for the cases $2 \leq p < n$ and $1 < p < 2$ separately and finally we show the optimality of the inequality (2.3).

Proof. Let us assume, $1 < p < n$ and $\gamma \geq 2$. Since the function $\gamma \mapsto (\log R/r)^{-\gamma}$, is monotonically decreasing on $[2, \infty)$, it is enough to prove the inequality (2.4) only for $\gamma = 2$. For both the cases $p \geq 2$ and $p < 2$, we shall first prove the inequality for smooth, positive and radially nonincreasing functions defined on a ball B, centered at 0 and of radius T. For $u \in C_0^2(B)$, $u > 0$, radially nonincreasing, we define

$$v(r) := u(r)\, r^{(n-p)/p}\,, \quad r = |x|. \tag{2.15}$$

Here without loss of generality we as well assume $u'(r) < 0$, (replacing u by $u + \epsilon(T - r)$ for $\epsilon > 0$, sufficiently small). Now we observe that,

$$\begin{aligned}\int_B |\nabla u|^p dx - \lambda_{n,p} \int_B \frac{|u(x)|^p}{|x|^p} dx &= \omega_n \int_0^T \left| \frac{n-p}{p} r^{-n/p} v - r^{1-n/p} v' \right|^p r^{n-1} dr - \lambda_{n,p}\, \omega_n \int_0^T \frac{v^p}{r}\, dr \\ &= \omega_n \lambda_{n,p} \int_0^T v^p(r) \left\{ \left| 1 - \frac{p v'(r)\, r}{(n-p) v(r)} \right|^p - 1 \right\} \frac{dr}{r}. \qquad (2.16)\end{aligned}$$

Since u is a decreasing function, we have from (2.15), $v'(r) - \dfrac{(n-p)v(r)}{p\, r} < 0$ and we call $x(r) := -\dfrac{p v'(r)\, r}{(n-p)v(r)}$ so that, $x(r) > -1$. Now we consider the following cases:

Case $p \geq 2$: By the inequality (2.9) in the Key-Lemma (2.4.1), and from (2.16), we obtain

$$\begin{aligned}\int_B |\nabla u|^p dx - \lambda_{n,p} \int_B \frac{|u(x)|^p}{|x|^p} dx &\geq C\, \omega_n \left(\frac{n-p}{p} \right)^{p-2} \int_0^T v^{p-2}(r) |v'(r)|^2 r\, dr \\ &\quad + B\, \omega_n \int_0^T |v'(r)|^p r^{p-1}\, dr - p\, \omega_n \int_0^T v^{p-1}(r) v'(r)\, dr \\ &= \frac{4C\, \omega_n}{p^2} \left(\frac{n-p}{p} \right)^{p-2} \int_0^T \left| \left(v^{p/2}(r) \right)' \right|^2 r\, dr \\ &\quad + B\, \omega_n \int_0^T |v'(r)|^p r^{p-1}\, dr. \qquad (2.17)\end{aligned}$$

because of the fact that $v(0) = v(T) = 0$. Now by 2-dimensional Hardy inequality (corollary (2.2.2)),

$$\begin{aligned}\int_0^T \left| \left(v^{p/2}(r) \right)' \right|^2 r\, dr &\geq \frac{1}{4} \int_0^T \left(\frac{v^{p/2}(r)}{r \log R/r} \right)^2 r\, dr \\ &= \frac{1}{4} \int_0^T \frac{u^p(r)}{r^p} \left(\log R/r \right)^{-2} r^{n-1} dr \\ &= \frac{1}{4} \int_B \frac{|u(x)|^p}{|x|^p} \left(\log R/|x| \right)^{-2} dx\,. \qquad (2.18)\end{aligned}$$

Now for $0 \leq \beta < p$, we have

$$\int_B \frac{|u(x)|^q}{|x|^\beta} dx = \omega_n \int_0^T |v(r)|^q r^{n-\beta-1-q(n-p)/p} dr.$$

Notice that, for C^1 radial function $v(r)$ vanishing at T,

$$\begin{aligned} |v(r)| &= \left| \int_r^T v'(t)dt \right| \\ &\leq \int_r^T \left| v'(t)t^{(p-1)/p} \right| t^{-(p-1)/p} dt \\ &\leq \left(\int_r^T |v'(t)|^p t^{p-1} dt \right)^{1/p} \left(\log \frac{T}{r} \right)^{(p-1)/p} \end{aligned}$$

and hence,

$$\begin{aligned} \int_0^T |v(r)|^q r^{\sigma-1} dr &\leq \left(\int_0^T |v'(r)|^p r^{p-1} dr \right)^{q/p} \int_0^T r^{\sigma-1} \left(\log \frac{T}{r} \right)^{q(p-1)/p} dr \\ &= \left(\int_0^T |v'(r)|^p r^{p-1} dr \right)^{q/p} \int_0^\infty e^{-\sigma r} r^{q(p-1)/p} dr, \end{aligned}$$

where $\sigma := n - \beta - q(n-p)/p$. Now the second integral on the right hand side is convergent iff $\sigma > 0$, i.e., $1 < q < \dfrac{p(n-\beta)}{n-p}$. Call

$$C_1 := B(\omega_n)^{1-p/q} \left(\int_0^\infty e^{-\sigma r} r^{q(p-1)/p} dr \right)^{-p/q}.$$

Then from these, we get

$$B\ \omega_n \int_0^T |v'(r)|^p r^{p-1} dr \geq C_1 \left(\int_B \frac{|u(x)|^q}{|x|^\beta} dx \right)^{p/q}.$$

Combining this with (2.17), and (2.18) we get the inequality (2.4) and hence (2.3) for radially decreasing, smooth positive functions. Now by density arguments, inequality (2.4) is valid for any $u \in W_0^{1,p}(B)$, $u \geq 0$, and radially nonincreasing. For a general domain Ω, we use symmetrization arguments. Let B_T be a ball having same volume as Ω with $T = (n|\Omega|_n/\omega_n)^{1/n}$, where $\omega_n = |S^{n-1}|_{n-1}$ and let $|u|^*$ be the symmetric decreasing rearrangement of the function $|u|$. Now observe that, for any $u \in W_0^{1,p}(\Omega)$, $|u|^* \in W_0^{1,p}(B_T)$ and $|u|^* > 0$ and radially nonincreasing, see [5, 45], and hence inequality (2.4) holds for $|u|^*$. It is well known that the symmetrization does not change the L^p- norm, decreases gradient norm and increases the integrals $\int\limits_\Omega \dfrac{|u(x)|^p}{|x|^p} dx$ and $\int\limits_\Omega \dfrac{|u(x)|^p}{|x|^p} \left(\log \dfrac{R}{|x|} \right)^{-2} dx$, since both the singular weights are decreasing functions of $|x|$ under our assumption on R, thanks to the Hardy's inequality for rearrangements. Hence (2.4) holds also for any $u \in W_0^{1,p}(\Omega)$.

Case $1 < p < 2$: Since for the case $1 < p < 2$, $(1+x)^p$, does not have global estimate in terms of either x^2 or $|x|^p$, we do the following decomposition of the interval $[0,T]$. Take $M \geq \max\left\{1, \frac{1}{(n-p)\,(\log\, R/T)}\right\}$ and denote, $\alpha := \frac{n-p}{p}$. Now we define the following sets, $A := \left\{r \in [0,T] : x(r) = -\frac{pv'(r)\,r}{(n-p)v(r)} < M\right\}$, $B := \{r \in [0,T] : x(r) > M\}$ and $\Gamma := \left\{r \in [0,T] : \frac{v'(r)}{v(r)} = -\frac{\alpha\, M}{r}\right\}$ so that $[0,T] = A \cup B \cup \Gamma$. Now observe that, on the set A, $\frac{v'(r)}{v(r)} > -\frac{\alpha\, M}{r}$, i.e., *iff* $\left(\log v(r)\, r^{\alpha\, M}\right)' > 0$, i.e, *iff* the function $v\, r^{\alpha\, M}$ is monotonically increasing. So, if we define the function $g(r)$ to be $v\, r^{\alpha\, M}$, for $0 \leq r \leq T$, then from the above facts we can write, $A = \{r \in [0,T] : g(r) \quad \text{is monotonically increasing}\}$ and the set $B = \{r \in [0,T] : g(r) \quad \text{is monotonically decreasing}\}$. We now prove the theorem through following steps:

Step I: Let us assume that, $\Gamma = \left\{r : \frac{v'(r)}{v(r)} = -\frac{\alpha\, M}{r}\right\} = \left\{r : \frac{u'(r)}{u(r)} = -\frac{\alpha(M+1)}{r}\right\}$ is a finite set. Now observe that critical points of the function $v\, r^{\alpha\, M}$ is precisely Γ and hence its cardinality will be odd. So let, $\Gamma = \{a_1, a_2 \cdots a_{2m+1}\}$ with $a_0 := 0 < a_1 < \cdots a_{2m+1} < a_{2m+2} := T$. Hence $[0,T]$ can be decomposed as $[0,T] = A \cup B \cup \Gamma$, with $A = \cup_{i=0}^{m}(a_{2i}, a_{2i+1})$ and $B = \cup_{i=0}^{m}(a_{2i+1}, a_{2i+2})$. We denote, $C_1(p) := 2^{p-4}p(p-1)$. Hence from (2.16) and by the inequality (2.10) in the Key-Lemma (2.4.1), we get

$$\begin{aligned}
X(u) &:= \int\limits_{B(0,T)} |\nabla u|^p dx - \lambda_{n,p} \int\limits_{B(0,T)} \frac{|u(x)|^p}{|x|^p} dx \\
&= \omega_n \lambda_{n,p} \left[\int\limits_A v^p(r) \left\{\left|1 - \frac{v'(r)\, r}{\alpha\, v(r)}\right|^p - 1\right\} \frac{dr}{r} + \int\limits_B v^p(r) \left\{\left|1 - \frac{v'(r)\, r}{\alpha\, v(r)}\right|^p - 1\right\} \frac{dr}{r}\right] \\
&\geq C_1(p)\, \omega_n \lambda_{n,p} \left[\frac{M^{p-2}}{\alpha^2} \int\limits_A v^{p-2}(r) \left(v'(r)\right)^2 r\, dr + \frac{2}{\alpha^p} \int\limits_B |v'(r)|^p\, r^{p-1} dr\right] \\
&= C_1(p)\, \omega_n \left[\frac{1}{(M\alpha)^{2-p}} \sum_{i=0}^{m} \int_{a_{2i}}^{a_{2i+1}} v^{p-2}(r) \left(v'(r)\right)^2 r\, dr + 2 \sum_{i=0}^{m} \int_{a_{2i+1}}^{a_{2i+2}} |v'(r)|^p\, r^{p-1} dr\right] \\
&= 2C_1(p)\, \omega_n \left[\frac{2}{p^2\, (M\alpha)^{2-p}} \sum_{i=0}^{m} \int_{a_{2i}}^{a_{2i+1}} \left|\left(v^{p/2}\right)'\right|^2 r\, dr + \sum_{i=0}^{m} \int_{a_{2i+1}}^{a_{2i+2}} |v'|^p\, r^{p-1} dr\right]. \quad (2.19)
\end{aligned}$$

For $r \in [0,T]$, define $w(r) := v^p(r) \left(\log\, R/r\right)^{-1}$. Now by Lemma (2.2.1) (with $k = 1$), we have for $0 \leq i \leq m$

$$\int_{a_{2i}}^{a_{2i+1}} \left|\left(v^{p/2}(r)\right)'\right|^2 r\, dr \geq \frac{1}{4} \int_{a_{2i}}^{a_{2i+1}} \frac{v^p(r)}{\left(r\, \log\, R/r\right)^2} r\, dr - \frac{1}{2} \left\{w(a_{2i+1}) - w(a_{2i})\right\}. \quad (2.20)$$

Therefore from (2.19) and (2.20), we obtain

$$\begin{aligned} X(u) &\geq \frac{\omega_n\, C_1(p)}{p^2\,(M\alpha)^{2-p}}\left[\sum_{i=0}^{m}\int_{a_{2i}}^{a_{2i+1}}\frac{v^p(r)}{(r\,\log\,R/r)^2}r\,dr - 2\sum_{i=0}^{m}\{w(a_{2i+1})-w(a_{2i})\}\right] \\ &\quad + 2\omega_n C_1(p)\sum_{i=0}^{m}\int_{a_{2i+1}}^{a_{2i+2}}|v'(r)|^p\,r^{p-1}dr \\ &= \frac{\omega_n\, C_1(p)}{p^2\,(M\alpha)^{2-p}}\left[\sum_{i=0}^{m}\int_{a_{2i}}^{a_{2i+1}}\frac{v^p(r)}{(r\,\log\,R/r)^2}r\,dr + 2\left\{\sum_{i=0}^{m-1}[w(a_{2i+2})-w(a_{2i+1})]-w(a_{2m+1})\right\}\right] \\ &\quad + 2\omega_n C_1(p)\sum_{i=0}^{m}\int_{a_{2i}}^{a_{2i+2}}|v'(r)|^p\,r^{p-1}dr. \end{aligned} \tag{2.21}$$

Let us observe that $w'(r) = pv^{p-1}(r)v'(r)\,(\log\,R/r)^{-1} + \dfrac{1}{r}v^p(r)\,(\log\,R/r)^{-2}$, and hence for $0 \leq i \leq m$, $(w(a_{2m+2}) = w(T) = 0)$ we have

$$\begin{aligned} w(a_{2i+2}) - w(a_{2i+1}) &= \int_{a_{2i+1}}^{a_{2i+2}} w'(r)\,dr \\ &= \int_{a_{2i+1}}^{a_{2i+2}} v^p(r)(r\,\log\,R/r)^{-2}r\,dr \\ &\quad + p\int_{a_{2i+1}}^{a_{2i+2}} v^{p-1}(r)v'(r)\,(\log\,R/r)^{-1}\,dr. \end{aligned} \tag{2.22}$$

Hence from (2.21) and (2.22), we have

$$\begin{aligned} X(u) &\geq \frac{\omega_n\, C_1(p)}{p^2\,(M\alpha)^{2-p}}\left[\sum_{i=0}^{m}\int_{a_{2i}}^{a_{2i+1}}\frac{v^p(r)}{(r\,\log\,R/r)^2}r\,dr + 2\sum_{i=0}^{m}\int_{a_{2i+1}}^{a_{2i+2}}\frac{v^p(r)}{(r\,\log\,R/r)^2}r\,dr\right] \\ &\quad + 2\omega_n C_1(p)\sum_{i=0}^{m}\int_{a_{2i+1}}^{a_{2i+2}}\left\{|v'(r)|^p r^{p-1} + \frac{(M\alpha)^{p-2}}{p}v^{p-1}(r)v'(r)\,(\log\,R/r)^{-1}\right\}dr \\ &\geq \frac{\omega_n\, C_1(p)}{p^2\,(M\alpha)^{2-p}}\int_0^T\frac{v^p(r)}{(r\,\log\,R/r)^2}r\,dr \\ &\quad + 2\omega_n C_1(p)\sum_{i=0}^{m}\int_{a_{2i+1}}^{a_{2i+2}}\left\{|v'|^p r^{p-1} + \frac{(M\alpha)^{p-2}}{p}v^{p-1}v'\,(\log\,R/r)^{-1}\right\}dr. \end{aligned} \tag{2.23}$$

Now, since $v'(r) < -\dfrac{\alpha\, M\, v(r)}{r}$, on (a_{2i+1}, a_{2i+2}) for each $0 \le i \le m$, we have

$$\begin{aligned} Y(v) &:= |v'(r)|^p r^{p-1} + \frac{(M\alpha)^{p-2}}{p} v^{p-1}(r) v'(r) \left(\log\, R/r\right)^{-1} \\ &= (-v'(r))^p\, r^{p-1} + \frac{(M\alpha)^{p-2}}{p} v^{p-1}(r) v'(r) \left(\log\, R/r\right)^{-1} \\ &= (-v'(r))\, v^{p-1}(r) \left[\left(\frac{-v'(r)\, r}{v(r)} \right)^{p-1} - \frac{(M\alpha)^{p-2}}{p} \left(\log\, R/r\right)^{-1} \right] \\ &\ge (-v'(r))\, v^{p-1}(r) \left[(M\alpha)^{p-1} - \frac{(M\alpha)^{p-2}}{p} \left(\log\, R/T\right)^{-1} \right] \\ &\ge 0 \quad \text{iff } (M\alpha)^{p-1} - \frac{(M\alpha)^{p-2}}{p} \left(\log\, R/T\right)^{-1} \ge 0. \end{aligned} \tag{2.24}$$

Since $M \ge \dfrac{1}{n-p} \left(\log\, R/T\right)^{-1} = \frac{1}{p\alpha} \left(\log\, R/T\right)^{-1}$, $Y(v) \ge 0$ on each of the intervals (a_{2i+1}, a_{2i+2}), $0 \le i \le m$, and hence from (2.23), we get

$$\begin{aligned} X(u) &\ge \omega_n \frac{(M\alpha)^{p-2}\, C_1(p)}{p^2} \int_0^T \frac{v^p(r)}{\left(r\, \log\, R/r\right)^2} r\, dr \\ &= \frac{(M\alpha)^{p-2}\, C_1(p)}{p^2} \int\limits_{B_T} \frac{|u(x)|^p}{|x|^p} \left(\log \frac{R}{|x|} \right)^{-2} dx. \end{aligned} \tag{2.25}$$

Step II: If the set Γ is not finite, we proceed as follows. Define the function $\phi(r) := r\, u'(r) + \delta\, u(r)$, for $r \in [0, T]$, $\delta := \alpha\,(M+1)$. Observe that $\phi \in C^1[0,T]$, $\phi(0) = \delta\, u(0) > 0$ and $\phi(T) = Tu'(T) < 0$ and $\Gamma = \phi^{-1}\{0\}$. Since $0 \notin \phi(\partial[0,T])$, 0 is a critical value of the function ϕ (i.e. $\phi(r_0) = 0$ with $\phi'(r_0) = 0$), as $\Gamma = \phi^{-1}\{0\}$ is not finite. By Sards Theorem we know that regular values of ϕ are dense in $\mathbb{R}$, so there exists $\epsilon_k\, \delta > 0$, regular values of ϕ such that $\epsilon_k \to 0$, as $k \to \infty$. Thus for $\epsilon_k < u(0)$, the set $\phi^{-1}\{\epsilon_k\, \delta\} = \left\{ r \in [0,T] : \dfrac{(u(r) - \epsilon_k)'}{u(r) - \epsilon_k} = -\dfrac{\delta}{r} \right\}$ is finite. Denoting $u_k(r) := u(r) - \epsilon_k$, then there exists $T_k > 0$ such that $u_k(T_k) = 0$, $u_k' < 0$ on $[0, T_k]$ and $T_k \to T$ as $k \to \infty$. Now by step I, we have the inequality (2.25) for u_k on $[0, T_k]$. Since the constant in (2.25) does not depend on k, passing through the limit as $k \to \infty$ we get the inequality (2.25) for any smooth, positive and strictly radially decreasing functions in $W_0^{1,p}(\Omega)$.

Step III: Now the remaining proof of the inequality (2.3) will follow from (2.25) together with the symmetrization arguments as in the case of $p \ge 2$.

Case $p = n$: In this case the inequality (2.3), follows immediately from the n-dimensional Hardy inequality (Lemma 2.3.1), together with symmetrization arguments.

Optimality Now suppose $1 < p < n$ and $0 \le \gamma < 2$. Since $\lambda_{n,p}$ is the best constant for the inequality (2.1), the Theorem follows for the case $\gamma = 0$. So we assume $0 < \gamma < 2$. The Theorem will follow if we can prove for the unit ball $B := B(0,1)$ that,

$$\inf_{\substack{u \in W_0^{1,p}(B) \\ u \neq 0}} \left(I_\gamma(u) := \frac{\int\limits_B \left(|\nabla u|^p - \lambda_{n,p} \frac{|u(x)|^p}{|x|^p} \right) dx}{\int\limits_B \frac{|u(x)|^p}{|x|^p} \left(\log R/|x| \right)^{-\gamma} dx} \right) = 0. \tag{2.26}$$

Now, our aim is to construct a family $u_\epsilon \in W_0^{1,p}(B)$, $\epsilon > 0$, such that

$$\lim_{\epsilon \to 0} I_\gamma(u_\epsilon) = 0 \tag{2.27}$$

and then (2.26) will follow. For $\epsilon > 0$, sufficiently small, let us define

$$u_\epsilon(r) := \begin{cases} 0, & \text{for } r \le \epsilon^2 \,, \\ \dfrac{\log\, r/\epsilon^2}{r^{(n-p)/p} \log\, 1/\epsilon}, & \text{for } \epsilon^2 \le r \le \epsilon \,, \\ \dfrac{\log\, 1/r}{r^{(n-p)/p} \log\, 1/\epsilon}, & \text{for } \epsilon \le r \le 1 \,. \end{cases} \tag{2.28}$$

Clearly, u_ϵ is continuous and differentiable a.e. and its derivative is given by

$$u_\epsilon'(r) = \begin{cases} 0, & \text{for } 0 \le r \le \epsilon^2 \,, \\ \dfrac{1}{r^{n/p} \log\, 1/\epsilon} \left[1 - \dfrac{n-p}{p} \log\, r/\epsilon^2 \right], & \text{for } \epsilon^2 \le r \le \epsilon \,, \\ -\dfrac{1}{r^{n/p} \log\, 1/\epsilon} \left[1 + \dfrac{n-p}{p} \log\, 1/r \right], & \text{for } \epsilon \le r \le 1 \,. \end{cases}$$

Then we have

$$\begin{aligned} \int\limits_B \frac{|u_\epsilon(x)|^p}{|x|^p}\, dx &= \frac{\omega_n}{(\log\, 1/\epsilon)^p} \left[\int_{\epsilon^2}^{\epsilon} \left(\log\, r/\epsilon^2 \right)^p \frac{dr}{r} + \int_\epsilon^1 (\log\, 1/r)^p \frac{dr}{r} \right] \\ &= \frac{\omega_n}{(p+1)\,(\log\, 1/\epsilon)^p} \left[\int_{\epsilon^2}^{\epsilon} \frac{d}{dr} \left(\log\, r/\epsilon^2 \right)^{p+1} dr - \int_\epsilon^1 \frac{d}{dr} (\log\, 1/r)^{p+1}\, dr \right] \\ &= \frac{2\omega_n}{(p+1)} (\log\, 1/\epsilon) \,. \end{aligned} \tag{2.29}$$

Since $\epsilon > 0$ is sufficiently small, we have the following estimates, after a change of variables and the use of Neumann series:

$$\begin{aligned}
\int_B |\nabla u_\epsilon|^p dx &= \frac{\omega_n}{(\log\, 1/\epsilon)^p}\Big[\int_{\epsilon^2}^{\epsilon} \left|\alpha \log\, r/\epsilon^2 - 1\right|^p \frac{dr}{r} + \int_{\epsilon}^{1} |1 + \alpha \log\, 1/r|^p \frac{dr}{r}\Big] \\
&= \frac{\omega_n}{\alpha\,(\log\, 1/\epsilon)^p}\Big[\int_0^1 r^p\, dr + \int_0^{\alpha \log\, 1/\epsilon - 1} r^p\, dr + \int_1^{\alpha \log\, 1/\epsilon + 1} r^p\, dr\Big] \\
&= \frac{\lambda_{n,p}\omega_n}{(p+1)} \log\, 1/\epsilon\Big[\Big(1 - \frac{1}{\alpha \log\, 1/\epsilon}\Big)^{p+1} + \Big(1 + \frac{1}{\alpha \log\, 1/\epsilon}\Big)^{p+1}\Big] \\
&= \frac{2\lambda_{n,p}\omega_n}{(p+1)} \log\, 1/\epsilon + O(\frac{1}{\log\, 1/\epsilon}). \qquad (2.30)
\end{aligned}$$

Therefore from (2.29) and (2.30), we conclude that

$$\int_B |\nabla u_\epsilon|^p dx - \left(\frac{n-p}{p}\right)^p \int_B \frac{|u_\epsilon(x)|^p}{|x|^p}\, dx = O\left(\frac{1}{\log\, 1/\epsilon}\right). \qquad (2.31)$$

Finally, for sufficiently small $\epsilon > 0$, let us estimate the following integral,

$$\begin{aligned}
\int_B \frac{|u_\epsilon(x)|^p}{|x|^p} (\log R/|x|)^{-\gamma}\, dx &= \frac{\omega_n}{(\log\, 1/\epsilon)^p}\Big[\int_{\epsilon^2}^{\epsilon} \frac{(\log\, r/\epsilon^2)^p}{r\,(\log\, R/r)^\gamma} dr + \int_{\epsilon}^{1} \frac{(\log\, 1/r)^p}{r\,(\log\, R/r)^\gamma} dr\Big] \\
&\geq \frac{\omega_n}{(\log\, 1/\epsilon)^p}\Big[\left(\log\, R/\epsilon^2\right)^{-\gamma} \int_{\epsilon^2}^{\epsilon} \left(\log\, r/\epsilon^2\right)^p \frac{dr}{r} \\
&\quad + (\log\, R/\epsilon)^{-\gamma} \int_{\epsilon}^{1} (\log\, 1/r)^p \frac{dr}{r}\Big] \\
&\geq \frac{\omega_n}{(p+1)} \log\, 1/\epsilon\Big[\left(\log\, R/\epsilon^2\right)^{-\gamma} + (\log\, R/\epsilon)^{-\gamma}\Big] \\
&\geq \frac{\omega_n}{(p+1)} \log\, 1/\epsilon\Big[\left(2 \log\, 1/\epsilon^2\right)^{-\gamma} + (\log\, 1/\epsilon)^{-\gamma}\Big] \\
&= C\,(\log\, 1/\epsilon)^{1-\gamma}\,. \qquad (2.32)
\end{aligned}$$

Since $0 < \gamma < 2$, from (2.31) and (2.32), we obtain $I_\gamma(u_\epsilon) \to 0$, as $\epsilon \to 0$ and hence the Theorem for the case $1 < p < n$.

