A DE GIORGI CONJECTURE ON THE REGULARITY OF MINIMIZERS OF CARTESIAN AREA IN 1D

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ABSTRACT. We prove a $C^{1,1}$-regularity of minimizers of the functional

$$\int_I \sqrt{1+|Du|^2} + \int_I |u-g|ds, \quad u \in BV(I),$$

provided $I \subset \mathbb{R}$ is a bounded open interval and $\|g\|_\infty$ is sufficiently small, thus partially establishing a De Giorgi conjecture in dimension one and codimension one. We also extend our result to a suitable anisotropic setting.

1. INTRODUCTION

The non-parametric minimal surfaces, more generally, the prescribed mean curvature surfaces, have been extensively studied in the literature from the variational perspective (see e.g. [11, 16, 17, 18, 14] and the references therein). Given an open set $\Omega \subset \mathbb{R}^n$ and a sufficiently regular function $H : \Omega \to \mathbb{R}$, the underlying equation is rewritten as

$$\operatorname{div} \frac{\nabla u}{\sqrt{1+|\nabla u|^2}} = H \quad \text{in } \Omega \tag{1.1}$$

with a prescribed Dirichlet or Neumann boundary condition, and corresponds to the Euler-Lagrange equation of the functional

$$\int_\Omega \sqrt{1+|\nabla u|^2} dx + \int_\Omega Hu\, dx, \quad u \in C^1(\Omega). \tag{1.2}$$

It is well-known that under suitable assumptions on Ω and H, the minimizers are in fact locally $C^{2+\alpha}$, and hence solve (1.1) in a classical sense.

In the context of functionals with linear growth, a related problem is the existence and regularity of minimizers of the (convex, but not strictly convex) functional

$$\mathscr{F}(u) := \int_\Omega \sqrt{1+|\nabla u|^2} dx + \int_\Omega |u-g|\, dx, \quad u \in C^1(\Omega),$$

where $g \in L^1(\Omega)$ is given, see [7]. In this case, the associated Euler-Lagrange equation becomes formally a differential inclusion of the form

$$\operatorname{div} \frac{\nabla u}{\sqrt{1+|\nabla u|^2}} \in \begin{cases} \{1\} & \text{in } \{u > g\}, \\ [-1,1] & \text{in } \{u = g\}, \\ \{-1\} & \text{in } \{u < g\}, \end{cases} \tag{1.3}$$

thus, in the sets $\{u > g\}$ and $\{u < g\}$, the subgraph of u has mean curvature equal to 1 and -1, respectively.

Unlike the minimizers of the functional in (1.2), the equation (1.3) may admit nonregular solutions, as observed in [7]. For instance, if $n = 1$, $\Omega = (-1,1)$ and

$$g(s) = \begin{cases} 2 & \text{if } s \in (0,1), \\ -2 & \text{if } s \in (-1,0), \end{cases}$$

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one can readily check that the functions

$$u_{a,b}(s) = \begin{cases} \sqrt{2s-s^2}+a & \text{if } s\in(0,1),\\ 0 & \text{if } s=0,\\ \sqrt{-2s-s^2}+b & \text{if } s\in(-1,0)\end{cases} \tag{1.4}$$

with $-1\le b\le a\le 1$ satisfy (1.3) and minimize $\mathscr{F}$, with $\mathscr{F}(u_{a,b}) = 4+\frac{\pi}{2}$. However, any $u_{a,b}$ is not continuously differentiable at $s=0$, even worse – it has a jump if $a>b$.

To study regularity of minimizers of $\mathscr{F}$, in [7] De Giorgi posed the following conjecture, which seems nontrivial even when $n=k=1$.

Conjecture 1.1. *For any $n,k\ge 1$, there exists $\sigma := \sigma(n,k)>0$ such that for any open ball $B\subset\mathbb{R}^n$ and $g\in L^\infty(B;\mathbb{R}^k)$ with $\|g\|_\infty\le\sigma$ the following minimum is achieved:*

$$\min\Big\{\int_B\sqrt{1+\sum|M_i(\nabla u)|^2}dx+\int_B|u-g|dx:\ u\in C^1(B;\mathbb{R}^k)\Big\}, \tag{1.5}$$

where the sum is taken over all minors $M_i(\nabla u)$ of the Jacobian matrix ∇u of u.

A variation of this conjecture for $n=1$ and $k\ge 1$ has been recently addressed in [5]: using the Sobolev regularity theory for the minimizers of an Ambrosio-Tortorelli-type functional [2], the authors have shown the existence of $\sigma := \sigma(k,|I|^{-1/2})>0$, such that for any $g\in L^\infty(I;\mathbb{R}^k)$ with $\|g\|_\infty\le\sigma$, the minimum problem

$$\min\Big\{\int_I\sqrt{1+|u'|^2}\,ds+\int_I(u-g)^2\,ds:\ u\in C^1(I;\mathbb{R}^k)\Big\} \tag{1.6}$$

admits a unique solution, where I is a bounded interval. This result does not solve Conjecture 1.1, due to the exponent 2 in the second integral of the functional and to the dependence of σ on the length $|I|$ of the interval I. To prove the existence of solutions, they observe that if $u\in W^{1,\infty}(I;\mathbb{R}^k)$ minimizes the Γ-limit F of a suitable sequence of approximating functionals, then it also minimizes the functional in (1.6), which turns out to be Sobolev regular provided that $\|g\|_\infty$ is small enough depending only on $|I|$. Next, they show that u is in fact a solution to the corresponding Euler-Lagrange equation with suitable boundary conditions, which yield the continuity of the derivative (here the quadratic term $(u-g)^2$ is important in the analysis of the Euler-Lagrange equation).

In the present paper we consider $n=k=1$ and generalize the functional in (1.5) to the anisotropic case with L^p-fidelity terms. Given an anisotropy (a norm) φ in $\mathbb{R}^2$, $p\in[1,+\infty)$, a bounded open interval $I\subset\mathbb{R}$ and $g\in L^\infty(I)$, we consider the functional

$$\mathscr{G}(u) = \int_I\varphi^o(-Du,1)+\int_I|u-g|^p ds,\quad u\in L^1(I), \tag{1.7}$$

where φ^o is the dual of φ and

$$\int_I\varphi^o(-Du,1) := \sup\Big\{\int_I(uh_1'+h_2)ds:\ (h_1,h_2)\in C_c^1(I;\mathbb{R}^2),\ \|\varphi(h_1,h_2)\|_\infty\le 1\Big\}$$

is the φ-total variation of $(-Du,\mathcal{L}^1)$ when $u\in BV(I)$. The main result of this paper reads as follows (see also Theorem 5.4).

Theorem 1.2. *Let φ be an anisotropy in $\mathbb{R}^2$ such that the unit ball $W^\varphi := \{\varphi\le 1\}$ is symmetric with respect to the coordinate axes and does not have vertical facets. Let $I\subset\mathbb{R}$ be a bounded open interval. Then there exists $\sigma := \sigma(\varphi,p,|I|)>0$ such that for any $g\in L^\infty(I)$ with $\|g\|_\infty<\sigma$ every minimizer of $\mathscr{G}$ is Lipschitz in I. Additionally, if φ is C^2 out of the origin and elliptic (see Definition 2.1), then minimizers are $C^{1,1}$ in I.*

In the Euclidean case $\varphi = |\cdot|$, Theorem 1.2 provides a positive solution to Conjecture 1.1 for $n=k=1$, except that our σ depends on $|I|$ (as in [5]); at the same time we gain an extra regularity of minimizers.

To prove Theorem 1.2, we begin by observing that if g is bounded, then every minimizer u of $\mathscr{G}$ is also bounded, with $\|u\|_\infty\le\|g\|_\infty$ (see Lemma 5.1). If, additionally, φ is even in each coordinate (equivalently, W^φ is symmetric with respect to the coordinate axes), then the subgraph and the epigraph of u are (γ,Λ)-local minimizers of the φ-perimeter, in the sense of Definition 4.5 below, with suitable constants γ, Λ (Proposition 5.3). Next in Proposition 4.3, we prove that if u satisfies an appropriate L^∞-bound depending only on Λ and $|I|$, a tangent φ-ball condition holds at each point x on the graph of u for all radii r up to $\frac{\alpha_0}{\Lambda} > 0$ for a constant $\alpha_0 > 0$ depending only on φ. The uniformity of the radii of these tangent balls allows to estimate the deviation of the generalized normals of the reduced boundary of the subgraph of u in $I \times \mathbb{R}$ from the vertical direction (see (4.14)) which is away from 0 provided that $\|u\|_\infty$ is small enough depending only on φ, Λ and $|I|$. In particular, this observation and [19, Lemma 3.10] imply that u is Lipschitz in I (Corollary 4.4). Finally, an explicit choice of σ is made, using the previous L^∞-bounds for u (Theorem 5.4). When φ is C^2 and elliptic, then the tangent φ-ball condition becomes equivalent to the classical tangent ball condition, and hence u must be $C^{1,1}$ in I.

Note that the function $u_{a,b}$ in (1.4) shows that in case $\|g\|_\infty$ is large, the validity of a tangent ball condition may not suffice for the regularity of minimizers of $\mathscr{F}$.

When $\Lambda = 0$ and $\gamma = +\infty$, (γ, Λ)-local minimizers coincide with classical local minimizers. In this case, assuming φ is symmetric with respect to the coordinate axes, we can characterize all possible Cartesian local minimizers of the φ-perimeter (see Theorem 3.1).

The paper is organized as follows. In Section 2 we introduce some preliminaries on anisotropies, Λ-local minimizers, and φ-ball condition for Cartesian Λ-local minimizers. In Section 3 we provide a characterization of local minimizers. Some regularity properties of Λ-local minimizers and their further generalizations are studied in Section 4. Finally, we prove Theorem 1.2 in Section 5.

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2. Some preliminaries

In what follows, by $\mathcal{L}^m$ (typically for $m = 1,2$) and $\mathcal{H}^t$ we denote the Lebesgue measure in $\mathbb{R}^m$ and the t-dimensional Hausdorff measure in $\mathbb{R}^2$. Depending on the context, we use $|\cdot|$ to denote the Euclidean norm of a vector in $\mathbb{R}^2$, the length of a bounded interval on $\mathbb{R}$ and the measure of a (Lebesgue) measurable set in $\mathbb{R}^m$ for $m = 1,2$. The scalar product in $\mathbb{R}^2$ is indicated by $\langle \cdot, \cdot \rangle$. The symmetric difference of sets A and B is denoted $A\Delta B$. The symbol $B_r(x)$ stands for the Euclidean ball in $\mathbb{R}^2$ centered at x and of radius $r > 0$. The topological closure, interior and boundary of $E \subset \mathbb{R}^2$ will be denoted by $\overline{E}$, $\mathring{E}$ and ∂E, respectively. Given an open interval $I \subset \mathbb{R}$, we write $\mathscr{O}(I)$ and $\mathscr{O}_b(I)$ to denote the collection of all open and all bounded open subsets of I, respectively.

2.1. Anisotropies.

Let $\varphi : \mathbb{R}^2 \to [0, +\infty)$ be an anisotropy, i.e., a positively one-homogeneous even convex function with

$$c \le \varphi \le \frac{1}{c} \quad \text{on the unit circle } \mathbb{S}^1 \tag{2.1}$$

for some $c \in (0,1]$. We denote by φ^o the dual of φ, defined as

$$\varphi^o(\xi) = \max_{\varphi(\eta)=1} \langle \xi, \eta \rangle,$$

which is also an anisotropy in $\mathbb{R}^2$. We say that φ is a C^k-anisotropy for some $k \ge 1$ provided that $\varphi \in C^k_{\mathrm{loc}}(\mathbb{R}^2 \setminus \{0\})$.

The unit φ-ball $W^\varphi := \{\varphi \le 1\}$ is sometimes called the Wulff shape of φ. We also introduce the Wulff shape of radius r centered at x as $W^\varphi_r(x) := \{\varphi(\cdot - x) \le r\}$; clearly $\mathring{W}^\varphi_r(x) = \{\varphi(\cdot - x) < r\}$. Given $\eta \in \partial W^\varphi$, we call any vector $\nu \in \partial \varphi^o(\eta)$ a *normal* to W^φ at η, where ∂ is the subdifferential. Note that if W^φ is not regular at η, for instance, it has a corner, its set of normals at η forms a nonempty closed convex cone.

