

EXTREMALS FOR POINCARÉ-SOBOLEV SHARP CONSTANTS IN STEINER SYMMETRIC SETS

LORENZO BRASCO, LUCA BRIANI, AND FRANCESCA PRINARI

ABSTRACT. We prove existence of minimizers for the sharp Poincaré-Sobolev constant in general Steiner symmetric sets, in the subcritical and superhomogeneous regime. The sets considered are not necessarily bounded, thus the relevant embeddings may suffer from a lack of compactness. We prove existence by means of an elementary compactness method. We also prove an exponential decay at infinity for minimizers, showing that in the case of Steiner symmetric sets the relevant estimates only depend on the underlying geometry. Finally, we illustrate the optimality of the existence result, by means of some examples.

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1. INTRODUCTION

1.1. A starting point: the two-dimensional strip. Motivated by obtaining existence of solitary waves for a two-dimensional, inviscid and incompressible flow, in [1, Theorem 7.5 & Corollary 7.6], Amick and Toland proved the following remarkable result: for every $q > 2$ and every $\lambda > 0$ the

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following boundary value problem for the *Lane-Emden equation*

$$(1.1) \quad \begin{cases} -\Delta u = \lambda |u|^{q-2} u, & \text{in } S := \mathbb{R} \times (-1, 1), \\ u = 0, & \text{on } \partial S \end{cases}$$

does admit a *positive* solution with finite energy, i.e. such that $u \in W_0^{1,2}(S)$. Concurrently with them, Bona, Bose and Turner obtained the same kind of result in [6, Theorem 5.1]: we refer the reader to their paper [6] for a thorough explanation of the physical model leading to this problem¹.

From the variational point of view, this result may appear a little bit surprising:

- at first, because the embedding $W_0^{1,2}(S) \hookrightarrow L^q(S)$ is only continuous, but not compact, due to the translation invariance of S in the direction $\mathbf{e}_1 = (1, 0)$. This lack of compactness prevents the application of standard methods to prove existence in Sobolev spaces. The two main ones would be: either to look for a critical point of the functional

$$\mathfrak{F}_{q,\lambda}(\varphi) := \frac{1}{2} \int_S |\nabla \varphi|^2 dx - \frac{\lambda}{q} \int_S |\varphi|^q dx,$$

by using the Mountain Pass Theorem, for example (see [44, Chapter II, Theorem 6.2]); or to look for a global minimizer of the following constrained problem

$$\lambda_{2,q}(S) := \inf_{\varphi \in W_0^{1,2}(S)} \left\{ \int_S |\nabla \varphi|^2 dx : \int_S |\varphi|^q dx = 1 \right\}.$$

Indeed, observe that a minimizer v of this problem (which can be taken to be positive, thanks to the fact that both the functional and the constraint are even), would solve

$$-\Delta v = \lambda_{2,q}(S) v^{q-1}, \quad \text{in } S.$$

Then, by taking the scaled function $u = (\lambda/\lambda_{2,q}(S))^{1/(2-q)} v$, we would get a positive solution of (1.1). This particular solution of the Lane-Emden equation is called *with minimal energy*, since it minimizes the free energy functional $\mathfrak{F}_{q,\lambda}$ among all nontrivial solutions (see for example the proof of [13, Corollary 1.4]);

- secondly, this existence result is *false* for $1 < q \leq 2$. Indeed, in the case $1 < q < 2$ we know that every positive solution $u \in W_0^{1,2}(S)$ to (1.1) would automatically be a minimizer of the functional $\mathfrak{F}_{q,\lambda}$ above (see for example [15, Proposition 3.5 & Remark 3.6]). However, it is not difficult to see that this functional is unbounded from below, due to the scale invariance of S along the \mathbf{e}_1 direction. Indeed, for every non-trivial $\varphi \in W_0^{1,2}(S)$ and every $\ell > 0$, by defining $\varphi_\ell(x_1, x_2) = \varphi(\ell x_1, x_2)$, after a change of variable we would get

$$\lim_{\ell \rightarrow 0^+} \mathfrak{F}_{q,\lambda}(\varphi_\ell) = \lim_{\ell \rightarrow 0^+} \left[\frac{\ell}{2} \int_S |\nabla \varphi|^2 dx - \frac{\lambda}{q} \frac{1}{\ell} \int_S |\varphi|^q dx \right] = -\infty.$$

The borderline case $q = 2$ is quite peculiar, since (1.1) boils down to the eigenvalue equation. In this case, the only possibility to have existence of a positive solution u (which thus would be a positive eigenfunction) would be that λ coincides with the first eigenvalue of the Dirichlet-Laplacian on S , i.e.

$$\lambda = \inf_{\varphi \in W_0^{1,2}(S)} \left\{ \int_S |\nabla \varphi|^2 dx : \int_S |\varphi|^2 dx = 1 \right\},$$

¹It is fair to recall that the equation considered in [1, 6] is slightly more general, since they admit a broader class of nonlinear right-hand sides: for the scopes of this paper, we will confine ourselves to the case of the simple equation above.

with u minimizer of this problem. However, it is well-known that on the strip S this problem does not admit a solution.

In light of the previous observations, at that time the existence result of [1] and [6] greatly stimulated the interest towards related results for semilinear or quasilinear PDEs in unbounded domains, where standard compactness methods may fail. Without any attempt of completeness, it is mandatory to recall the papers by Esteban and Lions [27, 28, 29], appeared at the same time as [1, 6] (see also [41, Section V.2]). There, the authors generalized this existence result by considering more general cylindrical sets of the form $\Omega = \omega \times \mathbb{R}^k \subseteq \mathbb{R}^N$, with $\omega \subseteq \mathbb{R}^{N-k}$ open bounded set and $1 \leq k \leq N - 1$. We also refer to [24] and [37] for related existence results for the same equation in some unbounded sets, under suitable geometric structural assumptions.

1.2. Main results. It is now time to present the main scopes and outcomes of our paper. We need at first to settle some notation. For $1 < p < N$, we define the Sobolev critical exponent

$$p^* = \frac{Np}{N-p},$$

as it is customary. Then, for every exponent $q > p$ such that

$$(1.2) \quad q \begin{cases} < p^*, & \text{if } p < N, \\ < \infty, & \text{if } p = N, \\ \leq \infty, & \text{if } p > N, \end{cases}$$

we define

$$(1.3) \quad \lambda_{p,q}(\Omega) = \inf_{\varphi \in C_0^\infty(\Omega)} \left\{ \int_{\Omega} |\nabla \varphi|^p dx : \|\varphi\|_{L^q(\Omega)} = 1 \right\}.$$

Observe that this coincides with the sharp constant for the embedding

$$(1.4) \quad \mathcal{D}_0^{1,p}(\Omega) \hookrightarrow L^q(\Omega),$$

where $\mathcal{D}_0^{1,p}(\Omega)$ is the *homogeneous Sobolev space*, given by the completion of $C_0^\infty(\Omega)$ with respect to the norm

$$C_0^\infty(\Omega) \ni \varphi \mapsto \|\nabla \varphi\|_{L^p(\Omega)}.$$

By keeping in mind the existence result on the two-dimensional strip, in this paper we will focus on proving existence of minimizers for (1.3) for sets which could be “very large” (so that the relevant Sobolev embedding may fail to be compact), but having some geometric properties which permit to retrieve a suitable form of compactness, at least for minimizing sequences. In turn, this will show existence of positive solutions with minimal energy to the following quasilinear Lane-Emden equation

$$(1.5) \quad -\Delta_p u = \lambda u^{q-1}, \quad \text{in } \Omega,$$

with the power q being *superhomogeneous* (i.e. $q > p$) and *subcritical* (in the sense of Sobolev embeddings). For the case $p = 2$, apart for the physical motivation described in [6], we recall that positive solutions of (1.5) describe the extinction profiles of positive solutions to the so-called *fast diffusion equation* with Dirichlet boundary conditions (see the pioneering result [3, Theorem 2] or the thorough discussion of [7, Section 9]).

We will also be interested in deriving some properties of minimizers for (1.3) (hereafter also called *extremals*), notably their behaviour at infinity.

More precisely, we will work with open *Steiner symmetric sets* $\Omega \subseteq \mathbb{R}^N$, i.e. sets such that:

- (S1) Ω is symmetric with respect to each hyperplane $\{x \in \mathbb{R}^N : \langle x, \mathbf{e}_i \rangle = 0\}$;
 (S2) Ω is convex in every direction \mathbf{e}_i .

Here $\{\mathbf{e}_1, \dots, \mathbf{e}_N\}$ stands for the canonical orthonormal basis of \mathbb{R}^N , as usual. We will show that, for these sets, the existence result of [1, 6] still holds, for every $1 < p < \infty$ and every q as above. Observe that our sets do not necessarily have a cylindrical structure. In this respect, the existence part of this paper only partially superposes with the aforementioned results by Esteban and Lions, for example. It should be also pointed out that our methods of proof will be *elementary* (for example, we will not appeal to any *concentration-compactness method*), as we will explain in the next subsections. Our paper can find [6] as its true ancestor, from the mathematical point of view.

We summarize below the main findings of the present paper: this will result by collecting together Theorems 5.1, 5.2 and 7.7 below.

Main Theorem. *Let $1 < p < \infty$ and assume that $q > p$ satisfies (1.2). Let $\Omega \subseteq \mathbb{R}^N$ be an open Steiner symmetric set such that $\Omega \neq \mathbb{R}^N$. Then, the infimum defining $\lambda_{p,q}(\Omega)$ is attained by some Steiner symmetric function $u \in W_0^{1,p}(\Omega) \setminus \{0\}$. Moreover, every Steiner symmetric extremal satisfies the following exponential decay estimate*

$$|u(x)| \leq C e^{-a|x|}, \quad \text{for } x \in \Omega,$$

with $a = a(p, q, \lambda_{p,p}(\Omega)) > 0$ and $C = C(N, p, q, \Omega) > 0$.

Observe that, under the standing assumptions of the Main Theorem, we admit unbounded open sets which could be quite “large”, as already anticipated. For example, these sets could be unbounded in every direction and have infinite volume. Thus, in principle, it is not even clear that $\lambda_{p,q}(\Omega) > 0$, i.e. that the continuous embedding (1.4) holds. Actually, this is the case, as we show in Section 3: more precisely, we prove at first that each open Steiner symmetric set $\Omega \neq \mathbb{R}^N$ has finite *inradius*. The latter is the quantity defined by

$$r_\Omega = \sup \left\{ r > 0 : \text{there exists a ball } B_r(x) \subseteq \Omega \right\}.$$

It is well-known that the condition $r_\Omega < +\infty$ is only a necessary one, for the embedding (1.4) to hold. We show that for a Steiner symmetric set *this is sufficient*, as well, through the following quantitative geometric lower bound

$$(1.6) \quad \lambda_{p,q}(\Omega) \geq \frac{1}{C_{N,p,q}} \left(\frac{1}{r_\Omega} \right)^{p-N+\frac{p}{q}N},$$

which is interesting in itself.

Before proceeding further with some comments on the proof of the Main Theorem, a list of remarks are in order.

Remark 1.1 (The case $q \leq p$). For $q \leq p$, the assumptions on Ω of the Main Theorem are not enough to ensure that $\lambda_{p,q}(\Omega)$ is attained. As a counter-example, we can take for simplicity $p = N = 2$ and consider again the strip

$$S = \mathbb{R} \times (-1, 1).$$

Then we recall that:

- (1) for $q < 2$, in general we have that the embedding $\mathcal{D}_0^{1,2}(\Omega) \hookrightarrow L^q(\Omega)$ is continuous if and only if this is compact (see [43, Theorem 15.6.2]). Since $\mathcal{D}_0^{1,2}(S) \hookrightarrow L^q(S)$ can not be compact (due to the translation invariance of the set along the direction \mathbf{e}_1), we must have $\lambda_{2,q}(S) = 0$;
- (2) for $p = q = 2$, we have that $\lambda_{2,2}(S)$ coincides with the bottom of the spectrum of the Dirichlet-Laplacian in S and we have already observed that this is not attained.

Remark 1.2 (Breaking Steiner symmetry). Our assumptions (S1) and (S2) on the open sets are, in a sense, necessary, whenever the embedding $\mathcal{D}_0^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ fails to be compact. We will show for example that there exist:

- *convex* sets, having only $N - 1$ orthogonal hyperplanes of symmetry;
- sets with N orthogonal hyperplanes of symmetry, but not convex along one of the coordinate directions;

such that $\lambda_{p,q}$ does not admit an extremal. We will also highlight that our existence result is quite unstable, with respect to small (smooth) deformations of the boundary: for example, while the strip $S = \mathbb{R} \times (-1, 1)$ admits an extremal, by making a small symmetric *inward* deformation we get a new open set, having two orthogonal axis of symmetry, such that $\lambda_{p,q}$ is not attained.

Remark 1.3 (Uniqueness). This is an interesting problem, which however is not tackled in this paper. It is well-known that the uniqueness of positive solutions to the Lane-Emden (1.5) is quite a delicate matter, in the case $q > p$. Even for solutions with minimal energy, i.e. positive extremals of $\lambda_{p,q}(\Omega)$, the known uniqueness results are very few. For example, we refer the interested reader to [13, Theorem 1.1], [22, Theorem 5], [23, Theorem 4.1], [35, Theorem 1.1 & Proposition 6.3] and [40, Lemma 3 & Theorem 1], for a positive answer in some particular cases, when Ω is *bounded*. However, extending these partial results to the case of unbounded sets as those considered in the Main Theorem does not seem to be straightforward. In the special case of the strip $S = \mathbb{R} \times (-1, 1)$ and for $p = 2$, uniqueness up to translation for positive solutions to (1.1) seems to belong to the folklore on the subject: we refer to [21, page 665], which contains a very brief sketch of the proof.

Finally, we recall that, in general, uniqueness fails already for some bounded sets, see for example [12, Section 4.2].

1.3. Some comments on the proofs. For the existence part, we will use the Direct Method in the Calculus of Variations. Here, we first point out that we do not prove that *every* minimizing sequence is sequentially precompact in $L^q(\Omega)$. We will rather construct a particular minimizing sequence which has this compactness property: this will be accomplished by considering the minimizers u_n of a suitably “perturbed” problem, depending on an index $n \in \mathbb{N}$.

Such a problem consists in adding a “vanishing confinement” term, i.e. we consider

$$(1.7) \quad \inf_{u \in W_0^{1,p}(\Omega)} \left\{ \int_{\Omega} |\nabla u|^p dx + \frac{1}{n+1} \int_{\Omega} V |u|^p dx : \|u\|_{L^q(\Omega)} = 1 \right\}.$$

The precise form of the positive potential V is not very important, but it must enjoy the following crucial properties:

- (V1) it has to be a *confining potential*, i.e. for every $t > 0$ the sublevel set $\{x \in \mathbb{R}^N : V(x) \leq t\}$, is bounded;

(V2) for every non-negative function $u \in L^p(\Omega)$, we have

$$\int_{\Omega} V u^p dx \geq \int_{\Omega} V (\mathcal{S}_{\mathbf{e}_i}(u))^p dx, \quad \text{for every } i \in \{1, \dots, N\},$$

where $\mathcal{S}_{\mathbf{e}_i}(u)$ stands for the *Steiner symmetrization of u with respect to the direction \mathbf{e}_i* , see Section 3 below.

For example, it turns out that the simple choice $V(x) = |x|$ will do the job. The property (V1) guarantees that a minimizer $u_n \in W_0^{1,p}(\Omega)$ for (1.7) actually exists; moreover, property (V2) assures that it can be taken to be Steiner symmetric, thanks to the so-called *Pólya-Szegő principle* for the Steiner symmetrization (see [20] or Section 3 below).

In order to prove that $\{u_n\}_{n \in \mathbb{N}}$ is sequentially precompact in $L^q(\Omega)$ and get existence of an extremal for our original problem, the first crucial step is excluding that $\{u_n\}_{n \in \mathbb{N}}$ weakly converges to the null function. To this aim, we use the equation solved by u_n and rely on a uniform (with respect to n) Harnack's inequality. This fact and the Steiner symmetry of each u_n permit to conclude that u_n is uniformly bounded from below on a fixed ball. This is enough to exclude the vanishing of the weak limit.

We then get the desired strong convergence by relying on the same idea as in the proof of [18, Lemma 4.2], which deals with existence of an extremal for the sharp constant in a Gagliardo-Nirenberg-Sobolev inequality on the whole space. Here, we exploit the classical *Brézis-Lieb Lemma* for the sequence $\{(u_n, \nabla u_n)\}_{n \in \mathbb{N}}$: as a technical point, we recall that in order to apply the Brézis-Lieb Lemma, the weak convergence is not enough when $p \neq 2$ (see [16, Remarks, point (iii), page 487]). Thus, we have to preliminary show that $\{(u_n, \nabla u_n)\}_{n \in \mathbb{N}}$ converges pointwise almost everywhere. For the functions themselves this is quite easy; for their gradients we can use a monotonicity trick, based on the ellipticity of the equation solved by each u_n . The last fact marks a difference with the case considered in [18], where $p = 2$ and one can be dispensed with this additional pointwise convergence. As a byproduct of the proof, we actually get that $\{u_n\}_{n \in \mathbb{N}}$ converges (up to a subsequence) strongly in $W_0^{1,p}(\Omega)$.

All these tools are quite robust and do not rely on higher regularity estimates (differently from the proofs of [1] and [6], for example). Thus, we can easily treat the case of a generic exponent $1 < p < \infty$.

We now want to make some comments on the decay estimate for extremals. We first remark that this is a consequence of a more general result, proved in Section 7: this concerns non-negative weak subsolutions $u \in W_0^{1,p}(\Omega)$ of the Lane-Emden equation, i.e.

$$-\Delta_p u \leq \lambda u^{q-1}, \quad \text{in } \Omega,$$

in a general open set $\Omega \subseteq \mathbb{R}^N$, not necessarily Steiner symmetric.

The result is obtained by using iterative methods *à la* De Giorgi-Moser. These are nonlinear methods in nature, thus here as well we can treat the case of a general exponent $1 < p < \infty$. We point out in particular that we do not rely on ODE methods (like for example in the classical result [4, Theorem 1]) or on the construction of suitable barrier functions (see for example [19, Proposition 1]). The first step of the proof is to get an $L^\infty - L^q$ a priori estimate “localized at infinity”, i.e. an estimate of the type

$$\|u\|_{L^\infty(\Omega \setminus B_{\varrho+1})} \lesssim \|u\|_{L^q(\Omega \setminus B_\varrho)}, \quad \text{for every } \varrho > 0.$$

By combining this estimate with the fact that $u \in L^q(\Omega)$, we get that u is arbitrarily small (in the sup norm) in the complement of a sufficiently large ball. More precisely, for every $\varepsilon > 0$ there exists

$R_\varepsilon > 0$ such that

$$(1.8) \quad \|u\|_{L^\infty(\Omega \setminus B_{R_\varepsilon})} < \varepsilon.$$

Thanks to the fact that $q > p$, this in turn entails that

$$-\Delta_p u \leq \lambda \varepsilon^{q-p} u^{p-1}, \quad \text{in } \Omega \setminus B_{R_\varepsilon}.$$

In other words, we get that u is a subsolution “at infinity” of the eigenvalue equation for the p -Laplacian, with an eigenvalue which can be taken as small as we wish. The exponential decay now easily follows from this fact, provided we assume that Ω supports the Poincaré inequality

$$c_\Omega \int_\Omega |\varphi|^p dx \leq \int_\Omega |\nabla \varphi|^p dx, \quad \text{for every } \varphi \in C_0^\infty(\Omega).$$

As the reader may imagine, there is a subtle point here: in order to be a subsolution with a small eigenvalue, we need to choose a sufficiently large radius R_ε . This choice heavily depends on the (sub)solution u itself and it affects the constant C appearing in the final decay estimate. Thus, in general the resulting a priori estimate is a bit implicit.

We will show in Theorem 7.7 that, for Steiner symmetric sets and Steiner symmetric extremals, such an estimate only depends on the open set Ω , i.e. the radius R_ε at which the extremal “starts to decay” may be taken to depend *only on the underlying geometry* of Ω , not on the extremal itself. More precisely, it depends on how the orthogonal sections of Ω “shrink” or “persist” at infinity (we refer the reader to Section 7 for a more precise discussion).

