Brinkman's law as Γ-limit of compressible low Mach Navier-Stokes equations and application to randomly perforated domains

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May 16, 2025

Abstract

We consider the time-dependent compressible Navier-Stokes equations in the low Mach number regime inside a family of domains $(\Omega_\varepsilon)_{\varepsilon>0}$ in $\mathbb{R}^3$. Assuming that $\lim_{\varepsilon\to 0}\Omega_\varepsilon = \Omega \subset \mathbb{R}^3$ in a suitable sense, we show that in the limit the fluid flow inside Ω is governed by the incompressible Navier-Stokes-Brinkman equations, provided the latter one admits a strong solution. The abstract convergence result is complemented with a stochastic homogenization result for randomly perforated domains in the critical regime.

Contents

1 Introduction

Homogenization of compressible and incompressible Navier-Stokes equations has gained a lot of interest during the last decades. The simplest setting one may consider is a smoothly bounded domain $\Omega \subset \mathbb{R}^3$ with small holes (representing a container with tiny obstacles), precisely for $\varepsilon > 0$ let Ω_ε be defined by

$$\Omega_\varepsilon = \Omega \setminus \bigcup_{i\in I_\varepsilon} B_{\varepsilon^\alpha}(\varepsilon z_i), \quad \alpha \geq 1, \quad z_i \in 2\mathbb{Z}^3, \quad I_\varepsilon = \{i : \overline{B_\varepsilon(\varepsilon z_i)} \subset \Omega\}. \tag{1.1}$$

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For given $T > 0$, we consider in $(0,T)\times\Omega_\varepsilon$ the compressible Navier-Stokes equations in the low Mach number regime $\mathrm{Ma}(\varepsilon)\to 0$:

$$\begin{cases} \partial_t\rho_\varepsilon + \mathrm{div}_x(\rho_\varepsilon \mathbf{u}_\varepsilon) = 0 & \text{in } (0,T)\times\Omega_\varepsilon, \\ \partial_t(\rho_\varepsilon\mathbf{u}_\varepsilon) + \mathrm{div}_x(\rho_\varepsilon\mathbf{u}_\varepsilon\otimes\mathbf{u}_\varepsilon) + \dfrac{1}{\mathrm{Ma}^2(\varepsilon)}\nabla_x p(\rho_\varepsilon) = \mathrm{div}_x\mathbb{S}(\nabla_x\mathbf{u}_\varepsilon) + \rho_\varepsilon\mathbf{f} & \text{in } (0,T)\times\Omega_\varepsilon, \\ \mathbb{S}(\nabla_x\mathbf{u}_\varepsilon) = \mu\left(\nabla_x\mathbf{u}_\varepsilon + \nabla_x^T\mathbf{u}_\varepsilon - \frac{2}{3}\mathrm{div}_x(\mathbf{u}_\varepsilon)\mathbb{1}\right) + \eta\,\mathrm{div}_x(\mathbf{u}_\varepsilon)\mathbb{1} & \text{in } (0,T)\times\Omega_\varepsilon, \end{cases} \tag{1.2}$$

where ρ_ε and $\mathbf{u}_\varepsilon$ are the fluid's density and velocity, respectively, p is the pressure, $\mathbf{f}$ the force, and $\mathbb{S}$ denotes the Newtonian viscous stress tensor with constant viscosity coefficients $\mu > 0$, $\eta \geq 0$. Further, we impose no-slip boundary conditions

$$\mathbf{u}_\varepsilon = 0 \qquad \text{on } (0,T)\times\partial\Omega_\varepsilon. \tag{1.3}$$

For the pressure p and the force $\mathbf{f}$ we assume

$$\begin{gathered} \mathbf{f}\in L^\infty(0,T;L^\infty(\mathbb{R}^3;\mathbb{R}^3)), \quad p\in C([0,\infty))\cap C^2((0,\infty)), \quad p'(\rho)>0 \text{ if } \rho>0, \quad p(0)=0, \\ \exists\gamma\geq 1,\ 0<\underline{p}\leq\overline{p}<\infty: \quad \underline{p}\leq \liminf_{\rho\to\infty}\frac{p(\rho)}{\rho^\gamma} \leq \limsup_{\rho\to\infty}\frac{p(\rho)}{\rho^\gamma}\leq\overline{p}. \end{gathered} \tag{1.4}$$

A prototypical example of such forces is gravity $\mathbf{f} = -(0,0,1)^T$. As for the pressure, one might think of the adiabatic pressure law $p(\rho) = a\rho^\gamma$ for some $a > 0$.

The flow behavior in the limit $\varepsilon\to 0$ is strongly affected by the parameter α, that is, by the size of the holes $B_{\varepsilon^\alpha}(\varepsilon z_i)$ in (1.1). In particular, if $1\leq\alpha<3$, then the holes create huge friction on the flow such that the latter eventually stops; a proper rescaling of the fluid's velocity leads to Darcy's law for the limiting system. If instead $\alpha>3$, then the holes do not influence the flow considerably and the limit flow is governed by the same Navier-Stokes equations we started with. Eventually, if $\alpha = 3$, which is the case we are concerned with in the present work, we are in the *critical regime*. Indeed, in the latter, the holes create some friction that in the limit $\varepsilon\to 0$ gives rise to an additional Brinkman term which resembles the so called Stokes drag law. Moreover, since we consider the case of a low Mach number $\mathrm{Ma}(\varepsilon)\to 0$, the fluid shall become *incompressible* in the limit $\varepsilon\to 0$. Gathering all these considerations, we shall expect for $\alpha = 3$ and $\mathrm{Ma}(\varepsilon)\to 0$ a limiting system of the form

$$\begin{cases} \mathrm{div}_x\mathbf{u} = 0 & \text{in } (0,T)\times\Omega, \\ \bar\rho(\partial_t\mathbf{u} + (\mathbf{u}\cdot\nabla_x\mathbf{u}) + \nabla_x\Pi + \mu\mathrm{M}\mathbf{u} = \mu\Delta_x\mathbf{u} + \bar\rho\mathbf{f} & \text{in } (0,T)\times\Omega, \\ \mathbf{u} = 0 & \text{on } (0,T)\times\partial\Omega, \end{cases} \tag{1.5}$$

where now $\bar\rho > 0$ is constant, $\mathrm{M}\in\mathbb{R}^{3\times 3}_{\mathrm{sym}}$ is a given positive definite symmetric matrix representing the Brinkman term, and Π the associated pressure (Lagrange multiplier) to the divergence-free condition.

In this paper we consider a more general setting, namely, we replace the perforated domains in (1.1) for $\alpha = 3$ with a family of domains $(\Omega_\varepsilon)_{\varepsilon>0}$ which satisfies mild assumptions and at the same time keeps the relevant features of the critical regime $\alpha = 3$. In particular, we will assume $(\Omega_\varepsilon)_{\varepsilon>0}$ converge in the sense of Mosco to a fixed domain Ω (see Definition 2.7). In a nutshell, our informal main result reads as follows:

Theorem 1.1 (See Theorem 2.8)**.** *Let $\varepsilon>0$ and $\Omega_\varepsilon,\Omega\subset D\subset\mathbb{R}^3$ be smoothly bounded domains confined to an overall bounded domain D. If $\lim_{\varepsilon\to 0}\Omega_\varepsilon = \Omega$ (in the sense of Definition 2.7) and $\mathrm{Ma}(\varepsilon)$ with $\lim_{\varepsilon\to 0}\mathrm{Ma}(\varepsilon) = 0$ complies with this convergence of domains, and if (1.5) exhibits a strong solution $(\bar\rho,\boldsymbol{u})$, then a sequence of weak solutions $(\rho_\varepsilon,\boldsymbol{u}_\varepsilon)$ to (1.2) converges to $(\bar\rho,\boldsymbol{u})$ as $\varepsilon\to 0$.*

We then complement our analysis by applying Theorem 1.1 to a specific choice of domains. More precisely, we consider a family of randomly perforated domains and show that, if the size of the holes is of order ε^3, they converge in the sense of Mosco (Theorem 4.1). We stress that the random setting is a generalization of the periodic one given in (1.1).

State of the art. Theorem 1.1 shares similarities with the one proven in [9]. The main crucial difference is that in our case an additional Brinkman term $\mu M\mathbf{u}$ pops up in the limit. The latter can be traced back to the fact that our reference domains $(\Omega_\varepsilon)_{\varepsilon>0}$ are essentially perforated domains with holes of critical size (that is, $\alpha = 3$ in (1.1)), whereas the holes in [9] are much tinier (i.e., $\alpha > 3$). Such Brinkman term appeared for the first time in the homogenization of the Dirichlet problem in [12] for holes of size ε^3. Later, Allaire found in [3, 4, 5] (for all dimensions $d \geq 2$ and periodically distributed holes) that there are three different regimes concerning the size of the holes ε^α, $\alpha \geq 1$, for the homogenization of the incompressible Stokes equations, which are (for $d = 3$) precisely the regimes $\alpha \in [1, 3)$, $\alpha = 3$, and $\alpha > 3$ mentioned above. The first result of homogenization of *compressible* fluids for a *critically* sized perforation was given by the first and last author in [10]. In the latter the authors proved that the Brinkman term also appears for the limit of the stationary Navier-Stokes equation in the low Mach number regime. Our aim in here is to generalize their result in two directions: to the time-dependent setting, and also to more general domains in the sense of Mosco (domain) convergence.

Concerning results on stochastic homogenization, that is, randomly perforated domains, we recall the result of Giunti, Höfer, and Velázquez [21], where the authors proved that the Brinkman term arises from the Poisson equation in a critically randomly perforated domain. Later, Giunti and Höfer [20] derived a similar result for the stationary incompressible Stokes equations. Results for compressible fluids under the presence of randomly distributed, but *tiny* holes, can be found in [11]. Γ-convergence for nonlinear Dirichlet problems on randomly perforated domains was studied in [41]. Similar problems in the context of stochastic homogenization from a variational point of view were treated for example in [6, 32, 33].
Besides the already mentioned references given above, there are many works dealing with homogenization of compressible as well as incompressible models, heat-conducting fluids, and transition in various ways. We refer the interested reader to the literature given in the references below.

Regarding incompressible models, Tartar gave the first result in [42]. He obtained a Darcy law, which was proposed more than 120 years earlier by Darcy [13]. Later, Mikelić [35] gave results for the evolutionary system, and obtained a Darcy law with memory effect. An incompressible system with non-constant density was investigated in [8, 27, 39], where the limiting systems are Darcy's law, the unperturbed Navier-Stokes equations, and Brinkman's law, respectively. Non-Newtonian fluids were considered in [26, 29]. Moreover, critical perforations in heat conducting fluids were investigated in [17]. The case of particles changing size when time continues was recently treated in [43].

The literature for compressible fluids focuses mostly on the case of tiny (subcritical) holes, or the case when the holes' mutual distance is proportional to their size such that rescaling arguments apply. More precisely, Masmoudi considered in his seminal paper [34] the latter case and obtained a compressible Darcy's law; convergence rates for the same type of problem were found recently in [23]. In [22] the authors obtained an incompressible Darcy's law, but starting from a compressible system with a low Mach number. The case of tiny holes were investigated in [15, 16, 36, 38] for different size of holes, values of the adiabatic exponent γ, and dimensions. Heat conducting compressible fluids were considered in [7, 28, 40] for a deterministic distribution of holes, and in [37] for randomly distributed ones.

In addition, it is worth mentioning that also other models of homogenization have been studied such as [24], who looked at homogenization from an operator theoretical point of view, and [14], who studied nematic liquid crystals.

Organization of the paper. The problem under consideration and our main result Theorem 2.8 can be found in Section 2. The proof of our main result is then carried out in Section 3. Finally, Section 4 is devoted to the application of Theorem 2.8 to randomly perforated domains.

Notation. Throughout the paper, we will make use of the following notations:

(a) $\mathbb{1}$ denotes the identity matrix in $\mathbb{R}^{3\times 3}$.

(b) We let χ_E be the characteristic function of the set E.

(c) $a \lesssim b$ denotes $a \leq Cb$ for some constant $C > 0$ which is independent of a, b, and ε.

(d) We distinguish vector quantities from scalar ones by using bold symbols, i.e., a is scalar whereas $\mathbf{a}$ is a vector.

(e) $\mathbf{a} \otimes \mathbf{b}$ and $\mathbf{a} \cdot \mathbf{b}$ denote the tensor product and scalar product between the vectors $\mathbf{a}$ and $\mathbf{b}$, respectively. $A : B = \sum_{i,j=1}^{d} A_{ij} B_{ij}$ denotes the scalar (Frobenius inner) product between the matrices A and B.

(f) We denote with φ, $\boldsymbol{\varphi}$ respectively ϕ, $\boldsymbol{\phi}$ test functions depending on the pair (t, x) and only on x, where $t \in (0, T)$ is the time variable and $x \in \mathbb{R}^3$ the space variable.

(g) $\mathbf{e}_k$ for $k \in \{1, 2, 3\}$ denotes the k-th element of the canonical basis in $\mathbb{R}^3$.

(h) Lebesgue and Sobolev spaces will be denoted as usual by L^p and $W^{k,p}$, respectively.

(i) $C_{\text{weak}}(0, T; L^2(\Omega; \mathbb{R}^3))$ denotes the space of functions $\mathbf{u}(t, \cdot) \in L^2(\Omega; \mathbb{R}^3)$ for all $t \in [0, T]$ such that

$$t \mapsto \int_\Omega \mathbf{u}(t, x) \cdot \boldsymbol{\phi}(x) \, \mathrm{d}x$$

is continuous for all $\boldsymbol{\phi} \in C_c^\infty(\Omega; \mathbb{R}^3)$;

(j) We denote by div_x and ∇_x the divergence and gradient in the variable x respectively. We denote with ∂_t the partial derivative with respect to t.

(k) $\mathbb{R}^{3\times 3}_{\text{sym}}$ is the space of 3×3 symmetric real matrices.

(l) $\mathcal{M}^+(\Omega; \mathbb{R}^{3\times 3}_{\text{sym}})$ denotes the space of finite matrix valued non-negative measures on Ω ranging to $\mathbb{R}^{3\times 3}_{\text{sym}}$.

(m) $\mathcal{H}^2$ denotes the 2-dimensional Hausdorff measure, and $\mathcal{H}^2 \llcorner E$ the 2-dimensional Hausdorff measure restricted to the set E.

2 Setting of the problem and main result

In this section we describe the setting of our problem and state our main result. To start, for $\varepsilon > 0$ we consider a family of domains $(\Omega_\varepsilon)_{\varepsilon>0}$ confined to a bounded subset $D \subset \mathbb{R}^3$. We let also $\Omega \subset D$ be a smoothly bounded domain. On $(0, T) \times \Omega_\varepsilon$ (with $T > 0$) we consider the system of equations in (1.2) coupled with no-slip boundary conditions (1.3), whereas on $(0, T) \times \Omega$ we consider the system of equations (1.5).

2.1 Suitable solution concepts

We will give some concepts of weak, dissipative, and strong solutions to the systems (1.2) and (1.5), respectively, suitable for our purposes.

Definition 2.1 (Finite energy weak solution to (1.2))**.** Let $\gamma \geq 1$. We say that $(\rho_\varepsilon, \mathbf{u}_\varepsilon)$ is a *finite energy weak solution* to system (1.2) with initial data

$$\rho_\varepsilon(0,\cdot) = \rho_{0,\varepsilon} \in L^\gamma(\Omega_\varepsilon), \ \rho_{0,\varepsilon} \geq 0, \qquad\qquad (\rho_\varepsilon \mathbf{u}_\varepsilon)(0,\cdot) = (\rho \mathbf{u})_{0,\varepsilon} \in L^{\frac{2\gamma}{\gamma+1}}(\Omega_\varepsilon;\mathbb{R}^3),$$

if the following hold:

(i) *Integrability.* We have

$$\begin{gathered}\rho_\varepsilon \in L^\infty(0,T;L^\gamma(\Omega_\varepsilon)), \qquad \rho_\varepsilon \geq 0, \qquad \int_{\Omega_\varepsilon} \rho_\varepsilon(\tau,\cdot)\,\mathrm{d}x = \int_{\Omega_\varepsilon} \rho_{0,\varepsilon}\,\mathrm{d}x \ \text{ for almost any } \ \tau \in [0,T], \\ \mathbf{u}_\varepsilon \in L^2(0,T;W_0^{1,2}(\Omega_\varepsilon;\mathbb{R}^3)), \qquad \rho_\varepsilon \mathbf{u}_\varepsilon \in L^\infty(0,T;L^{\frac{2\gamma}{\gamma+1}}(\Omega_\varepsilon;\mathbb{R}^3)).\end{gathered} \tag{2.1}$$

(ii) *Equation of continuity.* It holds

$$\int_0^T \int_{\Omega_\varepsilon} \rho_\varepsilon \partial_t \varphi + \rho_\varepsilon \mathbf{u}_\varepsilon \cdot \nabla_x \varphi \,\mathrm{d}x\,\mathrm{d}t = -\int_{\Omega_\varepsilon} \rho_{0,\varepsilon} \varphi(0,\cdot)\,\mathrm{d}x, \tag{2.2}$$

for any $\varphi \in C_c^1([0,T) \times \overline{\Omega}_\varepsilon)$.

(iii) *Momentum equation.* It holds

$$\begin{aligned}&\int_0^\tau \int_{\Omega_\varepsilon} \rho_\varepsilon \mathbf{u}_\varepsilon \cdot \partial_t \boldsymbol{\varphi} + \rho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon : \nabla_x \boldsymbol{\varphi} + \frac{1}{\mathrm{Ma}^2(\varepsilon)} p(\rho_\varepsilon) \mathrm{div}_x \boldsymbol{\varphi} \,\mathrm{d}x\,\mathrm{d}t \\ &\quad = \int_0^\tau \int_{\Omega_\varepsilon} \mathbb{S}(\nabla_x \mathbf{u}_\varepsilon) : \nabla_x \boldsymbol{\varphi} - \rho_\varepsilon \mathbf{f} \cdot \boldsymbol{\varphi} \,\mathrm{d}x\,\mathrm{d}t + \int_{\Omega_\varepsilon} (\rho_\varepsilon \mathbf{u}_\varepsilon)(\tau,\cdot) \cdot \boldsymbol{\varphi}(\tau,\cdot)\,\mathrm{d}x - \int_{\Omega_\varepsilon} (\rho \mathbf{u})_{0,\varepsilon} \cdot \boldsymbol{\varphi}(0,\cdot)\,\mathrm{d}x,\end{aligned} \tag{2.3}$$

for any $\tau \in [0,T]$ and any $\boldsymbol{\varphi} \in C_c^1([0,T) \times \Omega_\varepsilon;\mathbb{R}^3)$.

(iv) *Energy inequality.* For any $\tau \in [0,T]$, there holds

$$\begin{aligned}&\int_{\Omega_\varepsilon} \left[\frac{1}{2} \mathbb{1}_{\{\rho_\varepsilon > 0\}} \frac{|\rho_\varepsilon \mathbf{u}_\varepsilon|^2}{\rho_\varepsilon} + \frac{1}{\mathrm{Ma}^2(\varepsilon)} P(\rho_\varepsilon) \right](\tau,\cdot)\,\mathrm{d}x + \int_0^\tau \int_{\Omega_\varepsilon} \mathbb{S}(\nabla_x \mathbf{u}_\varepsilon) : \nabla_x \mathbf{u}_\varepsilon \,\mathrm{d}x\,\mathrm{d}t \\ &\quad \leq \int_{\Omega_\varepsilon} \frac{1}{2} \mathbb{1}_{\{\rho_{0,\varepsilon} > 0\}} \frac{|(\rho \mathbf{u})_{0,\varepsilon}|^2}{\rho_{0,\varepsilon}} + \frac{1}{\mathrm{Ma}^2(\varepsilon)} P(\rho_{0,\varepsilon})\,\mathrm{d}x + \int_0^\tau \int_{\Omega_\varepsilon} \rho_\varepsilon \mathbf{f} \cdot \mathbf{u}_\varepsilon \,\mathrm{d}x\,\mathrm{d}t,\end{aligned} \tag{2.4}$$

where

$$P'(\rho)\rho - P(\rho) = p(\rho), \quad P(1) = 0. \tag{2.5}$$

Remark 2.2. We remark that from (2.5), we may write $P(\rho)$ in compact form as

$$P(\rho) = \rho \int_1^\rho \frac{p(s)}{s^2}\,\mathrm{d}s.$$

Definition 2.3 (Dissipative solution to (1.5))**.** Given $\bar{\rho} > 0$, we say that

$$\mathbf{u} \in C_{\mathrm{weak}}([0,T];L^2(\Omega;\mathbb{R}^3)) \cap L^2(0,T;W_0^{1,2}(\Omega;\mathbb{R}^3))$$

is a *dissipative solution* to (1.5) with initial datum

$$\mathbf{u}(0,\cdot) = \mathbf{u}_0 \in L^2(\Omega), \quad \mathrm{div}_x \mathbf{u}_0 = 0,$$

if the following hold:

1. *Incompressibility.*

$$\mathrm{div}_x \mathbf{u} = 0 \quad \text{ a.e. in } (0,T) \times \Omega. \tag{2.6}$$

2. *Momentum equation.* There exists $\mathfrak{R} \in L^\infty(0,T;\mathcal{M}^+(\Omega;\mathbb{R}^{3\times 3}_{\text{sym}}))$ such that

$$\begin{aligned}\left[\int_\Omega \bar{\rho}\mathbf{u}\cdot\boldsymbol{\varphi}\,\mathrm{d}x\right]_{t=0}^{t=\tau} &= \int_0^\tau\int_\Omega \bar{\rho}\mathbf{u}\cdot\partial_t\boldsymbol{\varphi}\,\mathrm{d}x\,\mathrm{d}t - \mu\int_0^\tau\int_\Omega (\mathrm{M}\mathbf{u})\cdot\boldsymbol{\varphi}\,\mathrm{d}x\,\mathrm{d}t \\ &+ \int_0^\tau\int_\Omega (\bar{\rho}\mathbf{u}\otimes\mathbf{u}):\nabla_x\boldsymbol{\varphi} - \mu\nabla_x\mathbf{u}:\nabla_x\boldsymbol{\varphi} + \bar{\rho}\mathbf{f}\cdot\boldsymbol{\varphi} + \mathfrak{R}:\nabla_x\boldsymbol{\varphi}\,\mathrm{d}x\,\mathrm{d}t,\end{aligned} \tag{2.7}$$

for almost any $\tau \in [0,T]$ and any $\boldsymbol{\varphi} \in C_c^1([0,T]\times\Omega;\mathbb{R}^3)$ with $\mathrm{div}_x\boldsymbol{\varphi} = 0$.

3. *Energy inequality.* For $\mathfrak{R}$ as above it holds

$$\begin{aligned}&\int_\Omega \frac{1}{2}\bar{\rho}|\mathbf{u}|^2(\tau,\cdot)\,\mathrm{d}x + \int_\Omega \frac{1}{2}\mathrm{trace}[\mathfrak{R}](\tau,\cdot)\,\mathrm{d}x + \mu\int_0^\tau\int_\Omega |\nabla_x\mathbf{u}|^2\,\mathrm{d}x\,\mathrm{d}t + \mu\int_0^\tau\int_\Omega (\mathrm{M}\mathbf{u})\cdot\mathbf{u}\,\mathrm{d}x\,\mathrm{d}t \\ &\le \int_\Omega \frac{1}{2}\bar{\rho}|\mathbf{u}_0|^2\,\mathrm{d}x + \int_0^\tau\int_\Omega \bar{\rho}\mathbf{f}\cdot\mathbf{u}\,\mathrm{d}x\,\mathrm{d}t,\end{aligned} \tag{2.8}$$

for almost any $\tau \in [0,T]$.

We say that $(\bar{\rho},\mathbf{u})$ is a dissipative solution if $\mathbf{u}$ is a dissipative solution with respect to $\bar{\rho}$.

Remark 2.4. Note that in (2.7), (2.8) and in the following there is a little abuse of notation. Indeed we write $\mathfrak{R}:\nabla_x\boldsymbol{\varphi}\,\mathrm{d}x$ and $\mathrm{trace}[\mathfrak{R}](\tau,\cdot)\,\mathrm{d}x$, however in general $\mathfrak{R}$ is not absolutely continuous with respect to Lebesgue measure.

Definition 2.5 (Strong solution to (1.5))**.** Let $\bar{\rho} > 0$ be a given constant. We say that $\mathbf{u}$ is a *strong solution* to (1.5) with initial datum

$$\mathbf{u}(0,\cdot) = \mathbf{u}_0 \in \bigcap_{1\le p<\infty} W^{1,p}(\Omega;\mathbb{R}^3), \quad \mathrm{div}_x\mathbf{u}_0 = 0,$$

provided

$$\mathbf{u} \in C(0,T;W^{2,p}(\Omega;\mathbb{R}^3)) \cap C^1(0,T;L^q(\Omega;\mathbb{R}^3)) \text{ for all } p,q \in [1,\infty),$$

and $\mathbf{u}$ satisfies (1.5) pointwise. We say that the pair $(\bar{\rho},\mathbf{u})$ is a strong solution if $\mathbf{u}$ is a strong solution with respect to $\bar{\rho}$.

Definition 2.6 (Well-prepared initial data)**.** We say that the initial data $(\rho_{0,\varepsilon},(\rho\mathbf{u})_{0,\varepsilon})$ from Definition 2.1 are *well-prepared* if there is a constant $\bar{\rho} > 0$ and a velocity field

$$\mathbf{u}_0 \in \bigcap_{1\le p<\infty} W^{1,p}(\Omega;\mathbb{R}^3), \qquad \mathrm{div}_x\mathbf{u}_0 = 0 \ \text{ in } \Omega, \qquad \mathbf{u}_0 = 0 \ \text{ on } \partial\Omega, \tag{2.9}$$

such that

$$\lim_{\varepsilon\to 0}\frac{1}{\mathrm{Ma}^2(\varepsilon)}\int_{\Omega_\varepsilon}\left[P(\rho_{0,\varepsilon}) - P'(\bar{\rho})(\rho_{0,\varepsilon}-\bar{\rho}) - P(\bar{\rho})\right]\mathrm{d}x = 0, \tag{2.10}$$

and

$$(\rho\mathbf{u})_{0,\varepsilon} \to \bar{\rho}\mathbf{u}_0 \quad \text{in } L^{\frac{2\gamma}{\gamma+1}}(\Omega;\mathbb{R}^3), \tag{2.11}$$

$$\lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon}\mathbb{1}_{\{\rho_{0,\varepsilon}>0\}}\frac{|(\rho\mathbf{u})_{0,\varepsilon}|^2}{\rho_{0,\varepsilon}}\,\mathrm{d}x = \int_\Omega \bar{\rho}|\mathbf{u}_0|^2\,\mathrm{d}x. \tag{2.12}$$

2.2 Admissible domains and the main result

To generalize the class of domains given in (1.1), we now introduce the class of admissible domains $(\Omega_\varepsilon)_{\varepsilon>0}$.

Definition 2.7 (Assumptions on $(\Omega_\varepsilon)_{\varepsilon>0}$)**.** Let $\varepsilon > 0$ and $((\Omega_\varepsilon)_{\varepsilon>0},\Omega,D)$ be a family of bounded domains $\Omega_\varepsilon,\Omega \subset D \subset \mathbb{R}^3$ with Ω of class $C^{2+\nu}$ for some $\nu > 0$.

(M1) We say that $((\Omega_\varepsilon)_{\varepsilon>0}, \Omega, D)$ satisfies (M1) if for any $\boldsymbol{\phi} \in W_0^{1,2}(D;\mathbb{R}^3)$ and $\boldsymbol{\phi}_\varepsilon \in W_0^{1,2}(\Omega_\varepsilon;\mathbb{R}^3) \subset W_0^{1,2}(D;\mathbb{R}^3)$ such that $\boldsymbol{\phi}_\varepsilon \rightharpoonup \boldsymbol{\phi}$ weakly in $W_0^{1,2}(D;\mathbb{R}^3)$, there holds $\boldsymbol{\phi} \in W_0^{1,2}(\Omega;\mathbb{R}^3)$.

