BACH-PINCHED METRICS ON CLOSED MANIFOLDS

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Abstract. Exploiting the deformation method introduced by Aubin in his seminal work to construct constant negative scalar curvature metrics, we show the existence, on every closed manifold of dimension four, of a metric whose Bach tensor is pinched by the scalar curvature.

1. Introduction

Let (M, g) be a Riemannian manifold of dimension $n \geq 3$: it is well known that the Riemann curvature tensor Riem_g admits the decomposition

$$\mathrm{Riem}_g = \mathrm{W}_g + \frac{1}{2(n-1)} \mathrm{Ric}_g \owedge g - \frac{S_g}{2(n-1)(n-2)} g \owedge g,$$

where W_g is the *Weyl tensor*, Ric_g is the *Ricci tensor* and S_g is the *scalar curvature* and $\owedge$ denotes the Kulkarni-Nomizu product. A fundamental question in Riemannian Geometry is to understand the relations between the curvature and the topology of the underlying manifold: for instance, an example of this relation is provided by metrics with *positive scalar curvature* [17, 18, 25, 30], *non-positive sectional curvature* or by metrics which are *locally conformally flat*, i.e. $\mathrm{W}_g \equiv 0$ for $n \geq 4$ [4, 16, 24].
On the other hand, there are examples of curvature conditions that are unobstructed: in [2, 3] Aubin showed that, on every smooth n-dimensional closed (compact with empty boundary) manifold, there exists a smooth Riemannian metric with *constant negative scalar curvature*. This result was then extended to complete non-compact manifolds by Bland and Kalka in [6]. In particular, there are no topological obstructions to metrics with negative scalar curvature and, more in general, Lohkamp proved that on every smooth complete Riemannian manifold there are no obstructions to the existence of metrics with *negative Ricci curvature* [29].
Note that in [3] Aubin also proved that, if M is a closed Riemannian manifold of dimension $n \geq 4$, then there always exists a metric with non-vanishing Weyl curvature, i.e. $|\mathrm{W}_g|_g > 0$ everywhere. As a consequence, in [13] the authors showed the existence of *weak harmonic Weyl* metrics on every closed Riemannian four-manifold, i.e. critical points, in a conformal class, of the normalized L^2-norm of the Cotton tensor. Moreover, in [10] the second author extended Aubin's construction of metrics with negative scalar curvature proving that every n-dimensional closed manifold admits a Riemannian metric with *constant negative scalar-Weyl curvature*, i.e.

$$S_g + t|\mathrm{W}_g|_g \equiv -1,$$

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for every $t \in \mathbb{R}$ and, as a consequence, there are no topological obstructions to the existence of metrics with negative scalar curvature and *Weyl-pinched curvature*, generalizing a result of Seshadri in dimension four [31]: namely, on every closed manifold of dimension $n \geq 4$, for every $\varepsilon > 0$ there exists a Riemannian metric $g = g_\varepsilon$ such that

$$S_g < 0 \quad \text{and } |\mathrm{W}_g|_g^2 < \varepsilon S_g^2 \text{ on } M.$$

In this paper, we are interested in the Bach tensor B_g, which, if $n = 4$, is locally defined as

$$B_{ij} = W_{ikjl,lk} + \frac{1}{2} R_{kl} W_{ikjl}.$$

This geometric quantity, introduced by Bach in the context of Conformal Relativity [5], is a divergence-free, conformally covariant tensor, i.e., if $\widetilde{g} = e^{2u} g$ is a metric conformal to g, B_g satisfies [15]

$$\mathrm{B}_{\widetilde{g}} = e^{-2u} \, \mathrm{B}_g \, . \tag{1.1}$$

We say that a Riemannian metric g is *Bach-flat* if $\mathrm{B}_g \equiv 0$ on M: typical examples are provided by conformally Einstein metrics, locally conformally flat metrics, and, more generally, half conformally flat metrics. Furthermore, Bach-flat metrics are exactly the critical points of the so-called Weyl functional

$$\mathfrak{W}(g) = \int_M |\mathrm{W}_g|_g^2 dV_g.$$

Up to now, no topological obstructions to the existence of Bach-flat metrics on closed smooth four-manifolds are known, although some partial rigidity and classification results have been proven [8, 9, 14, 27, 32, 33]: we recall that one of the few examples of non-trivial Bach-flat metrics was constructed by Abbena, Garbiero and Salamon on a solvable Lie group [1]. In [22], Gursky and Viaclovsky developed a gluing method to construct B^t-flat metrics, i.e. critical points of the Weyl functional perturbed with a quadratic scalar curvature term, on connected sums of Einstein four-manifolds. On the other hand, it is possible to find unobstructed metrics whose Bach tensor satisfies curvature properties: indeed, using a special metric deformation introduced by Aubin [3], the second and the third author, together with P. Mastrolia, showed the existence of metrics with non-vanishing Bach tensor on every four-dimensional closed manifold [11].

Our main result is the following

Theorem 1.1. *On every smooth* 4*-dimensional closed manifold* M*, for every* $t \in \mathbb{R}$*, there exists a smooth Riemannian metric* $g = g_t$ *with*

$$S_g + t|\,\mathrm{B}_g\,|_g^{\frac{1}{2}} \equiv -1 \quad \text{on } M.$$

In particular, there are no topological obstructions for negative scalar-Bach curvature metrics.

Therefore, choosing $t = 1/\sqrt{\varepsilon}$, $\varepsilon > 0$, in Theorem 1.1 we obtain the following existence result for metrics with negative scalar curvature and *Bach-pinched curvature*:

Corollary 1.2. *On every smooth* n*-dimensional closed manifold, for every* $\varepsilon > 0$*, there exists a smooth Riemannian metric* $g = g_\varepsilon$ *with*

$$S_g < 0 \quad \text{and} \quad |\,\mathrm{B}_g\,|_g < \varepsilon S_g^2 \quad \text{on } M.$$

The naive idea of the proof of Theorem 1.1, similar to the one exploited in [2, 3, 10], would be to start from a reference metric g, to be suitably chosen, to construct a non-conformal Riemannian metric $\overline{g}$ such that

$$\int_M \left(S_{\overline{g}} + t|\mathrm{B}_{\overline{g}}|_{\overline{g}}^{\frac{1}{2}} \right) dV_{\overline{g}} < 0$$

and to apply the method used by Gursky in [19] to produce the desired constant negative scalar-Bach metric. Since we must deal with the modified conformal Laplacian of g

$$-6\Delta_g + S_g + t|\mathrm{B}_g|_g^{\frac{1}{2}},$$

to construct smooth (at least, C^4) metrics we have to take into account the lack of regularity of the operator on $\{\mathrm{B}_g = 0\}$: to overcome this difficulty, throughout the proof we often rely on the existence of Riemannian metrics with non-vanishing Bach tensor [11] (see Section 4 for further details). We also want to stress out the fact that, in order to have precise estimates, we had to compute the full variation formula of the Bach tensor under Aubin's deformation (see Section 5 for all the detailed computations).

2. The scalar-Bach curvature

In this section we focus on the variational and conformal aspects of the scalar-Bach curvature, which are analogous to those of the scalar-Weyl curvature, first studied by Gursky in [19]. Let (M, g) be a n-dimensional closed (compact with empty boundary) Riemannian manifold. We start by recalling the definition of the conformal Laplacian is the operator:

$$\mathcal{L}_g := -\frac{4(n-1)}{n-2}\Delta_g + S_g.$$

It satisfies the following well known conformal covariance property: if $\widetilde{g} = u^{4/(n-2)}g$, where u is a positive smooth function on M, then

$$\mathcal{L}_{\widetilde{g}}\phi = u^{-\frac{n+2}{n-2}}\mathcal{L}_g(\phi u), \quad \forall \phi \in C^2(M).$$

Observe that this operator plays a prominent role in the resolution of the Yamabe variational problem: indeed, the scalar curvature of the conformally related metric $\widetilde{g}$ is given by

$$S_{\widetilde{g}} = u^{-\frac{n+2}{n-2}}\mathcal{L}_g u.$$

In [19], Gursky introduced a modification of the conformal Laplacian, introducing a new term depending on the Weyl curvature. Given $t \in \mathbb{R}$, we recall the definition of scalar-Weyl curvature

$$F_g := S_g + t|\mathrm{W}_g|_g \tag{2.1}$$

and the associated modified conformal Laplacian

$$\mathcal{L}_g^t := -\frac{4(n-1)}{n-2}\Delta_g + F_g,$$

where

$$|\mathrm{W}_g|_g = \sqrt{W_{ijkl}W_{pqrs}g^{ip}g^{jq}g^{kr}g^{ls}}$$

denotes the norm of the Weyl curvature of g. It was proved in [19] that the pairs $(F_g, \mathcal{L}_g^t)$ and $(S_g, \mathcal{L}_g)$ share the same conformal properties. In fact, if $\widetilde{g} = u^{4/(n-2)}g$, then

$$\mathcal{L}_{\widetilde{g}}^t \phi = u^{-\frac{n+2}{n-2}} \mathcal{L}_g^t(\phi u), \quad \forall \phi \in C^2(M), \quad \text{and} \quad F_{\widetilde{g}} = u^{-\frac{n+2}{n-2}} \mathcal{L}_g^t u. \tag{2.2}$$

In an analogous way, for $n = 4$, given $t \in \mathbb{R}$ we define the *scalar-Bach curvature*

$$F_g^B := S_g + t|\mathrm{B}_g|_g^{\frac{1}{2}} \tag{2.3}$$

and the associated modified conformal Laplacian

$$\mathscr{L}_g^t := -6\Delta_g + F_g^B,$$

where $|\mathrm{B}_g|_g = \sqrt{B_{ij}B_{pq}g^{ip}g^{jq}}$ denotes the norm of the Bach tensor of g. A crucial observation is the fact that the pair $(F_g^B, \mathscr{L}_g^t)$ preserves the same conformal properties of $(S_g, \mathcal{L})$; indeed, let $\widetilde{g} = u^2 g$, then

$$u^2(\mathrm{B}_{\widetilde{g}})_{ij} = (\mathrm{B}_g)_{ij}$$

and

$$u^8|\mathrm{B}_{\widetilde{g}}|_{\widetilde{g}}^2 = |\mathrm{B}_g|_g^2,$$

therefore

$$\mathscr{L}_{\widetilde{g}}^t \phi = u^{-3} \mathscr{L}_g^t(\phi u), \quad \forall \phi \in C^4(M), \quad \text{and} \quad F_{\widetilde{g}}^B = u^{-3} \mathscr{L}_g^t u. \tag{2.4}$$

In particular, adapting the argument of [19, Proposition 3.2], we have the following:

Lemma 2.1. *Let (M, g) be a 4-dimensional closed Riemannian manifold with $|\mathrm{B}_g|_g > 0$. Then, there exists a smooth metric $\widetilde{g} \in [g]$ with either $F_{\widetilde{g}}^B > 0$, $F_{\widetilde{g}}^B < 0$, or $F_{\widetilde{g}}^B \equiv 0$. Moreover, these three possibilities are mutually exclusive.*

Proof. Let $\mu_t(g)$ denote the principle eigenvalue of $\mathscr{L}_g^t$ and let ϕ denote the eigenfunction relative to $\mu_t(g)$. By the maximum principle ϕ can be assumed to be positive. In particular ϕ satisfies

$$\mathscr{L}_g^t \phi = \mu_t(g)\phi,$$

that is equivalent to

$$-6\Delta\phi = -F_g^B \phi + \mu_t(g)\phi.$$

Note that, since $|\mathrm{B}_g|_g > 0$, then $F_g^B \in C^\infty(M)$ and thus $\phi \in C^\infty(M)$. Let us consider the conformal change $\widetilde{g} = \phi^2 g$, then $\widetilde{g} \in [g]$ is smooth and by (2.4)

$$F_{\widetilde{g}}^B = \mu_t(g)\phi^{-2}.$$

Therefore, F_g^B is either positive, negative or identically zero, depending on the sign of $\mu_t(g)$ and these possibilities are mutually exclusive because the sign of $\mu_t(g)$ is conformally invariant. $\square$

In analogy with the Yamabe problem, Gursky introduced the following modified fuctional

$$\widehat{Y}(u) := \frac{\int_M u\,\mathcal{L}_g^t u\,dV_g}{\left(\int_M u^{2n/(n-2)}\,dV_g\right)^{(n-2)/2}}$$

and

$$\widehat{Y}(M, [g]) := \inf_{u \in H^1(M)} \widehat{Y}(u),$$

which is conformally invariant. Following a classical subcritical regularization argument, he proved that, if $\widehat{Y}(M,[g]) \leq 0$, then the variational problem of finding a conformal metric $\widetilde{g} \in [g]$ with constant scalar-Weyl curvature F can be solved. See [19, Proposition 3.5] for the proof (in dimension four). In an analogous way, when $n=4$ and $|\mathrm{B}_g|_g > 0$, we can consider the functional

$$\widehat{Y}^B(u) := \frac{\int_M u\, \mathscr{L}_g^t u\, dV_g}{\left(\int_M u^4\, dV_g\right)}$$

and the conformal invariant

$$\widehat{Y}^B(M,[g]) := \inf_{u \in H^1(M)} \widehat{Y}^B(u).$$

By (2.4), it is easy to see that the functional $u \mapsto \widehat{Y}^B(u)$ is equivalent to the modified Einstein-Hilbert functional

$$\widetilde{g} = u^2 g \longmapsto \frac{\int_M F^B_{\widetilde{g}}\, dV_{\widetilde{g}}}{\mathrm{Vol}_{\widetilde{g}}(M)}.$$

In particular the following lemma holds:

Lemma 2.2. *Let (M,g) be a 4-dimensional closed Riemannian manifold with $|\mathrm{B}_g|_g > 0$. If there exists a metric $g' \in [g]$ such that*

$$\int_M F_{g'}\, dV_{g'} < 0,$$

then, there exists a (unique) C^∞ metric $\widetilde{g} \in [g]$ such that $F_{\widetilde{g}} \equiv -1$.

Proof. Since

$$\int_M F_{g'}\, dV_{g'} < 0,$$

arguing as in [28, Proposition 4.4], there exists $v \in H^1(M)$ which attains the minimum of $\widehat{Y}^B(M,[g])$. In particular,

$$-6\Delta v + F^B_g v = K v^{-3};$$

where K is a negative constant. Then, since $|\mathrm{B}_g|_g > 0$, F_g is smooth and, by elliptic regularity, we have that $v \in C^\infty(M)$ and $\widetilde{g} \in [g]$ such that $\widetilde{g} = v^{-2} g$ is smooth. ☐

Note that these techniques introduced by Gursky have been used in various contexts, such as [7, 10, 20, 21, 23, 26].