The difficult point is to handle the case when Ω contains some unbounded tubes of fixed width (the prototypical case being the two-dimensional strip). In this case, we can get the claimed uniform result by carefully adapting an argument from [6]. An explicit local Lipschitz estimate for extremals will be the key to get the result, by avoiding the stronger regularity used in [6] for the case of the Laplacian (this could not be available in the case $p \neq 2$ and for a general Steiner symmetric set).

1.4. Plan of the paper. In Sections 2 and 3 we introduce some notation and discuss some preliminary results, which will be used throughout the paper. In particular, in Proposition 3.4 we show the lower bound (1.6). In Section 4 we consider the minimizing sequence $\{u_n\}_{n \in \mathbb{N}}$ for the problem (1.3), obtained by solving (1.7); we also discuss its main features. Then, Section 5 is devoted to the existence of extremals, by distinguishing the cases $q < \infty$ (see Theorem 5.1) and $q = \infty$ (see Theorem 5.2). The optimality of the assumptions of these results are discussed in Section 6, by means of simple counterexamples. At last, in Section 7 we prove the decay properties of extremals: first, we show an estimate holding for every non-negative subsolution to the Lane-Emden equation, see Theorem 7.3; then, as discussed above, we refine such an estimate in the case of extremals of Steiner symmetric sets, see Theorem 7.7.

For completeness, in Appendix A we present some technical results concerning the case $q = \infty$, which are needed in the paper.

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2. PRELIMINARIES

2.1. Notation. For $r > 0$ and $x_0 \in \mathbb{R}^N$ we denote by $B_r(x_0)$ the N -dimensional open ball centered at x_0 , with radius $r > 0$. When $x_0 = 0$, we will simply write B_r . We also denote $\omega_N = |B_1|$. Occasionally, we will also need the notation

$$Q_r(x_0) = \prod_{i=1}^N (x_{0,i} - r, x_{0,i} + r) \quad \text{where } x_0 = (x_{0,1}, \dots, x_{0,N}),$$

for open N -dimensional hypercubes. As in the case of balls, we will simply write Q_r when the center x_0 coincides with the origin.

We define the *inradius* of an open set $\Omega \subseteq \mathbb{R}^N$ as

$$r_\Omega = \sup \left\{ r > 0 : \text{there exists a ball } B_r(x) \subseteq \Omega \right\}.$$

For an open set $\Omega \subseteq \mathbb{R}^N$ and an exponent $1 < p < \infty$, we will use the distinguished notation

$$\lambda_p(\Omega) = \inf_{u \in C_0^\infty(\Omega)} \left\{ \int_\Omega |\nabla u|^p dx : \|u\|_{L^p(\Omega)} = 1 \right\}.$$

This is a particular instance of (1.3), corresponding to the choice $q = p$.

We explicitly note that the infimum over $C_0^\infty(\Omega)$ in the definitions of both $\lambda_p(\Omega)$ and $\lambda_{p,q}(\Omega)$ can be equivalently performed on the whole $W_0^{1,p}(\Omega)$, see for example [14, Lemma 2.6]. Here, by $W_0^{1,p}(\Omega)$ we intend the closure of $C_0^\infty(\Omega)$ in the standard Sobolev space $W^{1,p}(\Omega)$.

2.2. Miscellaneous facts. The following simple result expresses in a quantitative way the sub-additivity of concave powers. It should be well-known, but we enclose the proof for the reader's convenience.

Lemma 2.1. *Let $0 < \alpha < 1$, for every $A, B \geq 0$ we have*

$$(A + B)^\alpha \leq A^\alpha + B^\alpha - \frac{1 - \alpha}{2^{\alpha+1}} \min\{A^\alpha, B^\alpha\}.$$

Proof. Without loss of generality, we can suppose that $B \leq A$. We can also suppose that $B > 0$, otherwise there is nothing to prove. If we set $t = B/A \in (0, 1]$, we just need to prove that

$$(2.1) \quad (1 + t)^\alpha \leq 1 + t^\alpha - \frac{1 - \alpha}{2^{\alpha+1}} t^\alpha.$$

To this aim, we introduce the function of one real variable

$$h(t) = \frac{1 + t^\alpha}{(1 + t)^\alpha}, \quad \text{for every } t \in (0, 1].$$

We have

$$h'(t) = \alpha \frac{t^{\alpha-1} - 1}{(1+t)^{\alpha+1}} \geq \frac{\alpha}{2^{\alpha+1}} (t^{\alpha-1} - 1), \quad \text{for every } t \in (0, 1].$$

Thus, we get

$$\begin{aligned} h(t) &= h(0) + \int_0^t h'(\tau) d\tau \geq 1 + \frac{\alpha}{2^{\alpha+1}} \int_0^t (\tau^{\alpha-1} - 1) d\tau \\ &= 1 + \frac{1}{2^{\alpha+1}} (t^\alpha - \alpha t) \geq 1 + \frac{1-\alpha}{2^{\alpha+1}} t^\alpha. \end{aligned}$$

By recalling the definition of h , this implies that

$$1 + t^\alpha \geq (1+t)^\alpha + (1+t)^\alpha \frac{1-\alpha}{2^{\alpha+1}} t^\alpha.$$

By using that $(1+t)^\alpha \geq 1$, we get (2.1), as desired. \square

We recall the following particular family of *Gagliardo-Nirenberg interpolation inequalities*

$$(2.2) \quad \|\psi\|_{L^r(\Omega)} \leq G_{N,p,r} \|\nabla \psi\|_{L^p(\Omega)}^\vartheta \|\psi\|_{L^p(\Omega)}^{1-\vartheta}, \quad \text{for every } \psi \in C_0^\infty(\Omega),$$

which hold for every $1 < p < \infty$ and $r > p$ satisfying (1.2), with $G_{N,p,r} > 0$ only depending on N, p and r . The exponent ϑ is simply dictated by scale invariance and given by

$$\vartheta = \frac{N}{p} - \frac{N}{r},$$

see for example [11, Theorem 3.8.1] or [38, Theorem 12.83].

In the next result, we recall that in the super-homogeneous regime $q > p$ we are interested in, all the Poincaré-Sobolev constants are equivalent.

Proposition 2.2. *Let $1 < p < \infty$ and $q > p$ satisfying (1.2). Then there exist $C_2 = C_2(N, p, q) > 0$ and $C_3 = C_3(N, p, q) > 0$ such that, for every open set $\Omega \subseteq \mathbb{R}^N$, it holds*

$$C_2 \left(\lambda_p(\Omega) \right)^{1 - \frac{N}{p} + \frac{N}{q}} \leq \lambda_{p,q}(\Omega) \leq C_3 \left(\lambda_p(\Omega) \right)^{1 - \frac{N}{p} + \frac{N}{q}}.$$

In particular, for every $q > p$ satisfying (1.2) we have

$$\lambda_p(\Omega) > 0 \quad \iff \quad \lambda_{p,q}(\Omega) > 0.$$

Proof. By using the Gagliardo-Nirenberg inequality (2.2) with $r = q$, we have that

$$\|u\|_{L^q(\Omega)}^p \leq (G_{N,p,q})^p \left(\int_\Omega |\nabla u|^p dx \right)^\vartheta \left(\int_\Omega |u|^p dx \right)^{1-\vartheta}, \quad \text{for every } u \in C_0^\infty(\Omega),$$

where the exponent $\vartheta \in (0, 1)$ is given by

$$\vartheta = \frac{N}{p} - \frac{N}{q}.$$

If we assume $\lambda_p(\Omega) > 0$, by using the Poincaré inequality on the right-hand side above, we get

$$\|u\|_{L^q(\Omega)}^p \leq (G_{N,p,q})^p \left(\int_\Omega |\nabla u|^p dx \right) \left(\frac{1}{\lambda_p(\Omega)} \right)^{1-\vartheta}, \quad \text{for every } u \in C_0^\infty(\Omega),$$

which in turn implies the desired lower bound for $\lambda_{p,q}(\Omega)$.

The upper bound can be deduced by [43, Theorem 15.4.1] (for an alternative proof, see [14, Corollary 5.9] for $p > N$ and [9, Theorem 6.1] for $1 < p \leq N$). \square

2.3. The Lane-Emden equation. In the case $q < \infty$, whenever the infimum defining $\lambda_{p,q}(\Omega)$ is attained by some function $u \in W_0^{1,p}(\Omega)$, we have that u satisfies the associated Euler-Lagrange equation

$$\int_{\Omega} \langle |\nabla u|^{p-2} \nabla u, \nabla \varphi \rangle dx = \lambda_{p,q}(\Omega) \int_{\Omega} |u|^{q-2} u \varphi dx, \quad \text{for every } \varphi \in C_0^\infty(\Omega).$$

In other words, u is a weak solution of the so-called *Lane-Emden equation*

$$-\Delta_p u = \lambda_{p,q}(\Omega) |u|^{q-2} u, \quad \text{in } \Omega,$$

with homogeneous Dirichlet boundary condition. In the case $p > N$ and $q = \infty$, one can show that the Euler-Lagrange equation takes the following form

$$\int_{\Omega} \langle |\nabla u|^{p-2} \nabla u, \nabla \varphi \rangle dx = \lambda_{p,\infty}(\Omega) |u(x_0)|^{p-2} u(x_0) \varphi(x_0), \quad \text{for every } \varphi \in C_0^\infty(\Omega),$$

where $x_0 \in \Omega$ satisfies $|u(x_0)| = \|u\|_{L^\infty(\Omega)}$. We refer to Appendix A below for this fact.

In particular, for every q as in (1.2), we have that every non-negative extremal u of $\lambda_{p,q}(\Omega)$ is a weakly p -superharmonic function, that is, u satisfies

$$\int_{\Omega} \langle |\nabla u|^{p-2} \nabla u, \nabla \varphi \rangle dx \geq 0, \quad \text{for every } \varphi \in C_0^\infty(\Omega) \text{ such that } \varphi \geq 0.$$

As a consequence, when Ω is a connected open set, every non-negative minimizer u of $\lambda_{p,q}(\Omega)$ is actually positive in Ω , thanks to the strong minimum principle. We also recall that any minimizer u of $\lambda_{p,q}(\Omega)$, if it does exist, has constant sign: for the case $q < \infty$ we refer to [13, Lemma 2.1], while for the case when $p > N$ and $q = \infty$, we refer to Proposition A.3.

Finally, we recall the following mild regularity result. The specific form of the a priori estimate will be important.

Lemma 2.3. *Let $1 < p < \infty$ and let $p \leq q < \infty$ be an exponent satisfying (1.2). Let $\Omega \subseteq \mathbb{R}^N$ be an open set such that $\lambda_p(\Omega) > 0$ and let $u \in W_0^{1,p}(\Omega)$ be a non-negative weak subsolution of*

$$-\Delta_p u = \lambda u^{q-1}, \quad \text{in } \Omega,$$

for some $\lambda > 0$. Then $u \in L^\infty(\Omega)$ and we have

$$\|u\|_{L^\infty(\Omega)} \leq C_1 \left(\lambda^{\frac{N}{p-q}} \|u\|_{L^q(\Omega)} \right)^{\frac{pq}{p-q-(q-p)N}},$$

for some $C_1 = C_1(N, p, q) > 0$.

Proof. The proof consists in reproducing verbatim that of [13, Proposition 2.4], thus it is omitted. For $q > p$, one just needs to preliminarily observe that $u \in L^q(\Omega)$, as well. Indeed, by Proposition 2.2 and the assumption $\lambda_p(\Omega) > 0$, the continuous embedding $W_0^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ holds. \square

3. STEINER SYMMETRIZATION

In this section we recall the main features of the classical Steiner symmetrization, referring the reader to [20], [34, Section 2.2], [36] and [39, Chapter 3] for more details.

3.1. Sets. For every direction $\mathbf{e} \in \mathbb{S}^{N-1}$, we denote by $\Pi_{\mathbf{e}} : \mathbb{R}^N \rightarrow \mathbb{R}^N$ the orthogonal projection onto the hyperplane $\langle \mathbf{e} \rangle^\perp := \{x \in \mathbb{R}^N : \langle x, \mathbf{e} \rangle = 0\}$, that is

$$\Pi_{\mathbf{e}}(x) = x - \langle x, \mathbf{e} \rangle \mathbf{e}, \quad \text{for every } x \in \mathbb{R}^N.$$

For every direction $\mathbf{e} \in \mathbb{S}^{N-1}$, we also define

$$\mathcal{R}_{\mathbf{e}}(x) = x - 2 \langle x, \mathbf{e} \rangle \mathbf{e}, \quad \text{for every } x \in \mathbb{R}^N,$$

the *reflection with respect to the hyperplane* $\langle \mathbf{e} \rangle^\perp$.

Let $E \subseteq \mathbb{R}^N$ be a set and $\mathbf{e} \in \mathbb{S}^{N-1}$ a given direction. For every $x \in \Pi_{\mathbf{e}}(E)$, we set

$$E_{\mathbf{e}}(x) = \{y \in E : \exists t \in \mathbb{R} \text{ such that } y = x + t\mathbf{e}\}.$$

This is nothing but the intersection between E and the line passing through x , with direction \mathbf{e} . The *Steiner symmetrization of E with respect to the direction \mathbf{e}* is the set $\mathcal{S}_{\mathbf{e}}(E)$ given by

$$\mathcal{S}_{\mathbf{e}}(E) = \left\{ x + t\mathbf{e} : x \in \Pi_{\mathbf{e}}(E), |t| < \frac{\mathcal{H}^1(E_{\mathbf{e}}(x))}{2} \right\}.$$

Here \mathcal{H}^1 is the one-dimensional Hausdorff measure.

We recall that if the set $E \subseteq \mathbb{R}^N$ is measurable, then $\mathcal{S}_{\mathbf{e}}(E)$ has the same property and there holds

$$|E| = |\mathcal{S}_{\mathbf{e}}(E)|,$$

for a proof see [32, Theorem 2.3]. Moreover, if E is an open set, then $\mathcal{S}_{\mathbf{e}}(E)$ is open, as well. In the following, we say that:

- E is *symmetric with respect to the hyperplane* $\langle \mathbf{e} \rangle^\perp$ if

$$\mathcal{R}_{\mathbf{e}}(E) = E.$$

In particular, for every $x \in \Pi_{\mathbf{e}}(E)$, if $x + t\mathbf{e} \in E$ for some $t \in \mathbb{R}$, then we have $x - t\mathbf{e} \in E$, as well;

- E is *convex in the direction \mathbf{e}* if

$$x + t_1\mathbf{e} \in E, x + t_2\mathbf{e} \in E \implies x + (st_1 + (1-s)t_2)\mathbf{e} \in E, \quad \begin{array}{l} \text{for every } x \in \Pi_{\mathbf{e}}(E), \\ t_1, t_2 \in \mathbb{R}, s \in (0, 1). \end{array}$$

Then, the following characterization holds:

$$\mathcal{S}_{\mathbf{e}}(E) = E \quad \iff \quad E \text{ is symmetric with respect to } \langle \mathbf{e} \rangle^\perp \text{ and convex in the direction } \mathbf{e}.$$

In the sequel, we will consider open sets with an orthogonal system of symmetries, according to the following definition.

Definition 3.1 (The class \mathfrak{S}^N). We say that an open set $\Omega \subseteq \mathbb{R}^N$ is *Steiner symmetric* if

- (S1) Ω is symmetric with respect to $\langle \mathbf{e}_j \rangle^\perp$, for every $j \in \{1, \dots, N\}$;
- (S2) Ω is convex in the direction \mathbf{e}_j , for every $j \in \{1, \dots, N\}$.

We will indicate by \mathfrak{S}^N the collection of all Steiner symmetric open subsets of \mathbb{R}^N .

Remark 3.2. In particular, every $\Omega \in \mathfrak{S}^N$ has the following property: if $\Sigma \subseteq \Omega$ is a convex set, then we have

$$(\Sigma \cup \mathcal{R}_{\mathbf{e}_j}(\Sigma))^{\text{ch}} \subseteq \Omega, \quad \text{for every } j \in \{1, \dots, N\},$$

that is Ω contains the *convex hull* of $\Sigma \cup \mathcal{R}_{\mathbf{e}_j}(\Sigma)$, as well.

The next result contains some basic geometric properties of Steiner symmetric sets.

Lemma 3.3. *Let $\Omega \in \mathfrak{S}^N$ with $\Omega \neq \mathbb{R}^N$. Then we have*

$$r_\Omega < +\infty \quad \text{and} \quad B_{r_\Omega} \subseteq \Omega.$$

Moreover, the following measure density condition holds

$$(3.1) \quad \inf \left\{ \frac{|B_r(x) \setminus \Omega|}{|B_r(x)|} : x \in \partial\Omega, r > 0 \right\} \geq \left(\frac{1}{2}\right)^N.$$

Proof. Let $\{r_n\}_{n \in \mathbb{N}} \subseteq (0, +\infty)$ be an increasing sequence converging to r_Ω . Accordingly, there exists a sequence of points $\{x_n\}_{n \in \mathbb{N}} \subseteq \Omega$ such that

$$B_{r_n}(x_n) \subseteq \Omega, \quad \text{for every } n \in \mathbb{N}.$$

By appealing to Remark 3.2 with $\Sigma = B_{r_n}(x_n)$ and $j = 1$, we get

$$\left(B_{r_n}(x_n) \cup \mathcal{R}_{\mathbf{e}_1}(B_{r_n}(x_n))\right)^{\text{ch}} \subseteq \Omega, \quad \text{for every } n \in \mathbb{N}.$$

By observing that

$$\left(B_{r_n}(x_n) \cup \mathcal{R}_{\mathbf{e}_1}(B_{r_n}(x_n))\right)^{\text{ch}} \supseteq B_{r_n}(\Pi_{\mathbf{e}_1}(x_n)),$$

we get

$$B_{r_n}(\Pi_{\mathbf{e}_1}(x_n)) \subseteq \Omega, \quad \text{for every } n \in \mathbb{N}.$$

We can repeat the above argument with the ball $B_{r_n}(\Pi_{\mathbf{e}_1}(x_n))$ and \mathbf{e}_2 , in place of $B_{r_n}(x_n)$ and \mathbf{e}_1 , by also getting that

$$B_{r_n}(\Pi_{\mathbf{e}_2} \circ \Pi_{\mathbf{e}_1}(x_n)) \subseteq \Omega, \quad \text{for every } n \in \mathbb{N}.$$

By applying this scheme recursively with respect to every direction \mathbf{e}_i for $i \in \{1, \dots, N\}$, we finally obtain that

$$B_{r_n} = B_{r_n}(\Pi_{\mathbf{e}_N} \circ \dots \circ \Pi_{\mathbf{e}_1}(x_n)) \subseteq \Omega, \quad \text{for every } n \in \mathbb{N}.$$

By recalling that $\{r_n\}_{n \in \mathbb{N}}$ converges to r_Ω and that $\Omega \neq \mathbb{R}^N$, the previous result guarantees that $r_\Omega < \infty$ and $B_{r_\Omega} \subseteq \Omega$, as claimed.

We now come to the proof of the measure density estimate (3.1). Let $x = (x_1, \dots, x_N) \in \partial\Omega$, thanks to the symmetries of Ω we have

$$|B_r(\bar{x}) \setminus \Omega| = |B_r(x) \setminus \Omega|,$$

where $\bar{x} = (|x_1|, \dots, |x_N|) \in \partial\Omega$. Thus, we can suppose without loss of generality that $x_i \geq 0$, for every $i \in \{1, \dots, N\}$. We claim that

$$(3.2) \quad \mathcal{C}_x := \prod_{i=1}^N [x_i, +\infty) \subseteq \mathbb{R}^N \setminus \Omega.$$

Observe that \mathcal{C}_x is nothing but a *hyperoctant* in \mathbb{R}^N , with vertex at x . The argument is similar to the one given in the first part of the proof. By contradiction, we suppose that there exists

$$z = (z_1, \dots, z_N) \in \Omega \cap \mathcal{C}_x.$$

By Remark 3.2, we get that Ω must contain the segment joining z and $\mathcal{R}_{\mathbf{e}_1}(z)$. In other words, we have

$$(t, z_2, \dots, z_N) \in \Omega, \quad \text{for every } -z_1 \leq t \leq z_1.$$

In particular, since $z \in \mathcal{C}_x$, we have $z_1 \geq x_1$ and thus with the choice $t = x_1$, we get

$$(x_1, z_2, \dots, z_N) \in \Omega.$$

We can now iterate and repeat this argument in the other coordinate directions. This would give us $x = (x_1, x_2, \dots, x_N) \in \Omega$, which is a contradiction. Therefore (3.2) holds true. As a consequence, for every $r > 0$ we have

$$|B_r(x) \setminus \Omega| \geq |B_r(x) \cap \mathcal{C}_x| = 2^{-N} |B_r(x)|,$$

that proves the desired estimate. \square

Thanks to the geometric properties encoded in Lemma 3.3, we deduce the following expedient result. This is interesting in itself.