For the case $p = n$, by taking the same test functions u_ϵ with $p = n$, it is easy to verify that, $\int_B |\nabla u_\epsilon|^n dx = \frac{2\omega_n}{(n+1)} (\log\, 1/\epsilon)^{1-n}$ and $\int_B \frac{|u_\epsilon(x)|^n}{|x|^n} (\log R/|x|)^{-\gamma}\, dx \geq C\,(\log\, 1/\epsilon)^{1-\gamma}$ and hence

$$I_\gamma(u_\epsilon) \leq \frac{2\omega_n}{C(n+1)} (\log 1/\epsilon)^{\gamma - n} \to 0 \quad \text{as } \epsilon \to 0$$

Therefore the optimality follows. ∎

Proof of corollary 2.1.2. In the proof of Theorem (2.1.1), we have used only the first term of the series in 2-dimensional Hardy inequality. For the case $p \geq 2$, we use the inequality (2.7) (corollary 2.2.2) in (2.18) to prove the corollary. For the case $1 < p < 2$, the proof is a bit more cumbersome because of the presence of more boundary terms in (2.20) and we give an out line the of proof. We first prove the corollary with $k = 2$. First we recall the inequality (2.19),

$$X(u) \geq 2C_1(p)\,\omega_n \left[\frac{2}{p^2\,(M\alpha)^{2-p}} \sum_{i=0}^{m} \int_{a_{2i}}^{a_{2i+1}} \left| \left(v^{p/2}\right)' \right|^2 r\,dr + \sum_{i=0}^{m} \int_{a_{2i+1}}^{a_{2i+2}} |v'|^p\, r^{p-1} dr \right]. \quad (2.33)$$

For $r \in [0,T]$ define, $w_1(r) := v^p(r)\,(\log\, R/r)^{-1}$, $w_2(r) := w_1(r)\left(\log^{(2)} R/r\right)^{-1}$ and $\psi_2(r) := w_1(r) + w_2(r)$. Now by Lemma (2.2.1), with $k = 2$, we have for $0 \leq i \leq m$

$$\begin{aligned} \int_{a_{2i}}^{a_{2i+1}} \left| \left(v^{p/2}(r)\right)' \right|^2 r\,dr \;&\geq\; \frac{1}{4} \sum_{j=1}^{2} \int_{a_{2i}}^{a_{2i+1}} \frac{v^p(r)}{\left(r\,\prod_{l=1}^{j} \log^{(l)}\, R/r\right)^2} r\,dr \\ &\quad - \frac{1}{2}\left\{\psi(a_{2i+1}) - \psi(a_{2i})\right\}. \end{aligned} \quad (2.34)$$

Now, from (2.33) and (2.34) we obtain

$$\begin{aligned} X(u) \;&\geq\; \frac{\omega_n\, C_1(p)}{p^2\,(M\alpha)^{2-p}} \sum_{j=1}^{2} \sum_{i=0}^{m} \int_{a_{2i}}^{a_{2i+1}} \frac{v^p(r)}{\left(r\,\prod_{l=1}^{j} \log^{(l)}\, R/r\right)^2} r\,dr \\ &\quad + \frac{2\omega_n\, C_1(p)}{p^2\,(M\alpha)^{2-p}} \left\{ \sum_{i=0}^{m-1} \left[\psi_2(a_{2i+2}) - \psi_2(a_{2i+1})\right] - \psi_2(a_{2m+1}) \right\} \\ &\quad + 2\omega_n C_1(p) \sum_{i=0}^{m} \int_{a_{2i}}^{a_{2i+2}} |v'(r)|^p\, r^{p-1} dr. \end{aligned} \quad (2.35)$$

Let us observe that $w_2'(r) = w_1'(r)\left(\log^{(2)} R/r\right)^{-1} + \dfrac{w_1(r)}{r}\,(\log\, R/r)^{-1}\left(\log^{(2)} R/r\right)^{-2}$, and hence for $0 \leq i \leq m$, ($\psi_2(a_{2m+2}) = \psi_2(T) = 0$) we have

$$\begin{aligned} \psi_2(a_{2i+2}) - \psi_2(a_{2i+1}) \;&=\; \int_{a_{2i+1}}^{a_{2i+2}} \psi_2'(r)\,dr \\ &=\; \int_{a_{2i+1}}^{a_{2i+2}} w_1'(r)\,dr + \int_{a_{2i+1}}^{a_{2i+2}} w_2'(r)\,dr \\ &=\; \int_{a_{2i+1}}^{a_{2i+2}} \left(1 + \left(\log^{(2)} R/r\right)^{-1}\right) w_1'(r)\,dr \\ &\quad + \int_{a_{2i+1}}^{a_{2i+2}} \frac{w_1(r)}{r}\,(\log\, R/r)^{-1}\left(\log^{(2)} R/r\right)^{-2} dr\,. \end{aligned}$$

Therefore,

$$\begin{aligned}\psi_2(a_{2i+2})-\psi_2(a_{2i+1}) &= \int_{a_{2i+1}}^{a_{2i+2}} v^p(r)\left(1+\left(\log^{(2)} R/r\right)^{-1}\right)(r\,\log\,R/r)^{-2} r\,dr \\ &\quad + p\int_{a_{2i+1}}^{a_{2i+2}} v^{p-1}(r)v'(r)\left(1+\left(\log^{(2)} R/r\right)^{-1}\right)(\log\,R/r)^{-1}\,dr \\ &\quad + \int_{a_{2i+1}}^{a_{2i+2}} v^p(r)\left(r\,\log\,R/r\,\log^{(2)} R/r\right)^{-2} r\,dr \\ &\geq \sum_{j=1}^{2}\int_{a_{2i+1}}^{a_{2i+2}} v^p(r)\left(r\prod_{l=1}^{j}\log^{(l)}\,R/r\right)^{-2} r\,dr \\ &\quad + p\int_{a_{2i+1}}^{a_{2i+2}} v^{p-1}v'\left(1+\left(\log^{(2)} R/r\right)^{-1}\right)(\log\,R/r)^{-1}\,dr\,. \quad (2.36)\end{aligned}$$

Hence from (2.35) and (2.36), we have

$$\begin{aligned}X(u) &\geq \frac{\omega_n\,C_1(p)}{p^2\,(M\alpha)^{2-p}}\sum_{j=1}^{2}\int_0^T v^p\left(r\prod_{l=1}^{j}\log^{(l)}\,R/r\right)^{-2} r\,dr \\ &\quad + 2\omega_n C_1(p)\sum_{i=0}^{m}\int_{a_{2i+1}}^{a_{2i+2}}|v'|^p r^{p-1}\,dr \\ &\quad + \frac{2\omega_n C_1(p)}{p\,(M\alpha)^{2-p}}\sum_{i=0}^{m}\int_{a_{2i+1}}^{a_{2i+2}} v^{p-1}v'\left(1+\left(\log^{(2)} R/r\right)^{-1}\right)(\log\,R/r)^{-1}\,dr\,. \quad (2.37)\end{aligned}$$

For suitable choice of M, (as it is shown in (2.24)), sum of the last two terms on the right hand side of the inequality (2.37) can be shown to be non-negative. Now by following the proof of Theorem (2.1.1), we have

$$\int_\Omega |\nabla u|^p dx \geq \left(\frac{n-p}{p}\right)^p \int_\Omega \frac{|u(x)|^p}{|x|^p}\,dx + C\sum_{j=1}^{2}\int_\Omega \frac{|u(x)|^p}{|x|^p}\left(\prod_{i=1}^{j}\log^{(i)}\frac{R}{|x|}\right)^{-2} dx\,. \quad (2.38)$$

We now proceed by induction. Suppose, the inequality (2.38) holds for $N = k-1$, then by defining the iterative functions $w_k(r) := w_{k-1}(r)\left(\log^{(k)}\,R/r\right)^{-1}$, $\psi_k(r) := w_{k-1}(r) + w_k(r)$, for $k \geq 3$, and following the similar calculations we prove the corollary. ∎

Proof of Corollary 2.1.4. If $f \in \Re_p$, then

$$\lim_{\epsilon\to 0}\sup_{x\in B_\epsilon} f(x)|x|^p(\log(1/|x|)^2 < \infty$$

and hence for sufficiently small ϵ

$$f(x) < \frac{C}{|x|^p(\log\,1/|x|)^2} \quad \text{in } B_\epsilon := B(0,\epsilon)\,.$$

Outside the ball B_ϵ, both are bounded functions and hence C can be so chosen that this inequality holds in Ω and hence (2.5) follows from (2.3).

If $f \notin \Re_p$ and if $|x|^p f(x) \left(\log \frac{1}{|x|}\right)^2$ tends to ∞ as $|x| \to 0$, then we can write $f(x) = h(x)/|x|^p(\log 1/|x|)^2$, where $h(x)$ tends to infinity as x tends to 0. Then from the calculations of Theorem (2.1.1), for $\epsilon > 0$ sufficiently small (u_ϵ as defined in (2.28))

$$\begin{aligned}
\int\limits_{B(0,1)} |u_\epsilon(x)|^p \, f(x) \, dx &= \int\limits_{B(0,1)} \frac{|u_\epsilon(x)|^p h(x)}{|x|^p \left(\log 1/|x|\right)^2} dx \\
&= \frac{\omega_n}{(\log 1/\epsilon)^p} \Big[\int\limits_{B(0,\epsilon)\setminus B(0,\epsilon^2)} \frac{(\log |x|/\epsilon^2)^p \; h(x)}{|x|^n \left(\log R/|x|\right)^2} \, dx \\
&\quad + \int\limits_{B(0,1)\setminus B(0,\epsilon)} \frac{(\log 1/|x|)^p \; h(x)}{|x|^n \left(\log R/|x|\right)^2} dx \Big] \\
&\geq \frac{\omega_n \; \inf_{B(0,\epsilon)\setminus B(0,\epsilon^2)} h(x)}{(\log 1/\epsilon)^p} \left(\log R/\epsilon^2\right)^{-2} \int_{\epsilon^2}^{\epsilon} \left(\log r/\epsilon^2\right)^p \frac{dr}{r} \\
&\geq \frac{\omega_n \; m(\epsilon) \log 1/\epsilon}{(p+1) \left(\log R/\epsilon^2\right)^2} \\
&\geq \frac{\omega_n \; m(\epsilon) \log 1/\epsilon}{(p+1) \left(2 \log 1/\epsilon^2\right)^2} \\
&= C \frac{m(\epsilon)}{\log 1/\epsilon} \, .
\end{aligned}$$

Here m_ϵ stands for $\inf_{B_\epsilon \setminus B_{\epsilon^2}} h(x)$. Since $m_\epsilon \to \infty$ as $\epsilon \to 0$, we conclude that $I_f(u_\epsilon) \to 0$ as $\epsilon \to 0$ (I_f, is a similar quotient as I_γ which we have defined in (2.26)) and hence the inequality (2.5) cannot hold for such $f \notin \Re_p$. ∎

§2.6 An estimate for the best constant

Let us assume $1 < p < n$ and $R \geq e^{2/p} \sup_\Omega |x|$. Now, we define the constant $C(n, p, R, \Omega)$, by

$$C(n, p, R, \Omega) := \inf_{\substack{u \in W_0^{1,p}(\Omega) \\ u \neq 0}} Q_{\Omega,R}(u), \tag{2.39}$$

where ,

$$Q_{\Omega,R}(u) := \frac{\int\limits_\Omega |\nabla u|^p \, dx - \lambda_{n,p} \int\limits_\Omega \frac{|u|^p}{|x|^p} \, dx}{\int\limits_\Omega \frac{|u(x)|^p}{|x|^p} \left(\log \frac{R}{|x|}\right)^{-2} dx} \, . \tag{2.40}$$

and by Theorem (2.1.1), $C(n, p, R, \Omega) > 0$, and we call it *second Hardy constant*. From the proof of the Theorem (2.1.1), we note that the second Hardy constant is again not achieved. In this section we prove the following result:

(2.6.1) Theorem. *The second Hardy constant $C(n,p,R,\Omega)$ is independent of the domain Ω and for $2 \le p < n$, $R \ge e^{2/p} \sup_\Omega |x|$*

$$\frac{p-1}{p^2}\left(\frac{n-p}{p}\right)^{p-2} \le C(n,p,R) \le \frac{p-1}{2}\left(\frac{n-p}{p}\right)^{p-2}. \tag{2.41}$$

Proof. We prove the independence of the second Hardy constant through the following steps.
Step 1. We first prove that if B_1 and B_2 are concentric balls centered at origin of radii T_1 and T_2 respectively then $C(n,p,R_1,B_1) = C(n,p,R_2,B_2)$, where $R_i = \alpha T_i$ with $\alpha \ge e^{2/p}$, $i = 1, 2$. Take $u \in W_0^{1,p}(B_2)$ and define, $v(x) = u\left(\frac{T_2}{T_1}x\right)$ for $|x| < T_1$. Then

$$\begin{aligned} Q_{B_1,R_1}(v) &= \frac{\int\limits_{B_1} |\nabla v|^p dx - \lambda_{n,p} \int\limits_{B_1} \frac{|v|^p}{|x|^p}\, dx}{\int\limits_{B_1} \frac{|v|^p}{|x|^p}\left(\log \frac{\alpha T_1}{|x|}\right)^{-2} dx} \\ &= \frac{\int\limits_{B_2} |\nabla u|^p dx - \lambda_{n,p} \int\limits_{B_2} \frac{|u|^p}{|x|^p}\, dx}{\int\limits_{B_2} \frac{|u|^p}{|x|^p}\left(\log \frac{\alpha T_2}{|x|}\right)^{-2} dx} \\ &= Q_{B_2,R_2}(u), \end{aligned}$$

and hence $C(n,p,R_1,B_1) = C(n,p,R_2,B_2)$.

Step 2. Now we prove that $C(n,p,R,\Omega) = C(n,p,R,\Omega^*)$, where $\Omega^* = B(0,T)$ is the ball of radius $T = \left(\frac{|\Omega|}{|B(0,1)|}\right)^{1/n}$, $|.|_n$ denotes the n-dimensional Lebesgue measure. Here we use the same argument as in the proof Theorem (2.1.1). Take Ω^* as above, then for any $u \in W_0^{1,p}(\Omega)$, $|u|^* \in W_0^{1,p}(\Omega^*)$, where $|u|^*$ be the symmetric decreasing rearrangement of the function $|u|$. By the standard symmetrization arguments (as in the proof of Theorem (2.1.1)), we conclude that for any $u \in W_0^{1,p}(\Omega)$, $Q_{\Omega,R}(u) \ge Q_{\Omega^*,R}(u^*)$ and hence $C(n,p,R,\Omega) \ge C(n,p,R,\Omega^*)$. To prove the other inequality, take $r > 0$ such that the ball $B_r = B(0,r) \subseteq \Omega$. Then clearly, $C(n,p,R,\Omega) \le C(n,p,R,B_r)$ and hence by step1, $C(n,p,R,\Omega) = C(n,p,R,\Omega^*)$.

Now if Ω_1 and Ω_2 are two bounded domains with $R_i \ge e^{2/p}\sup_{\Omega_i}|x|$, by step1 and step2, $C(n,p,R_1,\Omega_1) = C(n,p,R_2,\Omega_2)$ and hence the constant is independent of the domain. We shall denote this constant by $C(n,p,R)$, for a given bounded domain Ω when $R \ge e^{2/p}\sup_\Omega |x|$.

Observe that in the Key-Lemma (2.4.1), for the case $p \ge 2$, we can simply take $C = p-1$ and $B = 0$, i.e, $(1+x)^p \ge 1 + px + (p-1)x^2$ for all $x \ge -1$. Now applying this to the inequality (2.17), we obtain $\frac{p-1}{p^2}\left(\frac{n-p}{p}\right)^{p-2} \le C(n,p,R)$. Since $C(n,p,R)$ is independent of the domain we try to find an upper bound in the ball Ω^* and without loss of generality we assume $\Omega^* = B(0,1)$. Here our idea is to construct a family of functions in $W_0^{1,p}(\Omega^*)$ and

then estimate $Q_{\Omega^*,R}$ for such a family. We take the following family of functions (similar to u_ϵ in (2.28)) in the following manner. For any $0 < \epsilon < 1$ and for $k \geq 2$, an integer we define

$$u_{\epsilon,k}(r) := \begin{cases} 0, & \text{for } r \leq \epsilon^k\,, \\ \dfrac{\log r/\epsilon^k}{(k-1)r^{(n-p)/p}\log 1/\epsilon}, & \text{for } \epsilon^k \leq r \leq \epsilon\,, \\ \dfrac{\log 1/r}{r^{(n-p)/p}\log 1/\epsilon}, & \text{for } \epsilon \leq r \leq 1 \end{cases}$$

Clearly, $u_{\epsilon,k} \in W_0^{1,p}(\Omega)$ is continuous and differentiable a.e. and its derivative is given by

$$u'_{\epsilon,k}(r) = \begin{cases} 0, & \text{for } 0 \leq r \leq \epsilon^k\,, \\ \dfrac{1}{(k-1)r^{n/p}\log 1/\epsilon}\left[1 - \dfrac{n-p}{p}\log r/\epsilon^k\right], & \text{for } \epsilon^k \leq r \leq \epsilon\,, \\ -\dfrac{1}{r^{n/p}\log 1/\epsilon}\left[1 + \dfrac{n-p}{p}\log 1/r\right], & \text{for } \epsilon \leq r \leq 1 \end{cases}$$

Since $\epsilon > 0$ is sufficiently small, we have the following estimates, after a change of variables and the use of Neumann series:

$$\begin{aligned} \int\limits_{B(0,1)} |\nabla u_{\epsilon,k}|^p dx &= \frac{\omega_n}{(\log 1/\epsilon)^p}\Big[\frac{1}{(k-1)^p}\int_{\epsilon^k}^{\epsilon}\left|\frac{n-p}{p}\log\frac{r}{\epsilon^k} - 1\right|^p\frac{dr}{r} + \int_{\epsilon}^{1}\left|1 + \frac{n-p}{p}\log\frac{1}{r}\right|^p\frac{dr}{r}\Big] \\ &= \frac{\lambda_{n,p}\omega_n}{(p+1)}(\log 1/\epsilon)\left[(k-1)\left(1 - \frac{p}{(k-1)(n-p)\log 1/\epsilon}\right)^{p+1}\right. \\ &\quad \left. + \left(1 + \frac{p}{(n-p)\log 1/\epsilon}\right)^{p+1}\right] \\ &= \frac{\lambda_{n,p}\omega_n}{(p+1)}(\log 1/\epsilon)\left[(k-1) - \frac{p(p+1)}{(n-p)\log 1/\epsilon}\right. \\ &\quad + \frac{p(p+1)}{2(k-1)}\left(\frac{p}{(n-p)\log 1/\epsilon}\right)^2 + O\left(\frac{1}{(k-1)^2(\log 1/\epsilon)^3}\right) \\ &\quad \left. + 1 + \frac{p(p+1)}{(n-p)\log 1/\epsilon} + \frac{p(p+1)}{2}\left(\frac{p}{(n-p)\log 1/\epsilon}\right)^2 + O\left(\frac{1}{\log 1/\epsilon}\right)^3\right] \\ &= \frac{k\,\lambda_{n,p}\omega_n}{(p+1)}\log 1/\epsilon + \frac{k\,p\,\omega_n}{2(k-1)}\left(\frac{n-p}{p}\right)^{p-2}(\log 1/\epsilon)^{-1} \\ &\quad + O\left(\frac{1}{(k-1)\log 1/\epsilon}\right)^2 + O\left(\frac{1}{\log 1/\epsilon}\right)^2. \end{aligned} \tag{2.42}$$

Then we have

$$\begin{aligned}\int\limits_{B(0,1)} \frac{|u_{\epsilon,k}|^p}{|x|^p}\,dx &= \frac{\omega_n}{(\log 1/\epsilon)^p}\Big[\frac{1}{(k-1)^p}\int_{\epsilon^k}^{\epsilon}\Big(\log r/\epsilon^k\Big)^p\,\frac{dr}{r}+\int_{\epsilon}^{1}(\log 1/r)^p\,\frac{dr}{r}\Big]\\ &= \frac{\omega_n}{(p+1)\,(\log 1/\epsilon)^p}\Big[\frac{1}{(k-1)^p}\int_{\epsilon^k}^{\epsilon}\frac{d}{dr}\Big(\log r/\epsilon^k\Big)^{p+1}\,dr-\int_{\epsilon}^{1}\frac{d}{dr}(\log 1/r)^{p+1}\,dr\Big]\\ &= \frac{k\,\omega_n}{(p+1)}\,(\log 1/\epsilon)\,. \end{aligned} \tag{2.43}$$