We write $\operatorname{dist}_\varphi(x, S) := \inf\{\varphi(x - y) : \ y \in S\}$ to denote the φ-distance function from a nonempty set S.

Definition 2.1 (Elliptic anisotropy)**.** An anisotropy φ in $\mathbb{R}^2$ is *elliptic* provided that there exists $\lambda > 0$ such that $\phi - \lambda|\cdot|$ is also an anisotropy in $\mathbb{R}^2$.

For instance, any anisotropy induced by some positive definite quadratic form, is elliptic. The following proposition can be found in [13, Appendix A] and provides a characterization of elliptic C^2-anisotropies.

Proposition 2.2. *For any C^2-anisotropy the following assertions are equivalent:*
(a) φ is elliptic;
(b) φ^o is C^2 and elliptic;
(c) there exists $\bar r \in (0,1)$ such that for any $z \in \partial W^\varphi$ there exist $x_z, y_z \in \mathbb{R}^2$ such that

$$B_{\bar r}(x_z) \subset W^\varphi \subset \overline{B_{1/\bar r}(y_z)} \qquad \text{and} \qquad \partial B_{\bar r}(x_z) \cap \partial W^\varphi = \partial B_{1/\bar r}(y_z) \cap \partial W^\varphi = \{z\}.$$

Another interesting class of anisotropies is introduced in [4, Section 4]:

Definition 2.3. We say an anisotropy φ is *partially monotone* if

$$(x_1,y_1),(x_2,y_2) \in \mathbb{R}^2, \quad |x_1| \le |x_2|,\ |y_1| \le |y_2| \qquad \Longrightarrow \qquad \varphi(x_1,y_1) \le \varphi(x_2,y_2).$$

According to [4, Appendix A] the following statements are equivalent:

- φ is partially monotone;
- φ^o is partially monotone;
- $\varphi(x_1,x_2) = \varphi(|x_1|,|x_2|)$ for all $x_1,x_2 \in \mathbb{R}$.

Thus, φ is partially monotone if and only if it is even in each coordinates separately. Equivalently, φ is partially monotone if and only if its Wulff shape W^φ is symmetric with respect to the coordinates axes.

2.2. **Anisotropic total variation and perimeter.** Let φ be an anisotropy in $\mathbb{R}^2$ and $I \subseteq \mathbb{R}$ be an open interval. Recall that a function $u : I \to \mathbb{R}$ has locally bounded variation in I, and we write $u \in BV_{\rm loc}(I)$, if its distributional derivative Du is a Radon measure in I. If, additionally, $u \in L^1(I)$ and Du is a bounded Radon measure in I, then u is called a function of bounded variation and is denoted by $u \in BV(I)$.

Given $u \in BV_{\rm loc}(I)$, the anisotropic area of the graph of $u \in BV_{\rm loc}(I)$ is defined by the φ-total variation of the Radon measure $(-Du, \mathcal{L}^1)$ in an open set $J \Subset I$ as

$$\mathscr{A}_\varphi(u,J) := \int_J \varphi^o(-Du,\mathcal{L}^1) := \sup\Big\{ \int_J (uh_1' + h_2)ds :\ (h_1,h_2) \in C_c^1(J;\mathbb{R}^2),\ \|\varphi(h_1,h_2)\|_\infty \le 1 \}.$$

When $u \in W^{1,1}_{\rm loc}(I)$, the Radon-Nikodym theorem implies

$$\mathscr{A}_\varphi(u,J) = \int_J \varphi^o(-u',1)ds,$$

and hence, in the case of the Euclidean anisotropy,

$$\mathscr{A}_{|\cdot|}(u,J) = \int_J \sqrt{1+u'^2}\,ds.$$

Note that [6, p. 390]

$$\begin{aligned}\mathscr{A}_\varphi(u,I) &= \int_I \varphi^o(-u',1)ds + \varphi^o(D^s u,0)(I)\\ &= \int_I \varphi^o(-u',1)ds + \sum_{x\in J_u} \varphi^o(\mathbf{e}_1)|u^+(x) - u^-(x)| + \varphi^o(D^c u,0)(I),\end{aligned} \tag{2.2}$$

where $D^s u$ and $D^c u$ are the singular part of Du with respect to $\mathcal{L}^1$ and the Cantor part respectively, J_u is the jump set of u, $u^\pm(x)$ are the right and left traces of u at x and

$$\varphi^o(\mu,0)(I) = \sup\Big\{ \int_I \eta\, d\mu :\ \eta \in C_c(I),\ \|\varphi(\eta,0)\|_\infty \le 1 \Big\}$$

is the partial φ-total variation of a Radon measure μ in I.

A measurable set $E \subset \mathbb{R}^2$ is called of locally finite perimeter in an open set $\Omega \subseteq \mathbb{R}^2$, and denoted as $E \in BV_{\rm loc}(\Omega;\{0,1\})$, provided that the distributional derivative $D\chi_E$ of its characteristic function χ_E is a Radon measure in Ω. If, additionally, $D\chi_E$ is a bounded Radon measure in Ω, then E has finite perimeter. We denote by $\partial^* E$ and ν_E the reduced boundary and the generalized outer unit normal of E, respectively. If $\chi_E \in BV(\mathbb{R}^2)$, we write $E \in BV(\mathbb{R}^2;\{0,1\})$. We refer for instance to [1, 11, 15, 17] for more information on BV-functions and sets of finite perimeter.

We define the φ-perimeter of E in the open set $\Omega \subseteq \mathbb{R}^2$ as

$$\mathscr{P}_\varphi(E,\Omega) = \int_{\Omega\cap\partial^* E} \varphi^o(\nu_E)\, d\mathcal{H}^1.$$

We also set $\mathscr{P}_\varphi(E) := \mathscr{P}_\varphi(E,\mathbb{R}^2)$.

For a function $u : I \to \mathbb{R}$ we write

$$\mathrm{sg}(u) := \{(s,t) \in I\times\mathbb{R} :\ u(s) > t\} \subset \mathbb{R}^2$$

to denote the (strict) subgraph of u (sometimes called hypograph). There is a natural connection between the anisotropic area of the graph and the anisotropic perimeter of the subgraph.

Lemma 2.4 ([6]). *$u \in BV_{\rm loc}(I)$ if and only if $\mathrm{sg}(u)$ has locally finite perimeter in $I \times \mathbb{R}$. Moreover, in either case, for any $J \Subset I$,*

$$\mathscr{A}_\varphi(u,J) = \mathscr{P}_\varphi(\mathrm{sg}(u), J\times\mathbb{R}).$$

2.3. **Local minimizers.** In this section we recall the notion of Λ-local minimizer.

Definition 2.5. Let φ be an anisotropy on $\mathbb{R}^2$ and $\Lambda \geq 0$.

- We call a function $u \in BV_{\rm loc}(I)$ a Λ-local minimizer of $\mathscr{A}_\varphi$ in I if

$$\mathscr{A}_\varphi(u,J) \leq \mathscr{A}_\varphi(u+\psi,J) + \Lambda\int_J |\psi|\, ds$$

  whenever $J \in \mathscr{O}_b(I)$ and $\psi \in C^1_c(J)$.
- For an open set $\Omega \subseteq \mathbb{R}^2$ and $\Lambda \geq 0$, we call a set $E \in BV_{\rm loc}(\Omega;\{0,1\})$ a Λ-local minimizer of $\mathscr{P}_\varphi$ in Ω if

$$\mathscr{P}_\varphi(E,\Omega') \leq \mathscr{P}_\varphi(F,\Omega') + \Lambda|E\Delta F|$$

  for any open $\Omega' \Subset \Omega$ and $F \in BV_{\rm loc}(\Omega;\{0,1\})$ with $E\Delta F \Subset \Omega'$.

When $\Lambda = 0$, following the literature, we shortly call u (resp. E) a local minimizer.

By approximation, one can show that $u \in BV_{\rm loc}(I)$ is a Λ-local minimizer of $\mathscr{A}_\varphi$ in I if and only if

$$\mathscr{A}_\varphi(u,J) \leq \mathscr{A}_\varphi(v,J) + \Lambda\int_J |u-v|\, ds$$

whenever $J \in \mathscr{O}_b(I)$ and $v \in BV_{\rm loc}(I)$ with $\mathrm{supp}\,(u-v) \Subset J$.

Remark 2.6. If I is bounded and $u \in L^\infty(I)\cap BV_{\rm loc}(I)$ is a Λ-local minimizer of $\mathscr{A}_\varphi$ in I, then $u \in BV(I)$. Indeed, for any open interval $J \Subset I$ consider the test function $v = u\chi_{I\setminus J}$. Then

$$\mathscr{A}_\varphi(u,J) \leq \varphi(\mathbf{e}_2)|J| + 4\|u\|_\infty + 2\Lambda\|u\|_\infty|J|.$$

Now letting $J \nearrow I$ we find $\mathscr{A}_\varphi(u,I) < +\infty$ and hence $u \in BV(I)$. In particular, the traces of u on ∂I are well-defined.

These two notions are linked as follows.

Proposition 2.7. *Let $u \in BV_{\rm loc}(I)$.*

- *If the subgraph $\mathrm{sg}(u)$ is a Λ-local minimizer of $\mathscr{P}_\varphi$ in $I\times\mathbb{R}$, then u is a Λ-local minimizer of $\mathscr{A}_\varphi$ in I.*
- *If φ is partially monotone and u is a Λ-local minimizer of $\mathscr{A}_\varphi$ in I, then $\mathrm{sg}(u)$ is a Λ-local minimizer of $\mathscr{P}_\varphi$ in $I\times\mathbb{R}$.*

At the moment we do not have any explicit example showing the necessity of partial monotonicity in the second assertion of the proposition.

Proof. Let $\mathrm{sg}(u)$ be a Λ-local minimizer of $\mathscr{P}_\varphi$ in $I\times\mathbb{R}$, and fix $J\in\mathscr{O}_b(I)$ and $\psi\in C_c^1(J)$. Then $\mathrm{sg}(u)\Delta\mathrm{sg}(u+\psi)\Subset J\times\mathbb{R}$ and hence, for any bounded open set $\Omega'\Subset I\times\mathbb{R}$ compactly containing $\mathrm{sg}(u)\Delta\mathrm{sg}(u+\psi)$ we have

$$\mathscr{P}_\varphi(\mathrm{sg}(u),J\times\mathbb{R})-\mathscr{P}_\varphi(\mathrm{sg}(u+\psi),J\times\mathbb{R})=\mathscr{P}_\varphi(\mathrm{sg}(u),\Omega')-\mathscr{P}_\varphi(\mathrm{sg}(u+\psi),\Omega')\le\Lambda|\mathrm{sg}(u)\Delta\mathrm{sg}(u+\psi)|.$$

By Lemma 2.4 and the equality

$$\int_J|u-v|ds=|[\mathrm{sg}(u)\Delta\mathrm{sg}(v)]\cap[J\times\mathbb{R}]|,$$

the inequality above is equivalent to

$$\mathscr{A}_\varphi(u,J)-\mathscr{A}_\varphi(u+\psi,J)\le\Lambda\int_J|(u+\psi)-u|ds=\Lambda\int_J|\psi|ds,$$

and hence u is a Λ-local minimizer of $\mathscr{A}_\varphi$ in I.