(M2) We say that $((\Omega_\varepsilon)_{\varepsilon>0}, \Omega, D)$ satisfies (M2) if for any $\boldsymbol{\phi} \in C_c^\infty(\Omega;\mathbb{R}^3)$ with $\mathrm{div}_x \boldsymbol{\phi} = 0$ there exists a sequence of solenoidal functions $(\boldsymbol{\phi}_\varepsilon)_{\varepsilon>0}$ such that

$$\boldsymbol{\phi}_\varepsilon \in W_0^{1,2}(\Omega_\varepsilon,\mathbb{R}^3), \quad \mathrm{div}_x \boldsymbol{\phi}_\varepsilon = 0 \quad \text{in } \Omega_\varepsilon; \tag{2.13}$$

$$\boldsymbol{\phi}_\varepsilon \rightharpoonup \boldsymbol{\phi} \quad \text{weakly in } W_0^{1,2}(D;\mathbb{R}^3); \tag{2.14}$$

$$\nabla_x \boldsymbol{\phi}_\varepsilon \to \nabla_x \boldsymbol{\phi} \quad \text{strongly in } L^{\frac{3}{2}}(D;\mathbb{R}^3); \tag{2.15}$$

$$\limsup_{\varepsilon\to 0} \|\nabla_x(\boldsymbol{\phi}_\varepsilon - \boldsymbol{\phi})\|_{L^r(D;\mathbb{R}^3)} \mathrm{Ma}(\varepsilon)^{\frac{2}{\gamma}} = 0 \quad \text{with } r := \frac{3\gamma}{2\gamma-3}. \tag{2.16}$$

In addition, there exists a matrix $\mathrm{M} \in \mathbb{R}^{3\times 3}_{\mathrm{sym}}$ with the following properties:

Let $\mathbf{v} \in L^2(0,T;W_0^{1,2}(\Omega;\mathbb{R}^3))$ and let $\mathbf{v}_\varepsilon \in L^2(0,T;W_0^{1,2}(\Omega_\varepsilon;\mathbb{R}^3))$ satisfy

$$\mathbf{v}_\varepsilon \rightharpoonup \mathbf{v} \quad \text{weakly in } L^2(0,T;W^{1,2}(D;\mathbb{R}^3)), \tag{2.17}$$

$$\mathrm{div}_x \mathbf{v}_\varepsilon = \partial_t g_\varepsilon + \mathrm{div}_x \mathbf{h}_\varepsilon, \tag{2.18}$$

for some $g_\varepsilon \in L^\infty(0,T;L^\gamma(\Omega_\varepsilon))$, $\mathbf{h}_\varepsilon \in L^2(0,T;L^{\frac{6\gamma}{\gamma+6}}(\Omega_\varepsilon;\mathbb{R}^3))$ with the following properties: $g_\varepsilon \to \bar{g}$ in $L^\infty(0,T;L^\gamma(D))$ for some $\bar{g} \in \mathbb{R}$, and $\|\mathbf{h}_\varepsilon\|_{L^2(0,T;L^{\frac{6\gamma}{\gamma+6}}(\Omega_\varepsilon;\mathbb{R}^3))} \lesssim \mathrm{Ma}(\varepsilon)^{\min\{\frac{2}{\gamma},1\}}$. Assume further that there exist sets $M_\varepsilon \subset (0,T)\times\Omega_\varepsilon$ such that

$$\|g_\varepsilon \chi_{M_\varepsilon}\|_{L^\infty(0,T;L^\infty(D))} \lesssim 1, \tag{2.19}$$

$$\|g_\varepsilon(1-\chi_{M_\varepsilon})\|_{L^\infty(0,T;L^\gamma(D))} \lesssim \mathrm{Ma}(\varepsilon)^{\frac{2}{\gamma}}. \tag{2.20}$$

Then there holds for a.e. $\tau \in [0,T]$

$$\lim_{\varepsilon\to 0} \int_0^\tau \int_{\Omega_\varepsilon} \nabla_x \boldsymbol{\phi}_\varepsilon : \nabla_x \mathbf{v}_\varepsilon \,\mathrm{d}x\,\mathrm{d}t = \int_0^\tau \int_\Omega \nabla_x \boldsymbol{\phi} : \nabla_x \mathbf{v} \,\mathrm{d}x\,\mathrm{d}t + \int_0^\tau \int_\Omega (\mathrm{M}\boldsymbol{\phi}) \cdot \mathbf{v} \,\mathrm{d}x\,\mathrm{d}t, \tag{2.21}$$

$$\liminf_{\varepsilon\to 0} \int_0^\tau \int_{\Omega_\varepsilon} |\nabla_x \mathbf{v}_\varepsilon|^2 \,\mathrm{d}x\,\mathrm{d}t \geq \int_0^\tau \int_\Omega |\nabla_x \mathbf{v}|^2 \,\mathrm{d}x\,\mathrm{d}t + \int_0^\tau \int_\Omega (\mathrm{M}\mathbf{v}) \cdot \mathbf{v} \,\mathrm{d}x\,\mathrm{d}t. \tag{2.22}$$

If $((\Omega_\varepsilon)_{\varepsilon>0}, \Omega, D)$ satisfies (M1), (M2) we say that Ω_ε converge to Ω in the sense of Mosco.

In Section 4 we will consider an example of $((\Omega_\varepsilon)_{\varepsilon>0}, \Omega, D)$ where Ω_ε is obtained by removing random holes from Ω and $D = \Omega$, hence we will write $((\Omega_\varepsilon)_{\varepsilon>0}, \Omega)$ in place of $((\Omega_\varepsilon)_{\varepsilon>0}, \Omega, D)$. We are now in the position to state our main result.

Theorem 2.8. *Let $((\Omega_\varepsilon)_{\varepsilon>0}, \Omega, D)$ be a family of bounded domains as in Definition 2.7. Let $(\rho_\varepsilon, \boldsymbol{u}_\varepsilon)_{\varepsilon>0}$ be a family of finite energy weak solutions in the sense of Definition 2.1 to the Navier-Stokes system (1.2) under assumptions (1.4) in $(0,T)\times\Omega_\varepsilon$, emanating from the well-prepared initial data $(\bar{\rho}, \boldsymbol{u}_0)$ satisfying (2.9)–(2.12) for some $\bar{\rho} > 0$. Finally, let*

$$\gamma > \frac{3}{2}, \qquad \lim_{\varepsilon\to 0} \mathrm{Ma}(\varepsilon) = 0. \tag{2.23}$$

Then the following holds.

(a) Assume $((\Omega_\varepsilon)_{\varepsilon>0}, \Omega, D)$ obeys (M1). Then

$$\rho_\varepsilon \to \bar{\rho} \text{ strongly in } L^\infty(0,T;L^\gamma(D)),$$
$$\boldsymbol{u}_\varepsilon \rightharpoonup \boldsymbol{u} \text{ weakly in } L^2(0,T;W_0^{1,2}(D,\mathbb{R}^3)),$$

with $\boldsymbol{u}(0,\cdot)=\boldsymbol{u}_0$*. In particular,* $\boldsymbol{u}\in L^2(0,T;W_0^{1,2}(\Omega;\mathbb{R}^3))$*.*

(*b*) *Assume in addition that* $((\Omega_\varepsilon)_{\varepsilon>0},\Omega,D)$ *satisfies also (M2). Then* $(\bar\rho,\boldsymbol{u})$ *is a dissipative solution to* (1.5) *in the sense of Definition 2.3.*

(*c*) *Assume that there exists a strong solution to the system* (1.5) *in the sense of Definition 2.5 with initial datum* $\boldsymbol{u}_0$*. Then the latter coincides to* $(\bar\rho,\boldsymbol{u})$ *and* $\boldsymbol{u}_\varepsilon\to\boldsymbol{u}$ *strongly in* $L^2((0,T)\times D;\mathbb{R}^3)$*.*

3 Proof of Theorem 2.8

In this section we provide the proof of Theorem 2.8. For simplicity we divide the proof into two steps contained in Propositions 3.1 and 3.2 below. More precisely, in Proposition 3.1 we show 4.61 and that finite energy weak solutions to (1.2) converge to dissipative solutions of (1.5), from which we deduce (*b*), while in Proposition 3.2 we show the validity of the weak-strong uniqueness principle which in turn implies (*c*).

Proposition 3.1 (Convergence to dissipative solutions of (1.5))**.** *Let* $((\Omega_\varepsilon)_{\varepsilon>0},\Omega,D)$ *be a family of bounded domains that satisfies (M1). Let also* $(\rho_\varepsilon,\boldsymbol{u}_\varepsilon)_{\varepsilon>0}$ *be a family of finite energy weak solutions to the Navier-Stokes system* (1.2) *in* $(0,T)\times\Omega_\varepsilon$*, which satisfies* (2.9)–(2.12) *with* (1.4) *and* (2.23)*. Then*

$$\rho_\varepsilon\to\bar\rho \text{ strongly in } L^\infty(0,T;L^\gamma(D)), \tag{3.1}$$

$$\boldsymbol{u}_\varepsilon\rightharpoonup\boldsymbol{u} \text{ weakly in } L^2(0,T;W_0^{1,2}(D,\mathbb{R}^3)). \tag{3.2}$$

If in addition (M2) holds, then $(\bar\rho,\boldsymbol{u})$ *is a dissipative solution to* (1.5)*.*

Proof. The validity of (3.1) and (3.2) has already been proven in [9, Theorem 1.3]. Note that their assumption (M1) is actually exactly the same as ours such that their proof remains valid. Additionally, the energy inequality enforces (see [9, Section 2])

$$\|\sqrt{\rho_\varepsilon}\mathbf{u}_\varepsilon\|_{L^\infty(0,T;L^2(D;\mathbb{R}^3))}\lesssim 1, \tag{3.3}$$

as well as

$$\sup_{\tau\in[0,T]}\int_D\frac{1}{\mathrm{Ma}^2(\varepsilon)}(P(\rho_\varepsilon)-P'(\bar\rho)(\rho_\varepsilon-\bar\rho)-P(\bar\rho))\,\mathrm{d}x\lesssim 1. \tag{3.4}$$

Assume also (M2) holds. Thus, our task is to show that $(\bar\rho,\mathbf{u})$ is a dissipative solution to (1.5).

Step 1: Incompressibility. Let $\varphi\in C_c^1([0,T)\times\overline{\Omega})$. Using (2.2), we can show as in [30, Proposition 3.3] that

$$\int_0^T\int_D\rho_\varepsilon\partial_t\varphi+\rho_\varepsilon\mathbf{u}_\varepsilon\cdot\nabla_x\varphi\,\mathrm{d}x\,\mathrm{d}t=-\int_D\rho_{0,\varepsilon}\varphi(0,\cdot)\,\mathrm{d}x, \tag{3.5}$$

where we extended ρ_ε and $\mathbf{u}_\varepsilon$ by 0 to the whole of D.
With (3.1), (3.2), and (3.3) we get

$$\rho_\varepsilon\mathbf{u}_\varepsilon\overset{*}{\rightharpoonup}\bar\rho\mathbf{u}\text{ weakly}^*\text{ in }L^\infty(0,T;L^{\frac{2\gamma}{\gamma+1}}(D;\mathbb{R}^3)). \tag{3.6}$$

Indeed, for $\boldsymbol{\phi}\in L^1(0,T;L^{\frac{2\gamma}{\gamma-1}}(D;\mathbb{R}^3))$,

$$\int_0^T\int_D(\rho_\varepsilon\mathbf{u}_\varepsilon^\delta-\bar\rho\mathbf{u})\cdot\boldsymbol{\phi}\,\mathrm{d}x\,\mathrm{d}t=\int_0^T\int_D(\sqrt{\rho_\varepsilon}-\sqrt{\bar\rho})\sqrt{\rho_\varepsilon}\mathbf{u}_\varepsilon^\delta\cdot\boldsymbol{\phi}\,\mathrm{d}x\,\mathrm{d}t+\int_0^T\int_D\sqrt{\bar\rho}(\sqrt{\rho_\varepsilon}-\sqrt{\bar\rho})\mathbf{u}_\varepsilon^\delta\cdot\boldsymbol{\phi}\,\mathrm{d}x\,\mathrm{d}t$$
$$+\int_0^T\int_D\bar\rho(\mathbf{u}_\varepsilon^\delta-\mathbf{u})\cdot\boldsymbol{\phi}\,\mathrm{d}x\,\mathrm{d}t,$$

where the superscript δ denotes a convolution in time only to ensure $\mathbf{u}_\varepsilon^\delta \in L^\infty(-\delta, T+\delta; W_0^{1,2}(D;\mathbb{R}^3))$ such that all integrals above are well defined. Due to the strong convergence $\rho_\varepsilon \to \bar{\rho}$ in $L^\infty(0,T;L^\gamma(D))$, the weak convergence $\mathbf{u}_\varepsilon \rightharpoonup \mathbf{u}$ in $L^2(0,T;W_0^{1,2}(D;\mathbb{R}^3))$ that ensures as well $\mathbf{u}_\varepsilon^\delta \rightharpoonup \mathbf{u}^\delta$ weakly in $W_0^{1,2}(D;\mathbb{R}^3)$ uniformly in time, and the bound (3.3), we infer

$$\lim_{\delta\to 0}\lim_{\varepsilon\to 0}\int_0^T\int_D(\rho_\varepsilon \mathbf{u}_\varepsilon^\delta - \bar{\rho}\mathbf{u})\cdot \boldsymbol{\phi}\,\mathrm{d}x\,\mathrm{d}t = 0.$$

Using (3.6) and (3.1), we have for $\varepsilon \to 0$ in (3.5)

$$0 = \int_0^T\int_D \bar{\rho}\partial_t\varphi\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_D \bar{\rho}\mathbf{u}\cdot\nabla_x\varphi\,\mathrm{d}x\,\mathrm{d}t + \int_D \bar{\rho}\varphi(0,\cdot)\,\mathrm{d}x.$$

Now, we conclude with partial integration and $\bar{\rho}$ being a positive constant that

$$\begin{aligned}0 &= -\int_0^T\int_D(\partial_t\bar{\rho})\varphi\,\mathrm{d}x\,\mathrm{d}t - \int_D\bar{\rho}\varphi(0,\cdot)\,\mathrm{d}x + \int_0^T\int_D\bar{\rho}\mathbf{u}\cdot\nabla_x\varphi\,\mathrm{d}x\,\mathrm{d}t + \int_D\bar{\rho}\varphi(0,\cdot)\,\mathrm{d}x\\ &= \int_0^T\int_D\bar{\rho}\mathbf{u}\cdot\nabla_x\varphi\,\mathrm{d}x\,\mathrm{d}t,\end{aligned}$$

showing $\mathrm{div}_x\mathbf{u} = 0$.

Step 2: Momentum Equation. We consider a specific ansatz for the test functions, namely

$$\psi(t)\boldsymbol{\phi}(x), \quad \psi\in C_c^1([0,T)), \quad \boldsymbol{\phi}\in C_c^1(\Omega;\mathbb{R}^3), \quad \mathrm{div}_x(\boldsymbol{\phi}) = 0.$$

Due to (M2) there exists a sequence of solenoidal functions $(\boldsymbol{\phi}_\varepsilon)_{\varepsilon>0}$ approximating $\boldsymbol{\phi}$ such that (2.13)–(2.22) hold. We use that $\boldsymbol{\phi}_\varepsilon\psi \in C_c^1([0,T)\times\Omega_\varepsilon;\mathbb{R}^3)$ is a good test function to system (1.2) to get

$$\begin{aligned}0 = &\int_0^\tau\int_{\Omega_\varepsilon}\rho_\varepsilon\mathbf{u}_\varepsilon\cdot\partial_t(\boldsymbol{\phi}_\varepsilon\psi)\,\mathrm{d}x\,\mathrm{d}t + \int_0^\tau\int_{\Omega_\varepsilon}(\rho_\varepsilon\mathbf{u}_\varepsilon\otimes\mathbf{u}_\varepsilon):\nabla_x(\boldsymbol{\phi}_\varepsilon\psi)\,\mathrm{d}x\,\mathrm{d}t\\ &-\int_0^\tau\int_{\Omega_\varepsilon}\mathbb{S}(\nabla_x\mathbf{u}_\varepsilon):\nabla_x(\boldsymbol{\phi}_\varepsilon\psi)\,\mathrm{d}x\,\mathrm{d}t + \int_0^\tau\int_{\Omega_\varepsilon}\rho_\varepsilon\mathbf{f}\cdot(\boldsymbol{\phi}_\varepsilon\psi)\,\mathrm{d}x\,\mathrm{d}t\\ &+\int_{\Omega_\varepsilon}(\rho\mathbf{u})_{0,\varepsilon}\cdot(\boldsymbol{\phi}_\varepsilon\psi)(0,\cdot)\,\mathrm{d}x - \int_{\Omega_\varepsilon}(\rho_\varepsilon\mathbf{u}_\varepsilon)(\tau,\cdot)\cdot(\boldsymbol{\phi}_\varepsilon\psi)(\tau,\cdot)\,\mathrm{d}x\,.\end{aligned}$$

Now using that $\boldsymbol{\phi}_\varepsilon = (\boldsymbol{\phi}_\varepsilon - \boldsymbol{\phi}) + \boldsymbol{\phi}$ the above identity can be rewritten as

$$\begin{aligned}&\int_0^\tau\int_{\Omega_\varepsilon}\rho_\varepsilon\mathbf{u}_\varepsilon\cdot\partial_t(\boldsymbol{\phi}\psi) + (\rho_\varepsilon\mathbf{u}_\varepsilon\otimes\mathbf{u}_\varepsilon):\nabla_x(\boldsymbol{\phi}\psi) - \mathbb{S}(\nabla_x\mathbf{u}_\varepsilon):\nabla_x(\boldsymbol{\phi}\psi) + \rho_\varepsilon\mathbf{f}\cdot(\boldsymbol{\phi}\psi)\,\mathrm{d}x\,\mathrm{d}t\\ &\quad + \int_{\Omega_\varepsilon}(\rho\mathbf{u})_{0,\varepsilon}\cdot\boldsymbol{\phi}\psi(0)\,\mathrm{d}x - \int_{\Omega_\varepsilon}(\rho_\varepsilon\mathbf{u}_\varepsilon)(\tau,\cdot)\cdot\boldsymbol{\phi}\psi(\tau)\,\mathrm{d}x\\ &= \int_0^\tau\int_{\Omega_\varepsilon}\rho_\varepsilon\mathbf{u}_\varepsilon\cdot(\boldsymbol{\phi}-\boldsymbol{\phi}_\varepsilon)\partial_t\psi\,\mathrm{d}x\,\mathrm{d}t + \int_0^\tau\int_{\Omega_\varepsilon}(\rho_\varepsilon\mathbf{u}_\varepsilon\otimes\mathbf{u}_\varepsilon):\nabla_x(\boldsymbol{\phi}-\boldsymbol{\phi}_\varepsilon)\psi\,\mathrm{d}x\,\mathrm{d}t\\ &\quad - \int_0^\tau\int_{\Omega_\varepsilon}\mathbb{S}(\nabla_x\mathbf{u}_\varepsilon):\nabla_x(\boldsymbol{\phi}-\boldsymbol{\phi}_\varepsilon)\psi\,\mathrm{d}x\,\mathrm{d}t + \int_0^\tau\int_{\Omega_\varepsilon}\rho_\varepsilon\mathbf{f}\cdot(\boldsymbol{\phi}-\boldsymbol{\phi}_\varepsilon)\psi\,\mathrm{d}x\,\mathrm{d}t\\ &\quad + \int_{\Omega_\varepsilon}(\rho\mathbf{u})_{0,\varepsilon}\cdot(\boldsymbol{\phi}-\boldsymbol{\phi}_\varepsilon)\psi(0)\,\mathrm{d}x - \int_{\Omega_\varepsilon}(\rho_\varepsilon\mathbf{u}_\varepsilon)(\tau,\cdot)\cdot(\boldsymbol{\phi}-\boldsymbol{\phi}_\varepsilon)\psi(\tau)\,\mathrm{d}x =: \sum_{i=1}^6 I_i.\end{aligned}$$

We now show that $I_i \to 0$ as $\varepsilon\to 0$ for all i except I_3 which will eventually yield the Brinkman term. We start by estimating I_1. This term can be bounded using the Sobolev embedding and Hölder's inequality as

$$
\begin{aligned}
|I_1| &\leq \int_0^T |\partial_t \psi| \int_{\Omega_\varepsilon} |\rho_\varepsilon \mathbf{u}_\varepsilon \cdot (\boldsymbol{\phi} - \boldsymbol{\phi}_\varepsilon)| \,\mathrm{d}x \,\mathrm{d}t \\
&\lesssim \|\rho_\varepsilon\|_{L^\infty(0,T;L^\gamma(D))} \|\mathbf{u}_\varepsilon\|_{L^2(0,T;L^6(D;\mathbb{R}^3))} \|\boldsymbol{\phi} - \boldsymbol{\phi}_\varepsilon\|_{L^{\frac{6\gamma}{5\gamma-6}}(D;\mathbb{R}^3)}.
\end{aligned}
$$

By $\gamma > 3/2$, we have $6\gamma/(5\gamma - 6) < 6$ and thus $W_0^{1,2}$ is compactly embedded in $L^{\frac{6\gamma}{5\gamma-6}}$. Hence, by (2.14), (3.1), and (3.2), it follows that $|I_1| \to 0$.
To estimate the convective term I_2, we write

$$
\mathcal{M}_\varepsilon^{\mathrm{ess}} := \{(t,x) \in (0,T) \times \Omega_\varepsilon \mid \frac{1}{2}\bar{\rho} < \rho_\varepsilon(t,x) < \bar{\rho} + 1\}, \quad \mathcal{M}_\varepsilon^{\mathrm{res}} := ((0,T) \times \Omega_\varepsilon) \setminus \mathcal{M}_\varepsilon^{\mathrm{ess}},
$$

$$
\rho_\varepsilon = \rho_\varepsilon^{\mathrm{ess}} + \rho_\varepsilon^{\mathrm{res}}, \quad \rho_\varepsilon^{\mathrm{ess}} := \rho_\varepsilon \chi_{\mathcal{M}_\varepsilon^{\mathrm{ess}}}, \quad \rho_\varepsilon^{\mathrm{res}} := \rho_\varepsilon \chi_{\mathcal{M}_\varepsilon^{\mathrm{res}}}. \tag{3.7}
$$

Similarly, we split $I_2 = I_2^{\mathrm{ess}} + I_2^{\mathrm{res}}$, where I_2^{ess} and I_2^{res} denote the corresponding integrals on $\mathcal{M}_\varepsilon^{\mathrm{ess}}$ and $\mathcal{M}_\varepsilon^{\mathrm{res}}$, respectively. For the essential part we use $|\rho_\varepsilon^{\mathrm{ess}}| \lesssim 1$ and Sobolev embedding to conclude

$$
\begin{aligned}
|I_2^{\mathrm{ess}}| &\leq \int_0^T |\psi| \int_{\Omega_\varepsilon} |\rho_\varepsilon^{\mathrm{ess}}| |\mathbf{u}_\varepsilon|^2 |\nabla_x(\boldsymbol{\phi} - \boldsymbol{\phi}_\varepsilon)| \,\mathrm{d}x \,\mathrm{d}t \\
&\lesssim \|\mathbf{u}_\varepsilon\|^2_{L^2(0,T;L^6(D;\mathbb{R}^3))} \|\nabla_x(\boldsymbol{\phi} - \boldsymbol{\phi}_\varepsilon)\|_{L^{\frac{3}{2}}(D;\mathbb{R}^3)} \\
&\lesssim \|\nabla_x(\boldsymbol{\phi} - \boldsymbol{\phi}_\varepsilon)\|_{L^{\frac{3}{2}}(D;\mathbb{R}^3)}.
\end{aligned}
$$

The last term vanishes as $\varepsilon \to 0$ due to (2.15). Moreover, we find from (3.4) that

$$
\|\rho_\varepsilon^{\mathrm{ess}} - \bar{\rho}\|_{L^\infty(0,T;L^2(D))} \lesssim \mathrm{Ma}(\varepsilon), \tag{3.8}
$$

where we used that on $\mathcal{M}_\varepsilon^{\mathrm{ess}}$, the function $P(\rho_\varepsilon) - P'(\bar{\rho})(\rho_\varepsilon - \bar{\rho}) - P(\bar{\rho})$ is coercive in the sense that

$$
P(\rho_\varepsilon) - P'(\bar{\rho})(\rho_\varepsilon - \bar{\rho}) - P(\bar{\rho}) \gtrsim |\rho_\varepsilon - \bar{\rho}|^2,
$$

see [18, Lemma 5.1]. Due to (2.15) the essential part of the convective term vanishes for $\varepsilon \to 0$. For the residual part we use the following estimate from the proof of [9, Theorem 1.3]:

$$
\|\rho_\varepsilon^{\mathrm{res}}\|_{L^\infty(0,T;L^\gamma(D))} \lesssim \mathrm{Ma}(\varepsilon)^{\frac{2}{\gamma}}. \tag{3.9}
$$

From this we obtain

$$
\begin{aligned}
|I_2^{\mathrm{res}}| &\leq \int_0^T |\psi| \int_{\Omega_\varepsilon} |\rho_\varepsilon^{\mathrm{res}}| |\mathbf{u}_\varepsilon|^2 |\nabla_x(\boldsymbol{\phi} - \boldsymbol{\phi}_\varepsilon)| \,\mathrm{d}x \,\mathrm{d}t \\
&\lesssim \|\rho_\varepsilon^{\mathrm{res}}\|_{L^\infty(0,T;L^\gamma(D))} \|\mathbf{u}_\varepsilon\|^2_{L^2(0,T;L^6(D;\mathbb{R}^3))} \|\nabla_x(\boldsymbol{\phi} - \boldsymbol{\phi}_\varepsilon)\|_{L^r(D;\mathbb{R}^3)} \\
&\lesssim \mathrm{Ma}(\varepsilon)^{\frac{2}{\gamma}} \|\nabla_x(\boldsymbol{\phi} - \boldsymbol{\phi}_\varepsilon)\|_{L^r(D;\mathbb{R}^3)},
\end{aligned}
$$

where $r = \frac{3\gamma}{2\gamma-3} > \frac{3}{2}$. By (2.16), the right-hand side of the above inequality vanishes for $\varepsilon \to 0$, which in turn shows that $|I_2| \to 0$ as $\varepsilon \to 0$.