Proposition 3.4. *Let $1 < p < \infty$ and $q \geq p$ satisfying (1.2). Let $\Omega \in \mathfrak{S}^N$ be such that $\Omega \neq \mathbb{R}^N$. Then we have $\lambda_{p,q}(\Omega) > 0$. More precisely, the following lower bound holds*

$$c \left(\frac{1}{r_\Omega} \right)^{p-N+N\frac{p}{q}} \leq \lambda_{p,q}(\Omega),$$

for some $c = c(N, p, q) > 0$.

Proof. By applying Lemma 3.3, we know that $r_\Omega < +\infty$. For $p > N$, it is then sufficient to appeal to [43, Theorem 15.4.1] (see also [14, Corollary 5.9]). For $1 < p \leq N$, we can simply evoke [8, Corollary 5.6], thanks to the measure density condition (3.1). \square

In Section 7 we will need the following further definitions, aimed at describing the “local geometry at infinity” of a Steiner symmetric set. For an open set $\Omega \subseteq \mathbb{R}^N$, we define for every $i \in \{1, \dots, N\}$

$$\Omega_{i,t} = \{x \in \Omega : \langle x, \mathbf{e}_i \rangle = t\}, \quad \text{for } t \in \mathbb{R}.$$

These are the sections of Ω which are orthogonal to \mathbf{e}_i . Observe that these are open subsets of the hyperplane $\{x : \langle x, \mathbf{e}_i \rangle = t\} \simeq \mathbb{R}^{N-1}$, with respect to the induced topology. Accordingly, for every $i \in \{1, \dots, N\}$ we define

$$(3.3) \quad r_{\Omega_{i,t}} = \sup \left\{ r > 0 : \exists B_r(x_0) \subseteq \mathbb{R}^N \text{ with } x_0 \in \Omega_{i,t} \text{ s.t. } (B_r(x_0))_{i,t} \subseteq \Omega_{i,t} \right\}, \quad \text{for } t \in \mathbb{R}.$$

The quantity $r_{\Omega_{i,t}}$ is nothing but the inradius of the section $\Omega_{i,t}$. When $\Omega \in \mathfrak{S}^N$, thanks to Steiner symmetry, it is not difficult to see that it enjoys the following properties.

Lemma 3.5. *Let $\Omega \in \mathfrak{S}^N$ be such that $\Omega \neq \mathbb{R}^N$. Then for every $i \in \{1, \dots, N\}$*

$$(3.4) \quad r_{\Omega_{i,t}} = r_{\Omega_{i,-t}}, \text{ for every } t \in \mathbb{R} \quad \text{and} \quad r_{\Omega_{i,t}} \leq r_{\Omega_{i,s}}, \text{ for every } 0 \leq s < t.$$

Moreover, there exists a constant $\tau_\Omega > 0$ such that

$$(3.5) \quad \max_{i \in \{1, \dots, N\}} r_{\Omega_{i,t}} < +\infty, \quad \text{for every } t \geq \tau_\Omega.$$

Proof. The properties (3.4) follow quite easily from (S1) and (S2).

In order to prove (3.5), we observe that if $\Omega_{i,t} = \{x \in \mathbb{R}^N : \langle x, \mathbf{e}_i \rangle = t\}$ for every $t \in \mathbb{R}$, then Ω would coincide with the whole space \mathbb{R}^N . Thus, there must exist $t_i \in \mathbb{R}$ such that $\Omega_{i,t} \neq \{x \in \mathbb{R}^N : \langle x, \mathbf{e}_i \rangle = t\}$. With a slight abuse of notation, we observe that $\Omega_{i,t} \in \mathfrak{S}^{N-1}$. By applying Lemma 3.3 in dimension $N-1$, we get that $\Omega_{i,t}$ has a finite inradius and a maximal ball is centered at the point $t_i \mathbf{e}_i$. In light of the first property in (3.4), we can suppose that $t_i \geq 0$, while, by the second property in (3.4), we get that

$$r_{\Omega_{i,t}} \leq r_{\Omega_{i,t_i}}, \text{ for every } 0 \leq t_i < t.$$

By repeating this argument for every direction, we get the conclusion. \square

3.2. Functions. Let $1 \leq p < \infty$, let $E \subseteq \mathbb{R}^N$ a measurable set and let $u \in L^p(E)$ be a non-negative function. We extend u by setting $u = 0$ in $\mathbb{R}^N \setminus E$. We can define the *Steiner symmetrization of u with respect to the direction $\mathbf{e} \in \mathbb{S}^{N-1}$* as the non-negative function $\mathcal{S}_{\mathbf{e}}(u)$ given by

$$\mathcal{S}_{\mathbf{e}}(u)(y) = \int_0^{+\infty} 1_{\mathcal{S}_{\mathbf{e}}(E_u^+(t))}(y) dt,$$

where, for every $t \geq 0$, we indicated

$$E_u^+(t) = \{x \in E : u(x) > t\},$$

and 1_A stands for the characteristic function of a set A . We note that:

- $\mathcal{S}_{\mathbf{e}}(u) = 0$ almost everywhere in $\mathbb{R}^N \setminus \mathcal{S}_{\mathbf{e}}(E)$;
- by exploiting the definition above and the layer cake representation, it is easy to show that for almost every $t \in \mathbb{R}$, it holds

$$(3.6) \quad \mathcal{S}_{\mathbf{e}}(E)_{\mathcal{S}_{\mathbf{e}}(u)}^+(t) = \{y \in \mathcal{S}_{\mathbf{e}}(E) : \mathcal{S}_{\mathbf{e}}(u)(y) > t\} = \mathcal{S}_{\mathbf{e}}(E_u^+(t)).$$

Hence, $\mathcal{S}_{\mathbf{e}}(u)$ is a non-negative measurable function on \mathbb{R}^N which is equimeasurable with u . In particular, this implies that $\mathcal{S}_{\mathbf{e}}(u) \in L^p(\mathbb{R}^N)$ with

$$(3.7) \quad \|\mathcal{S}_{\mathbf{e}}(u)\|_{L^p(\mathbb{R}^N)} = \|u\|_{L^p(\mathbb{R}^N)};$$

- if v is a Lipschitz function compactly supported in an open set Ω , then $\mathcal{S}_{\mathbf{e}}(v)$ is a Lipschitz function with compact support in $\mathcal{S}_{\mathbf{e}}(\Omega)$, as well.

Finally, we recall that, if $u \in W^{1,p}(\mathbb{R}^N)$ with $1 < p < \infty$, then the celebrated Pölya-Szëgo inequality holds true

$$(3.8) \quad \int_{\mathbb{R}^N} |\nabla \mathcal{S}_{\mathbf{e}}(u)|^p dx \leq \int_{\mathbb{R}^N} |\nabla u|^p dx.$$

For a proof, we refer, for example, to [17, Proposition 3.1].

The following statement is well-known, but we have not been able to detect a proof in the literature. Due to its pivotal importance, we eventually decided to include it.

Lemma 3.6. *Let $1 < p < \infty$ and let $\Omega \subseteq \mathbb{R}^N$ be an open set. For every $\mathbf{e} \in \mathbb{S}^{N-1}$ and every $u \in W_0^{1,p}(\Omega)$, we have $\mathcal{S}_{\mathbf{e}}(u) \in W_0^{1,p}(\mathcal{S}_{\mathbf{e}}(\Omega))$.*

Proof. By definition, there exists a sequence $\{u_n\}_{n \in \mathbb{N}} \subseteq C_0^\infty(\Omega)$

$$\lim_{n \rightarrow \infty} \|u_n - u\|_{W^{1,p}(\Omega)} = 0.$$

We extend all the functions to be 0 outside Ω . The *contractivity property* of Steiner symmetrization (see [30, Propositions 1-2]) gives

$$\|\mathcal{S}_{\mathbf{e}}(u_n) - \mathcal{S}_{\mathbf{e}}(u)\|_{L^p(\mathbb{R}^N)} \leq \|u_n - u\|_{L^p(\mathbb{R}^N)}, \quad \text{for every } n \in \mathbb{N}.$$

Hence $\mathcal{S}_{\mathbf{e}}(u_n)$ converges to $\mathcal{S}_{\mathbf{e}}(u)$ strongly in $L^p(\mathbb{R}^N)$. Moreover, (3.8) gives that $\mathcal{S}_{\mathbf{e}}(u) \in W^{1,p}(\mathbb{R}^N)$. Still by (3.8) and the boundedness of $\{\nabla u_n\}_{n \in \mathbb{N}}$, we also get that $\{\nabla \mathcal{S}_{\mathbf{e}}(u_n)\}_{n \in \mathbb{N}}$ is bounded in $L^p(\mathbb{R}^N)$. Thus, we easily obtain that $\{\nabla \mathcal{S}_{\mathbf{e}}(u_n)\}_{n \in \mathbb{N}}$ weakly converges in $L^p(\mathbb{R}^N)$ to $\nabla \mathcal{S}_{\mathbf{e}}(u)$. Indeed,

by the definition of weak gradient and the strong convergence of $\{\mathcal{S}_e(u_n)\}_{n \in \mathbb{N}}$ in $L^p(\mathbb{R}^N)$, we have for every vector field $\phi \in C_0^\infty(\mathbb{R}^N; \mathbb{R}^N)$

$$\begin{aligned} \int_{\mathbb{R}^N} \langle \nabla \mathcal{S}_e(u), \phi \rangle dx &= - \int_{\mathbb{R}^N} \mathcal{S}_e(u) \operatorname{div} \phi dx = - \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} \mathcal{S}_e(u_n) \operatorname{div} \phi dx \\ &= \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} \langle \nabla \mathcal{S}_e(u_n), \phi \rangle dx. \end{aligned}$$

Note that each function $\mathcal{S}_e(u_n)$ is a Lipschitz function with compact support in $\mathcal{S}_e(\Omega)$, thus we have in particular that $\{\mathcal{S}_e(u_n)\}_{n \in \mathbb{N}} \subseteq W_0^{1,p}(\mathcal{S}_e(\Omega))$. Hence, we can conclude that the function $\mathcal{S}_e(u)$ is the weak limit of a sequence belonging to $W_0^{1,p}(\mathcal{S}_e(\Omega))$. The latter being weakly closed, we get the conclusion. \square

In the next section we will work with functions $u \in L^p(\Omega)$ satisfying the condition

$$\int_{\Omega} |x| |u|^p dx < +\infty.$$

We will show that the previous integral does not increase under the Steiner symmetrization. At first, we prove the following preliminary lemma. We also refer to [46, Lemma 2.11] for a stronger result in the case $\alpha = 2$.

Lemma 3.7. *Let $A \subseteq \mathbb{R}^N$ be a measurable set. For every $\alpha > 0$ and every $e \in \mathbb{S}^{N-1}$, we have*

$$\int_A |x|^\alpha dx \geq \int_{\mathcal{S}_e(A)} |x|^\alpha dx.$$

Proof. Without loss of generality, we can assume that

$$\int_A |x|^\alpha dx < +\infty.$$

Hence, by using the Markov-Chebyshev inequality, we get that

$$|\{x \in A : |x| > 1\}| \leq \int_A |x|^\alpha dx < +\infty,$$

which implies $|A| < +\infty$. Notice that for every $\tau > 0$ we have that

$$\{x \in A : |x|^\alpha > \tau\} = A \setminus \overline{B_{\tau^{1/\alpha}}}.$$

Moreover, since

$$\mathcal{S}_e(A \cap B_{\tau^{1/\alpha}}) \subseteq \mathcal{S}_e(A) \cap \mathcal{S}_e(B_{\tau^{1/\alpha}}) = \mathcal{S}_e(A) \cap B_{\tau^{1/\alpha}},$$

it holds

$$|A \cap B_{\tau^{1/\alpha}}| = |\mathcal{S}_e(A \cap B_{\tau^{1/\alpha}})| \leq |\mathcal{S}_e(A) \cap B_{\tau^{1/\alpha}}|.$$

To obtain the claimed result we can combine these facts and Cavalieri Formula as follows:

$$\begin{aligned} \int_A |x|^\alpha dx &= \int_0^{+\infty} |\{x \in A : |x|^\alpha > \tau\}| d\tau = \int_0^{+\infty} |A \setminus \overline{B_{\tau^{1/\alpha}}}| d\tau \\ &= \int_0^{+\infty} [|A| - |A \cap B_{\tau^{1/\alpha}}|] d\tau \\ &\geq \int_0^{+\infty} [|\mathcal{S}_e(A)| - |\mathcal{S}_e(A) \cap B_{\tau^{1/\alpha}}|] d\tau \\ &= \int_0^{+\infty} |\{x \in \mathcal{S}_e(A) : |x|^\alpha > \tau\}| d\tau = \int_{\mathcal{S}_e(A)} |x|^\alpha dx. \end{aligned}$$

This concludes the proof. \square

Proposition 3.8. *Let $A \subseteq \mathbb{R}^N$ be a measurable set and $1 \leq p < \infty$. For every $\alpha > 0$, $\mathbf{e} \in \mathbb{S}^{N-1}$ and for every non-negative function $u \in L^p(A)$ it holds*

$$(3.9) \quad \int_A |x|^\alpha u^p dx \geq \int_{\mathcal{S}_{\mathbf{e}}(A)} |x|^\alpha (\mathcal{S}_{\mathbf{e}}(u))^p dx.$$

Proof. Without loss of generality, we can assume that

$$\int_A |x|^\alpha u^p dx < +\infty.$$

By applying Lemma 3.7 together with Cavalieri formula for the measure $d\mu = |x|^\alpha dx$, we obtain

$$\int_A |x|^\alpha u^p dx = p \int_0^{+\infty} t^{p-1} \left(\int_{A_u^+(t)} |x|^\alpha dx \right) dt \geq p \int_0^{+\infty} t^{p-1} \left(\int_{\mathcal{S}_{\mathbf{e}}(A_u^+(t))} |x|^\alpha dx \right) dt.$$

Thanks to (3.6), the previous inequality implies that

$$\int_A |x|^\alpha u^p dx \geq p \int_0^{+\infty} t^{p-1} \left(\int_{\mathcal{S}_{\mathbf{e}}(A)_{\mathcal{S}_{\mathbf{e}}(u)}^+(t)} |x|^\alpha dx \right) dt = \int_{\mathcal{S}_{\mathbf{e}}(A)} |x|^\alpha (\mathcal{S}_{\mathbf{e}}(u))^p dx,$$

where the last identity again follows from Cavalieri formula. This concludes the proof. \square

We conclude this section with the following definition.

Definition 3.9 (Steiner symmetric function). For $\Omega \in \mathfrak{S}^N$, we say that a non-negative function $u \in L^p(\Omega)$ is *Steiner symmetric* if

$$u \equiv \mathcal{S}_{\mathbf{e}_j}(u), \quad \text{in } \Omega, \quad \text{for every } j \in \{1, \dots, N\}.$$

We notice that if $u \in L^p(\Omega)$ is Steiner symmetric, then it is non-increasing along every coordinate direction. In particular, if $u \in L^p(\Omega) \cap L^\infty(\Omega)$, we have

$$\|u\|_{L^\infty(\Omega)} = \|u\|_{L^\infty(B_r)}, \quad \text{for every } r > 0.$$

Finally, we note that if $u \in L^p(\Omega)$ is non-negative, then the function $\mathcal{S}_{\mathbf{e}_N} \circ \mathcal{S}_{\mathbf{e}_{N-1}} \circ \dots \circ \mathcal{S}_{\mathbf{e}_1}(u)$ is of course Steiner symmetric.

4. A MINIMIZING SEQUENCE

We will use the following notation

$$W_0^{1,p}(\Omega; |x|) := \left\{ u \in W_0^{1,p}(\Omega) : \int_\Omega |x| |u|^p dx < +\infty \right\}.$$

We start with a result holding for a class of open sets under minimal assumptions, not necessarily Steiner symmetric. This shows that $\lambda_{p,q}(\Omega)$ can be approximated by a problem containing a “vanishing confinement” term, thus giving a suitable gain of compactness. For completeness, we also include the case $q = p$, even if we will not use it in the sequel.

Lemma 4.1. *Let $1 < p < \infty$ and $q \geq p$ satisfying (1.2). Let $\Omega \subseteq \mathbb{R}^N$ be an open set such that $\lambda_p(\Omega) > 0$. For every $n \in \mathbb{N}$, we define*

$$\lambda_{p,q}(\Omega; V_n) := \inf_{u \in W_0^{1,p}(\Omega; |x|)} \left\{ \mathcal{G}_{p,n}(u) : \|u\|_{L^q(\Omega)} = 1 \right\},$$

where

$$\mathcal{G}_{p,n}(u) = \int_{\Omega} |\nabla u|^p dx + \int_{\Omega} V_n |u|^p dx \quad \text{and} \quad V_n(x) = \frac{|x|}{n+1}.$$

Then, we have

$$(4.1) \quad \lim_{n \rightarrow \infty} \lambda_{p,q}(\Omega; V_n) = \lambda_{p,q}(\Omega).$$

Moreover, the value $\lambda_{p,q}(\Omega; V_n)$ is attained by some function $u_n \in W_0^{1,p}(\Omega; |x|)$ with $\|u_n\|_{L^q(\Omega)} = 1$ and we have

$$(4.2) \quad \lim_{n \rightarrow \infty} \int_{\Omega} |\nabla u_n|^p dx = \lambda_{p,q}(\Omega).$$

Proof. Since we have $V_{n+1} \leq V_n$, we get

$$\mathcal{G}_{p,n}(u) \geq \mathcal{G}_{p,n+1}(u), \quad \text{for every } u \in W_0^{1,p}(\Omega; |x|).$$

Thus, it is not difficult to see that $n \mapsto \lambda_{p,q}(\Omega; V_n)$ is monotone decreasing and

$$\lim_{n \rightarrow \infty} \lambda_{p,q}(\Omega; V_n) \geq \lambda_{p,q}(\Omega).$$

In order to prove that this limit coincides with $\lambda_{p,q}(\Omega)$, for every $\varepsilon > 0$ and every $u \in C_0^\infty(\Omega)$ such that $\|u\|_{L^q(\Omega)} = 1$, we can choose $n_\varepsilon \in \mathbb{N}$ such that

$$\frac{1}{n_\varepsilon + 1} \int_{\Omega} |x| |u|^p dx \leq \varepsilon.$$

Accordingly, we get for every $n \geq n_\varepsilon$

$$\lambda_{p,q}(\Omega; V_n) \leq \lambda_{p,q}(\Omega; V_{n_\varepsilon}) \leq \int_{\Omega} |\nabla u|^p dx + \frac{1}{n_\varepsilon + 1} \int_{\Omega} |x| |u|^p dx \leq \int_{\Omega} |\nabla u|^p dx + \varepsilon.$$

By arbitrariness of both $\varepsilon > 0$ and u , we conclude that

$$\lim_{n \rightarrow \infty} \lambda_{p,q}(\Omega; V_n) \leq \lambda_{p,q}(\Omega),$$

as well.