3. Aubin's metric deformation

We recall the deformation of a Riemannian metric g, introduced by Aubin in [2, 3] and defined as

$$\overline{g} = g + df \otimes df, \quad f \in C^\infty(M); \tag{3.1}$$

throughout this section, the barred quantities are referred to the metric $\overline{g}$, while the unbarred ones are related to g. Locally, given an open chart $U \subset M$ with coordinate functions $dx^1, ..., dx^n$, we can rewrite (3.1) as

$$\overline{g}_{ij} = g_{ij} + f_i f_j, \tag{3.2}$$

where $f_i = \partial_i f = \frac{\partial f}{\partial x^i}$. The Levi-Civita connection is locally expressed by the Christoffel symbols Γ^k_{ij}, which, with respect to $\overline{g}$, are defined as

$$\overline{\Gamma}^l_{ij} = \Gamma^k_{ij} + \frac{f^l f_{ij}}{1+|\nabla f|^2}, \tag{3.3}$$

where $f^i = g^{ij} f_j$, $f_{ij} = \partial_j f_i - \Gamma^k_{ij} f_l$ and Γ^k_{ij} are the Christoffel symbols of the metric g. In particular, we have

$$\begin{aligned} dV_{\overline{g}} &= \left(1+|\nabla f|^2\right)^{\frac{1}{2}} dV_g; \\ \overline{g}^{ij} &= g^{ij} - \frac{f_i f_j}{1+|\nabla f|^2}. \end{aligned} \tag{3.4}$$

Similarly, starting from (3.3), we can compute the curvature components of the metric $\overline{g}$ (see [12, Chapter 2]): for instance, the local components of the $(0,4)$-Riemann tensor $\overline{\mathrm{Riem}}$ are written as

$$\overline{R}_{ijlt} = R_{ijlt} + \frac{1}{1+|\nabla f|^2}(f_{il} f_{jt} - f_{it} f_{jl}) =: R_{ijkl} + E^R_{ijkl}. \tag{3.5}$$

Tracing (3.5), we obtain the local expressions for the Ricci tensor $\overline{\mathrm{Ric}}$ and the scalar curvature $\overline{S}$:

$$\begin{aligned} \overline{R}_{ij} = R_{ij} &- \frac{1}{1+|\nabla f|^2} f^t f^l R_{itjl} + \frac{1}{1+|\nabla \phi|^2}(\Delta f \cdot f_{ij} - f_{it} f^t_j) + \\ &- \frac{1}{\left(1+|\nabla f|^2\right)^2} f^t f^l (f_{ij} f_{tl} - f_{it} f_{jl}) \\ &=: R_{ij} + F_{ij}; \end{aligned} \tag{3.6}$$

$$\begin{aligned} \overline{S} = S &- \frac{2}{1+|\nabla f|^2} R_{ij} f^i f^j + \frac{1}{1+|\nabla f|^2}\left[(\Delta f)^2 - |\mathrm{Hess}\, f|^2\right] + \\ &- \frac{2}{\left(1+|\nabla f|^2\right)^2}\left[\Delta f \cdot f^i f^j f_{ij} - f^i f_{ij} f^{jp} f_p\right] \\ &=: S + H. \end{aligned} \tag{3.7}$$

Note that the proof of (3.4) and that of the scalar curvature can be found in [3] while the other transformations can be found in [12, Chapter 2]. Moreover, on a 4-dimensional manifold, we have

$$B_{ij} = \frac{1}{2}\left[\Delta R_{ij} - \frac{1}{3} S_{ij} + 2R_{kl} R_{ikjl} - \frac{2}{3} S R_{ij} - \frac{1}{6}\Delta S g_{ij} - \frac{1}{2}\left(|\mathrm{Ric}|^2 - \frac{S^2}{3}\right) g_{ij}\right]. \tag{3.8}$$

Then

$$\overline{B}_{ij} = B_{ij} + E(f)_{ij}, \tag{3.9}$$

where

$$\begin{aligned} E(f)_{ij} := \frac{1}{2}\Bigg[&\overline{\Delta}\overline{R}_{ij} - \frac{1}{3}\overline{S}_{ij} + 2\overline{R}_{kl}\overline{R}_{ikjl} - \frac{2}{3}\overline{S}\overline{R}_{ij} - \frac{1}{6}\overline{\Delta}\overline{S}\overline{g}_{ij} - \frac{1}{2}\left(|\overline{\mathrm{Ric}}|^2_{\overline{g}} - \frac{\overline{S}^2}{3}\right)\overline{g}_{ij} \\ &- \Delta R_{ij} - \frac{1}{3} S_{ij} + 2R_{kl} R_{ikjl} - \frac{2}{3} S R_{ij} - \frac{1}{6}\Delta S g_{ij} - \frac{1}{2}\left(|\mathrm{Ric}|^2 - \frac{S^2}{3}\right) g_{ij}\Bigg]. \end{aligned} \tag{3.10}$$

Moreover, we point out that

$$\overline{S} = S - \frac{R_{ij} f^i f^j}{1 + |\nabla f|^2} + \nabla^i \left(\frac{\Delta f f_i - f_{ij} f^j}{1 + |\nabla f|^2} \right)$$

and thus

$$\int_M \overline{S}\, dV_g = \int_M S\, dV_g - \int_M \frac{R_{ij} f^i f^j}{1 + |\nabla f|^2}\, dV_g.$$

We now prove the validity of the following integral sufficient condition for the existence of a constant negative scalar-Bach curvature, in the conformal class $[g]$ of a metric g:

Lemma 3.1. *Let M be a 4-dimensional closed manifold. If there exists a positive smooth function $u \in C^\infty(M)$ such that for a Riemannian metric g on M, satisfying $|B_g|_g > 0$, it holds*

$$\int_M F_g^B\, u^2\, dV_g + 6 \int_M |\nabla u|^2\, dV_g < 0,$$

then there exists a (unique) C^∞ metric $\widetilde{g} \in [g]$ such that $F^B_{\widetilde{g}} \equiv -1$.

Proof. Arguing as in [10, Lemma 3.2], we consider the conformal metric $g'_{ij} = u^2 g$. By (2.4) we have

$$F^B_{g'} = S_{g'} + t|\,\mathrm{B}_{g'}|^{\frac{1}{2}}_{g'} = u^{-2} \left(S_g + t|\,\mathrm{B}_g|^{\frac{1}{2}}_g - 6\frac{\Delta u}{u} \right).$$

Therefore, since $dV_{g'} = u^4 dV_g$, using the assumption we obtain

$$\int_M F^B_{g'}\, dV_{g'} = \int_M F^B_g\, u^2\, dV_g + 6 \int_M |\nabla u|^2\, dV_g < 0.$$

The conclusion follows now by Lemma 2.2. ☐

Adapting the method described in [10] and using Lemma 3.1, we are able to find a sufficient condition for the existence of metrics with constant negative scalar-Bach curvature:

Lemma 3.2. *Let (M, g) be a 4-dimensional closed manifold. Suppose that there exists a smooth function $f \in C^\infty(M)$ such that, for some $t > 0$, it holds*

$$\int_M \left(S_g + t|\mathrm{B}_g + E_g(f)|^{\frac{1}{2}}_f \right) dV_g - \int_M \frac{R_{ij} f^i f^j}{1 + |\nabla f|^2}\, dV_g + \frac{3}{2} \int_M \left[\frac{f_{ip} f^p f_{iq} f^q}{(1 + |\nabla f|^2)^2} - \frac{|f_{ij} f^i f^j|^2}{(1 + |\nabla f|^2)^3} \right] dV_g < 0,$$

where $|\cdot|_f$ denotes the norm with respect of $g + df \otimes df$, $E_g(f)$ is defined as in (3.10) and

$$|\mathrm{B}_{g+df\otimes df}|_f = |\mathrm{B}_g + E_g(f)|_f > 0, \quad \text{on } M.$$

Then, there exists a (unique) $C^\infty(M)$ metric $\widetilde{g} \in [g + df \otimes df]$ such that $F_{\widetilde{g}} \equiv -1$.

4. Proof of Theorem 1.1

This section is dedicated to the proof of Theorem 1.1. We point out that that the technique we use takes strong inspiration from [2], [3] and [10]. Before we begin the proof, we state the following useful

Step 1. First, we focus on the case

$$t > 0.$$

From [11], we know that there exist a metric g_0 such that $|\mathrm{B}_{g_0}|_{g_0} > 0$ on M, hence we can choose a metric $g \in [g_0]$ such that

$$F_g^B \geq 0 \quad \text{on } M, \tag{4.1}$$

otherwise, Lemma 2.1 would imply the existence of a smooth metric $g \in [g_0]$ such that $F_g^B < 0$ and Theorem 1.1 would follow from Lemma 2.2. Consider a positive smooth function $\psi \in C^\infty(M)$, a positive constant $k > 0$ and define

$$g' := \psi g, \quad g'' := g' + d(k\psi) \otimes d(k\psi).$$

To prove the existence of a metric $\widetilde{g}$ such that $F_{\widetilde{g}}^B \equiv -1$, we will proceed as follows: first we will prove that

$$\int_M F_{\widetilde{g}''}^B dV_g < 0, \tag{4.2}$$

where $\widetilde{g}'' = (1 + k^2|\nabla\psi|^2)^{-\frac{1}{2}} g''$, for suitable choices of ψ and k. Then, up to a perturbation of k, we will prove that $\left|\mathrm{B}_{g''}\right|_{g''} > 0$ everywhere and the claim will follow by Lemma 3.2. To show (4.2), observe that

$$g'' = g' + d(k\psi) \otimes d(k\psi) = \psi\left[g + d(2k\sqrt{\psi}) \otimes d(2k\sqrt{\psi})\right] =: \psi\overline{g}.$$

Applying the same argument in the proof of [10, Lemma 3.3], we deduce that

$$\begin{aligned} \Phi_M &:= \int_M F_{\widetilde{g}''}^B dV_{\widetilde{g}''} \\ &= \int_M \left(S_{g'} + t|\,\mathrm{B}_{g'} + E_{g'}(k\psi)|_{k\psi}^{\frac{1}{2}}\right) dV_{g'} - \int_M \frac{R'_{ij}\nabla_{g'}^i\psi\nabla_{g'}^j\psi}{1/k^2 + |\nabla_{g'}\psi|_{g'}^2} dV_{g'} \\ &\quad + \frac{3}{2}\int_M \left[\frac{\nabla_{ip}^{g'}\psi\nabla_{g'}^p\psi\nabla_{iq}^{g'}\psi\nabla_{g'}^q\psi}{(1/k^2 + |\nabla_{g'}\psi|_{g'}^2)^2} - \frac{|\nabla_{ij}^{g'}\psi\nabla_{g'}^i\psi\nabla_{g'}^j\psi|^2}{(1/k^2 + |\nabla_{g'}\psi|_{g'}^2)^3}\right] dV_{g'}, \end{aligned} \tag{4.3}$$

where $|\cdot|_{k\psi}$ denotes the norm with respect to $g' + d(k\psi) \otimes d(k\psi)$.

With respect to the metric g, by standard formulas for conformal transformations (see [12]), we have

$$\begin{aligned} S_{g'} &= \frac{1}{\psi}\left(S_g - 3\frac{\Delta\psi}{\psi} + \frac{3}{2}\frac{|\nabla\psi|^2}{\psi^2}\right), \\ R'_{ij} &= R_{ij} - \frac{\psi_{ij}}{\psi} + \frac{3}{2}\frac{\psi_i\psi_j}{\psi^2} - \frac{1}{2}\frac{\Delta\psi}{\psi}g_{ij}, \\ B'_{ij} &= \frac{1}{\psi}B_{ij}, \\ dV_{g'} &= \psi^2\, dV_g, \\ \nabla_{ij}^{g'}\psi &= \psi_{ij} - \frac{1}{\psi}\left(\psi_i\psi_j - \frac{1}{2}|\nabla\psi|^2 g_{ij}\right). \end{aligned} \tag{4.4}$$

where R'_{ij}, B'_{ij} and R_{ij}, B_{ij} are relative to the metrics g' and g, respectively. On the other hand, observe that

$$g'' = g' + d(k\psi) \otimes d(k\psi) = \psi\left[g + d(2k\sqrt{\psi}) \otimes d(2k\sqrt{\psi})\right] =: \psi\overline{g};$$

moreover, in dimension four the Bach tensor is conformally covariant, namely

$$B'_{ij} = \frac{1}{\psi}B_{ij}.$$

Thus,

$$\begin{aligned}
B'_{ij} + E'(k\psi)_{ij} = B''_{ij} &= \frac{1}{\psi}\overline{B}_{ij} \\
&= \frac{1}{\psi}\Big(B_{ij} + E(2k\sqrt{\psi})_{ij}\Big) \\
&= B_{ij} + \frac{1}{\psi}E(2k\sqrt{\psi})_{ij}
\end{aligned}$$

which implies that the "error term" of the Bach tensor satisfies:

$$E_{g'}(k\psi) = \frac{1}{\psi}E_g(2k\sqrt{\psi}).$$

In particular, the following relation is satisfied

$$\big|\mathrm{B}_{g''}\big|^{\frac{1}{2}}_{k\psi} = \big|\mathrm{B}_{g'} + E_{g'}(k\psi)\big|^{\frac{1}{2}}_{k\psi} = \frac{1}{\psi^2}\Big|\mathrm{B}_g + E_g(2k\sqrt{\psi})\Big|^{\frac{1}{2}}_{\overline{g}}.$$

Then, following the computations in [3] and [10] we get

$$\begin{aligned}
\Phi_M = &\int_M \left(S_g + \frac{t}{\psi}|\,\mathrm{B}_g + E_g(2k\sqrt{\psi})|^{\frac{1}{2}}_{\overline{g}} - \frac{R_{ij}\psi_i\psi_j}{\psi/k^2 + |\nabla\psi|^2}\right)\psi\, dV_g \\
&+ \int_M \frac{\psi_{ij}\psi^i\psi^j}{\psi/k^2 + |\nabla\psi|^2}\, dV_g \\
&+ \frac{1}{k^2}\frac{3}{2}\int_M \frac{|\nabla\psi|^2}{\psi/k^2 + |\nabla\psi|^2}\, dV_g - \frac{1}{k^2}\int_M \frac{\psi\Delta\psi}{\psi/k^2 + |\nabla\psi|^2}\, dV_g \\
&+ \frac{3}{2}\int_M \left[\frac{\psi_{ip}\psi^p\psi_{iq}\psi^q}{(\psi/k^2 + |\nabla\psi|^2)^2} - \frac{|\psi_{ij}\psi^i\psi^j|^2}{(\psi/k^2 + |\nabla\psi|^2)^3}\right]\psi\, dV_g \\
&+ \frac{1}{k^2}\frac{3}{2}\int_M \frac{\frac{1}{4}|\nabla\psi|^6 - |\nabla\psi|^2(\psi_{ij}\psi^i\psi^j)\psi}{(\psi/k^2 + |\nabla\psi|^2)^3}\psi\, dV_g.
\end{aligned} \tag{4.5}$$

Step 2. Let $p \in M$ and consider a local, normal, geodesic polar coordinate system $\rho, \omega_1, ..., \omega_{n-1}$ defined in an open neighborhood V of p, in order to have

$$g_{\rho\rho} = 1, \quad g_{\rho i} = 0, \quad g_{ij} = \delta_{ij} + \rho^2 a_{ij}, \quad g^{\rho\rho} = 1$$

at p, where the index i corresponds to the coordinate ω_i, for $i = 1, ..., n-1$ and the coefficients a_{ij} are of order 1; from now on, we use the index convention

$$\alpha, \beta, \gamma, ... = 1, 2, 3, \rho, \quad i, j, k, ... = 1, 2, 3.$$

The Christoffel symbols of the Levi-Civita connection are written as

$$\Gamma^{\rho}_{\rho\rho} = \Gamma^{\rho}_{\rho i} = 0, \quad \Gamma^{\rho}_{ij} = -\frac{\rho}{2}(a_{ij} + \rho\partial_\rho a_{ij}). \tag{4.6}$$

Let B_r be the geodesic ball centered at p of radius $0 < r < r_0$, with r_0 such that $B_{r_0} \subset V$ and let $y = y(x)$ is a real C^4 function such that

$$\begin{cases} y(-x) = y(x), \ \forall x \in \mathbb{R} \\ y(x) = 1, \ \forall |x| \geq 1 \\ y(x) \geq \delta > 0, \ \forall x \in \mathbb{R} \\ y'(x) > 0, \ \forall 0 < x < 1 \\ y'(x) \geq 1, \ \forall (1/4)^{1/(n-1)} \leq x \leq (3/4)^{1/(n-1)} \\ |y''| \leq \frac{y'}{(1-x)} \quad \text{as } x \to 1 \end{cases}.$$

Let $B_r = B_r(p)$ be the geodesic ball centered at p of radius $0 < r < r_0$, with r_0 such that $B_r \subset M$. For $p' \in B_r$, we choose

$$\psi(p') := y\left(\frac{\rho}{r}\right), \quad \rho = \mathrm{dist}_g(p', p).$$

From now on, to simplify the expressions, we will omit arguments in the functions: it will be clear that if ψ, ψ_ρ, etc. are computed at $p' \in B_r$, then y, y', y'' will be computed at ρ/r with $\rho = \mathrm{dist}_g(p', p)$. Moreover, we will denote by $C = C(n, \delta, t, p) > 0$ some universal positive constant independent of r and k.

Step 3. Arguing as in [10], it is possible to obtain an estimate for the terms not involving the Bach tensor, when they are restricted to the ball B_r. In particular, applying the same argument of Step 3 of [10] we obtain that

$$\Phi_{B_r} \leq t \int_{B_r} |\,\mathrm{B}_g + E_g(2k\sqrt{\psi})|_g^{\frac{1}{2}}\, dV_g + C|B_r| + \frac{1}{r^2}\int_{B_r} y''\, dV_g + \frac{1}{k^2}\Theta, \tag{4.7}$$

where Φ_{B_r} denotes the quantity defined in (4.5) restricted to B_r. Note that this intermediate estimate, when $t = 0$, coincides with the one of Aubin in [3].