Next, we want to prove that I_3 converges to the Brinkman term. Therefore, we first show that $\mathbf{u}_\varepsilon$ and $\mathbf{u}$ satisfy the conditions of (2.17)–(2.20). (2.17) is clear from (3.2). Let us define

$$
\begin{aligned}
g_\varepsilon &:= -\frac{\rho_\varepsilon}{\bar{\rho}}, \\
\mathbf{h}_\varepsilon &:= \frac{(\bar{\rho} - \rho_\varepsilon)\mathbf{u}_\varepsilon}{\bar{\rho}}.
\end{aligned}
$$

Then, by the first equation of (1.2), we have

$$\mathrm{div}_x(\mathbf{u}_\varepsilon) = \frac{1}{\bar\rho}\mathrm{div}_x((\bar\rho - \rho_\varepsilon)\mathbf{u}_\varepsilon) + \frac{1}{\bar\rho}\mathrm{div}_x(\rho_\varepsilon \mathbf{u}_\varepsilon) = \mathrm{div}_x(\mathbf{h}_\varepsilon) + \partial_t g_\varepsilon$$

in the sense of distributions. Further, we have from (3.8) and (3.9) that

$$\|\mathbf{h}_\varepsilon\|_{L^2(0,T;L^{\frac{6\gamma}{\gamma+6}}(D;\mathbb{R}^3))} \lesssim \|\rho_\varepsilon - \bar\rho\|_{L^\infty(0,T;L^\gamma(D))}\|\mathbf{u}_\varepsilon\|_{L^2(0,T;L^6(D;\mathbb{R}^3))} \lesssim \mathrm{Ma}(\varepsilon)^{\min\{\frac{2}{\gamma},1\}},$$

and $g_\varepsilon \to -1$ in $L^\infty(0,T;L^\gamma(D))$ due to (3.1). In turn, we get (2.19) and (2.20) for $M_\varepsilon := \mathcal{M}_\varepsilon^{\mathrm{ess}}$ such that we can use (2.21) and (2.22) for $\mathbf{v}_\varepsilon = \mathbf{u}_\varepsilon$ and $\mathbf{v} = \mathbf{u}$. By the divergence-free assumption on $\boldsymbol\phi$ and (2.13), we have $\mathrm{div}_x(\boldsymbol\phi - \boldsymbol\phi_\varepsilon) = 0$. This together with (3.2) and (2.21) leads to

$$\begin{aligned}
I_3 &= -\int_0^\tau \int_{\Omega_\varepsilon} \mathbb{S}(\nabla_x \mathbf{u}_\varepsilon) : \nabla_x(\psi(\boldsymbol\phi - \boldsymbol\phi_\varepsilon))\,\mathrm{d}x\,\mathrm{d}t \\
&= -\int_0^\tau \psi \left(\mu \int_{\Omega_\varepsilon} \nabla_x \mathbf{u}_\varepsilon : \nabla_x(\boldsymbol\phi - \boldsymbol\phi_\varepsilon)\,\mathrm{d}x + \left(\frac{\mu}{3} + \eta\right)\int_{\Omega_\varepsilon} \mathrm{div}_x(\mathbf{u}_\varepsilon)\mathrm{div}_x(\boldsymbol\phi - \boldsymbol\phi_\varepsilon)\,\mathrm{d}x\right)\mathrm{d}t \\
&= -\int_0^\tau \psi\mu \left(\int_{\Omega_\varepsilon} \nabla_x \mathbf{u}_\varepsilon : \nabla_x \boldsymbol\phi\,\mathrm{d}x\right)\mathrm{d}t + \int_0^\tau \psi\mu\left(\int_{\Omega_\varepsilon} \nabla_x \mathbf{u}_\varepsilon : \nabla_x \boldsymbol\phi_\varepsilon\,\mathrm{d}x\right)\mathrm{d}t \\
&\xrightarrow{\varepsilon\to 0} -\int_0^\tau \psi\mu\left(\int_\Omega \nabla_x \mathbf{u} : \nabla_x\boldsymbol\phi\,\mathrm{d}x\right)\mathrm{d}t + \int_0^\tau \psi\mu\left(\int_\Omega \nabla_x\mathbf{u} : \nabla_x\boldsymbol\phi\,\mathrm{d}x + \int_\Omega (\mathrm{M}\boldsymbol\phi)\cdot\mathbf{u}\right)\mathrm{d}t \\
&= \mu\int_0^T \int_\Omega \psi(\mathrm{M}\boldsymbol\phi)\cdot\mathbf{u}\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}$$

Indeed, since $\mathrm{div}_x(\boldsymbol\phi - \boldsymbol\phi_\varepsilon) = 0$ and by smoothing $\boldsymbol\phi - \boldsymbol\phi_\varepsilon$ by convolution with some suitable functions $(\eta_\delta)_{\delta>0}$, we get

$$\begin{aligned}
\int_0^\tau \int_{\Omega_\varepsilon} \nabla_x^T \mathbf{u}_\varepsilon : \nabla_x(\psi(\boldsymbol\phi - \boldsymbol\phi_\varepsilon))\,\mathrm{d}x\,\mathrm{d}t &= \lim_{\delta\to 0}\int_0^\tau \psi\int_{\Omega_\varepsilon} \nabla_x^T\mathbf{u}_\varepsilon : \nabla_x((\boldsymbol\phi - \boldsymbol\phi_\varepsilon) * \eta_\delta)\,\mathrm{d}x\,\mathrm{d}t \\
&= -\lim_{\delta\to 0}\int_0^\tau \psi\int_{\Omega_\varepsilon} \mathbf{u}_\varepsilon \cdot \nabla_x(\mathrm{div}_x((\boldsymbol\phi - \boldsymbol\phi_\varepsilon) * \eta_\delta))\,\mathrm{d}x\,\mathrm{d}t = 0,
\end{aligned}$$

and

$$\int_0^\tau \int_{\Omega_\varepsilon} (\mathrm{div}_x \mathbf{u}_\varepsilon \mathbb{1}) : \nabla_x(\boldsymbol\phi - \boldsymbol\phi_\varepsilon)\,\mathrm{d}x\,\mathrm{d}t = \int_0^\tau \psi\int_{\Omega_\varepsilon} \mathrm{div}_x\mathbf{u}_\varepsilon\ \mathrm{div}_x(\boldsymbol\phi - \boldsymbol\phi_\varepsilon)\,\mathrm{d}x\,\mathrm{d}t = 0.$$

To estimate I_4 we use that $\mathbf{f} \in L^\infty(0,T;L^\infty(D;\mathbb{R}^3))$, which together with (3.1) and (2.14) yields

$$\begin{aligned}
|I_4| &\le \int_0^T |\psi|\int_{\Omega_\varepsilon} |\rho_\varepsilon \mathbf{f}||\boldsymbol\phi - \boldsymbol\phi_\varepsilon|\,\mathrm{d}x\,\mathrm{d}t \\
&\lesssim \|\rho_\varepsilon\|_{L^\infty(0,T;L^\gamma(D))}\|\boldsymbol\phi - \boldsymbol\phi_\varepsilon\|_{L^{\frac{\gamma}{\gamma-1}}(D;\mathbb{R}^3)} \to 0
\end{aligned}$$

by compact Sobolev embedding as $\gamma/(\gamma-1) < 6$ for any $\gamma > 6/5$. Finally, due to $2\gamma/(\gamma-1) < 6$ for any $\gamma > 3/2$, (2.11), and (2.14), we get

$$|I_5| \le |\psi(0)|\int_{\Omega_\varepsilon} |(\rho\mathbf{u})_{0,\varepsilon}\cdot(\boldsymbol\phi - \boldsymbol\phi_\varepsilon)|\,\mathrm{d}x \lesssim \|(\rho\mathbf{u})_{0,\varepsilon}\|_{L^\infty(0,T;L^{\frac{2\gamma}{\gamma+1}}(D;\mathbb{R}^3))}\|\boldsymbol\phi - \boldsymbol\phi_\varepsilon\|_{L^{\frac{2\gamma}{\gamma-1}}(D;\mathbb{R}^3)} \to 0.$$

By (3.1), (3.2), and (2.14), we have $I_6 \in L^1(0,T)$ and for every function $\xi \in L^\infty(0,T)$

$$\begin{aligned}
\left|\int_0^T \xi(\tau) I_6(\tau)\,\mathrm{d}\tau\right| &= \left|\int_0^T \int_{\Omega_\varepsilon} \xi(\tau)(\rho_\varepsilon \mathbf{u}_\varepsilon)(\tau,\cdot)\cdot(\boldsymbol\phi - \boldsymbol\phi_\varepsilon)\psi(\tau)\,\mathrm{d}x\,\mathrm{d}\tau\right| \\
&\lesssim \|\xi\|_{L^\infty(0,T)}\|\rho_\varepsilon\|_{L^\infty(0,T;L^\gamma(D))}\|\mathbf{u}_\varepsilon\|_{L^2(0,T;L^6(D;\mathbb{R}^3))}\|\boldsymbol\phi - \boldsymbol\phi_\varepsilon\|_{L^{\frac{6\gamma}{5\gamma-6}}(D;\mathbb{R}^3)} \to 0,
\end{aligned}$$

by the same arguments as for I_1. Consequently, we have $I_6 \rightharpoonup 0$ in $L^1(0,T)$ and therefore $I_6(\tau) \to 0$ for almost any $\tau \in [0,T]$. In total, we proved that

$$\lim_{\varepsilon\to 0} \sum_{i=1}^{6} I_i = \lim_{\varepsilon\to 0} I_3 = \mu \int_0^\tau \int_\Omega \psi(\mathrm{M}\boldsymbol{\phi}) \cdot \mathbf{u} \,\mathrm{d}x \,\mathrm{d}t. \tag{3.10}$$

We now look at the left-hand side of (**??**). Using (2.11), (3.1), (3.2), and the fact that we might prolong $\boldsymbol{\phi} \in C_c^1(\Omega;\mathbb{R}^3)$ by zero to the whole of D, we get

$$\begin{aligned}
\lim_{\varepsilon\to 0} \int_0^\tau \int_{\Omega_\varepsilon} \rho_\varepsilon \mathbf{f} \cdot (\boldsymbol{\phi}\psi) \,\mathrm{d}x \,\mathrm{d}t &= \lim_{\varepsilon\to 0} \int_0^\tau \int_D \rho_\varepsilon \mathbf{f} \cdot (\boldsymbol{\phi}\psi) \,\mathrm{d}x \,\mathrm{d}t = \int_0^\tau \int_D \bar{\rho} \mathbf{f} \cdot (\boldsymbol{\phi}\psi) \,\mathrm{d}x \,\mathrm{d}t \\
&= \int_0^\tau \int_\Omega \bar{\rho} \mathbf{f} \cdot (\boldsymbol{\phi}\psi) \,\mathrm{d}x \,\mathrm{d}t, \\
\lim_{\varepsilon\to 0} \int_0^\tau \int_{\Omega_\varepsilon} \mathbb{S}(\nabla_x \mathbf{u}_\varepsilon) : \nabla_x(\boldsymbol{\phi}\psi) \,\mathrm{d}x \,\mathrm{d}t &= \int_0^\tau \int_\Omega \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x(\boldsymbol{\phi}\psi) \,\mathrm{d}x \,\mathrm{d}t = \int_0^\tau \int_\Omega \mu \nabla_x \mathbf{u} : \nabla_x(\boldsymbol{\phi}\psi) \,\mathrm{d}x \,\mathrm{d}t, \\
\lim_{\varepsilon\to 0} \int_{\Omega_\varepsilon} (\rho\mathbf{u})_{0,\varepsilon} \cdot \boldsymbol{\phi}\psi(0) \,\mathrm{d}x &= \int_\Omega \bar{\rho}\mathbf{u}_0 \cdot \boldsymbol{\phi}\psi(0) \,\mathrm{d}x, \\
\lim_{\varepsilon\to 0} \int_0^\tau \partial_t \psi \int_{\Omega_\varepsilon} \rho_\varepsilon \mathbf{u}_\varepsilon \cdot \boldsymbol{\phi} \,\mathrm{d}x \,\mathrm{d}t &= \int_0^\tau \partial_t \psi \int_\Omega \bar{\rho}\mathbf{u} \cdot \boldsymbol{\phi} \,\mathrm{d}x \,\mathrm{d}t.
\end{aligned} \tag{3.11}$$

Moreover, for every function $\xi \in L^\infty(0,T)$, we have, using (3.1) and (3.2),

$$\lim_{\varepsilon\to 0} \int_0^T \int_{\Omega_\varepsilon} \xi(\tau)(\rho_\varepsilon \mathbf{u}_\varepsilon)(\tau,\cdot)\boldsymbol{\phi}\psi(\tau) \,\mathrm{d}x \,\mathrm{d}\tau = \int_0^T \int_\Omega \xi(\tau)(\bar{\rho}\mathbf{u})(\tau,\cdot)\boldsymbol{\phi}\psi(\tau) \,\mathrm{d}x \,\mathrm{d}\tau.$$

Consequently, we see

$$\lim_{\varepsilon\to 0} \int_{\Omega_\varepsilon} (\rho_\varepsilon \mathbf{u}_\varepsilon)(\tau,\cdot)\boldsymbol{\phi}\psi(\tau) \,\mathrm{d}x = \int_\Omega (\bar{\rho}\mathbf{u})(\tau,\cdot)\boldsymbol{\phi}\psi(\tau) \,\mathrm{d}x \tag{3.12}$$

for almost any $\tau \in [0,T]$.
To treat the convective term we use again the decomposition $\rho_\varepsilon = \rho_\varepsilon^{\mathrm{ess}} + \rho_\varepsilon^{\mathrm{res}}$. Recall from (3.9) that

$$\|\rho_\varepsilon^{\mathrm{res}}\|_{L^\infty(0,T;L^\gamma(D))} \lesssim \mathrm{Ma}(\varepsilon)^{\frac{2}{\gamma}}.$$

Consequently, we may estimate

$$\begin{aligned}
\left| \int_0^\tau \int_{\Omega_\varepsilon} \rho_\varepsilon^{\mathrm{res}} \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon : \nabla_x(\boldsymbol{\phi}\psi) \,\mathrm{d}x \,\mathrm{d}t \right| &\lesssim \int_0^T |\psi| \|\rho_\varepsilon^{\mathrm{res}}\|_{L^\gamma(D)} \|\mathbf{u}_\varepsilon\|_{L^6(D;\mathbb{R}^3)}^2 \,\mathrm{d}t \\
&\lesssim \mathrm{Ma}(\varepsilon)^{\frac{2}{\gamma}} \|\mathbf{u}_\varepsilon\|_{L^2(0,T;W^{1,2}(D;\mathbb{R}^3))} \to 0,
\end{aligned} \tag{3.13}$$

where we used (2.23) and (3.2).

Finally, we know that $\rho_\varepsilon^{\mathrm{ess}} \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon$ is uniformly bounded in $L^1(0,T;L^3(D;\mathbb{R}^{3\times 3}))$ and $L^\infty(0,T;L^1(D;\mathbb{R}^{3\times 3}))$. Hence, we may extract a subsequence (not relabeled) such that $\rho_\varepsilon^{\mathrm{ess}} \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon$ converges weakly in $L^{\frac{4}{3}}(0,T;L^2(D;\mathbb{R}^{3\times 3}))$, the weak limit of which we denote by $\overline{\rho\mathbf{u}\otimes\mathbf{u}}$. Thus,

$$\int_0^T \int_{\Omega_\varepsilon} (\rho_\varepsilon^{\mathrm{ess}} \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon) : \nabla_x(\boldsymbol{\phi}\psi) \,\mathrm{d}x \,\mathrm{d}t \to \int_0^T \int_\Omega \overline{\rho\mathbf{u}\otimes\mathbf{u}} : \nabla_x(\boldsymbol{\phi}\psi) \,\mathrm{d}x \,\mathrm{d}t. \tag{3.14}$$

A similar argument leads to

$$\sqrt{\rho_\varepsilon}\mathbf{u}_\varepsilon \overset{*}{\rightharpoonup} \sqrt{\bar{\rho}}\mathbf{u} \quad \text{weakly-}* \text{ in } L^\infty(0,T;L^2(D;\mathbb{R}^3)).$$

Since $\mathbf{v} \mapsto \mathbf{v} \otimes \mathbf{v}$ is convex, we have

$$\mathfrak{R} := \overline{\rho \mathbf{u} \otimes \mathbf{u}} - \bar{\rho}\mathbf{u} \otimes \mathbf{u} \geq 0 \quad \text{in the sense of symmetric matrices.} \tag{3.15}$$

Gathering (3.13), (3.14), and (3.15), we find

$$\lim_{\varepsilon \to 0} \int_0^\tau \int_{\Omega_\varepsilon} \rho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon : \nabla_x(\boldsymbol{\phi}\psi) \,\mathrm{d}x \,\mathrm{d}t = \int_0^\tau \int_\Omega \bar{\rho}\mathbf{u} \otimes \mathbf{u} : \nabla_x(\boldsymbol{\phi}\psi) \,\mathrm{d}x \,\mathrm{d}t + \int_0^\tau \int_\Omega \mathfrak{R} : \nabla_x(\boldsymbol{\phi}\psi) \,\mathrm{d}x \,\mathrm{d}t. \tag{3.16}$$

Combining (**??**), (3.10), (3.11), (3.12), and (3.16), we infer

$$\begin{aligned} \int_0^\tau \int_\Omega \bar{\rho}\mathbf{u} \cdot \partial_t \boldsymbol{\varphi} + \bar{\rho}\mathbf{u} \otimes \mathbf{u} : \nabla_x \boldsymbol{\varphi} - \mu \nabla_x \mathbf{u} : \nabla_x \boldsymbol{\varphi} + \bar{\rho}\mathbf{f} \cdot \boldsymbol{\varphi} + \mathfrak{R} : \nabla_x \boldsymbol{\varphi} \,\mathrm{d}x \,\mathrm{d}t \\ + \int_\Omega \bar{\rho}\mathbf{u} \cdot \boldsymbol{\varphi}(0,\cdot) \,\mathrm{d}x - \int_\Omega \bar{\rho}\mathbf{u} \cdot \boldsymbol{\varphi}(\tau,\cdot) \,\mathrm{d}x = \mu \int_0^\tau \int_\Omega \mathrm{M}\mathbf{u} \cdot \boldsymbol{\varphi} \,\mathrm{d}x \,\mathrm{d}t, \end{aligned} \tag{3.17}$$

for all $\boldsymbol{\varphi} \in C_c^1([0,T) \times \Omega)$, $\boldsymbol{\varphi}(t,x) = \psi(t)\boldsymbol{\phi}(x)$ with $\mathrm{div}_x \boldsymbol{\varphi} = 0$, and almost any $\tau \in [0,T]$. Eventually, by a density argument, (3.17) holds true for any $\boldsymbol{\varphi} \in C_c^1([0,T) \times \Omega)$ satisfying $\mathrm{div}_x \boldsymbol{\varphi} = 0$.

Step 3: Energy inequality. Since $(\rho_\varepsilon, \mathbf{u}_\varepsilon)$ are finite energy weak solutions, we have

$$\begin{aligned} &\int_{\Omega_\varepsilon} \left[\frac{1}{2} \mathbb{1}_{\{\rho_\varepsilon > 0\}} \frac{|\rho_\varepsilon \mathbf{u}_\varepsilon|^2}{\rho_\varepsilon} + \frac{1}{\mathrm{Ma}^2(\varepsilon)} P(\rho_\varepsilon) \right] (\tau,\cdot) \,\mathrm{d}x + \int_0^\tau \int_{\Omega_\varepsilon} \mathbb{S}(\nabla_x \mathbf{u}_\varepsilon) : \nabla_x \mathbf{u}_\varepsilon \,\mathrm{d}x \,\mathrm{d}t \\ &\leq \int_{\Omega_\varepsilon} \frac{1}{2} \mathbb{1}_{\{\rho_{0,\varepsilon} > 0\}} \frac{|(\rho\mathbf{u})_{0,\varepsilon}|^2}{\rho_{0,\varepsilon}} + \frac{1}{\mathrm{Ma}^2(\varepsilon)} P(\rho_{0,\varepsilon}) \,\mathrm{d}x + \int_0^\tau \int_{\Omega_\varepsilon} \rho_\varepsilon \mathbf{f} \cdot \mathbf{u}_\varepsilon \,\mathrm{d}x \,\mathrm{d}t \end{aligned} \tag{3.18}$$

for almost any $\tau \in [0,T]$. First, we want to control the limit of the second term of the energy inequality

$$\begin{aligned} \int_0^\tau \int_{\Omega_\varepsilon} \mathbb{S}(\nabla_x \mathbf{u}_\varepsilon) : \nabla_x \mathbf{u}_\varepsilon \,\mathrm{d}x \,\mathrm{d}t &= \int_0^\tau \int_{\Omega_\varepsilon} \mu \nabla_x \mathbf{u}_\varepsilon : \nabla_x \mathbf{u}_\varepsilon \,\mathrm{d}x \,\mathrm{d}t + \left(\frac{\mu}{3} + \eta \right) \int_0^\tau \int_{\Omega_\varepsilon} |\mathrm{div}_x(\mathbf{u}_\varepsilon)|^2 \,\mathrm{d}x \,\mathrm{d}t \\ &\geq \mu \int_0^\tau \int_{\Omega_\varepsilon} |\nabla_x \mathbf{u}_\varepsilon|^2 \,\mathrm{d}x \,\mathrm{d}t, \end{aligned}$$

where we used that with partial integration and the symmetry of second derivatives, we have

$$\begin{aligned} \int_0^T \int_{\Omega_\varepsilon} \nabla_x^T \mathbf{u}_\varepsilon : \nabla_x \mathbf{u}_\varepsilon \,\mathrm{d}x \,\mathrm{d}t &= \lim_{\delta \to 0} \int_0^T \int_{\Omega_\varepsilon} \nabla_x^T \mathbf{u}_\varepsilon : \nabla_x(\mathbf{u}_\varepsilon * \eta_\delta) \,\mathrm{d}x \,\mathrm{d}t = \lim_{\delta \to 0} - \int_0^T \int_{\Omega_\varepsilon} \mathbf{u}_\varepsilon \cdot \nabla_x \mathrm{div}_x(\mathbf{u}_\varepsilon * \eta_\delta) \,\mathrm{d}x \,\mathrm{d}t \\ &= \lim_{\delta \to 0} \int_0^T \int_{\Omega_\varepsilon} \mathrm{div}_x(\mathbf{u}_\varepsilon) \cdot \mathrm{div}_x(\mathbf{u}_\varepsilon * \eta_\delta) \,\mathrm{d}x \,\mathrm{d}t = \int_0^T \int_{\Omega_\varepsilon} |\mathrm{div}_x(\mathbf{u}_\varepsilon)|^2 \,\mathrm{d}x \,\mathrm{d}t, \end{aligned}$$

where $(\eta_\delta)_{\delta > 0}$ are suitable mollifiers. This leads to

$$\begin{aligned} &\int_{\Omega_\varepsilon} \left[\frac{1}{2} \mathbb{1}_{\{\rho_\varepsilon > 0\}} \frac{|\rho_\varepsilon \mathbf{u}_\varepsilon|^2}{\rho_\varepsilon} + \frac{1}{\mathrm{Ma}^2(\varepsilon)} P(\rho_\varepsilon) \right] (\tau,\cdot) \,\mathrm{d}x + \mu \int_0^\tau \int_{\Omega_\varepsilon} |\nabla_x \mathbf{u}_\varepsilon|^2 \,\mathrm{d}x \,\mathrm{d}t \\ &\leq \int_{\Omega_\varepsilon} \frac{1}{2} \mathbb{1}_{\{\rho_{0,\varepsilon} > 0\}} \frac{|(\rho\mathbf{u})_{0,\varepsilon}|^2}{\rho_{0,\varepsilon}} + \frac{1}{\mathrm{Ma}^2(\varepsilon)} P(\rho_{0,\varepsilon}) \,\mathrm{d}x + \int_0^\tau \int_{\Omega_\varepsilon} \rho_\varepsilon \mathbf{f} \cdot \mathbf{u}_\varepsilon \,\mathrm{d}x \,\mathrm{d}t \end{aligned}$$

for almost any $\tau \in [0,T]$. Let now $\xi \in C_c(0,T)$ with $\xi \geq 0$, $\int_0^T \xi(\tau) \,\mathrm{d}\tau = 1$. Multiplication by $\xi(\tau)$ and integration leads to

$$\begin{aligned} &\int_0^T \int_{\Omega_\varepsilon} \xi(\tau) \left[\frac{1}{2} \mathbb{1}_{\{\rho_\varepsilon > 0\}} \frac{|\rho_\varepsilon \mathbf{u}_\varepsilon|^2}{\rho_\varepsilon} + \frac{1}{\mathrm{Ma}^2(\varepsilon)} P(\rho_\varepsilon) \right] (\tau,\cdot) \,\mathrm{d}x \,\mathrm{d}\tau + \mu \int_0^T \xi(\tau) \int_0^\tau \int_{\Omega_\varepsilon} |\nabla_x \mathbf{u}_\varepsilon|^2 \,\mathrm{d}x \,\mathrm{d}t \,\mathrm{d}\tau \\ &\leq \int_0^T \int_{\Omega_\varepsilon} \xi(\tau) \left[\frac{1}{2} \mathbb{1}_{\{\rho_{0,\varepsilon} > 0\}} \frac{|(\rho\mathbf{u})_{0,\varepsilon}|^2}{\rho_{0,\varepsilon}} + \frac{1}{\mathrm{Ma}^2(\varepsilon)} P(\rho_{0,\varepsilon}) \right] \mathrm{d}x \,\mathrm{d}\tau + \int_0^T \xi(\tau) \int_0^\tau \int_{\Omega_\varepsilon} \rho_\varepsilon \mathbf{f} \cdot \mathbf{u}_\varepsilon \,\mathrm{d}x \,\mathrm{d}t \,\mathrm{d}\tau. \end{aligned} \tag{3.19}$$

Since $\mathbb{1}_{\rho_{0,\varepsilon}>0}|(\rho\mathbf{u})_{0,\varepsilon}|^2/\rho_{0,\varepsilon}$ is bounded in $L^1(0,T;L^3(D))$ and $L^\infty(0,T;L^1(D))$, we get

$$\mathbb{1}_{\rho_\varepsilon>0}\frac{|\rho_\varepsilon\mathbf{u}_\varepsilon|^2}{\rho_\varepsilon} \rightharpoonup \overline{\rho|\mathbf{u}|^2} \quad \text{weakly in } L^{\frac{4}{3}}(0,T;L^2(D)).$$

Additionally, by (2.1), we have

$$\int_{\Omega_\varepsilon}\rho_\varepsilon(\tau,\cdot)\,\mathrm{d}x = \int_{\Omega_\varepsilon}\rho_{0,\varepsilon}\,\mathrm{d}x \ \text{ for almost any } \ \tau\in[0,T],$$

hence

$$\begin{aligned}\int_{\Omega_\varepsilon}P(\rho_\varepsilon)(\tau,\cdot)\,\mathrm{d}x - \int_{\Omega_\varepsilon}P(\rho_{0,\varepsilon})\,\mathrm{d}x &= \int_{\Omega_\varepsilon}P(\rho_\varepsilon)(\tau,\cdot) - P'(\bar\rho)(\rho_\varepsilon-\bar\rho)(\tau,\cdot) - P(\bar\rho)\,\mathrm{d}x\\ &\quad - \int_{\Omega_\varepsilon}P(\rho_{0,\varepsilon}) - P'(\bar\rho)(\rho_{0,\varepsilon}-\bar\rho) - P(\bar\rho)\,\mathrm{d}x.\end{aligned}$$

Moreover, by definition of P (see (2.5)), we have $P''(\rho) = p'(\rho)/\rho \ge 0$ and hence almost everywhere

$$P(\rho_\varepsilon) - P'(\bar\rho)(\rho_\varepsilon-\bar\rho) - P(\bar\rho) \ge 0,$$

see also Lemma 5.1 in [18]. With this at hand, (2.10), (2.12), (2.22), (3.1), and (3.2), we can pass to the limit $\varepsilon\to 0$ in (3.19) to obtain

$$\begin{aligned}&\int_0^T\int_\Omega \frac{1}{2}\xi(\tau)\overline{\rho|\mathbf{u}|^2}(\tau,\cdot)\,\mathrm{d}x\,\mathrm{d}\tau + \mu\int_0^T\int_0^\tau\int_\Omega \xi(\tau)|\nabla_x\mathbf{u}|^2\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}\tau + \mu\int_0^T\int_0^\tau\int_\Omega \xi(\tau)(\mathrm{M}\mathbf{u})\cdot\mathbf{u}\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}\tau\\ &\le \int_\Omega\frac{1}{2}\bar\rho|\mathbf{u}_0|^2\,\mathrm{d}x + \int_0^T\int_0^\tau\int_\Omega \xi(\tau)\bar\rho\mathbf{f}\cdot\mathbf{u}\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}\tau.\end{aligned}$$

Since $\rho_\varepsilon|\mathbf{u}_\varepsilon|^2 = \mathrm{trace}[\rho_\varepsilon\mathbf{u}_\varepsilon\otimes\mathbf{u}_\varepsilon]$ and $\mathrm{trace}[\,\cdot\,]$ is a linear operator, we have for $\mathfrak{R}$ from (3.15) that

$$\mathrm{trace}[\mathfrak{R}] = \overline{\rho|\mathbf{u}|^2} - \bar\rho|\mathbf{u}|^2.$$

As the function ξ is arbitrary, we get

$$\begin{aligned}&\int_\Omega\frac{1}{2}\bar\rho|\mathbf{u}|^2(\tau,\cdot)\,\mathrm{d}x + \int_\Omega\frac{1}{2}\mathrm{trace}[\mathfrak{R}](\tau,\cdot)\,\mathrm{d}x + \mu\int_0^\tau\int_\Omega|\nabla_x\mathbf{u}|^2\,\mathrm{d}x\,\mathrm{d}t + \mu\int_0^\tau\int_\Omega(\mathrm{M}\mathbf{u})\cdot\mathbf{u}\,\mathrm{d}x\,\mathrm{d}t\\ &\le \int_\Omega\frac{1}{2}\bar\rho|\mathbf{u}_0|^2\,\mathrm{d}x + \int_0^\tau\int_\Omega\bar\rho\mathbf{f}\cdot\mathbf{u}\,\mathrm{d}x\,\mathrm{d}t,\end{aligned}$$

showing that $(\bar\rho,\mathbf{u})$ is a dissipative solution of the incompressible Navier-Stokes-Brinkman system (1.5).

□

To finally prove Theorem 2.8, we will make use of the weak-strong uniqueness principle (see, e.g., [1]), which is the content of the following proposition.

Proposition 3.2 (Weak-Strong uniqueness)**.** *Let $(\bar\rho,\boldsymbol{u})$ be a dissipative solution to (1.5) emanating from the initial datum $\boldsymbol{u}_0$ in (2.9). If there exists a strong solution to (1.5) emanating from the same initial datum $\boldsymbol{u}_0$, then it coincides with $(\bar\rho,\boldsymbol{u})$.*

Proof. We divide the proof into two steps. First, we recall the derivation of the relative energy inequality, and subsequently apply it to our setting.