To show that $\lambda_{p,q}(\Omega; V_n)$ is attained, we can assume that $W_0^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ is not compact, otherwise the proof would be straightforward. We take $\{u_{m,n}\}_{m \in \mathbb{N}}$ to be a minimizing sequence such that

$$\mathcal{G}_{p,n}(u_{m,n}) \leq \lambda_{p,q}(\Omega; V_n) + \frac{1}{m+1} \quad \|u_{m,n}\|_{L^q(\Omega)} = 1, \quad \text{for every } m \in \mathbb{N}.$$

Since $\lambda_p(\Omega) > 0$, there exists a subsequence (not relabeled) $\{u_{m,n}\}_{m \in \mathbb{N}}$ weakly converging to some function $u_n \in W_0^{1,p}(\Omega)$. Moreover, for every $R > 0$ and for every $m, n \in \mathbb{N}$, we have

$$\begin{aligned} \int_{\Omega \setminus B_R} |u_{m,n}|^p dx &\leq \frac{1}{R} \int_{\Omega \setminus B_R} |x| |u_{m,n}|^p dx \\ &\leq \frac{n+1}{R} \mathcal{G}_{p,n}(u_{m,n}) \leq \frac{n+1}{R} \left(\lambda_{p,q}(\Omega; V_n) + \frac{1}{m+1} \right). \end{aligned}$$

Thus, we get

$$\lim_{R \rightarrow +\infty} \int_{\Omega \setminus B_R} |u_{m,n}|^p dx = 0,$$

uniformly in m . Hence, the Riesz-Fréchet-Kolmogorov theorem ensures that $\{u_{m,n}\}_{m \in \mathbb{N}}$ converges to u_n strongly in $L^p(\Omega)$ as m goes to ∞ . In particular, by observing that the potential V_n is locally bounded, for every $k \in \mathbb{N}$ we have

$$\liminf_{m \rightarrow \infty} \int_{\Omega} V_n |u_{m,n}|^p dx \geq \liminf_{m \rightarrow \infty} \int_{\Omega \cap B_k} V_n |u_{m,n}|^p dx = \int_{\Omega \cap B_k} V_n |u_n|^p dx.$$

By arbitrariness of k , the Monotone Convergence Theorem gives that

$$\liminf_{m \rightarrow \infty} \int_{\Omega} V_n |u_{m,n}|^p dx \geq \int_{\Omega} V_n |u_n|^p dx.$$

By combining this fact and the lower semicontinuity of L^p norms with respect to the weak convergence, we get that

$$(4.3) \quad \lambda_{p,q}(\Omega; V_n) = \lim_{m \rightarrow \infty} \mathcal{G}_{p,n}(u_{m,n}) \geq \int_{\Omega} |\nabla u_n|^p dx + \int_{\Omega} V_n |u_n|^p dx,$$

which implies that $u_n \in W_0^{1,p}(\Omega; |x|)$ and

$$\mathcal{G}_{p,n}(u_n) \leq \lambda_{p,q}(\Omega; V_n).$$

In the case $q = p$, this is enough to conclude that u_n is a minimizer. For $q > p$, we still need to verify that u_n has unit $L^q(\Omega)$ norm. An application of the interpolation inequality (2.2) gives

$$\|u_{m,n} - u_n\|_{L^q(\Omega)} \leq G_{N,p,q} \|\nabla u_{m,n} - \nabla u_n\|_{L^p(\Omega)}^{\theta} \|u_{m,n} - u_n\|_{L^p(\Omega)}^{1-\theta}.$$

Thus, the sequence $\{u_{m,n}\}_{m \in \mathbb{N}}$ converges to u_n also in $L^q(\Omega)$. In particular $\|u_n\|_{L^q(\Omega)} = 1$ and u_n is feasible for the minimization problem which defines $\lambda_{p,q}(\Omega; V_n)$. In light of (4.3), we can finally infer that u_n is a minimizer.

In order to prove (4.2), we can simply observe that

$$\lambda_{p,q}(\Omega) \leq \int_{\Omega} |\nabla u_n|^p dx \leq \int_{\Omega} |\nabla u_n|^p dx + \int_{\Omega} V_n |u_n|^p dx = \lambda_{p,q}(\Omega; V_n).$$

By taking the limit as n goes to ∞ and using (4.1), we conclude. \square

Remark 4.2. The optimality condition for u_n is given by the weak form of the following Euler-Lagrange equation

$$(4.4) \quad -\Delta_p u_n + V_n |u_n|^{p-2} u_n = \lambda_{p,q}(\Omega; V_n) |u_n|^{q-2} u_n, \quad \text{in } \Omega.$$

More precisely, we have that

$$\int_{\Omega} \langle |\nabla u_n|^{p-2} \nabla u_n, \nabla \varphi \rangle dx + \int_{\Omega} V_n |u_n|^{p-2} u_n \varphi dx = \lambda_{p,q}(\Omega; V_n) \int_{\Omega} |u_n|^{q-2} u_n \varphi dx,$$

for every $\varphi \in C_0^\infty(\Omega)$. Observe that, thanks to the fact that V_n is locally bounded in \mathbb{R}^N , by a standard density argument the previous equation still holds for test functions $\varphi \in W_0^{1,p}(\Omega')$, with Ω' open bounded subset of Ω .

Remark 4.3. With the previous notation, let us suppose that $B_r \subseteq \Omega$. If we take $v \in W_0^{1,p}(B_r)$ to be an extremal function for $\lambda_{p,q}(B_r)$, by extending it equal to zero outside B_r , we get for $p < q < \infty$

$$\begin{aligned} \lambda_{p,q}(\Omega; V_n) &\leq \int_{B_r} |\nabla v|^p dx + \int_{B_r} \frac{|x|}{n+1} |v|^p dx \\ &\leq \lambda_{p,q}(B_r) + \frac{1}{n+1} \left(\int_{B_r} |x|^{\frac{q}{q-p}} dx \right)^{\frac{q-p}{q}} \\ &= r^{N-p-\frac{p}{q}N} \lambda_{p,q}(B_1) + \frac{1}{n+1} \left(\frac{N \omega_N}{N + \frac{q}{q-p}} \right)^{\frac{q-p}{q}} r^{N+1-\frac{p}{q}N}. \end{aligned}$$

In the second estimate we used Hölder's inequality and the fact that v has unit $L^q(B_r)$ norm. A similar estimate holds also in the case $q = \infty$, by simply using that $|v| \leq \|v\|_{L^\infty(B_r)} = 1$.

We then specialize the previous problem to the case of a Steiner symmetric set and give some mild regularity properties of the minimizers found in the previous result.

Proposition 4.4. *Let $1 < p < \infty$ and let $p < q < \infty$ satisfy (1.2). Let $\Omega \in \mathfrak{S}^N$ be such that $\Omega \neq \mathbb{R}^N$. With the notation of Lemma 4.1, the sequence $\{u_n\}_{n \in \mathbb{N}}$ can be chosen so that it has the following properties:*

(i) $u_n \geq 0$ and u_n is Steiner symmetric;

(ii) there exists $M_1 = M_1(N, p, q, r_\Omega) > 0$ such that

$$\|u_n\|_{W^{1,p}(\Omega)} \leq M_1, \quad \text{for every } n \in \mathbb{N};$$

(iii) $u_n \in L^\infty(\Omega)$ and there exists $M_2 = M_2(N, p, q, r_\Omega) > 0$ such that

$$\|u_n\|_{L^\infty(\Omega)} \leq M_2, \quad \text{for every } n \in \mathbb{N};$$

(iv) there exists $\mathfrak{m} = \mathfrak{m}(N, p, q, r_\Omega) > 0$ such that

$$\inf_{B_{r_\Omega/(3\sqrt{N})}} u_n \geq \mathfrak{m}, \quad \text{for every } n \in \mathbb{N}.$$

Proof. We first observe that $\lambda_p(\Omega) > 0$, thanks to Proposition 3.4. Thus, the results of Lemma 4.1 actually hold. The property (i) is quite straightforward, it is sufficient to observe that the minimization problem

$$\lambda_{p,q}(\Omega; V_n) := \inf_{u \in W_0^{1,p}(\Omega; |x|)} \left\{ \mathcal{G}_{p,n}(u) : \|u\|_{L^q(\Omega)} = 1 \right\},$$

is invariant by substituting u with $|u|$. Moreover, by repeatedly applying Lemma 3.6, Proposition 3.8 and (3.7), for every $u \geq 0$ admissible in the above minimization problem, we have that its Steiner symmetrization

$$\tilde{u} := \mathcal{S}_{\mathbf{e}_N} \circ \mathcal{S}_{\mathbf{e}_{N-1}} \circ \cdots \circ \mathcal{S}_{\mathbf{e}_1}(u)$$

is still admissible. Finally, by taking into account (3.9) and (3.8), we have

$$\mathcal{G}_{p,n}(\tilde{u}) \leq \mathcal{G}_{p,n}(u).$$

Thus, by minimality, we can suppose that each u_n has the claimed properties.

In order to bound uniformly the $W^{1,p}(\Omega)$ norm, we first observe that

$$u \mapsto \|\nabla u\|_{L^p(\Omega)},$$

is an equivalent norm on $W_0^{1,p}(\Omega)$, thanks to the fact that $\lambda_p(\Omega) > 0$. More precisely, by Proposition 3.4, we have that

$$\|u\|_{W^{1,p}(\Omega)} \leq C(N, p, r_\Omega) \|\nabla u\|_{L^p(\Omega)}, \quad \text{for every } u \in W_0^{1,p}(\Omega).$$

Now, we observe that, by minimality of u_n we have

$$(4.5) \quad \|\nabla u_n\|_{L^p(\Omega)}^p \leq \int_{\Omega} |\nabla u_n|^p dx + \int_{\Omega} V_n |u_n|^p dx = \lambda_{p,q}(\Omega; V_n).$$

By Lemma 3.3, we know that $r_\Omega < +\infty$ and $B_{r_\Omega} \subseteq \Omega$. We can apply the estimate of Remark 4.3 with the ball B_{r_Ω} and get

$$(4.6) \quad \lambda_{p,q}(\Omega; V_n) \leq r_\Omega^{N-p-\frac{p}{q}N} \lambda_{p,q}(B_1) + \frac{1}{n+1} \left(\frac{N \omega_N}{N + \frac{q}{q-p}} \right)^{\frac{q-p}{q}} r_\Omega^{N+1-\frac{p}{q}N}.$$

By combining this estimate with (4.5), we get a uniform upper bound on $\|\nabla u_n\|_{L^p(\Omega)}$, depending only on N, p, q and r_Ω . This concludes the proof of (ii).

We now consider point (iii). The optimality condition for our minimization problem is given by the Euler-Lagrange equation (4.4), in weak form. Since $u_n \geq 0$, this reads as

$$(4.7) \quad -\Delta_p u_n + V_n u_n^{p-1} = \lambda_{p,q}(\Omega; V_n) u_n^{q-1}, \quad \text{in } \Omega,$$

in weak form. In particular, by using that $V_n u_n^{p-1} \geq 0$, we get that each u_n is a non-negative weak subsolution of

$$-\Delta_p u_n \leq \lambda_{p,q}(\Omega; V_n) u_n^{q-1}, \quad \text{in } \Omega.$$

By appealing to Lemma 2.3, we get that $u_n \in L^\infty(\Omega)$, with the estimate

$$\|u_n\|_{L^\infty(\Omega)} \leq C_1 \left(\lambda_{p,q}(\Omega; V_n) \right)^{\frac{N}{p q - (q-p)N}}.$$

Observe that we also used that u_n has unit $L^q(\Omega)$ norm. On account of (4.6), by defining

$$M_2 := C_1 \left(r_\Omega^{N-p-\frac{p}{q}N} \lambda_{p,q}(B_1) + \left(\frac{N \omega_N}{N + \frac{q}{q-p}} \right)^{\frac{q-p}{q}} r_\Omega^{N+1-\frac{p}{q}N} \right)^{\frac{N}{p q - (q-p)N}},$$

we thus get the claimed uniform estimate in (iii), by observing that M_2 only depends on N, p, q and r_Ω .

We finally come to property (iv). We note that $\|u_n\|_{L^\infty(\Omega)}$ is uniformly bounded from below, as well. Indeed, point (ii) combined with a simple interpolation, gives

$$1 = \|u_n\|_{L^q(\Omega)} \leq \|u_n\|_{L^\infty(\Omega)}^{\frac{q-p}{q}} \|u_n\|_{L^p(\Omega)}^{\frac{p}{q}} \leq \|u_n\|_{L^\infty(\Omega)}^{\frac{q-p}{q}} M_1^{\frac{p}{q}},$$

which implies

$$\|u_n\|_{L^\infty(\Omega)} \geq M_1^{-\frac{p}{q-p}}, \quad \text{for every } n \in \mathbb{N},$$

and this lower bound only depends on N, p, q and r_Ω . We also observe that

$$\|u_n\|_{L^\infty(\Omega)} = \|u_n\|_{L^\infty(B_r)}, \quad \text{for every } r > 0,$$

thanks to the Steiner symmetry of u_n . We now wish to appeal to Harnack's inequality, in order to conclude. To this aim, we observe that, from (4.7), we get in particular that u_n is a non-negative weak solution of

$$-\operatorname{div} \mathbf{A}(\nabla u) = \mathbf{B}_n(x, u), \quad \text{in } B_{r_\Omega} \subseteq \Omega,$$

where

$$\mathbf{A}(\xi) = |\xi|^{p-2} \xi, \quad \mathbf{B}_n(x, t) = \left(\lambda_{p,q}(\Omega_n; V_n) u_n(x)^{q-p} - V_n(x) \right) |t|^{p-2} t.$$

We notice that for $x \in B_{r_\Omega}$ we have

$$\begin{aligned} |\mathbf{B}_n(x, t)| &\leq V_n(x) |t|^{p-1} + \lambda_{p,q}(\Omega_n; V_n) u_n(x)^{q-p} |t|^{p-1} \\ &\leq \left[r_\Omega + M_2^{q-p} \left(r_\Omega^{N-p-\frac{p}{q}N} \lambda_{p,q}(B_1) + \left(\frac{N \omega_N}{N + \frac{q}{q-p}} \right)^{\frac{q-p}{q}} r_\Omega^{N+1-\frac{p}{q}N} \right) \right] |t|^{p-1} \\ &=: b |t|^{p-1}, \end{aligned}$$

with $b = b(N, p, q, r_\Omega) > 0$. Observe that we used (4.6) and the uniform upper bound of point (iii). Accordingly, we can apply Harnack's inequality of [45, Theorem 1.1] in the cube $Q_{r_\Omega/\sqrt{N}} \subseteq B_{r_\Omega}$ so to get

$$\sup_{Q_{r_\Omega/(3\sqrt{N})}} u_n \leq H \inf_{Q_{r_\Omega/(3\sqrt{N})}} u_n,$$

where the constant H depends only on N, p, q and $r_\Omega > 0$, in light of the discussion above. By observing that $B_{r_\Omega/(3\sqrt{N})} \subseteq Q_{r_\Omega/(3\sqrt{N})}$, we finally get

$$H \inf_{B_{r_\Omega/(3\sqrt{N})}} u_n \geq \sup_{B_{r_\Omega/(3\sqrt{N})}} u_n = \|u_n\|_{L^\infty(\Omega)} \geq M_1^{-\frac{p}{q-p}},$$

which gives the desired uniform lower bound, by setting $\mathbf{m} = M_1^{-\frac{p}{q-p}}/H > 0$. \square

5. PROOF OF THE MAIN THEOREM: EXISTENCE

In this section, we are going to prove the existence part of the Main Theorem stated at the beginning. We start with the case $q < \infty$.

Theorem 5.1 (Case $q < \infty$). *Let $1 < p < \infty$ and assume that $q > p$ satisfies (1.2), with $q < \infty$. Let $\Omega \in \mathfrak{S}^N$ be such that $\Omega \neq \mathbb{R}^N$. Then, the infimum defining $\lambda_{p,q}(\Omega)$ is attained by some Steiner symmetric function $u \in W_0^{1,p}(\Omega) \setminus \{0\}$.*

Proof. First of all, by applying Proposition 3.4, we can assure that $\lambda_{p,q}(\Omega) > 0$. We wish to apply the Direct Method in the Calculus of Variations: unfortunately, under our standing assumptions, in general the embedding

$$W_0^{1,p}(\Omega) \hookrightarrow L^q(\Omega),$$

is not compact. However, we will show that it is possible to choose a particular minimizing sequence, which is precompact in $L^q(\Omega)$.

More precisely, we want to use the sequence $\{u_n\}_{n \in \mathbb{N}} \subseteq W_0^{1,p}(\Omega; |x|)$ obtained in Proposition 4.4. Observe that this is indeed a minimizing sequence for our problem, thanks to (4.2). In particular, this is a bounded sequence in $W_0^{1,p}(\Omega)$. Thus, it converges weakly to a function $u \in W_0^{1,p}(\Omega)$, up to a subsequence. It is not difficult to see that $u \neq 0$: indeed, by testing the weak convergence in $L^p(\Omega)$ against the characteristic function of the ball $B_{r_\Omega/(3\sqrt{N})}$, we get

$$\int_{B_{r_\Omega/(3\sqrt{N})}} u \, dx = \lim_{n \rightarrow \infty} \int_{B_{r_\Omega/(3\sqrt{N})}} u_n \, dx \geq \mathbf{m} \left| B_{r_\Omega/(3\sqrt{N})} \right| > 0,$$

in light of point (iv) of Proposition 4.4. This shows that $u \neq 0$, as claimed. In order to show that u is an extremal for $\lambda_{p,q}(\Omega)$ and conclude, it is sufficient to show that

$$(5.1) \quad \lim_{n \rightarrow \infty} \|u_n - u\|_{L^q(\Omega)} = 0,$$

possibly up to a subsequence.