Conversely, assume that φ is partially monotone and u is a Λ-local minimizer of $\mathscr{A}_\varphi$ in I. Let $F\in BV_{\mathrm{loc}}(I\times\mathbb{R};\{0,1\})$ be such that $\mathrm{sg}(u)\Delta F\Subset J\times(a,b)\Subset I\times\mathbb{R}$ for some $J\in\mathscr{O}_b(I)$ and $a,b\in\mathbb{R}$. Let v be the function, whose subgraph is the vertical rearrangement of F, i.e.,

$$v(s)=a+\mathcal{H}^1(\{x_2\in(a,b):\,(s,x_2)\in F\}),\quad s\in I.$$

Note that by the definition of the rearrangement, for a.e. $s\in I$, $v(s)$ satisfies

$$|u(s)-v(s)|=\mathcal{H}^1((\mathrm{sg}(u)\Delta F)\cap\{x_1=s\})$$

so that by the Fubini-Tonelli theorem,

$$\int_{J'}|u-v|ds=\int_{J'}\mathcal{H}^1\big((\mathrm{sg}(u)\Delta F)\cap\{x_1=s\}\big)ds=|(\mathrm{sg}(u)\Delta F)\cap(J'\times\mathbb{R})|\quad\text{for any } J'\in\mathscr{O}_b(I).\tag{2.3}$$

Repeating the same arguments of [4, Section 4] (see also [18]) we can show that $v\in BV^1_{\mathrm{loc}}(I)$, $\mathrm{supp}\,(u-v)\Subset J$,

$$\mathcal{L}^1(J')\le\int_{J'\times\mathbb{R}}|\langle D\chi_F,\mathbf{e}_2\rangle|\quad\text{and}\quad\int_{J'}|Dv|\le\int_{J'\times\mathbb{R}}|\langle D\chi_F,\mathbf{e}_1\rangle|\quad\text{for any } J'\in\mathscr{O}_b(I),\tag{2.4}$$

where the Radon measures $-\langle D\chi_F,\mathbf{e}_1\rangle$ and $-\langle D\chi_F,\mathbf{e}_2\rangle$ are the horizontal and vertical components of $D\chi_F$, which coincide with $\langle\nu_F,\mathbf{e}_1\rangle\,\mathcal{H}^1\llcorner\partial^*F$ and $\langle\nu_F,\mathbf{e}_2\rangle\,\mathcal{H}^1\llcorner\partial^*F$, respectively. Since φ^o is partially monotone, by (2.4)

$$\begin{aligned}\mathscr{A}_\varphi(v,J)=\int_{J'}\varphi^o(-Dv,1)=\int_{J'}\varphi^o(|Dv|,\mathcal{L}^1)\le\int_{J'\times\mathbb{R}}\varphi^o\big(|\langle D\chi_F,\mathbf{e}_1\rangle|,|\langle D\chi_F,\mathbf{e}_2\rangle|\big)\\=\int_{J'\times\mathbb{R}}\varphi^o\big(\langle D\chi_F,\mathbf{e}_1\rangle,\langle D\chi_F,\mathbf{e}_2\rangle\big)=\int_{J'\times\mathbb{R}}\varphi^o(D\chi_F)=\mathscr{P}_\varphi(F,J'\times\mathbb{R})\end{aligned}\tag{2.5}$$

for all $J'\in\mathscr{O}_b(I)$.

Now, by Lemma 2.4 and the Λ-local minimality of u,

$$\mathscr{P}_\varphi(\mathrm{sg}(u),J\times\mathbb{R})-\mathscr{P}_\varphi(\mathrm{sg}(v),J\times\mathbb{R})=\mathscr{A}_\varphi(u,J)-\mathscr{A}_\varphi(v,J)\le\Lambda\int_J|u-v|ds.\tag{2.6}$$

Applying (2.3) and (2.5) with $J'=J$ and recalling that $\mathrm{sg}(u)\Delta F\Subset J\times(a,b)$, from (2.6) we conclude

$$\begin{aligned}\mathscr{P}_\varphi(\mathrm{sg}(u),J\times(a,b))-\mathscr{P}_\varphi(F,J\times(a,b))=&\mathscr{P}_\varphi(\mathrm{sg}(u),J\times\mathbb{R})-\mathscr{P}_\varphi(F,J\times\mathbb{R})\\\le&\mathscr{P}_\varphi(\mathrm{sg}(u),J\times\mathbb{R})-\mathscr{P}_\varphi(\mathrm{sg}(v),J\times\mathbb{R})\le\Lambda|\mathrm{sg}(u)\Delta F|.\end{aligned}$$

Thus, by definition, $\mathrm{sg}(u)$ is a Λ-local minimizer of $\mathscr{P}_\varphi$ in $I\times\mathbb{R}$. □

Note that if φ is partially monotone, then $\mathscr{A}_\varphi(u,\cdot)=\mathscr{A}_\varphi(-u,\cdot)$, and hence u is Λ-local minimizer if and only if so is $-u$. Thus, from Proposition 2.7 we get the following corollary.

Corollary 2.8. *Let $u\in BV_{\mathrm{loc}}(I)$. For any partially monotone anisotropy φ, the following assertions are equivalent:*

- *u is a Λ-local minimizer of $\mathscr{A}_\varphi$ in I;*
- *$-u$ is a Λ-local minimizer of $\mathscr{A}_\varphi$ in I;*
- *the subgraph $\mathrm{sg}(u)$ of u is a Λ-local minimizer of $\mathscr{P}_\varphi$ in $I\times\mathbb{R}$;*
- *the (strict) epigraph $\mathrm{epi}(u) := \{(s,t)\in I\times\mathbb{R}:\ u(s)<t\}$ is a Λ-local minimizer of $\mathscr{P}_\varphi$ in $I\times\mathbb{R}$.*

2.4. **Density estimates.** The proof of the next lemma is well-known in the literature (see e.g. [8, 13]) and can be proven, for instance, using the filling-in or cutting-out with balls.

Lemma 2.9. *Given an anisotropy φ, $\Lambda\ge 0$ and an open set $\Omega\subseteq\mathbb{R}^2$, let $E\in BV_{\mathrm{loc}}(\Omega;\{0,1\})$ be a Λ-local minimizer of $\mathscr{P}_\varphi$ in Ω. Assume that $E=E^{(1)}$, i.e., E coincides with its Lebesgue points. Then for any $\Omega'\Subset\Omega$ there exist constants $r_0 := r_0(\varphi,\Lambda,\mathrm{dist}(\partial\Omega',\partial\Omega))>0$ and $q_0 := q_0(\varphi,\Lambda)\in(0,1/2)$ such that*

$$P(E,B_r(x)) \le \frac{r}{q_0},\quad x\in\Omega',\ r\in(0,r_0),$$

$$q_0 \le \frac{|E\cap B_r(x)|}{|B_r(x)|} \le 1-q_0,\quad x\in\partial E,\ r\in(0,r_0),$$

and

$$P(E,B_r(x)) \ge q_0 r,\quad x\in\partial E,\ r\in(0,r_0).$$

From Lemma 2.9 and a covering argument we immediate deduce that every Λ-local minimizer $E=E^{(1)}$ in Ω satisfies

$$\partial E = \overline{\partial^* E},\quad \mathcal{H}^1\big(\Omega'\cap(\partial E\setminus\partial^*E)\big)=0 \quad\text{and}\quad \mathcal{H}^1\big(\Omega'\cap(\overline{E}\setminus\mathring{E})\big)<+\infty \quad\text{for any open } \Omega'\Subset\Omega. \tag{2.7}$$

In particular, possibly changing a negligible set, E can be assumed open or closed.

Remark 2.10. Any Λ-local minimizer E of $\mathscr{P}_\varphi$ in Ω satisfies

$$\mathscr{P}_\varphi(E,B_\rho(x)) \le \mathscr{P}_\varphi(F,B_\rho(x)) + \Lambda\sqrt{\pi}\rho\sqrt{|E\Delta F|}$$

whenever $x\in\partial E$, $B_\rho(x)\Subset\Omega$ and $E\Delta F\Subset B_\rho(x)$. Thus, E is ω-minimal in the sense of [20] with $\omega(\rho)=\Lambda\sqrt{\pi}\rho$. In particular, by [20, Theorem 3.4], the set Σ of all points $x\in\Omega\cap\partial E$ around which $\Omega\cap\partial E$ is not a Lipschitz graph is discrete and is empty if W^φ is not a quadrilateral[1].

3. Classification of local minimizers

In this section we classify the minimizers of $\mathscr{A}_\varphi$, i.e., study functions $u\in BV_{\mathrm{loc}}(I)$ satisfying $\mathscr{A}_\varphi(u,J)\le\mathscr{A}_\varphi(v,J)$ for any open set $J\Subset I$ and $v\in BV_{\mathrm{loc}}(I)$ with $\mathrm{supp}\,(u-v)\Subset J$.

Theorem 3.1 (Characterization of local minimizers)**.** *Let φ be a partially monotone anisotropy, $I\subseteq\mathbb{R}$ be an interval and $u\in BV_{\mathrm{loc}}(I)$. Let $\Gamma_u := (I\times\mathbb{R})\cap\overline{\partial^*\mathrm{sg}(u)}$ be the generalized graph of u and $\nu_{\mathrm{sg}(u)}:\Gamma_u\to\mathbb{S}^1$ be the unit normal field, outer to $(I\times\mathbb{R})\cap\mathrm{sg}(u)$, defined $\mathcal{H}^1$-a.e. on Γ_u. Then u is a local minimizer of $\mathscr{A}_\varphi$ in I if and only if there exists a vector $N\in\mathbb{R}^2$ such that*

$$\varphi(N)=1 \qquad \text{and} \qquad \big\langle N,\nu_{\mathrm{sg}(u)}\big\rangle = \varphi^o(\nu_{\mathrm{sg}(u)}) \quad \mathcal{H}^1\text{-a.e. on } \Gamma_u. \tag{3.1}$$

Moreover, u is monotone in I.

In the literature the vector N satisfying (3.1) is sometimes called a Cahn-Hoffman vector field associated to the rectifiable curve Γ_u.

[1]In fact, ω-minimal sets are defined for any anisotropy, not necessarily even and [20, Theorem 3.4] shows that in general Σ is discrete. Moreover, if W^φ is neither a triangle nor a quadrilateral, then Σ is empty.

Proof. We expect this result to be well-known in the literature; for completeness we provide the proof.

$\Rightarrow$. We apply a calibration argument as in [4, Example 2.4]. Assume that there exists a vector N satisfying (3.1) and let $F \in BV_{\rm loc}(I\times\mathbb{R};\{0,1\})$ be such that $F\Delta {\rm sg}(u) \Subset J\times\mathbb{R}$ for some $J \Subset I$. Then

$$\mathscr{P}_\varphi(F,J\times\mathbb{R}) = \int_{(J\times\mathbb{R})\cap\partial^*F} \varphi^o(\nu_F)\,d\mathcal{H}^1 \ge \int_{(J\times\mathbb{R})\cap\partial^*F} \langle \nu_F, N\rangle\, d\mathcal{H}^1.$$

On the other hand, by the divergence theorem

$$0 = \int_{F\setminus{\rm sg}(u)} {\rm div} N\,dx - \int_{{\rm sg}(u)\setminus F} {\rm div} N\,dx = \int_{(J\times\mathbb{R})\cap\partial^*F} \langle \nu_F, N\rangle\, d\mathcal{H}^1 - \int_{(J\times\mathbb{R})\cap\partial^*{\rm sg}(u)} \left\langle \nu_{{\rm sg}(u)}, N\right\rangle d\mathcal{H}^1,$$

and thus

$$\int_{(J\times\mathbb{R})\cap\partial^*F} \langle \nu_F, N\rangle\, d\mathcal{H}^1 = \int_{(J\times\mathbb{R})\cap\partial^*{\rm sg}(u)} \left\langle \nu_{{\rm sg}(u)}, N\right\rangle d\mathcal{H}^1 = \int_{(J\times\mathbb{R})\cap\partial^*{\rm sg}(u)} \varphi^o(\nu_{{\rm sg}(u)})\,d\mathcal{H}^1 = \mathscr{P}_\varphi({\rm sg}(u),J\times\mathbb{R}).$$

Hence ${\rm sg}(u)$ is a local minimizer of $\mathscr{P}_\varphi$ in $I\times\mathbb{R}$. Then Corollary 2.8 implies that u is a local minimizer of $\mathscr{A}_\varphi$ in I.