Thus from (2.43) and (2.42)

$$\begin{aligned}\int\limits_{B(0,1)}|\nabla u_{\epsilon,k}|^p dx-\left(\frac{n-p}{p}\right)^p\int\limits_{B(0,1)}\frac{|u_{\epsilon,k}|^p}{|x|^p}\,dx &= \frac{k\,p\,\omega_n}{2(k-1)}\left(\frac{n-p}{p}\right)^{p-2}(\log 1/\epsilon)^{-1}\\ &\quad+O\left(\frac{1}{\log 1/\epsilon}\right)^2. \end{aligned} \tag{2.44}$$

Finally, let us try find a"good" estimate of the following integral

$$\begin{aligned} I_p: &= \int\limits_{B(0,1)}\frac{|u_{\epsilon,k}|^p}{|x|^p}\,(\log R/|x|)^{-2}\,dx\\ &= \frac{\omega_n}{(\log 1/\epsilon)^p}\Big[\frac{1}{(k-1)^p}\int_{\epsilon^k}^{\epsilon}\frac{\Big(\log r/\epsilon^k\Big)^p}{r\,(\log R/r)^2}dr+\int_{\epsilon}^{1}\frac{(\log 1/r)^p}{r\,(\log R/r)^2}dr\Big]. \end{aligned}$$

Now by change of variable, $r\mapsto \log R/r$ and denoting, $a_\epsilon := \log R/\epsilon$, $b_\epsilon := \log R/\epsilon^k$ and $c := \log R$, we have

$$I_p=\frac{\omega_n}{((k-1)\log 1/\epsilon)^p}\int_{a_\epsilon}^{b_\epsilon}\frac{\Big(\log\, R\,e^{-r}/\epsilon^k\Big)^p}{r^2}dr\;+\;\frac{\omega_n}{(\log 1/\epsilon)^p}\int_{c}^{a_\epsilon}\frac{(\log\, e^r/R)^p}{r^2}dr\,. \tag{2.45}$$

We now call the first and second integral of (2.45) by I_p^1 and I_p^2 respectively and we do the following estimations:

$$\begin{aligned} I_p^1 &= \int_{a_\epsilon}^{b_\epsilon}\Big(\log\, R/\epsilon^k-\,r\Big)^p\,\frac{dr}{r^2}\\ &= b_\epsilon^p\int_{a_\epsilon}^{b_\epsilon}\Big(1\,-\,\frac{r}{b_\epsilon}\Big)^p\,\frac{dr}{r^2}\\ &\geq b_\epsilon^p\int_{a_\epsilon}^{b_\epsilon}\left(1\,-\,\frac{pr}{b_\epsilon}\,+\,\frac{(p-1)r^2}{b_\epsilon^2}\right)\frac{dr}{r^2}\\ &= \frac{b_\epsilon^p}{a_\epsilon}\left[\left(1-\frac{a_\epsilon}{b_\epsilon}\right)\left(1+(p-1)\frac{a_\epsilon}{b_\epsilon}\right)\,-\,\frac{p\,a_\epsilon}{b_\epsilon}\log\frac{b_\epsilon}{a_\epsilon}\right], \end{aligned} \tag{2.46}$$

and

$$\begin{aligned} I_p^2 &= \int_c^{a_\epsilon} (r - \log R)^p \frac{dr}{r^2} \\ &= \int_c^{a_\epsilon} r^{p-2} \left(1 - \frac{c}{r}\right)^p dr \\ &\geq \int_c^{a_\epsilon} r^{p-2} \left(1 - \frac{pc}{r} + \frac{(p-1)c^2}{r^2}\right) dr \\ &= \begin{cases} a_\epsilon \left[\left(1 - \frac{c}{a_\epsilon}\right) - 2\frac{c}{a_\epsilon} \log \frac{a_\epsilon}{c} + o(1)\right], & \text{for } p = 2\,, \\ a_\epsilon^2 \left[\frac{1}{2}\left(1 - (\frac{c}{a_\epsilon})^2\right) + 2\left(\frac{c}{a_\epsilon}\right)^2 \log \frac{a_\epsilon}{c} + o(1)\right], & \text{for } p = 3\,, \\ a_\epsilon^{p-1} \left[\frac{1}{p-1}\left(1 - (\frac{c}{a_\epsilon})^{p-1}\right) + o(1)\right] & \text{for } p \neq 2,\ p \neq 3 \end{cases} \\ &= a_\epsilon^{p-1} \left[\frac{1}{p-1} + o(1)\right], \end{aligned} \tag{2.47}$$

where $o(1) \to 0$ as $\epsilon \to 0$. Thus from (2.46) and (2.47) we obtain

$$\begin{aligned} I_p \geq J_{k,\epsilon} : &= \frac{\omega_n}{((k-1)\log 1/\epsilon)^p} \frac{b_\epsilon^p}{a_\epsilon} \left[\left(1 - \frac{a_\epsilon}{b_\epsilon}\right)\left(1 + (p-1)\frac{a_\epsilon}{b_\epsilon}\right) - \frac{p\,a_\epsilon}{b_\epsilon} \log \frac{b_\epsilon}{a_\epsilon}\right] \\ &\quad + \frac{\omega_n}{(\log 1/\epsilon)^p} a_\epsilon^{p-1} \left[\frac{1}{p-1} + o(1)\right], \end{aligned}$$

and hence from (2.44), for each fixed $k \geq 2$ we obtain

$$\begin{aligned} Q_{\Omega^*,R}(u_{\epsilon,k}) &\leq \frac{p\,k}{2(k-1)} \left(\frac{n-p}{p}\right)^{p-2} (\log 1/\epsilon)^{p-1} \\ &\quad \times \left[\frac{b_\epsilon^p}{(k-1)^p a_\epsilon} \left\{\left(1 - \frac{a_\epsilon}{b_\epsilon}\right)\left(1 + (p-1)\frac{a_\epsilon}{b_\epsilon}\right)\right\} + a_\epsilon^{p-1} \left\{\frac{1}{p-1} + o(1)\right\}\right]^{-1} \\ &\quad + J_{k,\epsilon}^{-1} \left[O\left(\frac{1}{\log 1/\epsilon}\right)^2\right] \\ &= \frac{p\,k}{2(k-1)} \left(\frac{n-p}{p}\right)^{p-2} \\ &\quad \times \left[\frac{(k-1)^{-p} b_\epsilon^p}{a_\epsilon \left(\log \frac{1}{\epsilon}\right)^{p-1}} \left\{\left(1 - \frac{a_\epsilon}{b_\epsilon}\right)\left(1 + (1-p)\frac{a_\epsilon}{b_\epsilon}\right)\right\} + \left(\frac{a_\epsilon}{\log \frac{1}{\epsilon}}\right)^{p-1} \left\{\frac{1}{p-1} + o(1)\right\}\right]^{-1} \\ &\quad + J_{k,\epsilon}^{-1} \left[O\left(\frac{1}{\log 1/\epsilon}\right)^2\right]. \end{aligned} \tag{2.48}$$

Here we note that $b_\epsilon^p/a_\epsilon(\log 1/\epsilon)^{p-1} \to k^p$ as $\epsilon \to 0$ and hence $J_{k,\epsilon}^{-1}\left[O\left(\dfrac{1}{\log 1/\epsilon}\right)^2\right] \to 0$ as either $\epsilon \to 0$ or $k \to \infty$. Thus we have

$$\begin{aligned} Q_{\Omega^*,R}(u_{\epsilon,k}) &\to \frac{p\,k}{2(k-1)}\left(\frac{n-p}{p}\right)^{p-2} \\ &\quad \times \left[\left(\frac{k}{k-1}\right)^p \left\{\left(1-\frac{1}{k}\right)\left(1+\frac{p-1}{k}\right)+\frac{p}{k}\log\frac{1}{k}\right\}+\frac{1}{p-1}\right]^{-1}, \quad \text{as } \epsilon \to 0 \\ &\to \frac{p}{2}\left(\frac{n-p}{p}\right)^{p-2}\left[1+\frac{1}{p-1}\right]^{-1}, \quad \text{as } k \to \infty \\ &= \frac{p-1}{2}\left(\frac{n-p}{p}\right)^{p-2}. \end{aligned}$$

Since $C(n,p,R) \le Q_{\Omega^*,R}(u_{\epsilon,k})$, for all $k \ge 2$ and for any sufficiently small $\epsilon > 0$, we have by passing through the limits as $\epsilon \to 0$ and $k \to \infty$,

$$C(n,p,R) \le \frac{p-1}{2}\left(\frac{n-p}{p}\right)^{p-2}$$

and hence the theorem. ∎

Chapter 3

On the Schröndinger Operator $\Delta + V(x)$

§3.1 Introduction

Let Ω be a bounded open subset of $\mathbb{R}^n$, with $0 \in \Omega$ and $n \geq 3$. In the analysis of partial differential equations involving singular coefficients or certain degeneracies, an important tool is Hardy-Sobolev inequality. For example, in describing atomic models involving singular potentials like Coulomb potential, $1/|x|$ via Dirac or Schrödinger equation, we are led to the study of the equation

$$-\left(\Delta + V(x)\right) u = f(x, u)\,,$$

where $V(x) = \frac{1}{|x|^\alpha}$, $0 < \alpha \leq 2$, for example see [24]. A particular important potential is the so called *inverse-square potential*

$$V(x) = \frac{\mu}{|x|^2},$$

which appears in the linearized analysis of standard combustion models which lead to blow-up phenomena, see [9, 6, 12, 20, 25, 26, 32, 42].

It is well known that the operator $(-\Delta)^{-1}$, considered between suitable function spaces ($H_0^1(\Omega)$, for example) has a discrete spectrum, consisting of eigenvalues. The Hardy-Sobolev inequality can be used to show that the operator $L_\mu := -\Delta - \mu/|x|^2$, is invertible from H_0^1 to H_0^1 and L_μ^{-1} also has discrete spectrum if $\mu < \left(\frac{n-2}{2}\right)^2$, see for example [13, 31]. If $\mu \geq \left(\frac{n-2}{2}\right)^2$, L_μ can be shown to have only continuous spectrum, see [31]. Thus $\left(\frac{n-2}{2}\right)^2$ is a sort of transition point for the operator L_μ.

In many applications [12, 13], such as stability analysis, one needs to know the first eigenvalue of linear operators like L_μ. It is also interesting to know the behaviour of the first eigenvalue λ_μ^1 of L_μ, $0 < \mu < \left(\frac{n-2}{2}\right)^2$, when μ approaches $\left(\frac{n-2}{2}\right)^2$. In [31], using the radial equation for the first eigenfunction u_1, in a ball, it is shown that $\lambda_\mu^1 \to \lambda^1 > 0$ as $\mu \to \left(\frac{n-2}{2}\right)^2$. For a general domain Ω, the same result is shown in [12], using the improved Hardy-Sobolev inequality. Here, we consider the weighted eigenvalue problem for L_μ,

$$\begin{cases} L_\mu u &= \lambda u(x) f(x) & \text{in} \quad \Omega \\ u &= 0 & \text{on} \quad \partial\Omega, \end{cases}$$

where the positive weight function f satisfies

$$\lim_{|x|\to 0} f(x)|x|^2 \left(\log \frac{1}{|x|}\right)^2 < \infty\,. \tag{3.1}$$

Here we show that, for such f, $\lambda^1_\mu(f) \to \lambda^1(f) > 0$ as $\mu \to \lambda^* := \frac{(n-2)^2}{4}$ and if f is more singular than this, namely if $|x|^2 (\log 1/|x|)^2 \ f(x) \to \infty$, then $\lambda^1_\mu(f) \to \lambda^1(f) = 0$. Thus (3.1), indeed is the borderline separating the case $\lambda^1(f) > 0$ and $\lambda^1(f) = 0$. This follows from the existence of the improved Hardy-Sobolev inequality of the type (2.3) in Chapter 2 with weight function f, and the failure of it. In order to show that the eigenvalue problem does not admit any positive $H^1_0(\Omega)$ solutions, we will show more generally that the equation $L_\mu u \geq 0$ does not admit any H^1_0 supersolution. Precise statements are give in section 3.3.

§3.2 Preliminary Embeddings

Let us first recall the well known Hardy-Sobolev inequality. For $1 < p < n$, $D^{1,p}(\mathbb{R}^n)$ is embedded continuously in $L^p(\mathbb{R}^n, |x|^{-p})$ and

$$\int\limits_{\mathbb{R}^n} \frac{|u|^p}{|x|^p} dx \leq \left(\frac{p}{n-p}\right)^p \int\limits_{\mathbb{R}^n} |\nabla u|^p dx \qquad \forall\ u \in D^{1,p}(\mathbb{R}^n) \tag{3.2}$$

where $D^{1,p}(\mathbb{R}^n)$ is the completion of $C^\infty_0(\mathbb{R}^n;\mathbb{R})$ in the norm $\|u\|_{D^{1,p}} = \|\nabla u\|_{L^p}$ (See [27], for example). If $\Omega = B_R$, a ball, $R > 0$ then also (3.2) is true for any $u \in W^{1,p}_0(B_R)$. If Ω is any general bounded domain, using Schwartz's symmetrization, one can verify the validity of (3.2) for $W^{1,p}_0(\Omega)$. The constant $\lambda_{n,p} := \left(\frac{n-p}{p}\right)^p$ is the best for the Hardy-Sobolev embedding for any Ω.

To prove our main results in this Chapter, we will make use of the following two propositions which are immediate consequence of Hardy-Sobolev and Sobolev embeddings.

(3.2.1) Proposition. *Let Ω be a bounded domain in $\mathbb{R}^n$ and $1 < p < n$. Let $f \in \Im_p$, where*

$$\Im_p := \left\{ f : \Omega \to \mathbb{R}^+ \;\middle|\; \lim_{|x|\to 0} |x|^p f(x) = 0 \text{ with } f \in L^\infty_{\text{loc}}(\Omega \setminus \{0\}) \right\}.$$

Then the embedding $W^{1,p}_0(\Omega) \hookrightarrow L^p(\Omega, f dx)$ is compact.

Proof. Since $\lim_{|x|\to 0} |x|^p f(x) = 0$, for any $\epsilon > 0$, $\quad \exists\ \delta > 0$ such that

$$\sup_{B_\delta \subseteq \Omega} |x|^p f(x) \leq \epsilon \ \text{ and } \ f|_{\Omega \setminus B_\delta} \ \text{ is bounded}$$

where $B_\delta = B(0,\delta)$. Now the continuity of the embedding $W_0^{1,p}(\Omega) \hookrightarrow L^p(\Omega, f)$ will follow from the Hardy, Sobolev and Poincaré's inequalities. To show the compactness of the embedding let $(g_m) \subseteq W_0^{1,p}(\Omega)$ be a bounded sequence. By reflexivity of the space and the Sobolev embedding we have, for a subsequence (g_{m_k}) of (g_m) as $k \to \infty$

$$\begin{cases} g_{m_k} \rightharpoonup g \ \text{ weakly in } W_0^{1,p}(\Omega) \\ \\ g_{m_k} \to g \ \text{ strongly in } L^p(\Omega) \ . \end{cases} \tag{3.3}$$

Let $C_\delta := \|f\|_{L^\infty(\Omega \setminus B_\delta)}$, so we have

$$\begin{aligned} \int\limits_\Omega |g_{m_k} - g|^p f \ dx &\leq \int\limits_{B_\delta} |g_{m_k} - g|^p f(x) \ dx + C_\delta \|g_{m_k} - g\|^p_{L^p(\Omega)} \\ &\leq \epsilon \int\limits_{B_\delta} |g_{m_k} - g|^p |x|^{-p} dx + C_\delta \|g_{m_k} - g\|^p_{L^p(\Omega)} \ . \end{aligned}$$

Hence by Hardy inequality we have

$$\int_\Omega |g_{m_k} - g|^p f \ dx \leq \epsilon \left(\frac{p}{n-p}\right)^p \|\nabla(g_{m_k} - g)\|^p_{L^p(\Omega)} + C_\delta \|g_{m_k} - g\|^p_{L^p(\Omega)} \, .$$

Since $(g_{m_k}) \subseteq W_0^{1,p}(\Omega)$ is bounded,

$$\int_\Omega |g_{m_k} - g|^p f(x) \ dx \leq \epsilon M + C_\delta \|g_{m_k} - g\|^p_{L^p(\Omega)}$$

where $M > 0$ be a constant depending on n and p. By (3.3), we have

$$\lim_{k \to \infty} \int\limits_\Omega |g_{m_k} - g|^p f \ dx \leq \epsilon M \ .$$

As $\epsilon > 0$ is arbitrary, $\lim_{k\to\infty} \int\limits_\Omega |g_{m_k} - g|^p f \, dx = 0$. Hence $W_0^{1,p}(\Omega) \hookrightarrow L^p(\Omega, f)$ is compact.∎

(3.2.2) Remark. *The class of functions $f \in \Im_p$ has lower order singularity than $|x|^{-p}$ at the origin. Here we give some examples of such functions:*
(a) any bounded function.
(b) in a small neighbourhood of 0, f is $|x|^{-\beta}, 0 < \beta < p$.
(c) $f(x) = |x|^{-p} / (\log 1/|x|)^\gamma$, $\gamma > 0$, in a small neighbourhood of 0.

(3.2.3) Remark. *If either $\lim_{|x|\to 0} |x|^p f(x)$ does not exist or it is different from zero, then the embedding need not be compact. For example, take $f(x)$* to be either $c|x|^{-p}|\sin \frac{1}{|x|}|$, or $c|x|^{-p}/(1+|x|^p)$ *where $c > 0$, then the required limit does not exist in the first case and is $c > 0$, in the second case. Since $|\sin \frac{1}{|x|}|$* and $1/(1+|x|^p) \in L^\infty(\Omega)$ *and $W_0^{1,p}(\Omega) \hookrightarrow L^p(\Omega, |x|^{-p})$ is not compact, the embedding $W_0^{1,p}(\Omega) \hookrightarrow L^p(\Omega, f)$, is also noncompact.*

(3.2.4) Proposition. *Let Ω be a bounded domain and $1 < p < n$. Let $f \in \Re_{p,\beta}$ where*

$$\Re_{p,\beta} := \left\{ f \in \Im_p \;\Big|\; 0 < \lim_{|x|\to 0} |x|^\beta f(x) < \infty \right\}$$

for some $0 \le \beta < p$. Let $p^*_\beta := \dfrac{p(n-\beta)}{(n-p)}$. Then the embedding $W^{1,p}_0(\Omega) \hookrightarrow L^q(\Omega, f)$ is

$$\text{(i)} \quad \text{continuous} \quad \forall \quad p \le q \le p^*_\beta \ ,$$

$$\text{(ii)} \quad \text{compact} \quad \forall \quad p \le q < p^*_\beta \ .$$

Proof. (i) The cases $\beta = 0$ and $q = p$ follows from the Sobolev and Hardy's inequality respectively. Let $0 < \beta < p$ and $p < q \le p^*_\beta$. Since $f \in \Re_{p,\beta}$ there exists $\delta(f) > 0$ and $C_\delta > 0$ such that $f(x) \le \dfrac{C_\delta}{|x|^\beta}$ on $B(0,\delta)$. Then for $u \in W^{1,p}_0(\Omega)$ we have

$$\begin{aligned} \int_\Omega |u|^q f \, dx &\le C_\delta \int_{B_\delta} \frac{|u|^q}{|x|^\beta} \, dx + C(\delta) \int\limits_{\Omega \setminus B_\delta} |u|^q \, dx \\ &\le C_\delta \int_{B_\delta} \left(\frac{|u|^\beta}{|x|^\beta} \right) |u|^{q-\beta} \, dx + C(\delta) \int\limits_\Omega |u|^q dx \ . \end{aligned}$$

By Hölder, Hardy and Poincaré inequalities we have

$$\begin{aligned} \textstyle\int_\Omega |u|^q f \, dx &\le C_\delta \left(\int\limits_{B_\delta} \frac{|u|^p}{|x|^p} dx \right)^{\beta/p} \left(\int\limits_{B_\delta} |u|^{p(q-\beta)/(p-\beta)} dx \right)^{(p-\beta)/p} + C_{n,p,\delta} \|\nabla u\|^q_{L^p(\Omega)} \\ &\le C_{n,p,q,\delta} \|\nabla u\|^\beta_{L^p(\Omega)} \|\nabla u\|^{q-\beta}_{L^p(\Omega)} + C_{n,p,\delta} \|\nabla u\|^q_{L^p(\Omega)} \ . \end{aligned}$$

Here we have used the fact that, $p(q-\beta)/(p-\beta) \le p^* = np/(n-p)$ iff $q \le p^*_\beta$. Hence

$$\left(\int\limits_\Omega |u|^q f \, dx \right)^{1/q} \le C \|\nabla u\|_p$$

where $C > 0$ depending on n, p, q, δ and Ω. Hence $W^{1,p}_0(\Omega) \hookrightarrow L^q(\Omega, f)$ is continuous.

(ii) For $q = p$, by proposition (3.2.3) the embedding is compact. So assume $0 < \beta < p$ and $p < q < p^*_\beta$. Let $(g_m) \subseteq W^{1,p}_0(\Omega)$ be bounded. Then there exists a subsequence (g_{m_k}) of (g_m) such that as $k \to \infty$

$$\begin{cases} g_{m_k} \rightharpoonup g \quad \text{weakly in} \quad W^{1,p}_0(\Omega) \\ g_{m_k} \to g \quad \text{strongly in} \quad L^r(\Omega) \quad \forall \ p \le r < p^*. \end{cases} \tag{3.4}$$

We have

$$\int_\Omega |g_{m_k} - g|^q \, f(x) \, dx \le C_\delta \int_{B_\delta} \frac{|g_{m_k} - g|^q}{|x|^\beta} + C_\delta \|g_{m_k} - g\|^q_{L^p(\Omega)} \, . \tag{3.5}$$

Choose $r > 0$ such that $q < r < p^* := np/(n-p)$ and let $p_1 = r/q$. Then $p_1' = r/(r-q)$. By Sobolev embedding, $(g_{m_k} - g)^q \in L^{p_1}(\Omega)$. If $|x|^{-\beta} \in L^{p_1'}(\Omega)$, then we can use Hölder inequality in (3.5). Now, $|x|^{-\beta} \in L^{p_1'}(\Omega)$ iff $\beta p_1' < n$, i.e. iff $\beta r/(r-q) < n$. i.e. iff $r > nq/(n-\beta)$. Since $q < p(n-\beta)/(n-p)$ we have $nq/(n-\beta) < np/(n-p) = p^*$. Choose $r > q$ such that $nq/(n-\beta) < r < p^*$. Hence from (3.4) and (3.5), we have

$$\begin{aligned}
\int_\Omega |g_{m_k} - g|^q \, f(x) \, dx &\le C_\delta \left(\int_{B_\delta} |g_{m_k} - g|^r \right)^{q/r} \left(\int_{B_\delta} \frac{1}{|x|^{\beta r/(r-q)}} \right)^{(r-q)/r} + C_\delta \|g_{m_k} - g\|^q_{L^q(\Omega)} \\
&\le C \|g_{m_k} - g\|^q_{L^r(\Omega)} \\
&\to 0 \text{ as } k \to \infty \, .
\end{aligned}$$

Hence the proof follows. ∎

(3.2.5) Remark. *In the above proposition p^*_β is critical in the sense that if $q > p^*_\beta$, then we will not have any such embedding and for $q = p^*_\beta$ we can show easily the noncompactness of this embedding in the same way as in the case of Sobolev embedding $W^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$.*

(3.2.6) Proposition. For $1 < p < n$ and $0 < \alpha < p$, then $D^{1,p}(\mathbb{R}^n)$ is continuously embedded in $L^{p^*_\alpha}(\mathbb{R}^n, |x|^{-\alpha})$, where $p^*_\alpha := \dfrac{p(n-\alpha)}{n-p}$.