In what follows, when needed, we can always think of our functions $\{u_n\}_{n \in \mathbb{N}}$ and u as defined on the whole \mathbb{R}^N , by extending them to be 0 on the complement of Ω . Thus, we also have $\{u_n\}_{n \in \mathbb{N}} \subseteq W^{1,p}(\mathbb{R}^N)$ and in particular $\{u_n\}_{n \in \mathbb{N}} \subseteq W^{1,p}(B_R)$, for every $R > 0$. Recall that for every $R > 0$, the embedding $W^{1,p}(B_R) \hookrightarrow L^p(\Omega)$ is compact. By using this fact and the weak convergence in $W_0^{1,p}(\Omega)$ exposed above, a standard argument (see for example [11, Lemma 3.8.7 & Remark 3.9.5]) also gives that

$$(5.2) \quad \lim_{n \rightarrow \infty} \|u_n - u\|_{L^p(B_R)} = \lim_{n \rightarrow \infty} \|u_n - u\|_{L^p(\Omega \cap B_R)} = 0, \quad \text{for every } R > 0.$$

We are going to show that the same holds for the sequence of the gradients $\{\nabla u_n\}_{n \in \mathbb{N}}$. We use again the Euler-Lagrange equation (4.4). Let $R > 0$ and $\zeta \in C_0^\infty(\mathbb{R}^N)$ a cut-off function such that

$$0 \leq \zeta \leq 1, \quad \zeta = 1 \text{ on } B_R, \quad \zeta = 0 \text{ on } \mathbb{R}^N \setminus B_{2R}, \quad \|\nabla \zeta\|_\infty \leq \frac{C}{R},$$

for some dimensional constant $C > 0$. We test the weak formulation of (4.4) with $\zeta(u_n - u) \in W_0^{1,p}(\Omega \cap B_{2R})$, which is feasible by Remark 4.2. We obtain

$$\begin{aligned} \int_{\Omega} \langle |\nabla u_n|^{p-2} \nabla u_n, \nabla u_n - \nabla u \rangle \zeta \, dx &= \lambda_{p,q}(\Omega; V_n) \int_{\Omega} u_n^{q-1} (u_n - u) \zeta \, dx \\ &\quad - \frac{1}{n+1} \int_{\Omega} |x| u_n^{p-1} (u_n - u) \zeta \, dx \\ &\quad - \int_{\Omega} \langle |\nabla u_n|^{p-2} \nabla u_n, \nabla \zeta \rangle (u_n - u) \, dx. \end{aligned}$$

It is not difficult to see that the integrals in the right-hand side converge to 0, as n goes to ∞ . Indeed, by Proposition 4.4 and the properties of ζ , we have

$$\lambda_{p,q}(\Omega; V_n) \left| \int_{\Omega} u_n^{q-1} (u_n - u) \zeta \, dx \right| \leq \lambda_{p,q}(\Omega; V_n) M_2^{q-1} |B_{2R}|^{\frac{p-1}{p}} \left(\int_{B_{2R}} |u_n - u|^p \, dx \right)^{\frac{1}{p}},$$

$$\left| \frac{1}{n+1} \int_{\Omega} |x| u_n^{p-1} (u_n - u) \zeta \, dx \right| \leq 2R M_2^{p-1} |B_{2R}|^{\frac{p-1}{p}} \left(\int_{B_{2R}} |u_n - u|^p \, dx \right)^{\frac{1}{p}},$$

and

$$\left| \int_{\Omega} \langle |\nabla u_n|^{p-2} \nabla u_n, \nabla \zeta \rangle (u_n - u) \, dx \right| \leq \frac{C}{R} M_1^{p-1} \left(\int_{B_{2R}} |u_n - u|^p \, dx \right)^{\frac{1}{p}}.$$

If we use (5.2), we get the claim. In particular, by taking the limit as n goes to ∞ , we deduce that

$$\lim_{n \rightarrow \infty} \int_{\Omega} \langle |\nabla u_n|^{p-2} \nabla u_n, \nabla u_n - \nabla u \rangle \zeta \, dx = 0.$$

On the other hand, the previously inferred weak convergence of ∇u_n to ∇u gives also

$$\lim_{n \rightarrow \infty} \int_{\Omega} \langle |\nabla u|^{p-2} \nabla u, \nabla u_n - \nabla u \rangle \zeta \, dx = 0.$$

By subtracting these two identities we deduce that

$$\lim_{n \rightarrow \infty} \int_{\Omega} \langle |\nabla u_n|^{p-2} \nabla u_n - |\nabla u|^{p-2} \nabla u, \nabla u_n - \nabla u \rangle \zeta \, dx = 0.$$

We can now proceed in a standard way, by exploiting the monotonicity properties of the p -Laplacian (see for example the proof of [15, Lemma B.1]), so to obtain that

$$(5.3) \quad \lim_{n \rightarrow \infty} \|\nabla u_n - \nabla u\|_{L^p(B_R)} = \lim_{n \rightarrow \infty} \|\nabla u_n - \nabla u\|_{L^p(\Omega \cap B_R)} = 0, \quad \text{for every } R > 0.$$

Before proceeding further, we observe that from (5.2) and (5.3), by using a standard diagonal argument, we can infer existence of a subsequence (not relabelled) such that we also have

$$\lim_{n \rightarrow \infty} u_n(x) = u(x), \quad \lim_{n \rightarrow \infty} \nabla u_n(x) = \nabla u(x), \quad \text{for a. e. } x \in \Omega.$$

This in turn permits to apply the so-called *Brezis-Lieb Lemma* (see [16, Theorem 1]). We thus obtain

$$\lim_{n \rightarrow \infty} \left(\int_{\Omega} |\nabla u_n|^p \, dx - \int_{\Omega} |\nabla u_n - \nabla u|^p \, dx \right) = \int_{\Omega} |\nabla u|^p \, dx.$$

Observe that from (4.2), we know that

$$\lim_{n \rightarrow \infty} \int_{\Omega} |\nabla u_n|^p \, dx = \lambda_{p,q}(\Omega).$$

Thus, by joining the last two facts, we can conclude that

$$(5.4) \quad \lim_{n \rightarrow \infty} \int_{\Omega} |\nabla u_n - \nabla u|^p \, dx = \lambda_{p,q}(\Omega) - \int_{\Omega} |\nabla u|^p \, dx.$$

Analogously, from the almost everywhere convergence of u_n to u , we have

$$(5.5) \quad \lim_{n \rightarrow \infty} \left(\int_{\Omega} |u_n|^q \, dx - \int_{\Omega} |u_n - u|^q \, dx \right) = \int_{\Omega} |u|^q \, dx.$$

In particular, there exists $n_0 \in \mathbb{N}$ such that

$$A_n := \int_{\Omega} |u_n|^q \, dx - \int_{\Omega} |u_n - u|^q \, dx \geq \frac{1}{2} \int_{\Omega} |u|^q \, dx > 0, \quad \text{for every } n \geq n_0.$$

Observe that we used that $u \not\equiv 0$, by the first part of the proof. By taking into account that each u_n has unit $L^q(\Omega)$ norm, we obtain, for every $n \geq n_0$,

$$\begin{aligned} \lambda_{p,q}(\Omega) &= \lambda_{p,q}(\Omega) \|u_n\|_{L^q(\Omega)}^p \leq \lambda_{p,q}(\Omega) A_n^{\frac{p}{q}} + \lambda_{p,q}(\Omega) \|u_n - u\|_{L^q(\Omega)}^p \\ &\quad - c \lambda_{p,q}(\Omega) \min \left\{ A_n^{\frac{p}{q}}, \|u_n - u\|_{L^q(\Omega)}^p \right\}, \end{aligned}$$

with $c = c(p/q) > 0$, thanks to the enforced sub-additivity of the power $p/q < 1$, given by Lemma 2.1. By using the definition of $\lambda_{p,q}(\Omega)$, we then get, for every $n \geq n_0$,

$$\lambda_{p,q}(\Omega) \leq \lambda_{p,q}(\Omega) A_n^{\frac{p}{q}} + \int_{\Omega} |\nabla u_n - \nabla u|^p \, dx - c \lambda_{p,q}(\Omega) \min \left\{ A_n^{\frac{p}{q}}, \|u_n - u\|_{L^q(\Omega)}^p \right\}.$$

We take the limit as n goes to ∞ : in light of (5.4) and (5.5), this yields

$$0 \leq \lambda_{p,q}(\Omega) \left(\int_{\Omega} |u|^q \, dx \right)^{\frac{p}{q}} - \int_{\Omega} |\nabla u|^p \, dx - c \lambda_{p,q}(\Omega) \lim_{n \rightarrow \infty} \min \left\{ A_n^{\frac{p}{q}}, \|u_n - u\|_{L^q(\Omega)}^p \right\},$$

that is

$$\int_{\Omega} |\nabla u|^p dx + c \lambda_{p,q}(\Omega) \lim_{n \rightarrow \infty} \min \left\{ A_n^{\frac{p}{q}}, \|u_n - u\|_{L^q(\Omega)}^p \right\} \leq \lambda_{p,q}(\Omega) \left(\int_{\Omega} |u|^q dx \right)^{\frac{p}{q}}.$$

On the other hand, by definition of $\lambda_{p,q}(\Omega)$, we have

$$\lambda_{p,q}(\Omega) \left(\int_{\Omega} |u|^q dx \right)^{\frac{p}{q}} \leq \int_{\Omega} |\nabla u|^p dx.$$

By joining the last two equations, we get in particular that

$$\lim_{n \rightarrow \infty} \min \left\{ A_n^{\frac{p}{q}}, \|u_n - u\|_{L^q(\Omega)}^p \right\} = 0.$$

By recalling that A_n is uniformly bounded from below, the previous fact implies that $\{u_n\}_{n \in \mathbb{N}}$ converges strongly in $L^q(\Omega)$ to u , i.e. (5.1) holds. This proves that u is an extremal for $\lambda_{p,q}(\Omega)$.

Once we get that u is an extremal, we can go back to (5.4) and obtain that

$$\lim_{n \rightarrow \infty} \|\nabla u_n - \nabla u\|_{L^p(\Omega)} = 0.$$

On account of the assumption $\lambda_p(\Omega) > 0$, this in turn implies that u_n converges to u strongly in $L^p(\Omega)$, as well. Hence, the contractivity of the Steiner symmetrization and the Steiner symmetry of each u_n gives

$$\begin{aligned} 0 &= \lim_{n \rightarrow \infty} \|u_n - u\|_{L^p(\Omega)} \geq \limsup_{n \rightarrow \infty} \|\mathcal{S}_{\mathbf{e}_i}(u_n) - \mathcal{S}_{\mathbf{e}_i}(u)\|_{L^p(\Omega)} \\ &= \limsup_{n \rightarrow \infty} \|u_n - \mathcal{S}_{\mathbf{e}_i}(u)\|_{L^p(\Omega)} \geq 0, \quad \text{for every } i \in \{1, \dots, N\}. \end{aligned}$$

By uniqueness of the limit, we obtain that $u = \mathcal{S}_{\mathbf{e}_i}(u)$ for every $i \in \{1, \dots, N\}$, as desired. This eventually concludes the proof. \square

By a limit argument, we get the existence of an optimal function at the endpoint $q = \infty$, as well.

Theorem 5.2 (Case $q = \infty$). *Let $N < p < \infty$ and let $\Omega \in \mathfrak{S}^N$ be such that $\Omega \neq \mathbb{R}^N$. For every $p < q < \infty$, let $u_q \in W_0^{1,p}(\Omega)$ be a Steiner symmetric function such that*

$$\|u_q\|_{L^q(\Omega)} = 1, \quad \lambda_{p,q}(\Omega) = \int_{\Omega} |\nabla u_q|^p dx.$$

Then, there exists a Steiner symmetric function $u_{\infty} \in W_0^{1,p}(\Omega)$ such that

$$(5.6) \quad \lim_{q \rightarrow \infty} \|u_q - u_{\infty}\|_{W^{1,p}(\Omega)} = 0.$$

Moreover, the function u_{∞} is the unique Steiner symmetric extremal of $\lambda_{p,\infty}(\Omega)$. Finally, we have

$$\lambda_{p,q}(\Omega) (u_q)^{q-1} \xrightarrow{q \rightarrow \infty} \lambda_{p,\infty}(\Omega) \delta_0, \quad \text{in } \mathcal{D}'(\Omega).$$

Proof. First of all, we note that the existence of $u_q \in W_0^{1,p}(\Omega)$ as in the statement is ensured by Theorem 5.1. We also observe that $\lambda_p(\Omega) > 0$ thanks to Proposition 3.4. Thus, by Poincaré inequality we have that

$$u \mapsto \|\nabla u\|_{L^p(\Omega)},$$

is an equivalent norm on $W_0^{1,p}(\Omega)$. By Sobolev embeddings in the case $p > N$, we have that

$$W_0^{1,p}(\Omega) \hookrightarrow C^{0,\alpha_p}(\bar{\Omega}), \quad \text{with } \alpha_p = 1 - \frac{N}{p}.$$

More precisely, by Morrey's inequalities, for every $q > p$ we have that

$$(5.7) \quad |u_q(x) - u_q(y)| \leq \mathcal{M}_1 \|\nabla u_q\|_{L^p(\Omega)} |x - y|^{\alpha_p}, \quad \text{for every } x, y \in \mathbb{R}^N,$$

and

$$(5.8) \quad |u_q(x)| \leq \mathcal{M}_2 \|\nabla u_q\|_{L^p(\Omega)}^{\frac{N}{p}} \|u_q\|_{L^p(\Omega)}^{1 - \frac{N}{p}}, \quad \text{for every } x \in \mathbb{R}^N.$$

On account of the monotonicity of $\lambda_{p,q}(\Omega)$ with respect to the set inclusion and the fact that Ω has finite inradius (thanks to Lemma 3.3), we have that

$$\int_{\Omega} |\nabla u_q|^p dx = \lambda_{p,q}(\Omega) \leq \frac{\lambda_{p,q}(B_1)}{r_{\Omega}^{\frac{p+q}{q}N-N}}, \quad \text{for every } p < q < \infty.$$

By using that $q \mapsto \lambda_{p,q}(B_1)$ is continuous on every compact interval (see [2, Theorem 1] and [31]) and that by [14, Corollary 6.2]

$$\lim_{q \rightarrow \infty} \lambda_{p,q}(B_1) = \lambda_{p,\infty}(B_1) < +\infty,$$

we thus obtain that the family $\{u_q\}_{q>p}$ is bounded in $W_0^{1,p}(\Omega)$. By using this fact in conjunction with (5.7) and (5.8), we get that the family $\{u_q\}_{q>p}$ is equicontinuous and bounded in $C_b^0(\bar{\Omega})$, the space of continuous and bounded functions over $\bar{\Omega}$. By applying the Ascoli-Arzelà Theorem on the compact set $\bar{B}_{r_{\Omega}} \subseteq \bar{\Omega}$ and the reflexivity of the space $W_0^{1,p}(\Omega)$, we have that there exists a sequence $\{u_{q_n}\}_{n \in \mathbb{N}} \subseteq \{u_q\}_{q>p}$ and a function $u_{\infty} \in W_0^{1,p}(\Omega) \cap C^0(\bar{B}_{r_{\Omega}})$ such that $\{u_{q_n}\}_{n \in \mathbb{N}}$ weakly converges to u_{∞} in $W_0^{1,p}(\Omega)$ and uniformly on the compact set $\bar{B}_{r_{\Omega}}$.

By the properties of u_q , we have

$$\int_{\Omega} |\nabla u_q|^p dx = \lambda_{p,q}(\Omega) = \lambda_{p,q}(\Omega) \int_{\Omega} |u_q|^q dx \leq \lambda_{p,q}(\Omega) \|u_q\|_{L^{\infty}(\Omega)}^{q-p} \int_{\Omega} |u_q|^p dx.$$

By using the Poincaré inequality on the left-hand side and then simplifying the L^p norm of u_q , we obtain with simple manipulations

$$\left(\frac{\lambda_p(\Omega)}{\lambda_{p,q}(\Omega)} \right)^{\frac{1}{q-p}} \leq \|u_q\|_{L^{\infty}(\Omega)}.$$

By the Steiner symmetry, we know that u_q is maximal at the origin. In particular, we get that

$$u_{q_n}(0) \geq \left(\frac{\lambda_p(\Omega)}{\lambda_{p,q_n}(\Omega)} \right)^{\frac{1}{q_n-p}} \quad \text{for every } n \in \mathbb{N}.$$

By passing to the limit as n goes to ∞ and using [14, Corollary 6.2], we obtain

$$u_{\infty}(0) = \lim_{n \rightarrow \infty} u_{q_n}(0) \geq 1.$$

Hence $u_{\infty} \not\equiv 0$ and

$$(5.9) \quad \|u_{\infty}\|_{L^{\infty}(\Omega)} \geq 1.$$

On the other hand, by using the definition of $\lambda_{p,\infty}(\Omega)$, the weak convergence in $W^{1,p}(\Omega)$ of the sequence $\{u_{q_n}\}_{n \in \mathbb{N}}$ and once again [14, Corollary 6.2], we get

$$(5.10) \quad \lambda_{p,\infty}(\Omega) \|u_{\infty}\|_{L^{\infty}(\Omega)}^p \leq \int_{\Omega} |\nabla u_{\infty}|^p dx \leq \lim_{n \rightarrow \infty} \int_{\Omega} |\nabla u_{q_n}|^p dx = \lim_{n \rightarrow \infty} \lambda_{p,q_n}(\Omega) = \lambda_{p,\infty}(\Omega).$$

By combining (5.9) and (5.10), we deduce

$$\|u_\infty\|_{L^\infty(\Omega)} = 1 \quad \text{and} \quad \int_{\Omega} |\nabla u_\infty|^p dx = \lambda_{p,\infty}(\Omega),$$

showing at first that u_∞ is an extremal for $\lambda_{p,\infty}(\Omega)$.

We can now improve the convergence of the sequence $\{u_{q_n}\}_{n \in \mathbb{N}}$. Indeed, the minimality of u_∞ implies that all the inequalities in (5.10) are actually equalities. In particular, we get

$$\lim_{n \rightarrow \infty} \|\nabla u_{q_n}\|_{L^p(\Omega)} = \|\nabla u_\infty\|_{L^p(\Omega)}.$$

By Clarkson's inequalities, this in turn yields that

$$\lim_{n \rightarrow \infty} \|\nabla u_{q_n} - \nabla u_\infty\|_{L^p(\Omega)} = 0,$$

which gives the claimed compactness in $W_0^{1,p}(\Omega)$.

We also observe that, by the same argument used at the end of the proof of Theorem 5.1, we have that u_∞ is Steiner symmetric. We wish to prove that it is the *unique* Steiner symmetric extremal. By Lemma A.2, we have that it solves

$$\int_{\Omega} \langle |\nabla u_\infty|^{p-2} \nabla u_\infty, \nabla \varphi \rangle dx = \lambda_{p,\infty}(\Omega) \varphi(0), \quad \text{for every } \varphi \in W_0^{1,p}(\Omega),$$

and $x = 0$ is the unique maximum point of u_∞ (the fact that it coincides with the origin follows from the Steiner symmetry). Let us assume that there exists another $v \neq u_\infty$ which is a Steiner symmetric extremal of $\lambda_{p,\infty}(\Omega)$. Then

$$\left\| \frac{u_\infty + v}{2} \right\|_{L^\infty(\Omega)} = \frac{u_\infty(0) + v(0)}{2} = 1,$$

i.e. $(u_\infty + v)/2$ would be an admissible function for $\lambda_{p,\infty}(\Omega)$. By the strict convexity of the p -Dirichlet integral we would get a contradiction. Thus, we obtain that u_∞ is the unique Steiner symmetric extremal of $\lambda_{p,\infty}(\Omega)$: accordingly, every convergent sequence $\{u_{q_n}\}_{n \in \mathbb{N}} \subseteq \{u_q\}_{q > p}$ converges to the same limit u_∞ . This gives the convergence of the whole family, i.e. we have (5.6).

Finally, by using (5.6), we obtain for every $\varphi \in C_0^\infty(\Omega)$

$$\begin{aligned} \lim_{q \rightarrow \infty} \lambda_{p,q}(\Omega) \int_{\Omega} u_q^{q-1} \varphi dx &= \lim_{n \rightarrow \infty} \int_{\Omega} \langle |\nabla u_q|^{p-2} \nabla u_q, \nabla \varphi \rangle dx \\ &= \int_{\Omega} \langle |\nabla u|^{p-2} \nabla u, \nabla \varphi \rangle dx = \lambda_{p,\infty}(\Omega) \varphi(0), \end{aligned}$$

which gives that

$$\lambda_{p,q}(\Omega) (u_q)^{q-1} \xrightarrow{q \rightarrow \infty} \lambda_{p,\infty}(\Omega) \delta_0, \quad \text{in } \mathcal{D}'(\Omega).$$

This concludes the proof. \square

6. EXAMPLES AND COUNTER-EXAMPLES

We start by showing a couple of examples to which our main result applies.

Example 6.1. Let $1 \leq k \leq N - 1$, a noteworthy application of Theorem 5.1 is to open sets of the type

$$S = \mathbb{R}^{N-k} \times \omega,$$

where $\omega \subseteq \mathbb{R}^k$ is an open set such that $\omega \in \mathfrak{S}^k$ and $\omega \neq \mathbb{R}^k$. Clearly $S \in \mathfrak{S}^N$ with $S \neq \mathbb{R}^N$: by virtue of Theorem 5.1 and Theorem 5.2, we deduce that $\lambda_{p,q}(S)$ is attained, when $1 < p < \infty$ and

$q > p$ satisfies (1.2). In particular, for $q \neq \infty$ satisfying (1.2), Theorem 5.1 ensures the existence of a positive solution to the boundary value problem

$$-\Delta_p u = \lambda u^{q-1} \quad \text{in } S, \quad u \in W_0^{1,p}(S),$$

for every $\lambda > 0$. It is sufficient to use a simple scaling argument, as noticed in the introduction. This applies for example to the slab

$$S = \mathbb{R}^{N-1} \times (-1, 1),$$

thus generalizing the result obtained in [1], where existence is shown in the case $N = 2$, $k = 1$ and $p = 2 < q < \infty$. Notice that, in this example, the assumption $p < q$ is crucial, existence failing when $p = q$.

Example 6.2 (Infinite cross). Another example of application is given by the set

$$C = ((-1, 1) \times \mathbb{R}) \cup (\mathbb{R} \times (-1, 1)) \subseteq \mathbb{R}^2.$$

This is a two-dimensional Steiner symmetric open set: accordingly, there exists a minimizer for $\lambda_{p,q}(C)$, for every $1 < p < \infty$ and $q > p$ satisfying (1.2). As above, the superhomogeneous Lane-Emden equation

$$-\Delta_p u = \lambda u^{q-1} \quad \text{in } C,$$

admits a positive solution $u \in W_0^{1,p}(\Omega)$, for every $\lambda > 0$. Here the situation is slightly different with respect to Example 6.1: indeed, observe that for $p = q = 2$ we still have existence of an extremal, see for example [5, Example 3.2].