Step 4. We now give an estimate of the remaining terms, in which the Bach tensor appears. Since

$$\overline{g} = g + d(2k\sqrt{\psi}) \otimes d(2k\sqrt{\psi}),$$

for the sake of simplicity, we introduce

$$\eta := 2\sqrt{\psi}, \tag{4.8}$$

where

$$\psi(p') := y\Big(\frac{\rho}{r}\Big)$$

and we have

$$\overline{g} = g + k^2 d\eta \otimes d\eta. \tag{4.9}$$

From (3.4), we have

$$\overline{g}^{\rho\rho} = \frac{1}{1 + k^2\eta_\rho^2}, \quad \overline{g}^{\rho i} = 0, \quad \overline{g}^{ij} = g^{ij}.$$

In particular, since $\overline{g} \geq g$, for every $(0, p)$-tensor T we immediately get that

$$|\,\mathrm{T}_g\,|_{\overline{g}} \leq |\,\mathrm{T}_g\,|_g \leq C, \tag{4.10}$$

where C is a constant. Note that, by definition of η and (4.6),

$$\eta_i = 0, \quad \eta_\rho = \frac{1}{2r}\eta^{-1}y'\Big(\frac{\rho}{r}\Big);$$
$$\eta_{\rho\rho} = -\frac{1}{4}\left[\frac{1}{r}y'\Big(\frac{\rho}{r}\Big)\right]^2\eta^{-3} + \frac{1}{2r^2}\eta^{-1}y''\Big(\frac{\rho}{r}\Big);$$
$$\eta_{\rho i} = 0, \quad \eta_{ij} = \frac{1}{4r}\eta^{-1}\rho(a_{ij}+\rho\partial_\rho a_{ij})y'\Big(\frac{\rho}{r}\Big),$$

which imply that there exists $C \in \mathbb{R}$ such that

$$\frac{C^{-1}y'}{r} \le \eta_\rho \le \frac{Cy'}{r}; \qquad |\eta_{ij}| \le Cr|\eta_\rho| \le C \tag{4.11}$$

and, more in general, if r is sufficiently small,

$$\eta_\alpha \le \frac{C}{r}, \quad |\eta_{\alpha\beta}| \le \frac{C}{r^2}.$$

Observe that, since y only depends on ρ, differentiating η with respect to an angular coordinate does not raise the exponent of r at the denominator, while differentiating with respect to ρ produces an additional $1/r$ in the derivative: hence, one can easily note that

$$|\partial_{\alpha_1,\dots,\alpha_N}\eta| \le \frac{C}{r^M},$$

where $M = \#\{i = 1, ..., N : \alpha_i = \rho\}$. In particular, we have

(4.12)
$$|\eta_{\rho\rho}| \le \frac{C}{r^2}; \quad |\eta_{\rho\rho\rho}| \le \frac{C}{r^3}; \quad |\eta_{\rho\rho\rho\rho}| \le \frac{C}{r^4}; \quad |\eta_{ij}| \le C; \quad |\partial_\rho\eta_{ij}| \le \frac{C}{r}; \quad |\partial_t\partial_\rho\eta_{ij}| \le \frac{C}{r}; \quad |\partial_{i_1\dots i_l}\eta_{ij}| \le C.$$

Moreover, by assumption $|y''(x)| \le y'/(1-\frac{\rho}{r})$ and the definition of y, we exploit

$$|\eta_{\rho\rho}| \le C\left(\frac{(y')^2}{r^2} + \frac{|y''|}{r^2}\right) \le \frac{C|y'|}{r^2(\frac{\rho}{r}-1)} \le \frac{C|\eta_\rho|}{r(\frac{\rho}{r}-1)} \le \frac{C|\eta_\rho|}{r-\rho} \tag{4.13}$$

and

$$\begin{cases} |\partial_\rho\eta_{ij}| = \left|\partial_\rho\left[\frac{1}{4r}\eta^{-1}\rho(a_{ij}+\rho\partial_\rho a_{ij})y'\Big(\frac{\rho}{r}\Big)\right]\right| \le \frac{C|\eta_\rho|}{1-\frac{\rho}{r}} \le \frac{C|\eta_\rho|}{r-\rho}; \\ |\partial_t\partial_\rho\eta_{ij}| = \left|\partial_t\partial_\rho\left[\frac{1}{4r}\eta^{-1}\rho(a_{ij}+\rho\partial_\rho a_{ij})y'\Big(\frac{\rho}{r}\Big)\right]\right| \le \frac{C|\eta_\rho|}{1-\frac{\rho}{r}} \le \frac{C|\eta_\rho|}{r-\rho}; \end{cases} \tag{4.14}$$

summarizing, we have

$$|\partial_\rho\eta_p| = \left|\eta_{\rho\rho} + \Gamma^\alpha_{\rho\rho}\eta_\alpha\right| = |\eta_{\rho\rho}| \le \frac{C|\eta_\rho|}{r-\rho};$$
$$|\partial_\rho\eta_{\rho\rho}| = \left|\eta_{\rho\rho\rho} + 2\Gamma^\alpha_{\rho\rho}\eta_{\alpha\rho}\right| = |\eta_{\rho\rho\rho}| \le \frac{C}{r^3}\left(\left|(y')^3\right| + \left|y'y''\right| + \left|y'''\right|\right) \le \frac{C}{r^3};$$
$$|\partial_\rho\eta_{ij}| \le \frac{C|\eta_\rho|}{1-\frac{\rho}{r}} = \frac{Cr|\eta_\rho|}{r-\rho};$$
$$|\partial_\rho\partial_\rho\eta_{ij}| = \left|\frac{\eta_{\rho\rho}\rho y'L}{r} + \frac{\eta_\rho^2\rho y'L}{r} + \frac{\eta_\rho y'L}{r} + \frac{\eta_\rho\rho y''L}{r^2} + \frac{\eta_\rho y'L}{r} + \frac{Ly''}{r^2} + \frac{\eta_\rho\rho y''L}{r} + \frac{\rho y'''L}{r^3}\right| \tag{4.15}$$
$$\le \frac{C|y'''|}{r^2} \le \frac{C}{r^2};$$

$$|\partial_l \eta_{ij}| = \left| \frac{1}{4r} \eta^{-1} \rho (\partial_l a_{ij} + \rho \partial_l \partial_\rho a_{ij}) y' \left(\frac{\rho}{r}\right) \right| \leq Cr |\eta_\rho| \leq C;$$
$$|\partial_t \partial_l \eta_{ij}| = \left| \frac{1}{4r} \eta^{-1} \rho (\partial_t \partial_l a_{ij} + \rho \partial_t \partial_l \partial_\rho a_{ij}) y' \left(\frac{\rho}{r}\right) \right| \leq Cr \eta_\rho \leq C;$$
$$|\partial_t \partial_\rho \eta_{ij}| \leq C |\partial_\rho \eta_{ij}| \leq \frac{rC|\eta_\rho|}{r - \rho},$$

where L denotes a bounded quantity and C is a constant. Depending on the term we need to estimate, we are going to use (4.12) or (4.13) and (4.14).
To give an estimate of the remaining integral depending on the Bach tensor we exploit the following

Lemma 4.1. *We have*

$$\int_{B_r} \left| \mathrm{B}_g + E_g(2k\sqrt{\psi}) \right|_{\overline{g}}^{\frac{1}{2}} dV_g = \int_{B_r} |\mathrm{B}_{\overline{g}}|_{\overline{g}}^{\frac{1}{2}} dV_g \leq C|B_r| + \frac{C}{k}\Theta, \tag{4.16}$$

for some constant $C = C(\delta, t, p)$ and a continuous function $\Theta = \Theta(p, \frac{1}{k}, r) > 0$; here $|B_r|$ denotes the volume of B_r.

Proof. Given a tensor T in the metric g, we will denote as $\overline{\mathrm{T}}$ the same tensor with respect to the metric $\overline{g}$.
We recall that by (3.10), we have

(4.17)
$$|\overline{\mathrm{B}}|_{\overline{g}} = \left| \frac{1}{2} \left[\bar{\Delta} \overline{\mathrm{Ric}} - \frac{1}{3} \overline{\mathrm{Hess}}(\overline{S}) + 2 \overline{\mathrm{Ric}} * \overline{\mathrm{Riem}} - \frac{2}{3} \overline{S}\, \overline{\mathrm{Ric}} - \frac{1}{6} \bar{\Delta} \overline{S} \overline{g} - \frac{1}{2} \left(|\overline{\mathrm{Ric}}|_{\overline{g}}^2 - \frac{\overline{S}^2}{3} \right) \overline{g} \right] \right|_{\overline{g}}$$
$$\leq \frac{1}{2} \left[|\bar{\Delta} \overline{\mathrm{Ric}}|_{\overline{g}} + \frac{1}{3} |\overline{\mathrm{Hess}}(\overline{S})|_{\overline{g}} + 2 |\overline{\mathrm{Ric}} * \overline{\mathrm{Riem}}|_{\overline{g}} + \frac{2}{3} |\overline{S}\, \overline{\mathrm{Ric}}|_{\overline{g}} + \frac{1}{6} |\bar{\Delta} \overline{S} \overline{g}|_{\overline{g}} + \frac{1}{2} \left| \left(|\overline{\mathrm{Ric}}|_{\overline{g}}^2 - \frac{\overline{S}^2}{3} \right) \overline{g} \right|_{\overline{g}} \right],$$

where $*$ denotes the contraction $R_{\gamma\delta} R_{\alpha\gamma\beta\delta}$. The computations in Section 5 show that

$$\begin{cases} |\bar{\Delta} \overline{\mathrm{Ric}}|_{\overline{g}} \leq C\Big(1 + \frac{1}{r-\rho} + \frac{1}{k^2}\Theta\Big); \\ |\overline{\mathrm{Hess}}(\overline{S})|_{\overline{g}} \leq C\Big(1 + \frac{1}{r-\rho} + \frac{1}{k^2}\Theta\Big); \\ |\overline{\mathrm{Ric}} * \overline{\mathrm{Riem}}|_{\overline{g}} \leq C\big(1 + \frac{1}{k^2}\Theta\big); \\ |\overline{S}\, \overline{\mathrm{Ric}}|_{\overline{g}} \leq C\big(1 + \frac{1}{k^2}\Theta\big); \\ |\bar{\Delta} \overline{S} \overline{g}|_{\overline{g}} \leq C\Big(1 + \frac{1}{r-\rho} + \frac{1}{k^2}\Theta\Big); \\ \Big| \Big(|\overline{\mathrm{Ric}}|_{\overline{g}} - \frac{\overline{S}^2}{3} \Big) \overline{g} \Big|_{\overline{g}} \leq C\big(1 + \frac{1}{k^2}\Theta\big), \end{cases}$$

for some $C = C(n, \delta, t, p) > 0$ and $\Theta = \Theta(p, 1/k, r) > 0$; hence we have

$$|\overline{\mathrm{B}}|_{\overline{g}} \leq C \left(1 + \frac{1}{r - \rho} + \frac{1}{k^2} \Theta \right).$$

As a consequence, we get

$$|\overline{\mathrm{B}}|_{\overline{g}}^{\frac{1}{2}} \leq C + \frac{C}{(r - \rho)^{\frac{1}{2}}} + \frac{C}{k} \Theta. \tag{4.18}$$

Note that, when $\rho \to r$, we have that $1/(r-\rho)^{\frac{1}{2}}$ is integrable and thus (4.18) implies

$$t\int_{B_r} |\overline{\mathrm{B}}|_{\overline{g}}^{\frac{1}{2}} \leq C|B_r| + \frac{C}{k}\Theta.$$

□

Remark 4.2. We point out that, in [10, Lemma 4.1], there is a misprint: the power of k should be -1 instead of -2, which, however, does not affect the validity of the arguments.

Step 5. Using Lemma 4.1 in (4.7), we obtain

$$\Phi_{B_r} \leq C|B_r| + \frac{1}{r^2}\int_{B_r} y''\, dV_g + \frac{1}{k}\Theta. \tag{4.19}$$

Now, we have to make sure that $|\mathrm{B}_{\overline{g}}|_{\overline{g}} > 0$ on M: in order to do so, assume that there exists a point $p' \in B_r$ such that $|\mathrm{B}_{\overline{g}}|_{\overline{g}} = 0$ vanishes at p'. If we evaluate the components of $\mathrm{B}_{\overline{g}}$ at p', we obtain that the equation $\overline{B}_{\alpha\beta}(p') = 0$ is a polynomial equation of finite degree $N_{\alpha,\beta}$ in the variable k, for $\alpha, \beta = 1, 2, 3, \rho$, up to multiplying the equation for $(1 + k^2\eta_\rho^2)^\lambda$, where $\lambda = \lambda(\alpha, \beta)$ is the highest power of $(1 + k^2\eta_\rho^2)$ appearing in the denominators of the expression of $\overline{B}_{\alpha\beta}$ (this can be done since all the denominators in these expressions are of the form $(1 + k^2\eta_\rho)^\gamma$, as can be seen in the previous computations and in Section 5: this means that $|\mathrm{B}_{\overline{g}}|_{\overline{g}} = 0$ at p' if and only if k is a root of all the polynomials, which implies that

$$k \in A := \{k_1, ..., k_L\}, \quad L \leq N := \min_{\alpha,\beta=1,2,3,\rho} N_{\alpha,\beta}$$

where $k_1, ..., k_L$ are the common roots of the polynomials. We observe that, since k is a fixed constant in (4.9), the roots of the polynomials $\overline{B}_{\alpha\beta}(q)$ have to be contained in A, for every $q \in B_r$ such that $|\mathrm{B}_{\overline{g}}|_{\overline{g}} = 0$ at q. Therefore, in order to have that $\mathrm{B}_{\overline{g}}$ does not vanish on B_r, it is sufficient to choose k outside of A in (4.9): hence, we can conclude that $|\mathrm{B}_{\overline{g}}|_{\overline{g}} \neq 0$ on B_r.

In order to conclude the proof, we apply the same argument as the one in Step 5 of [10]: for the sake of completeness, we include it here. Using (4.19) and that, by assumption, $y'(x) \geq 1$ for all $(1/4)^{1/(n-1)} \leq x \leq (3/4)^{1/(n-1)}$, we obtain

$$\begin{aligned}\Phi_{B_r} &\leq C\left(1 + \frac{1}{r}\right)|B_r| - \frac{n-1}{r}|\mathbb{S}^{n-1}| \inf_M \sqrt{\det g_{ij}} \int_{r(\frac{1}{4})^{1/(n-1)}}^{r(\frac{3}{4})^{1/(n-1)}} \rho^{n-2}\, d\rho + \frac{1}{k}\Theta \\ &\leq C\left(1 + \frac{1}{r}\right)|B_r| - \frac{C_2}{r^2}|B_r| + \frac{1}{k}\Theta,\end{aligned}$$

where we used the fact that $|B_r| \sim cr^n$ as $r \to 0$. In particular, there exist a continuous function $\lambda(p) > 0$ and, for $p \in M$ fixed, a continuous function $\Theta_p(r) > 0$ in r, for $0 < r < r_0$, such that

$$\Theta(p, 1/k, t) \leq \Theta_p(r),$$

and

$$\Phi_{B_r} \leq \left[C\left(1 + \frac{1}{r}\right) - \frac{\lambda}{r^2}\right]|B_r| + \frac{1}{k}\Theta_p(r). \tag{4.20}$$

Since, by (4.1), $F^B_g = S_g + t|\,\mathrm{B}_g\,|_g^{\frac{1}{2}} \geq 0$, given $\nu > 0$, there exists a positive radius $0 < r_1 < r_0$ such that

$$\frac{\lambda}{r_1^2} - C\left(1 + \frac{1}{r_1}\right) - 1 \geq \nu \widehat{F}^B_g, \tag{4.21}$$

where $\widehat{F}^B_g := \left(\int_M F^B_g \, dV_g\right) / \mathrm{Vol}_g(M)$.

We consider h disjoint geodesic balls $B^j_{r_1}(p_j)$ of radius $r = r_1$, $j = 1, ..., h$, together with the corresponding function $\psi^{[j]}$, as constructed above, such that, for ν sufficiently large,

$$\sum_{j=1}^{h} \left|B^j_{r_1}(p_j)\right| > \frac{1}{\nu} \mathrm{Vol}_g(M);$$

we define A_j as before and, on B^j, we choose k_j such that the Bach tensor of the deformed metric does not vanish on B^j and $k_j \notin A_i$ for every $i = 1, ..., h$.