$\Leftarrow$. Assume that u is a local minimizer of $\mathscr{A}_\varphi$ in I. Since $|Du|(J) < +\infty$ for any open interval $J \Subset I$, we have $u \in L^\infty(J)$. In particular, $u \in BV_{\rm loc}(I)$.

Let $J \Subset I$ be any interval, whose boundary points are not on the jump set of u; such an interval exists because u has at most countably many jumps. Let v be the function such that $v = u$ in $I\setminus J$ and linear in J such that the traces of u and v on ∂J coincide. By the local boundedness of u, ${\rm sg}(u)\Delta{\rm sg}(v) \Subset I\times\mathbb{R}$. Then by the local minimality of u and the anisotropic minimality of segments [10],

$$\mathscr{P}_\varphi({\rm sg}(u),J\times\mathbb{R}) \ge \mathscr{P}_\varphi({\rm sg}(v),J\times\mathbb{R}) \ge \mathscr{P}_\varphi({\rm sg}(u),J\times\mathbb{R}),$$

and hence $\mathscr{P}_\varphi({\rm sg}(u),J\times\mathbb{R}) = \mathscr{P}_\varphi({\rm sg}(v),J\times\mathbb{R})$. Choose a $N\in\mathbb{R}^2$ satisfying $\varphi(N) = 1$ and $\left\langle \nu_{[p,q]}, N\right\rangle = \varphi^o(\nu_{[p,q]})$ on $[p,q]$. As above, by the divergence formula

$$0 = \int_{{\rm sg}(v)\setminus{\rm sg}(u)} {\rm div} N\,dx - \int_{{\rm sg}(u)\setminus{\rm sg}(v)} {\rm div} N\,dx = \int_{(J\times\mathbb{R})\cap\partial^*{\rm sg}(v)} \langle \nu_F, N\rangle\, d\mathcal{H}^1 - \int_{(J\times\mathbb{R})\cap\partial^*{\rm sg}(u)} \left\langle \nu_{{\rm sg}(u)}, N\right\rangle d\mathcal{H}^1.$$

Thus,

$$\begin{aligned}\mathscr{P}_\varphi({\rm sg}(v),J\times\mathbb{R}) = \mathscr{P}_\varphi({\rm sg}(u),J\times\mathbb{R}) &= \int_{(J\times\mathbb{R})\cap\partial^*{\rm sg}(u)} \varphi^o(\nu_{{\rm sg}(u)})d\mathcal{H}^1 \ge \int_{(J\times\mathbb{R})\cap\partial^*{\rm sg}(u)} \langle \nu_F, N\rangle\, d\mathcal{H}^1\\ &= \int_{(J\times\mathbb{R})\cap\partial^*{\rm sg}(v)} \left\langle \nu_{{\rm sg}(v)}, N\right\rangle d\mathcal{H}^1 = \int_{(J\times\mathbb{R})\cap\partial^*{\rm sg}(v)} \varphi^o(\nu_{{\rm sg}(v)})\,d\mathcal{H}^1 = \mathscr{P}_\varphi({\rm sg}(v),J\times\mathbb{R}),\end{aligned} \tag{3.2}$$

where in the fourth equality we used that v is linear in J. Thus, all inequalities in (3.2) are in fact equalities. Since $\varphi^o(\nu_{{\rm sg}(u)}) \ge \left\langle \nu_{{\rm sg}(u)}, N\right\rangle$ $\mathcal{H}^1$-a.e. on $(J\times\mathbb{R})\cap\partial^*{\rm sg}(u)$ and

$$\int_{(J\times\mathbb{R})\cap\partial^*{\rm sg}(u)} \varphi^o(\nu_{{\rm sg}(u)})d\mathcal{H}^1 = \int_{(J\times\mathbb{R})\cap\partial^*{\rm sg}(u)} \langle \nu_F, N\rangle\, d\mathcal{H}^1,$$

from the Chebyshev inequality it follows that $\varphi^o(\nu_{{\rm sg}(u)}) = \langle \nu_F, N\rangle$ $\mathcal{H}^1$-a.e. on $(J\times\mathbb{R})\cap\partial^*{\rm sg}(u)$. Now, consider a sequence $J_k \nearrow I$ of open relatively compact intervals and the associated constant vectors $N_k \in \partial W^\varphi$. Notice that each N_k satisfies

$$\varphi(N_k) = 1 \qquad \text{and} \qquad \varphi^o(\nu_{{\rm sg}(u)}) = \langle \nu_F, N_k\rangle \quad \mathcal{H}^1\text{-a.e. on } (J_k\times\mathbb{R})\cap\partial^*{\rm sg}(u). \tag{3.3}$$

Since ∂W^φ is compact, there is no loss of generality in assuming $N_k \to N$ for some $N\in\partial W^\varphi$. Note that, given $\bar k\in\mathbb{N}$, all N_k with $k\ge\bar k$ satisfy (3.3) in $J_{\bar k}$. Since N_k appear linearly in the second relation of (3.3), it follows that any vector in the closed convex hull $K_{\bar k}$ of $\cup_{k\ge\bar k}N_k$ also satisfies (3.3). Clearly, N belongs to $K_{\bar k}$ for all $\bar k$. As $J_k\nearrow I$, it follows that N satisfies (3.1).

Finally, let us show that u is monotone, i.e., it admits a monotone representative. Indeed, suppose that there exist $(a,b) \Subset I$ and $t \in \mathbb{R}$ such that $(a,t),(b,t) \in \Gamma_u$. Let us define the competitor $v = u\chi_{I\setminus(a,b)} + t\chi_{(a,b)}$. By the local minimality of u, for any open interval J with $(a,b) \Subset J \Subset I$ we have

$$0 \le \mathscr{A}_\varphi(v,J) - \mathscr{A}_\varphi(u,J) = \int_{(a,b)} \Big(\varphi^o(0,1)d\mathcal{L}^1 - \varphi^o(-Du,1)\Big) \\ + \varphi^o(\mathbf{e}_1)\Big(|v^+(a)-v^-(a)| - |u^+(a)-u^-(a)| + |v^+(b)-v^-(b)| - |u^+(b)-u^-(b)|\Big), \quad (3.4)$$

where in the equality we used (2.2). By the definition of v and the choice of t, $u^-(a) = v^-(a)$, $u^+(b) = v^+(b)$, $v^+(a) = v^-(b) = t$ and

$$|v^+(a)-v^-(a)| \le |u^+(a)-u^-(a)|, \qquad |v^+(b)-v^-(b)| \le |u^+(b)-u^-(b)|.$$

Moreover, by the partial monotonicity of φ^o we have

$$\int_a^b \varphi^o(-u',1)ds \ge \int_a^b \varphi^o(0,1)ds,$$

and hence, by (3.4) and (2.2) we have

$$0 \le \varphi^o(\mathbf{e}_1)\Big(|u^+(a)-u^-(a)| - |v^+(a)-v^-(a)| + |u^+(b)-u^-(b)| - |v^+(b)-v^-(b)|\Big) \\ \le \int_a^b \varphi^o(0,1)ds - \int_a^b \varphi^o(-u',1)ds - \varphi^o(D^su,0)(a,b) \le 0.$$

Thus, all inequalities are in fact equalities, $u^+(a) = u^-(b) = t$, $u' = 0$ a.e. in (a,b) and $D^su = 0$. This implies $u = v$ in (a,b). This observation shows that for any $\lambda \in \mathbb{R}$ the set $\{u = \lambda\}$ is either empty, or one point or an interval. Therefore, u is monotone. □

Example 3.2 (Strictly convex anisotropies)**.** Assume that φ^o is strictly convex, i.e.,

$$\varphi^o(x+y) < \varphi^o(x) + \varphi^o(y) \quad \text{whenever } |x| = |y| \text{ with } x \ne \pm y.$$

Then for any interval $I \subseteq \mathbb{R}$, the function $u \in BV_{\rm loc}(I)$ is a local minimizer of $\mathscr{A}_\varphi$ if and only if u is linear. Indeed, by the strict convexity of φ, for any $N \in \partial W^\varphi$ there exists a unique $\nu \in \mathbb{S}^1$ such that $\langle N, \nu\rangle = \varphi^o(\nu)$. Thus, by Theorem 3.1 u is a local minimizer of $\mathscr{A}_\varphi$ in I if any only if Γ_u admits a constant unit normal $\mathcal{H}^1$-a.e., which is equivalent to say that u is linear.

Example 3.3 (Square anisotropy)**.** Let $W^\varphi = [-1,1]^2$ and $I \subseteq \mathbb{R}$ be an interval, Then u is a local minimizer of $\mathscr{A}_\varphi$ if and only if u is monotone. Indeed, by Theorem 3.1 every local minimizer is monotone. Conversely, consider any nondecreasing function $u : I \to \mathbb{R}$. By monotonicity, the unit normals ν_u to Γ_u lie in the smaller closed arc of $\mathbb{S}^1$ between $-\mathbf{e}_1$ (jump part) and $\mathbf{e}_2$ (constant part). Thus, any constant vector $N = (-1,1) \in \partial W^\varphi$ satisfies

$$\langle N, \nu_u\rangle = |\langle \nu_u, \mathbf{e}_1\rangle| + |\langle \nu_u, \mathbf{e}_2\rangle| = \varphi^o(\nu_u) \quad \mathcal{H}^1\text{-a.e. on } \Gamma_u.$$

Hence, by Theorem 3.1, u is a local minimizer.

Example 3.4 (Lens-shaped anisotropies)**.** Given $a > 0$, let $\gamma \in C^1([-a,0])$ be a strictly increasing concave function with $\gamma(-a) = 0$ and $\gamma'(0) = 0$. Let W^φ be the convex set symmetric with respect to the coordinate axes such that $((-\infty,0)\times(0,+\infty)) \cap \partial W^\varphi$ is the graph of γ. Let $I \subseteq \mathbb{R}$ be an interval. Then $u \in BV_{\rm loc}(I)$ is a local minimizer of $\mathscr{A}_\varphi$ in I if any only if either u is linear, or u is monotone and piecewise linear, and all segments/half-lines of its graph are tangent[2] to W^φ at exactly one of the two points $\pm\frac{\mathbf{e}_1}{\varphi(\mathbf{e}_1)}$.

[2] I.e., their normal belongs to $\partial\varphi(\mathbf{e}_1)$.

4. Regularity of Λ-minimizers

Now consider the case $\Lambda > 0$. In this case a general characterization of Λ-local minimizers as in Theorem 3.1 seems not available. In this section, under some assumptions of φ, we show that if the L^∞-norm of a Λ-minimizer of $\mathscr{A}_\varphi$ in I is sufficiently small, then u is Lipschitz in I.

Theorem 4.1 (Regularity of Λ-minimizers)**.** *Let φ be a partially monotone anisotropy such that W^φ does not have vertical facets (so that $\pm\mathbf{e}_1$ is an "outer normal" to W^φ only at $\frac{\pm\mathbf{e}_1}{\varphi(\mathbf{e}_1)}$). Given a bounded open interval $I \subset \mathbb{R}$ and $\Lambda > 0$, let $u \in BV_{\rm loc}(I)$ be a Λ-local minimizer of $\mathscr{A}_\varphi$ in I satisfying*

$$\|u\|_\infty < \min\Big\{\tfrac{\alpha_0\varphi(\mathbf{e}_1)}{4\Lambda}, \tfrac{\alpha_0}{2\Lambda\varphi(\mathbf{e}_2)}, \tfrac{|I|\varphi(\mathbf{e}_1)}{4\Lambda\varphi(\mathbf{e}_2)}\Big\}, \tag{4.1}$$

with

$$\alpha_0 = \alpha_0(\varphi) := \tfrac{\mathscr{P}_\varphi(W^\varphi)}{(2\mathscr{P}_\varphi(W^\varphi)+1)\sqrt{|W^\varphi|}} > 0. \tag{4.2}$$

Then u is Lipschitz in I. Moreover, if φ is C^2 and elliptic, then $u \in C^{1,1}(I)$, that is, u is continuously differentiable and its derivative u' is Lipschitz in I.