Proof. For $u \in D^{1,p}(\mathbb{R}^n)$, we have $\displaystyle\int_{\mathbb{R}^n} \frac{|u|^{p^*_\alpha}}{|x|^\alpha} \, dx = \int_{\mathbb{R}^n} \frac{|u|^\alpha}{|x|^\alpha} |u|^{p^*_\alpha - \alpha} \, dx$. Since $0 < \alpha < p$, by Hölder and Hardy-Sobolev inequality, we have

$$\begin{aligned}
\int_{\mathbb{R}^n} \frac{|u|^{p^*_\alpha}}{|x|^\alpha} \, dx &\le \left(\int_{\mathbb{R}^n} \frac{|u|^p}{|x|^p} \, dx \right)^{\alpha/p} \left(\int_{\mathbb{R}^n} |u|^{p(p^*_\alpha - \alpha)} \, dx \right)^{(p-\alpha)/p} \\
&\le \left(\frac{p}{n-p} \right)^\alpha \left(\int_{\mathbb{R}^n} |\nabla u|^p \, dx \right)^{\alpha/p} \left(\int_{\mathbb{R}^n} |u|^{p^*} \, dx \right)^{(p-\alpha)/p} \\
&\le \left(\frac{p}{n-p} \right)^\alpha \left(\int_{\mathbb{R}^n} |\nabla u|^p \, dx \right)^{\alpha/p} \left(\int_{\mathbb{R}^n} |\nabla u|^p \, dx \right)^{p^*(p-\alpha)/p} \\
&= \left(\frac{p}{n-p} \right)^\alpha \left(\int_{\mathbb{R}^n} |\nabla u|^p \, dx \right)^{p^*_\alpha/p}
\end{aligned}$$

and therefore

$$\left(\int\limits_{\mathbb{R}^n} \frac{|u|^{p^*_\alpha}}{|x|^\alpha}\,dx\right)^{1/p^*_\alpha} \le \left(\frac{p}{n-p}\right)^{\alpha/p^*_\alpha} \left(\int\limits_{\mathbb{R}^n} |\nabla u|^p\,dx\right)^{1/p},$$

and hence the proposition. ∎

§3.3 Weighted Eigenvalue Problem

In this section we prove the following theorems:

(3.3.1) Theorem. *Let* $0 \le \mu < \lambda^* := \dfrac{(n-2)^2}{4}$, $\lambda \in \mathbb{R}^+$ *and* $f \in \Im_2$, *where*

$$\Im_2 := \left\{ f : \Omega \to \mathbb{R}^+ \;\middle|\; \lim_{|x|\to 0} |x|^2 f(x) = 0 \text{ with } f \in L^\infty_{\text{loc}}(\Omega \setminus \{0\}) \right\}.$$

The eigenvalue problem

$$\left\{ \begin{array}{rcll} -\left(\Delta + \frac{\mu}{|x|^2}\right) u &=& \lambda u f(x) & \text{in } \Omega \\ u &=& 0 & \text{on } \partial\Omega \end{array} \right. \tag{3.6}$$

admits nontrivial weak solutions in $H^1_0(\Omega)$ *corresponding to* $\lambda \in (\lambda^k_\mu(f))_{k=1}^\infty$, *where* $0 < \lambda^1_\mu(f) < \lambda^2_\mu(f) \le \lambda^3_\mu(f) \le \ldots \to +\infty$. *If* Ω *is* $C^{1,1}$, *then any weak solution of (3.6) is in* $H^1_0(\Omega) \cap W^{2,r}(\Omega)$ *for all* r, *with* $1 < r < 2n/(n+2)$.
Furthermore, if $f \in \Re_2 := \left\{ f \in \Im_2 \;\middle|\; \limsup_{|x|\to 0} |x|^2 f(x) \left(\log \frac{1}{|x|}\right)^2 < \infty \right\}$, *then* $\lambda^1_\mu(f) \to \lambda(f) > 0$ *as* $\mu \to \lambda^*$. *If* $f \notin \Re_2$ *and* $|x|^2 f(x) \left(\log \frac{1}{|x|}\right)^2$ *tends to* ∞ *as* $|x| \to 0$, *then the limit* $\lambda(f) = 0$.

Proof.
Existence : For any $0 \le \mu < \lambda^*$, define $a_\mu : H^1_0(\Omega) \times H^1_0(\Omega) \to \mathbb{R}$, by

$$a_\mu(u,v) := \int\limits_\Omega \langle \nabla u, \nabla v \rangle\,dx - \mu \int\limits_\Omega \frac{u}{|x|^2} v\,dx \quad \text{for } u,\, v \in H^1_0(\Omega).$$

By Hardy inequality, it follows that a_μ is a continuous symmetric bilinear mapping. Moreover,

$$a_\mu(u,u) \ge (1 - \frac{\mu}{\lambda^*})\|\nabla u\|_2 \quad \text{for all } v \in H^1_0(\Omega),$$

i.e. the operator, $L_\mu = -(\Delta + \dfrac{\mu}{|x|^2})$ on $H^1_0(\Omega)$ is coercive, for all $0 \le \mu < \lambda^*$. Hence by Lax-Milgram lemma, for any $g \in H^{-1}(\Omega)$

$$\left\{ \begin{array}{rcll} L_\mu u &=& g & \text{in } \Omega \\ u &=& 0 & \text{on } \partial\Omega \end{array} \right.$$

has unique solution, and $L_\mu^{-1} : H^{-1}(\Omega) \to H_0^1(\Omega)$ exists and continuous. Let us observe that for $u \in H_0^1(\Omega)$, $uf \in H^{-1}(\Omega)$, where for $v \in H_0^1(\Omega)$,

$$\begin{aligned}
\langle uf, v\rangle : &= \int_\Omega uv\, f\, dx \\
&\leq \left(\int_\Omega u^2\, f\, dx\right)^{1/2} \left(\int_\Omega v^2\, f\, dx\right)^{1/2} \\
&\leq C \left(\int_\Omega |\nabla u|^2\, dx\right)^{1/2} \left(\int_\Omega |\nabla v|^2\, dx\right)^{1/2}.
\end{aligned}$$

Hence by proposition (3.2.3), it follows that

$$\begin{array}{rccc} M_f : & H_0^1(\Omega) & \to & H^{-1}(\Omega) \\ & u & \longmapsto & uf \end{array}$$

is a compact operator. Since L_μ^{-1} is positive definite and self-adjoint,

$$S_\mu = L_\mu^{-1} \circ M_f : H_0^1(\Omega) \to H_0^1(\Omega)$$

is a compact, self-adjoint, positive definite operator. Hence the problem can be written as

$$\left\{ \begin{array}{rcll} S_\mu u &=& \lambda^{-1} u & \text{in } \Omega \\ u &=& 0 & \text{on } \partial\Omega \end{array} \right.$$

and has nontrivial solutions corresponding to $\lambda \in (\lambda_\mu^k(f))$ where $0 < \lambda_\mu^1(f) \leq \lambda_\mu^2(f) \leq \lambda_\mu^3(f) \leq \ldots \to +\infty$, and they are characterized by the usual Rayleigh quotients.

Let $u \in H_0^1(\Omega)$ be a first eigenfunction i.e,

$$L_\mu u = \lambda_\mu^1 uf\,.$$

Since $u^+, u^- \in H_0^1(\Omega)$, (where $u^+ := \max\{u, 0\}$ and $u^- := \max\{-u, 0\}$) we have ,

$$\left\{ \begin{array}{rcll} -\Delta u^+ &\geq& 0 & \text{in } \Omega, \\ -\Delta u^- &\geq& 0 & \text{on } \Omega\,. \end{array} \right.$$

Here we can not appeal to the classical maximum principle because f need not be smooth enough. So we intend to use weak version of the strong maximum principle. Let $u^+ = 0$ on $\Omega_1 \subseteq \Omega$ with $|\Omega_1| > 0$. Then there exists a ball $B \subset\subset \Omega$ with $|\Omega_1 \cap B| > 0$. Therefore,

$$0 = \operatorname{ess}\inf_{\Omega_1 \cap B} u^+ = \operatorname{ess}\inf_B u^+ = \operatorname{ess}\inf_\Omega u^+.$$

Hence by Theorem 8.19 in [28] we conclude that either $u^+ \equiv 0$ a.e on Ω or $u^+ > 0$ a.e on Ω. Thus u cannot change sign.

If possible, let $u, v \in H_0^1(\Omega)$ be two orthogonal eigenfunctions corresponding to $\lambda = \lambda_\mu^1$. Using the equations for u and v we have

$$\int_\Omega u\, v\, (\frac{\mu}{|x|^2} + \lambda_\mu^1 f)\, dx = 0$$

which is a contradiction, because u, v do not change sign in Ω. Hence λ_μ^1 is simple. ∎

Regularity: Without loss of generality let $v \in H_0^1(\Omega)$ be a solution of (1.18) corresponding to $\lambda = \lambda_\mu^1$. Let

$$h(x) = \frac{\mu}{|x|^2} v(x) + \lambda_\mu^1 v(x) f(x) \ \text{ on } \ \Omega.$$

Since $|x|^{-2} \in L^r, \ \forall\, r < n/2$, and $v(x) \in L^{2^*}$ we have $v(x)|x|^{-2} \in L^r \ \forall\, 1 < r < 2n/(n+2)$. There exists a constant $c > 0$ such that $|f(x)| < c|x|^{-2}$. Hence it follows that $h(x) \in L^r$ for all $r \in (1, 2n/(n+2))$. By Theorem (9.15) in [28] it follows that $v \in W^{2,r}(\Omega)$ for all r $1 < r < 2n/(n+2)$. ∎

(3.3.2) Remark. *This H_0^1 eigenfunction need not be in L^∞. In fact it need not be even in L^p for $p > 2^*$. The eigenfunctions in a ball, calculated in [13] for the case $f \equiv 1$ and μ in $(0, (n-2/2)^2)$ behave like $r^{1-(n/2)+\nu}$ near 0, with $\nu = \sqrt{((n-2/2)^2 - \mu)}$. Such eigenfunctions lie in L^p for $p < 2n/(n-2-2\nu)$ only. However, if $f \in C^\infty(\Omega \setminus \{0\})$ then by the standard elliptic regularity theory, see corollary 8.11 in [28] any eigenfunction will be in $C^\infty(\Omega \setminus \{0\})$.*

Limit of λ_μ^1 as $\mu \to \lambda^*$: For $f \in \Im_2$ we have for all $0 \le \mu < \lambda^*$,

$$\lambda_\mu^1(f) = \min_{\substack{u \in H_0^1(\Omega) \\ u \neq 0}} \frac{\int_\Omega (|\nabla u|^2 - \mu \frac{|u|^2}{|x|^2})\, dx}{\int_\Omega |u|^2\, f\, dx}. \tag{3.7}$$

Let μ_k be increasing to λ^* as $k \to \infty$. Then $(\lambda_{\mu_k}^1(f))$ is a non-increasing sequence, bounded above by λ_0^1, the (first eigenvalue of $-\Delta$ with zero Dirichlet boundary value) and below by 0. Let

$$\lambda_{\mu_k}^1(f) \to \lambda(f) \ \text{ as } \ k \to \infty\,. \tag{3.8}$$

Claim: $\lambda(f) > 0$ for $f \in \Re_2$.

For any $f \in \Re_2$, from Corollary (2.1.4), we have

$$C(f) \ = \ \inf_{\substack{u \in H_0^1(\Omega) \\ u \neq 0}} \frac{\int_\Omega (|\nabla u|^2 - \lambda^* \frac{|u|^2}{|x|^2})\, dx}{\int_\Omega |u|^2 f(x)\, dx} \ > \ 0\ .$$

Since for all $u \in H_0^1(\Omega)$ and for every $k \geq 1$,

$$\lambda^1_{\mu_k}(f) \leq \frac{\int\limits_\Omega (|\nabla u|^2 - \mu_k \frac{|u|^2}{|x|^2})\, dx}{\int\limits_\Omega |u|^2 f(x)\, dx} \tag{3.9}$$

therefore by taking $k \to \infty$, we have for every $u \in H_0^1(\Omega)$,

$$\lambda(f) \leq \frac{\int\limits_\Omega (|\nabla u|^2 - \lambda^* \frac{|u|^2}{|x|^2}) dx}{\int\limits_\Omega |u|^2 f(x)\, dx} \tag{3.10}$$

and hence

$$\lambda(f) \leq C(f).$$

To prove the other way inequality, observe that for all $u \in H_0^1(\Omega)$ the right hand side of (3.9) is greater than the right hand side of the inequality (3.10) and therefore for each $\mu < \lambda^*$

$$\lambda^1_\mu(f) \geq C(f)\,.$$

Hence

$$\lambda^1_{\mu_k} \to \lambda(f) = C(f) > 0 \quad \text{as } k \to \infty\,.$$

If $f \notin \Re_2$, then again by the Corollary (2.1.4), inequality of the type (2.5) cannot hold and hence $\lambda(f) = 0$. ∎

(3.3.3) Theorem. *For $\mu > \lambda^*$, there exists no $u \in H_0^1(\Omega)$ with $u \geq 0$, $u \not\equiv 0$ such that*

$$-\Delta u \geq \frac{\mu}{|x|^2}\, u \quad \text{in} \quad \Omega$$

i.e., there does not exist any non-negative $u \in H_0^1(\Omega)$ with , $u \not\equiv 0$ such that

$$\int\limits_\Omega \nabla u.\nabla \phi dx \geq \mu \int\limits_\Omega \frac{u}{|x|^2}\, \phi\, dx$$

for all $\phi \in H_0^1(\Omega), \quad \phi \geq 0.$

Proof. Let $\mu > \lambda^*$ and if possible let there exist nonnegative $u \in H_0^1(\Omega)$, $u \not\equiv 0$ such that for all $\phi \in H_0^1(\Omega),\ \phi \geq 0$

$$\int\limits_\Omega \nabla u.\nabla \phi\, dx \geq \mu \int\limits_\Omega \frac{u}{|x|^2} \phi\, dx\,. \tag{3.11}$$

Then by strong maximum principle for weak solutions (as in the proof of Theorem (3.3.1)), $u > 0$ a.e., in Ω. Since $0 \in \Omega$ there exists $0 < r < 1$ such that $B_r = B(0,r) \subseteq \Omega$. From Theorem (3.3.1), for any $0 \leq \beta < 2$, there exists $\phi_\beta > 0$ in $H_0^1(B_r)$ and $\lambda(\beta) > 0$, such that

$$-\Delta\phi_\beta = \frac{\lambda(\beta)}{|x|^\beta}\phi_\beta \quad \text{in} \quad B_r\,. \tag{3.12}$$

It is easy to check that, $\beta \mapsto \lambda(\beta)$ is a non-increasing function on $[0,2)$. By Hardy inequality and (3.12), we have $\lambda(\beta) \geq \lambda^*$; for any $\beta < 2$. Hence

$$\lim_{\beta\to 2} \lambda(\beta) \geq \lambda^*. \tag{3.13}$$

Since the inequality (3.11) is true for any $\phi \in H_0^1(B_r),\ \phi \geq 0$, we get

$$-\int_{B_r} u\Delta\phi_\beta\,dx + \int_{\partial B(0,r)} \frac{\partial\phi_\beta}{\partial\nu}d\sigma \geq \mu\int_{B_r} \frac{u}{|x|^2}\phi_\beta\,dx\,.$$

From above inequality and (3.12),

$$\lambda(\beta)\int_{B_r} \frac{u}{|x|^\beta}\phi_\beta\,dx + \int_{\partial B(0,r)} \frac{\partial\phi_\beta}{\partial\nu}d\sigma \geq \mu\int_{B_r} \frac{u}{|x|^2}\phi_\beta\,dx\,,$$

By Hopf Lemma [28], $\frac{\partial\phi_\beta}{\partial\nu} < 0$ on $\partial B(0,r)$ and hence we have

$$\mu < \lambda(\beta) \quad \text{for all} \quad \beta \in [0,2). \tag{3.14}$$

Since for any $\beta < 2$,

$$\lambda(\beta) \leq \frac{\int_{B_r} |\nabla v|^2 dx}{\int_{B_r} \frac{v^2}{|x|^\beta}dx}$$

for all $v \in H_0^1(B_r)$, we conclude that

$$\lim_{\beta\to 2} \lambda(\beta) \leq \lambda^*. \tag{3.15}$$

Hence, from (3.13) and (3.15) we obtain

$$\lim_{\beta\to 2} \lambda(\beta) = \lambda^*. \tag{3.16}$$

Since $\mu > \lambda^*$, (3.14) and (3.16) lead to a contradiction. Hence the theorem. ∎

In the following theorem we prove a more general nonexistence result.

(3.3.4) Theorem. *Let $\Omega \subset \mathbb{R}^n$, $n \geq 3$, be a bounded domain. Suppose that u is a non-negative function in $H_0^1(\Omega)$ such that*

$$-\Delta u - \frac{(n-2)^2}{4|x|^2}u \geq -\lambda \frac{\phi}{|x|^2}u\,, \quad in \quad \Omega$$

where ϕ is a continuous, non-negative function in $\bar{\Omega}$ such that

$$\lim_{|x|\to 0} \phi(x) \left(\log \frac{1}{|x|}\right)^{\gamma} = 0. \tag{3.17}$$

Then for any $\lambda \in \mathbb{R}$ and $\gamma \geq 2$, $u \equiv 0$.

Proof. Assume $\gamma \geq 2$ and if possible let there exists u, non-negative function in $H_0^1(\Omega) \cap C(\Omega)$, $u \not\equiv 0$ such that

$$-\Delta u - \frac{(n-2)^2}{4|x|^2}u \geq -\lambda \frac{\phi}{|x|^2}u, \quad \text{in} \quad \Omega\,.$$

where ϕ satisfies the condition (3.17). Then by maximum principle $u > 0$ in Ω. Our aim is to construct a subsolution in a suitable domain to get a contradiction. For $k > 0$, consider the following test function

$$\psi_k(r) := r^{-\frac{n-2}{2}} \left(\log 2/r\right)^{-k}, \qquad \text{where} \quad r = |x|,\ |x| < 1\,. \tag{3.18}$$

Call $\sigma := \frac{n-2}{2}$ and $X = X(r) := \log 2/r$. Since

$$\psi_k'(r) = -\sigma\, r^{-\sigma-1} X^{-k} \,+\, k\, r^{-\sigma-1} X^{-k-1}$$

we have

$$\begin{aligned}
\int\limits_{B(0,1)} |\nabla \psi_k(x)|^2\, dx &= \int_0^1 r^{-2(\sigma+1)} X^{-2k} \left(\frac{k}{X} - \sigma\right)^2 r^{n-1} dr \\
&= \int_0^1 \left(\frac{k^2}{X^2} - \frac{2k\sigma}{X} + \sigma^2\right) X^{-2k} \frac{dr}{r} \\
&= \int_{\log 2}^{\infty} \left(\frac{k^2}{r^2} - \frac{2k\sigma}{r} + \sigma^2\right) r^{-2k}\, dr \\
&< \infty \quad \text{iff} \quad k > 1/2.
\end{aligned}$$

Therefore $\psi_k \in H^1(B(0,1))$, iff $k > \frac{1}{2}$. Now, let us observe that

$$\begin{aligned}
\psi_k''(r) &= \sigma(\sigma+1)\, r^{-\sigma-2} X^{-k} \;\sigma\, k\, r^{-\sigma-2} X^{-k-1} \\
&- \quad (\sigma+1)\, k\, r^{-\sigma-2} X^{-k-1} \,+\, k(k+1)\, r^{-\sigma-2} X^{-k-2}.
\end{aligned}$$

Thus

$$\begin{aligned} -\Delta\psi_k &= \psi_k''(r) + \frac{n-1}{r}\psi_k'(r) \\ &= \frac{(n-2)^2}{4|x|^2}\psi_k - k(k+1)\frac{\psi_k}{|x|^2}\left(\log 2/|x|\right)^{-2}. \end{aligned}$$

Hence

$$\begin{aligned} L\psi_k &:= -\left[\Delta\psi_k + \frac{(n-2)^2}{4|x|^2}\psi_k\right] + \lambda\frac{\phi}{|x|^2}\psi_k \\ &= \frac{\psi_k}{|x|^2}\left(\log 2/|x|\right)^{-\gamma}\left[-k(k+1)\left(\log 2/|x|\right)^{\gamma-2} + \lambda\phi\left(\log 2/|x|\right)^{\gamma}\right] \quad \text{in } \Omega\,. \end{aligned} \tag{3.19}$$

By the condition on ϕ, for any $\lambda \in \mathbb{R}$, the most dominant term on the right hand side of (3.19) is $-k(k+1)\left(\log 2/|x|\right)^{\gamma-2}$. Thus we can choose $\delta > 0$ independent of k, such that

$$-\left[\Delta\psi_k + \frac{(n-2)^2}{4|x|^2}\psi_k\right] + \lambda\frac{\phi}{|x|^2}\psi_k \le 0 \quad \text{in} \quad B(0,\delta). \tag{3.20}$$

Take $\epsilon > 0$, depending only on δ such that $\epsilon\,\psi_k \le u$ on the boundary of the ball $B(0,\delta)$. Now define, $w_k := \epsilon\,\psi_k - u$ on $B(0,\delta)$, then $w_k^+ \in H_0^1(B_\delta)$ and

$$Lw_k \le 0 \qquad \text{on} \quad B_\delta.$$

Now by multiplying (testing) the above inequality by w_k^+ and integrating by parts we get

$$\int_{B_\delta} |\nabla w_k^+|^2 - \frac{(n-2)^2}{4}\int_{B_\delta}\left(\frac{w_k^+}{|x|}\right)^2 + \lambda\int_{B_\delta}\left(\frac{w_k^+}{|x|}\right)^2\phi \le 0$$

Hence by Hardy inequality $w_k^+ \equiv 0$ on B_δ i.e., $\epsilon\,\psi_k \le u$ on B_δ, for all $k > 1/2$. Hence $\epsilon\,\psi_{1/2} \le u$ on B_δ, which contradicts the fact that $u \in H^1(\Omega)$ because, $\int\limits_{B_\delta}\frac{|\psi_{1/2}|^2}{|x|^2} = \infty$. ∎

(3.3.5) Remark. *If $\phi(x) = \left(\log 1/|x|\right)^{-2}$ in $B_\delta \subset \Omega$ for some $\delta > 0$, then for any $\lambda < 3/4$, the inequality (3.20) is still valid and hence a similar non-existence will follow.*

§3.4 Nonlinear Weighted Eigenvalue Problems

In this section we first study the following semilinear problem:

(3.4.1) Theorem. *The problem*

$$\left\{\begin{array}{rcll} -\left(\Delta + \frac{\mu}{|x|^2}\right) u &=& u^{q-1} f(x) & \text{in } \Omega \\ u &>& 0 & \text{in } \Omega \\ u &=& 0 & \text{on } \partial\Omega\,, \end{array}\right. \tag{3.21}$$

where $0 \le \mu < \lambda^*$, $f \in \Re_{2,\beta} := \left\{ f \in \Im_2 \;\middle|\; 0 < \lim_{|x|\to 0} |x|^\beta f(x) < \infty \right\}$ *for some* $0 \le \beta < 2$ *and* $2 < q < 2^*_\beta$, *admits a weak solution in* $H^1_0(\Omega)$.