Step 1: Relative Energy Inequality. We test (2.7) with $\boldsymbol{\varphi} = \mathbf{U} \in C_c^1([0,T]\times\Omega;\mathbb{R}^3)$ such that $\mathrm{div}_x\mathbf{U} = 0$, and add on both sides the term $\int_0^\tau \int_\Omega \bar\rho \partial_t \mathbf{U}\cdot\mathbf{U}\,\mathrm{d}x\,\mathrm{d}t$. Using that

$$\int_0^\tau \int_\Omega \bar\rho \partial_t \mathbf{U}\cdot\mathbf{U}\,\mathrm{d}x\,\mathrm{d}t = \left[\int_\Omega \frac{1}{2}\bar\rho|\mathbf{U}|^2\,\mathrm{d}x\right]_{t=0}^{t=\tau},$$

and summing with the energy inequality (2.8), we obtain

$$\begin{aligned}\left[\int_\Omega \frac{1}{2}\bar\rho|\mathbf{u}-\mathbf{U}|^2\,\mathrm{d}x\right]_{t=0}^{t=\tau} &+ \int_\Omega \frac{1}{2}\mathrm{trace}[\mathfrak{R}](\tau,\cdot)\,\mathrm{d}x + \mu\int_0^\tau\int_\Omega |\nabla_x\mathbf{u}|^2\,\mathrm{d}x\,\mathrm{d}t + \mu\int_0^\tau\int_\Omega \mathrm{M}\mathbf{u}\cdot\mathbf{u}\,\mathrm{d}x\,\mathrm{d}t\\ &\le \int_0^\tau\int_\Omega \bar\rho\partial_t\mathbf{U}\cdot\mathbf{U}\,\mathrm{d}x\,\mathrm{d}t + \int_0^\tau\int_\Omega \bar\rho\mathbf{f}\cdot\mathbf{u}\,\mathrm{d}x\,\mathrm{d}t + \mu\int_0^\tau\int_\Omega \mathrm{M}\mathbf{u}\cdot\mathbf{U}\,\mathrm{d}x\,\mathrm{d}t\\ &- \int_0^\tau\int_\Omega \bar\rho\mathbf{u}\cdot\partial_t\mathbf{U} + (\bar\rho\mathbf{u}\otimes\mathbf{u}):\nabla_x\mathbf{U} - \mu\nabla_x\mathbf{u}:\nabla_x\mathbf{U} + \bar\rho\mathbf{f}\cdot\mathbf{U} + \mathfrak{R}:\nabla_x\mathbf{U}\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

By regrouping some terms, we infer

$$\begin{aligned}&\left[\int_\Omega \frac{1}{2}\bar\rho|\mathbf{u}-\mathbf{U}|^2\,\mathrm{d}x\right]_{t=0}^{t=\tau} + \int_\Omega \frac{1}{2}\mathrm{trace}[\mathfrak{R}](\tau,\cdot)\,\mathrm{d}x + \mu\int_0^\tau\int_\Omega \nabla_x\mathbf{u}:\nabla_x(\mathbf{u}-\mathbf{U})\,\mathrm{d}x\,\mathrm{d}t + \mu\int_0^\tau\int_\Omega \mathrm{M}\mathbf{u}\cdot(\mathbf{u}-\mathbf{U})\,\mathrm{d}x\,\mathrm{d}t\\ &\le \int_0^\tau\int_\Omega \bar\rho\partial_t\mathbf{U}\cdot(\mathbf{U}-\mathbf{u}) - (\bar\rho\mathbf{u}\otimes\mathbf{u}):\nabla_x\mathbf{U}\,\mathrm{d}x\,\mathrm{d}t + \int_0^\tau\int_\Omega \bar\rho\mathbf{f}\cdot(\mathbf{u}-\mathbf{U})\,\mathrm{d}x\,\mathrm{d}t - \int_0^\tau\int_\Omega \mathfrak{R}:\nabla_x\mathbf{U}\,\mathrm{d}x\,\mathrm{d}t.\end{aligned} \tag{3.20}$$

By the incompressibility condition (2.6) it follows that

$$\begin{aligned}\int_0^\tau\int_\Omega \bar\rho\mathbf{u}\otimes\mathbf{u}:\nabla_x\mathbf{U}\,\mathrm{d}x\,\mathrm{d}t &= \int_0^\tau\int_\Omega \bar\rho\mathbf{u}\otimes(\mathbf{u}-\mathbf{U}):\nabla_x\mathbf{U}\,\mathrm{d}x\,\mathrm{d}t\\ &= \int_0^\tau\int_\Omega \bar\rho(\mathbf{u}-\mathbf{U})\otimes(\mathbf{u}-\mathbf{U}):\nabla_x\mathbf{U}\,\mathrm{d}x\,\mathrm{d}t + \int_0^\tau\int_\Omega \bar\rho\mathbf{U}\otimes(\mathbf{u}-\mathbf{U}):\nabla_x\mathbf{U}\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

Thus, we rewrite (3.20) to get the relative energy inequality as

$$\begin{aligned}\left[\int_\Omega \frac{1}{2}\bar\rho|\mathbf{u}-\mathbf{U}|^2\,\mathrm{d}x\right]_{t=0}^{t=\tau} &+ \int_\Omega \frac{1}{2}\mathrm{trace}[\mathfrak{R}](\tau,\cdot)\,\mathrm{d}x + \mu\int_0^\tau\int_\Omega |\nabla_x(\mathbf{u}-\mathbf{U})|^2 + \mathrm{M}(\mathbf{u}-\mathbf{U})\cdot(\mathbf{u}-\mathbf{U})\,\mathrm{d}x\,\mathrm{d}t\\ &\le \int_0^\tau\int_\Omega \bar\rho(\partial_t\mathbf{U} + \mathbf{U}\cdot\nabla_x\mathbf{U})\cdot(\mathbf{U}-\mathbf{u})\,\mathrm{d}x\,\mathrm{d}t + \int_0^\tau\int_\Omega \bar\rho\mathbf{f}\cdot(\mathbf{u}-\mathbf{U})\,\mathrm{d}x\,\mathrm{d}t\\ &- \int_0^\tau\int_\Omega \mathfrak{R}:\nabla_x\mathbf{U}\,\mathrm{d}x\,\mathrm{d}t - \int_0^\tau\int_\Omega \bar\rho(\mathbf{u}-\mathbf{U})\otimes(\mathbf{u}-\mathbf{U}):\nabla_x\mathbf{U}\,\mathrm{d}x\,\mathrm{d}t\\ &- \mu\int_0^\tau\int_\Omega \nabla_x\mathbf{U}:\nabla_x(\mathbf{u}-\mathbf{U})\,\mathrm{d}x\,\mathrm{d}t - \mu\int_0^\tau\int_\Omega \mathrm{M}\mathbf{U}\cdot(\mathbf{u}-\mathbf{U})\,\mathrm{d}x\,\mathrm{d}t.\end{aligned} \tag{3.21}$$

By density we can extend (3.21) to any $\mathbf{U} \in L^q(0,T;W^{1,q}(\Omega;\mathbb{R}^3)) \cap W^{1,q}(0,T;L^q(\Omega;\mathbb{R}^3))$ satisfying $\mathrm{div}_x\mathbf{U} = 0$ and $\mathbf{U}|_{\partial\Omega} = 0$, provided $q > 1$ is large enough such that all occurring integrals are well-defined.

Step 2: Weak-strong uniqueness. Let $\hat{\mathbf{u}}$ be a strong solution to (1.5) emanating from the initial datum $\mathbf{u}_0$. Then, since $\mathrm{div}_x(\mathbf{u}-\hat{\mathbf{u}}) = 0$ a.e. in $(0,T)\times\Omega$, it holds

$$\int_0^\tau\int_\Omega (\bar\rho\partial_t\hat{\mathbf{u}} + \bar\rho\hat{\mathbf{u}}\cdot\nabla_x\hat{\mathbf{u}})\cdot(\hat{\mathbf{u}}-\mathbf{u})\,\mathrm{d}x\,\mathrm{d}t = \int_0^\tau\int_\Omega (-\mu\mathrm{M}\hat{\mathbf{u}} + \mu\Delta_x\hat{\mathbf{u}} + \bar\rho\mathbf{f})\cdot(\hat{\mathbf{u}}-\mathbf{u})\,\mathrm{d}x\,\mathrm{d}t.$$

Combining this with (3.21) for $\mathbf{U} = \hat{\mathbf{u}}$, we obtain

$$\left[\int_\Omega \frac{1}{2}\bar{\rho}|\mathbf{u}-\hat{\mathbf{u}}|^2\,\mathrm{d}x\right]_{t=0}^{t=\tau} + \int_\Omega \frac{1}{2}\mathrm{trace}[\mathfrak{R}](\tau,\cdot)\,\mathrm{d}x + \mu\int_0^\tau\int_\Omega |\nabla_x(\mathbf{u}-\hat{\mathbf{u}})|^2 + \mathrm{M}(\mathbf{u}-\hat{\mathbf{u}})\cdot(\mathbf{u}-\hat{\mathbf{u}})\,\mathrm{d}x\,\mathrm{d}t$$
$$\leq \int_0^\tau\int_\Omega (-\mu\mathrm{M}\hat{\mathbf{u}} + \mu\Delta_x\hat{\mathbf{u}} + \bar{\rho}\mathbf{f})\cdot(\hat{\mathbf{u}}-\mathbf{u})\,\mathrm{d}x\,\mathrm{d}t + \int_0^\tau\int_\Omega \bar{\rho}\mathbf{f}\cdot(\mathbf{u}-\hat{\mathbf{u}})\,\mathrm{d}x\,\mathrm{d}t$$
$$- \int_0^\tau\int_\Omega \mathfrak{R} : \nabla_x\hat{\mathbf{u}}\,\mathrm{d}x\,\mathrm{d}t - \int_0^\tau\int_\Omega \bar{\rho}(\mathbf{u}-\hat{\mathbf{u}})\otimes(\mathbf{u}-\hat{\mathbf{u}}) : \nabla_x\hat{\mathbf{u}}\,\mathrm{d}x\,\mathrm{d}t$$
$$- \mu\int_0^\tau\int_\Omega \nabla_x\hat{\mathbf{u}} : \nabla_x(\mathbf{u}-\hat{\mathbf{u}})\,\mathrm{d}x\,\mathrm{d}t - \mu\int_0^\tau\int_\Omega \mathrm{M}\hat{\mathbf{u}}\cdot(\mathbf{u}-\hat{\mathbf{u}})\,\mathrm{d}x\,\mathrm{d}t,$$

which in turn reduces to

$$\int_\Omega \frac{1}{2}\bar{\rho}|\mathbf{u}-\hat{\mathbf{u}}|^2(\tau,\cdot)\,\mathrm{d}x + \int_\Omega \frac{1}{2}\mathrm{trace}[\mathfrak{R}](\tau,\cdot)\,\mathrm{d}x + \mu\int_0^\tau\int_\Omega |\nabla_x\mathbf{u}-\nabla_x\hat{\mathbf{u}}|^2 + \mathrm{M}(\mathbf{u}-\hat{\mathbf{u}})\cdot(\mathbf{u}-\hat{\mathbf{u}})\,\mathrm{d}x\,\mathrm{d}t$$
$$\leq -\int_0^\tau\int_\Omega \mathfrak{R} : \nabla_x\hat{\mathbf{u}}\,\mathrm{d}x\,\mathrm{d}t - \int_0^\tau\int_\Omega \bar{\rho}(\mathbf{u}-\hat{\mathbf{u}})\otimes(\mathbf{u}-\hat{\mathbf{u}}) : \nabla_x\hat{\mathbf{u}}\,\mathrm{d}x\,\mathrm{d}t.$$

Being the second and the third term on the left-hand side both non-negative, we infer

$$\int_\Omega \frac{1}{2}\bar{\rho}|\mathbf{u}-\hat{\mathbf{u}}|^2(\tau,\cdot)\,\mathrm{d}x + \int_\Omega \frac{1}{2}\mathrm{trace}[\mathfrak{R}](\tau,\cdot)\,\mathrm{d}x \leq -\int_0^\tau\int_\Omega \mathfrak{R} : \nabla_x\hat{\mathbf{u}}\,\mathrm{d}x\,\mathrm{d}t - \int_0^\tau\int_\Omega \bar{\rho}(\mathbf{u}-\hat{\mathbf{u}})\otimes(\mathbf{u}-\hat{\mathbf{u}}) : \nabla_x\hat{\mathbf{u}}\,\mathrm{d}x\,\mathrm{d}t,$$

and thus

$$\int_\Omega \frac{1}{2}\bar{\rho}|\mathbf{u}-\hat{\mathbf{u}}|^2(\tau,\cdot)\,\mathrm{d}x + \int_\Omega \frac{1}{2}\mathrm{trace}[\mathfrak{R}](\tau,\cdot)\,\mathrm{d}x \leq \|\nabla_x\hat{\mathbf{u}}\|_{L^\infty((0,\tau)\times\Omega;\mathbb{R}^3)}\int_0^\tau\int_\Omega |\mathfrak{R}|\,\mathrm{d}x\,\mathrm{d}t$$
$$+ \|\nabla_x\hat{\mathbf{u}}\|_{L^\infty((0,\tau)\times\Omega;\mathbb{R}^3)}\int_0^\tau\int_\Omega \bar{\rho}|\mathbf{u}-\hat{\mathbf{u}}|^2\,\mathrm{d}x\,\mathrm{d}t,$$

for a.e. $\tau \in [0,T]$. Eventually, since $\mathfrak{R}$ is positive semidefinite, it follows $|\mathfrak{R}| \lesssim \mathrm{trace}[\mathfrak{R}]$ and thus by Grönwall's lemma we deduce $\mathbf{u} = \hat{\mathbf{u}}$ and $\mathfrak{R} = 0$. This finishes the proof of Theorem 2.8. □

4 Application to randomly perforated domains

This section is devoted to provide an application of Theorem 2.8. More precisely, we consider a family of randomly perforated domains $(\Omega_\varepsilon)_{\varepsilon>0}$, whose holes are balls distributed through a Poisson point process, and whose radii are identically and independently distributed (i.i.d.) random variables that scale as ε^3. Note that this corresponds to the randomized counterpart of our initial example of domains given in (1.1). We will show that, if $\mathrm{Ma}(\varepsilon)$ satisfies suitable assumptions (see (4.3) below), then $((\Omega_\varepsilon)_{\varepsilon>0},\Omega)$ satisfy (M1) and (M2). Thus, in particular, Theorem 2.8 applies. Moreover, in this way we are able to generalize the outcomes of [10] to the time-dependent setting.

We let $\Omega \subset \mathbb{R}^3$ be a smoothly bounded open set which is star-shaped with respect to the origin. We then define the family of randomly perforated domains $(\Omega_\varepsilon)_{\varepsilon>0}$ as

$$\Omega_\varepsilon := \Omega \setminus H^\varepsilon, \quad H^\varepsilon := \bigcup_{I_\varepsilon} T_i^\varepsilon = \bigcup_{I_\varepsilon} B_{\varepsilon^3 r_i}(\varepsilon z_i), \quad I_\varepsilon := \{z_i \in \Phi \cap \varepsilon^{-1}\Omega : \overline{B_{\varepsilon^3 r_i}(\varepsilon z_i)} \subset \Omega\}, \tag{4.1}$$

where Φ is a Poisson point process on $\mathbb{R}^3$ with homogeneous intensity rate $\lambda > 0$. Such a process is characterized by the following two properties:

- For each measurable set $S \subset \mathbb{R}^3$ with finite measure, the probability of finding exactly n points from Φ inside S is $(\lambda|S|)^n\exp(-\lambda|S|)/n!$;
- for two disjoint measurable sets $S_1, S_2 \subset \mathbb{R}^3$, the random variables $S_1 \cap \Phi$ and $S_2 \cap \Phi$ are independent.

The radii $\mathcal{R} := \{r_i\}_{z_i \in \Phi} \subset [R_0, \infty)$ for some $R_0 > 0$ are i.i.d. random variables. Further, we assume that there exists a constant $\beta > 0$ such that the radii r_i satisfy the moment bound

$$\langle r^{1+\beta} \rangle < \infty, \tag{4.2}$$

where $\langle f \rangle$ is the expected value of the random variable f. Let $(\tilde{\Omega}, \mathcal{F}, \mathbb{P})$ be a probability space associated to the marked point process $(\Phi, \mathcal{R})$, i.e., the joint process of centers and radii distributed as above. For more details on marked Poisson point processes, we refer the reader to [25].

Theorem 4.1. *Let $(\Omega_\varepsilon)_{\varepsilon>0}$ be as in (4.1). Then there exists $\delta = \delta(\beta) > 0$ and an exponent $p(\delta, \gamma) > 0$ such that, if $\mathrm{Ma}(\varepsilon)$ vanishes fast enough such that*

$$\limsup_{\varepsilon \to 0} \varepsilon^{-p(\delta,\gamma)} \mathrm{Ma}(\varepsilon)^{\min\{1, \frac{2}{\gamma}\}} = 0, \tag{4.3}$$

then $((\Omega_\varepsilon)_{\varepsilon>0}, \Omega)$ satisfy (M1) and (M2). The exponent $p(\delta, \gamma)$ can be taken to be

$$p(\delta, \gamma) = \max\left\{3,\ 1 + \frac{6}{\gamma},\ 3\frac{6-\gamma}{2\gamma - 3}\right\}.$$

The rest of this section is devoted to the proof of Theorem 4.1. Clearly the family $((\Omega_\varepsilon)_{\varepsilon>0}, \Omega)$ with $(\Omega_\varepsilon)_{\varepsilon>0}$ given in (4.1) satisfies (M1). Thus, we have to prove that also (M2) is satisfied, that is (2.13)–(2.16) and (2.21)-(2.22) hold true. We start with a general result Lemma 4.3 (see Section 4.1) which says that if $(\Omega_\varepsilon)_{\varepsilon>0}$ is a family of perforated domains such that (M1) and (H1)–(H5) below hold, then also (2.22) holds. Afterwards we prove that (H1)–(H5) are satisfied when $(\Omega_\varepsilon)_{\varepsilon>0}$ are defined as in (4.1) and $\mathrm{Ma}(\varepsilon)$ obeys (4.3) (see Section 4.2). Eventually, for a given $\boldsymbol{\phi} \in C_c^\infty(\Omega; \mathbb{R}^3)$ with $\mathrm{div}_x \boldsymbol{\phi} = 0$ we construct a family of test functions with the properties (2.13)–(2.16) and (2.21) (see Section 4.3).

Remark 4.2. The authors believe that the assumption $\mathcal{R} \subset [R_0, \infty)$ for some $R_0 > 0$ can be relaxed to

$$\langle r^{-m} \rangle \le C < \infty,$$

for some $C > 0$ and any $m > 0$. However, we wish not to unnecessarily complicate the calculations, hence we focus on the case where $r_i \ge R_0 > 0$ for any i.

4.1 General perforated domains and special test functions

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with smooth boundary. Let $N(\varepsilon) \in \mathbb{N}$, $N(\varepsilon) \to \infty$ as $\varepsilon \to 0$, and let $(T_i^\varepsilon)_{1 \le i \le N(\varepsilon)} \subset \Omega$ be a family of closed smooth sets with non-empty interior (the holes). We consider the family of perforated domains $(\Omega_\varepsilon)_{\varepsilon>0}$ given by

$$\Omega_\varepsilon := \Omega \setminus \bigcup_{i=1}^{N(\varepsilon)} T_i^\varepsilon. \tag{4.4}$$

We assume that the holes T_i^ε are such that (M1) holds. Moreover, we assume that there exist pairs of functions $(\boldsymbol{\omega}_k^\varepsilon, \boldsymbol{\mu}_k)_{1 \le k \le 3}$ with the following properties:

(H1) $\boldsymbol{\omega}_k^\varepsilon \in W^{1,2}(\Omega; \mathbb{R}^3)$;

(H2) $\mathrm{div}_x(\boldsymbol{\omega}_k^\varepsilon) = 0$ in Ω and $\boldsymbol{\omega}_k^\varepsilon = 0$ in the holes T_i^ε;

(H3) $\boldsymbol{\omega}_k^\varepsilon \rightharpoonup \mathbf{e}_k$ weakly in $W^{1,2}(\Omega; \mathbb{R}^3)$;

(H4) $\boldsymbol{\mu}_k \in W^{-1,\infty}(\Omega; \mathbb{R}^3)$;

(H5) For all $\mathbf{v} \in L^2(0,T;W^{1,2}(\Omega;\mathbb{R}^3))$, $\mathbf{v}_\varepsilon \in L^2(0,T;W^{1,2}(\Omega_\varepsilon;\mathbb{R}^3))$ that satisfy (2.17)–(2.20), there holds

$$\lim_{\varepsilon\to 0}\int_0^\tau\int_{\Omega_\varepsilon}\nabla_x\boldsymbol{\omega}_k^\varepsilon : \nabla_x(\psi\mathbf{v}_\varepsilon)\,\mathrm{d}x\,\mathrm{d}t = \int_0^\tau\int_\Omega \boldsymbol{\mu}_k\psi\mathbf{v}\,\mathrm{d}x\,\mathrm{d}t,$$

for every $\psi \in C_c^\infty((0,T)\times\Omega)$ and a.e. $\tau\in[0,T]$.

Given the hypotheses (H1)–(H5), we can show:

Lemma 4.3. *Let $\Omega\subset\mathbb{R}^3$ be a smoothly bounded domain, and let Ω_ε be as in (4.4). Assume that (M1) and (H1)–(H5) are satisfied for $((\Omega_\varepsilon)_{\varepsilon>0},\Omega)$ and some $(\boldsymbol{\omega}_k^\varepsilon,\boldsymbol{\mu}_k)_{1\le k\le 3}$. Then for all $\boldsymbol{v}\in L^2(0,T;W_0^{1,2}(\Omega;\mathbb{R}^3))$, $\boldsymbol{v}_\varepsilon\in L^2(0,T;W_0^{1,2}(\Omega_\varepsilon;\mathbb{R}^3))$ that satisfy (2.17)–(2.20), we have*

$$\liminf_{\varepsilon\to 0}\int_0^\tau\int_{\Omega_\varepsilon}|\nabla_x\boldsymbol{v}_\varepsilon|^2\,\mathrm{d}x\,\mathrm{d}t \ge \int_0^\tau\int_\Omega|\nabla_x\boldsymbol{v}|^2\,\mathrm{d}x\,\mathrm{d}t + \int_0^\tau\int_\Omega(\mathrm{M}\boldsymbol{v})\cdot\boldsymbol{v}\,\mathrm{d}x\,\mathrm{d}t \quad \forall\tau\in[0,T],$$

with $\mathrm{M}_{ij}=\mu_i^j=\boldsymbol{\mu}_i\cdot\boldsymbol{e}_j$.

Proof. The proof we give here is the same for $d=3$ and $d\ge 3$, so let us prove the Lemma in the general setting with obvious changes on the assumptions if $d>3$. Let $\boldsymbol{\varphi}=(\varphi_1,\dots,\varphi_d)\in C_c^\infty((0,T)\times\Omega;\mathbb{R}^d)$. Then

$$\begin{aligned}
0 &\le \int_0^\tau\int_\Omega|\nabla_x(\mathbf{v}_\varepsilon-\sum_{k=1}^d\varphi_k\boldsymbol{\omega}_k^\varepsilon)|^2\,\mathrm{d}x\,\mathrm{d}t\\
&= \int_0^\tau\int_\Omega|\nabla_x\mathbf{v}_\varepsilon|^2\,\mathrm{d}x\,\mathrm{d}t + \sum_{1\le i,k\le d}\Big[\int_0^\tau\int_\Omega\varphi_k\varphi_i\nabla_x\boldsymbol{\omega}_k^\varepsilon:\nabla_x\boldsymbol{\omega}_i^\varepsilon\,\mathrm{d}x\,\mathrm{d}t + \int_0^\tau\int_\Omega(\nabla_x\varphi_k\otimes\boldsymbol{\omega}_k^\varepsilon):(\nabla_x\varphi_i\otimes\boldsymbol{\omega}_i^\varepsilon)\,\mathrm{d}x\,\mathrm{d}t\Big]\\
&\quad + 2\sum_{1\le i,k\le d}\int_0^\tau\int_\Omega\varphi_k\nabla_x\boldsymbol{\omega}_k^\varepsilon:(\nabla_x\varphi_i\otimes\boldsymbol{\omega}_i^\varepsilon)\,\mathrm{d}x\,\mathrm{d}t - 2\sum_{1\le k\le d}\int_0^\tau\int_\Omega\nabla_x\mathbf{v}_\varepsilon:(\nabla_x\varphi_k\otimes\boldsymbol{\omega}_k^\varepsilon)\,\mathrm{d}x\,\mathrm{d}t\\
&\quad - 2\sum_{1\le k\le d}\int_0^\tau\int_\Omega\nabla_x\boldsymbol{\omega}_k^\varepsilon:(\varphi_k\nabla_x\mathbf{v}_\varepsilon)\,\mathrm{d}x\,\mathrm{d}t\\
&= \int_0^\tau\int_\Omega|\nabla_x\mathbf{v}_\varepsilon|^2\,\mathrm{d}x\,\mathrm{d}t + \sum_{1\le i,k\le d}\int_0^\tau\int_\Omega\nabla_x\boldsymbol{\omega}_k^\varepsilon:\nabla_x(\varphi_k\varphi_i\boldsymbol{\omega}_i^\varepsilon)\,\mathrm{d}x\,\mathrm{d}t\\
&\quad - \sum_{1\le i,k\le d}\int_0^\tau\int_\Omega\nabla_x\boldsymbol{\omega}_k^\varepsilon:(\boldsymbol{\omega}_i^\varepsilon\otimes\nabla_x(\varphi_k\varphi_i))\,\mathrm{d}x\,\mathrm{d}t + \sum_{1\le i,k\le d}\int_0^\tau\int_\Omega(\nabla_x\varphi_k\otimes\boldsymbol{\omega}_k^\varepsilon):(\nabla_x\varphi_i\otimes\boldsymbol{\omega}_i^\varepsilon)\,\mathrm{d}x\,\mathrm{d}t\\
&\quad + 2\sum_{1\le i,k\le d}\int_0^\tau\int_\Omega\varphi_k\nabla_x\boldsymbol{\omega}_k^\varepsilon:(\nabla_x\varphi_i\otimes\boldsymbol{\omega}_i^\varepsilon)\,\mathrm{d}x\,\mathrm{d}t - 2\sum_{1\le k\le d}\int_0^\tau\int_\Omega\nabla_x\mathbf{v}_\varepsilon:(\nabla_x\varphi_k\otimes\boldsymbol{\omega}_k^\varepsilon)\,\mathrm{d}x\,\mathrm{d}t\\
&\quad - 2\sum_{1\le k\le d}\int_0^\tau\int_\Omega\nabla_x\boldsymbol{\omega}_k^\varepsilon:\nabla_x(\varphi_k\mathbf{v}_\varepsilon)\,\mathrm{d}x\,\mathrm{d}t + 2\sum_{1\le k\le d}\int_0^\tau\int_\Omega\nabla_x\boldsymbol{\omega}_k^\varepsilon:(\nabla_x\varphi_k\otimes\mathbf{v}_\varepsilon)\,\mathrm{d}x\,\mathrm{d}t. \qquad (4.5)
\end{aligned}$$

Now, we use (H1)–(H3) to infer that we can choose $\mathbf{v}_\varepsilon=\boldsymbol{\omega}_k^\varepsilon$ in (H5) to obtain

$$\int_0^\tau\int_\Omega\nabla_x\boldsymbol{\omega}_k^\varepsilon:\nabla_x(\varphi_k\varphi_i\boldsymbol{\omega}_i^\varepsilon)\,\mathrm{d}x\,\mathrm{d}t \;\to\; \int_0^\tau\int_\Omega\boldsymbol{\mu}_k\cdot(\varphi_k\varphi_i\mathbf{e}_i)\,\mathrm{d}x\,\mathrm{d}t.$$

Since $\boldsymbol{\omega}_k^\varepsilon\rightharpoonup\mathbf{e}_k$ weakly in $W^{1,2}(\Omega;\mathbb{R}^d)$, we get $\boldsymbol{\omega}_k^\varepsilon\to\mathbf{e}_k$ strongly in $L^2(\Omega;\mathbb{R}^d)$ such that

$$\begin{aligned}
\int_0^\tau\int_\Omega\nabla_x\boldsymbol{\omega}_k^\varepsilon:(\boldsymbol{\omega}_i^\varepsilon\otimes\nabla_x(\varphi_k\varphi_i))\,\mathrm{d}x\,\mathrm{d}t &\;\to\; 0,\\
\int_0^\tau\int_\Omega(\nabla_x\varphi_k\otimes\boldsymbol{\omega}_k^\varepsilon):(\nabla_x\varphi_i\otimes\boldsymbol{\omega}_i^\varepsilon)\,\mathrm{d}x\,\mathrm{d}t &\;\to\; \int_0^\tau\int_\Omega(\nabla_x\varphi_k\otimes\mathbf{e}_k):(\nabla_x\varphi_i\otimes\mathbf{e}_i)\,\mathrm{d}x\,\mathrm{d}t,\\
\int_0^\tau\int_\Omega\varphi_k\nabla_x\boldsymbol{\omega}_k^\varepsilon:(\nabla_x\varphi_i\otimes\boldsymbol{\omega}_i^\varepsilon)\,\mathrm{d}x\,\mathrm{d}t &\;\to\; 0.
\end{aligned}$$

With $\mathbf{v}_\varepsilon \rightharpoonup \mathbf{v}$ weakly in $L^2(0,T;W_0^{1,2}(\Omega;\mathbb{R}^d))$ and $\boldsymbol{\omega}_k^\varepsilon \to \mathbf{e}_k$ strongly in $L^2(\Omega;\mathbb{R}^d)$, we infer

$$\int_0^\tau \int_\Omega \nabla_x \mathbf{v}_\varepsilon : (\nabla_x \varphi_k \otimes \boldsymbol{\omega}_k^\varepsilon)\,\mathrm{d}x\,\mathrm{d}t \;\to\; \int_0^\tau \int_\Omega \nabla_x \mathbf{v} : (\nabla_x \varphi_k \otimes \mathbf{e}_k)\,\mathrm{d}x\,\mathrm{d}t.$$

Due to (H5), we see that

$$\int_0^\tau \int_\Omega \nabla_x \boldsymbol{\omega}_k^\varepsilon : \nabla_x(\varphi_k \mathbf{v}_\varepsilon)\,\mathrm{d}x\,\mathrm{d}t \;\to\; \int_0^\tau \int_\Omega \boldsymbol{\mu}_k \cdot (\varphi_k \mathbf{v})\,\mathrm{d}x\,\mathrm{d}t,$$

since $\mathbf{v}_\varepsilon \rightharpoonup \mathbf{v}$ weakly in $L^2(0,T;L^6(\Omega;\mathbb{R}^d))$. Moreover, boundedness of $\mathbf{v}_\varepsilon$ in $L^2(0,T;W_0^{1,2}(\Omega;\mathbb{R}^d))$ and the fact that $\boldsymbol{\omega}_k^\varepsilon \to \mathbf{e}_k$ strongly in $L^2(\Omega;\mathbb{R}^d)$ leads by partial integration to

$$\int_0^\tau \int_\Omega \nabla_x \boldsymbol{\omega}_k^\varepsilon : (\nabla_x \varphi_k \otimes \mathbf{v}_\varepsilon)\,\mathrm{d}x\,\mathrm{d}t = -\int_0^\tau \int_\Omega (\boldsymbol{\omega}_k^\varepsilon - \mathbf{e}_k) \cdot \mathrm{div}_x(\nabla_x \varphi_k \otimes \mathbf{v}_\varepsilon)\,\mathrm{d}x\,\mathrm{d}t \to 0.$$

Next, it is clear that there exists a subsequence, still denoted by $\mathbf{v}_\varepsilon$, such that

$$\lim_{\varepsilon\to 0} \int_0^\tau \int_\Omega |\nabla_x \mathbf{v}_\varepsilon|^2\,\mathrm{d}x\,\mathrm{d}t = \liminf_{\varepsilon\to 0} \int_0^\tau \int_\Omega |\nabla_x \mathbf{v}_\varepsilon|^2\,\mathrm{d}x\,\mathrm{d}t.$$

Passing to the limits in (4.5) leads to

$$\begin{aligned} 0 \le \liminf_{\varepsilon\to 0} \int_0^\tau \int_\Omega |\nabla_x \mathbf{v}_\varepsilon|^2\,\mathrm{d}x\,\mathrm{d}t + \sum_{1\le k\le d} \int_0^\tau \int_\Omega \boldsymbol{\mu}_k \cdot (\varphi_k \boldsymbol{\varphi})\,\mathrm{d}x\,\mathrm{d}t \\ + \int_0^\tau \int_\Omega |\nabla_x \boldsymbol{\varphi}|^2\,\mathrm{d}x\,\mathrm{d}t - 2\int_0^\tau \int_\Omega \nabla_x \mathbf{v} : \nabla_x \boldsymbol{\varphi}\,\mathrm{d}x\,\mathrm{d}t - 2\sum_{1\le k\le d} \int_0^\tau \int_\Omega \boldsymbol{\mu}_k \cdot (\varphi_k \mathbf{v})\,\mathrm{d}x\,\mathrm{d}t. \end{aligned}$$

Now we choose a sequence $\boldsymbol{\varphi}^{(n)} \in C_c^\infty((0,T)\times\Omega;\mathbb{R}^d)$ with $\boldsymbol{\varphi}^{(n)} \to \mathbf{v}$ strongly in $L^2(0,T;W_0^{1,2}(\Omega;\mathbb{R}^d))$ and pass to the limit to obtain

$$\begin{aligned} 0 \le \liminf_{\varepsilon\to 0} \int_0^\tau \int_\Omega |\nabla_x \mathbf{v}_\varepsilon|^2\,\mathrm{d}x\,\mathrm{d}t + \int_0^\tau \int_\Omega (\mathrm{M}\mathbf{v}) \cdot \mathbf{v}\,\mathrm{d}x\,\mathrm{d}t + \int_0^\tau \int_\Omega |\nabla_x \mathbf{v}|^2\,\mathrm{d}x\,\mathrm{d}t \\ - 2\int_0^\tau \int_\Omega |\nabla_x \mathbf{v}|^2\,\mathrm{d}x\,\mathrm{d}t - 2\int_0^\tau \int_\Omega (\mathrm{M}\mathbf{v}) \cdot \mathbf{v}\,\mathrm{d}x\,\mathrm{d}t. \end{aligned}$$

Rearranging leads to

$$\int_0^\tau \int_\Omega |\nabla_x \mathbf{v}|^2\,\mathrm{d}x\,\mathrm{d}t + \int_0^\tau \int_\Omega (\mathrm{M}\mathbf{v}) \cdot \mathbf{v}\,\mathrm{d}x\,\mathrm{d}t \le \liminf_{\varepsilon\to 0} \int_0^\tau \int_\Omega |\nabla_x \mathbf{v}_\varepsilon|^2\,\mathrm{d}x\,\mathrm{d}t.$$

□

4.2 Validity of (H1)–(H5) in the random setting

Coming from the rather general class of domains (4.4) back to randomly perforated domains, from now on let $(\Omega_\varepsilon)_{\varepsilon>0}$ be defined by (4.1). We then construct a family $(\boldsymbol{\omega}_k^\varepsilon, \boldsymbol{\mu}_k)_{1\le k\le 3}$ verifying (H1)–(H5) under the assumption (4.3). We start by collecting some useful notation.