On every B^j, we set

$$k \geq \max\left\{1, \sup_{j=1,\ldots,h} \frac{\Theta_{p_j}(r_1)}{|B^j_{r_1}(p_j)|}\right\}, \quad k \notin \bigcup_{j=1}^{h} A_j,$$

which is possible since $\bigcup_{j=1}^h A_j$ is a finite set. From (4.20) and (4.21), for all $j = 1, \ldots, h$, we get

$$\Phi_{B^j_{r_1}} \leq -\nu \widehat{F}^B_g |B^j_{r_1}(p_j)| - |B^j_{r_1}(p_j)| + \frac{1}{k}\Theta_{p_j}(r_1) \leq -\nu \widehat{F}^B_g |B^j_{r_1}(p_j)|.$$

If we define

$$\psi := \begin{cases} 1, & \text{on } M \setminus \bigcup_{j=1}^h B^j \\ \psi^{[j]}, & \text{on } B^j, \end{cases}$$

we obtain

$$\Phi_M \leq \int_M F^B_g \, dV_g - \nu \widehat{F}^B_g \sum_{j=1}^{h} |B^j_{r_1}(p_j)| = \widehat{F}^B_g \left(\mathrm{Vol}_g(M) - \nu \sum_{j=1}^{h} |B^j_{r_1}(p_j)|\right) < 0;$$

furthermore, by our choice of k and the fact that $|\mathrm{B}_g|_g > 0$ on M, we obtain that $|\mathrm{B}_{\overline{g}}|_{\overline{g}} > 0$ on M. By Lemma 3.2, there exists a metric $\widetilde{g} \in [\overline{g}]$ such that $F^B_{\widetilde{g}} \equiv -1$.

Finally, we consider the case $t \leq 0$; by [2, 3] we know that, on a closed 4-dimensional manifold, there exists a Riemannian metric g' with constant scalar curvature -1, which is constructed *via* the same deformation we exploited in the previous case, starting from a reference metric. Hence, let g be a Riemannian metric on M such that $|\mathrm{B}_g|_g > 0$ at every point of M: exploiting Aubin's proof and the previous argument on the choice of k, we can produce a metric $\overline{g}$ such that $\int_M S_{\overline{g}} dV_{\overline{g}} < 0$ and $|\mathrm{B}_{\overline{g}}|_{\overline{g}} > 0$ on M. Therefore, since $t \leq 0$, obviously $\int_M F^B_{\overline{g}} dV_{\overline{g}} < 0$ and, by Lemma 2.2, there exists a metric $\widetilde{g} \in [\overline{g}]$ such that $F^B_{\widetilde{g}} \equiv -1$.

This concludes the proof of Theorem 1.1.

□

5. Estimates on the deformed Bach tensor

In this section we prove the estimate (4.18) that we used in the fourth step of the proof of Theorem 1.1: for the sake of simplicity, we will write $\mathrm{T} = \mathrm{T}_g$ and $\overline{\mathrm{T}} = \mathrm{T}_{\overline{g}}$ for every considered tensor T. We recall that, for the metric $\overline{g}$ defined in (4.9), we have

$$(5.1)\qquad \left|\overline{\mathrm{B}}\right|_{\overline{g}} = \left|\frac{1}{2}\left[\overline{\Delta}\overline{\mathrm{Ric}} - \frac{1}{3}\overline{\mathrm{Hess}}(\overline{S}) + 2\overline{\mathrm{Ric}} * \overline{\mathrm{Riem}} - \frac{2}{3}\overline{S}\,\overline{\mathrm{Ric}} - \frac{1}{6}\overline{\Delta}\overline{S}\overline{g} - \frac{1}{2}\left(\left|\overline{\mathrm{Ric}}\right|^2_{\overline{g}} - \frac{\overline{S}^2}{3}\right)\overline{g}\right]\right|_{\overline{g}}.$$

To prove the validity of (4.18), we analyze the components of $\overline{B}$ separately, proving that each term in (5.1) satisfies

$$(5.2)\qquad |\cdot|_{\overline{g}} \leq C\left(1 + \frac{1}{r-\rho} + \frac{1}{k}\Theta\right) \quad \text{in } B_r,$$

where C is a positive constant and $\Theta = \Theta(p, \frac{1}{k}, r)$ is a positive continuous function. For the sake of simplicity, given a $(0,q)$-tensor T, we will denote

$$\left|T_{\alpha_1\dots\alpha_q}\right|^2_{\overline{g}} = T_{\alpha_1\dots\alpha_q}T_{\beta_1\dots\beta_q}\overline{g}^{\alpha_1\beta_1}\dots\overline{g}^{\alpha_q\beta_q};$$

for instance, on a $(0,2)$-tensor we have

$$\begin{aligned}
|T_{ij}|^2_{\overline{g}} &= T_{ij}T_{lt}\overline{g}^{il}\overline{g}^{jt} = |T_{ij}|^2_g \leq C|T_{ij}|^2;\\
|T_{i\rho}|^2_{\overline{g}} &= T_{i\rho}T_{j\rho}\overline{g}^{ij}\overline{g}^{\rho\rho} = \frac{1}{1+k^2\eta_\rho^2}|T_{i\rho}|^2_g \leq \frac{C}{1+k^2\eta_\rho^2}|T_{i\rho}|^2;\\
|T_{\rho\rho}|^2_{\overline{g}} &= T_{\rho\rho}T_{\rho\rho}\overline{g}^{\rho\rho}\overline{g}^{\rho\rho} = \frac{1}{(1+k^2\eta_\rho^2)^2}|T_{\rho\rho}|^2_g = \frac{1}{(1+k^2\eta_\rho^2)^2}|T_{\rho\rho}|^2.
\end{aligned}$$

Note that when we consider Aubin's deformation, we have (see Section 3)

$$\begin{aligned}
\overline{R}_{\alpha\beta\gamma\delta} &= R_{\alpha\beta\gamma\delta} + E^R_{\alpha\beta\gamma\delta};\\
\overline{R}_{\alpha\beta} &= R_{\alpha\beta} + F_{\alpha\beta};\\
\overline{S} &= S + H;
\end{aligned}$$

where

$$\begin{aligned}
E^R_{ijlt} =& \frac{k^2}{1+k^2\eta_\rho^2}(\eta_{il}\eta_{jt} - \eta_{it}\eta_{jl});\\
E^R_{i\rho lt} =& 0;\\
E^R_{i\rho l\rho} =& \frac{k^2\eta_{\rho\rho}\eta_{il}}{1+k^2\eta_\rho^2};\\
F_{ij} =& -\frac{k^2\eta_\rho^2 R_{i\rho j\rho}}{1+k^2\eta_\rho^2} + \frac{Ck^2(\eta_{ANG}\eta_{ANG})_{ij}}{1+k^2\eta_\rho^2} + \frac{k^2\eta_{\rho\rho}^2\eta_{ij}}{(1+k^2\eta_\rho^2)^2};\\
F_{i\rho} =& 0;\\
F_{\rho\rho} =& \frac{k^2\eta_i^i\eta_{\rho\rho}}{1+k^2\eta_\rho^2};\\
H =& -\frac{2k^2}{1+k^2\eta_\rho^2}R_{\rho\rho}\eta_\rho^2 + \frac{k^2}{1+k^2\eta_\rho^2}\Big((\Delta\eta)^2 - |\mathrm{Hess}(\eta)|^2\Big) - \frac{2k^4}{(1+k^2\eta_\rho^2)^2}\left(\Delta\eta\eta_\rho^2\eta_{\rho\rho} - \eta_\rho^2\eta_{\rho\rho}^2\right)
\end{aligned}$$

$$=L+\frac{L}{1+k^2\eta_\rho^2}+\frac{Ck^2\eta_{ANG}\eta_{ANG}}{1+k^2\eta_\rho^2}+\frac{2k^2\eta_{\rho\rho}\eta_{ll}}{(1+k^2\eta_\rho^2)^2}.$$

We use the notation η_{ANG} to denote terms of the type η_{ij}, where i,j are angular coordinates (with this notation, we have coupled the angular terms $\eta_{ll}\eta_{ij}$ and $\eta_{il}\eta^l_j$ in F_{ij} and the terms η^2_{ll} and $\eta_{ij}\eta_{ij}$ in H, since for small r, they have the same behavior). Then, by (4.12) and (4.11) we get:

(5.3)

$$\begin{aligned}
|E^R_{ijlt}| =&\left|\frac{k^2}{1+k^2\eta_\rho^2}(\eta_{il}\eta_{jt}-\eta_{it}\eta_{jl})\right| \leq \left|\frac{k^2Lr^2\eta_\rho^2}{k^2\eta_\rho^2}\right| \leq Cr^2 \leq C;\\
|E^R_{i\rho lt}| =&0;\\
|E^R_{i\rho l\rho}| =&\left|\frac{k^2\eta_{\rho\rho}\eta_{il}}{1+k^2\eta_\rho^2}\right| \leq \left|\frac{k^2L}{k^2\eta_\rho^2}\frac{r\eta_\rho^2}{r^2}\right| \leq \frac{C}{r} \leq C\Theta;\\
|F_{ij}| =&\left|-\frac{k^2\eta_\rho^2R_{i\rho j\rho}}{1+k^2\eta_\rho^2}+\frac{Ck^2(\eta_{ANG}\eta_{ANG})_{ij}}{1+k^2\eta_\rho^2}+\frac{k^2\eta_{\rho\rho}^2\eta_{ij}}{(1+k^2\eta_\rho^2)^2}\right|\\
\leq&\left|\frac{Lk^2\eta_\rho^2}{k^2\eta_\rho^2}\right|+\left|\frac{Lk^2\eta_\rho^2r^2}{k^2\eta_\rho^2}\right|+\left|\frac{k^2L}{r^2\frac{k^4}{r^4}\left(\frac{r^2}{k^2}+\frac{1}{4}\eta^{-2}(y')^2\right)^2}\right| \leq C+C+\frac{C}{k^2}\Theta \quad \leq C\left(1+\frac{1}{k^2}\Theta\right);\\
|F_{i\rho}| =&0;\\
|F_{\rho\rho}| =&\left|\frac{k^2\eta_i^i\eta_{\rho\rho}}{1+k^2\eta_\rho^2}\right| \leq C\Theta;\\
|H| =&\left|L+\frac{L}{1+k^2\eta_\rho^2}+\frac{Ck^2\eta_{ANG}\eta_{ANG}}{1+k^2\eta_\rho^2}+\frac{2k^2\eta_{\rho\rho}\eta_{ll}}{(1+k^2\eta_\rho^2)^2}\right|\\
\leq& C+C+C+r^2\frac{C}{k^2}\Theta \leq C\left(1+\frac{1}{k^2}\Theta\right).
\end{aligned}$$

Note that we have not used (4.13) and (4.14), however il will be useful to have an estimate of $F_{\rho\rho}$ in terms of $\frac{1}{r-\rho}$ (since we are going to use it later to compute some of the remainders of $\overline{\Delta \mathrm{Ric}}$):

$$|F_{\rho\rho}| \leq \left|\frac{k^2\eta_\rho^2\frac{r}{r-\rho}}{r\big(1+k^2\eta_\rho^2\big)}\right| \leq \frac{C}{r-\rho}. \tag{5.4}$$

It follows

(5.5)

$$\begin{aligned}
|E^R_{ijlt}|_{\overline{g}} =&|E^R_{ijlt}|_g \leq C|E^R_{ijlt}| \leq Cr^2 \leq C;\\
|E^R_{i\rho lt}|_{\overline{g}} =&0;\\
|E^R_{i\rho l\rho}|_{\overline{g}} =&\frac{1}{1+k^2\eta_\rho^2}\left|\frac{k^2\eta_{\rho\rho}\eta_{il}}{1+k^2\eta_\rho^2}\right|_g \leq \frac{C}{1+k^2\eta_\rho^2}\left|\frac{k^2\eta_{\rho\rho}\eta_{il}}{1+k^2\eta_\rho^2}\right| \leq \frac{k^2C}{r^2\frac{k^4}{r^4}\left(\frac{r^2}{k^2}+\frac{\eta^{-2}}{4}(y')^2\right)^2} \leq r^2\frac{C}{k^2}\Theta \leq \frac{C}{k^2}\Theta;\\
|F_{ij}|_{\overline{g}} =&|F_{ij}|_g \leq C|F_{ij}| \leq C\left(1+\frac{1}{k^2}\Theta\right);\\
|F_{i\rho}|_{\overline{g}} =&0;
\end{aligned}$$

$$
\begin{aligned}
|F_{\rho\rho}|_{\overline{g}} =& \frac{1}{1+k^2\eta_\rho^2}\left|\frac{k^2\eta_i^i\eta_{\rho\rho}}{1+k^2\eta_\rho^2}\right|_g = \frac{1}{1+k^2\eta_\rho^2}\left|\frac{k^2\eta_i^i\eta_{\rho\rho}}{1+k^2\eta_\rho^2}\right| \leq \frac{C}{k^2}\Theta;\\
|H|_{\overline{g}} =& |H|_g = |H| \leq C\Big(1+\frac{1}{k^2}\Theta\Big).
\end{aligned}
$$

As a consequence, exploiting (5.5), a straightforward computation proves that the terms of (4.17) in which $\mathrm{Riem}, \mathrm{Ric}$ and S appear satisfy (5.2). For instance,

$$
\begin{aligned}
\big|\overline{S}\,\overline{\mathrm{Ric}}\big|_{\overline{g}} =& |SR_{\alpha\beta} + HR_{\alpha\beta} + SF_{\alpha\beta} + HF_{\alpha\beta}|_{\overline{g}}\\
=& |SR_{ij} + HR_{ij} + SF_{ij} + HF_{ij} + SR_{i\rho} + HR_{i\rho} + SR_{\rho\rho} + HR_{\rho\rho} + SF_{\rho\rho} + HF_{\rho\rho}|_{\overline{g}}\\
\leq& |S|_{\overline{g}}|R_{ij}|_{\overline{g}} + |H|_{\overline{g}}|R_{ij}|_{\overline{g}} + |S|_{\overline{g}}|F_{ij}|_{\overline{g}} + |H|_{\overline{g}}|F_{ij}|_{\overline{g}} + |S|_{\overline{g}}|R_{i\rho}|_{\overline{g}} + |H|_{\overline{g}}|R_{i\rho}|_{\overline{g}}\\
&+ |S|_{\overline{g}}|R_{\rho\rho}|_{\overline{g}} + |H|_{\overline{g}}|R_{\rho\rho}|_{\overline{g}} + |S|_{\overline{g}}|F_{\rho\rho}|_{\overline{g}} + |H|_{\overline{g}}|F_{\rho\rho}|_{\overline{g}}\\
\leq& C + C\Big(1+\frac{1}{k^2}\Theta\Big) + C\Big(1+\frac{1}{k^2}\Theta\Big) + C\Big(1+\frac{1}{k^2}\Theta\Big) + C + C\Big(1+\frac{1}{k^2}\Theta\Big)\\
&+ C + C\Big(1+\frac{1}{k^2}\Theta\Big) + \frac{C}{k^2}\Theta + \frac{C}{k^2}\Theta\\
\leq& C\Big(1+\frac{1}{k^2}\Theta\Big),
\end{aligned}
\tag{5.6}
$$

A similar computation shows that

$$
\big|\overline{\mathrm{Ric}}\big|_{\overline{g}} \leq C\Big(1+\frac{1}{k^2}\Theta\Big); \qquad \big|\overline{S}\big|_{\overline{g}} \leq C\Big(1+\frac{1}{k^2}\Theta\Big)
$$

and, in particular:

$$
\begin{cases}
\big|\overline{\mathrm{Ric}} * \overline{\mathrm{Riem}}\big|_{\overline{g}} \leq C\big(1+\frac{1}{k^2}\Theta\big);\\
\Big|\Big(\big|\overline{\mathrm{Ric}}\big|_{\overline{g}} - \frac{\overline{S}^2}{3}\Big)\overline{g}\Big|_{\overline{g}} \leq C\big(1+\frac{1}{k^2}\Theta\big).
\end{cases}
\tag{5.7}
$$