Proof. For $0 \le \mu < \lambda^*$, we define the energy functional, $J_\mu : H^1_0(\Omega) \to \mathbb{R}$ by

$$J_\mu(u) = \frac{1}{2}\int_\Omega (|\nabla u|^2 - \mu \frac{|u|^2}{|x|^2})\, dx - \frac{1}{q}\int_\Omega |u|^q f(x)\, dx \qquad \forall\ u \in H^1_0(\Omega)\,.$$

By Proposition (3.2.4) and Hardy's inequality it follows that $J_\mu \in C^1(H^1_0(\Omega), \mathbb{R})$ and for any $u, v \in H^1_0(\Omega)$ and we have

$$_{H^{-1}}\langle DJ_\mu(u), v\rangle_{H^1} = \int_\Omega [\nabla u.\nabla v - \mu\, \frac{u}{|x|^2} v\, - |u|^{q-2} u\, v\, f(x)]\, dx$$

where DJ_μ is the Fréchet derivative of J_μ and $_{H^{-1}}\langle .,. \rangle_{H^1}$ denotes the duality between $H^{-1}(\Omega)$ and $H^1_0(\Omega)$. To get a nontrivial solution of the problem (3.21), we look for critical points of J_μ. We first show that J_μ satisfies the geometric conditions of the mountain pass lemma [4] and Palais-Smale $(P.S)$ condition. We will take $\|\nabla u\|_{L^2(\Omega)}$ as the norm in $H^1_0(\Omega)$ (because by Poincaré inequality, the norm on $H^1_0(\Omega)$ is equivalent to the L^2 norm of the gradient). Now we check the following geometric conditions:

(i) Clearly, $J_\mu(0) = 0$.
(ii) To prove there exists $\delta > 0$ and $\rho > 0$ such that $J_\mu(u) > \rho$ wherever $\|u\|_{H^1_0(\Omega)} = \delta$, observe that,

$$J_\mu(u) \ge \frac{1}{2}(1 - \lambda^{-1}_{n,2}\, \mu)\, \|u\|^2_{H^1_0(\Omega)} - \frac{1}{q}\int_\Omega |u|^q f(x)\, dx. \tag{3.22}$$

By Proposition (3.2.4) and inequality (3.22), we have for all $u \in H^1_0(\Omega)$

$$J_\mu(u) \ge C_1 \|u\|^2_{H^1_0(\Omega)} - C_2 \|u\|^q_{H^1_0(\Omega)} \tag{3.23}$$

where $C_1 = \frac{1}{2}(1 - \mu\lambda_{n,2}^{-1}) > 0$ and $C_2 > 0$ are constants. So for sufficiently small $\delta > 0$ there exists $\rho > 0$ such that $J_\mu(u) > \rho$ whenever $\|u\|_{H_0^1(\Omega)} = \delta$.
(iii) Since $q > 2$, from (3.23), it follows that $\exists\ v \in H_0^1(\Omega)$ with $\|v\|_{H_0^1(\Omega)} > \delta$ such that $J_\mu(v) < 0$. Hence J_μ satisfies the geometric conditions of the mountain pass theorem.
We prove that, J_μ satisfies $(P.S)_C$, for every $C > 0$. So, let $(u_m) \subset H_0^1(\Omega)$ be a $P.S$ sequence at the energy lavel C, i.e., as $m \to \infty$

$$J_\mu(u_m) \to C,\ DJ_\mu(u_m) \to 0 \text{ strongly in } H^{-1}(\Omega)\,. \tag{3.24}$$

Observe that

$$J_\mu(v) - \frac{1}{2}\,{}_{H^{-1}}\langle DJ_\mu(v), v\rangle_{H^1} = \left(\frac{1}{2} - \frac{1}{q}\right)\int_\Omega |v|^q f\,dx \qquad \forall\ \ v \in H_0^1(\Omega)\,. \tag{3.25}$$

This, together with (3.24) implies that (u_m) is bounded in $H_0^1(\Omega)$. By Proposition (3.2.4) and reflexivity, there exists a subsequence (u_{m_k}) of (u_m) such that as $k \to \infty$

$$\begin{cases} u_{m_k} \rightharpoonup u \ \text{ weakly in } H_0^1(\Omega) \\ u_{m_k} \rightharpoonup u \ \text{ weakly in } L^2(\Omega, |x|^{-2}) \\ u_{m_k} \to u \ \text{ strongly in } L^q(\Omega, f)\,. \end{cases} \tag{3.26}$$

Since $J_\mu(u_{m_k}) \to C$ as $k \to \infty$, we have

$$\begin{aligned} C &= J_\mu(u_{m_k}) + o(1) \\ &= \frac{1}{2}(\|\nabla u_{m_k}\|_2^2 - \mu\|u_{m_k}\|_{L^2(|x|^{-2})}^2) - \frac{1}{q}\|u_{m_k}\|_{L^q(f)}^q + o(1) \\ &= \frac{1}{2}\left(\|\nabla u\|_2^2 - \mu\|u\|_{L^2(|x|^{-2})}^2\right) - \frac{1}{q}\|u\|_{L^q(f)}^q \\ &\quad + \frac{1}{2}\left(\|\nabla(u_{m_k} - u)\|_2^2 - \mu\|u_{m_k} - u\|_{L^2(|x|^{-2})}^2\right) + o(1) \\ &\geq J_\mu(u) + \frac{1}{2}(1 - \mu\beta_{n,2}^{-1})\|\nabla(u_{m_k} - u)\|_2^2 + o(1) \end{aligned} \tag{3.27}$$

where where $o(1) \to 0$ as $k \to \infty$. We will now prove that $J_\mu(u) = C$. From (3.24) and (3.26), we have

$${}_{H^{-1}}\langle DJ_\mu(u), v\rangle_{H^1} = 0 \qquad \forall\ \ v \in H_0^1(\Omega)\,. \tag{3.28}$$

Therefore,

$$\begin{aligned} C &= \lim_{k\to\infty} J_\mu(u_{m_k}) \\ &= \lim_{k\to\infty} \left[J_\mu(u_{m_k}) - \frac{1}{2}\, {}_{H^{-1}}\langle DJ_\mu(u_{m_k}),\, u_{m_k}\rangle_{H^1} \right] \qquad \text{by} \quad (3.24) \\ &= \left(\frac{1}{2} - \frac{1}{q}\right) \int_\Omega |u|^q f\, dx \qquad \text{by} \quad (3.25) \quad \text{and} \quad (3.26) \\ &= J_\mu(u) - \frac{1}{2}\, {}_{H^{-1}}\langle DJ_\mu(u), u\rangle_{H^1} \qquad \text{by (3.25)} \\ &= J_\mu(u) \qquad \text{by (3.28).} \end{aligned}$$

Therefore from (3.27), we have $\|\nabla(u_{m_k} - u)\|^2_{L^2(\Omega)} \to 0$ as $k \to \infty$ and hence J_μ satisfies $(P.S)_C$. Now define,

$$\Gamma = \left\{ \gamma \in C([0,1], H^1_0(\Omega)) : \gamma(0) = 0, \gamma(1) = v \right\}.$$

Then, $d = \inf_{\gamma\in\Gamma} \max_{0\le t\le 1} J_\mu(\gamma(t)) \ge \rho > 0$. By the mountain pass theorem of Ambrosetti-Rabinowitz [4], d is a critical value i.e. there exists $u \in H^1_0(\Omega)$ such that $J'_\mu(u) = 0$ and $J_\mu(u) = d > 0$. Since $J_\mu(u) = d = J_\mu(|u|)$, the problem (3.21) admits a solution. ∎

(3.4.2) Remark. (i) *If Ω is a bounded starshaped domain with respect to 0 and $q \ge 2^*_\beta$, $f(x) = |x|^{-\beta}$ in Ω, $0 \le \beta < 2$, then by using a Pohozaev type identity [39], one can easily show that the problem (3.21) does not admit any solution. Thus 2^*_β is critical for this problem.*

(ii) *It is therefore natural to ask that, if one relaxes the condition on the local behaviour of $f(x)$, can there exist a solution to the problem (3.21). The counterexample below shows that the relaxation of the condition may not be possible. Consider the domain Ω to be the ball $B(0,r)$, $0 < r << 1$, $q > 2$ and let $f(x) = |x|^{-2}/|\log|x||$. Then, by using a Pohozaev type identity, one can easily show that the problem (3.21) does not admit any solution.*

Now we consider the following weighted eigenvalue problem with a singular weight,

$$\begin{cases} -\left(\operatorname{div}(|\nabla u|^{p-2}\nabla u) + \dfrac{\mu}{|x|^p}|u|^{p-2}u\right) = \lambda|u|^{p-2}u\ f & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega\,, \end{cases} \tag{3.29}$$

where $f \in \Im_p$, $\Im_p := \left\{ f : \Omega \to \mathbb{R}^+ \;\middle|\; \lim_{|x|\to 0} |x|^p f(x) = 0 \,\text{with}\, f \in L^\infty_{\text{loc}}(\Omega \setminus \{0\}) \right\}$, $1 < p < n$, $0 \le \mu < \lambda_{n,p}$, and $\lambda \in \mathbb{R}$. We look for a weak solution $u \in W^{1,p}_0(\Omega)$ of this problem and study the asymptotic behaviour of the first eigenvalues for different singular weights as μ increases to $\lambda_{n,p}$, after which the operator L_μ is no more bounded from below.

(3.4.3) Theorem. *The above problem, admits a positive weak solution $u \in W_0^{1,p}(\Omega)$ for all $1 < p < n$, corresponding to the first eigenvalue $\lambda = \lambda_\mu^1(f) > 0$. Moreover, as μ increases to $\lambda_{n,p}$; $\lambda_\mu^1(f) \to \lambda(f) \geq 0$, for all $f \in \Im_p$ and the limit $\lambda(f) > 0$ if* $f \in \Re_p := \left\{ f : \Omega \to \mathbb{R}^+ \;\middle|\; f \in L^\infty_{\text{loc}}(\Omega \setminus \{0\}) \text{ with } \limsup_{|x|\to 0} |x|^p f(x) \left(\log \frac{1}{|x|}\right)^2 < \infty \right\}$. *If $f \notin \Re_p$ and if $|x|^p f(x) \left(\log \frac{1}{|x|}\right)^2$ tends to ∞ as $|x| \to 0$, then the limit $\lambda(f) = 0$.*

For the case $p = 2$, problem (3.29), has been studied in Theorem (3.3.1).

In defining the eigenvalues for the operator $(\text{div}(|\nabla u|^{p-2}\nabla u) + \frac{\mu}{|x|^p}|u|^{p-2}u)$, in a given domain $\Omega \subseteq \mathbb{R}^n$, we interpret the first equation in (3.29) in the following weak sense:

$$\int_\Omega |\nabla u|^{p-2}\langle \nabla u, \nabla \phi \rangle dx = \lambda \int_\Omega |u|^{p-2} u\phi \, dx \;, \tag{3.30}$$

whenever $\phi \in W_0^{1,p}(\Omega)$.

In order to prove the theorem we need the following two lemmas, the first one is due to Boccardo and Murat [8] and the second one is a standard result from measure theory (see for example, [41], Chapter 1, section 4).

(3.4.4) Lemma. *If $(u_m)_{m\in\mathbb{N}} \subset W_0^{1,p}(\Omega)$ be such that as $k \to \infty$, $u_m \rightharpoonup u$ weakly in $W_0^{1,p}(\Omega)$ and satisfy*

$$-\Big(\text{div}(|\nabla u_m|^{p-2}\nabla u_m)\Big) = f_m + g_m$$

in $D'(\Omega)$, where $f_m \to 0$ strongly in $W^{-1,p'}(\Omega)$ and g_m is bounded in $M(\Omega)$, the space of Radon measures, i.e

$$| < g_m, \phi > | \leq C_K \, \|\phi\|_\infty$$

for all $\phi \in D(\Omega)$ with supp(ϕ) $\subset K$, then there exists a subsequence say, u_{m_k} such that

$$u_{m_k} \to u \quad \text{strongly in } W_0^{1,q}(\Omega) \quad \forall \;\; q < p.$$

(3.4.5) Lemma. *Let $(g_m)_{m\in\mathbb{N}} \subset L^p(\Omega)$, $1 \leq p < \infty$, be such that, as $m \to \infty$, $g_m \rightharpoonup g$ weakly in $L^p(\Omega)$, and $g_m(x) \to g(x)$ a.e. in Ω, then*

$$\lim_{m\to\infty} \Big[\|g_m\|_p^p - \|g_m - g\|_p^p \Big] = \|g\|_p^p$$

Proof of Theorem (3.4.3) Let us consider the functional J_μ on $W_0^{1,p}(\Omega)$, defined by

$$J_\mu(u) := \int_\Omega |\nabla u|^p dx - \mu \int_\Omega \frac{|u(x)|^p}{|x|^p} dx$$

which is continuous, Gateaux differentiable and coercive on $W_0^{1,p}(\Omega)$, thanks to Hardy-Sobolev inequality. We minimize this functional J_μ over the manifold,

$$M := \left\{ u \in W_0^{1,p}(\Omega) \;\Big|\; \int_\Omega |u|^p f dx = 1 \right\}$$

and let λ_μ^1 be the infimum. It is clear that $\lambda_\mu^1 > 0$. Since J_μ is coercive, because of the fact that $J_\mu(u) \geq \left(1 - \frac{\mu}{\lambda_{n,p}}\right) \|\nabla u\|_p^p$, by the *Variational principle of Ekeland*, see [21], we can choose a sequence $(u_m)_{m\in I\!N} \subset M$ such that

$$J_\mu(u_m) \to \inf_M J_\mu := \lambda_\mu^1 \quad \text{and} \quad DJ_\mu(u_m) \to 0 \text{ strongly in } W^{-1,p'}(\Omega) \text{ as } m \to \infty$$

where $p' := \frac{p}{p-1}$. The coercivity of J_μ implies that $(u_m)_{m\in I\!N}$, is a bounded sequence and hence we have for a subsequence, as $k \to \infty$

$$\begin{cases} u_{m_k} \rightharpoonup u \;\; \text{weakly in } W_0^{1,p}(\Omega) \\ u_{m_k} \rightharpoonup u \;\; \text{weakly in } L^p(\Omega, |x|^{-p}) \\ u_{m_k} \to u \;\; \text{strongly in } L^p(\Omega)\,. \end{cases} \tag{3.31}$$

Since by proposition (3.2.1), $W_0^{1,p}(\Omega)$ is compactly embedded in $L^p(\Omega, f)$, it follows that M is weakly closed and hence $u \in M$. Further u_m satisfies

$$-\left(\operatorname{div}(|\nabla u_m|^{p-2}\nabla u_m) + \frac{\mu}{|x|^p}|u_m|^{p-2}u_m\right) = \lambda_m(|u_m|^{p-2}u_m\, f\,) + f_m$$

in the sense of distributions (in $D'(\Omega)$), where $f_m \to 0$ strongly in $W^{-1,p'}(\Omega)$ and $\lambda_m \to \lambda$ as $m \to \infty$. Calling $g_m = \frac{\mu}{|x|^p}|u_m|^{p-2}u_m + \lambda_m(|u_m|^{p-2}u_m\, f\,)$, one can check that g_m is bounded in $M(\Omega)$. Then we use lemma (3.4.4) to conclude

$$\nabla u_{m_k}(x) \to \nabla u(x) \quad \text{a.e.} \quad \text{in } \Omega\,. \tag{3.32}$$

Now by Lemma (3.4.5) and (3.31)-(3.32), we have for large k

$$\begin{cases} \|\nabla u_{m_k}\|_p^p &= \|\nabla(u_{m_k} - u)\|_p^p + \|\nabla u\|_p^p + o(1) \\ \|u_{m_k}\|_{L^p(\Omega,|x|^{-p})}^p &= \|\nabla(u_{m_k} - u)\|_{L^p(\Omega,|x|^{-p})}^p + \|\nabla u\|_{L^p(\Omega,|x|^{-p})}^p + o(1) \end{cases}$$

where, $o(1) \to 0$ as $k \to \infty$. Now, note that for k sufficiently large (as $u \in M$)

$$\begin{aligned}
\lambda_\mu^1 &= \|\nabla u_{m_k}\|_p^p + \|u_{m_k}\|_{L^p(\Omega,|x|^{-p})}^p + o(1) \\
&= \|\nabla(u_{m_k} - u)\|_p^p - \mu\|u_{m_k} - u\|_{L^p(\Omega,|x|^{-p})}^p + \|\nabla u\|_p^p - \mu\|u\|_{L^p(\Omega,|x|^{-p})}^p + o(1) \\
&\geq \left(1 - \frac{\mu}{\lambda_{n,p}}\right)\|\nabla(u_{m_k} - u)\|_p^p + \|\nabla u\|_p^p - \mu\|u\|_{L^p(\Omega,|x|^{-p})}^p + o(1) \\
&\geq \left(1 - \frac{\mu}{\lambda_{n,p}}\right)\|\nabla(u_{m_k} - u)\|_p^p + \lambda_\mu^1 + o(1)\,.
\end{aligned}$$

Since $\mu < \lambda_{n,p}$, we conclude that

$$\|\nabla(u_{m_k} - u)\|_{L^p(\Omega)}^p \to 0 \text{ as } k \to \infty \text{ and } \|(u_{m_k} - u)\|_{L^p(\Omega,|x|^{-p})}^p \to 0 \text{ as } k \to \infty\,.$$

Hence we have $J_\mu(u) = \lambda_\mu^1$ and $\lambda = \lambda_\mu^1$. Since $J_\mu(|u|) = J_\mu(u)$, we can take $u > 0$ in Ω. Then using lemma (3.4.4), u is a distribution solution of (3.30) and since $u \in W_0^{1,p}(\Omega)$, it is a weak solution to the eigenvalue problem (3.29), corresponding to $\lambda = \lambda_\mu^1$.
Moreover, if $f \in \Re_p$, by corollary (2.1.4), in Chapter 2, we have

$$\lambda_\mu^1(f) \to \lambda(f) = \inf_{\substack{u \in W_0^{1,p}(\Omega) \\ u \neq 0}} \frac{\int_\Omega \left(|\nabla u|^p - \lambda_{n,p}\frac{|u(x)|^p}{|x|^p}\right)dx}{\int_\Omega |u(x)|^p f(x)dx} > 0 \text{ as } \mu \to \lambda_{n,p}.$$

If $f \notin \Re_p$, then again by corollary (2.1.4), inequality of the type (2.5) cannot hold and hence $\lambda(f) = 0$. ∎

Chapter 4

A Brezis-Nirenberg Type Problem

§4.1 Introduction

Let Ω be a bounded open subset of $\mathbb{R}^n$, $n \geq 3$ with $0 \in \Omega$ and $0 \leq \alpha < 2$. For the case $0 < \alpha < 2$, positive solutions in $H_0^1(\Omega)$ exist for the problem

$$-\Delta u = \frac{u}{|x|^\alpha}|u|^{2^*_\alpha - 2} \text{ in } \Omega$$

where $2^*_\alpha := \dfrac{2(n-\alpha)}{(n-2)}$, only when Ω is invariant under scaling centered at $x = 0$, for example $\Omega = \mathbb{R}^n$ or $\mathbb{R}^n_+$; (see [31], Theorem 2). Using a similar idea of Brezis-Nirenberg [11], it can be easily shown that the problem

$$\left\{ \begin{array}{rcll} -\Delta u & = & \frac{u^{2^*_\alpha - 1}}{|x|^\alpha} + \lambda u & \text{in } \Omega \\ u & > & 0 & \text{in } \Omega \\ u & = & 0 & \text{on } \partial\Omega \end{array} \right. ,$$

in bounded domain Ω, admits a solution in $H_0^1(\Omega)$, for all $\lambda > 0$, when $n \geq 4$ and for all $\lambda \in (0, \lambda_0)$, when $n = 3$, where $\lambda_0 \in (0, \lambda_1)$, λ_1 is the first eigenvalue of the operator $-\Delta$ in $H_0^1(\Omega)$. We study a similar problem with a superlinear lower order term $g(u)$ and a singular coefficient, where an interesting balance between the singularity and nonlinearity is essential for the existence of a solution. In this Chapter we prove the following theorem.

(4.1.1) Theorem. *The problem*

$$\left\{ \begin{array}{rcll} -\Delta u & = & \frac{u^{2^*_\alpha - 1}}{|x|^\alpha} + \lambda u^{q-1} f(x) & \text{in } \Omega \\ u & > & 0 & \text{in } \Omega \\ u & = & 0 & \text{on } \partial\Omega \end{array} \right. , \tag{4.1}$$

where $0 \leq \alpha < 2$, $f \in \Re_{2,\beta}$; for some $0 \leq \beta < 2$, $2 < q < 2^*_\beta$ and $\lambda \in \mathbb{R}^+$, admits a weak solution : (a) For all $n \geq 4$ and $\lambda > 0$.
(b) When $n = 3$, in the following different cases:

(i) For all $1 \leq \beta < 2$ *and* $\lambda > 0$.
(ii) For all $0 \leq \beta < 1$ *with* $2(2-\beta) < q < 2^*_\beta$ *and for all* $\lambda > 0$.
(iii) For all $0 \leq \beta < 1$ *with* $2 < q \leq 2(2-\beta)$ *and for sufficiently large* $\lambda > 0$.

Before going to the next section, we recall the propositions (3.2.4) and (3.2.5) from the previous Chapter.

(4.1.2) Proposition. *Let* Ω *be a bounded domain. Let* $f \in \Re_{2,\beta}$ *where*

$$\Re_{2,\beta} := \left\{ f \in \Im_p \;\middle|\; 0 < \lim_{|x|\to 0} |x|^\beta f(x) < \infty \right\}$$

for some $0 \leq \beta < 2$ *. Let* $2^*_\beta := \dfrac{2(n-\beta)}{(n-2)}$*, then the embedding* $H^1_0(\Omega) \hookrightarrow L^q(\Omega, f)$ *is*

$$\text{(i)} \quad \text{continuous} \quad \forall \quad 2 \leq q \leq 2^*_\beta\,,$$

$$\text{(ii)} \quad \text{compact} \quad \forall \quad 2 \leq q < 2^*_\beta\,.$$

(4.1.3) Proposition. For $0 < \alpha < 2$, then $D^{1,2}(\mathbb{R}^n)$ is continuously embedded in $L^{2^*_\alpha}(\mathbb{R}^n, |x|^{-\alpha})$.