For a Poisson point process Φ on $\mathbb{R}^3$ and any bounded set $E \subset \mathbb{R}^3$ that is star-shaped with respect to the origin, we define the random variables

$$\Phi(E) := \Phi \cap E, \quad \Phi^\varepsilon(E) := \Phi \cap (\varepsilon^{-1}E), \quad \mathcal{N}(E) := \#\Phi(E), \quad \mathcal{N}^\varepsilon(E) := \#\Phi^\varepsilon(E).$$

For each $\eta > 0$ we define the thinned process as

$$\Phi_\eta := \{x \in \Phi \colon \min_{y\in\Phi, y\neq x} |x-y| \geq \eta\},$$

and $\Phi_\eta(E)$, $\Phi_\eta^\varepsilon(E)$, $\mathcal{N}_\eta(E)$, $\mathcal{N}_\eta^\varepsilon(E)$ accordingly. Furthermore, we recall two lemmas ([20, Lemmas 3.1 and 3.2]) which guarantee that the holes H^ε can be decomposed as $H^\varepsilon = H_{\mathrm{g}}^\varepsilon \cup H_{\mathrm{b}}^\varepsilon$, where

- $H_{\mathrm{g}}^\varepsilon$ contains the good holes, which are small and well separated, and
- $H_{\mathrm{b}}^\varepsilon$ contains the bad holes, which are big and possibly overlapping.

Lemma 4.4 ([20, Lemma 3.1]). *There exists $\delta = \delta(\beta) > 0$ such that for almost every $\omega \in \tilde{\Omega}$ and all $\varepsilon \leq \varepsilon_0 = \varepsilon_0(\omega)$, there exists a partition $H^\varepsilon = H_{\mathrm{g}}^\varepsilon \cup H_{\mathrm{b}}^\varepsilon$ and a set $\Omega_{\mathrm{b}}^\varepsilon \subset \mathbb{R}^3$ such that $H_{\mathrm{b}}^\varepsilon \subset \Omega_{\mathrm{b}}^\varepsilon$ and*

$$\mathrm{dist}(H_{\mathrm{g}}^\varepsilon; \Omega_{\mathrm{b}}^\varepsilon) > \varepsilon^{1+\delta}, \quad |\Omega_{\mathrm{b}}^\varepsilon| \to 0.$$

Furthermore, $H_{\mathrm{g}}^\varepsilon$ is a union of disjoint balls centered in $n^\varepsilon \subset \Phi^\varepsilon(\Omega)$, namely

$$H_{\mathrm{g}}^\varepsilon = \bigcup_{z_i \in n^\varepsilon} B_{\varepsilon^3 r_i}(\varepsilon z_i), \quad \varepsilon^3 \# n^\varepsilon \to \lambda|\Omega|, \quad \min_{z_i \neq z_j \in n^\varepsilon} \varepsilon|z_i - z_j| \geq 2\varepsilon^{1+\frac{\delta}{2}}, \quad \varepsilon^3 r_i \leq \varepsilon^{1+2\delta}. \tag{4.6}$$

Finally, for any $\eta > 0$ we have

$$\lim_{\varepsilon\to 0} \varepsilon^3 \#(\{z_i \in \Phi_{2\eta}^\varepsilon(\Omega) \mid \mathrm{dist}(\varepsilon z_i, \Omega_{\mathrm{b}}^\varepsilon) \leq \eta\varepsilon\}) = 0.$$

We write $\mathcal{I}^\varepsilon := \Phi^\varepsilon(\Omega) \setminus n^\varepsilon$, i.e., the set of centers of the balls in $H_{\mathrm{b}}^\varepsilon$.

Lemma 4.5 ([20, Lemma 3.2]). *Let $\theta > 1$ be fixed. Then, for almost every $\omega \in \tilde{\Omega}$ and $\varepsilon \leq \varepsilon_0(\omega, \beta, \theta)$, we may choose $H_{\mathrm{g}}^\varepsilon$, $H_{\mathrm{b}}^\varepsilon$ of Lemma 4.4 in such a way that the following holds:*

- *There exist $\Lambda(\beta) > 1$, a sub-collection $J^\varepsilon \subset \mathcal{I}^\varepsilon$, and constants $\{\lambda_l^\varepsilon\}_{z_l \in J^\varepsilon} \subset [1, \Lambda]$ such that*

$$H_{\mathrm{b}}^\varepsilon \subset \bar{H}_{\mathrm{b}}^\varepsilon := \bigcup_{z_j \in J^\varepsilon} B_{\lambda_j^\varepsilon \varepsilon^3 r_j}(\varepsilon z_j), \quad \lambda_j^\varepsilon \varepsilon^3 r_j \leq \Lambda \varepsilon^{6\delta}.$$

- *There exists $k_{\max} = k_{\max}(\beta) > 0$ such that we may partition*

$$\mathcal{I}^\varepsilon = \bigcup_{k=-3}^{k_{\max}} \mathcal{I}_k^\varepsilon, \qquad J^\varepsilon = \bigcup_{k=-3}^{k_{\max}} J_k^\varepsilon,$$

  with $J_k^\varepsilon \subset \mathcal{I}_k^\varepsilon$ for all $k = 1, ..., k_{\max}$ and

$$\bigcup_{z_i \in \mathcal{I}_k^\varepsilon} B_{\varepsilon^3 r_i}(\varepsilon z_i) \subset \bigcup_{z_i \in J_k^\varepsilon} B_{\lambda_i^\varepsilon \varepsilon^3 r_i}(\varepsilon z_i);$$

- *For all $k = -3, ..., k_{\max}$ and every $z_i, z_j \in J_k^\varepsilon$, $z_i \neq z_j$,*

$$B_{\theta^2 \lambda_i^\varepsilon \varepsilon^3 r_i}(\varepsilon z_i) \cap B_{\theta^2 \lambda_j^\varepsilon \varepsilon^3 r_j}(\varepsilon z_j) = \emptyset;$$

- *For each $k = -3, ..., k_{\max}$ and $z_i \in \mathcal{I}_k^\varepsilon$, and for all $z_j \in \bigcup_{l=-3}^{k-1} J_l^\varepsilon$ we have*

$$B_{\varepsilon^3 r_i}(\varepsilon z_i) \cap B_{\theta \lambda_j^\varepsilon \varepsilon^3 r_j}(\varepsilon z_j) = \emptyset.$$

Finally, the set $\Omega_{\mathrm{b}}^{\varepsilon}$ *of Lemma 4.4 may be chosen as*

$$\Omega_{\mathrm{b}}^{\varepsilon} = \bigcup_{z_i \in J^{\varepsilon}} B_{\theta \lambda_i^{\varepsilon} \varepsilon^3 r_i}(\varepsilon z_i).$$

Let us moreover recall the Strong Law of Large Numbers (in our particular setting), which can be found, e.g., in [25, Theorem 8.14].

Lemma 4.6 (Strong Law of Large Numbers)**.** *Let* $E \subset \mathbb{R}^3$ *be a measurable bounded set, and* $(\Phi, \{r_i\}_{z_i \in \Phi})$ *be a marked Poisson point process with*

$$\langle r^m \rangle < \infty$$

for some $m > 0$*. Then, almost surely,*

$$\lim_{\varepsilon \to 0} \varepsilon^3 \mathcal{N}(\varepsilon^{-1} E) = \lambda |E|, \qquad \lim_{\varepsilon \to 0} \varepsilon^3 \sum_{z_i \in \varepsilon^{-1} E} r_i^m = \lambda \langle r^m \rangle |E|.$$

In the following, we will extend each function which is defined on Ω_ε by zero in H^ε. For $r > 0$ we set $B_r := B_r(0) \subset \mathbb{R}^3$. For each $z_i \in n^\varepsilon$ we define

$$a_{\varepsilon,i} := \varepsilon^3 r_i, \quad d_{\varepsilon,i} := \min \Big\{ \mathrm{dist}(\varepsilon z_i, \Omega_{\mathrm{b}}^{\varepsilon}), \ \frac{1}{2} \min_{\substack{z_j \in n^\varepsilon \\ z_j \neq z_i}} (\varepsilon |z_i - z_j|), \ \varepsilon \Big\}.$$

Since $z_i \in n^\varepsilon$, Lemma 4.4 guarantees the existence of some $\delta > 0$ such that

$$a_{\varepsilon,i} \le \varepsilon^{1+2\delta}, \quad \varepsilon^{1+\delta} \le d_{\varepsilon,i} \le \varepsilon. \tag{4.7}$$

For legibility, we drop the dependence on ε and set for $z_i \in n^\varepsilon$ and $\varepsilon > 0$ small enough such that $a_{\varepsilon,i} < \frac{1}{2} d_{\varepsilon,i}$

$$\begin{aligned}
&T_i := B_{a_{\varepsilon,i}}(\varepsilon z_i), \quad C_i := B_{\frac{1}{2} d_{\varepsilon,i}}(\varepsilon z_i) \setminus B_{a_{\varepsilon,i}}(\varepsilon z_i), \quad D_i := B_{d_{\varepsilon,i}}(\varepsilon z_i) \setminus B_{\frac{1}{2} d_{\varepsilon,i}}(\varepsilon z_i), \\
&A_i := B_{d_{\varepsilon,i}}(\varepsilon z_i) \setminus B_{\frac{1}{4} d_{\varepsilon,i}}(\varepsilon z_i), \quad E_i := B_{a_{\varepsilon,i}^{-1} d_{\varepsilon,i}}(\varepsilon z_i) \setminus B_{\frac{1}{2} a_{\varepsilon,i}^{-1} d_{\varepsilon,i}}(\varepsilon z_i), \\
&B_{1,i} := B_{\frac{1}{2} d_{\varepsilon,i}}(\varepsilon z_i), \quad B_{2,i} := B_{d_{\varepsilon,i}}(\varepsilon z_i), \\
&A_i^0 := B_{d_{\varepsilon,i}}(0) \setminus B_{\frac{1}{4} d_{\varepsilon,i}}(0), \quad E_i^0 := B_{a_{\varepsilon,i}^{-1} d_{\varepsilon,i}}(0) \setminus B_{\frac{1}{2} a_{\varepsilon,i}^{-1} d_{\varepsilon,i}}(0).
\end{aligned} \tag{4.8}$$

Remark 4.7. By [20, Lemma C.1 and Lemma C.2], it follows that

$$\lim_{\varepsilon \to 0} \varepsilon^3 \sum_{z_i \in \Phi^\varepsilon(\Omega)} r_i = \lambda \langle r \rangle |\Omega| \quad \text{and} \quad \lim_{\varepsilon \to 0} \varepsilon^3 \sum_{z_i \in \mathcal{I}_\varepsilon} r_i = 0 \quad \text{a.s.}$$

Thus, we have a.s.

$$\lim_{\varepsilon \to 0} \sum_{z_i \in n^\varepsilon} a_{\varepsilon,i} = \lim_{\varepsilon \to 0} \varepsilon^3 \sum_{z_i \in \Phi^\varepsilon(\Omega)} r_i - \lim_{\varepsilon \to 0} \varepsilon^3 \sum_{z_i \in \mathcal{I}_\varepsilon} r_i = \lambda \langle r \rangle |\Omega|. \tag{4.9}$$

Let $\theta > 1$ be fixed. Let $J^\varepsilon = \bigcup_{i=-3}^{k_{\max}} J_i^\varepsilon$, $\{\lambda_l^\varepsilon\}_{z_l \in J^\varepsilon}$ be given by Lemma 4.5. For each $z_i \in J^\varepsilon$ we define

$$R_i := \lambda_i^\varepsilon r_i, \quad B_{R,i} := B_{\varepsilon^3 R_i}(\varepsilon z_i), \quad B_{\theta,i} := B_{\varepsilon^3 \theta R_i}(\varepsilon z_i), \quad W_i := B_{\theta,i} \setminus B_{R,i}. \tag{4.10}$$

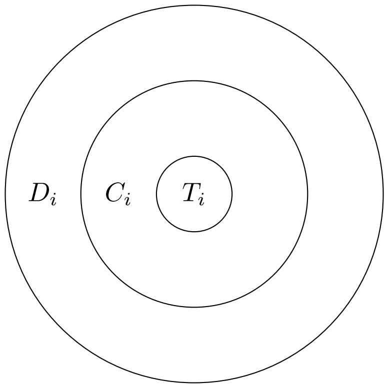

Figure 1: Cells for the construction of $(\boldsymbol{\omega}_k^\varepsilon, q_k^\varepsilon)$.

For each $k \in \{1,2,3\}$ let also $(\boldsymbol{\omega}_k, q_k)$ be the unique weak solution to the Stokes problem

$$\begin{cases} \nabla_x q_k - \Delta \boldsymbol{\omega}_k = 0 & \text{in } \mathbb{R}^3 \setminus B_1, \\ \operatorname{div}_x(\boldsymbol{\omega}_k) = 0 & \text{in } \mathbb{R}^3 \setminus B_1, \\ \boldsymbol{\omega}_k = 0 & \text{on } \partial B_1, \\ \boldsymbol{\omega}_k = \mathbf{e}_k & \text{at infinity.} \end{cases}$$

In the next lemma we use $(\boldsymbol{\omega}_k, q_k)$ to construct pairs of functions $(\boldsymbol{\omega}_k^{\varepsilon,g}, q_k^{\varepsilon,g})_{\varepsilon>0}$ in the spirit of Allaire's functions in [4], which vanish on the good holes $H_{\rm g}^\varepsilon$ and such that we have good control of their L^p and $W^{1,p}$ norm, respectively. In particular, the functions $\boldsymbol{\omega}_k^{\varepsilon,g}$ are the ones for which we will find $\boldsymbol{\mu}_k$ such that $(\boldsymbol{\omega}_k^{\varepsilon,g}, \boldsymbol{\mu}_k)_{1\le k\le 3}$ satisfy (H1)–(H5).

Lemma 4.8. *For each $\varepsilon > 0$ and $k \in \{1,2,3\}$ we consider the pair of functions $(\boldsymbol{\omega}_k^{\varepsilon,g}, q_k^{\varepsilon,g})$ given by*

$$(\boldsymbol{\omega}_k^{\varepsilon,g}, q_k^{\varepsilon,g}) := \begin{cases} (\boldsymbol{e}_k, 0) & \text{in } \Omega \setminus \bigcup_{z_i \in n^\varepsilon} B_{2,i}, \\ (\tilde{\boldsymbol{\omega}}_k^{\varepsilon,g}, \tilde{q}_k^{\varepsilon,g}) & \text{in } \bigcup_{z_i\in n^\varepsilon} D_i, \\ \left(\boldsymbol{\omega}_k\left(\frac{\cdot - \varepsilon z_i}{a_{\varepsilon,i}}\right), \frac{1}{a_{\varepsilon,i}} q_k\left(\frac{\cdot - \varepsilon z_i}{a_{\varepsilon,i}}\right)\right) & \text{in } C_i \text{ for } z_i \in n^\varepsilon, \\ (0,0) & \text{in } \bigcup_{z_i\in n^\varepsilon} T_i, \end{cases}$$

where $(\tilde{\boldsymbol{\omega}}_k^{\varepsilon,g}, \tilde{q}_k^{\varepsilon,g})$ is the solution to the Stokes problem

$$\begin{cases} \nabla_x \tilde{q}_k^{\varepsilon,g} - \Delta \tilde{\boldsymbol{\omega}}_k^{\varepsilon,g} = 0 & \text{in } D_i, \\ \operatorname{div}_x(\tilde{\boldsymbol{\omega}}_k^{\varepsilon,g}) = 0 & \text{in } D_i, \\ (\tilde{\boldsymbol{\omega}}_k^{\varepsilon,g}, \tilde{q}_k^{\varepsilon,g}) = (\boldsymbol{e}_k, 0) & \text{on } \partial B_{2,i}, \\ (\tilde{\boldsymbol{\omega}}_k^{\varepsilon,g}, \tilde{q}_k^{\varepsilon,g}) = \left(\boldsymbol{\omega}_k\left(\frac{\cdot - \varepsilon z_i}{a_{\varepsilon,i}}\right), \frac{1}{a_{\varepsilon,i}} q_k\left(\frac{\cdot - \varepsilon z_i}{a_{\varepsilon,i}}\right)\right) & \text{on } \partial B_{1,i}. \end{cases}$$

Then for all $p > \frac{3}{2}$ it holds

$$\|\nabla_x q_k^{\varepsilon,g}\|_{L^p(\cup_{z_i\in n^\varepsilon} C_i;\mathbb{R}^3)} \lesssim \varepsilon^{6(\frac{1}{p}-1)}, \tag{4.11}$$

$$\|\nabla_x \boldsymbol{\omega}_k^{\varepsilon,g}\|_{L^p(\cup_{z_i\in n^\varepsilon} C_i;\mathbb{R}^{3\times 3})} + \|q_k^{\varepsilon,g}\|_{L^p(\cup_{z_i\in n^\varepsilon} C_i)} \lesssim \begin{cases} \varepsilon^{(1+2\delta)(2-p)} & \text{if } p \le 2 \\ \varepsilon^{3(2-p)} & \text{if } p > 2 \end{cases}, \tag{4.12}$$

$$\|\nabla_x \boldsymbol{\omega}_k^{\varepsilon,g}\|_{L^p(\cup_{z_i\in n^\varepsilon} A_i;\mathbb{R}^{3\times 3})} + \|q_k^{\varepsilon,g}\|_{L^p(\cup_{z_i\in n^\varepsilon} A_i)} \lesssim \begin{cases} \varepsilon^{\delta(1-\frac{1}{p})}\varepsilon^{\frac{2}{p}-1} & \textit{if } p\le 2, \\ \varepsilon^{\delta(1-\frac{1}{p})}\varepsilon^{-(1+\delta)(1-\frac{2}{p})} & \textit{if } p>2. \end{cases}, \tag{4.13}$$

$$\|\nabla_x \boldsymbol{\omega}_k^{\varepsilon,g}\|_{L^p(\Omega;\mathbb{R}^{3\times 3})} + \|q_k^{\varepsilon,g}\|_{L^p(\Omega)} \lesssim \begin{cases} (\varepsilon^{\delta(1-\frac{1}{p})} + \varepsilon^{2\delta(\frac{2}{p}-1)})\varepsilon^{\frac{2}{p}-1} & \textit{if } p<2, \\ \varepsilon^{\delta(1-\frac{1}{p})}\varepsilon^{-(1+\delta)(1-\frac{2}{p})} + \varepsilon^{3(\frac{2}{p}-1)} & \textit{if } p\ge 2. \end{cases} \tag{4.14}$$

Proof. We use standard regularity theory for Stokes equations giving $|\nabla_x^{l+1}\boldsymbol{\omega}_k(x)| + |\nabla_x^l q_k(x)| \lesssim |x|^{-(l+2)}$, and $p>\frac{3}{2}$ to conclude

$$\|\nabla_x q_k\|^p_{L^p(\mathbb{R}^3\setminus B_1)} \lesssim 1, \tag{4.15}$$

$$\|\nabla_x \boldsymbol{\omega}_k\|^p_{L^p(\mathbb{R}^3\setminus B_1;\mathbb{R}^{3\times 3})} + \|q_k\|^p_{L^p(\mathbb{R}^3\setminus B_1)} \lesssim 1, \tag{4.16}$$

$$\|\nabla_x \boldsymbol{\omega}_k\|^p_{L^p(E_i^0;\mathbb{R}^{3\times 3})} + \|q_k\|^p_{L^p(E_i^0)} \lesssim \left(\frac{a_{\varepsilon,i}}{d_{\varepsilon,i}}\right)^{2p-3}, \tag{4.17}$$

$$\|\nabla_x \boldsymbol{\omega}_k\|^p_{L^p(B_{\frac{1}{2}d_{\varepsilon,i}a_{\varepsilon,i}^{-1}}\setminus B_{\frac{1}{4}d_{\varepsilon,i}a_{\varepsilon,i}^{-1}};\mathbb{R}^{3\times 3})} + \|q_k\|^p_{L^p(B_{\frac{1}{2}d_{\varepsilon,i}a_{\varepsilon,i}^{-1}}\setminus B_{\frac{1}{4}d_{\varepsilon,i}a_{\varepsilon,i}^{-1}})} \lesssim \left(\frac{a_{\varepsilon,i}}{d_{\varepsilon,i}}\right)^{2p-3}. \tag{4.18}$$

Rescaling together with (4.15), $r_i^{-1} < R_0^{-1} \lesssim 1$, and (4.9) leads to

$$\begin{aligned} \|\nabla_x q_k^{\varepsilon,g}\|^p_{L^p(\bigcup_{z_i\in n^\varepsilon} C_i;\mathbb{R}^3)} &= \sum_{z_i\in n^\varepsilon} a_{\varepsilon,i}^{3-2p}\|\nabla_x q_k\|^p_{L^p(B_{\frac{1}{2}d_{\varepsilon,i}a_{\varepsilon,i}^{-1}}\setminus B_1;\mathbb{R}^3)} \lesssim \sum_{z_i\in n^\varepsilon} a_{\varepsilon,i}^{3-2p} \\ &= \sum_{z_i\in n^\varepsilon} a_{\varepsilon,i}\varepsilon^{6(1-p)} r_i^{2(1-p)} \lesssim \varepsilon^{6(1-p)}, \end{aligned}$$

and (4.11) follows.

We now prove (4.12), (4.13), and (4.14). We start by noticing that, since $(\nabla_x \boldsymbol{\omega}_k^{\varepsilon,g}, q_k^{\varepsilon,g}) = (0,0)$ in $[\Omega \setminus (\cup_{z_i\in n^\varepsilon} B_{2,i})] \cup (\cup_{z_i\in n^\varepsilon} T_i)$, we have

$$\begin{aligned} \|\nabla_x \boldsymbol{\omega}_k^{\varepsilon,g}\|^p_{L^p(\Omega;\mathbb{R}^{3\times 3})} + \|q_k^{\varepsilon,g}\|^p_{L^p(\Omega)} &= \|\nabla_x \boldsymbol{\omega}_k^{\varepsilon,g}\|^p_{L^p(\cup_{z_i\in n^\varepsilon} C_i;\mathbb{R}^{3\times 3})} + \|q_k^{\varepsilon,g}\|^p_{L^p(\cup_{z_i\in n^\varepsilon} C_i)} \\ &\quad + \|\nabla_x \boldsymbol{\omega}_k^{\varepsilon,g}\|^p_{L^p(\cup_{z_i\in n^\varepsilon} D_i;\mathbb{R}^{3\times 3})} + \|q_k^{\varepsilon,g}\|^p_{L^p(\cup_{z_i\in n^\varepsilon} D_i)}. \end{aligned}$$

From (4.9), (4.16), the fact that $a_{\varepsilon,i}^{2-p} = (\varepsilon^3)^{(2-p)}(r_i)^{(2-p)} \le \varepsilon^{(1+2\delta)(2-p)}$, and $r_i \ge R_0 > 0$, we find

$$\begin{aligned} \|\nabla_x \boldsymbol{\omega}_k^{\varepsilon,g}\|^p_{L^p(\cup_{z_i\in n^\varepsilon} C_i;\mathbb{R}^{3\times 3})} + \|q_k^{\varepsilon,g}\|^p_{L^p(\cup_{z_i\in n^\varepsilon} C_i)} &= \sum_{z_i\in n^\varepsilon} a_{\varepsilon,i}^{3-p}(\|\nabla_x \boldsymbol{\omega}_k\|^p_{L^p(B_{\frac{1}{2}d_{\varepsilon,i}a_{\varepsilon,i}^{-1}}\setminus B_1;\mathbb{R}^{3\times 3})} + \|q_k\|^p_{L^p(B_{\frac{1}{2}d_{\varepsilon,i}a_{\varepsilon,i}^{-1}}\setminus B_1)}) \\ &\lesssim \sum_{z_i\in n^\varepsilon} a_{\varepsilon,i}^{3-p} = \sum_{z_i\in n^\varepsilon} a_{\varepsilon,i}a_{\varepsilon,i}^{2-p} \lesssim \begin{cases} \varepsilon^{(1+2\delta)(2-p)} & \text{if } p\le 2 \\ \varepsilon^{3(2-p)} & \text{if } p>2 \end{cases}. \end{aligned} \tag{4.19}$$

This proves (4.12).