Thus, it remains to analyze the covariant derivatives of $\overline{\mathrm{Ric}}$ and $\overline{S}$. Let

$$
G^\alpha_{\beta\gamma} := \overline{\Gamma}^\alpha_{\beta\gamma} - \Gamma^\alpha_{\beta\gamma} = \frac{k^2\eta^\alpha\eta_{\beta\gamma}}{1+k^2\eta_\rho^2},
\tag{5.8}
$$

then by (4.11), (4.12) and (5.8) we deduce

$$
\begin{aligned}
&G^i_{jk} = G^i_{j\rho} = G^i_{\rho\rho} = 0;\\
&\big|G^\rho_{ij}\big| = \left|\frac{k^2\eta_\rho\eta_{ij}}{1+k^2\eta_\rho^2}\right| \leq \frac{k^2Cr|\eta_\rho|^2}{1+k^2|\eta_\rho|^2} \leq rC;\\
&|G^\rho_{\rho\rho}| = \left|\frac{k^2\eta_\rho\eta_{\rho\rho}}{1+k^2\eta_\rho^2}\right| \leq \frac{k^2C}{r^2\frac{k^2}{r^2}\Big(\frac{r^2}{k^2} - \frac{\eta^{-2}}{4}|y'|^2\Big)} \leq C\Theta,
\end{aligned}
\tag{5.9}
$$

By definition, we need to compute

$$
\begin{aligned}
\bar{\Delta}\overline{\mathrm{Ric}} =& \overline{g}^{\rho\rho}\overline{\nabla}_\rho\overline{\nabla}_\rho(\overline{R}_{\rho\rho} + \overline{R}_{i\rho} + \overline{R}_{ij}) + \overline{g}^{lt}\overline{\nabla}_l\overline{\nabla}_t(\overline{R}_{\rho\rho} + \overline{R}_{i\rho} + \overline{R}_{ij})\\
=& \frac{1}{1+k^2\eta_\rho^2}\overline{\nabla}_\rho\overline{\nabla}_\rho(\overline{R}_{\rho\rho} + \overline{R}_{i\rho} + \overline{R}_{ij}) + g^{lt}\overline{\nabla}_l\overline{\nabla}_t(\overline{R}_{\rho\rho} + \overline{R}_{i\rho} + \overline{R}_{ij}),
\end{aligned}
\tag{5.10}
$$

where

$$\overline{\nabla}_\gamma \overline{R}_{\alpha\beta} = \nabla_\gamma R_{\alpha\beta} - G^\delta_{\gamma\alpha} R_{\delta\beta} - G^\delta_{\gamma\beta} R_{\alpha\delta} - G^\delta_{\gamma\alpha} F_{\delta\beta} - G^\delta_{\gamma\beta} F_{\alpha\delta} \tag{5.11}$$

and

$$\begin{aligned}\overline{\nabla}_\sigma \overline{\nabla}_\gamma \overline{R}_{\alpha\beta} =& \nabla_\sigma \nabla_\gamma R_{\alpha\beta} + \nabla_\sigma \nabla_\gamma F_{\alpha\beta} - G^\tau_{\sigma\alpha}(\nabla_\gamma R_{\tau\beta}) - G^\tau_{\sigma\beta}(\nabla_\gamma R_{\alpha\tau}) - G^\tau_{\sigma\gamma}(\nabla_\tau R_{\alpha\beta}) \\ & - G^\tau_{\sigma\alpha}(\nabla_\gamma F_{\tau\beta}) - G^\tau_{\sigma\beta}(\nabla_\gamma F_{\alpha\tau}) - G^\tau_{\sigma\gamma}(\nabla_\tau F_{\alpha\beta}) \\ & - \partial_\sigma G^\delta_{\gamma\alpha} R_{\delta\beta} - \partial_\sigma G^\delta_{\gamma\beta} R_{\alpha\delta} - \partial_\sigma G^\delta_{\gamma\alpha} F_{\delta\beta} - \partial_\sigma G^\delta_{\gamma\beta} F_{\alpha\delta} \\ & - G^\delta_{\gamma\alpha} \partial_\sigma R_{\delta\beta} - G^\delta_{\gamma\beta} \partial_\sigma R_{\alpha\delta} - G^\delta_{\gamma\alpha} \partial_\sigma F_{\delta\beta} - G^\delta_{\gamma\beta} \partial_\sigma F_{\alpha\delta} + \Gamma^\tau_{\sigma\gamma}(G^\delta_{\tau\alpha} R_{\delta\beta}) \\ & + \Gamma^\tau_{\sigma\alpha}(G^\delta_{\gamma\tau} R_{\delta\beta}) + \Gamma^\tau_{\sigma\beta}(G^\delta_{\gamma\alpha} R_{\delta\tau}) + \Gamma^\tau_{\sigma\gamma}(G^\delta_{\tau\beta} R_{\alpha\delta}) + \Gamma^\tau_{\sigma\beta}(G^\delta_{\gamma\tau} R_{\alpha\delta}) \\ & + \Gamma^\tau_{\sigma\alpha}(G^\delta_{\gamma\beta} R_{\tau\delta}) + \Gamma^\tau_{\sigma\gamma}(G^\delta_{\tau\alpha} F_{\delta\beta}) + \Gamma^\tau_{\sigma\alpha}(G^\delta_{\gamma\tau} F_{\delta\beta}) + \Gamma^\tau_{\sigma\beta}(G^\delta_{\gamma\alpha} F_{\delta\tau}) \\ & + \Gamma^\tau_{\sigma\gamma}(G^\delta_{\tau\beta} F_{\alpha\delta}) + \Gamma^\tau_{\sigma\beta}(G^\delta_{\gamma\tau} F_{\alpha\delta}) + \Gamma^\tau_{\sigma\alpha}(G^\delta_{\gamma\beta} F_{\tau\delta}).\end{aligned} \tag{5.12}$$

Now we give an estimate of $\left|\partial_\delta G^\alpha_{\beta\gamma}\right|$ in terms of suitable C and Θ, where C is a constant and $\Theta = \Theta(1/k, r, p)$ denotes a continuous function depending on $1/k, r$ and p; to do so we use the bounds on $\eta_\rho, \eta_{\rho\rho}, \eta_{ij}$ and on their partial derivatives in ρ together with

$$\left|\Gamma^\rho_{ij}\right| \le C\rho; \qquad \left|\Gamma^i_{\rho j}\right| \le C\rho; \qquad \left|\Gamma^i_{jt}\right| \le C\rho^2; \qquad \Gamma^\rho_{\rho i} = \Gamma^i_{\rho\rho} = \Gamma^\rho_{\rho\rho} = 0. \tag{5.13}$$

Then, using (4.11) and (4.12) we have

$$\begin{aligned}\left|\partial_\rho G^\rho_{ij}\right| &= \left|\frac{k^2\eta_{\rho\rho}\eta_{ij}}{1+k^2\eta_\rho^2} + \frac{k^2\eta_\rho\partial_\rho\eta_{ij}}{1+k^2\eta_\rho^2} - \frac{k^4 2\eta_\rho^2\eta_{\rho\rho}\eta_{ij}}{(1+k^2\eta_\rho^2)^2}\right| \\ &\le \frac{Ck^2}{r^2\frac{k^2}{r^2}\left|\frac{r^2}{k^2} - \frac{\eta^{-2}}{4}(y')^2\right|} + \frac{Ck^2}{r^2\frac{k^2}{r^2}\left|\frac{r^2}{k^2} - \frac{\eta^{-2}}{4}(y')^2\right|} + \frac{Ck^4}{r^4\frac{k^4}{r^4}\left|\frac{r^2}{k^2} - \frac{\eta^{-2}}{4}(y')^2\right|^2} \\ &\le C\Theta + C\Theta + C\Theta \le C\Theta; \\ \left|\partial_l G^\rho_{ij}\right| &= \left|\frac{k^2\eta_\rho\partial_l\eta_{ij}}{1+k^2\eta_\rho^2}\right| \le \left|\frac{k^2 r\eta_\rho^2}{k^2\eta_\rho^2}\right| \le Cr \le C; \\ \left|\partial_\rho G^\rho_{\rho\rho}\right| &= \left|\frac{k^2\eta_{\rho\rho}^2}{1+k^2\eta_\rho^2} + \frac{k^2\eta_\rho\eta_{\rho\rho\rho}}{1+k^2\eta_\rho^2} - \frac{2k^4\eta_\rho^2\eta_{\rho\rho}^2}{(1+k^2\eta_\rho^2)^2}\right| \\ &\le \frac{k^2 C}{r^4\frac{k^2}{r^2}\left|\frac{r^2}{k^2} - \frac{\eta^{-2}}{4}(y')\right|} + \frac{k^2 C}{r^4\frac{k^2}{r^2}\left|\frac{r^2}{k^2} - \frac{\eta^{-2}}{4}(y')\right|} + \frac{k^4 C}{r^6\frac{k^4}{r^4}\left(\frac{r^2}{k^2} - \frac{\eta^{-2}}{4}(y')\right)^2} \\ &\le \frac{C}{r^2}\Theta + \frac{C}{r^2}\Theta + \frac{C}{r^2}\Theta \le C\Theta; \\ \partial_l G^\rho_{\rho\rho} &= \partial_l G^i_{jt} = \partial_\rho G^i_{jt} = \partial_\rho G^i_{j\rho} = \partial_l G^i_{j\rho} = \partial_\rho G^i_{\rho\rho} = \partial_l G^i_{\rho\rho} = \partial_\rho G^\rho_{\rho i} = \partial_l G^\rho_{\rho i} = 0.\end{aligned}$$

We start computing $|\partial_\gamma F_{\alpha\beta}|$ and $|\nabla_\gamma F_{\alpha\beta}|$: we use (4.11) and (4.12) in order to obtain

(5.14)

$$|\partial_\rho F_{\rho\rho}| = \left|\frac{k^2\eta_{\rho\rho\rho}\eta^i_i}{1+k^2\eta_\rho^2} + \frac{k^2\eta_{\rho\rho}\partial_\rho\eta^i_i}{1+k^2\eta_\rho^2} - \frac{2k^4\eta_\rho\eta_{\rho\rho}^2\eta^i_i}{(1+k^2\eta_\rho^2)^2}\right|$$

$$
\begin{aligned}
&\le \frac{C}{r}\Theta + \frac{C}{r}\Theta + \frac{C}{r}\Theta \le C\Theta;\\
|\partial_l F_{\rho\rho}| =& \left|\frac{k^2\eta_{\rho\rho}\partial_l\eta_i^i}{1+k^2\eta_\rho^2}\right| \le \frac{k^2 C}{r^2\frac{k^2}{r^2}\left|\frac{r^2}{k^2}+\frac{\eta^{-2}}{4}(y')^2\right|} = C\Theta;\\
|\partial_\rho F_{ij}| =& \Bigg| -\frac{k^2\eta_\rho\eta_{\rho\rho}L}{1+k^2\eta_\rho^2} - \frac{k^2\eta_\rho^2\partial_\rho L}{1+k^2\eta_\rho^2} + \frac{2k^4\eta_\rho^3\eta_{\rho\rho}L}{(1+k^2\eta_\rho^2)^2} + \frac{Ck^2(\partial_\rho\eta_{ANG}\eta_{ANG})_{ij}}{1+k^2\eta_\rho^2} + \frac{Ck^4\eta_\rho\eta_{\rho\rho}(\eta_{ANG}\eta_{ANG})_{ij}}{(1+k^2\eta_\rho^2)^2}\\
&+ \frac{k^2\eta_{\rho\rho\rho}\eta_{ij}}{(1+k^2\eta_\rho^2)^2} + \frac{k^2\eta_{\rho\rho}\partial_\rho\eta_{ij}}{(1+k^2\eta_\rho^2)^2} - \frac{8k^4\eta_\rho\eta_{\rho\rho}^2\eta_{ij}}{(1+k^2\eta_\rho^2)^3}\Bigg|\\
=& \Bigg| -\frac{k^2\eta_\rho\eta_{\rho\rho}L}{(1+k^2\eta_\rho^2)^2} - \frac{k^2\eta_\rho^2\partial_\rho L}{1+k^2\eta_\rho^2} + \frac{Ck^2(\partial_\rho\eta_{ANG}\eta_{ANG})_{ij}}{1+k^2\eta_\rho^2} + \frac{Ck^4\eta_\rho\eta_{\rho\rho}(\eta_{ANG}\eta_{ANG})_{ij}}{(1+k^2\eta_\rho^2)^2}\\
&+ \frac{k^2\eta_{\rho\rho\rho}\eta_{ij}}{(1+k^2\eta_\rho^2)^2} + \frac{k^2\eta_{\rho\rho}\partial_\rho\eta_{ij}}{(1+k^2\eta_\rho^2)^2} - \frac{8k^4\eta_\rho\eta_{\rho\rho}^2\eta_{ij}}{(1+k^2\eta_\rho^2)^3}\Bigg|\\
\le& \frac{C}{k^2}\Theta + C + C\Theta + C\Theta + \frac{C}{k^2}\Theta + \frac{C}{k^2}\Theta + \frac{C}{k^2}\Theta \le C(1+\Theta)\\
|\partial_l F_{ij}| =& \left| -\frac{k^2\eta_\rho^2\partial_l R_{i\rho j\rho}}{1+k^2\eta_\rho^2} + \frac{Ck^2(\partial_l\eta_{ANG}\eta_{ANG})_{ij}}{1+k^2\eta_\rho^2} + \frac{k^2\eta_{\rho\rho}\partial_l\eta_{ij}}{(1+k^2\eta_\rho^2)^2}\right| \le C + C + \frac{C}{k^2}\Theta \le C\left(1+\frac{1}{k^2}\Theta\right);\\
\partial_\rho F_{i\rho} =& \partial_l F_{i\rho} = 0.
\end{aligned}
$$

Note that in the above estimates we have not used (4.13) and (4.14), although it is useful to exploit them and $|\eta_{ij}| \le Cr|\eta_\rho|$, $|\partial_l\eta_{ij}| \le Cr|\eta_\rho|$ (see (4.15)) to estimate $|\partial_l F_{\rho\rho}|$ and $|\partial_\rho F_{ij}|$:

$$
\begin{cases}
|\partial_l F_{\rho\rho}| & \le \frac{C}{r-\rho};\\
|\partial_\rho F_{ij}| & \le C + \frac{C}{r-\rho} + \frac{C}{k^2}\Theta.
\end{cases}
\tag{5.15}
$$

Combining (5.3), (5.13) with (5.14) we deduce

$$
\begin{aligned}
|\nabla_\rho F_{\rho\rho}| &= |\partial_\rho F_{\rho\rho}| \le C\Theta;\\
|\nabla_l F_{\rho\rho}| &= |\partial_l F_{\rho\rho}| \le C\Theta;\\
|\nabla_\rho F_{ij}| &= \left|\partial_\rho F_{ij} - \Gamma_{\rho i}^l F_{lj} - \Gamma_{\rho j}^l F_{il}\right|\\
&\le |\partial_\rho F_{ij}| + \left|\Gamma_{\rho i}^l F_{lj}\right| + \left|\Gamma_{\rho j}^l F_{il}\right|\\
&\le |\partial_\rho F_{ij}| + \left|\Gamma_{\rho i}^l\right||F_{lj}| + \left|\Gamma_{\rho j}^l\right||F_{il}|\\
&\le C(1+\Theta) + C\rho\left(1+\frac{1}{k^2}\right) + C\rho\left(1+\frac{1}{k^2}\right) \le C(1+\Theta);\\
|\nabla_l F_{ij}| &= \left|\partial_l F_{ij} - \Gamma_{li}^t F_{tj} - \Gamma_{lj}^t F_{it}\right|\\
&\le |\partial_l F_{ij}| + \left|\Gamma_{li}^t F_{tj}\right| + \left|\Gamma_{lj}^t F_{it}\right|\\
&\le |\partial_l F_{ij}| + \left|\Gamma_{li}^t\right||F_{tj}| + \left|\Gamma_{lj}^t\right||F_{it}|\\
&\le C\left(1+\frac{1}{k^2}\Theta\right) + C\rho\left(1+\frac{1}{k^2}\Theta\right) + C\rho\left(1+\frac{1}{k^2}\Theta\right) \le C\left(1+\frac{1}{k^2}\Theta\right);
\end{aligned}
\tag{5.16}
$$

$$\begin{aligned}
|\nabla_l F_{i\rho}| &= \left|-\Gamma^{\rho}_{ij} F_{\rho\rho} - \Gamma^{l}_{j\rho} F_{il}\right| \\
&\le \left|\Gamma^{\rho}_{ij}\right| |F_{\rho\rho}| + \left|\Gamma^{l}_{j\rho}\right| |F_{il}| \\
&\le C\rho\Theta + C\rho\left(1 + \frac{1}{k^2}\Theta\right) \le C(1+\Theta); \\
\nabla_\rho F_{i\rho} &= 0.
\end{aligned}$$