§4.2 Proof of Theorem (4.1.1)

For $\lambda \geq 0$, define, $J_\lambda : H^1_0(\Omega) \to \mathbb{R}$ by

$$J_\lambda(u) := \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \frac{1}{2^*_\alpha}\int_\Omega |u|^{2^*_\alpha}|x|^{-\alpha}dx - \frac{\lambda}{q}\int_\Omega |u|^q f\, dx \;\; \forall\, u \in H^1_0(\Omega).$$

By Proposition (4.1.2), $J_\lambda \in C^1(H^1_0, \mathbb{R})$, and for any $u, v \in H^1_0(\Omega)$ we have

$${}_{H^{-1}}\langle DJ_\lambda(u), v\rangle_{H^1} = \int_\Omega \left(\nabla u.\nabla v - |u|^{2^*_\alpha - 2}uv|x|^{-\alpha} - \lambda|u|^{q-2}uvf\right) dx$$

where where DJ_μ is the Fréchet derivative of J_μ and ${}_{H^{-1}}\langle .,. \rangle_{H^1}$ denotes the duality between $H^{-1}(\Omega)$ and $H^1_0(\Omega)$. Any nontrivial critical point of J_λ will give us a solution to the problem (4.1). For $\alpha = 0$ and $\beta = 0$, the existence of solution for the problem (4.1) was shown by Brezis-Nirenberg,
[11]. They showed that J_λ satisfies $P.S$ condition below the energy level $\frac{1}{n}S^{n/2}$, where S is the best Sobolev constant. In our case also we will get a certain energy level to conclude J_λ satisfies $P.S$ condition below that level.

For $0 \leq \alpha < 2$ the embedding $D^{1,2}(\mathbb{R}^n) \hookrightarrow L^{2^*_\alpha}(\mathbb{R}^n, |x|^{-\alpha})$ is continuous, (Proposition (4.1.3)). Define

$$S_\alpha := \inf_{\substack{u \in D^{1,2}(\mathbb{R}^n) \\ u \neq 0}} \frac{\|\nabla u\|^2_{L^2(\mathbb{R}^n)}}{\|u\|^2_{L^{2^*_\alpha}(\mathbb{R}^n, |x|^{-\alpha})}} \,. \tag{4.2}$$

Since $f \in \Re_{2,\beta}$, there exist $r > 0$ and $K_1(f),\ K_2(f) > 0$ such that $K_1|x|^{-\beta} \leq f(x) \leq K_2|x|^{-\beta}$ on $B(0,r)$. For any $\epsilon > 0$, the family u_ϵ on $\mathbb{R}^n$ defined by,

$$u_\epsilon(x) := \frac{C_\epsilon}{(\epsilon^{2-\alpha} + |x|^{2-\alpha})^{(n-2)/(2-\alpha)}}, \quad C_\epsilon = [\epsilon^{2-\alpha}(n-2)(n-\alpha)]^{(n-2)/2(2-\alpha)} \tag{4.3}$$

satisfies

$$-\Delta u_\epsilon = u_\epsilon^{2^*_\alpha - 1}|x|^{-\alpha} \text{ in } \mathbb{R}^n \,,$$

and is a minimizer for (4.2). Hence we have

$$\|\nabla u_\epsilon\|^2_{L^2(\mathbb{R}^n)} = \|u_\epsilon\|^{2^*_\alpha}_{L^{2^*_\alpha}(\mathbb{R}^n, |x|^{-\alpha})} = S_\alpha^{(n-\alpha)/(2-\alpha)} \,. \tag{4.4}$$

Choose $\phi \in C_0^\infty(\Omega)$ such that $\phi \equiv 1$ on B_r. For $\epsilon > 0$, define the family $\tilde{u}_\epsilon$ on Ω by

$$\tilde{u}_\epsilon(x) := \phi(x) u_\epsilon(x) \,.$$

By (4.3) and (4.4), it can be easily shown that

$$\|\nabla \tilde{u}_\epsilon\|^2_{L^2(\Omega)} = S_\alpha^{(n-\alpha)/(2-\alpha)} + O(\epsilon^{n-2}) \,, \tag{4.5}$$

$$\|\tilde{u}_\epsilon\|^{2^*_\alpha}_{L^{2^*_\alpha}(\Omega, |x|^{-\alpha})} = S_\alpha^{(n-\alpha)/(2-\alpha)} + O(\epsilon^{n-\alpha}) \,. \tag{4.6}$$

Define $v_\epsilon : \Omega \to \mathbb{R}$, by

$$v_\epsilon(x) = \frac{\tilde{u}_\epsilon(x)}{\|\tilde{u}_\epsilon\|_{L^{2^*_\alpha}(\Omega, |x|^{-\alpha})}} \tag{4.7}$$

so from (4.5) and (4.6) we have,

$$\|v_\epsilon\|_{L^{2^*_\alpha}(\Omega, |x|^{-\alpha})} = 1 \,, \tag{4.8}$$

and

$$\begin{aligned} \|\nabla v_\epsilon\|^2_{L^2(\Omega)} &= \left(S_\alpha^{(n-\alpha)/(2-\alpha)} + O(\epsilon^{n-\alpha})\right)^{-2/2^*_\alpha} \left(S_\alpha^{(n-\alpha)/(2-\alpha)} + O(\epsilon^{n-2})\right) \\ &= S_\alpha \left(1 + O(\epsilon^{n-\alpha})\right)^{-2/2^*_\alpha} \left(1 + O(\epsilon^{n-2})\right) \\ &= S_\alpha + O(\epsilon^{n-2}) \,. \end{aligned} \tag{4.9}$$

We can show that for $\lambda \geq 0$, J_λ satisfies the geometric conditions of mountain pass lemma as in the proof of Theorem (3.4.1). We will use the Theorem of Brezis-Nirenberg, (see Theorem 2.2 in [11]) to get a nontrivial critical point for J_λ. We need the following two lemmas which are similar to Theorem 2.1 and Lemma 2.1 in [11] or see lemma 2.3 in [40]. We indicate the proof here as the calculations are quite different and are important to understand the new phenomenon here, namely the balance between the singularity and the nonlinearity.

(4.2.1) Lemma. *Let $(u_m) \subseteq H_0^1(\Omega)$ be a sequence such that*

$$J_\lambda(u_m) \to \beta_\lambda < \frac{(2-\alpha)}{2(n-\alpha)} S_\alpha^{(n-\alpha)/(2-\alpha)} \text{ and } DJ_\lambda(u_m) \to 0 \text{ strongly} \quad \text{in } H^{-1}(\Omega)\,. \tag{4.10}$$

Then (u_m) is relatively compact in $H_0^1(\Omega)$ i.e. J_λ satisfies $(P.S)_{\beta_\lambda}$

Proof. Let us prove first (u_m) is bounded. From (4.10) we have for all $m \geq k$,

$$\begin{cases} J_\lambda(u_m) = \beta_\lambda + o(1) \\ \\ {}_{H^{-1}}\langle DJ_\lambda(u_m),\ u_m\rangle_{H^1} \leq o(1)\|u_m\|_{H_0^1(\Omega)}\ . \end{cases} \tag{4.11}$$

Now we have

$$2J_\lambda(u_m) - {}_{H^{-1}}\langle DJ_\lambda(u_m), u_m\rangle_{H^1} = \left(1 - \frac{2}{2_\alpha^*}\right) \int\limits_\Omega |u|^{2_\alpha^*} |x|^{-\alpha} dx + \lambda \left(1 - \frac{2}{q}\right) \int\limits_\Omega |u_m|^q f\, dx\ .$$

Since $\lambda \geq 0,\ 2_\alpha^* > 2,\ q > 2$, we have

$$\begin{cases} 2J_\lambda(u_m) - {}_{H^{-1}}\langle DJ_\lambda(u_m), u_m\rangle_{H^1} \geq \left(1 - \frac{2}{2_\alpha^*}\right) \int\limits_\Omega |u|^{2_\alpha^*} |x|^{-\alpha} dx \\ \\ 2J_\lambda(u_m) - {}_{H^{-1}}\langle DJ_\lambda(u_m), u_m\rangle_{H^1} \geq \lambda \left(1 - \frac{2}{q}\right) \int\limits_\Omega |u_m|^q f\, dx\ . \end{cases} \tag{4.12}$$

By (4.11) and (4.12), we have for all $m \geq 1$

$$\begin{cases} \int\limits_\Omega |u|^{2_\alpha^*} |x|^{-\alpha} \leq C_1 + o(1)\|u_m\|_{H_0^1(\Omega)} \\ \\ \int\limits_\Omega |u_m|^q f\, dx \leq C_2 + o(1)\|u_m\|_{H_0^1(\Omega)} \end{cases} \tag{4.13}$$

where C_1, C_2 are positive constants independent of m. Since

$$\|\nabla u_m\|_2^2 = 2J_\lambda(u_m) + \frac{2}{2_\alpha^*} \int\limits_\Omega \frac{|u_m|^{2_\alpha^*}}{|x|^\alpha} dx + \frac{2\lambda}{q} \int\limits_\Omega |u_m|^q f\, dx$$

from (4.10) and (4.13), we conclude

$$\|u_m\|^2_{H^1_0(\Omega)} \leq C + o(1)\|u_m\|^2_{H^1_0(\Omega)}$$

and hence (u_m) is bounded in $H^1_0(\Omega)$. By proposition (4.1.2) and the reflexivity of the spaces there exists a subsequence (u_{m_k}) of (u_m) such that as $k \to \infty$

$$\begin{cases} u_{m_k} \rightharpoonup u \ \text{ weakly in } H^1_0(\Omega) \\ u_{m_k} \to u \ \text{ strongly in } L^p(\Omega, |x|^{-\alpha}),\ \forall\ 2 \leq p < 2^*_\alpha \\ u_{m_k} \rightharpoonup u \ \text{ weakly in } L^{2^*_\alpha}(\Omega, |x|^{-\alpha}) \\ u_{m_k} \to u \ \text{ strongly in } L^q(\Omega, f),\ \ \forall\, 2 < q < 2^*_\beta \\ u_{m_k} \to u \ \text{ a.e. } [|x|^{-\alpha}dx] \ \text{ in } \ \Omega\ . \end{cases} \tag{4.14}$$

We write, $d\mu(x) = \frac{dx}{|x|^\alpha}$. Since $DJ_\lambda(u_m) \to 0$ strongly in $H^{-1}(\Omega)$, by (4.14) we have

$${}_{H^{-1}}\langle DJ_\lambda(u),\ v\rangle_{H^1} = 0\ \forall\, v \in H^1_0(\Omega)\ .$$

In particular ${}_{H^{-1}}\langle DJ_\lambda(u),\ u\rangle_{H^1} = 0$, and hence we have from (4.12)

$$J_\lambda(u) = \left(\frac{1}{2} - \frac{1}{2^*_\alpha}\right)\int\limits_\Omega |u|^{2^*_\alpha} d\mu(x) + \lambda\left(\frac{1}{2} - \frac{1}{q}\right)\int\limits_\Omega |u|^q\, f\, dx \geq 0\ . \tag{4.15}$$

Again by (4.14) we have

$$\int\limits_\Omega |\nabla u_{m_k}|^2 dx = \int\limits_\Omega |\nabla(u_{m_k} - u)|^2 dx + \int\limits_\Omega |\nabla u|^2 dx + o(1)\ , \tag{4.16}$$

$$\int\limits_\Omega |u_{m_k}|^{2^*_\alpha} d\mu(x) = \int\limits_\Omega |u_{m_k} - u|^{2^*_\alpha} d\mu(x) + \int\limits_\Omega |u|^{2^*_\alpha} d\mu(x) + o(1)\ , \tag{4.17}$$

and

$$\int\limits_\Omega \left(u_{m_k}|u_{m_k}|^{2^*_\alpha - 2} - u|u|^{2^*_\alpha - 2}\right)(u_{m_k} - u) d\mu(x) = \int\limits_\Omega |u_{m_k} - u|^{2^*_\alpha} d\mu(x) + o(1)\ . \tag{4.18}$$

By (4.16) - (4.17) we have

$$J_\lambda(u_{m_k}) = J_\lambda(u) + J_0(u_{m_k} - u) + o(1) \tag{4.19}$$

and by (4.18)

$$\begin{aligned} o(1) &= {}_{H^{-1}}\langle DJ_\lambda(u_{m_k}), u_{m_k} - u\rangle_{H^1} = {}_{H^{-1}}\langle DJ_\lambda(u_{m_k}) - DJ_\lambda(u), u_{m_k} - u\rangle_{H^1} \\ &= \int\limits_\Omega |\nabla(u_{m_k} - u)|^2 dx - \int\limits_\Omega |u_{m_k} - u|^{2^*_\alpha} d\mu(x) + o(1)\ . \end{aligned} \tag{4.20}$$

Hence it follows from (4.20) that

$$J_0(u_{m_k} - u) = \frac{(2-\alpha)}{2(n-\alpha)} \int\limits_{\Omega} |\nabla(u_{m_k} - u)|^2 dx + o(1) \tag{4.21}$$

and thus from (4.10), (4.15) and (4.19)

$$J_0(u_{m_k} - u) \leq J_\lambda(u_{m_k}) + o(1) \leq \beta_\lambda < \frac{(2-\alpha)}{2(n-\alpha)} S_\alpha^{\frac{n-\alpha}{2-\alpha}} \quad \forall\ k \geq k_0\ .$$

Combining (4.21) with this, we have $\forall\ k \geq k_0$

$$\|\nabla(u_{m_k} - u)\|_2^2 < S_\alpha^{\frac{n-\alpha}{2-\alpha}} \tag{4.22}$$

and hence

$$\left(1 - S_\alpha^{-2_\alpha^*/2} \|u_{m_k} - u\|_{H_0^1}^{2_\alpha^* - 2}\right) > 0\,. \tag{4.23}$$

Now (4.2) implies

$$\|w\|^2_{L^{2_\alpha^*}(\Omega, |x|^{-\alpha})} \leq S_\alpha^{-1} \|\nabla w\|_2^2 \qquad \forall\ \ w \in H_0^1(\Omega)\,.$$

Thus

$$\begin{aligned} \|u_{m_k} - u\|_{H_0^1}^2 \left(1 - S_\alpha^{-2_\alpha^*/2} \|u_{m_k} - u\|_{H_0^1}^{2_\alpha^* - 2}\right) &\leq \int\limits_{\Omega} |\nabla(u_{m_k} - u)|^2 dx - \int\limits_{\Omega} |u_{m_k} - u|^{2_\alpha^*} d\mu(x) \\ &= o(1)\ . \end{aligned} \tag{4.24}$$

From (4.23) and (4.24), it follows that $u_{m_k} \to u$ strongly in $H_0^1(\Omega)$. ∎

(4.2.2) Lemma. *There exists some $v \in H_0^1(\Omega), v \neq 0$ such that*

$$\sup_{t \geq 0} J_\lambda(tv) < \frac{(2-\alpha)}{2(n-\alpha)} S_\alpha^{\frac{n-\alpha}{2-\alpha}}$$

holds:
(a) For all $n \geq 4$ and $\lambda > 0$.
(b) When $n = 3$, in the following three cases:
(i) For all $1 \leq \beta < 2$, $2 < q < 2_\beta^*$ and $\lambda > 0$.
(ii) For all $0 \leq \beta < 1$ with $2(2-\beta) < q < 2_\beta^*$ and for all $\lambda > 0$.
(iii) For all $0 \leq \beta < 1$ with $2 < q \leq 2(2-\beta)$ and for sufficiently large $\lambda > 0$.

Proof. Take $\{v_\epsilon\}$, for $\epsilon > 0$ as defined in (4.7). Call $X_\epsilon := \|\nabla v_\epsilon\|^2_{L^2(\Omega)}$.

$$J_\lambda(tv_\epsilon) = \frac{t^2}{2}X_\epsilon - \frac{t^{2^*_\alpha}}{2^*_\alpha} - \frac{\lambda t^q}{q}\int\limits_\Omega |v_\epsilon|^q f\, dx\,. \tag{4.25}$$

Since $q,\ 2^*_\alpha > 2$, from (4.25) it follows that $J_\lambda(tv_\epsilon) \to -\infty$ on $t \to +\infty$. Hence $\sup_{t\geq 0} J_\lambda(tv_\epsilon)$ achieves at some $t_\epsilon > 0$ (If $t_\epsilon = 0$, then we are done). Therefore

$$\frac{d}{dt}J_\lambda(tv_\epsilon)\Big|_{t=t_\epsilon} = X_\epsilon - t_\epsilon^{2^*_\alpha - 2} - \lambda t_\epsilon^{q-2}\int\limits_\Omega |v_\epsilon|^q f(x)dx = 0 \tag{4.26}$$

and hence

$$t_\epsilon \leq X_\epsilon^{\frac{1}{2^*_\alpha - 2}}.$$

Let $Y_\epsilon = \sup_{t\geq 0} J_\lambda(tv_\epsilon) = J_\lambda(t_\epsilon v_\epsilon)$. Now observe that the function $g(t) := \dfrac{t^2}{2}X_\epsilon - \dfrac{t^{2^*_\alpha}}{2^*_\alpha}$ attains its maximum at $t = X_\epsilon^{\frac{1}{2^*_\alpha - 2}}$ and hence g is non-decreasing on $[0, X_\epsilon^{\frac{1}{2^*_\alpha - 2}}]$. Therefore from (4.25), we have

$$Y_\epsilon \leq \frac{(2-\alpha)}{2(n-\alpha)}X_\epsilon^{(n-\alpha)/(2-\alpha)} - \frac{\lambda t_\epsilon^q}{q}\int\limits_\Omega |v_\epsilon|^q f\, dx\,.$$

Hence by (4.9), we have

$$Y_\epsilon \leq \frac{(2-\alpha)}{2(n-\alpha)}S_\alpha^{(n-\alpha)/(2-\alpha)} + O(\epsilon^{n-2}) - \frac{\lambda t_\epsilon^q}{q}\int\limits_\Omega |v_\epsilon|^q f\, dx\,. \tag{4.27}$$

The aim is to show that the third term in (4.27) is small and dominates the second term. We will first show that

$$t_\epsilon \to S_\alpha^{1/(2^*_\alpha - 2)} = S_\alpha^{(n-2)/2(2-\alpha)} \quad as \quad \epsilon \to 0\,. \tag{4.28}$$

Since $X_\epsilon \to S_\alpha$ as $\epsilon \to 0$, it is enough to prove that

$$\int\limits_\Omega |v_\epsilon|^q f(x)dx \to 0 \ \text{ as } \ \epsilon \to 0\,. \tag{4.29}$$

From (4.3), (4.7) and (4.8), we have

$$\int\limits_\Omega |v_\epsilon|^q f(x)dx \leq C\epsilon^{q(n-2)/2}\int\limits_0^r \frac{s^{n-1-\beta}}{[\epsilon^{2-\alpha} + s^{2-\alpha}]^{q(n-2)/(2-\alpha)}}ds + O(\epsilon^{q(n-2)/2}) \tag{4.30}$$

where $C > 0$ is a constant independent of ϵ. Now, for $\epsilon < r$ we have

$$\begin{aligned}\int_0^r \frac{s^{n-1-\beta}}{\left[\epsilon^{2-\alpha}+s^{2-\alpha}\right]^{q(n-2)/(2-\alpha)}}ds &= \int_0^\epsilon \frac{s^{n-1-\beta}}{\left[\epsilon^{2-\alpha}+s^{2-\alpha}\right]^{q(n-2)/(2-\alpha)}}ds + \int_\epsilon^r \frac{s^{n-1-\beta}}{\left[\epsilon^{2-\alpha}+s^{2-\alpha}\right]^{q(n-2)/(2-\alpha)}}ds \\ &\leq \epsilon^{n-1-\beta}\int_0^\epsilon \frac{ds}{\epsilon^{q(n-2)}} + \int_\epsilon^r s^{n-1-\beta-q(n-2)ds} \\ &= \epsilon^{n-\beta-q(n-2)} + I_{\epsilon,n}\ , \end{aligned} \tag{4.31}$$

where

$$I_{\epsilon,n} = \int_\epsilon^r s^{n-\beta-1-q(n-2)}ds\ .$$

For all $n \geq 4$, we have $n-\beta-q(n-2) < 0$ and for $n = 3$, $n-\beta-q(n-2) = 3-\beta-q$. Therefore,

$$I_{\epsilon,n} = \begin{cases} |\log\epsilon| - |\log r|\ , & \text{for} \quad n=3,\ 0 \leq \beta < 1 \ \text{ with } \ q = 3-\beta\ , \\ C_1(\epsilon^{3-\beta-q} - C_1')\ , & \text{for} \quad n=3,\ q \neq 3-\beta\ , \\ C_2(\epsilon^{n-\beta-q(n-2)} - C_2')\ , & \text{for all} \quad n \geq 4\ . \end{cases} \tag{4.32}$$

where $C_1', C_2, C_2' > 0$ and $C_1 \geq 0$ are constants. Since $q < 2_\beta^*$, i.e., $q(n-2)/2 < (n-\beta)$, from (4.30)- (4.32), we have

$$\int_\Omega |v_\epsilon| f(x)dx \to 0 \text{ as } \epsilon \to 0\,.$$

Now to prove,

$$Y_\epsilon < \frac{(2-\alpha)}{2(n-\alpha)} S_\alpha^{(n-\alpha)/(2-\alpha)}\ , \tag{4.33}$$

for sufficiently small $\epsilon > 0$, it is enough to prove that $\lambda \int_\Omega |v_\epsilon|^q f\ dx$, goes zero slower than ϵ^{n-2} as $\epsilon \to 0$, or equivalently

$$\lambda \lim_{\epsilon\to 0} \epsilon^{2-n} \int_\Omega |v_\epsilon|^q f\ dx = +\infty\,. \tag{4.34}$$

From (4.3), (4.7) , we have

$$\begin{aligned}\int_\Omega |v_\epsilon|^q f(x) &\geq \int_{B_\epsilon} |v_\epsilon|^q f(x)dx \\ &= \int_{B_\epsilon} |v_\epsilon|^q |x|^{-\beta} dx \\ &\geq C\epsilon^{q(n-2)/2}\epsilon^{n-\beta-q(n-2)} \\ &= C\epsilon^{n-\beta-q(n-2)/2} .\end{aligned}$$

Therefore,

$$\epsilon^{2-n}\int_\Omega |v_\epsilon|^q f(x)dx \geq C\epsilon^{2-\beta-q(n-2)/2} , \tag{4.35}$$

where $C > 0$ is a constant. From (4.35), we have

Case (a): When $n \geq 4$, we have $(2-\beta) - q(n-2)/2 < 0$, because $q > 2$ and hence (4.34) holds for any $\lambda > 0$.

Case (b): When $n = 3$, $2 - \beta - q(n-2)/2 = 2 - \beta - q/2$.

(i) Since for $1 \leq \beta < 2$ we have $2 - \beta - q/2 < 0$; (4.34) holds for all $\lambda > 0$.

(ii) For $0 \leq \beta < 1$ with $2(2-\beta) < q < 2^*_\beta$, we have $2 - \beta - q/2 < 0$; so (4.34) holds for all $\lambda > 0$.