Note that every partially monotone elliptic anisotropy satisfies the assumption of the theorem. Moreover, by Remark 2.6, $u \in BV(I)$. Furthermore, by the partial monotonicity of φ and Corollary 2.8, the subgraph and epigraph of u are Λ-local minimizers of $\mathscr{P}_\varphi$ in $I\times\mathbb{R}$. Notice also that when W^φ is not a quadrilateral, Remark 2.10 ensures that the boundaries of these sets in $I\times\mathbb{R}$ are locally given by a Lipschitz graph. However, this graphicality property of ${\rm sg}(u)$ and ${\rm epi}(u)$ does not yield that u itself is Lipschitz; for instance, u may have jump discontinuities (see the function $u_{a,b}$ in (1.4)).

The proof of Theorem 4.1 is postponed to the end of the section after some ancillary results. We start with the following property of Wulff shapes.

Lemma 4.2. *Let a bounded $D \in BV(\mathbb{R}^2;\{0,1\})$ and a Wulff shape W_r^φ with $r > 0$ be such that $W_r^\varphi \cap D = \emptyset$ and $\mathcal{H}^1(\partial^*D\cap\partial W_r^\varphi) > 0$. Then*

$$\int_{\partial^*D\setminus\partial W_r^\varphi}\varphi^o(\nu_D)\,d\mathcal{H}^1 - \int_{\partial^*D\cap\partial W_r^\varphi}\varphi^o(\nu_D)\,d\mathcal{H}^1 \ge \alpha_0\min\Big\{\sqrt{|D|}, \frac{|D|}{r}\Big\}, \tag{4.3}$$

where $\alpha_0 > 0$ is given in (4.2).

Proof. The proof is similar to [10, Proposition 6.3]. Recall that the isoperimetric inequality (see e.g. [9]) says

$$\mathscr{P}_\varphi(E) \ge c_\varphi\sqrt{|E|},\quad E \in BV(\mathbb{R}^2;\{0,1\}), \tag{4.4}$$

where $c_\varphi := \frac{\mathscr{P}_\varphi(W^\varphi)}{\sqrt{|W^\varphi|}}$, and the equality in (4.4) holds if and only if $E = x + rW^\varphi = W_r^\varphi(x)$ for some $x \in \mathbb{R}^2$ and $r \ge 0$. When φ is Euclidean, $c_\varphi = \sqrt{4\pi}$.

Fix any $\alpha > 1$. First consider the case

$$\int_{\partial^*D\setminus\partial W_r^\varphi}\varphi^o(\nu_D)\,d\mathcal{H}^1 \le \alpha\int_{\partial^*D\cap\partial W_r^\varphi}\varphi^o(\nu_D)\,d\mathcal{H}^1 \tag{4.5}$$

Since $W_r^\varphi\cap D = \emptyset$, by [15, Theorem 16.3] one has $\partial^*(D\cup W_r^\varphi) \approx_{\mathcal{H}^1} [\partial^*D\setminus W_r^\varphi]\cup[\partial^*W_r^\varphi\setminus\partial^*D]$, where $A \approx_\mu B$ stands for $\mu(A\Delta B) = 0$. Therefore,

$$\begin{aligned}\int_{\partial^*D\setminus\partial W_r^\varphi}\varphi^o(\nu_D)\,d\mathcal{H}^1 - \int_{\partial^*D\cap\partial W_r^\varphi}\varphi^o(\nu_D)\,d\mathcal{H}^1 = \int_{\partial^*(D\cup W_r^\varphi)}\varphi^o(\nu_{D\cup W_r^\varphi})\,d\mathcal{H}^1 - \int_{\partial W_r^\varphi}\varphi^o(\nu_{W_r^\varphi})\,d\mathcal{H}^1\\ = \mathscr{P}_\varphi(D\cup W_r^\varphi) - \mathscr{P}_\varphi(W_r^\varphi) \ge c_\varphi\Big(\sqrt{|D\cup W_r^\varphi|} - \sqrt{|W_r^\varphi|}\Big) = \tfrac{c_\varphi|D|}{\sqrt{|D\cup W_r^\varphi|}+\sqrt{|W_r^\varphi|}},\end{aligned} \tag{4.6}$$

where in the first inequality we used the isoperimetric inequality (4.4). Moreover, by (4.5) and again by the isoperimetric inequality and the assumption $D\cap W_r^\varphi=\emptyset$,

$$\begin{aligned}c_\varphi\sqrt{|D\cup W_r^\varphi|}\le P_\varphi(D\cup W_r^\varphi)=&\int_{\partial^*D\setminus\partial W_r^\varphi}\varphi^o(\nu_D)\,d\mathcal{H}^1+\int_{\partial^*W_r^\varphi\setminus\partial D}\varphi^o(\nu_{W_r^\varphi})\,d\mathcal{H}^1\\ \le&\,\alpha\int_{\partial^*D\cap\partial W_r^\varphi}\varphi^o(\nu_D)\,d\mathcal{H}^1+\int_{\partial^*W_r^\varphi\setminus\partial D}\varphi^o(\nu_{W_r^\varphi})\,d\mathcal{H}^1\le\alpha\int_{\partial^*W_r^\varphi}\varphi^o(\nu_{W_r^\varphi})d\mathcal{H}^1=\alpha\mathscr{P}_\varphi(W_r^\varphi).\end{aligned}$$

Thus, recalling $c_\varphi\sqrt{|W_r^\varphi|}=\mathscr{P}_\varphi(W_r^\varphi)$, from (4.6) we get

$$\int_{\partial^*D\setminus\partial W_r^\varphi}\varphi^o(\nu_D)\,d\mathcal{H}^1-\int_{\partial^*D\cap\partial W_r^\varphi}\varphi^o(\nu_D)\,d\mathcal{H}^1\ge\frac{c_\varphi|D|}{(\alpha+1)\mathscr{P}_\varphi(W_r^\varphi)}=\frac{|D|}{(\alpha+1)r\sqrt{|W^\varphi|}}.\tag{4.7}$$

Now consider the case

$$\int_{\partial^*D\setminus\partial W_r^\varphi}\varphi^o(\nu_D)\,d\mathcal{H}^1>\alpha\int_{\partial^*D\cap\partial W_r^\varphi}\varphi^o(\nu_D)\,d\mathcal{H}^1\tag{4.8}$$

so that

$$\int_{\partial^*D\setminus\partial W_r^\varphi}\varphi^o(\nu_D)\,d\mathcal{H}^1-\int_{\partial^*D\cap\partial W_r^\varphi}\varphi^o(\nu_D)\,d\mathcal{H}^1>\Big(1-\frac{1}{\alpha}\Big)\int_{\partial^*D\setminus\partial W_r^\varphi}\varphi^o(\nu_D)\,d\mathcal{H}^1.\tag{4.9}$$

Then by the isoperimetric inequality, (4.9) and (4.8) we get

$$\begin{aligned}c_\varphi\,|D|^{1/2}\le&\mathscr{P}_\varphi(D)=\int_{\partial^*D\setminus\partial W_r^\varphi}\varphi^o(\nu_D)\,d\mathcal{H}^1+\int_{\partial^*D\cap\partial W_r^\varphi}\varphi^o(\nu_D)\,d\mathcal{H}^1\\ \le&\Big(1+\frac{1}{\alpha}\Big)\int_{\partial^*D\setminus\partial W_r^\varphi}\varphi^o(\nu_D)\,d\mathcal{H}^1\le\frac{\alpha+1}{\alpha-1}\Big(\int_{\partial^*D\setminus\partial W_r^\varphi}\varphi^o(\nu_D)\,d\mathcal{H}^1-\int_{\partial^*D\cap\partial W_r^\varphi}\varphi^o(\nu_D)\,d\mathcal{H}^1\Big).\end{aligned}$$

Combining this inequality with (4.7) we deduce

$$\int_{\partial^*D\setminus\partial W_r^\varphi}\varphi^o(\nu_D)\,d\mathcal{H}^1-\int_{\partial^*D\cap\partial W_r^\varphi}\varphi^o(\nu_D)\,d\mathcal{H}^1\ge\min\Big\{\frac{(\alpha-1)c_\varphi\sqrt{|D|}}{\alpha+1},\frac{|D|}{(\alpha+1)r\sqrt{|W^\varphi|}}\Big\}.$$

Now, choosing $\alpha:=1+\frac{1}{\mathscr{P}_\varphi(W^\varphi)}$ we get (4.3). □

Next we "improve" the regularity of subgraph and epigraph of Λ-minimizers u. For simplicity, let E_u and F_u be the open representatives of $\mathrm{sg}(u)$ and $\mathrm{epi}(u)$, respectively (see Lemma 2.9), and let

$$\Gamma_u:=(I\times\mathbb{R})\cap\partial E_u=(I\times\mathbb{R})\cap\partial F_u$$

be the (generalized) graph of u in I. By Remark 2.10 Γ_u is a locally Lipschitz curve[3] (thus an arcwise connected set) and, as the traces of u on ∂I are well-defined (see Remark 2.6), its topological closure $\overline{\Gamma}_u$ consists of the union of Γ_u and two points on ∂I, whose vertical coordinates correspond to the traces of u.

Proposition 4.3 (Contact φ-ball condition)**.** *Let φ be a partially monotone anisotropy in $\mathbb{R}^2$, $I\subset\mathbb{R}$ be a bounded open interval, $\Lambda>0$ and $u\in BV_{\rm loc}(I)$ be a Λ-local minimizer of $\mathscr{A}_\varphi$ in I with*

$$\|u\|_\infty\le\frac{\alpha_0\varphi(\mathbf{e}_1)}{4\Lambda},\tag{4.10}$$

where α_0 is given by (4.2). Then for any $r\in(0,\frac{\alpha_0}{\Lambda})$:

(a) if $\mathring{W}_r^\varphi(y)\cap F_u=\emptyset$ with $\Gamma_u\cap\partial\mathring{W}_r^\varphi(y)\ne\emptyset$, then $\overline{\Gamma}_u\cap\partial W_r^\varphi(y)$ is connected (possibly singletons);
(b) if $\mathring{W}_r^\varphi(z)\cap E_u=\emptyset$ with $\Gamma_u\cap\partial\mathring{W}_r^\varphi(z)\ne\emptyset$, then $\overline{\Gamma}_u\cap\partial W_r^\varphi(z)$ is connected (possibly a singleton);
(c) for any $x\in\Gamma_u$ there exist φ-balls $W_r^\varphi(y)$ and $W_r^\varphi(z)$ such that $\mathring{W}_r^\varphi(y)\cap E_u=\emptyset$, $\mathring{W}_r^\varphi(z)\cap F_u=\emptyset$, and $\Gamma_u\cap\partial W_r^\varphi(y)$ and $\Gamma_u\cap\partial W_r^\varphi(z)$ are connected sets (possibly a singleton) containing x.

[3] Possibly out of a discrete set when W^φ is a quadrilateral.

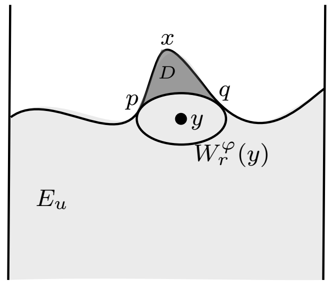

FIG. 1.

Note that the φ-balls $\mathring{W}_r^\varphi(y)$ and $\mathring{W}_r^\varphi(z)$ need not to lie in $I\times\mathbb{R}$. The assertion (c) is related to the rW^φ-condition in the literature, see e.g. [3, Definition 5].

Proof. (a) Assume by contradiction that there exist $r\in(0,\frac{\alpha_0}{\Lambda})$ and a φ-ball $W_r^\varphi(y)$ such that $\mathring{W}_r^\varphi(y)\cap F_u=\emptyset$ and the intersection $\overline{\Gamma}_u\cap\partial\mathring{W}_r^\varphi(y)$ is not connected. Let us denote by $p\neq q$ the upmost left and upmost right points of this intersection; in case there are several upmost left and/or upmost right points we select those with the smallest vertical coordinate. Note that $p,q\in\overline{I}\times\mathbb{R}$.