Note that using (5.15), we obtain

$$(5.17) \qquad \begin{cases} |\nabla_l F_{\rho\rho}| \le \frac{C}{r-\rho}; \\ |\nabla_\rho F_{ij}| \le C + \frac{C}{r-\rho} + \frac{C}{k^2}\Theta; \\ |\nabla_l F_{i\rho}| \le C + \frac{C}{r-\rho} + \frac{C}{k^2}\Theta. \end{cases}$$

We are now ready to compute $|\partial_\gamma \partial_\delta F_{\alpha\beta}|$ and $|\nabla_\gamma \nabla_\delta F_{\alpha\beta}|$: using (4.11), (4.12) and the definition of $F_{\alpha\beta}$ we get

(5.18)

$$\begin{aligned}
|\partial_\rho\partial_\rho F_{\rho\rho}| =& \Bigg| \frac{k^2\eta_{\rho\rho\rho\rho}\eta^i_i}{1+k^2\eta_\rho^2} + \frac{k^2\eta_{\rho\rho\rho}\partial_\rho\eta^i_i}{1+k^2\eta_\rho^2} - \frac{2k^4\eta_\rho\eta_{\rho\rho}\eta_{\rho\rho\rho}\eta^i_i}{(1+k^2\eta_\rho^2)^2} + \frac{k^2\eta_{\rho\rho\rho}\partial_\rho\eta^i_i}{1+k^2\eta_\rho^2} \\
&+ \frac{k^2\eta_{\rho\rho}\partial_\rho\partial_\rho\eta^i_i}{1+k^2\eta_\rho^2} - \frac{2k^4\eta_\rho\eta_{\rho\rho}^2\partial_\rho\eta^i_i}{(1+k^2\eta_\rho^2)^2} - \frac{2k^4\eta_{\rho\rho}^3\eta^i_i}{(1+k^2\eta_\rho^2)^2} \\
&- \frac{4k^4\eta_\rho\eta_{\rho\rho}\eta_{\rho\rho\rho}\eta^i_i}{(1+k^2\eta_\rho^2)^2} - \frac{2k^4\eta_\rho\eta_{\rho\rho}^2\partial_\rho\eta^i_i}{(1+k^2\eta_\rho^2)^2} + \frac{8k^6\eta_\rho^2\eta_{\rho\rho}^3\eta^i_i}{(1+k^2\eta_\rho^2)^3} \Bigg| \\
\le& C\Theta + C\Theta + C\Theta + C\Theta + C\Theta + C\Theta + C\Theta + C\Theta + C\Theta + C\Theta \le C\Theta \\
|\partial_t\partial_l F_{\rho\rho}| =& \left| \frac{k^2\eta_{\rho\rho}\partial_t\partial_l\eta^i_i}{1+k^2\eta_\rho^2} \right| \le C\Theta; \\
|\partial_\rho\partial_\rho F_{ij}| =& \Bigg| \frac{k^2\eta_{\rho\rho}^2 L}{1+k^2\eta_\rho^2} + \frac{k^2\eta_\rho\eta_{\rho\rho\rho} L}{1+k^2\eta_\rho^2} + \frac{k^4\eta_\rho^2\eta_{\rho\rho}^2 L}{(1+k^2\eta_\rho^2)^2} + \frac{k^2\eta_\rho\eta_{\rho\rho}\partial_\rho L}{1+k^2\eta_\rho^2} + \frac{k^2\eta_\rho^2\partial_\rho\partial_\rho L}{1+k^2\eta_\rho^2} \\
&+ \frac{k^4\eta_\rho^3\eta_{\rho\rho}\partial_\rho L}{(1+k^2\eta_\rho^2)^2} + \frac{k^4\eta_\rho^2\eta_{\rho\rho}^2 L}{(1+k^2\eta_\rho^2)^2} + \frac{k^4\eta_\rho^3\eta_{\rho\rho\rho} L}{(1+k^2\eta_\rho^2)^2} + \frac{k^6\eta_\rho^4\eta_{\rho\rho}^2 L}{(1+k^2\eta_\rho^2)^3} + \frac{k^4\eta_\rho^2\eta_{\rho\rho}^2\partial_\rho L}{(1+k^2\eta_\rho^2)^2} \\
&+ \frac{Ck^2(\partial_\rho\partial_\rho\eta_{ANG}\eta_{ANG})_{ij}}{1+k^2\eta_\rho^2} + \frac{Ck^2(\partial_\rho\eta_{ANG}\partial_\rho\eta_{ANG})_{ij}}{1+k^2\eta_\rho^2} - \frac{2Ck^4\eta_\rho\eta_{\rho\rho}(\eta_{ANG}\partial_\rho\eta_{ANG})_{ij}}{(1+k^2\eta_\rho^2)^2} \\
&+ \frac{Ck^4\eta_{\rho\rho}^2(\eta_{ANG}\eta_{ANG})_{ij}}{(1+k^2\eta_\rho^2)^2} + \frac{Ck^4\eta_\rho\eta_{\rho\rho\rho}(\eta_{ANG}\eta_{ANG})_{ij}}{(1+k^2\eta_\rho^2)^2} + \frac{Ck^4\eta_\rho\eta_{\rho\rho}(\eta_{ANG}\partial_\rho\eta_{ANG})_{ij}}{(1+k^2\eta_\rho^2)^2} \\
&+ \frac{Ck^6\eta_\rho^2\eta_{\rho\rho}^2(\eta_{ANG}\eta_{ANG})_{ij}}{(1+k^2\eta_\rho^2)^3} + \frac{k^2\eta_{\rho\rho\rho\rho}\eta_{ij}}{(1+k^2\eta_\rho^2)^2} + \frac{k^2\eta_{\rho\rho\rho}\partial_\rho\eta_{ij}}{(1+k^2\eta_\rho^2)^2} - \frac{2k^4\eta_\rho\eta_{\rho\rho}\eta_{\rho\rho\rho}\eta_{ij}}{(1+k^2\eta_\rho^2)^3} \\
&+ \frac{k^2\eta_{\rho\rho\rho}\partial_\rho\eta_{ij}}{(1+k^2\eta_\rho^2)^2} + \frac{k^2\eta_{\rho\rho}\partial_\rho\partial_\rho\eta_{ij}}{(1+k^2\eta_\rho^2)^2} - \frac{2k^4\eta_\rho\eta_{\rho\rho}^2\partial_\rho\eta_{ij}}{(1+k^2\eta_\rho^2)^3} + \frac{Ck^4\eta_{\rho\rho}^3\eta_{ij}}{(1+k^2\eta_\rho^2)^3} \\
&+ \frac{Ck^4\eta_\rho\eta_{\rho\rho}\eta_{\rho\rho\rho}\eta_{ij}}{(1+k^2\eta_\rho^2)^3} + \frac{Ck^4\eta_\rho\eta_{\rho\rho}^2\partial_\rho\eta_{ij}}{(1+k^2\eta_\rho^2)^3} + \frac{Ck^6\eta_\rho^2\eta_{\rho\rho}^3\eta_{ij}}{(1+k^2\eta_\rho^2)^4} \Bigg|
\end{aligned}$$

$$
\begin{aligned}
&\leq C\Theta + C\Theta + C\Theta + C\Theta + C + C\Theta + C\Theta + C\Theta + C\Theta + C\Theta + C\Theta \\
&\quad + C\Theta + C\Theta + C\Theta + C\Theta + C\Theta + C\Theta + \frac{C}{k^2}\Theta + \frac{C}{k^2}\Theta + \frac{C}{k^2}\Theta + \frac{C}{k^2}\Theta \\
&\quad + \frac{C}{k^2}\Theta + \frac{C}{k^2}\Theta + \frac{C}{k^2}\Theta + \frac{C}{k^2}\Theta + \frac{C}{k^2}\Theta + \frac{C}{k^2}\Theta \leq C(1+\Theta); \\
|\partial_t\partial_l F_{ij}| &= \Bigg| - \frac{k^2\eta_\rho^2 \partial_t\partial_l R_{i\rho j\rho}}{1+k^2\eta_\rho^2} + \frac{2k^2(\partial_t\partial_l \eta_{ANG}\eta_{ANG})_{ij}}{(1+k^2\eta_\rho^2)^2} + \frac{2k^2\eta_{\rho\rho}^2(\partial_t\partial_l \eta_{ANG}\eta_{ANG})_{ij}}{(1+k^2\eta_\rho^2)^2} \\
&\quad + \frac{2k^2(\partial_t \eta_{ANG}\partial_l \eta_{ANG})_{ij}}{(1+k^2\eta_\rho^2)^2} + \frac{2k^2\eta_{\rho\rho}^2(\partial_t \eta_{ANG}\partial_l \eta_{ANG})_{ij}}{(1+k^2\eta_\rho^2)^2} \Bigg| \\
&\leq C + C + \frac{C}{k^2}\Theta + C + \frac{C}{k^2}\Theta \\
&\leq C\left(1 + \frac{1}{k^2}\Theta\right); \\
|\partial_\rho\partial_\rho F_{i\rho}| &= |\partial_t\partial_l F_{i\rho}| = 0.
\end{aligned}
$$

Using (5.3), (5.13), (5.14), (5.16) and (5.18), we deduce

(5.19)

$$
\begin{aligned}
|\nabla_\rho\nabla_\rho F_{\rho\rho}| &= |\partial_\rho\partial_\rho F_{\rho\rho}| \leq C\Theta; \\
|\nabla_t\nabla_l F_{\rho\rho}| &= \left|\partial_t\nabla_l F_{\rho\rho} - \Gamma_{tl}^\rho(\nabla_\rho F_{\rho\rho}) - \Gamma_{tl}^i(\nabla_i F_{\rho\rho})\right| = \left|\partial_t\partial_l F_{\rho\rho} - \Gamma_{tl}^\rho(\nabla_\rho F_{\rho\rho}) - \Gamma_{tl}^i(\nabla_i F_{\rho\rho})\right| \\
&\leq C\Theta + C\Theta + \rho^2 C\Theta \leq C\Theta; \\
|\nabla_\rho\nabla_\rho F_{ij}| &= \left|\partial_\rho\nabla_\rho F_{ij} - \Gamma_{\rho i}^\alpha \nabla_\rho F_{\alpha j} - \Gamma_{\rho j}^\alpha \nabla_\rho F_{i\alpha}\right| \\
&= \left|\partial_\rho\Big(\partial_\rho F_{ij} - \Gamma_{\rho i}^l F_{lj} - \Gamma_{\rho j}^l F_{il}\Big) - \Gamma_{\rho i}^l \nabla_\rho F_{lj} - \Gamma_{\rho j}^l \nabla_\rho F_{il}\right| \\
&= \left|\partial_\rho\partial_\rho F_{ij} - \partial_\rho\Gamma_{\rho i}^l F_{lj} + \Gamma_{\rho i}^l \partial_\rho F_{lj} - \partial_\rho\Gamma_{\rho j}^l F_{il} + \Gamma_{\rho j}^l \partial_\rho F_{il} - \Gamma_{\rho i}^l \nabla_\rho F_{lj} - \Gamma_{\rho j}^l \nabla_\rho F_{il}\right| \\
&\leq C(1+\Theta) + C\left(1 + \frac{1}{k^2}\Theta\right) + C\rho\left(1 + \frac{1}{k^2}\Theta\right) + C\left(1 + \frac{1}{k^2}\Theta\right) \\
&\quad + C\rho\left(1 + \frac{1}{k^2}\Theta\right) + C\rho(1+\Theta) + C\rho(1+\Theta) \\
&\leq C(1+\Theta); \\
|\nabla_t\nabla_l F_{ij}| &= \Big|\partial_t\nabla_l F_{ij} - \Gamma_{ti}^s(\nabla_l F_{sj}) - \Gamma_{tj}^s(\nabla_l F_{si}) - \Gamma_{tl}^s(\nabla_s F_{ij}) \\
&\quad - \Gamma_{tl}^\rho(\nabla_\rho F_{ij}) - \Gamma_{ti}^\rho(\nabla_l F_{\rho j}) - \Gamma_{tj}^\rho(\nabla_l F_{i\rho})\Big| \\
&= \Big|\partial_t(\partial_l F_{ij} - \Gamma_{li}^s F_{sj} - \Gamma_{lj}^s F_{is}) - \Gamma_{ti}^s(\nabla_l F_{sj}) - \Gamma_{tj}^s(\nabla_l F_{si}) \\
&\quad - \Gamma_{tl}^s(\nabla_s F_{ij}) - \Gamma_{tl}^\rho(\nabla_\rho F_{ij}) - \Gamma_{ti}^\rho(\nabla_l F_{\rho j}) - \Gamma_{tj}^\rho(\nabla_l F_{i\rho})\Big| \\
&= \Big|\partial_t\partial_l F_{ij} - \partial_t\Gamma_{li}^s F_{sj} - \Gamma_{li}^s\partial_t F_{sj} - \partial_t\Gamma_{lj}^s F_{is} - \Gamma_{lj}^s\partial_t F_{is} \\
&\quad - \Gamma_{ti}^s(\nabla_l F_{sj}) - \Gamma_{tj}^s(\nabla_l F_{si}) - \Gamma_{tl}^s(\nabla_s F_{ij}) \\
&\quad - \Gamma_{tl}^\rho(\nabla_\rho F_{ij}) - \Gamma_{ti}^\rho(\nabla_l F_{\rho j}) - \Gamma_{tj}^\rho(\nabla_l F_{i\rho})\Big| \\
&\leq |\partial_t\partial_l F_{ij}| + |\partial_t\Gamma_{li}^s F_{sj}| + |\Gamma_{li}^s\partial_t F_{sj}| + \left|\partial_t\Gamma_{lj}^s F_{is}\right| + \left|\Gamma_{lj}^s\partial_t F_{is}\right| \\
&\quad + |\Gamma_{ti}^s(\nabla_l F_{sj})| + \left|\Gamma_{tj}^s(\nabla_l F_{si})\right| + |\Gamma_{tl}^s(\nabla_s F_{ij})|
\end{aligned}
$$

$$
\begin{aligned}
&+\left|\Gamma^\rho_{tl}(\nabla_\rho F_{ij})\right| + |\Gamma^\rho_{ti}(\nabla_l F_{\rho j})| + \left|\Gamma^\rho_{tj}(\nabla_l F_{i\rho})\right| \\
\leq & C\left(1+\frac{1}{k^2}\Theta\right) + C\left(1+\frac{1}{k^2}\Theta\right) + C\rho\left(1+\frac{1}{k^2}\Theta\right) \\
&+ C\rho\left(1+\frac{1}{k^2}\Theta\right) + \rho^2 C\left(1+\frac{1}{k^2}\Theta\right) + \rho^2\left(C+\frac{C}{k^2}\Theta\right) \\
&+ \left|\Gamma^\rho_{ll}(\nabla_\rho F_{ij})\right| + \left|\Gamma^\rho_{li}(\nabla_l F_{\rho j})\right| + \left|\Gamma^\rho_{lj}(\nabla_l F_{i\rho})\right|
\end{aligned}
$$

$$
\begin{aligned}
|\nabla_\rho\nabla_\rho F_{i\rho}| =& \left|\partial_\rho\nabla_\rho F_{i\rho} - \Gamma^\alpha_{\rho i}\nabla_\rho F_{\alpha\rho} - \Gamma^\alpha_{\rho\rho}\nabla_\rho F_{i\alpha} - \Gamma^\alpha_{\rho\rho}\nabla_\alpha F_{i\rho}\right| = 0; \\
|\nabla_t\nabla_l F_{i\rho}| =& \left|\partial_t\nabla_l F_{i\rho} - \Gamma^\alpha_{ti}\nabla_\rho F_{\alpha\rho} - \Gamma^\alpha_{t\rho}\nabla_\rho F_{i\alpha} - \Gamma^\alpha_{tl}\nabla_\alpha F_{i\rho}\right|; \\
=& \left|\partial_t\left(\partial_l F_{i\rho} - \Gamma^\rho_{li}F_{\rho\rho} - \Gamma^j_{l\rho}F_{ij}\right) - \Gamma^\alpha_{ti}\nabla_\rho F_{\alpha\rho} - \Gamma^\alpha_{t\rho}\nabla_\rho F_{i\alpha} - \Gamma^\alpha_{tl}\nabla_\alpha F_{i\rho}\right| \\
=& \left|\partial_t\Gamma^\rho_{li}F_{\rho\rho} - \Gamma^\rho_{li}\partial_t F_{\rho\rho} - \partial_t\Gamma^j_{l\rho}F_{ij} - \Gamma^j_{l\rho}\partial_t F_{ij} - \Gamma^j_{ti}\nabla_l F_{j\rho} - \Gamma^\rho_{ti}\nabla_l F_{\rho\rho} - \Gamma^j_{t\rho}\nabla_l F_{ij} - \Gamma^j_{tl}\nabla_j F_{i\rho}\right| \\
\leq& \left|\partial_t\Gamma^\rho_{li}F_{\rho\rho}\right| + \left|\Gamma^\rho_{li}\partial_t F_{\rho\rho}\right| + C\left(1+\frac{1}{k^2}\Theta\right) + C\rho\left(1+\frac{1}{k^2}\Theta\right) + \left|\Gamma^j_{ti}\nabla_l F_{j\rho}\right| \\
&+ |\Gamma^\rho_{ti}\nabla_l F_{\rho\rho}| + C\rho\left(1+\frac{1}{k^2}\Theta\right) + \left|\Gamma^j_{tl}\nabla_j F_{i\rho}\right|;
\end{aligned}
$$