We now wish to show the optimality of the assumptions (S1) and (S2) in Definition 3.1, in the case of unbounded sets for which the embedding $W_0^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$ fails to be compact. Namely, we give two simple counterexamples, aiming at showing that if we drop one of the assumptions, then the conclusion of Theorem 5.1 may fail to hold.

Example 6.3 (Sets satisfying (S2), but not satisfying (S1)). We consider a set of the form

$$S = \omega \times \mathbb{R},$$

where the open set $\omega \subsetneq \mathbb{R}^{N-1}$ is such that $\omega \in \mathfrak{S}^{N-1}$. We also set

$$S^+ = \omega \times (0, +\infty),$$

which satisfies the second assumption (S2) of Definition 3.1, but it does not satisfy the first one (S1). We claim that $\lambda_{p,q}(S^+)$ is not attained in $W_0^{1,p}(S^+)$, when $p < q$ satisfy (1.2). To this aim, we start by noticing that

$$(6.1) \quad \lambda_{p,q}(S) = \lambda_{p,q}(S^+).$$

Indeed, the fact that $\lambda_{p,q}(S) \leq \lambda_{p,q}(S^+)$ simply follows by the inclusion $S^+ \subseteq S$ and the monotonicity of $\lambda_{p,q}$. On the other hand, for every $\varepsilon > 0$ there exists $\phi_\varepsilon \in C_0^\infty(S)$ such that

$$\lambda_{p,q}(S) + \varepsilon \geq \int_S |\nabla \phi_\varepsilon|^p dx, \quad \|\phi_\varepsilon\|_{L^q(S)} = 1.$$

Since ϕ_ε is compactly supported, its support is contained in $\omega \times (-L, +\infty)$, for $L > 0$ large enough. Therefore, by translation invariance of the sharp Poincaré-Sobolev sharp constant, we have

$$\lambda_{p,q}(S) + \varepsilon \geq \lambda_{p,q}(\omega \times (-L, +\infty)) = \lambda_{p,q}(S^+).$$

By the arbitrariness of $\varepsilon > 0$, we deduce that $\lambda_{p,q}(S) \geq \lambda_{p,q}(S^+)$, as well.

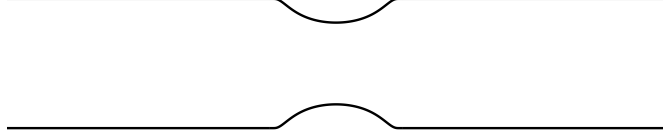


FIGURE 1. The pinched slab Ω_ε^- of Example 6.4, with inward pinching. The constant $\lambda_{p,q}(\Omega_\varepsilon^-)$ is not attained.

By exploiting (6.1), it is not difficult to obtain that $\lambda_{p,q}(S^+)$ does not admit an extremal. It is sufficient to reproduce the same argument of [10, Remark 5.5]: any positive extremal u for $\lambda_{p,q}(S_+)$ would be an extremal for $\lambda_{p,q}(S)$, as well, thanks to (6.1). In particular, $u \not\equiv 0$ would be a p -superharmonic function in S , identically vanishing on the set $\omega \times (-\infty, 0)$. This would violate the minimum principle.

Example 6.4 (Sets satisfying (S1), but not satisfying (S2)). By keeping the same notation as in Example 6.3, we first observe that for every open set $\Omega \subseteq \mathbb{R}^N$ such that

$$(6.2) \quad y_0 + S^+ \subseteq \Omega \subsetneq S \quad \text{and} \quad |S \setminus \Omega| > 0,$$

the constant $\lambda_{p,q}(\Omega)$ can not be attained. Indeed, by monotonicity we have

$$\lambda_{p,q}(S^+) \leq \lambda_{p,q}(\Omega) \leq \lambda_{p,q}(S),$$

and thus $\lambda_{p,q}(\Omega) = \lambda_{p,q}(S)$, thanks to (6.1). By exploiting the same argument based on the minimum principle as in Example 6.3, we get that $\lambda_{p,q}(\Omega)$ can not admit an extremal.

As an explicit example of this situation, we can take the following *pinched slab*

$$\Omega_\varepsilon^- = \left\{ (x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R} : |x_N| < 1 - \varepsilon \varphi \left(\frac{|x'|}{\varepsilon} \right) \right\}, \quad 0 < \varepsilon < 1,$$

where $\varphi \in C_0^\infty((-1, 1) \setminus \{0\})$ is non-negative even function, monotone decreasing on $[0, 1)$ and with $\varphi(0) = 1$. Observe that the set Ω_ε satisfies the first assumption (S1) of Definition 3.1, but it fails to satisfy (S2): indeed, it is not convex in the directions $\mathbf{e}_1, \dots, \mathbf{e}_{N-1}$, because of the vertical inward pinching. It is not difficult to see that Ω_ε^- satisfies (6.2), thus the constant $\lambda_{p,q}(\Omega_\varepsilon^-)$ is not attained.

On the contrary, it is worth observing that the set

$$\Omega_\varepsilon^+ = \left\{ (x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R} : |x_N| < 1 + \varepsilon \varphi \left(\frac{|x'|}{\varepsilon} \right) \right\}, \quad 0 < \varepsilon < 1,$$

satisfies both (S1) and (S2), thus $\lambda_{p,q}(\Omega_\varepsilon^+)$ is attained, for $q > p$ satisfying (1.2).

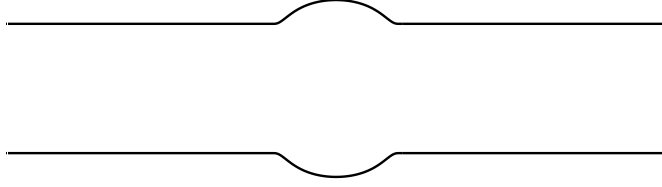


FIGURE 2. The pinched slab Ω_ε^+ of Example 6.4, with outward pinching. This is a Steiner symmetric set and the constant $\lambda_{p,q}(\Omega_\varepsilon^+)$ is attained, thanks to Theorems 5.1 and 5.2.

7. PROOF OF THE MAIN THEOREM: DECAY AT INFINITY

In this section, we prove that the positive extremals of $\lambda_{p,q}(\Omega)$ exponentially decay to zero at infinity. This will be the consequence of a more general result, valid for non-negative subsolutions of the Lane-Emden equation, in generic open sets.

7.1. Subsolutions of the Lane-Emden equation. The following result is a classical $L^\infty - L^q$ estimate, but “localized at infinity”. It implies a weak decay at infinity in uniform norm.

Lemma 7.1. *Let $1 < p < \infty$ and $p \leq q < \infty$ satisfying (1.2). Let $\Omega \subseteq \mathbb{R}^N$ be an open set such that $\lambda_p(\Omega) > 0$. Let $u \in W_0^{1,p}(\Omega)$ be a non-negative weak subsolution of*

$$-\Delta_p u = \lambda u^{q-1}, \quad \text{in } \Omega,$$

for some $\lambda > 0$. Then for every $\varrho > 0$, we have

$$(7.1) \quad \|u\|_{L^\infty(\Omega \setminus B_{\varrho+1})} \leq C_4 \left(1 + \lambda \|u\|_{L^\infty(\Omega)}^{q-p}\right)^{\frac{N}{p-q}} \|u\|_{L^q(\Omega \setminus B_\varrho)},$$

for some $C_4 = C_4(N, p, q) > 0$.

Proof. By using Lemma 2.3, we already know that $u \in L^\infty(\Omega)$. Thus, we get that u verifies in particular

$$(7.2) \quad \int_{\Omega} \langle |\nabla u|^{p-2} \nabla u, \nabla \varphi \rangle dx \leq \lambda \|u\|_{L^\infty(\Omega \setminus B_r)}^{q-p} \int_{\Omega} u^{p-1} \varphi dx,$$

for every $\varphi \in W_0^{1,p}(\Omega \setminus \overline{B_r})$ such that $\varphi \geq 0$ in Ω . We will get the desired $L^\infty - L^q$ estimate by using a standard Moser iteration.

We fix $\varrho > 0$ as in the statement and, for every $\varrho \leq r < R$, we take a radially symmetric Lipschitz cut-off function η such that

$$0 \leq \eta \leq 1, \quad \eta \equiv 0 \text{ on } B_r, \quad \eta \equiv 1 \text{ on } \mathbb{R}^N \setminus B_R, \quad \|\nabla \eta\|_{L^\infty(\mathbb{R}^N)} = \frac{1}{R-r}.$$

For every $\beta \geq 1$ we insert in (7.2) the test function

$$\varphi = \eta^p u^\beta,$$

which is feasible. With simple algebraic manipulations, we get

$$(7.3) \quad \begin{aligned} \beta \left(\frac{p}{p+\beta-1} \right)^p \int_{\Omega} \left| \nabla \left(u^{\frac{p+\beta-1}{p}} \right) \right|^p \eta^p dx &\leq p \int_{\Omega} |\nabla u|^{p-1} |\nabla \eta| \eta^{p-1} u^\beta dx \\ &+ \lambda \|u\|_{L^\infty(\Omega \setminus B_r)}^{q-p} \int_{\Omega} u^{\beta+p-1} \eta^p dx. \end{aligned}$$

By using Young's inequality, we get for every $\delta > 0$

$$(7.4) \quad \begin{aligned} p \int_{\Omega} |\nabla u|^{p-1} |\nabla \eta| \eta^{p-1} u^\beta dx &\leq (p-1) \delta \int_{\Omega} |\nabla u|^p u^{\beta-1} \eta^p + \delta^{1-p} \int_{\Omega} |\nabla \eta|^p u^{\beta+p-1} dx \\ &= (p-1) \left(\frac{p}{\beta+p-1} \right)^p \delta \int_{\Omega} \left| \nabla \left(u^{\frac{\beta+p-1}{p}} \right) \right|^p \eta^p \\ &+ \delta^{1-p} \int_{\Omega} |\nabla \eta|^p u^{\beta+p-1} dx. \end{aligned}$$

We choose $\delta = \beta/(2(p-1))$, thus from (7.3) and (7.4) we get

$$\begin{aligned} \frac{\beta}{2} \left(\frac{p}{p+\beta-1} \right)^p \int_{\Omega} \left| \nabla \left(u^{\frac{p+\beta-1}{p}} \right) \right|^p \eta^p dx &\leq \left(\frac{2(p-1)}{\beta} \right)^{p-1} \int_{\Omega} |\nabla \eta|^p u^{\beta+p-1} dx \\ &+ \lambda \|u\|_{L^\infty(\Omega \setminus B_r)}^{q-p} \int_{\Omega} u^{\beta+p-1} \eta^p dx. \end{aligned}$$

This can be rewritten as

$$(7.5) \quad \begin{aligned} \int_{\Omega} \left| \nabla \left(u^{\frac{p+\beta-1}{p}} \right) \right|^p \eta^p dx &\leq \frac{2}{\beta} \left(\frac{p+\beta-1}{p} \right)^p \left(\frac{2(p-1)}{\beta} \right)^{p-1} \int_{\Omega} |\nabla \eta|^p u^{\beta+p-1} dx \\ &+ \lambda \|u\|_{L^\infty(\Omega \setminus B_r)}^{q-p} \frac{2}{\beta} \left(\frac{p+\beta-1}{p} \right)^p \int_{\Omega} u^{\beta+p-1} \eta^p dx. \end{aligned}$$

We add on both sides the quantity

$$\int_{\Omega} |\nabla \eta|^p u^{\beta+p-1} dx,$$

and then use that

$$\int_{\Omega} \left| \nabla \left(u^{\frac{p+\beta-1}{p}} \right) \right|^p \eta^p dx + \int_{\Omega} |\nabla \eta|^p u^{\beta+p-1} dx \geq \frac{1}{2^{p-1}} \int_{\Omega} \left| \nabla \left(\eta u^{\frac{\beta+p-1}{p}} \right) \right|^p dx.$$

On account of (7.5) we obtain

$$\begin{aligned} \frac{1}{2^{p-1}} \int_{\Omega} \left| \nabla \left(\eta u^{\frac{\beta+p-1}{p}} \right) \right|^p dx &\leq \left[1 + \frac{2}{\beta} \left(\frac{p+\beta-1}{p} \right)^p \left(\frac{2(p-1)}{\beta} \right)^{p-1} \right] \int_{\Omega} |\nabla \eta|^p u^{\beta+p-1} dx \\ &+ \lambda \|u\|_{L^\infty(\Omega \setminus B_r)}^{q-p} \frac{2}{\beta} \left(\frac{p+\beta-1}{p} \right)^p \int_{\Omega} u^{\beta+p-1} \eta^p dx. \end{aligned}$$

Before proceeding further, we observe that

$$1 \leq \frac{1}{\beta} \left(\frac{p+\beta-1}{p} \right)^p \leq \beta^{p-1} \quad \text{and} \quad \frac{2(p-1)}{\beta} \leq 2p.$$

Thus, from the estimate above we also get

$$(7.6) \quad \int_{\Omega} \left| \nabla \left(\eta u^{\frac{\beta+p-1}{p}} \right) \right|^p dx \leq 4^p \beta^{p-1} p^{p-1} \int_{\Omega} |\nabla \eta|^p u^{\beta+p-1} dx \\ + \lambda \|u\|_{L^\infty(\Omega \setminus B_r)}^{q-p} 2^p \beta^{p-1} \int_{\Omega} u^{\beta+p-1} \eta^p dx.$$

By recalling the properties of η , with simple manipulations we get

$$(7.7) \quad \int_{\Omega} \left| \nabla \left(\eta u^{\frac{\beta+p-1}{p}} \right) \right|^p dx \leq 4^p \beta^{p-1} p^{p-1} \left[\frac{1}{(R-r)^p} + \lambda \|u\|_{L^\infty(\Omega \setminus B_r)}^{q-p} \right] \int_{\Omega \setminus B_r} u^{\beta+p-1} dx.$$

We now need to distinguish three cases, depending on whether $p < N$, $p = N$ or $p > N$.

Case $p < N$: we use the Sobolev inequality on the left-hand side of (7.7), while we simply estimate

$$\|u\|_{L^\infty(\Omega \setminus B_r)}^{q-p} \leq \|u\|_{L^\infty(\Omega)}^{q-p},$$

on the right-hand side. We thus end up with

$$(7.8) \quad T_{N,p} \left(\int_{\Omega \setminus B_R} u^{\frac{\beta+p-1}{p} p^*} dx \right)^{\frac{p}{p^*}} \leq 4^p \beta^{p-1} p^{p-1} \left[\frac{1}{(R-r)^p} + \lambda \|u\|_{L^\infty(\Omega)}^{q-p} \right] \int_{\Omega \setminus B_r} u^{\beta+p-1} dx.$$

For ease of notation, we introduce the parameter

$$\vartheta = \frac{\beta + p - 1}{p},$$

and observe that $\beta \leq p\vartheta$. Thus, from (7.8) we get

$$(7.9) \quad \left(\int_{\Omega \setminus B_R} u^{\vartheta p^*} dx \right)^{\frac{1}{p^* \vartheta}} \leq \left(\frac{4^p p^{2(p-1)}}{T_{N,p}} \right)^{\frac{1}{p \vartheta}} \left(\vartheta^{\frac{1}{\vartheta}} \right)^{\frac{p-1}{p}} \\ \times \left[\frac{1}{(R-r)^p} + \lambda \|u\|_{L^\infty(\Omega)}^{q-p} \right]^{\frac{1}{p \vartheta}} \left(\int_{\Omega \setminus B_r} u^{\vartheta p} dx \right)^{\frac{1}{p \vartheta}}.$$

We want to use (7.9) with the following choices

$$(7.10) \quad r = r_i = \varrho + \left(1 - \frac{1}{2^i} \right), \quad R = r_{i+1} = \varrho + \left(1 - \frac{1}{2^{i+1}} \right),$$

and

$$\vartheta_0 = \frac{q}{p}, \quad \vartheta_{i+1} = \frac{p^*}{p} \vartheta_i = \left(\frac{N}{N-p} \right)^{i+1} \frac{q}{p}, \quad \text{for every } i \in \mathbb{N}.$$

Starting from $i = 0$ and iterating (7.9) infinitely many times, we now get the claimed conclusion, by noticing that the sequence $\{\vartheta_i\}_{i \in \mathbb{N}}$ diverges to $+\infty$, together with the fact that

$$\sum_{i=0}^{\infty} \frac{1}{p \vartheta_i} = \frac{1}{q} \sum_{i=0}^{\infty} \left(\frac{N-p}{N} \right)^i = \frac{N}{pq},$$

and

$$\lim_{n \rightarrow \infty} \prod_{i=0}^n \vartheta_i^{\frac{1}{\vartheta_i}} = \exp \left(\sum_{i=0}^{\infty} \frac{\log \vartheta_i}{\vartheta_i} \right) < +\infty.$$

Case $p = N$: in this case, on the left-hand side of (7.7) we can apply the Gagliardo-Nirenberg inequality (2.2) with $p = N$ and $r = 2N$. This gives

$$L_N \left(\int_{\Omega} |\varphi|^{2N} dx \right)^{\frac{1}{2}} \leq \left(\int_{\Omega} |\nabla \varphi|^N dx \right)^{\frac{1}{2}} \left(\int_{\Omega} |\varphi|^N dx \right)^{\frac{1}{2}}, \quad \text{for every } \varphi \in W_0^{1,N}(\Omega).$$

By introducing the parameter

$$\vartheta = \frac{\beta + N - 1}{N},$$

we get

$$\begin{aligned} L_N \left(\int_{\Omega \setminus B_R} u^{2\vartheta N} dx \right)^{\frac{1}{2}} &\leq \left\{ 4^p \beta^{N-1} N^{N-1} \left(\frac{1}{(R-r)^N} + \lambda \|u\|_{L^\infty(\Omega)}^{q-N} \right) \right\}^{\frac{1}{2}} \\ &\quad \times \left(\int_{\Omega \setminus B_r} u^{\vartheta N} dx \right). \end{aligned}$$

The proof proceeds as in the previous case, with the choices (7.10) and

$$\vartheta_0 = \frac{q}{N}, \quad \vartheta_{i+1} = 2\vartheta_i = 2^{i+1} \frac{q}{N}, \quad \text{for every } i \in \mathbb{N}.$$

Case $p > N$: in this case, we apply the second Morrey inequality

$$\|\varphi\|_{L^\infty(\Omega)} \leq C_{N,p} \left(\int_{\Omega} |\nabla \varphi|^p dx \right)^{\frac{N}{p^2}} \left(\int_{\Omega} |\varphi|^p dx \right)^{\frac{1}{p} - \frac{N}{p^2}}, \quad \text{for every } \varphi \in W_0^{1,p}(\Omega),$$

to the function $\varphi = \eta u^{q/p}$. Hence, we combine the resulting estimate with the inequality (7.7) for $\beta = q - p + 1$, $r = \varrho$ and $R = \varrho + 1$. This implies the desired conclusion

$$\|u\|_{L^\infty(\Omega \setminus B_{\varrho+1})} \leq (C_{N,p})^{\frac{p}{q}} (4^p (q-p+1)^{p-1} p^{p-1})^{\frac{N}{p^2}} \left(1 + \lambda \|u\|_{L^\infty(\Omega)}^{q-p} \right)^{\frac{N}{p^2}} \|u\|_{L^q(\Omega \setminus B_\varrho)}.$$

The proof is over. \square

Remark 7.2. It is not difficult to see that, with exactly the same proof, one can also get

$$(7.11) \quad \|u\|_{L^\infty(\Omega \setminus \{x \in \mathbb{R}^N : \langle x, \mathbf{e} \rangle > t+1\})} \leq C_4 \left(1 + \lambda \|u\|_{L^\infty(\Omega)}^{q-p} \right)^{\frac{N}{p^2}} \|u\|_{L^q(\Omega \setminus \{x \in \mathbb{R}^N : \langle x, \mathbf{e} \rangle > t\})},$$

which holds for every $t > 0$ and every $\mathbf{e} \in \mathbb{S}^{N-1}$. The only difference is in the choice of the cut-off function: once fixed $t > 0$, for every $t \leq \tau < T$ it is sufficient to take a Lipschitz cut-off function η such that

$$\begin{aligned} 0 \leq \eta \leq 1, \quad \eta &\equiv 0 \text{ on } \{x \in \mathbb{R}^N : \langle x, \mathbf{e} \rangle \leq \tau\}, \quad \eta \equiv 1 \text{ on } \{x \in \mathbb{R}^N : \langle x, \mathbf{e} \rangle \geq T\}, \\ \|\nabla \eta\|_{L^\infty(\mathbb{R}^N)} &= \frac{1}{T - \tau}. \end{aligned}$$

We leave the details to the reader.