Let D be the nonempty open set enclosed by the subcurves of $\overline{\Gamma}_u$ and $\partial\mathring{W}_r^\varphi(y)$ between p and q, not intersecting $\mathring{W}_r^\varphi(y)$, see Fig. 1. Since W^φ is symmetric with respect to the coordinate axes, $y\pm r\frac{\mathbf{e}_1}{\varphi(\mathbf{e}_1)}$ are the upmost left and upmost right points of $W_r^\varphi(y)$ and hence,

$$\Big\langle y-r\tfrac{\mathbf{e}_1}{\varphi(\mathbf{e}_1)},\mathbf{e}_1\Big\rangle\le\min\{\langle p,\mathbf{e}_1\rangle,\langle q,\mathbf{e}_1\rangle\}\le\max\{\langle p,\mathbf{e}_1\rangle,\langle q,\mathbf{e}_1\rangle\}\le\Big\langle y-r\tfrac{\mathbf{e}_1}{\varphi(\mathbf{e}_1)},\mathbf{e}_1\Big\rangle.$$

Thus, recalling that p and q lie on the graph of u,

$$D\subset\Big[\langle y,\mathbf{e}_1\rangle-\tfrac{r}{\varphi(\mathbf{e}_1)},\langle y,\mathbf{e}_1\rangle+\tfrac{r}{\varphi(\mathbf{e}_1)}\Big]\times[-\|u\|_\infty,\|u\|_\infty]$$

and therefore

$$0<|D|\le\frac{4\|u\|_\infty r}{\varphi(\mathbf{e}_1)}. \tag{4.11}$$

First assume that

$$D\Subset I\times\mathbb{R}.$$

Then by Λ-minimality, for any open set $\Omega'\Subset I\times\mathbb{R}$, compactly contaning $D\subset E_u$, we have

$$\mathscr{P}_\varphi(E_u,\Omega')\le\mathscr{P}_\varphi(E_u\setminus D,\Omega')+\Lambda|D|. \tag{4.12}$$

Since $D\cap W_r^\varphi(y)=\emptyset$ and $\mathcal{H}^1(\partial^*D\cap\partial W_r^\varphi(y))>0$ (because $W_r^\varphi(y)$ touches ∂E_u at two different points), we can apply Lemma 4.2 to get

$$\mathscr{P}_\varphi(E_u,\Omega')-\mathscr{P}_\varphi(E_u\setminus D,\Omega')=\int_{\partial^*D\setminus\partial W_r^\varphi(y)}\varphi^o(\nu_D)\,d\mathcal{H}^1-\int_{\partial^*D\cap\partial W_r^\varphi(y)}\varphi^o(\nu_D)\,d\mathcal{H}^1\ge\alpha_0\min\Big\{\sqrt{|D|},\frac{|D|}{r}\Big\},$$

and thus

$$\alpha_0\min\Big\{\sqrt{|D|},\frac{|D|}{r}\Big\}\le\Lambda|D|. \tag{4.13}$$

Now, if $\sqrt{|D|}\le\frac{|D|}{r}$, then by (4.13), (4.11) and the assumption $r<\frac{\alpha_0}{\Lambda}$ we have

$$\alpha_0\le\Lambda\sqrt{|D|}<\Lambda\sqrt{\tfrac{4\|u\|_\infty\alpha_0}{\Lambda\varphi(\mathbf{e}_1)}}\qquad\text{so that}\qquad\|u\|_\infty>\tfrac{\alpha_0\varphi(\mathbf{e}_1)}{4\Lambda},$$

which contradicts (4.10). On the other hand, if $\frac{|D|}{r}<\sqrt{|D|}$, then again by (4.13) and the choice of r,

$$\frac{\alpha_0}{\Lambda}\le r<\frac{\alpha_0}{\Lambda},$$

a contradiction.

In case

$$D\cap(\partial I\times\mathbb{R})\neq\emptyset,$$

we fix $\varepsilon>0$ and choose an interval $J\Subset I$ with $|I\setminus J|<\varepsilon$ and replace D with $D_\varepsilon:=D\cap(J\times\mathbb{R})$. Note that for small ε, D_ε is non-empty and satisfies $D_\varepsilon\cap W_r^\varphi(y)=\emptyset$ and $\mathcal{H}^1(\partial^*D_\varepsilon\cap\partial W_r^\varphi(y))>0$. Thus, we can use $E_u\setminus D_\varepsilon$ as a competitor in (4.12) to get (4.13) with D_ε in place of D. Now letting $\varepsilon\to0^+$ we conclude (4.13) and the remaining part of the contradictory argument runs as above.

(b) is proven as (a).

(c) Fix any $x \in \Gamma_u$, $r \in (0, \frac{\alpha_0}{\Lambda})$ and consider the set

$$\Sigma := \{y \in \mathbb{R}^2 :\ \mathrm{dist}_\varphi(y, F_u) = \mathrm{dist}_\varphi(y, \Gamma_u) = r\}.$$

Note that Σ contains two little arcs outside the strip $I \times \mathbb{R}$, and hence, tangent balls may have centers outside it. Note that for any $y \in \Sigma$ the φ-ball $\mathring{W}_r^\varphi(y)$ is "tangent" to $\overline{\Gamma}_u$ at some point and does not intersect F_u. In view of (a) and (b), it suffices to show that there exists $\bar{y} \in \Sigma$ such that $x \in \partial W_r^\varphi(\bar{y})$. Indeed, otherwise, as Γ_u is a graph (an arcwise connected set), we could find $y \in \Sigma$ such that $\Gamma_u \cap \partial W_r^\varphi(y)$ contains two distinct points p and q, and x lies in the relative interior of the subcurve of Γ_u with endpoints p and q, but does not belong to $\partial W_r^\varphi(y)$. However, by (a), the set $\overline{\Gamma}_u \cap \partial W_r^\varphi(y)$ is connected, and hence $x \in \partial W_r^\varphi(y)$, a contradiction.

For a similar reason, for any $x \in \Gamma_u$ and $r \in (0, \frac{\alpha_0}{\Lambda})$ there exists $W_r^\varphi(y)$ such that $\mathring{W}_r^\varphi(y) \cap E_u = \emptyset$ and $x \in \Gamma_u \cap \partial W_r^\varphi(y)$. □

One corollary of Proposition 4.3 is the following lipschitzianity of u.

Corollary 4.4. *Let φ be a partially monotone anisotropy such that W^φ does not have vertical facets and, given a bounded open interval $I \subset \mathbb{R}$ and $\Lambda > 0$, let $u \in BV_{\mathrm{loc}}(I)$ be a Λ-local minimizer of $\mathscr{A}_\varphi$ in I satisfying* (4.1)*. Then u is Lipschitz in I.*

Proof. For simplicity, suppose $I = (-a, a)$ for some $a > 0$. We claim that there exists $\lambda > 0$ such that

$$\langle \nu_{E_u}, \mathbf{e}_2 \rangle \ge \lambda \quad \mathcal{H}^1\text{-a.e. on } (I \times \mathbb{R}) \cap \partial^* E_u, \tag{4.14}$$

where ν_{E_u} is the generalized outer unit normal to E_u. Indeed, by contradiction, assume that there exists a sequence $(x_k) \subset (I \times \mathbb{R}) \cap \partial^* E_u$ such that $\langle \nu_{E_u}(x_k), \mathbf{e}_2 \rangle \to 0$. Possibly passing to a not relabelled subsequence, replacing u with $-u$ and changing the orientation of I (i.e., using the mirror symmetry with respect to the vertical axis) if necessary, we may assume $(x_k) \subset (-a, 0] \times \mathbb{R}$ and $\nu_{E_u}(x_k) \cdot \mathbf{e}_1 \to -1$ as $k \to +\infty$.

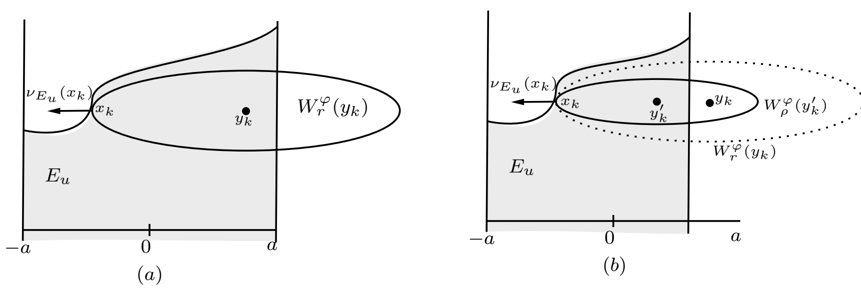

FIG. 2.

Given $\varepsilon \in (0,1)$, from Proposition 4.3 (c) select $r := \frac{\alpha_0}{(1+\varepsilon)\Lambda}$ and $W_r^\varphi(y_k)$ so that $\mathring{W}_r^\varphi(y_k) \cap F_u = \emptyset$ and $x_k \in \partial\Gamma_u \cap \partial W_r^\varphi(y_k)$. Since $\nu_{E_u}(x_k)$ is an "outer normal" also to $W_r^\varphi(y_k)$ at x_k and W^φ has no vertical facets, it follows that

$$\langle x_k - y_k, \mathbf{e}_2 \rangle \to 0.$$

In particular,

$$x_k - y_k = s_k \mathbf{e}_1 + t_k \mathbf{e}_2 \quad \text{with } s_k \to \tfrac{r}{\varphi(\mathbf{e}_1)} \text{ and } t_k \to 0. \tag{4.15}$$

First assume that, up to a not relabelled subsequence, $y_k \in (-a, a] \times \mathbb{R}$ for all $k \ge 1$, see Fig. 2 (a). Then, recalling that $\mathring{W}_r^\varphi(y_k) \cap (I \times \mathbb{R}) \subset E_u$, we have

$$\|u\|_\infty \ge \Big\langle y_k + \tfrac{r\mathbf{e}_2}{\varphi(\mathbf{e}_2)}, \mathbf{e}_2 \Big\rangle = \langle y_k, \mathbf{e}_2 \rangle + \tfrac{r}{\varphi(\mathbf{e}_2)} = \langle x_k, \mathbf{e}_2 \rangle + \tfrac{r}{\varphi(\mathbf{e}_2)} - t_k, \tag{4.16}$$

where $y_k + \frac{r\mathbf{e}_2}{\varphi(\mathbf{e}_2)}$ is the top point of $W_r^\varphi(y_k)$ in the vertical direction and in the last equality we used (4.15). As x_k lies on the graph of u, $\langle x_k, \mathbf{e}_2 \rangle \ge -\|u\|_\infty$, and thus, from (4.16) and the definition of r we deduce

$$2\|u\|_\infty \ge \tfrac{\alpha_0}{(1+\varepsilon)\Lambda\varphi(\mathbf{e}_2)} - t_k.$$

Now first letting $k \to +\infty$ and then $\varepsilon \to 0^+$ we deduce

$$\|u\|_\infty \ge \tfrac{\alpha_0}{2\Lambda\varphi(\mathbf{e}_2)},$$

which contradicts (4.1).