(iii) For $0 \leq \beta < 1$ with $2 < q \leq 2(2-\beta)$, we have $2 - \beta - q/2 \geq 0$. Hence we can conclude the validity of (4.34) only for large $\lambda > 0$.
Hence the lemma. ∎

Proof of Theorem (4.1.1): Choose t_0 sufficiently large such that $J_\lambda(t_0 v_\epsilon) < \rho = \inf_{\|u\|=\delta} J_\lambda(u)$. Call $v_0 = t_0 v_\epsilon$ and define,

$$\Gamma = \left\{\gamma \in C([0,1], H^1_0(\Omega)) : \gamma(0) = 0, \gamma(1) = v_0\right\} ,$$

and

$$\beta_\lambda = \inf_{\gamma\in\Gamma}\max_{t\in I} J_\lambda(\gamma(t)) \geq \rho > 0.$$

Therefore, we have

$$\beta_\lambda \leq \max_{t\geq 0} J_\lambda(tv_\epsilon) < \frac{(2-\alpha)}{2(n-\alpha)} S_\alpha^{(n-\alpha)/(2-\alpha)}. \tag{4.36}$$

By Theorem 2.2 of Brezis-Nirenberg [11] there exists a sequence $(u_m) \subseteq H^1_0(\Omega)$ such that as $m \to \infty$

$$J_\lambda(u_m) \to \beta_\lambda \text{ and } DJ_\lambda(u_m) \to 0 \text{ strongly in } H^{-1}(\Omega).$$

Since β_λ satisfies (4.36), using Lemma (4.2.1) and Lemma (4.2.2) we complete the proof. ∎

Chapter 5

On an Evolution Problem Involving Hardy-Sobolev Operator

§5.1 Introduction

Let Ω be bounded open subset in $\mathbb{R}^n$, $n \geq 3$ and $0 \in \Omega$. In 1984, Baras and Goldstein in their classical paper [7], considered the following heat problem

$$\left\{ \begin{array}{rcll} u_t - \Delta u & = & V(x)u & \text{in} \quad \Omega \times (0,\infty) \\ u(x,0) & = & u_0(x) & \text{in} \quad \Omega \\ u(x,t) & = & 0 & \text{on} \quad \partial\Omega \times (0,\infty). \end{array} \right. \tag{5.1}$$

For the critical potential $V(x) = \dfrac{\mu}{|x|^2}$, they have proved that the Cauchy-Dirichlet problem (5.1) has a global solution (*in the sense of distributions*) if $\mu \leq \lambda^* := \dfrac{(n-2)^2}{4}$ and no solution, even locally in time, if $\mu > \lambda^*$. It is well understood that, existence of solutions for the problem (5.1) is deeply connected to the Hardy-Sobolev inequality

$$\int_\Omega |\nabla u|^2 \, dx \geq \frac{(n-2)^2}{4} \int_\Omega \frac{|u|^2}{|x|^2} \, dx \tag{5.2}$$

and the associated spectral problem $-\Delta u - \dfrac{\mu}{|x|^2} u = \lambda u$. By the above inequality, it is easy to show that the problem (5.1) admits solution in $C\left([0,\infty); L^2(\Omega)\right) \cap L^2\left(0,\infty; H_0^1(\Omega)\right)$, if $\mu < \lambda^*$. But a strong change of behaviour takes place at the transition value $\mu = \lambda^*$. In this case the behaviour of the solution is intimately related to the improved form of the classical Hardy inequality (5.2). Here we recall the improved Hardy-Sobolev inequality from Chapter 2, which states that for $n \geq 3$ and for any $R \geq e \sup_\Omega |x|$, there exist positive constants C_1 and C_2 such that,

$$\int_\Omega |\nabla u|^2 dx \geq \left(\frac{n-2}{2}\right)^2 \int_\Omega \frac{|u|^2}{|x|^2} \, dx + C_1 \int_\Omega \frac{|u|^2}{|x|^2} \left(\log \frac{R}{|x|}\right)^{-2} dx + C_2 \int_\Omega |u|^2 \, dx \tag{5.3}$$

holds for all $u \in H_0^1(\Omega)$.

But when $n = 2$ and $R \geq e^{e^e} \sup_{\Omega} |x|$, then there exists a constant $C > 0$ such that, the following 2-dimensional Hardy-Sobolev inequality

$$\int_{\Omega} |\nabla u|^2 dx \geq \frac{1}{4} \int_{\Omega} \frac{|u|^2}{|x|^2} \left(\log \frac{R}{|x|} \right)^{-2} \left[1 + \left(\log \log \frac{R}{|x|} \right)^{-2} \right] dx + C \int_{\Omega} |u|^2 \, dx \ , \tag{5.4}$$

holds for all $u \in H_0^1(\Omega)$. For other types of improved Hardy inequality, see [43].

For the case $n \geq 3$ and $\mu = \lambda^*$, Vazquez and Zuazua [43] proved that the solution $u(t)$ of the problem (5.1) with $u_0 \in L^2(\Omega)$ is in the Hilbert space $H(\Omega)$, for all time $t > 0$ but not in $H_0^1(\Omega)$, even locally in time, where $H(\Omega)$ is the completion of $H_0^1(\Omega)$ with respect to the *energy norm* $\|u\|_H := \left(\int_{\Omega} \{ |\nabla u|^2 - \lambda^* \frac{u^2}{|x|^2} \} dx \right)^{1/2}$. It is interesting to notice that $H_0^1(\Omega) \subset H(\Omega) \subset \cap_{q<2} W_0^{1,q}(\Omega)$ and the inclusions are proper. So naturally a question arises: Is it possible to perturb the *Hardy-Sobolev operator* $L_{\lambda^*} := -\Delta - \frac{\lambda^*}{|x|^2}$, in such a way that the heat equation admits a non-negative solution $u(t) \in H_0^1(\Omega)$ for all $t > 0$ with $u_0 \geq 0$? Aim of this Chapter is to answer this question.

Take q and η measurable functions on Ω satisfying the following hypotheses :

(H_1) $\quad 0 \leq q \leq 1 \, ,$

(H_2) $\quad \eta \geq 0, \quad \eta \in L^{\infty}_{\mathrm{loc}}(\Omega \setminus \{0\}) \, ,$

(H_3)

$$\begin{cases} \lim\limits_{x \to 0} |x|^2 \left(\log \frac{1}{|x|} \right)^2 \eta(x) &= 0, \quad n \geq 3 \, , \\ \lim\limits_{x \to 0} |x|^2 \left(\log \frac{1}{|x|} \right)^2 \left(\log \log \frac{1}{|x|} \right)^2 \eta(x) &= 0, \quad n = 2. \end{cases}$$

The strange looking hypothesis (H_3) comes from the optimality on the existence of the improved Hardy-Sobolev inequality (5.3) and (5.4). Now, let us define the following constant,

$$C(q,\eta) := \begin{cases} \inf\limits_{H_0^1(\Omega)} \left\{ \int_{\Omega} \left[|\nabla u|^2 - \lambda^* \frac{|u|^2 q(x)}{|x|^2} \right] dx \ \Big| \ \int_{\Omega} |u|^2 \eta(x) \, dx = 1 \right\}, & \text{for } n \geq 3 \\ \inf\limits_{H_0^1(\Omega)} \left\{ \int_{\Omega} \left[|\nabla u|^2 - \frac{1}{4} \frac{|u|^2 q(x)}{\left(|x| \log R/|x| \right)^2} \right] dx \ \Big| \ \int_{\Omega} |u|^2 \eta(x) \, dx = 1 \right\} & \text{for } n = 2. \end{cases}$$

From the inequalities (5.3) and (5.4), it follows that $C(q,\eta) > 0$.

(5.1.1) Theorem. *Assume $n \geq 3$ and let q, η satisfy hypotheses (H_1) to (H_3). Then the problem*

$$\begin{cases} u_t - \Delta u - \lambda^* \dfrac{q(x)}{|x|^2}\, u &= \lambda \eta u \quad \text{in} \quad \Omega \times (0,\infty) \\ u(x,0) &= u_0(x) \quad \text{in} \quad \Omega \\ u(x,t) &= 0 \quad \text{on} \quad \partial\Omega \times (0,\infty), \end{cases} \tag{5.5}$$

where $u_0 \in L^\infty(\Omega)$, non-negative, $u_0 \not\equiv 0$; admits a unique global non-negative solution $u \in C\left([0,\infty); L^2(\Omega)\right) \cap L^2\left(0,\infty; H_0^1(\Omega)\right)$ with $u' \in L^2\left(0,\infty; H^{-1}(\Omega)\right)$, $u \not\equiv 0$; for all λ, $0 \leq \lambda \leq C(q,\eta)$, provided q satisfies

$$\liminf_{x\to 0} \left(\log \frac{1}{|x|}\right)^2 (1 - q(x)) > \frac{3}{(n-2)^2}. \tag{5.6}$$

(5.1.2) Remark. *By taking $q = 1 - \dfrac{C}{(\log R/|x|)^\alpha}$, and $\eta \equiv 1$, we can see that the problem*

$$\begin{cases} u_t - L_{\lambda^*}\, u &= \lambda u - \dfrac{\lambda^* C}{|x|^2((\log R/|x|)^\alpha} u \quad \text{in} \quad \Omega \times (0,\infty) \\ u(x,0) &= u_0(x) \quad \text{in} \quad \Omega \\ u(x,t) &= 0 \quad \text{on} \quad \partial\Omega \times (0,\infty), \end{cases}$$

has a unique global solution in $H_0^1(\Omega)$, for all $C > 0$, if $0 < \alpha < 2$, and for all $C > 3/(n-2)^2$ if $\alpha = 2$.

(5.1.3) Theorem. *Assume $n = 2$ and let q, η as in hypotheses $(H_1)-(H_3)$. Then the problem*

$$\begin{cases} u_t - \Delta u - \dfrac{1}{4} \dfrac{q(x)}{(|x|\, \log\, R/|x|)^2}\, u &= \lambda \eta u \quad \text{in} \quad \Omega \times (0,\infty) \\ u(x,0) &= u_0(x) \quad \text{in} \quad \Omega \\ u(x,t) &= 0 \quad \text{on} \quad \partial\Omega \times (0,\infty), \end{cases} \tag{5.7}$$

where $u_0 \in L^\infty(\Omega)$, non-negative $u_0 \not\equiv 0$; admits a unique global non-negative solution $u \in C\left([0,\infty); L^2(\Omega)\right) \cap L^2\left(0,\infty; H_0^1(\Omega)\right)$ with $u' \in L^2\left(0,\infty; H^{-1}(\Omega)\right)$, $u \not\equiv 0$; for all λ, $0 \leq \lambda \leq C(q,\eta)$, provided q satisfies

$$\liminf_{x\to 0} \left(\log\log \frac{1}{|x|}\right)^2 (1 - q(x)) > 3. \tag{5.8}$$

§5.2 Formulation of Weak Solutions

In this section we give the meaning of the solution which we have stated in the introduction of this Chapter. Here, we give some basic prerequisite and more details can be found in the book of Dautray and Lions [17]. Let $(X, \|.\|_X)$, be a real Hilbert space. Now, we define the following function spaces :

Definition 1. We denote by $L^p(0,\infty;X)$, $(1 \le p < \infty)$ the space of (classes of) functions $u : (0,\infty) \to X$, $t \mapsto u(t)$, such that,

$$\begin{cases} (i) \quad u \ \text{ is measurable for dt}\,, \\ \\ (ii) \quad \|u\|_{L^p(0,\infty;X)} \ := \ \left(\int_0^\infty \|u(t)\|_X^p dt\right)^{1/p} \ < \ +\infty\ . \end{cases}$$

Definition 2. Let $u \in L^1(0,\infty;X)$. We say $v \in L^1(0,\infty;X)$ is the weak derivative of u, written as

$$u' = v \quad \text{or} \quad \frac{d}{dt}u = v\ ,$$

provided

$$\int_0^\infty \phi'(t)u(t)\,dt = -\int_0^\infty \phi(t)v(t)\,dt\ ,$$

for all scalar test functions $\phi \in C_c^\infty(0,\infty)$. The above integral is in the sense of Bochner.

Definition 3. Let us take the Hilbert space $H_0^1(\Omega)$ and $H^{-1}(\Omega)$ be its dual, with the usual operator norm $\|.\|_{H^{-1}}$. It is known that

$$\begin{array}{ccccc} H_0^1(\Omega) & \hookrightarrow & L^2(\Omega) & \hookrightarrow & H^{-1}(\Omega) \\ u & \mapsto & u & \mapsto & U \end{array} \tag{5.9}$$

where $U(\phi) := \int\limits_\Omega u(x)\phi(x)\,dx$, for any $\phi \in H_0^1(\Omega)$; each space being dense in the following ($\hookrightarrow$ denotes continuous injection). We denote by $W = W\left(0,\infty;H_0^1(\Omega),H^{-1}(\Omega)\right)$ the space:

$$W\left(0,\infty;H_0^1(\Omega),H^{-1}(\Omega)\right) := \left\{u \;\middle|\; u \in L^2\left(0,\infty;H_0^1(\Omega)\right),\, u' \in L^2\left(0,\infty;H^{-1}(\Omega)\right)\right\}\ ,$$

equipped with the norm (by the identification (5.9), the space W make sense)

$$\begin{aligned} \|u\|_W \ &:= \ \left(\|u\|^2_{L^2\left(0,\infty;H_0^1(\Omega)\right)} + \|u\|^2_{L^2(0,\infty;H^{-1}(\Omega))}\right)^{1/2} \\ &= \ \left(\int_0^\infty \left(\|\nabla u(t)\|^2 + \|u'(t)\|^2_{H^{-1}}\right)dt\right)^{1/2}. \end{aligned}$$

The following Trace Theorem is well known see [17].

(5.2.1) Trace Theorem. *Every u in the Hilbert space $W\left(0,\infty;H_0^1(\Omega),H^{-1}(\Omega)\right)$ is almost everywhere equal to a continuous function of $[0,\infty]$ in $L^2(\Omega)$. Further,*

$$W\left(0,\infty;H_0^1(\Omega),H^{-1}(\Omega)\right)\hookrightarrow C\left([0,\infty];L^2(\Omega)\right)$$

where, $C\left([0,\infty];L^2(\Omega)\right)$ is the space of continuous functions over $[0,\infty]$ with values in $L^2(\Omega)$ equipped with the topology of uniform convergence.

We denote the duality between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$ by ${}_{H^{-1}}\langle\,.\,,\,.\,\rangle_{H^1}$. From the Definition 3, we note that ${}_{H^{-1}}\langle u,v\rangle_{H^1}$ reduces to the L^2 inner product ${}_{L^2}\langle u,v\rangle_{L^2}$, if $u\in L^2(\Omega)$. It is easy to show (see for example, [17]) that: for $u\in W\left(0,\infty;H_0^1(\Omega),H^{-1}(\Omega)\right)$, then for any $v\in H_0^1(\Omega)$

$$ {}_{H^{-1}}\langle\frac{d}{dt}u(.),v\rangle_{H^1}=\frac{d}{dt}\,{}_{H^{-1}}\langle u(.),v\rangle_{H^1}\quad\text{in } D'(0,\infty)\,, \tag{5.10}$$

in the sense

$$\int_0^\infty {}_{H^{-1}}\langle u'(t),v\rangle_{H^1}\,\phi(t)\,dt=-\int_0^\infty {}_{H^{-1}}\langle u(t),v\rangle_{H^1}\,\phi'(t)\,dt\ ,$$

for all scalar test functions $\phi\in C_c^\infty(0,\infty)$.

Now we are in a position to define a weak solution for the problems (5.5) and (5.7).

Definition 4. A function $u\in W\left(0,\infty;H_0^1(\Omega),H^{-1}(\Omega)\right)$, is said to be a solution to the problem (5.5) (similar for (5.7)), if for all $v\in H_0^1(\Omega)$

$$\frac{d}{dt}\,{}_{H^{-1}}\langle u(.),v\rangle_{H^1}+{}_{H^{-1}}\langle Lu(.),v\rangle_{H^1}=0\,, \tag{5.11}$$

in the sense of $D'(0,\infty)$ and $u(0)=u_0$ (this make sense because of the Trace Theorem). Here,

$$L:=-\Delta-\lambda^*\frac{q(x)}{|x|^2}-\lambda\eta\ ,$$

and ${}_{H^{-1}}\langle L\,.\,,\,.\,\rangle_{H^1}:H_0^1(\Omega)\times H_0^1(\Omega)\to\mathbb{R}$, is the bilinear functional defined by

$${}_{H^{-1}}\langle Lu,v\rangle_{H^1}:=\int_\Omega\langle\nabla u,\nabla v\rangle\,dx-\lambda^*\int_\Omega\frac{u\,q(x)}{|x|^2}v\,dx-\lambda\int_\Omega uv\,\eta(x)\,dx\,,\quad\forall\ u,\,v\in H_0^1(\Omega)\ .$$

§5.3 Proof of Theorem (5.1.1)

To prove the Theorem (5.1.1) we need the following Lemma. This Lemma is essentially contained in [3] and we use a similar idea of Adimurthi- Sandeep to give a proof.

(5.3.1) Lemma. *Let Ω be a bounded domain in $\mathbb{R}^n$, $n \geq 3$ with $0 \in \Omega$ and take q, η, $C(q,\eta)$ as defined in section 5.1, with the extra assumption (5.6) on q. Then for any $0 \leq \lambda \leq C(q,\eta)$, there exists $u \in H_0^1(\Omega)$, $u > 0$, satisfying the following equation*

$$-\Delta u - \lambda^* \frac{q(x)}{|x|^2}\, u - \lambda \eta u = \lambda^1(\Omega, q, \eta) u \qquad \text{in} \quad \Omega \tag{5.12}$$

for some $\lambda^1(\Omega, q, \eta) > 0$, depending on n, q, η and Ω.

Proof. Let λ be such that $0 \leq \lambda \leq C(q,\eta)$. For any s, $0 < s < 1$, we define the operator L_s by $L_s := -\Delta - s\left(\lambda^* \frac{q(x)}{|x|^2} + \lambda\eta\right)$. Then for any $u \in H_0^1(\Omega)$, by the inequality (5.3) we have

$$\begin{aligned}
{}_{H^{-1}}\langle L_s u, u\rangle_{H^1} &= \int_\Omega |\nabla u|^2 dx - s\left(\int_\Omega \lambda^* \frac{|u|^2 q(x)}{|x|^2} dx + \lambda \int_\Omega |u|^2 \eta(x)\, dx\right) \\
&= (1-s)\int_\Omega |\nabla u|^2 dx + s\left(\int_\Omega |\nabla u|^2 dx - \int_\Omega \lambda^* \frac{|u|^2 q(x)}{|x|^2} dx - \lambda \int_\Omega |u|^2 \eta(x)\, dx\right) \\
&\geq (1-s)\int_\Omega |\nabla u|^2 dx
\end{aligned}$$

where, ${}_{H^{-1}}\langle L_s\, .\, ,\, .\,\rangle_{H^1} : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$, is the bilinear functional defined by

$${}_{H^{-1}}\langle L_s u, v\rangle_{H^1} := \int_\Omega \langle \nabla u, \nabla v\rangle\, dx - s\left(\int_\Omega \lambda^* \frac{u\, q(x)}{|x|^2} v\, dx + \lambda \int_\Omega uv\, \eta(x)\, dx\right), \quad \forall\ u,\, v \in H_0^1(\Omega).$$

Thus, for each $0 < s < 1$, $\left(\int_\Omega |\nabla u|^2 dx - s\lambda^* \int_\Omega \frac{|u|^2 q(x)}{|x|^2} dx - s\lambda \int_\Omega |u|^2 \eta(x)\, dx\right)^{1/2}$, is equivalent to the standard norm of $H_0^1(\Omega)$ and therefore, the operator, $L_s : H_0^1(\Omega) \to H^{-1}(\Omega)$ defines an isometry. Hence

$$\lambda_s^1(\Omega, q, \eta) := \inf_{H_0^1(\Omega)} \left\{ \int_\Omega |\nabla u|^2 dx - s\lambda^* \int_\Omega \frac{|u|^2 q(x)}{|x|^2} dx - s\lambda \int_\Omega |u|^2 \eta(x)\, dx \;\middle|\; \int_\Omega |u|^2\, dx = 1 \right\},$$

is achieved and $\lambda_s^1(\Omega, q, \eta) > 0$. Therefore, there exists $u_s \in H_0^1(\Omega)$, $u_s > 0$ and $\int_\Omega |\nabla u_s|^2\, dx = 1$, satisfying the linear equation

$$-\Delta u_s - s\left(\lambda^* \frac{q(x)}{|x|^2} + \lambda\eta\right) u_s = \lambda_s^1(\Omega, q, \eta)\, u_s \qquad \text{in} \quad \Omega\,. \tag{5.13}$$

As we have shown in the the Theorem (3.3.1); by the inequality (5.3), we have $\lambda_s^1(\Omega, q, \eta) \to \lambda^1(\Omega, q, \eta) > 0$ as $s \to 1$ and

$$\lambda^1(\Omega, q, \eta) = \inf_{H_0^1(\Omega)} \left\{ \int_\Omega |\nabla u|^2 dx - \lambda^* \int_\Omega \frac{|u|^2 q}{|x|^2} dx - \lambda \int_\Omega |u|^2 \eta \, dx \;\; \Big| \int_\Omega |u|^2 \, dx = 1 \right\}. \quad (5.14)$$

Since $(u_s)_{0<s<1}$ is bounded in $H_0^1(\Omega)$, by passing through a subsequence, if necessary, as $s \to 1$, we have

$$\begin{cases} u_s \rightharpoonup u \quad \text{weakly in} \quad H_0^1(\Omega) \\ u_s \to u \quad \text{strongly in} \quad L^2(\Omega) \\ u_s \rightharpoonup u \quad \text{strongly in} \quad L^2(\Omega, \eta) \\ u_s \rightharpoonup u \quad \text{weakly in} \quad L^2(\Omega, |x|^{-2}), \end{cases} \quad (5.15)$$

and hence u satisfies the equation (5.12). So, it only remains to prove that $u > 0$. The idea of the proof is to show that the quantity, $\dfrac{|u_s|^2}{|x|^2}$ is bounded by a L^1 function (independent of s) so that we can pass through the limit in the integral as $s \to 1$. For this, we take the test function $\psi_k(x)$ as in (3.18) of Chapter 3, i.e.

$$\psi_k(x) := |x|^{-\frac{n-2}{2}} \left(\log R/|x|\right)^{-k}, \qquad k > 1/2.$$

Since q satisfies the condition (5.6), we can choose $k > 1/2$, such that

$$\liminf_{x \to 0} \left(\log \frac{R}{|x|} \right)^2 (1 - q(x)) > \frac{4k(k+1)}{(n-2)^2}. \quad (5.16)$$

Now, we have (exactly as in the proof of Theorem (3.3.4))

$$\begin{aligned} L^s \psi_k : &= -\left(\Delta + \lambda^* \frac{q(x)}{|x|^2} + \lambda\eta - \lambda_s^1(\Omega, q, \eta) \right) \psi_k \\ &= \lambda^* \frac{\psi_k}{|x|^2 \left(\log R/|x|\right)^2} \left[(1 - q(x)) \left(\log R/|x|\right)^2 - \frac{k(k+1)}{\lambda^*} \right. \\ &\quad \left. + \frac{\lambda}{\lambda^*} \eta(x) |x|^2 \left(\log R/|x|\right)^2 + \frac{\lambda_s^1(\Omega, q, \eta)}{\lambda^*} |x|^2 \left(\log R/|x|\right)^2 \right]. \end{aligned} \quad (5.17)$$