We can now proceed similarly as in the proof of [5, Theorem 5.1] to get the claimed exponential decay, at first in L^q norm and then in the sup norm. Here we crucially need the restriction $q > p$.

Theorem 7.3. *Let $1 < p < \infty$ and $p < q < \infty$ satisfying (1.2). Let $\Omega \subseteq \mathbb{R}^N$ be an open set such that $\lambda_p(\Omega) > 0$. Let $u \in W_0^{1,p}(\Omega)$ be a non-negative weak subsolution of*

$$-\Delta_p u = \lambda u^{q-1}, \quad \text{in } \Omega,$$

for some $\lambda > 0$. Then, there exist $a = a(p, q, \lambda_p(\Omega)) > 0$ and a constant $C_5 > 0$, depending on $N, p, q, \lambda, \lambda_p(\Omega)$ and u , such that

$$(7.12) \quad |u(x)| \leq C_5 \|u\|_{L^q(\Omega)} e^{-a|x|}, \quad \text{for a.e. } x \in \Omega.$$

Proof. We claim that it is sufficient to prove the existence of $a = a(p, q, \lambda_p(\Omega)) > 0$ and a constant $C_6 > 0$, depending on $N, p, q, \lambda, \lambda_p(\Omega)$ and u , such that

$$(7.13) \quad \|u\|_{L^q(\Omega \setminus B_R)} \leq C_6 \|u\|_{L^q(\Omega)} e^{-aR}, \quad \text{for every } R > 0.$$

Indeed, assume that (7.13) holds. For almost every $x \in \Omega \cap B_1$, by Lemma 2.3 we have

$$\begin{aligned} |u(x)| &\leq \|u\|_{L^\infty(\Omega)} \leq C_1 \left(\lambda^{\frac{N}{p-q}} \|u\|_{L^q(\Omega)} \right)^{\frac{pq}{p-q-(q-p)N}} \\ &\leq \left(C_1 e^a \lambda^{\frac{N}{p-q-(q-p)N}} \|u\|_{L^q(\Omega)}^{\frac{(q-p)N}{p-q-(q-p)N}} \right) \|u\|_{L^q(\Omega)} e^{-a|x|}. \end{aligned}$$

Moreover, for every $n \in \mathbb{N}$ and for almost every $x \in \Omega \cap (B_{n+2} \setminus B_{n+1})$, from Lemma 7.1 and (7.13), we would get

$$\begin{aligned} |u(x)| &\leq \|u\|_{L^\infty(\Omega \setminus B_{n+1})} \leq C_4 \left(1 + \lambda \|u\|_{L^\infty(\Omega)}^{q-p} \right)^{\frac{N}{p-q}} \|u\|_{L^q(\Omega \setminus B_n)} \\ &\leq \left(C_6 C_4 \left(1 + \lambda \|u\|_{L^\infty(\Omega)}^{q-p} \right)^{\frac{N}{p-q}} \right) \|u\|_{L^q(\Omega)} e^{-an} \\ &\leq \left(C_6 C_4 e^{2a} \left(1 + \lambda \|u\|_{L^\infty(\Omega)}^{q-p} \right)^{\frac{N}{p-q}} \right) \|u\|_{L^q(\Omega)} e^{-a|x|}. \end{aligned}$$

Then, (7.12) would follow by combining these estimates, with the constant C_5 equal to the maximum of the two constants above.

In order to show (7.13), we first observe that for every $\varepsilon > 0$ there exists a radius $r_\varepsilon > 0$ such that

$$(7.14) \quad \|u\|_{L^\infty(\Omega \setminus B_R)} \leq \varepsilon, \quad \text{for every } R \geq r_\varepsilon.$$

This follows from (7.1) and the fact that $u \in L^q(\Omega)$. For every $R \geq r_\varepsilon$, we take a radially symmetric cut-off Lipschitz function η such that

$$0 \leq \eta \leq 1, \quad \eta \equiv 0 \text{ on } B_R, \quad \eta \equiv 1 \text{ on } \mathbb{R}^N \setminus B_{R+1}, \quad \|\nabla \eta\|_{L^\infty(\mathbb{R}^N)} \leq 1.$$

From (7.6) with the choice $\beta = q - p + 1$, we thus get

$$\begin{aligned} \int_{\Omega} \left| \nabla \left(u^{\frac{q}{p}} \eta \right) \right|^p dx &\leq 4^p (p(q-p+1))^{p-1} \int_{\Omega} |\nabla \eta|^p u^q dx \\ &\quad + 2^p (q-p+1)^{p-1} \lambda \|u\|_{L^\infty(\Omega \setminus B_R)}^{q-p} \int_{\Omega} u^q \eta^p dx. \end{aligned}$$

By using that $u^{q/p} \eta$ is a feasible test function for $\lambda_p(\Omega)$ and (7.14), we get in particular

$$\lambda_p(\Omega) \int_{\Omega} u^q \eta^p dx \leq C_7 \int_{\Omega} |\nabla \eta|^p u^q dx + C_7 \lambda \varepsilon^{q-p} \int_{\Omega} u^q \eta^p dx,$$

where $C_7 = C_7(p, q) = 4^p (p(q-p+1))^{p-1}$. Thanks to the properties of η , this entails that

$$\begin{aligned} \lambda_p(\Omega) \int_{\Omega \setminus B_{R+1}} u^q dx &\leq C_7 \int_{\Omega \cap (B_{R+1} \setminus B_R)} u^q dx + C_7 \lambda \varepsilon^{q-p} \int_{\Omega \setminus B_R} u^q dx \\ &= C_7 (1 + \lambda \varepsilon^{q-p}) \int_{\Omega \cap (B_{R+1} \setminus B_R)} u^q dx \\ &\quad + C_7 \lambda \varepsilon^{q-p} \int_{\Omega \setminus B_{R+1}} u^q dx. \end{aligned}$$

This can be rearranged into

$$(\lambda_p(\Omega) - C_7 \lambda \varepsilon^{q-p}) \int_{\Omega \setminus B_{R+1}} u^q dx \leq C_7 (1 + \lambda \varepsilon^{q-p}) \int_{\Omega \cap (B_{R+1} \setminus B_R)} u^q dx.$$

It is finally time for declaring our choice of $\varepsilon > 0$: we take

$$\varepsilon = \left(\frac{\lambda_p(\Omega)}{2C_7\lambda} \right)^{\frac{1}{q-p}}.$$

Then, by setting $r_0 := r_\varepsilon$, the previous inequality implies that

$$\int_{\Omega \setminus B_{R+1}} u^q dx \leq K \int_{\Omega \cap (B_{R+1} \setminus B_R)} u^q dx, \quad \text{for every } R \geq r_0,$$

where we set

$$K := 1 + \frac{2C_7}{\lambda_p(\Omega)}.$$

If we introduce the notation

$$A(R) = \int_{\Omega \setminus B_R} u^q dx,$$

the previous estimate can be rewritten as

$$A(R+1) \leq K(A(R) - A(R+1)), \quad \text{for every } R \geq r_0.$$

This can be rearranged into the following recursive inequality

$$A(R+1) \leq \frac{K}{K+1} A(R), \quad \text{for every } R \geq r_0.$$

By taking into account that the map $r \mapsto A(r)$ is non-increasing, by suitably iterating this fact we get

$$A(R+1) \leq \left(\frac{K}{K+1} \right)^{R-r_0} A(r_0) \quad \text{for every } R \geq r_0,$$

with $A(r_0) \leq \|u\|_{L^q(\Omega)}^q$. By setting

$$a = \log \left(\frac{K+1}{K} \right)^{\frac{1}{q}},$$

we get

$$\|u\|_{L^q(\Omega \setminus B_R)} \leq \left(\frac{K}{K+1} \right)^{-\frac{r_0+1}{q}} \|u\|_{L^q(\Omega)} e^{-aR}, \quad \text{for every } R \geq r_0 + 1.$$

On the other hand, for $0 < R < r_0 + 1$ it is sufficient to note that

$$\|u\|_{L^q(\Omega \setminus B_R)} \leq \|u\|_{L^q(\Omega)} \leq e^{a(r_0+1)} \|u\|_{L^q(\Omega)} e^{-aR}.$$

Thus, by defining

$$C_6 = \max \left\{ e^{a(r_0+1)}, \left(\frac{K}{K+1} \right)^{-\frac{r_0+1}{q}} \right\},$$

we finally get the claimed estimate (7.13). \square

Remark 7.4. While the exponent a in the (7.12) is independent of the particular subsolution u , we remark that the constant C_5 does depend on u itself. Let us make this precise: an inspection of the previous proof reveals that C_5 depends on u through

- its $L^q(\Omega)$ norm;
- the radius r_0 such that

$$\|u\|_{L^\infty(\Omega \setminus B_{r_0})} \leq \left(\frac{\lambda_p(\Omega)}{2C_7\lambda} \right)^{\frac{1}{q-p}},$$

where $C_7 > 0$ only depends on p, q .

More precisely, the previous proof shows that for non-negative subsolutions such that

$$\|u\|_{L^q(\Omega)} \leq M \quad \text{and} \quad r_0 \leq \bar{R},$$

then we can obtain (7.12) with the constant C_5 depending only on $N, p, q, \lambda, \lambda_p(\Omega), M$ and \bar{R} .

7.2. Extremals.

Corollary 7.5. *Let $1 < p < \infty$ and $p < q$ satisfying (1.2). Let $\Omega \subseteq \mathbb{R}^N$ be an open set such that $\lambda_p(\Omega) > 0$. Assume that there exists a non-negative minimizer $u \in W_0^{1,p}(\Omega)$ of the following problem*

$$\lambda_{p,q}(\Omega) = \min_{u \in W_0^{1,p}(\Omega)} \left\{ \int_{\Omega} |\nabla u|^p dx : \|u\|_{L^q(\Omega)} = 1 \right\}.$$

Then, u satisfies (7.12).

Proof. We need to distinguish two cases, depending on whether $q < \infty$ or $q = \infty$.

Case $q < \infty$. By minimality, we have that u is a non-negative weak solution of

$$-\Delta_p u = \lambda_{p,q}(\Omega) u^{q-1}, \quad \text{in } \Omega.$$

The conclusion then follows directly from Theorem 7.3.

Case $q = \infty$. This case requires to be discussed only when $p > N$ and is simpler. Indeed, by applying Lemma A.2, we have that

$$\int_{\Omega} \langle |\nabla u|^{p-2} \nabla u, \nabla \varphi \rangle dx = \lambda_{p,\infty}(\Omega) |u(x_0)|^{p-2} u(x_0) \varphi(x_0), \quad \text{for every } \varphi \in W_0^{1,p}(\Omega),$$

where $x_0 \in \Omega$ is the unique point such that $|u(x_0)| = \|u\|_{L^\infty(\Omega)} = 1$. In particular if $R > r_0 := |x_0|$, we have that u is a p -harmonic function in $\Omega \setminus \bar{B}_R$, namely

$$\int_{\Omega} \langle |\nabla u|^{p-2} \nabla u, \nabla \varphi \rangle dx = 0, \quad \text{for every } \varphi \in C_0^\infty(\Omega \setminus \bar{B}_R).$$

We insert the test function $\varphi = u \eta^p$, where η is the same Lipschitz cut-off function as in the proof of Theorem 7.3, i.e.

$$0 \leq \eta \leq 1, \quad \eta \equiv 0 \text{ on } B_R, \quad \eta \equiv 1 \text{ on } \mathbb{R}^N \setminus B_{R+1}, \quad \|\nabla \eta\|_{L^\infty(\mathbb{R}^N)} \leq 1.$$

By proceeding as in the proof of (7.7), we get

$$\int_{\Omega} |\nabla(\eta u)|^p dx \leq 4^p p^{p-1} \int_{\Omega \cap (B_{R+1} \setminus B_R)} u^p dx.$$

By using that $u\eta$ is a feasible test function for $\lambda_{p,\infty}(\Omega)$, we get

$$(7.15) \quad \lambda_{p,\infty}(\Omega) \|u\|_{L^\infty(\Omega \setminus B_{R+1})}^p \leq 4^p p^{p-1} \int_{\Omega \cap (B_{R+1} \setminus B_R)} u^p dx.$$

This gives a $L^\infty - L^p$ decay estimate analogous to that of Lemma 7.1. On the other hand, by using that $u\eta$ is a feasible test function for $\lambda_p(\Omega)$, we get

$$\lambda_p(\Omega) \int_{\Omega \setminus B_{R+1}} u^p dx \leq 4^p p^{p-1} \int_{\Omega \cap (B_{R+1} \setminus B_R)} u^p dx.$$

We can use this iterative estimate and proceed as in the proof of Theorem 7.3, so to get

$$(7.16) \quad \|u\|_{L^p(\Omega \setminus B_R)} \leq C'_6 \|u\|_{L^p(\Omega)} e^{-aR}, \quad \text{for every } R > 0,$$

for suitable $a, C'_6 > 0$. Finally, the claimed exponential decay in uniform norm can be obtained by combining (7.16) and (7.15). We leave the details to the reader. \square

Remark 7.6 (Decay estimate for $q = \infty$). From the previous proof, in the case of a non-negative extremal u for $\lambda_{p,\infty}(\Omega)$, we see that the constant C_5 depends on u itself through the radius $r_0 = |x_0|$. Here x_0 is the unique maximum point of u . Thus, if we can prove that

$$|x_0| \leq \Lambda,$$

then we can obtain (7.12) with the constant C_5 depending only on $N, p, q, \lambda_{p,\infty}(\Omega), \lambda_p(\Omega)$ and Λ .

In the case of Steiner symmetric extremals, the quality of the previous decay estimate can be improved: the estimate *does not* depend on the extremal. This is the content of the following

Theorem 7.7 (Decay of extremals: Steiner case). *Let $1 < p < \infty$ and $p < q$ satisfying (1.2). Let $\Omega \in \mathfrak{S}^N$ be such that $\Omega \neq \mathbb{R}^N$. Then every Steiner symmetric extremal of $\lambda_{p,q}(\Omega)$ satisfies (7.12), with a constant C_5 depending only on N, p, q and Ω .*

Proof. Thanks to Theorem 5.1, we get that there exists at least a positive Steiner symmetric extremal. Let us take u a positive Steiner symmetric extremal. We preliminary notice that we can assume that u is continuous on $\bar{\Omega}$: for $p > N$, this simply follows from the Sobolev-type embedding

$$W_0^{1,p}(\Omega) \hookrightarrow C^{0,\alpha_p}(\bar{\Omega}),$$

previously recalled; for $1 < p \leq N$, this follows from classical regularity results for solutions of elliptic PDEs, thanks to the measure density condition of Lemma 3.3 (see for example [45, Theorem 4.2], [33, Theorem 2.7] and [42, Theorem 4.9]).

We already know by Corollary 7.5 that u decays exponentially fast at infinity. Since u is Steiner symmetric, we know that it has a maximum point at the origin: then, according to Remark 7.6, in the case $q = \infty$ there is nothing to prove.

Let us consider the case $q < \infty$. Thanks to the fact that u has unit $L^q(\Omega)$ norm, in light of Remark 7.4 we only need to show that it is possible to choose \bar{R} only depending N, p, q and Ω , but *not* on u itself, such that

$$\|u\|_{L^\infty(\Omega \setminus B_r)} \leq \left(\frac{\lambda_p(\Omega)}{2C_7 \lambda_{p,q}(\Omega)} \right)^{\frac{1}{q-p}} =: \mathcal{E}_\Omega, \quad \text{for every } r \geq \bar{R}.$$

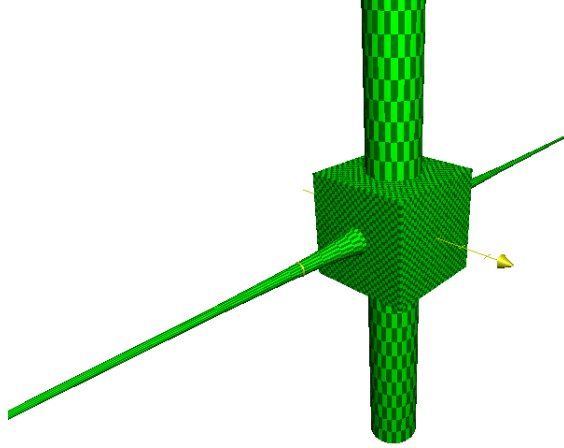


FIGURE 3. An unbounded Steiner symmetric set in \mathbb{R}^3 , exhibiting three different behaviours “at infinity”, according to the coordinate directions. In particular, we have that Ω shrinks at infinity in direction \mathbf{e}_1 ; it is bounded in direction \mathbf{e}_2 and it is tubular at infinity in direction \mathbf{e}_3 .

More precisely, by taking into account the symmetries of u , this is the same as

$$\max_{i \in \{1, \dots, N\}} \left(\sup_{t \geq \bar{R}} u(t \mathbf{e}_i) \right) \leq \mathcal{E}_\Omega.$$

In order to show existence of such a radius \bar{R} , we need to analyze the “behaviour at infinity” of our set Ω . To this aim, we recall the notation

$$\Omega_{i,t} = \{x \in \Omega : \langle x, \mathbf{e}_i \rangle = t\}, \quad \text{for } t \in \mathbb{R},$$

and the definition (3.3) of $r_{\Omega_{i,t}}$, for its (relative) inradius. Then, by recalling Lemma 3.5, for every fixed $i \in \{1, \dots, N\}$ we may have three different possibilities (see Figure 3):

- there exists $t_i > 0$ such that $r_{\Omega_{i,t}} = 0$ for every $|t| \geq t_i$. In this case, our set Ω is bounded in direction \mathbf{e}_i ;
- we have $r_{\Omega_{i,t}} > 0$ for every $t \in \mathbb{R}$ and

$$\lim_{|t| \rightarrow +\infty} r_{\Omega_{i,t}} = 0.$$

In this case, we say that Ω *shrinks at infinity in direction* \mathbf{e}_i ;

- finally, it may happen that

$$\lim_{|t| \rightarrow +\infty} r_{\Omega_{i,t}} > 0.$$

In this case, we say that Ω *is tubular at infinity in direction* \mathbf{e}_i .

Accordingly, we now divide the rest of the proof in three cases.