Now assume that $y_k \in (a, +\infty) \times \mathbb{R}$ for all $k \ge 1$, see Fig. 2 (b). In this case, by the partial monotonicity of φ, we have $r > \rho := a\varphi(\mathbf{e}_1)$ and hence $\rho < \frac{\alpha_0}{\Lambda}$. Applying Proposition 4.3 (c) with $r = \rho$ we can find $W_\rho^\varphi(y_k')$ such that $\mathring{W}_\rho^\varphi(y_k') \cap F_u = \emptyset$ with $x_k \in \Gamma_u \cap \partial W_\rho^\varphi(y_k')$. As above, $\langle x_k - y_k', \mathbf{e}_2 \rangle \to 0$, i.e.,

$$x_k - y_k' = s_k'\mathbf{e}_1 + t_k'\mathbf{e}_2 \quad \text{with } s_k' \to \tfrac{\rho}{\varphi(\mathbf{e}_1)} \text{ and } t_k' \to 0.$$

Since $\mathring{W}_\rho^\varphi(y_k') \cap (I \times \mathbb{R}) \subset E_u$, we can repeat the same arguments leading to (4.16) to get

$$\|u\|_\infty \ge \Big\langle y_k' + \tfrac{\rho\mathbf{e}_2}{\varphi(\mathbf{e}_2)}, \mathbf{e}_2 \Big\rangle = \langle y_k', \mathbf{e}_2 \rangle + \tfrac{\rho}{\varphi(\mathbf{e}_2)} = \langle x_k, \mathbf{e}_2 \rangle + \tfrac{\rho}{\varphi(\mathbf{e}_2)} - t_k' \ge -\|u\|_\infty + \tfrac{\rho}{\varphi(\mathbf{e}_2)} - t_k'.$$

Now letting $k \to +\infty$ and recalling the definition of ρ we get

$$\|u\|_\infty \ge \tfrac{a\varphi(\mathbf{e}_1)}{2\Lambda\varphi(\mathbf{e}_2)} = \tfrac{|I|\varphi(\mathbf{e}_1)}{4\Lambda\varphi(\mathbf{e}_2)},$$

which again contradicts (4.1).

These contradictions show the validity of (4.14). In view of (4.14) and [19, Lemma 3.10], it follows that $\Gamma_u := (I \times \mathbb{R}) \cap \partial E_u$ is the graph of a Lipschitz function (with a Lipschitz constant $\sqrt{1/\lambda^2 - 1}$) in the vertical direction. Thus, u admits a Lipschitz representative. □

Now we are ready to prove the regularity of Λ-local minimizers.

Proof of Theorem 4.1. By Corollary 4.4, u is Lipschitz in I. Assume now φ is C^2, elliptic and partially monotone. Then so is φ^0. Moreover, the boundary of W^φ is a closed C^2-curve without segments. By Proposition 4.3, the subgraph $E_u := \mathrm{sg}(u)$ of u satisfies uniform[4] interior and exterior φ-ball conditions at every point of $(I \times \mathbb{R}) \cap \partial E_u$. In view of Proposition 2.2 (c) this implies that E_u and F_u satisfies the classical ball condition of radius $\rho > 0$ with a suitable $\rho > 0$ depending only on α_0, Λ and the constant $\bar r$ in Proposition 2.2 (c). This allows us to obtain an L^∞-bound for the second derivative of u in terms of $1/\rho$, which yields u' is also Lipschitz in I, see for instance [12, Section 2] for details. This and the lipschitzianity of u imply $u \in C^{1,1}(I)$. □

4.1. **Some generalizations.** In this section we relax the Λ-local minimality assumption on u in Theorem 4.1. To this aim, we start with the following

Definition 4.5 ((γ, Λ)-local minimizer)**.** Given an anisotropy φ in $\mathbb{R}^2$, $\gamma > 0$, $\Lambda \ge 0$, and a bounded open interval $I \subset \mathbb{R}$, we say a function $u \in BV_{\mathrm{loc}}(I) \cap L^\infty(I)$ is a (γ, Λ)-*local minimizer* provided that its subgraph $E_u := \mathrm{sg}(u)$ satisfies

$$\mathscr{P}_\varphi(E_u, \Omega) \le \mathscr{P}_\varphi(F, \Omega) + \Lambda |E_u \Delta F| \tag{4.17}$$

for any open set $\Omega \Subset I \times (-\gamma, \gamma)$ and $F \in BV_{\mathrm{loc}}(I \times \mathbb{R}; \{0,1\})$ with $E_u \Delta F \Subset \Omega$.

Note that (γ, Λ)-local minimizers are not necessarily Λ-local minimizers, as local perturbations are taken only in $I \times (-\gamma, \gamma)$. Still, we can readily check that the density estimates in Lemma 2.9 and properties in (2.7) hold, and therefore we can speak about closed and open representatives of E_u. Moreover, in case I and u are bounded, we can apply (4.17) with the set D in the proof of Proposition 4.3 provided for instance

$$\gamma > \frac{\alpha_0 \varphi(\mathbf{e}_1)}{2\Lambda}; \tag{4.18}$$

for such γ, if φ is partially monotone and u satisfies (4.10), all assertions of Proposition 4.3 are valid. This was sufficient to prove Theorem 4.1. Thus, we have shown:

[4]The radii of the tangent Wulff shapes can be choosen a constant $r \in (0, \frac{\alpha_0}{\Lambda})$ along the graph of u.

Theorem 4.6. *Let φ be a partially monotone anisotropy in $\mathbb{R}^2$ such that W^φ does not have vertical facets, $\gamma > 0$ satisfy (4.18), $\Lambda > 0$ and $I \subset \mathbb{R}$ be a bounded interval. Then every (γ,Λ)-local minimizer u of $\mathscr{A}_\varphi$ in I, satisfying the L^∞-bound (4.1), is Lipschitz in I. If, additionally, φ is elliptic and C^2, then $u \in C^{1,1}(I)$.*

5. APPLICATIONS

5.1. **Minimizers of the perturbed area.** Let φ be an anisotropy in $\mathbb{R}^2$, $I \subset \mathbb{R}$ be a bounded open interval and $g \in L^\infty(I)$. Given $p \geq 1$, consider the functional in (1.7), i.e.,

$$\mathscr{G}(u) := \mathscr{A}_\varphi(u,I) + \int_I |u-g|^p ds, \quad u \in L^1(I),$$

where we set $\mathscr{G}(u) = +\infty$ if $u \notin BV(I)$ or $u - g \notin L^p(I)$.

Lemma 5.1. *There exists a minimizer $u \in L^1(I)$ of $\mathscr{G}$. Moreover, $u \in BV(I) \cap L^\infty(I)$ and $\|u\|_\infty \leq \|g\|_\infty$. Finally, if $p > 1$, then u is unique.*

Note that if $p = 1$, then in general minimizers are not unique, see the Introduction.

Proof. The proof is standard and we provide it for completeness. Let $(u_k) \subset L^1(I)$ be a minimizing sequence. We may assume $\mathscr{G}(u_k) \leq \mathscr{G}(0)$ for all k, therefore,

$$\mathscr{A}_\varphi(u_k,I) \leq \mathscr{G}(0) \quad \text{and} \quad \int_I |u-g|^p ds \leq \mathscr{G}(0).$$

By the convexity of φ^o and (2.1) we have

$$\mathscr{G}(0) \geq \mathscr{A}_\varphi(u_k,I) \geq \mathscr{V}_\varphi(u_k,I) - \varphi^o(\mathbf{e}_2)|I| \geq c\int_I |Du_k| - \varphi^o(\mathbf{e}_2)|I|.$$

Moreover, by the Hölder inequality,

$$\int_I |u_k| ds \leq \int_I |u_k - g| ds + \|g\|_\infty |I| \leq \Big(\int_I |u_k - g|^p ds\Big)^{1/p} |I|^{1-\frac{1}{p}} \|g\|_\infty |I| \leq (\mathscr{G}(0))^{1/p} + \|g\|_\infty |I|.$$

Thus, the sequence $(u_k)_k$ is bounded in $BV(I)$ and by the L^1-compactness in BV, up to a not relabelled subsequence, $u_k \to u$ in $L^1(I)$ for some $u \in BV(I)$. By the Riesz-Fischer lemma, we may also assume $u_k \to u$ a.e. in I. Then by the $L^1_{\mathrm{loc}}(I)$-lower semicontinuity of $\mathscr{A}_\varphi(\cdot,I)$,

$$\liminf_{k\to+\infty} \mathscr{A}_\varphi(u_k,I) \geq \mathscr{A}_\varphi(u,I)$$

and by the Fatou's lemma

$$\liminf_{k\to+\infty} \int_I |u_k - g|^p ds \geq \int_I |u-g|^p ds.$$

Thus, $u \in BV(I)$ is a minimizer of $\mathscr{G}$.

To show that $\|u\|_\infty \leq \|g\|_\infty$, let $v := \max\{u, -\|g\|_\infty\}$. Since $|u-g| \geq |v-g|$ a.e. in I, we have

$$\int_I |u-g|^p ds \geq \int_I |v-g|^p ds$$

with the strict inequality if the set $\{u < -\|g\|_\infty\}$ has positive measure. Moreover, by Lemma 2.4

$$\int_I \varphi^o(-Du,1) = \mathscr{P}_\varphi(\mathrm{sg}(u), I\times\mathbb{R}) = \mathscr{P}_\varphi(\mathrm{epi}(u), I\times\mathbb{R}).$$

Since

$$\mathscr{P}_\varphi(\mathrm{epi}(v), I\times\mathbb{R}) = \mathscr{P}_\varphi(\mathrm{epi}(u)\cap[I\times(-\|g\|_\infty,+\infty)], I\times\mathbb{R}) \leq \mathscr{P}_\varphi(\mathrm{epi}(u), I\times\mathbb{R}),$$

where in the last inequality we used a cutting with half-spaces argument, see e.g. [4], it follows that

$$\mathscr{A}_\varphi(u,I) = \mathscr{P}_\varphi(\mathrm{epi}(u), I\times\mathbb{R}) \geq \mathscr{P}_\varphi(\mathrm{epi}(v), I\times\mathbb{R}) \geq \mathscr{A}_\varphi(v,I).$$

Thus $\mathscr{G}(u) \geq \mathscr{G}(v)$ with the strict inequality if the set $\{u < -\|g\|_\infty\}$ has positive measure. Then the minimality of u implies $u \geq -\|g\|_\infty$ a.e. in I. Similarly, we can show $u \leq \|g\|_\infty$ a.e. in I.

The uniqueness of u in case $p > 1$ directly follows from the strict convexity of the L^p-norm. □

Remark 5.2. In fact,

$$-\|g^-\|_\infty \le -\|u^-\|_\infty \le \|u^+\|_\infty \le \|g^+\|_\infty,$$

where $a^\pm = \max\{\pm a, 0\}$ are the positive and negative parts of a.

The next proposition establishes a bridge between minimizers of $\mathscr{G}$ and (γ,Λ)-minimizers of $\mathscr{A}_\varphi$.

Proposition 5.3. *Assume that φ is partially monotone and let $u \in BV(I)\cap L^\infty(I)$ be a minimizer of $\mathscr{G}$. Then u is a (γ,Λ)-minimizer of $\mathscr{A}_\varphi$ in I for any $\gamma > 2\|g\|_\infty$, where*

$$\Lambda := p(\gamma + \|g\|_\infty)^{p-1}.$$

Proof. By Lemma 5.1, $\|u\|_\infty \le \|g\|_\infty$. Let $E_u := \mathrm{sg}(u)$ be the subgraph of u and consider any $F \in BV_{\mathrm{loc}}(I\times \mathbb{R};\{0,1\})$ with $E_u\Delta F \Subset I\times(-\gamma,\gamma)$. Let v be the vertical rearrangement of F as in the proof of Proposition 2.7. By construction, $v \in BV_{\mathrm{loc}}(I)$, $\mathrm{supp}\,(u-v)\Subset I$ and hence $v\in BV(I)$. In addition, by the Fubini-Tonelli theorem and the partial monotonicity of φ,

$$|E_u\Delta F| = \int_I |u-v|ds \quad \text{and} \quad \mathscr{P}_\varphi(F, I\times\mathbb{R}) \ge \mathscr{P}_\varphi(\mathrm{sg}(v), I\times\mathbb{R}) = \mathscr{A}_\varphi(v,I), \tag{5.1}$$

see for instance the proof of Proposition 2.7. Moreover, since $\gamma > 2\|u\|_\infty$ and F does not cross the horizontal sides of the rectangle $I\times(-\gamma,\gamma)$, we have $\|v\|_\infty < \gamma$. Thus, by the minimality of u and (5.1),

$$\begin{aligned}\mathscr{P}_\varphi(E_u, I\times\mathbb{R}) = \mathscr{A}_\varphi(u,I) &\le \mathscr{A}_\varphi(v,I) + \int_I \Big(|v-g|^p - |u-g|^p\Big)ds\\ &\le \mathscr{P}_\varphi(F,I\times\mathbb{R}) + p\max\{|v-g|^{p-1}, |u-g|^{p-1}\}\int_I |u-v|ds \le \mathscr{P}_\varphi(F,I\times\mathbb{R}) + \Lambda|E_u\Delta F|,\end{aligned} \tag{5.2}$$

where in the last inequality we used

$$\begin{aligned}\max\{|v-g|^{p-1}, |u-g|^{p-1}\} \le& \max\{(\|v\|_\infty + \|g\|_\infty)^{p-1}, (\|u\|_\infty + \|g\|_\infty)^{p-1}\}\\ \le& \max\{(\gamma + \|g\|_\infty)^{p-1}, (2\|g\|_\infty)^{p-1}\} \le (\gamma+\|g\|_\infty)^{p-1}.\end{aligned}$$