Let now $(\mathbf{v}_k^\varepsilon, \pi_k^\varepsilon)$ be the solution of the Stokes system

$$\begin{cases} \nabla_x \pi_k^\varepsilon - \Delta \mathbf{v}_k^\varepsilon = 0 & \text{in } B_2\setminus B_1, \\ \operatorname{div}_x(\mathbf{v}_k^\varepsilon) = 0 & \text{in } B_2\setminus B_1, \\ \mathbf{v}_k^\varepsilon = \boldsymbol{\omega}_k(\frac{1}{2}d_{\varepsilon,i}a_{\varepsilon,i}^{-1}\cdot) - \mathbf{e}_k & \text{on } \partial B_1, \\ \mathbf{v}_k^\varepsilon = 0 & \text{on } \partial B_2. \end{cases}$$

Thus, we have that the tuple $(\mathbf{v}_k^\varepsilon + \mathbf{e}_k,\, 2d_{\varepsilon,i}^{-1}\pi_k^\varepsilon)(2d_{\varepsilon,i}^{-1}(\cdot - \varepsilon z_i))$ solves the Stokes equation in D_i with boundary data

$$\mathbf{v}_k^\varepsilon(2d_{\varepsilon,i}^{-1}(\cdot-\varepsilon z_i))+\mathbf{e}_k=\boldsymbol{\omega}_k\left(\frac{\cdot-\varepsilon z_i}{a_{\varepsilon,i}}\right) \text{ on } \partial B_{1,i},$$
$$\mathbf{v}_k^\varepsilon(2d_{\varepsilon,i}^{-1}(\cdot-\varepsilon z_i))+\mathbf{e}_k=\mathbf{e}_k \text{ on } \partial B_{2,i}.$$

Due to the uniqueness of the solution of the Stokes equation and by definition of $(\boldsymbol{\omega}_k^{\varepsilon,g},q_k^{\varepsilon,g})$ we get

$$\begin{aligned}\boldsymbol{\omega}_k^{\varepsilon,g}-\mathbf{e}_k&=\mathbf{v}_k^\varepsilon(2d_{\varepsilon,i}^{-1}(\cdot-z_i)) &&\text{in } D_i,\\ q_k^{\varepsilon,g}&=\frac{2}{d_{\varepsilon,i}}\pi_k^\varepsilon(2d_{\varepsilon,i}^{-1}(\cdot-z_i)) &&\text{in } D_i.\end{aligned}$$

By rescaling and appealing to [19, Theorem II.4.3 and Theorem IV.6.1], it follows that

$$\begin{aligned}&\|\nabla_x\boldsymbol{\omega}_k^{\varepsilon,g}\|^p_{L^p(\cup_{z_i\in n^\varepsilon}D_i;\mathbb{R}^{3\times3})}+\|q_k^{\varepsilon,g}\|^p_{L^p(\cup_{z_i\in n^\varepsilon}D_i)}\lesssim\sum_{z_i\in n^\varepsilon}d_{\varepsilon,i}^{3-p}\|\nabla_x\mathbf{v}_k^\varepsilon\|^p_{L^p(B_2\setminus B_1;\mathbb{R}^{3\times3})}+d_{\varepsilon,i}^{3-p}\|\pi_k^\varepsilon\|^p_{L^p(B_2\setminus B_1)}\\&\lesssim\sum_{z_i\in n^\varepsilon}d_{\varepsilon,i}^{3-p}\|\boldsymbol{\omega}_k(2^{-1}d_{\varepsilon,i}a_{\varepsilon,i}^{-1}\cdot)-\mathbf{e}_k\|^p_{W^{1-\frac1p,p}(\partial B_1;\mathbb{R}^3)}\\&\lesssim\sum_{z_i\in n^\varepsilon}d_{\varepsilon,i}^{3-p}\|(\eta(\boldsymbol{\omega}_k-\mathbf{e}_k))(2^{-1}d_{\varepsilon,i}a_{\varepsilon,i}^{-1}\cdot)\|^p_{W^{1,p}(B_2\setminus B_1;\mathbb{R}^3)}\\&=\sum_{z_i\in n^\varepsilon}a_{\varepsilon,i}^3d_{\varepsilon,i}^{-p}\left(\|\eta(\boldsymbol{\omega}_k-\mathbf{e}_k)\|^p_{L^p(E_i^0;\mathbb{R}^3)}+\left(\frac12\frac{d_{\varepsilon,i}}{a_{\varepsilon,i}}\right)^p\|\nabla_x\eta(\boldsymbol{\omega}_k-\mathbf{e}_k)+\eta\nabla_x\boldsymbol{\omega}_k\|^p_{L^p(E_i^0;\mathbb{R}^{3\times3})}\right),\end{aligned}\tag{4.20}$$

where $0\le\eta\le1$ is a cut-off function in $B_{d_{\varepsilon,i}a_{\varepsilon,i}^{-1}}\setminus B_{\frac12 d_{\varepsilon,i}a_{\varepsilon,i}^{-1}}$ such that $\eta=0$ on $\partial B_{d_{\varepsilon,i}a_{\varepsilon,i}^{-1}}$, $\eta=1$ on $B_{\frac12 d_{\varepsilon,i}a_{\varepsilon,i}^{-1}}$, and $|\nabla_x\eta|\lesssim\frac{a_{\varepsilon,i}}{d_{\varepsilon,i}}$. Combining this with (4.17) gives

$$\begin{aligned}&\|\eta(\boldsymbol{\omega}_k-\mathbf{e}_k)\|^p_{L^p(E_i^0;\mathbb{R}^3)}+\left(\frac12\frac{d_{\varepsilon,i}}{a_{\varepsilon,i}}\right)^p\|\nabla_x\eta(\boldsymbol{\omega}_k-\mathbf{e}_k)+\eta\nabla_x\boldsymbol{\omega}_k\|^p_{L^p(E_i^0;\mathbb{R}^{3\times3})}\\&\lesssim\|\boldsymbol{\omega}_k-\mathbf{e}_k\|^p_{L^p(E_i^0;\mathbb{R}^3)}+\left(\frac{d_{\varepsilon,i}}{a_{\varepsilon,i}}\right)^p\|\nabla_x\boldsymbol{\omega}_k\|^p_{L^p(E_i^0;\mathbb{R}^{3\times3})}+\left(\frac12\frac{d_{\varepsilon,i}}{a_{\varepsilon,i}}\right)^p\|\nabla_x\eta\|^p_{L^\infty(E_i^0;\mathbb{R}^3)}\|\boldsymbol{\omega}_k-\mathbf{e}_k\|^p_{L^p(E_i^0)}\\&\lesssim\|\boldsymbol{\omega}_k-\mathbf{e}_k\|^p_{L^p(E_i^0;\mathbb{R}^3)}+\left(\frac{d_{\varepsilon,i}}{a_{\varepsilon,i}}\right)^p\left(\frac{d_{\varepsilon,i}}{a_{\varepsilon,i}}\right)^{3-2p}.\end{aligned}\tag{4.21}$$

Now we use $|\boldsymbol{\omega}_k(x)-\mathbf{e}_k|\le|x|^{-1}$ to infer

$$\|\boldsymbol{\omega}_k-\mathbf{e}_k\|^p_{L^p(E_i^0;\mathbb{R}^3)}\lesssim\int_{E_i^0}\frac{1}{|x|^p}\,\mathrm{d}x\lesssim\left(\frac{d_{\varepsilon,i}}{a_{\varepsilon,i}}\right)^{3-p}.$$

This together with (4.20) and (4.21) gives us

$$\begin{aligned}\|\nabla_x\boldsymbol{\omega}_k^{\varepsilon,g}\|^p_{L^p(\cup_{z_i\in n^\varepsilon}D_i;\mathbb{R}^{3\times3})}+\|q_k^{\varepsilon,g}\|^p_{L^p(\cup_{z_i\in n^\varepsilon}D_i)}&\lesssim\sum_{z_i\in n^\varepsilon}d_{\varepsilon,i}^{-p}a_{\varepsilon,i}^3\left(\frac{d_{\varepsilon,i}}{a_{\varepsilon,i}}\right)^{3-p}=\sum_{z_i\in n^\varepsilon}a_{\varepsilon,i}\left(\frac{a_{\varepsilon,i}}{d_{\varepsilon,i}}\right)^{p-1}d_{\varepsilon,i}^{2-p}\\&\lesssim\begin{cases}\varepsilon^{\delta p-1}\varepsilon^{2-p} & \text{if } p\le2\\ \varepsilon^{\delta p-1}\varepsilon^{-(1+\delta)(p-2)} & \text{if } p>2\end{cases},\end{aligned}\tag{4.22}$$

where the last inequality follows from (4.9) and the fact that by (4.7), we have

$$\left(\frac{a_{\varepsilon,i}}{d_{\varepsilon,i}}\right)^{p-1}d_{\varepsilon,i}^{2-p}\lesssim\frac{\varepsilon^{(1+2\delta)(p-1)}}{\varepsilon^{(\delta+1)(p-1)}}\varepsilon^{2-p}$$

for $p>2$. Combining (4.19) with (4.22) we find (4.14). To prove (4.13), we observe that by rescaling and (4.18) we get

$$
\begin{aligned}
&\|\nabla_x \boldsymbol{\omega}_k^{\varepsilon,g}\|^p_{L^p(\bigcup_{z_i\in n^\varepsilon} B_{\frac12 d_{\varepsilon,i}}(\varepsilon z_i)\setminus B_{\frac14 d_{\varepsilon,i}}(\varepsilon z_i);\mathbb{R}^{3\times3})} + \|q_k^{\varepsilon,g}\|^p_{L^p(\bigcup_{z_i\in n^\varepsilon} B_{\frac12 d_{\varepsilon,i}}(\varepsilon z_i)\setminus B_{\frac14 d_{\varepsilon,i}}(\varepsilon z_i))}\\
&\lesssim \sum_{z_i\in n^\varepsilon} a_{\varepsilon,i}^{3-p}\Big(\|\nabla_x\boldsymbol{\omega}_k\|^p_{L^p(B_{\frac12 d_{\varepsilon,i}a_{\varepsilon,i}^{-1}}\setminus B_{\frac14 d_{\varepsilon,i}a_{\varepsilon,i}^{-1}};\mathbb{R}^{3\times3})} + \|q_k\|^p_{L^p(B_{\frac12 d_{\varepsilon,i}a_{\varepsilon,i}^{-1}}\setminus B_{\frac14 d_{\varepsilon,i}a_{\varepsilon,i}^{-1}})}\Big)\\
&\lesssim \sum_{z_i\in n^\varepsilon} a_{\varepsilon,i}^{3-p}\left(\frac{d_{\varepsilon,i}}{a_{\varepsilon,i}}\right)^{3-2p} \lesssim \sum_{z_i\in n^\varepsilon} a_{\varepsilon,i}\left(\frac{a_{\varepsilon,i}}{d_{\varepsilon,i}}\right)^{p-1} d_{\varepsilon,i}^{2-p}\\
&\lesssim \begin{cases} \varepsilon^{\delta(p-1)}\varepsilon^{2-p} & \text{if } p\le 2\\ \varepsilon^{\delta(p-1)}\varepsilon^{-(1+\delta)(p-2)} & \text{if } p>2\end{cases}.
\end{aligned}
$$

This together with (4.22) and

$$
\bigcup_{z_i\in n^\varepsilon} A_i = \Big(\bigcup_{z_i\in n^\varepsilon} D_i\Big)\cup\Big(\bigcup_{z_i\in n^\varepsilon} B_{\frac12 d_{\varepsilon,i}}(\varepsilon z_i)\setminus B_{\frac14 d_{\varepsilon,i}}(\varepsilon z_i)\Big)
$$

implies

$$
\|\nabla_x \boldsymbol{\omega}_k^{\varepsilon,g}\|^p_{L^p(\cup_{z_i\in n^\varepsilon} A_i;\mathbb{R}^{3\times3})} + \|q_k^{\varepsilon,g}\|^p_{L^p(\cup_{z_i\in n^\varepsilon} A_i)} \lesssim \begin{cases} \varepsilon^{\delta(p-1)}\varepsilon^{2-p} & \text{if } p\le 2\\ \varepsilon^{\delta(p-1)}\varepsilon^{-(1+\delta)(p-2)} & \text{if } p>2\end{cases}.
$$

□

Definition 4.9. For $N\in\mathbb{N}$ we define

$$
n_N^\varepsilon := \left\{z_i\in n^\varepsilon \mid d_{\varepsilon,i}\ge \frac{\varepsilon}{N}\right\},\qquad r_{i,N} := \min\{r_i,N\},\qquad \mathcal{R}_N := \{r_{i,N}\}_{z_i\in n_N^\varepsilon},
$$

and we let $(\boldsymbol{\omega}_{k,N}^{\varepsilon,g}, q_{k,N}^{\varepsilon,g})$ be the analogue of $(\boldsymbol{\omega}_k^{\varepsilon,g}, q_k^{\varepsilon,g})$ when n^ε is replaced by n_N^ε.

Note that by definition we have for any $z_i\in n_N^\varepsilon$ that

$$
a_{\varepsilon,i}\le N\varepsilon^3 \quad\text{and}\quad d_{\varepsilon,i}^{-1}\le \frac{N}{\varepsilon}. \tag{4.23}
$$

Remark 4.10. From $n_N^\varepsilon\subset n^\varepsilon$ and the definition of $(\boldsymbol{\omega}_{k,N}^{\varepsilon,g}, q_{k,N}^{\varepsilon,g})$ and $(\boldsymbol{\omega}_k^{\varepsilon,g}, q_k^{\varepsilon,g})$, it readily follows that for all $p>\frac32$ we have uniformly in N

$$
\|\nabla_x q_{k,N}^{\varepsilon,g}\|_{L^p(\cup_{z_i\in n_N^\varepsilon} C_i;\mathbb{R}^3)} \lesssim \varepsilon^{6(\frac1p-1)},\quad \|q_{k,N}^{\varepsilon,g}\|_{L^p(\cup_{z_i\in n_N^\varepsilon} C_i)} \lesssim \begin{cases}\varepsilon^{(1+2\delta)(2-p)} & \text{if } p\le 2\\ \varepsilon^{3(2-p)} & \text{if } p>2\end{cases}. \tag{4.24}
$$

$$
\|\nabla_x \boldsymbol{\omega}_{k,N}^{\varepsilon,g}\|_{L^p(\cup_{z_i\in n_N^\varepsilon} A_i;\mathbb{R}^{3\times3})} + \|q_{k,N}^{\varepsilon,g}\|_{L^p(\cup_{z_i\in n_N^\varepsilon} A_i)} \lesssim \begin{cases}\varepsilon^{\delta(1-\frac1p)}\varepsilon^{\frac2p-1} & \text{if } p\le 2,\\ \varepsilon^{\delta(1-\frac1p)}\varepsilon^{-(1+\delta)(1-\frac2p)} & \text{if } p>2.\end{cases} \tag{4.25}
$$

$$
\|\nabla_x \boldsymbol{\omega}_{k,N}^{\varepsilon,g}\|_{L^p(\Omega;\mathbb{R}^{3\times3})} + \|q_{k,N}^{\varepsilon,g}\|_{L^p(\Omega)} \lesssim \begin{cases}(\varepsilon^{\delta(1-\frac1p)} + \varepsilon^{2\delta(\frac2p-1)})\varepsilon^{\frac2p-1} & \text{if } p<2,\\ \varepsilon^{\delta(1-\frac1p)}\varepsilon^{-(1+\delta)(1-\frac2p)} + \varepsilon^{3(\frac2p-1)} & \text{if } p\ge 2.\end{cases} \tag{4.26}
$$

Before we show that the functions $(\boldsymbol{\omega}_k^{\varepsilon,g})_{1\le k\le 3}$ from Lemma 4.8 satisfy (H5), we need some preliminary lemmas, which we collect in the following:

Lemma 4.11. *Let $c\in(0,1]$ be a fixed real number, and $\delta_i^{cd_{\varepsilon,i}}$ be the measure concentrated on the sphere $\partial B_{cd_{\varepsilon,i}}(\varepsilon z_i)$, that is, $\delta_i^{cd_{\varepsilon,i}} = \mathcal{H}^2\llcorner\partial B_{cd_{\varepsilon,i}}(\varepsilon z_i)$. Let also σ_3 be the area of the unit sphere in $\mathbb{R}^3$. Then, almost surely,*

$$
\sum_{z_i\in n_N^\varepsilon} (cd_{\varepsilon,i})^{-2} a_{\varepsilon,i}\delta_i^{cd_{\varepsilon,i}} \to \sigma_3\langle\mathcal{N}_{\frac2N}(\Omega)\rangle\langle r_N\rangle \quad \textit{strongly in } W^{-1,2}(\Omega), \tag{4.27}
$$

$$\sum_{z_i\in n_N^\varepsilon}(cd_{\varepsilon,i})^{-2}a_{\varepsilon,i}\delta_i^{cd_{\varepsilon,i}}(\boldsymbol{e}_k\cdot\boldsymbol{e}^i)\boldsymbol{e}^i\to\frac{\sigma_3}{3}\langle\mathcal{N}_{\frac{2}{N}}(\Omega)\rangle\langle r_N\rangle\boldsymbol{e}_k\quad \textit{strongly in } W^{-1,2}(\Omega), \tag{4.28}$$

where $\boldsymbol{e}^i\colon x\mapsto\frac{x-\varepsilon z_i}{|x-\varepsilon z_i|}$.

Proof. To prove Lemma 4.11 we argue as in [12, Lemma 2.3] and [2, Lemma II.3.5].

Step 1: Proof of (4.27). Let $p_i^\varepsilon\colon\overline{B_{cd_{\varepsilon,i}}(\varepsilon z_i)}\to\mathbb{R}$ be the solution of

$$\begin{cases}-\Delta p_i^\varepsilon=-3(cd_{\varepsilon,i})^{-3}a_{\varepsilon,i} & \text{in } B_{cd_{\varepsilon,i}}(\varepsilon z_i),\ z_i\in n^\varepsilon,\\ \dfrac{\partial p_i^\varepsilon}{\partial n}=(cd_{\varepsilon,i})^{-2}a_{\varepsilon,i} & \text{on } \partial B_{cd_{\varepsilon,i}}(\varepsilon z_i).\end{cases}$$

Then p^ε satisfies

$$\int_{B_{cd_{\varepsilon,i}}(\varepsilon z_i)}-\Delta p_i^\varepsilon\,\mathrm{d}x=-a_{\varepsilon,i}\sigma_3=-\int_{\partial B_{cd_{\varepsilon,i}}(\varepsilon z_i)}\frac{\partial p_i^\varepsilon}{\partial n}\mathrm{d}\mathcal{H}^2.$$

Hence, there exists a unique solution with $p^\varepsilon=0$ on $\partial B_{cd_{\varepsilon,i}}(\varepsilon z_i)$, given by

$$p_i^\varepsilon(x)=3(cd_{\varepsilon,i})^{-3}a_{\varepsilon,i}\left(\frac{s^2}{2}-\frac{c^2d_{\varepsilon,i}^2}{2}\right)\qquad\text{for } x\in\overline{B_{cd_{\varepsilon,i}}(\varepsilon z_i)},$$

where $s=|x-\varepsilon z_i|$. We have

$$\frac{\partial p_i^\varepsilon}{\partial s}=3(cd_{\varepsilon,i})^{-3}a_{\varepsilon,i}s\quad\text{in } B_{cd_{\varepsilon,i}}(\varepsilon z_i). \tag{4.29}$$

We define the function $p_N^\varepsilon\colon\Omega\to\mathbb{R}$ via

$$p_N^\varepsilon(x):=\begin{cases}p_i^\varepsilon(x) & \text{if } x\in B_{cd_{\varepsilon,i}}(\varepsilon z_i), z_i\in n_N^\varepsilon,\\ 0 & \text{if } x\in\Omega\setminus\bigcup_{z_i\in n_N^\varepsilon}B_{cd_{\varepsilon,i}}(\varepsilon z_i).\end{cases}$$

Hence, from (4.23), (4.29), and the definition of n_N^ε and $r_{i,N}$ it follows that

$$|\nabla_x p_N^\varepsilon|\le 3C(cd_{\varepsilon,i})^{-2}a_{\varepsilon,i}\le C_N\varepsilon.$$

As a consequence we have for any fixed $N\in\mathbb{N}$

$$\begin{aligned}p_N^\varepsilon&\to 0\quad\text{strongly in } W^{1,\infty}(\Omega),\\ \Delta p_N^\varepsilon&\to 0\quad\text{strongly in } W^{-1,\infty}(\Omega).\end{aligned}\tag{4.30}$$

By definition we have in the sense of $W^{-1,\infty}(\Omega)$

$$-\Delta p_N^\varepsilon=-\sum_{z_i\in n_N^\varepsilon}3c^{-3}d_{\varepsilon,i}^{-3}a_{\varepsilon,i}\chi_{B_{cd_{\varepsilon,i}}(\varepsilon z_i)}-\sum_{z_i\in n_N^\varepsilon}c^{-2}d_{\varepsilon,i}^{-2}a_{\varepsilon,i}\delta^{cd_{\varepsilon,i}}.$$

Let now

$$\begin{aligned}\eta_N^\varepsilon&=\sum_{z_i\in n_N^\varepsilon}c^{-3}d_{\varepsilon,i}^{-3}\varepsilon^3 r_{i,N}\chi_{B_{cd_{\varepsilon,i}}(\varepsilon z_i)},\\ \tilde\eta_N^\varepsilon&=\sum_{z_i\in\Phi^\varepsilon_{\frac{2}{N}}}c^{-3}d_{\varepsilon,i}^{-3}\varepsilon^3 r_{i,N}\chi_{B_{cd_{\varepsilon,i}}(\varepsilon z_i)}.\end{aligned}$$

By [21, Lemma 5.3], it follows that for every $\xi \in C_0^1(\Omega)$ almost surely

$$\lim_{\varepsilon\to 0}\int_\Omega \tilde{\eta}_N^\varepsilon \xi \,\mathrm{d}x = \lim_{\varepsilon\to 0}\sum_{z_i\in\Phi^\varepsilon_{\frac{2}{N}}}\left(\frac{\varepsilon}{cd_{\varepsilon,i}}\right)^3 r_{i,N}\int_{B_{cd_{\varepsilon,i}}(\varepsilon z_i)}\xi\,\mathrm{d}x$$
$$= \frac{\sigma_3}{3}\langle\mathcal{N}_{\frac{2}{N}}(\Omega)\rangle\langle r_N\rangle\int_\Omega \xi\,\mathrm{d}x.$$

Thus, we proved that

$$\tilde{\eta}_N^\varepsilon \overset{*}{\rightharpoonup} \frac{\sigma_3}{3}\langle\mathcal{N}_{\frac{2}{N}}(\Omega)\rangle\langle r_N\rangle \quad \text{in } L^\infty(\Omega) \text{ almost surely.} \tag{4.31}$$

Furthermore,

$$\tilde{\eta}_N^\varepsilon - \eta_N^\varepsilon \overset{*}{\rightharpoonup} 0 \quad \text{in } L^\infty(\Omega) \text{ almost surely.} \tag{4.32}$$

Indeed, we have $n_N^\varepsilon \subset \Phi^\varepsilon_{\frac{2}{N}}$ and $\mathrm{dist}(\varepsilon z_i, \Omega_b^\varepsilon) < \frac{\varepsilon}{N}$ for each $z_i \in \Phi^\varepsilon_{\frac{2}{N}} \setminus n_N^\varepsilon$. Let $\xi \in C_0^1(\Omega)$, then we get

$$\left|\int_\Omega(\tilde{\eta}_N^\varepsilon - \eta_N^\varepsilon)\xi\,\mathrm{d}x\right| \lesssim \sum_{z_i\in\Phi^\varepsilon_{\frac{2}{N}}\setminus n_N^\varepsilon} c^{-3}d_{\varepsilon,i}^{-3}\varepsilon^3 r_{i,N}\int_{B_{cd_{\varepsilon,i}}(\varepsilon z_i)}\xi\,\mathrm{d}x \lesssim \sum_{z_i\in\Phi^\varepsilon_{\frac{2}{N}}\setminus n_N^\varepsilon}\|\xi\|_{L^\infty(\Omega)}\varepsilon^3 N$$
$$\lesssim N\varepsilon^3 \#\{z_i\in\Phi^\varepsilon_{\frac{2}{N}} \mid \mathrm{dist}(\varepsilon z_i,\Omega_b^\varepsilon)\le \frac{\varepsilon}{N}\}.$$

Lemma 3.1 in [20] implies that the right-hand side vanishes as $\varepsilon \to 0$. By (4.31) and (4.32), we deduce

$$\eta_N^\varepsilon \overset{*}{\rightharpoonup} \frac{\sigma_3}{3}\langle\mathcal{N}_{\frac{2}{N}}(\Omega)\rangle\langle r_N\rangle \quad \text{in } L^\infty(\Omega) \text{ almost surely and hence strongly in } W^{-1,\infty}(\Omega). \tag{4.33}$$

By the strong convergences in (4.30), we get for fixed $N \in \mathbb{N}$

$$\sum_{z_i\in n_N^\varepsilon} c^{-2}d_{\varepsilon,i}^{-2}a_{\varepsilon,i}\delta^{cd_{\varepsilon,i}} \to \sigma_3\langle\mathcal{N}_{\frac{2}{N}}(\Omega)\rangle\langle r_N\rangle \quad \text{in } W^{-1,\infty}(\Omega) \text{ strongly almost surely,}$$

which shows (4.27).

Step 2: Proof of (4.28). Let $\mathbf{p}_{i,k}^\varepsilon : \overline{B_{cd_{\varepsilon,i}}(\varepsilon z_i)} \to \mathbb{R}^3$ be the solution of

$$\begin{cases} -\Delta \mathbf{p}_{i,k}^\varepsilon = -(cd_{\varepsilon,i})^{-3}a_{\varepsilon,i}\mathbf{e}_k & \text{in } B_{cd_{\varepsilon,i}}(\varepsilon z_i),\\ \dfrac{\partial \mathbf{p}_{i,k}^\varepsilon}{\partial n} = (cd_{\varepsilon,i})^{-2}a_{\varepsilon,i}(\mathbf{e}_k\cdot\mathbf{e}^i)\mathbf{e}^i & \text{on } \partial B_{cd_{\varepsilon,i}}(\varepsilon z_i).\end{cases}$$

Then $\mathbf{p}_{i,k}^\varepsilon$ satisfies

$$\int_{B_{cd_{\varepsilon,i}}(\varepsilon z_i)} -\Delta\mathbf{p}_{i,k}^\varepsilon\,\mathrm{d}x = -a_{\varepsilon,i}\frac{\sigma_3}{3}\mathbf{e}_k = -\int_{\partial B_{cd_{\varepsilon,i}}(\varepsilon z_i)}\frac{\partial\mathbf{p}_{i,k}^\varepsilon}{\partial n}\,\mathrm{d}s.$$

Thus, there exists a unique solution, up to an additive constant, given by

$$\mathbf{p}_{i,k}^\varepsilon(x) = \frac{1}{2}(cd_{\varepsilon,i})^{-3}a_{\varepsilon,i}(x-\varepsilon z_i)_k r\mathbf{e}^i(x) = \frac{1}{2}(cd_{\varepsilon,i})^{-3}a_{\varepsilon,i}s^2(\mathbf{e}_k\cdot\mathbf{e}^i)\mathbf{e}^i \quad \text{for } x\in\overline{B_{cd_{\varepsilon,i}}(\varepsilon z_i)},$$

where $s = |x - \varepsilon z_i|$. We get

$$\frac{\partial \mathbf{p}^\varepsilon_{i,k}}{\partial s} = (cd_{\varepsilon,i})^{-3} a_{\varepsilon,i} s (\mathbf{e}_k \cdot \mathbf{e}^i)\mathbf{e}^i = (cd_{\varepsilon,i})^{-2} a_{\varepsilon,i} (\mathbf{e}_k \cdot \mathbf{e}^i)\mathbf{e}^i \quad \text{on } \partial B_{cd_{\varepsilon,i}}(\varepsilon z_i).$$

Next, we are searching a function $\mathbf{t}^\varepsilon_{i,k} : B_{\varepsilon d_{\varepsilon,i}}(\varepsilon z_i) \to \mathbb{R}^3$ satisfying

$$\begin{aligned} \mathbf{p}^\varepsilon_{i,k} &= \mathbf{t}^\varepsilon_{i,k} \quad \text{on } \partial B_{cd_{\varepsilon,i}}(\varepsilon z_i), \\ \frac{\partial \mathbf{t}^\varepsilon_{i,k}}{\partial n} &= 0 \quad \text{on } \partial B_{cd_{\varepsilon,i}}(\varepsilon z_i). \end{aligned} \tag{4.34}$$

The unique solution of this system is given by

$$\mathbf{t}^\varepsilon_{i,k} = (cd_{\varepsilon,i})^{-3} a_{\varepsilon,i} \frac{s^2(3cd_{\varepsilon,i} - 2s)}{2cd_{\varepsilon,i}} (\mathbf{e}_k \cdot \mathbf{e}^i)\mathbf{e}^i.$$

For its Laplacian we compute

$$\Delta \mathbf{t}^\varepsilon_{i,k} = ((3cd_{\varepsilon,i} - 2s)\mathbf{e}_k - 6s(\mathbf{e}_k \cdot \mathbf{e}^i)\mathbf{e}^i) c^{-4} d^{-4}_{\varepsilon,i} a_{\varepsilon,i}.$$

Hence, from (4.23) we conclude

$$|\Delta \mathbf{t}^\varepsilon_{i,k}| \le 9 c^{-3} d^{-3}_{\varepsilon,i} a_{\varepsilon,i} \lesssim C_N. \tag{4.35}$$