Case 1: Ω is bounded in direction \mathbf{e}_i . This is the simplest case. In this case, we obviously have $u(t\mathbf{e}_i) = 0$ for every $t \geq t_i$ and thus

$$\sup_{t \geq t_i} u(t\mathbf{e}_i) = 0 \leq \mathcal{E}_\Omega;$$

Case 2: Ω shrinks at infinity in direction \mathbf{e}_i . Observe that, with a slight abuse, we can view the set $\Omega_{i,t}$ as a Steiner symmetric open set in \mathbb{R}^{N-1} , that is $\Omega_{i,t} \in \mathfrak{S}^{N-1}$. Moreover, thanks to Lemma 3.5, we know that for every $t \geq \tau_\Omega > 0$, such a section has finite inradius. In particular, in view of Lemma 3.3, Proposition 3.4 with $p = q$ and the definition of $r_{\Omega_{i,t}}$, for every $t \geq \tau_\Omega$ we have

$$(7.17) \quad \frac{1}{C} \left(\frac{1}{r_{\Omega_{i,t}}} \right)^p \int_{\Omega_{i,t}} |\psi|^p d\widehat{x}_i \leq \int_{\Omega_{i,t}} |\nabla \psi|^p d\widehat{x}_i, \quad \text{for every } \psi \in C_0^\infty(\Omega_{i,t}),$$

for some $C = C(N, p) > 0$. We used the notation $d\widehat{x}_i = dx_1 \dots dx_{i-1} dx_{i+1} \dots dx_N$. Therefore, for every $\varphi \in C_0^\infty(\Omega)$, by using Fubini-Tonelli's theorem and (7.17), we deduce the following Poincaré inequality

$$\begin{aligned} \int_{\{x \in \Omega : \langle x, \mathbf{e}_i \rangle > t\}} |\varphi|^p dx &= \int_t^\infty \left(\int_{\Omega_{i,x_i}} |\varphi|^p d\widehat{x}_i \right) dx_i \\ &\leq C \int_t^\infty (r_{\Omega_{i,x_i}})^p \left(\int_{\Omega_{i,x_i}} \left(\sum_{j \neq i} \left| \frac{\partial \varphi}{\partial x_j} \right|^2 \right)^{p/2} d\widehat{x}_i \right) dx_i \\ &\leq C (r_{\Omega_{i,t}})^p \int_\Omega |\nabla \varphi|^p dx. \end{aligned}$$

Observe that in the last inequality we used (3.4). By density, this estimate extends to $W_0^{1,p}(\Omega)$, as well. In particular, when applied to our extremal u , it gives the following decay for the L^p norm in direction \mathbf{e}_i

$$\int_{\{x \in \Omega : \langle x, \mathbf{e}_i \rangle > t\}} |u|^p dx \leq C \lambda_{p,q}(\Omega) (r_{\Omega_{i,t}})^p.$$

In terms of the L^q norm, we easily deduce

$$\|u\|_{L^q(\{x \in \Omega : \langle x, \mathbf{e}_i \rangle > t\})} \leq \|u\|_{L^\infty(\Omega)}^{\frac{q-p}{q}} \left(\int_{\{x \in \Omega : \langle x, \mathbf{e}_i \rangle > t\}} |u|^p dx \right)^{\frac{1}{q}} \leq \left(C \lambda_{p,q}(\Omega) \right)^{\frac{1}{q}} (r_{\Omega_{i,t}})^{\frac{p}{q}} \|u\|_{L^\infty(\Omega)}^{\frac{q-p}{q}}.$$

In turn, by Lemma 2.3 and using the fact that $\|u\|_{L^q(\Omega)} = 1$, we obtain

$$\|u\|_{L^q(\{x \in \Omega : x_i > t\})} \leq C_1^{\frac{q-p}{q}} \left(C \lambda_{p,q}(\Omega)^{\frac{N(q-p)}{p q - (q-p)N} + 1} \right)^{\frac{1}{q}} (r_{\Omega_{i,t}})^{\frac{p}{q}}.$$

In order to reach the conclusion, we may evoke the $L^\infty - L^q$ estimate (7.11) of Remark 7.2 (again in conjunction with Lemma 2.3), to get

$$u((t+1)\mathbf{e}_i) = \|u\|_{L^\infty(\{x \in \Omega : x_i > t+1\})} \leq C_8 (r_{\Omega_{i,t}})^{\frac{p}{q}},$$

for some $C_8 = C_8(N, p, q, \lambda_{p,q}(\Omega)) > 0$. Our standing assumptions on $r_{\Omega_{i,t}}$ provide the desired conclusion.

Case 3: Ω is tubular at infinity in direction \mathbf{e}_i . This is the most delicate case. We will adapt and generalize a trick taken from [6, Lemma 4.9], the latter being concerned with the case $p = N = 2$ and $\Omega \subseteq \mathbb{R}^2$ an infinite strip. Let us set

$$R_i = \lim_{|t| \rightarrow +\infty} r_{\Omega_i, t}.$$

Thanks to properties (S1) and (S2) in Definition 3.1, we have that Ω contains the infinite tube $\mathcal{T}_i(R_i)$, where we set

$$\mathcal{T}_i(r) = \left\{ x = (x_1, \dots, x_N) \in \mathbb{R}^N : |x - x_i \mathbf{e}_i| < r \right\}, \quad \text{for } r > 0.$$

From standard Elliptic Regularity, we know that u is locally Lipschitz continuous on Ω . More precisely, by [25, Theorem 1.1] we have the following local estimate

$$(7.18) \quad \|\nabla u\|_{L^\infty(B_{r/2}(x_0))} \leq C \left(\int_{B_r(x_0)} |\nabla u|^p dx \right)^{\frac{1}{p}} + C \|\mathbf{P}(\cdot, r)\|_{L^\infty(B_r(x_0))}^{\frac{1}{p-1}},$$

for every $B_r(x_0) \Subset \Omega$, where $C = C(N, p) > 0$ and $\mathbf{P}(x, r)$ is a non-linear potential depending on the right-hand side of the Lane-Emden equation. In our situation, it reads as

$$\mathbf{P}(x, r) = \int_0^r \left(\frac{\|\lambda_{p,q}(\Omega) u^{q-1}\|_{L^2(B_\varrho(x))}^2}{\varrho^{N-2}} \right)^{\frac{1}{2}} \frac{d\varrho}{\varrho}.$$

By using Lemma 2.3, the latter can be easily estimated as follows

$$\begin{aligned} \mathbf{P}(x, r) &\leq \lambda_{p,q}(\Omega) \|u\|_{L^\infty(\Omega)}^{q-1} \int_0^r \left(\frac{\omega_N \varrho^N}{\varrho^{N-2}} \right)^{\frac{1}{2}} \frac{d\varrho}{\varrho} \\ &= \lambda_{p,q}(\Omega) \|u\|_{L^\infty(\Omega)}^{q-1} \sqrt{\omega_N} r \leq C_1^{q-1} \sqrt{\omega_N} (\lambda_{p,q}(\Omega))^{1+\frac{N(q-1)}{p q - (q-p)N}} r. \end{aligned}$$

On the other hand, by the minimality of u we have

$$\int_{B_r(x_0)} |\nabla u|^p dx \leq \frac{1}{|B_r(x_0)|} \int_{\Omega} |\nabla u|^p dx = \frac{\lambda_{p,q}(\Omega)}{|B_r(x_0)|}.$$

Thus, from (7.18) we get in particular

$$\|\nabla u\|_{L^\infty(B_{r/2}(x_0))} \leq \frac{C_8}{r^{\frac{N}{p}}} + C_8 r^{\frac{1}{p-1}},$$

for every $B_r(x_0) \Subset \Omega$, with $C_8 > 0$ depending only on N, p, q and $\lambda_{p,q}(\Omega)$. We use this estimate with $r = R_i/2$ and $x_0 = t \mathbf{e}_i$, for an arbitrary $t \in \mathbb{R}$. With the notation above, we obtain

$$\|\nabla u\|_{L^\infty(\mathcal{T}_i(R_i/4))} = \sup_{t \in \mathbb{R}} \|\nabla u\|_{L^\infty(B_{R_i/4}(t \mathbf{e}_i))} \leq C_8 \left(\left(\frac{2}{R_i} \right)^{\frac{N}{p}} + \left(\frac{R_i}{2} \right)^{\frac{1}{p-1}} \right) =: C_9.$$

This gives a uniform Lipschitz estimate on u in the tube $\mathcal{T}_i(R_i/4)$. In particular, by using this fact we have

$$(7.19) \quad u(x) \geq u(x_i \mathbf{e}_i) - C_9 |x - x_i \mathbf{e}_i|, \quad \text{for } x \in \mathcal{T}_i(R_i/4).$$

We define the positive quantity

$$\varphi(t) = \min \left\{ \frac{R_i}{4}, \frac{u(t \mathbf{e}_i)}{2C_9} \right\},$$

then, from (7.19), we get

$$u(x) \geq \frac{u(x_i \mathbf{e}_i)}{2}, \quad \text{for every } x \text{ such that } |x - x_i \mathbf{e}_i| \leq \varphi(x_i).$$

By using this fact, the normalization on the $L^q(\Omega)$ norm and Fubini-Tonelli's Theorem, we get for every $t > 0$

$$\begin{aligned} 1 &= \int_{\mathbb{R}^N} u^q dx \geq \int_0^t \left(\int_{\{\hat{x}_i \in \mathbb{R}^{N-1}: |\hat{x}_i| \leq \varphi(x_i)\}} u^q d\hat{x}_i \right) dx_i \\ &\geq \frac{\omega_{N-1}}{2^q} \int_0^t \varphi(x_i)^{N-1} u(x_i \mathbf{e}_i)^q dx_i \geq \frac{\omega_{N-1}}{2^q} t \varphi(t)^{N-1} u(t \mathbf{e}_i)^q. \end{aligned}$$

In the last inequality we also used that both $t \mapsto \varphi(t)$ and $t \mapsto u(t \mathbf{e}_i)$ are non-increasing. Thus, if we define the increasingly diverging function

$$\Phi(\tau) = \left(\min \left\{ \frac{R_i}{4}, \frac{\tau}{2C_9} \right\} \right)^{N-1} \tau^q, \quad \text{for } \tau \geq 0,$$

we have obtained the following decay estimate

$$\Phi(u(t \mathbf{e}_i)) \leq \frac{2^q}{\omega_{N-1}} \frac{1}{t} \quad \text{that is} \quad u(t \mathbf{e}_i) \leq \Phi^{-1} \left(\frac{2^q}{\omega_{N-1}} \frac{1}{t} \right).$$

By observing that

$$\lim_{L \rightarrow 0^+} \Phi^{-1}(L) = 0,$$

and that the function Φ only depends on R_i, N, p, q and $\lambda_{p,q}(\Omega)$, we get the desired conclusion in this case, as well. \square

Remark 7.8. With reference to the last part of the previous proof, we observe that in general we can not expect our extremals to be *globally* Lipschitz continuous. This is already false in the case $p = N = 2$: for example, by considering the following Steiner symmetric set (see Figure 4)

$$E = \{(x, y) \in \mathbb{R}^2 : |y| < \psi(x)\}, \quad \text{with } \psi(x) = \sum_{i=0}^{\infty} \frac{1}{i+1} 1_{[i, i+1)}(x) + \sum_{j=0}^{\infty} \frac{1}{|j|+1} 1_{(j-1, j]}(x),$$

we notice that it has countably many boundary points at which we can center a cone with opening $3/2\pi$, entirely contained in E . These are the points of the form

$$\left\{ \left(i, \pm \frac{1}{|i|+1} \right) \right\}_{i \in \mathbb{Z} \setminus \{0\}}.$$

By using a standard barrier argument², we can easily see that the gradient of a positive solution $u \in W_0^{1,2}(E)$ to

$$-\Delta u = \lambda u^{q-1}, \quad \text{in } E,$$

must blow-up when approaching these points.

²A lower barrier can be constructed by taking a positive homogeneous harmonic function, as in [11, Lemma 6.7.1] for example.

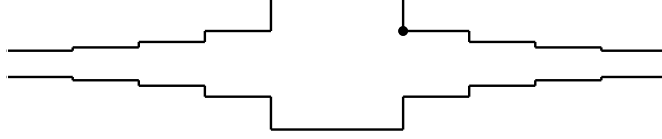


FIGURE 4. The set E in Remark 7.8. Near the boundary point marked by a black dot, the gradient of the solution blows-up.

APPENDIX A. THE ENDPOINT CASE $q = \infty$

We start with the following simple result, about decay at infinity of Sobolev functions.

Lemma A.1. *Let $p > N$ and let $\varphi \in W^{1,p}(\mathbb{R}^N)$. Then we have*

$$\lim_{R \rightarrow +\infty} \|\varphi\|_{L^\infty(\mathbb{R}^N \setminus B_R)} = 0.$$

In particular, there exists $x_0 \in \mathbb{R}^N$ such that

$$|\varphi(x_0)| = \|\varphi\|_{L^\infty(\mathbb{R}^N)}.$$

Proof. We first observe that

$$W^{1,p}(\mathbb{R}^N) \hookrightarrow C^{0,1-\frac{N}{p}}(\mathbb{R}^N),$$

by Morrey's inequalities. Thus, the second part of the statement will easily follow, once we will establish the claimed uniform decay property.

For every $x, y \in \mathbb{R}^N$, by the first Morrey inequality, we have

$$|\varphi(x)| \leq |\varphi(x) - \varphi(y)| + |\varphi(y)| \leq C \|\nabla \varphi\|_{L^p(\mathbb{R}^N)} |x - y|^{1-\frac{N}{p}} + |\varphi(y)|.$$

We integrate the previous estimate with respect to $y \in B_r(x)$, for an arbitrary radius $r > 0$. This gives

$$\omega_N r^N |\varphi(x)| \leq C \|\nabla \varphi\|_{L^p(\mathbb{R}^N)} \int_{B_r(x)} |x - y|^{1-\frac{N}{p}} dy + \int_{B_r(x)} |\varphi(y)| dy.$$

We divide by $\omega_N r^N$ and use a change of variable in the first integral. We get

$$|\varphi(x)| \leq \frac{C r^{1-\frac{N}{p}}}{\omega_N} \|\nabla \varphi\|_{L^p(\mathbb{R}^N)} \int_{B_1(0)} |z|^{1-\frac{N}{p}} dz + \int_{B_r(x)} |\varphi(y)| dy.$$

In particular, by using Jensen's inequality, we get

$$(A.1) \quad |\varphi(x)| \leq \frac{C r^{1-\frac{N}{p}}}{\omega_N} \|\nabla \varphi\|_{L^p(\mathbb{R}^N)} \int_{B_1(0)} |z|^{1-\frac{N}{p}} dz + \left(\int_{B_r(x)} |\varphi(y)|^p dy \right)^{\frac{1}{p}}.$$

We now take $0 < r < R$ and observe that for every $|x| > R$ we have

$$B_r(x) \subseteq \mathbb{R}^N \setminus B_{R-r}.$$

Thus, we obtain

$$\lim_{R \rightarrow +\infty} \left(\sup_{|x| > R} \int_{B_r(x)} |\varphi|^p dy \right) \leq \lim_{R \rightarrow +\infty} \int_{\mathbb{R}^N \setminus B_{R-r}} |\varphi|^p dy = 0,$$

thanks to the fact that $\varphi \in L^p(\mathbb{R}^N)$. By using this result in (A.1), we get

$$\lim_{R \rightarrow +\infty} \left(\sup_{|x| > R} |\varphi(x)| \right) \leq \frac{C r^{1-\frac{N}{p}}}{\omega_N} \|\nabla \varphi\|_{L^p(\mathbb{R}^N)} \int_{B_1(0)} |z|^{1-\frac{N}{p}} dz.$$

By arbitrariness of $r > 0$, we get the desired conclusion. \square

The next result is concerned with the Euler-Lagrange equation for $\lambda_{p,\infty}(\Omega)$.

Lemma A.2. *Let $p > N$ and let $\Omega \subseteq \mathbb{R}^N$ be an open set such that the following problem*

$$\lambda_{p,\infty}(\Omega) = \inf_{\varphi \in W_0^{1,p}(\Omega)} \left\{ \int_{\Omega} |\nabla \varphi|^p dx : \|\varphi\|_{L^\infty(\Omega)} = 1 \right\},$$

admits a solution $u \in W_0^{1,p}(\Omega)$. Then we have

$$\int_{\Omega} \langle |\nabla u|^{p-2} \nabla u, \nabla \varphi \rangle dx = \lambda_{p,\infty}(\Omega) |u(x_0)|^{p-2} u(x_0) \varphi(x_0), \quad \text{for every } \varphi \in W_0^{1,p}(\Omega),$$

where $x_0 \in \Omega$ is such that

$$(A.2) \quad |u(x_0)| = \|u\|_{L^\infty(\Omega)} = 1.$$

Moreover, such a point is unique.

Proof. In the case of an open bounded set, the result is already contained in [26, Theorem 2.5] and [35, Lemma 2.1]. We will give here a slightly different argument, which permits to handle a more general class of open sets.

We start by observing that the existence of a point $x_0 \in \Omega$ satisfying (A.2) follows from Lemma A.1. Also, by definition of $\lambda_{p,\infty}(\Omega)$, we have

$$\frac{1}{p} \int_{\Omega} |\nabla \varphi|^p dx - \frac{\lambda_{p,\infty}(\Omega)}{p} \|\varphi\|_{L^\infty(\Omega)}^p \geq 0, \quad \text{for every } \varphi \in W_0^{1,p}(\Omega).$$

In particular, by observing that $|\varphi(x_0)| \leq \|\varphi\|_{L^\infty(\Omega)}$, we also have

$$\mathcal{F}(\varphi) := \frac{1}{p} \int_{\Omega} |\nabla \varphi|^p dx - \frac{\lambda_{p,\infty}(\Omega)}{p} |\varphi(x_0)|^p \geq 0, \quad \text{for every } \varphi \in W_0^{1,p}(\Omega).$$

Moreover, we have $\mathcal{F}(u) = 0$, thanks to the minimality of u for $\lambda_{p,\infty}(\Omega)$ and to the property of x_0 . In other words, the function u is a minimizer for \mathcal{F} over $W_0^{1,p}(\Omega)$. Thus, by computing the Euler-Lagrange equation for this functional, we end up with the claimed equation.

Finally, let us suppose that there exists a second point $x_1 \in \Omega$ such that

$$|u(x_1)| = \|u\|_{L^\infty(\Omega)} = 1.$$

Thus, the function u must satisfy the same equation as before, with x_1 in place of x_0 . In particular, if we take $\varphi \in W_0^{1,p}(\Omega)$ such that $\varphi(x_0) = 0$ and $\varphi(x_1) \neq 0$, we would get

$$\lambda_{p,\infty}(\Omega) |u(x_1)|^{p-2} u(x_1) \varphi(x_1) = \int_{\Omega} \langle |\nabla u|^{p-2} \nabla u, \nabla \varphi \rangle dx = \lambda_{p,\infty}(\Omega) |u(x_0)|^{p-2} u(x_0) \varphi(x_0).$$

Thanks to the choice of φ , this gives

$$|u(x_1)|^{p-2} u(x_1) \varphi(x_1) = 0 \quad \text{that is} \quad u(x_1) = 0,$$

which would give a contradiction. \square

The next result extends to general open sets, not necessarily bounded, the result of [35, Corollary 2.2]. Our proof uses a simpler argument.

Proposition A.3. *Let $p > N$ and let $\Omega \subseteq \mathbb{R}^N$ be an open set. For $\lambda > 0$ and $x_0 \in \Omega$, let $u \in W_0^{1,p}(\Omega)$ be a non-trivial weak solution of*

$$-\Delta_p u = \lambda |u(x_0)|^{p-2} u(x_0) \delta_{x_0}, \quad \text{in } \Omega.$$

Then u must have constant sign.

Proof. Without loss of generality, we assume that $u^+ := \max\{u, 0\} \neq 0$. It is sufficient to show that $u = u^+$. By testing the weak formulation of the equation with $\varphi = u_+ \in W_0^{1,p}(\Omega)$, we get

$$0 < \int_{\Omega} |\nabla u_+|^p dx = \lambda |u(x_0)|^{p-2} u(x_0) u_+(x_0).$$

This implies that $u(x_0) = u_+(x_0) > 0$. Hence, by testing the weak formulation with $\varphi = u_-$, we obtain this time

$$\int_{\Omega} |\nabla u_-|^p dx = \lambda |u(x_0)|^{p-2} u(x_0) u_-(x_0) = 0.$$

Since $u_- \in W_0^{1,p}(\Omega)$, the previous identity shows that $u_- = 0$. \square

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(L. Brasco) DIPARTIMENTO DI MATEMATICA E INFORMATICA
UNIVERSITÀ DEGLI STUDI DI FERRARA
VIA MACHIAVELLI 35, 44121 FERRARA, ITALY
E-mail address: lorenzo.brasco@unife.it

(L. Briani) SCHOOL OF COMPUTATION, INFORMATION AND TECHNOLOGY,
TECHNICAL UNIVERSITY OF MUNICH
BOLTZMANNSTRASSE 3, 85748 GARCHING BEI MÜNCHEN, GERMANY
E-mail address: luca.briani@tum.de

(F. Prinari) DIPARTIMENTO DI SCIENZE AGRARIE, ALIMENTARI E AGRO-AMBIENTALI
UNIVERSITÀ DI PISA
VIA DEL BORGHETTO 80, 56124 PISA, ITALY
E-mail address: francesca.prinari@unipi.it