To obtain a final estimate for $|\nabla_t\nabla_l F_{ij}|$ and $|\nabla_t\nabla_l F_{i\rho}|$, we use equations (5.4) and (5.17) to finally deduce:

$$
|\nabla_t\nabla_l F_{ij}| \leq C + \frac{C}{r-\rho} + \frac{C}{k^2}\Theta, \qquad |\nabla_t\nabla_l F_{i\rho}| \leq C + \frac{C}{r-\rho} + \frac{C}{k^2}\Theta. \tag{5.20}
$$

We are now ready to give an estimate of the components of the remainders of $\bar\Delta\overline{\mathrm{Ric}}$:

$$
\begin{aligned}
\left|\overline{\nabla}_\rho\overline{\nabla}_\rho\overline{R}_{\rho\rho}\right| =& \left|\nabla_\rho\nabla_\rho R_{\rho\rho} + \nabla_\rho\nabla_\rho F_{\rho\rho} - 3G^\rho_{\rho\rho}\nabla_\rho R_{\rho\rho}\right. \\
&\left. - 3G^\rho_{\rho\rho}\nabla_\rho F_{\rho\rho} - 2\partial_\rho G^\rho_{\rho\rho}R_{\rho\rho} - 2\partial_\rho G^\rho_{\rho\rho}F_{\rho\rho}\right| \\
&\leq C + C\Theta + C\Theta + C\Theta + C\Theta + C\Theta \leq C(1+\Theta); \\
\left|\overline{\nabla}_t\overline{\nabla}_l\overline{R}_{\rho\rho}\right| =& \left|\nabla_t\nabla_l R_{\rho\rho} + \nabla_t\nabla_l F_{\rho\rho} - G^\rho_{tl}\nabla_\rho R_{\rho\rho} - G^\rho_{tl}\nabla_\rho F_{\rho\rho} + 2\Gamma^\rho_{tl}\big(G^\rho_{\rho\rho}R_{\rho\rho}\big)\right. \\
&\left. + 2\Gamma^s_{t\rho}\big(G^\rho_{ls}R_{\rho\rho}\big) + 2\Gamma^\rho_{tl}\big(G^\rho_{\rho\rho}F_{\rho\rho}\big) + 2\Gamma^s_{t\rho}\big(G^\rho_{ls}F_{\rho\rho}\big)\right| \\
&\leq C + C\Theta + C + C\Theta + C\Theta + C + C\Theta + C\Theta \leq C(1+\Theta); \\
\left|\overline{\nabla}_\rho\overline{\nabla}_\rho\overline{R}_{ij}\right| =& \left|\nabla_\rho\nabla_\rho R_{ij} + \nabla_\rho\nabla_\rho F_{ij} - G^\rho_{\rho\rho}\nabla_\rho R_{ij} - G^\rho_{\rho\rho}\nabla_\rho F_{ij}\right| \\
\leq& C + C(1+\Theta) + C\Theta + C\Theta \leq C(1+\Theta); \\
\left|\overline{\nabla}_t\overline{\nabla}_l\overline{R}_{ij}\right| =& \left|\nabla_t\nabla_l R_{ij} + \nabla_t\nabla_l F_{ij} - G^\rho_{ti}\nabla_l R_{\rho j} - G^\rho_{tj}\nabla_l R_{i\rho} - G^\rho_{tl}\nabla_\rho R_{ij}\right. \\
&- G^\rho_{ti}\nabla_l F_{\rho j} - G^\rho_{tj}\nabla_l F_{i\rho} - G^\rho_{tl}\nabla_\rho F_{ij} \\
&- \partial_t G^\rho_{li}R_{\rho j} - \partial_t G^\rho_{lj}R_{i\rho} - G^\rho_{li}\partial_t R_{\rho j} - G^\rho_{lj}\partial_t R_{i\rho} \\
&+ \Gamma^s_{tl}(G^\rho_{si}R_{\rho j}) + \Gamma^s_{ti}\big(G^\rho_{ls}R_{\rho j}\big) + \Gamma^\rho_{tj}\big(G^\rho_{li}R_{\rho\rho}\big) + \Gamma^s_{tj}\big(G^\rho_{li}R_{\rho s}\big) + \Gamma^s_{tl}\left(G^\rho_{sj}R_{i\rho}\right) \\
&\left. + \Gamma^s_{tj}\big(G^\rho_{ls}R_{i\rho}\big) + \Gamma^\rho_{ti}\left(G^\rho_{lj}R_{\rho\rho}\right) + \Gamma^s_{ti}\left(G^\rho_{lj}R_{\rho s}\right) + \Gamma^\rho_{tj}\left(G^\rho_{lj}F_{\rho\rho}\right) + \Gamma^\rho_{ti}\big(G^\rho_{li}F_{\rho\rho}\big)\right| \\
&\leq C + C\left(1 + \frac{1}{r-\rho} + \frac{1}{k^2}\Theta\right) + C + C + C + C\left(1 + \frac{1}{r-\rho} + \frac{1}{k^2}\Theta\right)
\end{aligned}
\tag{5.21}
$$

$$
\begin{aligned}
&+C\left(1+\frac{1}{r-\rho}+\frac{1}{k^2}\Theta\right)+C\left(1+\frac{1}{r-\rho}+\frac{1}{k^2}\Theta\right)+C\left(1+\frac{1}{r-\rho}+\frac{1}{k^2}\right)\\
&+C+C+C+C+C+C+C+C+C+C+C+C+\frac{C}{r-\rho}+\frac{C}{r-\rho}\\
&\leq C+\frac{C}{k^2}\Theta+\frac{C}{r-\rho};
\end{aligned}
$$

$$
\begin{aligned}
\left|\overline{\nabla}_\rho\overline{\nabla}_\rho\overline{R}_{i\rho}\right| =&\left|\nabla_\rho\nabla_\rho R_{i\rho}-2G^\rho_{\rho\rho}\nabla_\rho R_{i\rho}-\partial_\rho G^\rho_{\rho\rho}R_{i\rho}-G^\rho_{\rho\rho}\partial_\rho R_{i\rho}+\Gamma^t_{\rho i}\left(G^\rho_{\rho\rho}R_{t\rho}\right)\right|\\
&\leq C+C\Theta+C\Theta+C\Theta+C\Theta\leq C(1+\Theta);\\
\left|\overline{\nabla}_t\overline{\nabla}_l\overline{R}_{i\rho}\right| =&\left|\nabla_t\nabla_l R_{i\rho}-\nabla_t\nabla_l F_{i\rho}-G^\rho_{ti}\nabla_l R_{\rho\rho}-G^\rho_{tl}\nabla_\rho R_{i\rho}\right.\\
&-G^\rho_{ti}\nabla_l F_{\rho\rho}-\partial_t G^\rho_{li}R_{\rho\rho}-\partial_t G^\rho_{li}F_{\rho\rho}-G^\rho_{li}\partial_t R_{\rho\rho}-G^\rho_{li}\partial_t F_{\rho\rho}\\
&+\Gamma^s_{tl}(G^\rho_{si}R_{\rho\rho})+\Gamma^s_{ti}\left(G^\rho_{ls}R_{\rho\rho}\right)+\Gamma^s_{t\rho}\left(G^\rho_{li}R_{\rho s}\right)+\Gamma^\rho_{tl}\left(G^\rho_{\rho\rho}R_{i\rho}\right)\\
&\left.+\Gamma^s_{t\rho}\left(G^\rho_{ls}R_{i\rho}\right)+\Gamma^s_{tl}(G^\rho_{si}F_{\rho\rho})+\Gamma^s_{ti}\left(G^\rho_{ls}F_{\rho\rho}\right)\right|\\
\leq&C+C\left(1+\frac{1}{r-\rho}+\frac{1}{k^2}\Theta\right)+C+C+\frac{C}{r-\rho}\\
&+C\frac{C}{r-\rho}+C+C+C+C+C+\frac{C}{r-\rho}+\frac{C}{r-\rho}\\
\leq&C\left(1+\frac{1}{r-\rho}+\frac{1}{k^2}\Theta\right).
\end{aligned}
$$

Note that we have used equations (4.10), (5.3), (5.13), (5.14), (5.16) and (5.19) to estimate $\left|\overline{\nabla}_\rho\overline{\nabla}_\rho\overline{R}_{\rho\rho}\right|$, $\left|\overline{\nabla}_t\overline{\nabla}_l\overline{R}_{\rho\rho}\right|$ $\left|\overline{\nabla}_\rho\overline{\nabla}_\rho\overline{R}_{ij}\right|$ and $\left|\overline{\nabla}_\rho\overline{\nabla}_\rho\overline{R}_{i\rho}\right|$ and we have used (4.10), (5.3), (5.4), (5.13), (5.15), (5.17) and (5.20) to estimate $\left|\overline{\nabla}_t\overline{\nabla}_l\overline{R}_{ij}\right|$ and $\left|\overline{\nabla}_t\overline{\nabla}_l\overline{R}_{i\rho}\right|$. Therefore, using the equations in (5.21) in (5.12) we conclude

(5.22)

$$
\begin{aligned}
\left|\bar{\Delta}\overline{\mathrm{Ric}}\right|_{\bar{g}} =&\left|\frac{1}{1+k^2\eta_\rho^2}\overline{\nabla}_\rho\overline{\nabla}_\rho(\overline{R}_{\rho\rho}+\overline{R}_{i\rho}+\overline{R}_{ij})+g^{lt}\overline{\nabla}_l\overline{\nabla}_t(\overline{R}_{\rho\rho}+\overline{R}_{i\rho}+\overline{R}_{ij})\right|_{\bar{g}}\\
\leq&\frac{1}{1+k^2\eta_\rho^2}\left|\overline{\nabla}_\rho\overline{\nabla}_\rho\overline{R}_{\rho\rho}\right|_{\bar{g}}+\frac{1}{1+k^2\eta_\rho^2}\left|\overline{\nabla}_\rho\overline{\nabla}_\rho\overline{R}_{i\rho}\right|_{\bar{g}}+\frac{1}{1+k^2\eta_\rho^2}\left|\overline{\nabla}_\rho\overline{\nabla}_\rho\overline{R}_{ij}\right|_{\bar{g}}\\
&+\left|g^{lt}\overline{\nabla}_l\overline{\nabla}_t\overline{R}_{\rho\rho}\right|_{\bar{g}}+\left|g^{lt}\overline{\nabla}_l\overline{\nabla}_t\overline{R}_{i\rho}\right|_{\bar{g}}+\left|g^{lt}\overline{\nabla}_l\overline{\nabla}_t\overline{R}_{ij}\right|_{\bar{g}}\\
=&\frac{1}{(1+k^2\eta_\rho^2)^2}\left|\overline{\nabla}_\rho\overline{\nabla}_\rho\overline{R}_{\rho\rho}\right|_g+\frac{1}{(1+k^2\eta_\rho^2)^{\frac{3}{2}}}\left|\overline{\nabla}_\rho\overline{\nabla}_\rho\overline{R}_{i\rho}\right|_g+\frac{1}{1+k^2\eta_\rho^2}\left|\overline{\nabla}_\rho\overline{\nabla}_\rho\overline{R}_{ij}\right|_g\\
&+\frac{1}{1+k^2\eta_\rho^2}\left|g^{lt}\overline{\nabla}_l\overline{\nabla}_t\overline{R}_{\rho\rho}\right|_g+\frac{1}{(1+k^2\eta_\rho^2)^{\frac{1}{2}}}\left|g^{lt}\overline{\nabla}_l\overline{\nabla}_t\overline{R}_{i\rho}\right|_g+\left|g^{lt}\overline{\nabla}_l\overline{\nabla}_t\overline{R}_{ij}\right|_g\\
\leq&\frac{1}{(1+k^2\eta_\rho^2)^2}\left|\overline{\nabla}_\rho\overline{\nabla}_\rho\overline{R}_{\rho\rho}\right|+\frac{C}{(1+k^2\eta_\rho^2)^{\frac{3}{2}}}\left|\overline{\nabla}_\rho\overline{\nabla}_\rho\overline{R}_{i\rho}\right|+\frac{C}{1+k^2\eta_\rho^2}\left|\overline{\nabla}_\rho\overline{\nabla}_\rho\overline{R}_{ij}\right|\\
&+\frac{1}{1+k^2\eta_\rho^2}\left|g^{lt}\overline{\nabla}_l\overline{\nabla}_t\overline{R}_{\rho\rho}\right|+\frac{C}{(1+k^2\eta_\rho^2)^{\frac{1}{2}}}\left|g^{lt}\overline{\nabla}_l\overline{\nabla}_t\overline{R}_{i\rho}\right|+C\left|g^{lt}\overline{\nabla}_l\overline{\nabla}_t\overline{R}_{ij}\right|\\
\leq&\frac{1}{(1+k^2\eta_\rho^2)^2}C(1+\Theta)+\frac{1}{(1+k^2\eta_\rho^2)^{\frac{3}{2}}}C(1+\Theta)+\frac{1}{1+k^2\eta_\rho^2}C(1+\Theta)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{1+k^2\eta_\rho^2}C(1+\Theta)+\frac{1}{(1+k^2\eta_\rho^2)^{\frac12}}C\left(1+\frac{1}{r-\rho}+\frac{1}{k^2}\Theta\right)+C\left(1+\frac{1}{r-\rho}+\frac{1}{k^2}\Theta\right)\\
\leq& C\left(1+\frac{1}{k^2}\Theta\right)+C\left(1+\frac{1}{k^2}\Theta\right)+C\left(1+\frac{1}{k^2}\Theta\right)\\
&+C\left(1+\frac{1}{k^2}\Theta\right)+C\left(1+\frac{1}{r-\rho}+\frac{1}{k^2}\Theta\right)+C\left(1+\frac{1}{r-\rho}+\frac{1}{k^2}\Theta\right)\\
\leq& C\left(1+\frac{1}{r-\rho}+\frac{1}{k^2}\Theta\right).
\end{aligned}
$$

Now we compute $|\partial_\alpha H|$ using estimates (4.11) and (4.12):

$$
\begin{aligned}
|\partial_\rho H| =&\left|\partial_\rho L+\frac{\partial_\rho L}{1+k^2\eta_\rho^2}-\frac{2k^2L\eta_\rho\eta_{\rho\rho}}{(1+k^2\eta_\rho^2)^2}+\frac{Ck^2\partial_\rho\eta_{ANG}\eta_{ANG}}{1+k^2\eta_\rho^2}-\frac{Ck^4\eta_\rho\eta_{\rho\rho}\eta_{ANG}\eta_{ANG}}{(1+k^2\eta_\rho^2)^2}\right.\\
&\left.+\frac{2k^2\eta_{\rho\rho\rho}\eta_{ll}}{(1+k^2\eta_\rho^2)^2}+\frac{2k^2\eta_{\rho\rho}\partial_\rho\eta_{ll}}{(1+k^2\eta_\rho^2)^2}-\frac{8k^4\eta_\rho\eta_{\rho\rho}^2\eta_{ll}}{(1+k^2\eta_\rho^2)^3}\right|\\
\leq& C+C+C\Theta+C\Theta+C\Theta+\frac{C}{k^2}\Theta+\frac{C}{k^2}\Theta+\frac{C}{k^2}\Theta\\
\leq& C(1+\Theta);\\
|\partial_i H| =&\left|\partial_i L+\frac{\partial_i L}{1+k^2\eta_\rho^2}+\frac{Ck^2\partial_i\eta_{ANG}\eta_{ANG}}{1+k^2\eta_\rho^2}+\frac{2k^2\eta_{\rho\rho}\partial_i\eta_{ll}}{(1+k^2\eta_\rho^2)^2}\right|\\
\leq& C\left(1+\frac{1}{k^2}\Theta\right).
\end{aligned} \tag{5.23}
$$