By (5.16) and the hypothesis (H_3), the most dominant term on the right hand side of (5.17) is $(1 - q(x)) \left(\log R/|x|\right)^2$. Thus we can choose $\delta > 0$ independent of k, such that

$$L^s \psi_k \geq 0 \qquad \text{in} \quad B_\delta := B(0, \delta). \quad (5.18)$$

By the standard elliptic estimate applied to the domain $B_\delta \setminus \bar{B}_{\delta/2}$, we can find an $M > 0$ (independent of k), such that $u_s \leq M\,\psi_k$ on $|x| = \delta$, for all $s \in (0,1)$. Let us define the function v_s on B_δ, by

$$v_s(x) := u_s(x) - M\psi_k(x)\,,$$

then $v_s^+ \in H_0^1(B_\delta)$ for all $s \in (0,1)$. From (5.13) and (5.18), we obtain

$$\begin{aligned} L_s v_s - \lambda_s^1(\Omega,q,\eta)v_s &= \Big(L_s u_s - \lambda_s^1(\Omega,q,\eta)u_s\Big) - M\,L_s\psi_k - M\lambda_s^1(\Omega,q,\eta)\psi_k \\ &= -M\,L^s\psi_k - M\lambda^*(1-s)\frac{q\psi_k}{|x|^2} - M\lambda(1-s)\eta\psi_k \\ &\leq 0 \qquad \text{in} \quad B_\delta. \end{aligned}$$

Now, by multiplying (testing) the above inequality by v_s^+ and integrating by parts, we obtain

$$\int\limits_{B_\delta} |\nabla v_s^+|^2 dx - \lambda^* s \int\limits_{B_\delta} \frac{q(x)|v_s^+|^2}{|x|^2}\,dx - \lambda s \int\limits_{B_\delta} \eta(x)|v_s^+|^2\,dx - \lambda_s^1(\Omega,q,\eta)\int\limits_{B_\delta} |v_s^+|^2\,dx \leq 0\ .$$

We observe that the function

$$V_s(x) := \begin{cases} v_s^+(x) & \text{in } B_\delta\,, \\ \\ 0 & \text{in } \Omega \setminus B_\delta\,, \end{cases}$$

is in $H_0^1(\Omega)$ and hence from the definition of $\lambda_s^1(\Omega,q,\eta)$,

$$\int\limits_{B_\delta} |\nabla v_s^+|^2 dx = \lambda^* s \int\limits_{B_\delta} \frac{q(x)|v_s^+|^2}{|x|^2}\,dx + \lambda s \int\limits_{B_\delta} \eta(x)|v_s^+|^2\,dx + \lambda_s^1(\Omega,q,\eta)\int\limits_{B_\delta} |v_s^+|^2\,dx\ . \quad (5.19)$$

Now by the Corollary (2.1.2) in Chapter 2, with $k = 2$, (here we take δ to be sufficiently small) we have

$$\begin{aligned} \int\limits_{B_\delta} |\nabla v_s^+|^2 dx \ \geq\ & \lambda^* \int\limits_{B_\delta} \frac{|v_s^+|^2}{|x|^2}\,dx + C_1 \int\limits_{B_\delta} \frac{|v_s^+|^2}{|x|^2}\,(\log 1/|x|)^{-2}\,dx \\ & + C_2 \int\limits_{B_\delta} \frac{|v_s^+|^2}{|x|^2}\,(\log 1/|x|)^{-2}\,(\log\,\log 1/|x|)^{-2}\,dx \qquad (5.20) \end{aligned}$$

where C_1 and C_2 are positive constants. Now comparing (5.19) and (5.20), we conclude that $v^+ \equiv 0$ on B_δ. Hence

$$u_s(x) \leq M\psi_k(x) \qquad \text{in } B_\delta \quad \forall\ s \in (0,1)\,.$$

But, $\frac{|u_s|^2}{|x|^2} \leq M^2 \frac{|\psi_k|^2}{|x|^2} \in L^1(\Omega)$, and hence by the dominated convergence theorem, we have

$$\int_\Omega \frac{|u_s|^2 q(x)}{|x|^2}\, dx \to \int_\Omega \frac{|u|^2 q(x)}{|x|^2}\, dx \qquad \text{as} \quad s \to 1\,. \tag{5.21}$$

Similarly, we have $u_s \to u$ strongly in the weighted space $L^2(\Omega, \eta)$ as $s \to 1$. Now by testing the equation (5.13) with u_s and integrating by parts, we have

$$1 - \lambda^* s \int_\Omega \frac{q(x)|u_s|^2}{|x|^2}\, dx - \lambda s \int_\Omega \eta(x)|u_s|^2\, dx = \lambda_s^1(\Omega, q, \eta) \int_\Omega |u_s|^2\, dx\,. \tag{5.22}$$

Now from (5.15), (5.21) and (5.22), we have as $s \to 1$

$$\lambda^* \int_\Omega \frac{q(x)|u|^2}{|x|^2}\, dx + \lambda \int_\Omega \eta(x)|u|^2\, dx + \lambda^1(\Omega, q, \eta) \int_\Omega |u|^2\, dx = 1\ .$$

and hence $u \geq 0$, $u \not\equiv 0$, and $\lambda^1(\Omega)$ is achieved for the function u. Then by strong maximum principle for weak solutions (as we have shown in the proof of the Theorem (3.3.1)) we conclude that, $u > 0$ in Ω. ∎

Proof of the Theorem (5.1.1). We observe that, $L_s^{-1} : L^2(\Omega) \to L^2(\Omega)$, for any $0 < s < 1$, is compact, self-adjoint and positive definite operator. Hence, there exist an orthonormal basis $(e_k)_{k\geq 1}$ of $L^2(\Omega)$ constituted by eigenfunctions of the operator L_s, with eigenvalue sequence, $0 < \lambda_s^1 \leq \lambda_s^2 \leq \cdots \to +\infty$. So, for all $k \geq 1$, we have $L_s e_k = \lambda_s^k e_k$ in Ω, with $e_k = 0$ on $\partial\Omega$. By the standard semigroup theory [17], there exists a unique function $u_s \in W\left(0, \infty; H_0^1(\Omega), H^{-1}(\Omega)\right)$, $u_s \geq 0$, $u_s \not\equiv 0$ satisfying the following heat equation

$$(u_s)_t + L_s u_s = 0 \qquad \text{in} \quad \Omega \times (0, \infty) \tag{5.23}$$

with the initial data

$$u_s(x, 0) = u_0(x) \qquad \text{in } \Omega\,,$$

in the sense of Definition 4 of Section 5.2. Now by taking $v = u_s(t)$ in (5.11) and using the fact in (5.10), we have

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |u_s(t)|^2 dx = -\int_\Omega |\nabla u_s(t)|^2 dx + \lambda^* s \int_\Omega \frac{q|u_s(t)|^2}{|x|^2} dx + \lambda s \int_\Omega \eta |u_s(t)|^2 dx\ . \tag{5.24}$$

Now our aim to show that the sequence $(u_s)_{0<s<1}$ is bounded in the function space $W\left(0, \infty; H_0^1(\Omega), H^{-1}(\Omega)\right)$. But this does not follow immediately from (5.23). So we plan to find a suitable function ϕ such that u_s is pointwise bounded above by the function ϕ. To obtain such ϕ, we proceed as the following :

Let Ω_1, be a bounded open subset of $\mathbb{R}^n$ such that $\bar{\Omega}$ is properly contained in Ω_1. Take $v \in H_0^1(\Omega_1)$, $v > 0$ satisfying the equation (5.12) in the domain Ω_1, where $\lambda^1(\Omega_1, q, \eta)$ as in (5.14) and we simply denote, $\lambda^1(\Omega_1, q, \eta)$ by $\lambda^1(\Omega_1)$. Define the function ϕ on $\Omega_1 \times (0, \infty)$, by

$$\phi(x,t) := e^{-\lambda^1(\Omega_1)\,t} v(x)\,,$$

then $\phi \in W\left(0,\infty; H_0^1(\Omega_1), H^{-1}(\Omega_1)\right)$, $\phi > 0$ in $\Omega \times (0,\infty)$ and

$$\phi_t + L\phi = e^{-\lambda^1(\Omega_1)\,t}\left(-\lambda^(\Omega_1)v + Lv\right) = 0 \qquad \text{in} \quad \Omega_1 \times (0,\infty),$$

where $L := -\Delta - \lambda^* \dfrac{q(x)}{|x|^2} - \lambda\eta$. Hence ϕ is a solution to the problem

$$\left\{\begin{array}{rcll} \phi_t + L\phi & = & 0 & \text{in} \quad \Omega_1 \times (0,\infty) \\ \phi(x,0) & = & v(x) & \text{in} \quad \Omega_1 \\ \phi(x,t) & = & 0 & \text{on} \quad \partial\Omega_1 \times (0,\infty). \end{array}\right. \tag{5.25}$$

Now we define

$$w_s(x,t) := M\phi(x,t) - u_s(x,t) \qquad \text{on } \Omega \times (0,\infty)\,.$$

Since $u_s(x,0) = u_0(x) \in L^\infty(\Omega)$, we can choose some $M > 0$ such that

$$w_s(x,0) := w_s(0) = Mv(x) - u_0(x) \geq 0 \qquad \text{in } \Omega\,,$$

(we have taken $\Omega \subset\subset \Omega_1$ and $v \in H_0^1(\Omega_1)$, just to ensure that $v > 0$, on $\partial\Omega$, so that $w_s(0) \geq 0$ in Ω). From (5.23) and (5.25), we have

$$\begin{aligned} (w_s)_t + Lw_s &= M\left(\phi_t + L\phi\right) - (u_s)_t - Lu_s \\ &= L_s u_s - Lu_s \\ &= (1-s)\left(\frac{\lambda^* q}{|x|^2} + \lambda\eta\right) u_s \\ &\geq 0 \qquad \text{in } \Omega \times (0,\infty) \end{aligned} \tag{5.26}$$

Therefore, w_s satisfies

$$\left\{\begin{array}{rcll} (w_s)_t + Lw_s & \geq & 0 \quad \text{in} & \Omega \times (0,\infty) \\ w_s(x,0) & \geq & 0 \quad \text{in} & \Omega \\ w_s(x,t) & = & 0 \quad \text{on} & \partial\Omega \times (0,\infty). \end{array}\right.$$

Our aim is to prove that $w_s \geq 0$ in $\Omega \times (0,\infty)$. So, it is enough to prove that for all $t > 0$, $w_s^-(t) := \max\{-w_s(t), 0\} \equiv 0$ in Ω. Since $w_s^-(t) \in H_0^1(\Omega)$, $w_s^-(0) \equiv 0$ in Ω, using our basic definitions of Section 5.2; testing (5.26) by $w_s^-(t)$ and integrating from 0 to t, we obtain

$$-\frac{1}{2}\int_\Omega |w_s^-(t)|^2 dx - \int_0^t\int_\Omega |\nabla w_s^-(t)|^2 dx\,dt + \lambda^* \int_0^t\int_\Omega \frac{q|w_s^-(t)|^2}{|x|^2} dx\,dt + \lambda \int_0^t\int_\Omega \eta |w_s^-(t)|^2 dx\,dt \geq 0\,.$$

Again by the inequality (5.3) we get, for all $t > 0$

$$\int_\Omega |w_s^-(t)|^2 dx \;\leq\; 0\,.$$

Therefore, for all $t \geq 0$, $w_s^-(t) \equiv 0$ in Ω, i.e., $w_s \geq 0$ in $\Omega \times (0,\infty)$. In other words, we have

$$u_s \leq M e^{-\lambda^1(\Omega_1)\,t} v(x) \qquad \text{in} \quad \Omega \times (0,\infty). \tag{5.27}$$

To show that u_s is bounded in $L^2\left(0,\infty; H_0^1(\Omega)\right)$, we observe that, by using (5.27) in (5.24) and integrating from 0 to t

$$\begin{aligned}
\frac{1}{2}\int_\Omega |u_s(t)|^2 dx + \int_0^t\int_\Omega |\nabla u_s(t)|^2 dx\,dt &= \lambda^* s \int_0^t\int_\Omega \frac{q|u_s(t)|^2}{|x|^2} dx\,dt \\
&\quad + \lambda\, s \int_0^t\int_\Omega \eta |u_s(t)|^2 dx\,dt + \frac{1}{2}\int_\Omega |u_0(x)|^2 dx \\
&\leq \frac{M^2}{\lambda^1(\Omega_1)}\left(\lambda^* \int_\Omega \frac{|v|^2}{|x|^2} dx + \lambda \int_\Omega \eta(x)|v|^2 dx\right) \\
&\quad + \frac{1}{2}\int_\Omega |u_0(x)|^2 dx \\
&\leq \frac{M^2}{\lambda^1(\Omega_1)}\int_\Omega |\nabla v(t)|^2 dx + \frac{1}{2}\int_\Omega |u_0(x)|^2 dx
\end{aligned}$$

and hence, u_s is bounded in $L^2\left(0,\infty; H_0^1(\Omega)\right)$. Now from (5.23), we have for all $w \in H_0^1(\Omega)$ with $\|\nabla w\|_2 \leq 1$, by Hölder and Hardy inequality, we have

$$\begin{aligned}
|{}_{H^{-1}}\langle u_s'(t), w\rangle_{H^1}| &\leq \left(\int_\Omega |\nabla u_s(t)|^2 dx\right)^{1/2} + \lambda^* \left(\int_\Omega \frac{|u_s(t)|^2}{|x|^2} dx\right)^{1/2} \left(\int_\Omega \frac{|w|^2}{|x|^2} dx\right)^{1/2} \\
&\quad + \lambda \left(\int_\Omega \eta |u_s(t)|^2 dx\right)^{1/2} \left(\int_\Omega \eta |w|^2 dx\right)^{1/2} \\
&\leq C \left(\int_\Omega |\nabla u_s(t)|^2 dx\right)^{1/2}
\end{aligned}$$

for some constant $C > 0$ (independent of s).

Hence

$$\|u_s'(t)\|_{H^{-1}(\Omega)} := \sup_{\|\nabla w\|_2=1} |{}_{H^{-1}}\langle u_s'(t), w\rangle_{H^1}| \leq C\left(\int_\Omega |\nabla u_s(t)|^2 dx\right)^{1/2}. \tag{5.28}$$

Thus,

$$\int_0^\infty \|u_s'(t)\|^2_{H^{-1}(\Omega)} dt \leq C\int_0^\infty \int_\Omega |\nabla u_s(t)|^2 dx\, dt$$

which shows that u_s' is bounded in $L^2\left(0,\infty; H^{-1}(\Omega)\right)$. Therefore, as in the limit $s \to 1$, we have

$$\begin{cases} u_s \rightharpoonup u \quad \text{weakly in} \quad L^2\left(0,\infty; H_0^1(\Omega)\right) \\ \\ u_s' \rightharpoonup u' \quad \text{weakly in} \quad L^2\left(0,\infty; H^{-1}(\Omega)\right). \end{cases} \tag{5.29}$$

As $u \in W\left(0,\infty; H_0^1(\Omega), H^{-1}(\Omega)\right)$, by the Trace Theorem (5.2.1), $u(x,0) = u(0) \in L^2(\Omega)$. Since u_s satisfies the equation (5.23), we have for all $w \in H_0^1(\Omega)$ and for all $\zeta \in C_c^\infty(0,\infty)$,

$$-\int_0^\infty {}_{H^{-1}}\langle u_s(t), w\rangle_{H^1}\, \zeta'(t) dt + \int_0^\infty {}_{H^{-1}}\langle L_s u_s(t), w\rangle_{H^1}\, \zeta(t)\, dt = 0\ , \tag{5.30}$$

and hence by (5.29), we have as $s \to 1$

$$-\int_0^\infty {}_{H^{-1}}\langle u(t), w\,\zeta'(t)\,\rangle_{H^1} dt + \int_0^\infty {}_{H^{-1}}\langle Lu, w\,\zeta(t)\rangle_{H^1}\, dt = 0 \tag{5.31}$$

for all $w \in H_0^1(\Omega)$ and for all $\zeta \in C_c^\infty(0,\infty)$. Therefore u satisfies the equation

$$u_t + Lu = 0 \qquad \text{in } \Omega \times (0,\infty)\ ,$$

with the initial data $u(x,0) = u(0)$ in Ω. Our Theorem will be proved if we can show that $u(0)$ is the given initial data u_0. For this, let ζ be a C^∞ function on $[0,\infty)$ such that $\zeta(0) \neq 0$ and $\zeta(T) = 0$ for some large $T > 0$. Then for any $w \in H_0^1(\Omega)$ (exactly as in (5.31)), we have

$$-\int_0^\infty {}_{H^{-1}}\langle u(t), w\,\zeta'(t)\rangle_{H^1} dt + \int_0^\infty {}_{H^{-1}}\langle Lu, w\,\zeta(t)\rangle_{H^1}\, dt = \zeta(0)\,{}_{H^{-1}}\langle u(0), w\rangle_{H^1}\ , \tag{5.32}$$

and like in (5.30), we obtain

$$\begin{aligned} -\int_0^\infty {}_{H^{-1}}\langle u_s(t), w\,\zeta'(t)\rangle_{H^1}\, dt + \int_0^\infty {}_{H^{-1}}\langle L_s u_s(t), w\,\zeta(t)\rangle_{H^1}\, dt &= \zeta(0)\,{}_{H^{-1}}\langle u_s(0), w\rangle_{H^1} \\ &= \zeta(0)_{H^{-1}}\langle u_0, w\rangle_{H^1} \end{aligned}$$

Now by passing to the limit as $s \to 1$ in the above identity and using (5.29), we have

$$-\int_0^\infty {}_{H^{-1}}\langle u(t), w\,\zeta'(t)\rangle_{H^1}\,dt + \int_0^\infty {}_{H^{-1}}\langle Lu(t), w\,\zeta(t)\rangle_{H^1}\,dt = \zeta(0)\,{}_{H^{-1}}\langle u_0, w\rangle_{H^1} \qquad (5.33)$$

Comparing (5.32)- (5.33), we obtain $u(0) = u_0$ in Ω. Now we observe that for any $t > 0$, the solution u satisfies

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |u(t)|^2 dx = -\int_\Omega |\nabla u(t)|^2 dx + \lambda^* \int_\Omega \frac{q|u(t)|^2}{|x|^2} dx + \lambda \int_\Omega \eta |u(t)|^2 dx\ .$$

By the definition of $\lambda^1(\Omega, q, \eta)$ in (5.14) and invoking Gronwall's Lemma, we have

$$\int_\Omega |u(t)|^2 dx \le e^{-2\lambda^1(\Omega,q,\eta)\,t} \int_\Omega |u_0(x)|^2 dx \qquad (5.34)$$

Hence, the uniqueness and non-negativity (as $u_0 \ge 0$) follows from (5.34). ∎

§5.4 Proof of Theorem (5.1.3)

We prove the Theorem by establishing the following Lemma, which is essentially taken from [3].

(5.4.1) Lemma. *Let Ω be a bounded domain in $\mathbb{R}^2$, with $0 \in \Omega$ and $R \ge e^{e^e} \sup_\Omega |x|$. Take q, η, $C(q,\eta)$ as defined in section 5.1 with the extra assumption (5.8) on q. Then for any $0 \le \lambda \le C(q,\eta)$, there exists $u \in H_0^1(\Omega)$, $u > 0$, satisfying the following equation*

$$-\Delta u - \frac{1}{4}\frac{q(x)}{|x|^2\left(\log R/|x|\right)^2}\,u - \lambda\eta u = \lambda^1(\Omega, q, \eta)u \qquad \text{in} \quad \Omega \qquad (5.35)$$

where,

$$\lambda^1(\Omega, q, \eta) = \inf_{H_0^1(\Omega)} \left\{ \int_\Omega |\nabla u|^2 - \frac{1}{4}\int_\Omega \frac{q|u|^2}{|x|^2((\log R/|x|)^2} - \lambda \int_\Omega |u|^2\eta \ \ \Big| \int_\Omega |u|^2 = 1 \right\}. \qquad (5.36)$$

Proof. As in the proof of Lemma (5.3.1), for $0 < s < 1$, we define the operator $T_s := -\Delta - \frac{s}{4}\frac{q(x)}{|x|^2\left(\log R/|x|\right)^2} - \lambda s\eta$. By using the 2-dimensional Hardy-Sobolev inequality (5.4), we can show that there exists $u_s \in H_0^1(\Omega)$, $\|\nabla u_s\|_2 = 1$, $u_s > 0$, such that

$$T_s u_s = \lambda_s^1(\Omega, q, \eta)u_s \qquad \text{in} \quad \Omega \qquad (5.37)$$

where

$$\lambda_s^1(\Omega, q, \eta) = \inf_{H_0^1(\Omega)} \left\{ \int_\Omega |\nabla u|^2 - \frac{s}{4} \int_\Omega \frac{q|u|^2}{|x|^2((\log R/|x|)^2} - s\lambda \int_\Omega |u|^2 \eta \quad \Big| \int_\Omega |u|^2 = 1 \right\}.$$

It is easy to see that the function

$$\psi_k(x) := (\log R/|x|)^{1/2} (\log\log R/|x|)^{-k}$$

is in $H^1(\Omega)$ if and only if $k > 1/2$ (here we mention that, the above function is used by Adimurthi and Sandeep in [3]). We take $k > 1/2$, such that

$$\liminf_{x\to 0} \left(\log\log \frac{1}{|x|}\right)^2 (1 - q(x)) > 4k(k+1)\,. \tag{5.38}$$

As it is shown in [3] that,

$$\Delta\psi_k = \frac{\psi_k}{|x|^2 (\log R/|x|)^2} \left(\frac{1}{4} + \frac{k(1+k)}{(\log\log R/|x|)^2}\right)$$

therefore, by the hypothesis (H_3) and (5.38), we obtain

$$\begin{aligned} T^s\psi_k : &= -\Delta\psi_k - \frac{1}{4}\frac{q(x)}{|x|^2 (\log R/|x|)^2}\,\psi_k - \lambda\eta\psi_k - \lambda_s^1(\Omega, q, \eta)\psi_k \\ &= \frac{\psi_k}{4|x|^2 (\log R/|x|)^2} \Big[(1 - q(x)) - \frac{4k(1+k)}{(\log\log R/|x|)^2} \\ &\quad 4\lambda\eta\psi_k|x|^2 (\log R/|x|)^2 - 4\lambda_s^1(\Omega, q, \eta)\psi_k|x|^2 (\log R/|x|)^2\Big] \\ &\geq 0 \qquad \text{on} \quad B_\delta := B(0,\delta) \quad \text{for sufficiently small } \delta > 0. \end{aligned}$$

Now, exactly as in the proof of Lemma (5.3.1), using the inequality (5.4); we can show that $u_s \leq M\psi_k$ on B_δ, for all $s \in (0,1)$, $u_s \to u$ strongly in $L^2(\Omega, |x|^2)$ as $s \to 1$, which ensure $u > 0$ and satisfies the equation (5.35). ∎

Now, is easy to complete the proof by following proof of Theorem (5.1.1). Here, the only differences are, we have to use Lemma (5.4.1) instead of Lemma (5.4.1) and inequality (5.4) instead of (5.3) in the respective places . ∎

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