Let $\mathbf{r}^\varepsilon_{N,k} : \Omega \to \mathbb{R}^3$ be given by

$$\mathbf{r}^\varepsilon_{N,k} := \begin{cases} \mathbf{p}^\varepsilon_{i,k} - \mathbf{t}^\varepsilon_{i,k} & \text{in } B_{cd_{\varepsilon,i}}(\varepsilon z_i),\ z_i \in n^\varepsilon_N, \\ 0 & \text{in } \Omega \setminus \bigcup_{i \in n^\varepsilon_N} B_{cd_{\varepsilon,i}}(\varepsilon z_i), \end{cases}$$

in other words,

$$\mathbf{r}^\varepsilon_{N,k}(x) = s^2 c^{-3} d^{-3}_{\varepsilon,i} a_{\varepsilon,i} \left(\frac{r}{cd_{\varepsilon,i}} - 1 \right) (\mathbf{e}_k \cdot \mathbf{e}^i)\mathbf{e}^i \quad \text{in } B_{cd_{\varepsilon,i}}(\varepsilon z_i).$$

Using (4.23) we get

$$|\nabla_x \mathbf{r}^\varepsilon_{N,k}| \lesssim d^{-2}_{\varepsilon,i} a_{\varepsilon,i} \lesssim C_N \varepsilon.$$

As a consequence we have for any fixed $N \in \mathbb{N}$

$$\begin{aligned} \mathbf{r}^\varepsilon_{N,k} &\to 0 \quad \text{strongly in } W^{1,\infty}(\Omega; \mathbb{R}^3), \\ \Delta \mathbf{r}^\varepsilon_{N,k} &\to 0 \quad \text{strongly in } W^{-1,\infty}(\Omega; \mathbb{R}^3). \end{aligned} \tag{4.36}$$

Moreover, by definition we find in the sense of $W^{-1,\infty}(\Omega; \mathbb{R}^3)$

$$-\Delta \mathbf{r}^\varepsilon_{N,k} = -\eta^\varepsilon_N \mathbf{e}_k - \sum_{z_i \in n^\varepsilon_N} \chi^\varepsilon_{B_{cd_{\varepsilon,i}}(\varepsilon z_i)} \Delta \mathbf{t}^\varepsilon_{i,k} + \sum_{z_i \in n^\varepsilon_N} c^{-2} d^{-2}_{\varepsilon,i} a_{\varepsilon,i} \delta^{cd_{\varepsilon,i}} (\mathbf{e}_k \cdot \mathbf{e}^i)\mathbf{e}^i. \tag{4.37}$$

With (4.34) and (4.35), we have $\sum_{z_i \in n^\varepsilon_N} \chi^\varepsilon_{B_{cd_{\varepsilon,i}}(\varepsilon z_i)} \Delta \mathbf{t}^\varepsilon_{i,k} \in L^\infty(\Omega; \mathbb{R}^3)$ uniformly in $\varepsilon > 0$ and

$$\int_\Omega \sum_{z_i \in n^\varepsilon_N} \chi^\varepsilon_{B_{cd_{\varepsilon,i}}(\varepsilon z_i)} \Delta \mathbf{t}^\varepsilon_{i,k} \, \mathrm{d}x = \sum_{z_i \in n^\varepsilon_N} \int_{B_{cd_{\varepsilon,i}}(\varepsilon z_i)} \Delta \mathbf{t}^\varepsilon_{i,k} \, \mathrm{d}x = \sum_{z_i \in n^\varepsilon_N} \int_{\partial B_{cd_{\varepsilon,i}}(\varepsilon z_i)} \frac{\partial \mathbf{t}^\varepsilon_{i,k}}{\partial n} = 0.$$

Thus,

$$\sum_{z_i\in n_N^\varepsilon}\chi^\varepsilon_{B_{cd_{\varepsilon,i}}(\varepsilon z_i)}\Delta\mathbf{t}^\varepsilon_{i,k}\overset{*}{\rightharpoonup}0\quad\text{in } L^\infty(\Omega;\mathbb{R}^3)\text{ almost surely,}$$
$$\sum_{z_i\in n_N^\varepsilon}\chi^\varepsilon_{B_{cd_{\varepsilon,i}}(\varepsilon z_i)}\Delta\mathbf{t}^\varepsilon_{i,k}\to 0\quad\text{strongly in } W^{-1,\infty}(\Omega;\mathbb{R}^3). \tag{4.38}$$

We conclude (4.28) with (4.33), (4.36), (4.37), and (4.38) since a.s.

$$\sum_{z_i\in n_N^\varepsilon}c^{-2}d_{\varepsilon,i}^{-2}a_{\varepsilon,i}\delta^{cd_{\varepsilon,i}}(\mathbf{e}_k\cdot\mathbf{e}^i)\mathbf{e}^i\ \to\ \frac{\sigma_3}{3}\langle\mathcal{N}_{\frac{2}{N}}(\Omega)\rangle\langle r_N\rangle\mathbf{e}_k\quad\text{strongly in } W^{-1,\infty}(\Omega;\mathbb{R}^3).$$

□

Lemma 4.12. *Assume that* $\mathrm{Ma}(\varepsilon)$ *obeys* (4.3). *Let* $\boldsymbol{v}_\varepsilon\in L^2(0,T;W_0^{1,2}(\Omega_\varepsilon;\mathbb{R}^3))$ *be such that* (2.17)–(2.20) *hold for some* $\boldsymbol{v}\in L^2(0,T;W_0^{1,2}(\Omega;\mathbb{R}^3))$. *Then, for every* $N\in\mathbb{N}$ *and every* $\psi\in C_c^\infty((0,T)\times\Omega)$, *we have*

$$\lim_{\varepsilon\to 0}\int_0^\tau\int_\Omega q^{\varepsilon,g}_{k,N}\psi\,\mathrm{div}_x(\boldsymbol{v}_\varepsilon)\,\mathrm{d}x\,\mathrm{d}t=0\qquad\text{for a.e. }\tau\in[0,T].$$

Proof. Let $\xi\in C_c^\infty(\mathbb{R}^3)$ be a cut-off function such that $\xi\equiv 1$ in $B_{\frac14}$ and $\xi\equiv 0$ in $\mathbb{R}^3\setminus B_{\frac12}$. Then, we define

$$\xi_\varepsilon(x):=\begin{cases}\xi\big(\frac{x-\varepsilon z_i}{d_{\varepsilon,i}}\big) & \text{in } B_{1,i},\ z_i\in n_N^\varepsilon,\\ 0 & \text{in } \Omega\setminus\big(\bigcup_{z_i\in n_N^\varepsilon}B_{1,i}\big),\end{cases} \tag{4.39}$$

hence $|\nabla_x\xi_\varepsilon|\lesssim d_{\varepsilon,i}^{-1}$. By the assumed form of $\mathrm{div}_x\mathbf{v}_\varepsilon$ in (2.17) we may write

$$\begin{aligned}\int_0^\tau\int_\Omega q^{\varepsilon,g}_{k,N}\mathrm{div}_x(\mathbf{v}_\varepsilon)\psi\,\mathrm{d}x\,\mathrm{d}t&=\int_0^\tau\int_\Omega q^{\varepsilon,g}_{k,N}\xi_\varepsilon\mathrm{div}_x(\mathbf{v}_\varepsilon)\psi\,\mathrm{d}x\,\mathrm{d}t+\int_0^\tau\int_\Omega q^{\varepsilon,g}_{k,N}(1-\xi_\varepsilon)\mathrm{div}_x(\mathbf{v}_\varepsilon)\psi\,\mathrm{d}x\,\mathrm{d}t\\&=-\int_0^\tau\int_\Omega\nabla_x(q^{\varepsilon,g}_{k,N}\xi_\varepsilon\psi)\mathbf{h}_\varepsilon\,\mathrm{d}x\,\mathrm{d}t-\int_0^\tau\int_\Omega q^{\varepsilon,g}_{k,N}\xi_\varepsilon g_\varepsilon\partial_t\psi\,\mathrm{d}x\,\mathrm{d}t\\&\quad+\int_\Omega q^{\varepsilon,g}_{k,N}\xi_\varepsilon g_\varepsilon(\tau,\cdot)\psi(\tau,\cdot)\,\mathrm{d}x+\int_0^\tau\int_\Omega q^{\varepsilon,g}_{k,N}(1-\xi_\varepsilon)\mathrm{div}_x(\mathbf{v}_\varepsilon)\psi\,\mathrm{d}x\,\mathrm{d}t.\end{aligned} \tag{4.40}$$

Note that (4.3) in particular implies $\lim_{\varepsilon\to 0}\mathrm{Ma}(\varepsilon)=0$, hence from (2.17) we have $\|\mathrm{div}_x(\mathbf{v}_\varepsilon)\|_{L^2(0,T;L^2(\Omega))}\lesssim 1$. This together with (4.25) yields

$$\begin{aligned}\left|\int_0^\tau\int_\Omega q^{\varepsilon,g}_{k,N}(1-\xi_\varepsilon)\mathrm{div}_x(\mathbf{v}_\varepsilon)\psi\,\mathrm{d}x\,\mathrm{d}t\right|&\lesssim\int_0^\tau\int_{\Omega\setminus\bigcup_{z_i\in n_N^\varepsilon}B_{\frac14 d_{\varepsilon,i}}(\varepsilon z_i)}|q^{\varepsilon,g}_{k,N}||\mathrm{div}_x(\mathbf{v}_\varepsilon)|\,\mathrm{d}x\,\mathrm{d}t\\&\lesssim\|q^{\varepsilon,g}_{k,N}\|_{L^2(\bigcup_{z_i\in n_N^\varepsilon}A_i)}\|\mathrm{div}_x(\mathbf{v}_\varepsilon)\|_{L^2(0,T;L^2(\Omega))}\lesssim\varepsilon^{\frac{\delta}{2}},\end{aligned}$$

which in turn implies that the last term on the right-hand side of (4.40) vanishes as $\varepsilon\to 0$.
Since ξ_ε is bounded in $L^\infty(\Omega)$ and by (2.19), (2.20), we find for the second and third term in (4.40) and for $\gamma<2$

$$\begin{aligned}&\left|\int_0^\tau\int_\Omega q^{\varepsilon,g}_{k,N}\xi_\varepsilon g_\varepsilon\partial_t\psi\,\mathrm{d}x\,\mathrm{d}t\right|+\left|\int_\Omega q^{\varepsilon,g}_{k,N}\xi_\varepsilon g_\varepsilon(\tau,\cdot)\psi(\tau,\cdot)\,\mathrm{d}x\right|\\&\quad\lesssim\|\xi_\varepsilon\|_{L^{\frac73}(\Omega)}\|g_\varepsilon\chi_{M_\varepsilon}\|_{L^\infty((0,T)\times\Omega)}\|q^{\varepsilon,g}_{k,N}\|_{L^{\frac74}(\Omega)}+\|\xi_\varepsilon\|_{L^\infty(\Omega)}\|g_\varepsilon\chi_{M_\varepsilon^c}\|_{L^\infty(0,T;L^\gamma(\Omega))}\|q^{\varepsilon,g}_{k,N}\|_{L^{\frac{\gamma}{\gamma-1}}(\Omega)}\\&\quad\lesssim\|q^{\varepsilon,g}_{k,N}\|_{L^{\frac74}(\Omega)}+\|q^{\varepsilon,g}_{k,N}\|_{L^{\frac{\gamma}{\gamma-1}}(\Omega)}\mathrm{Ma}(\varepsilon)^{\frac{2}{\gamma}}\\&\quad\lesssim(\varepsilon^{\frac{\delta}{\gamma}}\varepsilon^{-\frac{(1+\delta)(2-\gamma)}{\gamma}}+\varepsilon^{-\frac{3(2-\gamma)}{\gamma}})\mathrm{Ma}(\varepsilon)^{\frac{2}{\gamma}}\lesssim\varepsilon^{-\frac{3(2-\gamma)}{\gamma}}\mathrm{Ma}(\varepsilon)^{\frac{2}{\gamma}},\end{aligned}$$

where $\frac{\gamma}{\gamma-1}>2>\frac32$ for any $\frac32<\gamma<2$, and the last inequality comes from the fact that $(\delta-\delta(2-\gamma))=\delta(\gamma-1)>0$. Consequently, seeing that also $-3\gamma^{-1}(2-\gamma)\in(-1,0)$ for any $\gamma\in(\frac32,2)$, the right-hand side vanishes due to (4.3).

For $\gamma \geq 2$ we have

$$\begin{aligned}
&\left|\int_0^\tau \int_\Omega q_{k,N}^{\varepsilon,g} \xi_\varepsilon g_\varepsilon \partial_t \psi \,\mathrm{d}x\,\mathrm{d}t\right| + \left|\int_\Omega q_{k,N}^{\varepsilon,g} \xi_\varepsilon g_\varepsilon(\tau,\cdot)\psi(\tau,\cdot)\,\mathrm{d}x\right| \\
&\quad\lesssim \|\xi_\varepsilon\|_{L^{\frac{7}{3}}(\Omega)} \|g_\varepsilon \chi_{M_\varepsilon}\|_{L^\infty((0,T)\times\Omega)} \|q_{k,N}^{\varepsilon,g}\|_{L^{\frac{7}{4}}(\Omega)} + \|\xi_\varepsilon\|_{L^{\frac{2\gamma}{\gamma-2}}(\Omega)} \|g_\varepsilon \chi_{M_\varepsilon^c}\|_{L^\infty(0,T;L^\gamma(\Omega))} \|q_{k,N}^{\varepsilon,g}\|_{L^2(\Omega)} \\
&\quad\lesssim \varepsilon^{\frac{3\delta}{7}} + \varepsilon^{\frac{1+2\delta}{7}} + \mathrm{Ma}(\varepsilon)^{\frac{2}{\gamma}},
\end{aligned}$$

and the right-hand side vanishes as $\varepsilon \to 0$.
Focusing on the first term in (4.40), for $\gamma < 3$, we can use (4.24), since $\frac{6\gamma}{5\gamma-6} \geq 2 > \frac{3}{2}$, and $\|\mathbf{h}_\varepsilon\|_{L^2(0,T;L^{\frac{6\gamma}{\gamma+6}}(\Omega_\varepsilon;\mathbb{R}^3))} \lesssim \mathrm{Ma}(\varepsilon)^{\min\{\frac{2}{\gamma},1\}}$ to get

$$\begin{aligned}
\left|\int_0^\tau \int_\Omega \nabla_x(q_{k,N}^{\varepsilon,g}\xi_\varepsilon\psi)\mathbf{h}_\varepsilon \,\mathrm{d}x\,\mathrm{d}t\right| &\lesssim \|\nabla_x(q_{k,N}^{\varepsilon,g}\xi_\varepsilon\psi)\|_{L^2(0,T;L^{\frac{6\gamma}{5\gamma-6}}(\bigcup_{z_i\in n_N^\varepsilon} C_i))} \|\mathbf{h}_\varepsilon\|_{L^2(0,T;L^{\frac{6\gamma}{6+\gamma}}(\Omega;\mathbb{R}^3))} \\
&\lesssim \left(\|\nabla_x q_{k,N}^{\varepsilon,g}\|_{L^{\frac{6\gamma}{5\gamma-6}}(\bigcup_{z_i\in n_N^\varepsilon} C_i;\mathbb{R}^3)} + \frac{1}{d_{\varepsilon,i}}\|q_{k,N}^{\varepsilon,g}\|_{L^{\frac{6\gamma}{5\gamma-6}}(\bigcup_{z_i\in n_N^\varepsilon} C_i)}\right) \mathrm{Ma}(\varepsilon)^{\min\{\frac{2}{\gamma},1\}} \\
&\lesssim \left(\varepsilon^{-\frac{\gamma+6}{\gamma}} + \varepsilon^{-(1+\delta)}\varepsilon^6\right) \mathrm{Ma}(\varepsilon)^{\min\{\frac{2}{\gamma},1\}} \lesssim \varepsilon^{-\frac{\gamma+6}{\gamma}} \mathrm{Ma}(\varepsilon)^{\min\{\frac{2}{\gamma},1\}},
\end{aligned} \tag{4.41}$$

where we used that on each C_i

$$|\nabla_x(q_{k,N}^{\varepsilon,g}\xi_\varepsilon\psi)| \lesssim |\xi_\varepsilon||\psi||\nabla_x q_{k,N}^{\varepsilon,g}| + |\nabla_x\xi_\varepsilon||\psi||q_{k,N}^{\varepsilon,g}| + |\xi_\varepsilon||\nabla_x\psi||q_{k,N}^{\varepsilon,g}| \lesssim |\nabla_x q_{k,N}^{\varepsilon,g}| + \frac{1}{d_{\varepsilon,i}}|q_{k,N}^{\varepsilon,g}|.$$

By assumption (4.3) the right-hand side of (4.41) vanishes for $\varepsilon \to 0$.
Finally, for $\gamma \geq 3$, we can use (4.24), $\frac{6\gamma}{5\gamma-6} \leq 2$, and $\|\mathbf{h}_\varepsilon\|_{L^2(0,T;L^{\frac{6\gamma}{\gamma+6}}(\Omega_\varepsilon;\mathbb{R}^3))} \lesssim \mathrm{Ma}(\varepsilon)^{\min\{\frac{2}{\gamma},1\}}$ to get

$$\begin{aligned}
\left|\int_0^\tau \int_\Omega \nabla_x(q_{k,N}^{\varepsilon,g}\xi_\varepsilon\psi)\mathbf{h}_\varepsilon \,\mathrm{d}x\,\mathrm{d}t\right| &\lesssim \|\nabla_x(q_{k,N}^{\varepsilon,g}\xi_\varepsilon\psi)\|_{L^2(0,T;L^{\frac{6\gamma}{5\gamma-6}}(\bigcup_{z_i\in n_N^\varepsilon} C_i;\mathbb{R}^3))} \|\mathbf{h}_\varepsilon\|_{L^2(0,T;L^{\frac{6\gamma}{6+\gamma}}(\Omega;\mathbb{R}^3))} \\
&\lesssim \left(\|\nabla_x q_{k,N}^{\varepsilon,g}\|_{L^{\frac{6\gamma}{5\gamma-6}}(\bigcup_{z_i\in n_N^\varepsilon} C_i;\mathbb{R}^3)} + \frac{1}{d_{\varepsilon,i}}\|q_{k,N}^{\varepsilon,g}\|_{L^{\frac{6\gamma}{5\gamma-6}}(\bigcup_{z_i\in n_N^\varepsilon} C_i)}\right) \mathrm{Ma}(\varepsilon)^{\min\{\frac{2}{\gamma},1\}} \\
&\lesssim \left(\|\nabla_x q_{k,N}^{\varepsilon,g}\|_{L^2(\bigcup_{z_i\in n_N^\varepsilon} C_i;\mathbb{R}^3)} + \frac{1}{d_{\varepsilon,i}}\|q_{k,N}^{\varepsilon,g}\|_{L^2(\bigcup_{z_i\in n_N^\varepsilon} C_i)}\right) \mathrm{Ma}(\varepsilon)^{\min\{\frac{2}{\gamma},1\}} \\
&\lesssim \left(\varepsilon^{-3} + \varepsilon^{-(1+\delta)}\right) \mathrm{Ma}(\varepsilon)^{\min\{\frac{2}{\gamma},1\}} \lesssim \varepsilon^{-3}\mathrm{Ma}(\varepsilon)^{\min\{\frac{2}{\gamma},1\}},
\end{aligned} \tag{4.42}$$

By assumption (4.3) the right-hand side of (4.42) vanishes for $\varepsilon \to 0$ and therefore the right-hand side of (4.40) does as well. □

Having the above lemmas at hand, we can now show:

Lemma 4.13. *Assume that* $\mathrm{Ma}(\varepsilon)$ *obeys* (4.3). *Let* $(\boldsymbol{\omega}_k^{\varepsilon,g}, q_k^{\varepsilon,g})_{1\leq k\leq 3}$ *be the pairs of functions from Lemma 4.8. Then there exist* $\boldsymbol{\mu}_k \in W^{-1,\infty}(\Omega;\mathbb{R}^3)$, $k \in \{1,2,3\}$, *such that the pairs* $(\boldsymbol{\omega}_k^{\varepsilon,g}, \boldsymbol{\mu}_k)_{1\leq k\leq 3}$ *satisfy (H5).*

Proof.

Step 1: First, we show that there exist $\boldsymbol{\mu}_{k,N}$ such that (H5) holds for $(\boldsymbol{\omega}^{\varepsilon,g}_{k,N}, \boldsymbol{\mu}_{k,N})$, where $\boldsymbol{\omega}^{\varepsilon,g}_{k,N}$ is given in Definition 4.9. To this end, we start in rewriting

$$\begin{aligned}\int_0^\tau \int_\Omega \nabla_x \boldsymbol{\omega}^{\varepsilon,g}_{k,N} : \nabla_x(\psi \mathbf{v}_\varepsilon)\,\mathrm{d}x\,\mathrm{d}t &= \int_0^\tau \int_\Omega \nabla_x \boldsymbol{\omega}^{\varepsilon,g}_{k,N} : \nabla_x(\psi \mathbf{v}_\varepsilon) - q^{\varepsilon,g}_{k,N} \mathrm{div}_x(\psi \mathbf{v}_\varepsilon)\,\mathrm{d}x\,\mathrm{d}t \\ &\quad + \int_0^\tau \int_\Omega q^{\varepsilon,g}_{k,N} \mathbf{v}_\varepsilon \cdot \nabla_x \psi\,\mathrm{d}x\,\mathrm{d}t + \int_0^\tau \int_\Omega \psi q^{\varepsilon,g}_{k,N} \mathrm{div}_x(\mathbf{v}_\varepsilon)\,\mathrm{d}x\,\mathrm{d}t.\end{aligned} \tag{4.43}$$

The first term of the right-hand side becomes

$$\begin{aligned}\int_0^\tau \int_\Omega \nabla_x \boldsymbol{\omega}^{\varepsilon,g}_{k,N} : \nabla_x(\psi \mathbf{v}_\varepsilon) - q^{\varepsilon,g}_{k,N} \mathrm{div}_x(\psi \mathbf{v}_\varepsilon)\,\mathrm{d}x\,\mathrm{d}t &= -\int_0^\tau \int_{\bigcup_{z_i \in n^\varepsilon_N} C_i} (\Delta \boldsymbol{\omega}^{\varepsilon,g}_{k,N} - \nabla_x q^{\varepsilon,g}_{k,N}) \cdot (\psi \mathbf{v}_\varepsilon)\,\mathrm{d}x\,\mathrm{d}t \\ &\quad + \int_0^\tau \int_\Omega \left(\sum_{z_i \in n^\varepsilon_N} \left(\frac{\partial \boldsymbol{\omega}^{\varepsilon,g}_{k,N}}{\partial s_i} - \mathbf{e}^i q^{\varepsilon,g}_{k,N} \right) \delta_i^{\frac{d_{\varepsilon,i}}{2}} \right) \cdot (\psi \mathbf{v}_\varepsilon)\,\mathrm{d}x\,\mathrm{d}t \\ &\quad + \int_0^\tau \int_{\bigcup_{z_i \in n^\varepsilon_N} D_i} (\nabla_x \boldsymbol{\omega}^{\varepsilon,g}_{k,N} : \nabla_x(\psi \mathbf{v}_\varepsilon) - q^{\varepsilon,g}_{k,N} \mathrm{div}_x(\psi \mathbf{v}_\varepsilon))\,\mathrm{d}x\,\mathrm{d}t,\end{aligned}$$

where $s_i = |x - \varepsilon z_i|$ is the radial coordinate and $\mathbf{e}^i \colon x \mapsto \frac{x - \varepsilon z_i}{|x - \varepsilon z_i|}$ the unit vector of $x \in B_{cd_{\varepsilon,i}}(\varepsilon z_i)$ in $B_{cd_{\varepsilon,i}}(\varepsilon z_i)$. Due to the definition of $\boldsymbol{\omega}^{\varepsilon,g}_{k,N}$ and $q^{\varepsilon,g}_{k,N}$, the first term on the right-hand side above vanishes identically. For the last term we get from (4.25)

$$\begin{aligned}&\left| \int_0^\tau \int_{\bigcup_{z_i \in n^\varepsilon_N} D_i} \nabla_x \boldsymbol{\omega}^{\varepsilon,g}_{k,N} : \nabla_x(\psi \mathbf{v}_\varepsilon) - q^{\varepsilon,g}_{k,N} \mathrm{div}_x(\psi \mathbf{v}_\varepsilon)\,\mathrm{d}x\,\mathrm{d}t \right| \\ &\qquad \lesssim (\|\nabla_x \boldsymbol{\omega}^{\varepsilon,g}_{k,N}\|_{L^2(\Omega;\mathbb{R}^3)} + \|q^{\varepsilon,g}_{k,N}\|_{L^2(\Omega)}) \|\mathbf{v}_\varepsilon\|_{L^2(0,T;W^{1,2}(\Omega;\mathbb{R}^3))} \lesssim \varepsilon^{\frac{\delta}{2}}.\end{aligned}$$

Hence, this term vanishes for $\varepsilon \to 0$. Next, we define

$$\boldsymbol{\mu}^\varepsilon_{k,N} := \sum_{z_i \in n^\varepsilon_N} \left(\frac{\partial \boldsymbol{\omega}^{\varepsilon,g}_{k,N}}{\partial s_i} - \mathbf{e}^i q^{\varepsilon,g}_{k,N} \right) \delta_i^{\frac{d_{\varepsilon,i}}{2}},$$

and show that

$$\int_0^\tau \int_\Omega \boldsymbol{\mu}^\varepsilon_{k,N} \cdot (\psi \mathbf{v}_\varepsilon)\,\mathrm{d}x\,\mathrm{d}t \to \int_0^\tau \int_\Omega \boldsymbol{\mu}_{k,N} \cdot (\psi \mathbf{v})\,\mathrm{d}x\,\mathrm{d}t, \tag{4.44}$$

where

$$\boldsymbol{\mu}_{k,N} = 2\sigma_3 \langle \mathcal{N}_{\frac{2}{N}}(\Omega) \rangle \langle r_N \rangle \mathbb{F}_k, \qquad \mathbb{F}_k = \int_{\partial B_1} \left(\frac{\partial \boldsymbol{\omega}_k}{\partial \mathbf{n}} - q_k \mathbf{n} \right) \mathrm{d}\mathcal{H}^2.$$

By invoking [4, Lemma 2.3.5] we get

$$\left(\frac{\partial \boldsymbol{\omega}^{\varepsilon,g}_{k,N}}{\partial s_i} - \mathbf{e}^i q^{\varepsilon,g}_{k,N} \right) \Bigg|_{s_i = \frac{d_{\varepsilon,i}}{2}} = \frac{1}{a_{\varepsilon,i}} \left(\frac{\partial \boldsymbol{\omega}_k}{\partial s_i} - \mathbf{e}^i q_k \right) \Bigg|_{s_i = \frac{d_{\varepsilon,i}}{2a_{\varepsilon,i}}} = \frac{2a_{\varepsilon,i}}{\sigma_3 d^2_{\varepsilon,i}} [\mathbb{F}_k + 3(\mathbb{F}_k \cdot \mathbf{e}^i)\mathbf{e}^i] + \mathbf{R}_\varepsilon,$$

where

$$\|\mathbf{R}_\varepsilon\|_{L^\infty(\Omega)} \lesssim \frac{a^2_{\varepsilon,i}}{d^3_{\varepsilon,i}} \lesssim \varepsilon^\delta d^{-2}_{\varepsilon,i} a_{\varepsilon,i}.$$

Now by Lemma 4.11 it follows that

$$\sum_{z_i \in n^\varepsilon_N} \left(\frac{d_{\varepsilon,i}}{2} \right)^{-2} a_{\varepsilon,i} [\mathbb{F}_k + 3(\mathbb{F}_k \cdot \mathbf{e}^i) \cdot \mathbf{e}^i] \delta_i^{\frac{d_{\varepsilon,i}}{2}} \to 2\sigma_3 \langle \mathcal{N}_{\frac{2}{N}}(\Omega) \rangle \langle r_N \rangle \mathbb{F}_k \quad \text{strongly in } W^{-1,2}(\Omega;\mathbb{R}^3) \text{ a.s.}$$

For the remaining term we have

$$-4C\sum_{z_i\in n_N^\varepsilon}\varepsilon^\delta d_{\varepsilon,i}^{-2}a_{\varepsilon,i}\delta_i^{d_{\varepsilon,i}/2}\le\sum_{z_i\in n_N^\varepsilon}\mathbf{R}_\varepsilon(x)\cdot\mathbf{e}_k\delta_i^{d_{\varepsilon,i}/2}\le 4C\sum_{z_i\in n_N^\varepsilon}\varepsilon^\delta d_{\varepsilon,i}^{-2}a_{\varepsilon,i}\delta_i^{d_{\varepsilon,i}/2}.$$

By adding $4C\sum_{z_i\in n_N^\varepsilon}\varepsilon^\delta d_{\varepsilon,i}^{-2}a_{\varepsilon,i}\delta_i^{d_{\varepsilon,i}/2}$ on both sides we get

$$0\le\sum_{z_i\in n_N^\varepsilon}\mathbf{R}_\varepsilon(x)\cdot\mathbf{e}_k\delta_i^{d_{\varepsilon,i}/2}+4C\sum_{z_i\in n_N^\varepsilon}\varepsilon^\delta d_{\varepsilon,i}^{-2}a_{\varepsilon,i}\delta_i^{d_{\varepsilon,i}/2}\le 8C\sum_{z_i\in n_N^\varepsilon}\varepsilon^\delta d_{\varepsilon,i}^{-2}a_{\varepsilon,i}\delta_i^{d_{\varepsilon,i}/2}.$$

Note that Lemma 4.11 enforces

$$4\sum_{z_i\in n_N^\varepsilon}\varepsilon^\delta d_{\varepsilon,i}^{-2}a_{\varepsilon,i}\delta_i^{d_{\varepsilon,i}/2}\to 0\quad\text{strongly in } W^{-1,2}(\Omega)\text{ a.s.}$$

Using [4, Lemma 2.3.8] we deduce that

$$\sum_{z_i\in n_N^\varepsilon}\mathbf{R}_\varepsilon(x)\cdot\mathbf{e}_k\delta_i^{d_{\varepsilon,i}/2}+4C\sum_{z_i\in n_N^\varepsilon}d_{\varepsilon,i}^{-2}a_{\varepsilon,i}\delta_i^{d_{\varepsilon,i}/2}\to 0\quad\text{strongly in } W^{-1,2}(\Omega)\text{ a.s.},$$

and finally

$$\sum_{z_i\in n_N^\varepsilon}\mathbf{R}_\varepsilon(x)\cdot\mathbf{e}_k\delta_i^{d_{\varepsilon,i}/2}\to 0\quad\text{strongly in } W^{-1,2}(\Omega)\text{ a.s.}$$

Therefore, we find

$$\boldsymbol{\mu}_{k,N}^\varepsilon\to\boldsymbol{\mu}_{k,N}\quad\text{strongly in } W^{-1,2}(\Omega;\mathbb{R}^3)\text{ a.s.},$$

and (4.44) is proven. Using moreover $q_{k,N}^{\varepsilon,g}\to 0$ strongly in $L^{\frac{6}{5}}(\Omega)$ and $\mathbf{v}_\varepsilon\rightharpoonup\mathbf{v}$ weakly in $L^2(0,T;W^{1,2}(\Omega;\mathbb{R}^3))$, we additionally get that

$$\int_0^\tau\int_\Omega q_{k,N}^{\varepsilon,g}\nabla_x\psi\cdot\mathbf{v}_\varepsilon\,\mathrm{d}x\,\mathrm{d}t\to 0.$$

Combining this with Lemma 4.12, the last two terms in (4.43) vanish, which concludes the proof of this first step.