Note that using (4.13), (4.14), and the fact that $|\eta_{ij}|\leq Cr|\eta_\rho|$ we deduce another estimate for $|\partial_\rho H|$, which we will use in the final inequality of $\left|\overline{\nabla}_i\overline{\nabla}_j\overline{S}\right|$:

$$
|\partial_\rho H|\leq C+\frac{C}{k^2}\Theta+\frac{C}{r-\rho}. \tag{5.24}
$$

Therefore, by (5.23) we obtain

$$
|\nabla_\rho H|=|\partial_\rho H|\leq C+C\Theta;\qquad |\nabla_i H|=|\partial_i H|\leq C+\frac{C}{k^2}\Theta \tag{5.25}
$$

and by (5.24)

$$
|\nabla_\rho H|\leq C+\frac{C}{r-\rho}+\frac{C}{k^2}\Theta. \tag{5.26}
$$

We now compute $\partial_\alpha\partial_\beta H$, which we will later need to compute $\mathrm{Hess}(H)$ and ΔH. Towards this aim we use (4.11) and (4.12) in $|\partial_\rho\partial_\rho H|$ and (4.13), (4.14) and $|\eta_{ij}|\leq Cr|\eta_\rho|$, $|\partial_l\eta_{ij}|\leq Cr|\eta_\rho|$ in the remaining terms:

$$
\begin{aligned}
|\partial_\rho\partial_\rho H| =&\left|\partial_\rho\partial_\rho L+\frac{\partial_\rho\partial_\rho L}{1+k^2\eta_\rho^2}-\frac{Ck^2\eta_\rho\eta_{\rho\rho}\partial_\rho L}{(1+k^2\eta_\rho^2)^2}-\frac{2k^2L\eta_{\rho\rho}^2}{(1+k^2\eta_\rho^2)^2}-\frac{2k^2L\eta_{\rho\rho\rho}\eta_\rho}{(1+k^2\eta_\rho^2)^2}\right.\\
&+\frac{8k^4\eta_\rho^2\eta_{\rho\rho}^2L}{(1+k^2\eta_\rho^2)^3}+\frac{Ck^2\partial_\rho\partial_\rho\eta_{ANG}\eta_{ANG}}{1+k^2\eta_\rho^2}+\frac{Ck^2\partial_\rho\eta_{ANG}\partial_\rho\eta_{ANG}}{1+k^2\eta_\rho^2}\\
&-\frac{Ck^4\eta_\rho\eta_{\rho\rho}\partial_\rho\eta_{ANG}\eta_{ANG}}{(1+k^2\eta_\rho^2)^2}-\frac{Ck^4\eta_{\rho\rho}^2\eta_{ANG}\eta_{ANG}}{(1+k^2\eta_\rho^2)^2}-\frac{Ck^4\eta_\rho\eta_{\rho\rho\rho}\eta_{ANG}\eta_{ANG}}{(1+k^2\eta_\rho^2)^2}
\end{aligned} \tag{5.27}
$$

$$
\begin{aligned}
&+\frac{Ck^6\eta_\rho^2\eta_{\rho\rho}^2\eta_{ANG}\eta_{ANG}}{(1+k^2\eta_\rho^2)^3}+\frac{2k^2\eta_{\rho\rho\rho\rho}\eta_{ll}}{(1+k^2\eta_\rho^2)^2}+\frac{2k^2\eta_{\rho\rho\rho}\partial_\rho\eta_{ll}}{(1+k^2\eta_\rho^2)^2}-\frac{8k^4\eta_\rho\eta_{\rho\rho}\eta_{\rho\rho\rho}\eta_{ll}}{(1+k^2\eta_\rho^2)^3}\\
&+\frac{2k^2\eta_{\rho\rho}\partial_\rho\partial_\rho\eta_{ll}}{(1+k^2\eta_\rho^2)^2}+\frac{2k^2\eta_{\rho\rho\rho}\partial_\rho\eta_{ll}}{(1+k^2\eta_\rho^2)^2}-\frac{8k^4\eta_\rho\eta_{\rho\rho}^2\partial_\rho\eta_{ll}}{(1+k^2\eta_\rho^2)^3}-\frac{8k^4\eta_{\rho\rho}^3\eta_{ll}}{(1+k^2\eta_\rho^2)^3}\\
&-\frac{16k^4\eta_{\rho\rho\rho}\eta_{\rho\rho}\eta_\rho\eta_{ll}}{(1+k^2\eta_\rho^2)^3}-\frac{8k^4\eta_\rho\eta_{\rho\rho}^2\partial_\rho\eta_{ll}}{(1+k^2\eta_\rho^2)^3}+\frac{48k^6\eta_\rho^2\eta_{\rho\rho}^3\eta_{ll}}{(1+k^2\eta_\rho^2)^4}\Bigg|\\
\le& C+C+\frac{C}{k^2}\Theta+\frac{C}{k^2}\Theta+\frac{C}{k^2}\Theta+\frac{C}{k^2}\Theta+C\Theta+C\Theta+C\Theta+C\Theta+C\Theta+C\Theta\\
&+\frac{C}{k^2}\Theta+\frac{C}{k^2}\Theta+\frac{C}{k^2}\Theta+\frac{C}{k^2}\Theta+\frac{C}{k^2}\Theta+\frac{C}{k^2}\Theta+\frac{C}{k^2}\Theta+\frac{C}{k^2}\Theta+\frac{C}{k^2}\Theta+\frac{C}{k^2}\Theta\\
\le& C+\frac{C}{k^2}\Theta+C\Theta;
\end{aligned}
$$

$$
\begin{aligned}
|\partial_l\partial_\rho H|=&\Bigg|\partial_l\partial_\rho L+\frac{\partial_l\partial_\rho L}{1+k^2\eta_\rho^2}-\frac{2k^2\partial_l L\eta_\rho\eta_{\rho\rho}}{(1+k^2\eta_\rho^2)^2}+\frac{Ck^2\partial_l\partial_\rho\eta_{ANG}\eta_{ANG}}{1+k^2\eta_\rho^2}+\frac{Ck^2\partial_\rho\eta_{ANG}\partial_l\eta_{ANG}}{1+k^2\eta_\rho^2}\\
&-\frac{Ck^4\eta_\rho\eta_{\rho\rho}\partial_l\eta_{ANG}\eta_{ANG}}{(1+k^2\eta_\rho^2)^2}+\frac{2k^2\eta_{\rho\rho\rho}\partial_l\eta_{tt}}{(1+k^2\eta_\rho^2)^2}+\frac{2k^2\eta_{\rho\rho}\partial_l\partial_\rho\eta_{tt}}{(1+k^2\eta_\rho^2)^2}-\frac{8k^4\eta_\rho\eta_{\rho\rho}^2\partial_l\eta_{tt}}{(1+k^2\eta_\rho^2)^3}\Bigg|\\
\le& C+C+\frac{C}{k^2}\Theta+\frac{C}{r-\rho}+\frac{C}{r-\rho}+\frac{C}{r-\rho}+\frac{C}{k^2}\Theta+\frac{C}{k^2}\Theta+\frac{C}{k^2}\Theta\\
\le& C+\frac{C}{k^2}\Theta+\frac{C}{r-\rho};
\end{aligned}
$$

$$
\begin{aligned}
|\partial_t\partial_i H|=&\Bigg|\partial_t\partial_i L+\frac{\partial_t\partial_i L}{1+k^2\eta_\rho^2}+\frac{Ck^2\partial_t\partial_i\eta_{ANG}\eta_{ANG}}{1+k^2\eta_\rho^2}+\frac{Ck^2\partial_i\eta_{ANG}\partial_t\eta_{ANG}}{1+k^2\eta_\rho^2}+\frac{2k^2\eta_{\rho\rho}\partial_t\partial_i\eta_{ll}}{(1+k^2\eta_\rho^2)^2}\Bigg|\\
&\le C+C+C+C+\frac{C}{k^2}\Theta\\
\le& C+\frac{C}{k^2}\Theta.
\end{aligned}
$$

It follows by (5.27),

$$
\begin{aligned}
|\nabla_\rho\nabla_\rho H|=&|\partial_\rho\partial_\rho H|\le C(1+\Theta)\\
|\nabla_\rho\nabla_l H|=&|\nabla_l\nabla_\rho H|=\left|\partial_l\partial_\rho H-\Gamma_{l\rho}^t\nabla_t H\right|\\
\le&|\partial_l\partial_\rho H|+\left|\Gamma_{l\rho}^t\nabla_t H\right|\\
\le& C\left(1+\frac{1}{r-\rho}+\frac{1}{k^2}\Theta\right)+C\rho\left(1+\frac{1}{k^2}\Theta\right)\le C\left(1+\frac{1}{r-\rho}+\frac{1}{k^2}\Theta\right);\\
|\nabla_t\nabla_i H|=&|\partial_t\partial_i H-\Gamma_{ti}^\alpha\nabla_\alpha H|=\left|\partial_t\partial_i H-\Gamma_{ti}^\rho\nabla_\rho H-\Gamma_{ti}^l\nabla_l H\right|\\
\le& C\left(1+\frac{1}{k^2}\Theta\right)+C\rho\left(1+\frac{1}{r-\rho}+\frac{1}{k^2}\Theta\right)+\rho\left(1+\frac{1}{k^2}\Theta\right)\\
\le& C\left(1+\frac{1}{r-\rho}+\frac{1}{k^2}\Theta\right)
\end{aligned}
\tag{5.28}
$$

where in the last two inequality we have used (5.13) and (5.26). The Hessian of $\overline{S}$ is given by

$$
\overline{\nabla}_\alpha\overline{\nabla}_\beta\overline{S}=\nabla_\alpha\nabla_\beta S+\nabla_\alpha\nabla_\beta H-G_{\alpha\beta}^\tau\nabla_\tau S+G_{\alpha\beta}^\tau\nabla_\tau H,
$$

then, to get an estimate on its norm with respect to the metric $\overline{g}$, we need to compute

$$\big|\overline{\nabla}_\rho\overline{\nabla}_\rho\overline{S}\big|_{\overline{g}};\quad \big|\overline{\nabla}_i\overline{\nabla}_\rho\overline{S}\big|_{\overline{g}};\quad \big|\overline{\nabla}_i\overline{\nabla}_j\overline{S}\big|_{\overline{g}},$$

that are

$$\begin{aligned}
\big|\overline{\nabla}_\rho\overline{\nabla}_\rho\overline{S}\big|_{\overline{g}} &= \frac{1}{1+k^2\eta_\rho}\big|\overline{\nabla}_\rho\overline{\nabla}_\rho\overline{S}\big|_g = \frac{1}{1+k^2\eta_\rho}\big|\overline{\nabla}_\rho\overline{\nabla}_\rho\overline{S}\big|\\
&= \frac{1}{(1+k^2\eta_\rho)^2}\big|\nabla_\rho\nabla_\rho S+\nabla_\rho\nabla_\rho H - G^\rho_{\rho\rho}\nabla_\tau S + G^\rho_{\rho\rho}\nabla_\rho H\big|\\
&\le \frac{1}{1+k^2\eta_\rho}[C+C\Theta+C\Theta+C\Theta]\\
&\le C\Big(1+\frac{1}{k^2}\Theta\Big);\\
\big|\overline{\nabla}_i\overline{\nabla}_\rho\overline{S}\big|_{\overline{g}} &= \frac{1}{(1+k^2\eta_\rho^2)^{\frac12}}\big|\overline{\nabla}_i\overline{\nabla}_\rho\overline{S}\big|_g \le \frac{C}{(1+k^2\eta_\rho^2)^{\frac12}}\big|\overline{\nabla}_i\overline{\nabla}_\rho\overline{S}\big|\\
&= \frac{1}{(1+k^2\eta_\rho^2)^{\frac12}}\big|\nabla_i\nabla_\rho S+\nabla_i\nabla_\rho H - G^\tau_{i\rho}\nabla_\tau S + G^\tau_{i\rho}\nabla_\tau H\big|\\
&= \frac{1}{(1+k^2\eta_\rho^2)^{\frac12}}|\nabla_i\nabla_\rho S+\nabla_i\nabla_\rho H|\\
&\le \frac{1}{(1+k^2\eta_\rho^2)^{\frac12}}\Big[C\Big(1+\frac{1}{k^2}\Theta\Big)+C\Big(1+\frac{1}{r-\rho}+\frac{1}{k^2}\Theta\Big)\Big]\\
&\le C\Big(1+\frac{1}{r-\rho}+\frac{1}{k^2}\Theta\Big);\\
\big|\overline{\nabla}_i\overline{\nabla}_j\overline{S}\big|_{\overline{g}} &= \big|\overline{\nabla}_i\overline{\nabla}_j\overline{S}\big|_g \le C\big|\overline{\nabla}_i\overline{\nabla}_j\overline{S}\big|\\
&= \Big|\nabla_i\nabla_j S+\nabla_i\nabla_j H - G^\rho_{ij}\nabla_\rho S + G^\rho_{ij}\nabla_\rho H\Big|\\
&\le C + C\Big(1+\frac{1}{r-\rho}+\frac{1}{k^2}\Theta\Big)+C\rho+C\Big(1+\frac{1}{r-\rho}+\frac{1}{k^2}\Theta\Big)\\
&\le C\Big(1+\frac{1}{r-\rho}+\frac{1}{k^2}\Theta\Big),
\end{aligned}$$

where we have used equations (4.10), (5.9), (5.25) and (5.28) in the estimate of $\big|\overline{\nabla}_\rho\overline{\nabla}_\rho\overline{S}\big|_{\overline{g}}$ and (4.10), (5.9), (5.26) and (5.28) in the the estimates of $\big|\overline{\nabla}_i\overline{\nabla}_\rho\overline{S}\big|_{\overline{g}}$ and $\big|\overline{\nabla}_i\overline{\nabla}_j\overline{S}\big|_{g}$. Therefore, we obtain

$$\big|\overline{\operatorname{Hess}}\,\overline{S}\big|_{\overline{g}} = \big|\overline{\nabla}_\rho\overline{\nabla}_\rho\overline{S}\big|_{\overline{g}}+\big|\overline{\nabla}_\rho\overline{\nabla}_i\overline{S}\big|_{\overline{g}}+\big|\overline{\nabla}_i\overline{\nabla}_\rho\overline{S}\big|_{\overline{g}}+\big|\overline{\nabla}_i\overline{\nabla}_j\overline{S}\big|_{\overline{g}} \le C\Big(1+\frac{1}{r-\rho}+\frac{1}{k^2}\Theta\Big) \tag{5.29}$$

and, by (5.28) and the definition of $\overline{g}$, we also have

$$\big|\bar{\Delta}\overline{S}\big|_{\overline{g}} = \Big|\overline{g}^{\alpha\beta}\overline{\nabla}_\alpha\overline{\nabla}_\beta\overline{S}\Big|_{\overline{g}} = \big|\overline{g}^{\rho\rho}\overline{\nabla}_\rho\overline{\nabla}_\rho\overline{S}+\overline{g}^{ti}\overline{\nabla}_t\overline{\nabla}_i\overline{S}\big|_{\overline{g}} \le C\Big(1+\frac{1}{r-\rho}+\frac{1}{k^2}\Theta\Big). \tag{5.30}$$

As a consequence, putting together (5.6), (5.7), (5.22), (5.29) and (5.30), we deduce:

$$\big|\overline{\mathrm{B}}\big|_{\overline{g}} \le C\Big(1+\frac{1}{r-\rho}+\frac{1}{k^2}\Theta\Big), \tag{5.31}$$

which implies (4.18).

Acknowledgements

The authors would like to thank Professor P. Mastrolia for his useful help regarding the computations in Section 5 and his suggestions about the whole manuscript. All three authors are members of the Gruppo Nazionale per le Strutture Algebriche, Geometriche e loro Applicazioni (GNSAGA) of INdAM (Istituto Nazionale di Alta Matematica).

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