Since $E_u\Delta F \Subset I\times(-\gamma,\gamma)$, comparing (5.2) with (4.17) we conclude that u is a (γ,Λ)-local minimizer of $\mathscr{A}_\varphi$. □

From Proposition 5.3 and Theorem 4.6 we deduce the following

Theorem 5.4 (Regularity of minimizers)**.** *Let φ be a partially monotone anisotropy in $\mathbb{R}^2$ such that W^φ does not have vertical facets, $I\subset\mathbb{R}$ be a bounded open interval and $p\ge 1$. Let*

$$\sigma := \Big(\tfrac{1}{4^{p-1}p}\min\Big\{\tfrac{\alpha_0\varphi(\mathbf{e}_1)}{4}, \tfrac{\alpha_0}{2\varphi(\mathbf{e}_2)}, \tfrac{|I|\varphi(\mathbf{e}_1)}{4\varphi(\mathbf{e}_2)}\Big\}\Big)^{1/p}, \tag{5.3}$$

where $\alpha_0 > 0$ is defined in (4.2). Let $g \in L^\infty(I)$ be such that

$$\|g\|_\infty < \sigma. \tag{5.4}$$

Then every minimizer u of the functional $\mathscr{G}$ in (1.7) is Lipschitz in I. Moreover, if φ is elliptic and C^2, then $u\in C^{1,1}(I)$.

Proof. Let $\gamma := 3\sigma$ and $\Lambda := (\gamma+\sigma)^{p-1}p = (4\sigma)^{p-1}p > 0$. By (5.3),

$$\sigma = \min\Big\{\tfrac{\alpha_0\varphi(\mathbf{e}_1)}{4\Lambda}, \tfrac{\alpha_0}{2\Lambda\varphi(\mathbf{e}_2)}, \tfrac{|I|\varphi(\mathbf{e}_1)}{4\Lambda\varphi(\mathbf{e}_2)}\Big\}.$$

Thus, γ satisfies (4.18). Since $\|g\|_\infty < \sigma = \gamma/3$, by Proposition 5.3 u is a (γ,Λ)-local minimizer of $\mathscr{A}_\varphi$ in I. Moreover, by Lemma 5.1, $\|u\|_\infty \le \|g\|_\infty$ and hence, from (5.3) and (5.4) it follows that u satisfies (4.1). Now the assertions directly follow from Theorem 4.6. □

When φ is Euclidean and $p=1$, (5.3) reads as

$$\sigma = \frac{1}{4}\min\Big\{\alpha_0, |I|\Big\}, \tag{5.5}$$

where $\alpha_0 = \frac{2\sqrt{\pi}}{4\pi+1}$. Thus, Theorem 5.4 implies that every minimizer of $\mathscr{G}$ belongs to $C^{1,1}(I)$ provided that $\|g\|_\infty < \sigma$. This positively answers to Conjecture 1.1 in case $n=k=1$, except for the dependence of σ on $|I|$. Note that our σ depends on $|I|$, while the σ of [5] (with $p=2$) depends only on $1/\sqrt{|I|}$.

The following example shows that in general the local $C^{1,1}$-regularity of u in Theorem 5.4 cannot be improved.

Example 5.5. Let $I=(-1,1)$, φ be Euclidean and

$$g(s) = \begin{cases} a & \text{if } s\in(0,1), \\ -a & \text{if } s\in(-1,0) \end{cases}$$

for some $a\in(0,1)$. Then the $C^{1,1}(I)\setminus C^2(I)$-function (see Fig. 3 (a))

$$u(s) = \begin{cases} a & \text{if } s\in[\sqrt{2a-a^2},1), \\ a-1+\sqrt{1-(s-\sqrt{2a-a^2})^2} & \text{if } s\in[0,\sqrt{2a-a^2}], \\ 1-a-\sqrt{1-(s+\sqrt{2a-a^2})^2} & \text{if } s\in[-\sqrt{2a-a^2},0], \\ -a & \text{if } s\in(-1,-\sqrt{2a-a^2}] \end{cases}$$

is the unique minimizer of $\mathscr{G}$. Indeed, for simplicity, set

$$h(s) := \frac{u'(s)}{\sqrt{1+u'(s)^2}} = \begin{cases} 0 & \text{if } s\in[\sqrt{2a-a^2},1] \\ -s+\sqrt{2a-a^2} & \text{if } s\in[0,\sqrt{2a-a^2}] \\ s+\sqrt{2a-a^2} & \text{if } s\in[-\sqrt{2a-a^2},0] \\ 0 & \text{if } s\in[-1,-\sqrt{2a-a^2}] \end{cases}$$

so that $h\in \mathrm{Lip}([-1,1])\cap C^\infty([-1,1]\setminus\{0,\pm\sqrt{2a-a^2}\})$ with

$$h' = \begin{cases} 0 & \text{a.e. in } \{u=g\}, \\ -1 & \text{a.e. in } \{u<g\}, \\ 1 & \text{a.e. in } \{u>g\}. \end{cases} \tag{5.6}$$

As $h(\pm 1)=0$, for any $v\in BV(-1,1)$ by integrating by parts we have

$$\begin{aligned} \int_{-1}^1 (u-v)h'ds &= (u-v)h\Big|_{-1}^1 - \int_{-1}^1 hu'ds + \int_{-1}^1 h\,dDv \\ &= -\int_{-1}^1 \frac{u'^2 ds}{\sqrt{1+u'^2}} + \int_{-1}^1 \frac{u'dDv}{\sqrt{1+u'^2}} = -\int_{-1}^1 \sqrt{1+u'^2}ds + \int_{-1}^1 \frac{ds+u'dDv}{\sqrt{1+u'^2}}. \end{aligned}$$

On the other hand, by the explicit expression of h',

$$\int_{-1}^1 (u-v)h'ds = \int_{-1}^1 (u-g)h'ds + \int_{-1}^1 (g-v)h'ds = \int_{-1}^1 |u-g|ds + \int_{-1}^1 (g-v)h'ds.$$

Combining these two equalities, we deduce

$$\int_{-1}^1 \sqrt{1+u'^2}ds + \int_{-1}^1 |u-g|ds = \int_{-1}^1 \frac{ds+u'dDv}{\sqrt{1+u'^2}} + \int_{-1}^1 (v-g)h'ds. \tag{5.7}$$

By the Hölder inequality[5] and the bound $\|h'\|_\infty \le 1$,

$$\int_{-1}^{1} \frac{ds + u' dDv}{\sqrt{1+u'^2}} \le \int_{-1}^{1} \sqrt{1+|Dv|^2} \quad \text{and} \quad \int_{-1}^{1} (v-g)h' ds \le \int_{-1}^{1} |v-g| ds.$$

Therefore, from (5.7) we deduce $\mathscr{G}(u) \le \mathscr{G}(v)$.

Next, let us show the uniqueness of u. Let $v \in BV(-1,1)$ be any other minimizer of $\mathscr{G}$. By Remark 5.2,

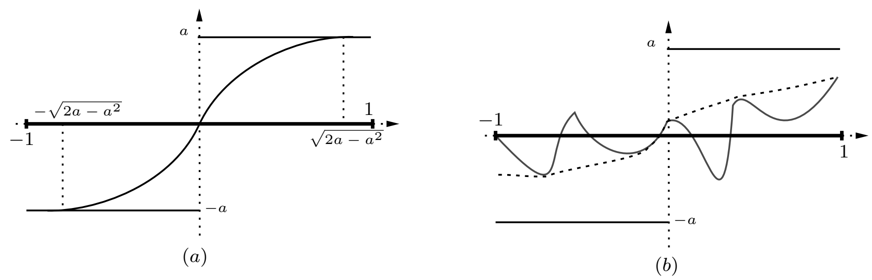

FIG. 3. The graph of u in (a) and convex/concave nondecreasing envelope of v in (b).

$-a \le v \le a$. Let $v^* : [0,1] \to \mathbb{R}$ be the smallest nondecreasing concave function with $v^* \ge v$ a.e. in $[0,1]$ and v_* be the largest nondecreasing convex function with $v_* \le v$ a.e. in $[0,1]$. Set $\bar v := v^* \chi_{(0,1]} + v_* \chi_{[-1,0)}$. As $g - v \ge g - \bar v \ge 0$ in $(0,1)$ and $v - g \ge \bar v - g \ge 0$ in $(-1,0)$, we have

$$\int_{-1}^{1} |g-v| ds \ge \int_{-1}^{1} |g - \bar v| ds$$

with strict inequality if $\{v \ne \bar v\}$ has positive Lebesgue measure. Moreover, as we are replacing nonconcave/nonconvex parts of the graph of v with line segments (see Fig. 3 (b)), $\mathscr{A}_\varphi(v, I) \ge \mathscr{A}_\varphi(\bar v, I)$. Thus, $\mathscr{G}(\bar v) \le \mathscr{G}(v)$. This inequality shows that we may assume $v = \bar v$. In this case, by (2.2)

$$\int_{-1}^{1} \frac{u' \, dDv}{\sqrt{1+u'^2}} = \int_{-1}^{1} \frac{u'v' ds}{\sqrt{1+u'^2}} + \sqrt{2a-a^2}\,(v^+(0) - v^-(0)).$$

Thus, as above, from (5.7), the Hölder inequality and (5.6) we get

$$\begin{aligned} \mathscr{G}(u) = \int_{-1}^{1} \sqrt{1+u'^2} ds + \int_{-1}^{1} |u-g| ds &= \int_{-1}^{1} \frac{(1+u'v')ds}{\sqrt{1+u'^2}} + \sqrt{2a-a^2}\,(v^+(0) - v^-(0)) + \int_{-1}^{1} |v-g| ds \\ &\le \int_{-1}^{1} \sqrt{1+v'^2} ds + \sqrt{2a-a^2}\,(v^+(0)-v^-(0)) + \int_{-1}^{1} |v-g| ds \le \mathscr{G}(v), \end{aligned} \tag{5.8}$$

where in the last inequality we used $\sqrt{2a-a^2} < 1$. Since both u and v are minimizers, all inequalities in (5.8) are in fact equalities. In particular, $u' = v'$ a.e. in $(-1,1)$ and $v^+(0) = v^-(0)$. This implies $u = v + C$ for some real constant C. Then, recalling $u(\pm 1) = \pm a$ and $-a \le v \le a$, we deduce $C = 0$, i.e., $u = v$.

Notice that Example 5.5 shows that the threshold σ in (5.5) is not optimal, in general.

Data availability. The paper has no associated data.

[5] If λ and μ are bounded Radon measures in a bounded open set $I \subset \mathbb{R}^n$ and $p, q \in C(\bar I)$, there holds

$$\int_I p d\mu + q d\lambda \le \int_I \sqrt{p^2+q^2}\, d\sqrt{\mu^2+\lambda^2},$$

where

$$\int_U \sqrt{\mu^2+\lambda^2} := \sup\Big\{ \int_U \phi d\mu + \psi d\lambda : \ (\phi,\psi) \in C_c(U;\mathbb{R}^2),\ \|\phi^2+\psi^2\|_\infty \le 1 \Big\}$$

is the total variation of (μ,λ) in the open set $U \subset I$. We apply this inequality with $p = (1+u'^2)^{-1/2}$, $q = u'(1+u'^2)^{-1/2}$, $\mu = \mathscr{L}^1$ and $\lambda = Dv$.

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