Step 2: We prove that (H5) holds true for $\boldsymbol{\omega}_k^{\varepsilon,g}$ and

$$\boldsymbol{\mu}_k=2\sigma_3\lambda\langle r\rangle\mathbb{F}_k,\qquad\qquad \mathbb{F}_k=\int_{\partial B_1}\left(\frac{\partial\boldsymbol{\omega}_k}{\partial\mathbf{n}}-q_k\mathbf{n}\right)\mathrm{d}\mathcal{H}^2. \tag{4.45}$$

This follows arguing as in the proof of [20, Lemma 2.5]. Indeed, for each $N\in\mathbb{N}$ we have

$$\begin{aligned}
&\limsup_{\varepsilon\to0}\left|\int_0^\tau\int_\Omega\nabla_x\boldsymbol{\omega}_k^{\varepsilon,g}:\nabla_x(\psi\mathbf{v}_\varepsilon)\,\mathrm{d}x\,\mathrm{d}t-\int_0^\tau\int_\Omega\boldsymbol{\mu}_k\cdot(\psi\mathbf{v})\,\mathrm{d}x\,\mathrm{d}t\right|\\
&\lesssim\left|\int_0^\tau\int_\Omega(\boldsymbol{\mu}_{k,N}-\boldsymbol{\mu}_k)\cdot(\psi\mathbf{v})\,\mathrm{d}x\,\mathrm{d}t\right|+\limsup_{\varepsilon\to0}\left|\int_0^\tau\int_\Omega\nabla_x(\boldsymbol{\omega}_k^{\varepsilon,g}-\boldsymbol{\omega}_{k,N}^{\varepsilon,g}):\nabla_x(\psi\mathbf{v}_\varepsilon)\,\mathrm{d}x\,\mathrm{d}t\right|\\
&\quad+\limsup_{\varepsilon\to0}\left|\int_0^\tau\int_\Omega\nabla_x\boldsymbol{\omega}_{k,N}^{\varepsilon,g}:\nabla_x(\psi\mathbf{v}_\varepsilon)\,\mathrm{d}x\,\mathrm{d}t-\int_0^\tau\int_\Omega\boldsymbol{\mu}_{k,N}\cdot(\psi\mathbf{v})\,\mathrm{d}x\,\mathrm{d}t\right|\\
&\lesssim\left|\int_0^\tau\int_\Omega(\boldsymbol{\mu}_{k,N}-\boldsymbol{\mu}_k)\cdot(\psi\mathbf{v})\,\mathrm{d}x\,\mathrm{d}t\right|+\limsup_{\varepsilon\to0}\sum_{z_i\in n^\varepsilon}\|\nabla_x(\boldsymbol{\omega}_k^{\varepsilon,g}-\boldsymbol{\omega}_{k,N}^{\varepsilon,g})\|_{L^2(B_{d_{\varepsilon,i}}(\varepsilon z_i))}.
\end{aligned}$$

In the last inequality we used that (H5) holds for $\boldsymbol{\omega}_{k,N}^{\varepsilon,g}$ and that due to the definition of $\boldsymbol{\omega}_k^{\varepsilon,g}$ and $\boldsymbol{\omega}_{k,N}^{\varepsilon,g}$ we have

$$\begin{aligned} \operatorname{supp}(\boldsymbol{\omega}_k^{\varepsilon,g} - \boldsymbol{\omega}_{k,N}^{\varepsilon,g}) &\subset \bigcup_{\substack{z_i \in n_N^\varepsilon \\ r_i \geq N}} B_{d_{\varepsilon,i}}(\varepsilon z_i) \cup \bigcup_{z_i \in n^\varepsilon \setminus n_N^\varepsilon} B_{d_{\varepsilon,i}}(\varepsilon z_i), \\ \boldsymbol{\omega}_k^{\varepsilon,g} - \boldsymbol{\omega}_{k,N}^{\varepsilon,g} &= \boldsymbol{\omega}_k^{\varepsilon,g} - \mathbf{e}_k \quad \text{in} \bigcup_{z_i \in n^\varepsilon \setminus n_N^\varepsilon} B_{d_{\varepsilon,i}}(\varepsilon z_i). \end{aligned}$$

Therefore, it follows along with Lemma 4.8 that

$$\begin{aligned} &\sum_{z_i \in n^\varepsilon} \|\nabla_x(\boldsymbol{\omega}_k^{\varepsilon,g} - \boldsymbol{\omega}_{k,N}^{\varepsilon,g})\|_{L^2(B_{d_{\varepsilon,i}}(\varepsilon z_i);\mathbb{R}^{3\times3})} \\ &\lesssim \sum_{\substack{z_i \in n_N^\varepsilon \\ r_i \geq N}} (\|\nabla_x \boldsymbol{\omega}_k^{\varepsilon,g}\|_{L^2(B_{d_{\varepsilon,i}}(\varepsilon z_i);\mathbb{R}^{3\times3})} + \|\nabla_x \boldsymbol{\omega}_{k,N}^{\varepsilon,g}\|_{L^2(B_{d_{\varepsilon,i}}(\varepsilon z_i);\mathbb{R}^{3\times3})}) + \sum_{z_i \in n^\varepsilon \setminus n_N^\varepsilon} \|\nabla_x \boldsymbol{\omega}_k^{\varepsilon,g}\|_{L^2(B_{d_{\varepsilon,i}}(\varepsilon z_i);\mathbb{R}^{3\times3})} \\ &\lesssim \sum_{\substack{z_i \in n_N^\varepsilon \\ r_i \geq N}} \varepsilon^3 (r_i + r_{i,N}) + \sum_{z_i \in n^\varepsilon \setminus n_N^\varepsilon} \varepsilon^3 r_i \lesssim \sum_{z_i \in n^\varepsilon} \varepsilon^3 r_i \mathbb{1}_{r_i \geq N} + \sum_{z_i \in n^\varepsilon \setminus n_N^\varepsilon} \varepsilon^3 r_i. \end{aligned} \tag{4.46}$$

The Strong Law of Large Numbers (Lemma 4.6) yields that

$$\lim_{\varepsilon \to 0} \varepsilon^3 \sum_{z_i \in n^\varepsilon} r_i \mathbb{1}_{r_i \geq N} = \langle r \mathbb{1}_{r \geq N} \rangle.$$

We use [4, Lemma C.1], (4.6) and $n_N^\varepsilon \subset n^\varepsilon$ to conclude

$$\lim_{N \to \infty} \lim_{\varepsilon \to 0} \varepsilon^3 \#(n^\varepsilon \setminus n_N^\varepsilon) = \lim_{N \to \infty} \lambda |\Omega| - \langle \mathcal{N}_{\frac{2}{N}}(\Omega) \rangle = 0, \tag{4.47}$$

showing that (4.46) vanishes for $\varepsilon \to 0$ and $N \to \infty$. We conclude Step 2 by arguing that almost surely

$$\lim_{N \to \infty} \left| \int_0^\tau \int_\Omega (\boldsymbol{\mu}_{k,N} - \boldsymbol{\mu}_k) \cdot (\psi \mathbf{v}) \, dx \, dt \right| = 0. \tag{4.48}$$

Indeed, with [4, Lemma C.1] we get

$$\lim_{N \to \infty} \langle \mathcal{N}_{\frac{2}{N}}(\Omega) \rangle = \lambda |\Omega|,$$

and by (4.2) we have

$$\lim_{N \to \infty} \langle r_N \rangle = \langle r \rangle.$$

This proves that

$$\boldsymbol{\mu}_{k,N} \xrightarrow{N \to \infty} \boldsymbol{\mu}_k \quad \text{weakly in } W^{-1,2}(\Omega; \mathbb{R}^3),$$

which in turn shows (4.48). □

In order to define finally the functions $\boldsymbol{\omega}_k^\varepsilon$, we still need to define the "bad" functions $\boldsymbol{\omega}_k^{\varepsilon,b}$. Let $\boldsymbol{\Psi} \in C^\infty(\overline{\Omega}; \mathbb{R}^3)$ with $\operatorname{div}_x \boldsymbol{\Psi} = 0$. We recall from Lemma 4.5 that

$$\Omega_{\mathrm{b}}^\varepsilon = \bigcup_{z_i \in J^\varepsilon} B_{\theta \lambda_i^\varepsilon \varepsilon^3 r_i}(\varepsilon z_i)$$

and $J^\varepsilon = \bigcup_{k=-3}^{k_{\max}} J_k^\varepsilon$, where for each k the set of centers J_k^ε is such that the corresponding balls are pairwise disjoint. Let then $R_j, B_{R,j}, B_{\theta,j}, W_j$ be defined as in (4.10). We will define the "bad" functions $\boldsymbol{\Psi}_{\mathrm{b}}^\varepsilon$ recursively:

For all $0 \le i \le k_{\max}+3$ let $\mathbf{\Psi}_\varepsilon^{-1} := \mathbf{\Psi}$ and $\mathbf{\Psi}_\varepsilon^i$ be given by

$$\mathbf{\Psi}_\varepsilon^i := \begin{cases} \mathbf{\Psi}_\varepsilon^{i-1} & \text{in } \Omega \setminus \bigcup_{z_j \in J^\varepsilon_{k_{\max}-i}} B_{\theta,j}, \\ \mathbf{\Psi}_\varepsilon^{i-1} + \mathbf{\Psi}_{j,\varepsilon}^i\left(\frac{\cdot - \varepsilon z_j}{\varepsilon^3 R_j}\right) & \text{in } W_j \text{ for all } z_j \in J^\varepsilon_{k_{\max}-i}, \\ 0 & \text{in } B_{R,j} \text{ for all } z_j \in J^\varepsilon_{k_{\max}-i}, \end{cases}$$

where $(\mathbf{\Psi}_{j,\varepsilon}^i, \pi_{j,\varepsilon}^i)$ is the weak solution of

$$\begin{cases} \Delta \mathbf{\Psi}_{j,\varepsilon}^i - \nabla_x \pi_{j,\varepsilon}^i = 0 & \text{in } B_\theta \setminus B_1, \\ \operatorname{div}_x \mathbf{\Psi}_{j,\varepsilon}^i = 0 & \text{in } B_\theta \setminus B_1, \\ \mathbf{\Psi}_{j,\varepsilon}^i = -\mathbf{\Psi}_\varepsilon^{i-1}(\varepsilon^3 R_j \cdot + \varepsilon z_j) & \text{on } \partial B_1, \\ \mathbf{\Psi}_{j,\varepsilon}^i = 0 & \text{on } \partial B_\theta. \end{cases}$$

Finally, set $\mathbf{\Psi}_{\mathrm{b}}^\varepsilon := \mathbf{\Psi}_\varepsilon^{k_{\max}+3}$. With this definition, we can show:

Lemma 4.14. $\Psi_{\mathrm{b}}^\varepsilon$ *satisfies*

$$\|\nabla_x(\mathbf{\Psi}_{\mathrm{b}}^\varepsilon - \mathbf{\Psi})\|^p_{L^p(\Omega;\mathbb{R}^{3\times 3})} \lesssim \begin{cases} \varepsilon^{6(2-p)\delta} & \text{for } 1 < p \le 2 \\ \varepsilon^{3(2-p)} & \text{for } p > 2 \end{cases} \qquad \text{almost surely,} \tag{4.49}$$

and

$$\lim_{\varepsilon \to 0} \|\nabla_x(\mathbf{\Psi}_{\mathrm{b}}^\varepsilon - \mathbf{\Psi})\|^2_{L^2(\Omega;\mathbb{R}^{3\times 3})} = 0 \qquad \text{almost surely.} \tag{4.50}$$

Proof. We observe that for all $0 \le i \le k_{\max}+3$

$$\|\mathbf{\Psi}_\varepsilon^i\|_{C^0(\Omega;\mathbb{R}^3)} \lesssim \|\mathbf{\Psi}\|_{C^0(\Omega;\mathbb{R}^3)}. \tag{4.51}$$

In the next step we show by induction that for all $0 \le i \le k_{\max}+3$ there holds

$$\|\nabla_x(\mathbf{\Psi} - \mathbf{\Psi}_\varepsilon^i)\|^p_{L^p(\Omega;\mathbb{R}^{3\times 3})} \lesssim \sum_{z_j \in \bigcup_{k=0}^i J^\varepsilon_{k_{\max}-k}} (\|\nabla_x \mathbf{\Psi}\|^p_{L^p(B_{\theta,j};\mathbb{R}^{3\times 3})} + (\varepsilon^3 R_j)^{3-p}\|\mathbf{\Psi}\|^p_{C^0(\Omega;\mathbb{R}^3)}). \tag{4.52}$$

Let $i = 0$, then we have by definition for all $z_j \in J^\varepsilon_{k_{\max}}$

$$\|\nabla_x(\mathbf{\Psi} - \mathbf{\Psi}_\varepsilon^0)\|_{L^p(B_{R,j};\mathbb{R}^{3\times 3})} = \|\nabla_x \mathbf{\Psi}\|_{L^p(B_{R,j};\mathbb{R}^{3\times 3})}, \qquad \|\nabla_x(\mathbf{\Psi} - \mathbf{\Psi}_\varepsilon^0)\|_{L^p(\Omega \setminus \bigcup_{z_j \in J^\varepsilon_{k_{\max}}} B_{\theta,j};\mathbb{R}^{3\times 3})} = 0.$$

Let η be a cut-off function on W_i with $\eta = 0$ on $\partial B_{\theta,i}$, $\eta = 1$ on $\partial B_{R,j}$, and $|\nabla_x \eta| \lesssim (\varepsilon^3 R_j)^{-1}$. By [19, Theorem II.4.3. and Theorem IV.6.1.] and (4.51) we have

$$\begin{aligned} \|\nabla_x(\mathbf{\Psi} - \mathbf{\Psi}_\varepsilon^0)\|^p_{L^p(W_j;\mathbb{R}^{3\times 3})} &= (\varepsilon^3 R_j)^{3-p}\|\nabla_x \mathbf{\Psi}_{j,\varepsilon}^0\|^p_{L^p(B_\theta \setminus B_1;\mathbb{R}^{3\times 3})} \\ &\lesssim (\varepsilon^3 R_j)^{3-p}\|(\eta\mathbf{\Psi})(\varepsilon^3 R_j \cdot + \varepsilon z_i)\|^p_{W^{1-\frac{1}{p},p}(\partial(B_\theta \setminus B_1);\mathbb{R}^3)} \\ &\lesssim (\varepsilon^3 R_j)^{3-p}\|(\eta\mathbf{\Psi})(\varepsilon^3 R_j \cdot + \varepsilon z_i)\|^p_{W^{1,p}(B_\theta \setminus B_1;\mathbb{R}^3)} \\ &\lesssim (\varepsilon^3 R_j)^{-p}\|\eta\mathbf{\Psi}\|^p_{L^p(W_j;\mathbb{R}^3)} + \|\nabla_x(\eta\mathbf{\Psi})\|^p_{L^p(W_j;\mathbb{R}^3)} \\ &\lesssim (\varepsilon^3 R_j)^{-p}\|\mathbf{\Psi}\|^p_{L^p(W_j;\mathbb{R}^3)} + \|\nabla_x \mathbf{\Psi}\|^p_{L^p(W_j;\mathbb{R}^3)} \\ &\lesssim (\varepsilon^3 R_j)^{3-p}\|\mathbf{\Psi}\|^p_{C^0(\Omega;\mathbb{R}^3)} + \|\nabla_x \mathbf{\Psi}\|^p_{L^p(W_j;\mathbb{R}^3)}, \end{aligned} \tag{4.53}$$

where we used that $\nabla_x \mathbf{\Psi}_{j,\varepsilon}^0$ and $(\eta\mathbf{\Psi})(\varepsilon^3 R_j \cdot + \varepsilon z_i)$ fulfill the same boundary conditions in $B_\theta \setminus B_1$. Thus,

$$\|\nabla_x(\mathbf{\Psi} - \mathbf{\Psi}_\varepsilon^0)\|^p_{L^p(\Omega;\mathbb{R}^{3\times 3})} \lesssim \sum_{z_j \in J^\varepsilon_{k_{\max}}} (\|\nabla_x \mathbf{\Psi}\|^p_{L^p(B_{\theta,j};\mathbb{R}^{3\times 3})} + (\varepsilon^3 R_j)^{3-p}\|\mathbf{\Psi}\|^p_{C^0(\Omega;\mathbb{R}^3)}).$$

Assume now (4.52) being true for $i-1$ and we want to prove it for i. By definition we have for all $z_j \in J^\varepsilon_{k_{\max}-i}$

$$\|\nabla_x(\mathbf{\Psi}^{i-1}_\varepsilon - \mathbf{\Psi}^i_\varepsilon)\|_{L^p(B_{R,j};\mathbb{R}^{3\times3})} = \|\nabla_x \mathbf{\Psi}^{i-1}_\varepsilon\|_{L^p(B_{R,j};\mathbb{R}^{3\times3})}, \quad \|\nabla_x(\mathbf{\Psi}^{i-1}_\varepsilon - \mathbf{\Psi}^i_\varepsilon)\|_{L^p(\Omega\setminus\bigcup_{z_j\in J^\varepsilon_{k_{\max}-i}} B_{\theta,j};\mathbb{R}^{3\times3})} = 0.$$

Using (4.51) and the same arguments as in (4.53), we have for the difference between the gradients of $\mathbf{\Psi}^{i-1}_\varepsilon$ and $\mathbf{\Psi}^i_\varepsilon$

$$\begin{aligned}\|\nabla_x(\mathbf{\Psi}^{i-1}_\varepsilon - \mathbf{\Psi}^i_\varepsilon)\|^p_{L^p(W_j;\mathbb{R}^{3\times3})} &= (\varepsilon^3 R_j)^{3-p}\|\nabla_x \mathbf{\Psi}^i_{j,\varepsilon}\|^p_{L^p(B_\theta\setminus B_1;\mathbb{R}^{3\times3})}\\ &\lesssim (\varepsilon^3 R_j)^{3-p}\|\mathbf{\Psi}^{i-1}_\varepsilon\|^p_{C^0(\Omega;\mathbb{R}^3)} + \|\nabla_x \mathbf{\Psi}^{i-1}_\varepsilon\|^p_{L^p(W_j;\mathbb{R}^{3\times3})}\\ &\lesssim (\varepsilon^3 R_j)^{3-p}\|\mathbf{\Psi}\|^p_{C^0(\Omega;\mathbb{R}^3)} + \|\nabla_x \mathbf{\Psi}^{i-1}_\varepsilon\|^p_{L^p(W_j;\mathbb{R}^{3\times3})}.\end{aligned}$$

This leads to

$$\|\nabla_x(\mathbf{\Psi}^{i-1}_\varepsilon - \mathbf{\Psi}^i_\varepsilon)\|^p_{L^p(\Omega;\mathbb{R}^{3\times3})} \lesssim \sum_{z_j\in J^\varepsilon_{k_{\max}-i}} (\|\nabla_x \mathbf{\Psi}^{i-1}_\varepsilon\|^p_{L^p(B_{\theta,j};\mathbb{R}^{3\times3})} + (\varepsilon^3 R_j)^{3-p}\|\mathbf{\Psi}\|^p_{C^0(\Omega;\mathbb{R}^3)})$$

such that we get for the difference between the gradients of $\mathbf{\Psi}$ and $\mathbf{\Psi}^i_\varepsilon$

$$\begin{aligned}\|\nabla_x(\mathbf{\Psi} - \mathbf{\Psi}^i_\varepsilon)\|^p_{L^p(\Omega;\mathbb{R}^{3\times3})} &\lesssim \|\nabla_x(\mathbf{\Psi} - \mathbf{\Psi}^{i-1}_\varepsilon)\|^p_{L^p(\Omega;\mathbb{R}^{3\times3})}\\ &\quad + \sum_{z_j\in J^\varepsilon_{k_{\max}-i}} (\|\nabla_x \mathbf{\Psi}^{i-1}_\varepsilon\|^p_{L^p(B_{\theta,j};\mathbb{R}^{3\times3})} + (\varepsilon^3 R_j)^{3-p}\|\mathbf{\Psi}\|^p_{C^0(\Omega;\mathbb{R}^3)})\\ &\lesssim \|\nabla_x(\mathbf{\Psi} - \mathbf{\Psi}^{i-1}_\varepsilon)\|^p_{L^p(\Omega;\mathbb{R}^{3\times3})}\\ &\quad + \sum_{z_j\in J^\varepsilon_{k_{\max}-i}} (\|\nabla_x(\mathbf{\Psi}^{i-1}_\varepsilon - \mathbf{\Psi})\|^p_{L^p(B_{\theta,j};\mathbb{R}^{3\times3})} + (\varepsilon^3 R_j)^{3-p}\|\mathbf{\Psi}\|^p_{C^0(\Omega;\mathbb{R}^3)} + \|\nabla_x \mathbf{\Psi}\|^p_{L^p(B_{\theta,j};\mathbb{R}^{3\times3})})\\ &\lesssim \|\nabla_x(\mathbf{\Psi} - \mathbf{\Psi}^{i-1}_\varepsilon)\|^p_{L^p(\Omega;\mathbb{R}^{3\times3})} + \sum_{z_j\in J^\varepsilon_{k_{\max}-i}} (\|\nabla_x \mathbf{\Psi}\|^p_{L^p(B_{\theta,j};\mathbb{R}^{3\times3})} + (\varepsilon^3 R_j)^{3-p}\|\mathbf{\Psi}\|^p_{C^0(\Omega;\mathbb{R}^3)}).\end{aligned}$$

Since (4.52) holds for $i-1$ we deduce

$$\|\nabla_x(\mathbf{\Psi} - \mathbf{\Psi}^i_\varepsilon)\|^p_{L^p(\Omega;\mathbb{R}^{3\times3})} \lesssim \sum_{z_j\in\bigcup_{k=0}^i J^\varepsilon_{k_{\max}-k}} (\|\nabla_x \mathbf{\Psi}\|^p_{L^p(B_{\theta,j};\mathbb{R}^{3\times3})} + (\varepsilon^3 R_j)^{3-p}\|\mathbf{\Psi}\|^p_{C^0(\Omega;\mathbb{R}^3)}) \tag{4.54}$$

and (4.52) is proven. For $i = k_{\max}+3$, being $r_j \geq R_0 > 0$, $\varepsilon^3 R_j \leq \Lambda\varepsilon^{6\delta}$, and $\lambda^\varepsilon_j \in [1,\Lambda]$, we have

$$\begin{aligned}\|\nabla_x(\mathbf{\Psi}^\varepsilon_{\mathrm{b}} - \mathbf{\Psi})\|^p_{L^p(\Omega;\mathbb{R}^{3\times3})} &= \|\nabla_x(\mathbf{\Psi}^{k_{\max}+3}_\varepsilon - \mathbf{\Psi})\|^p_{L^p(\Omega;\mathbb{R}^{3\times3})}\\ &\lesssim \sum_{z_j\in J^\varepsilon} (\|\nabla_x \mathbf{\Psi}\|^p_{L^p(B_{\theta,j};\mathbb{R}^{3\times3})} + (\varepsilon^3 R_j)^{3-p}\|\mathbf{\Psi}\|^p_{C^0(\Omega;\mathbb{R}^3)})\\ &\lesssim \varepsilon^{12\delta}\varepsilon^3 \sum_{z_j\in J^\varepsilon} r_j + \varepsilon^{3(2-p)}\varepsilon^3 \sum_{z_j\in J^\varepsilon} r_j^{3-p}\\ &\lesssim \varepsilon^3 \sum_{z_j\in J^\varepsilon} r_j \cdot \begin{cases} (\varepsilon^{12\delta} + \varepsilon^{6(2-p)\delta}) & \text{for } 1 < p \leq 2,\\ (\varepsilon^{12\delta} + \varepsilon^{3(2-p)}) & \text{for } p > 2.\end{cases}\end{aligned}$$

Since $J^\varepsilon \subset \mathcal{I}^\varepsilon$ and $n^\varepsilon = \Phi^\varepsilon(\Omega)\setminus\mathcal{I}^\varepsilon$ we get from Lemma 4.4 that $\varepsilon^3 \# J^\varepsilon \to 0$ almost surely if $\varepsilon \to 0$. This together with the Strong Law of Large Numbers Lemma 4.6 implies that

$$\lim_{\varepsilon\to0} \varepsilon^3 \sum_{z_i\in J^\varepsilon} r_i = 0 \quad \text{almost surely.}$$

Consequently, we obtain

$$\|\nabla_x(\mathbf{\Psi}_{\rm b}^\varepsilon - \mathbf{\Psi})\|^p_{L^p(\Omega;\mathbb{R}^{3\times 3})} \lesssim \begin{cases} \varepsilon^{6(2-p)\delta} & \text{for } 1 < p \le 2 \\ \varepsilon^{3(2-p)} & \text{for } p > 2 \end{cases} \qquad \text{almost surely,}$$

and

$$\lim_{\varepsilon\to 0} \|\nabla_x(\mathbf{\Psi}_{\rm b}^\varepsilon - \mathbf{\Psi})\|^2_{L^2(\Omega;\mathbb{R}^{3\times 3})} = 0 \quad \text{almost surely.}$$

□

Definition 4.15. We choose $\mathbf{\Psi}(x) = \mathbf{e}_k$ in Lemma 4.14 and define $\boldsymbol{\omega}_k^{\varepsilon,b} := \mathbf{\Psi}_{\rm b}^\varepsilon$ and $\boldsymbol{\omega}_k^\varepsilon := \boldsymbol{\omega}_k^{\varepsilon,g} + \boldsymbol{\omega}_k^{\varepsilon,b} - \mathbf{e}_k$.

Proposition 4.16. *Assume that* $\mathrm{Ma}(\varepsilon)$ *obeys* (4.3)*. The functions* $(\boldsymbol{\omega}_k^\varepsilon, \boldsymbol{\mu}_k)$ *satisfy (H1)–(H5), where* $\boldsymbol{\mu}_k$ *is given by* (4.45)*.*

Proof. (H1) and (H2) are valid through the definition of $\boldsymbol{\omega}_k^{\varepsilon,g}$ and $\boldsymbol{\omega}_k^{\varepsilon,b}$. Since $\boldsymbol{\omega}_k^{\varepsilon,g} \rightharpoonup \mathbf{e}_k$ weakly in $W^{1,2}(\Omega;\mathbb{R}^3)$ (Lemma 4.8) and $\boldsymbol{\omega}_k^{\varepsilon,b} \to \mathbf{e}_k$ strongly in $W^{1,2}(\Omega;\mathbb{R}^3)$ (Lemma 4.14) hold, (H3) is valid. (H4) is true due to the definition of $\boldsymbol{\mu}_k$. Finally, (H5) holds due to Lemma 4.13 and (4.50).

□

4.3 Construction of test functions in the random setting

In this last section we prove that, if $\mathrm{Ma}(\varepsilon)$ obeys (4.3), then the following holds: for a given $\boldsymbol{\phi} \in C_c^\infty(\Omega;\mathbb{R}^3)$ with $\mathrm{div}_x\boldsymbol{\phi} = 0$ there exists a sequence $(\boldsymbol{\phi}_\varepsilon)_{\varepsilon>0}$ that satisfies (M2) and (2.21) with $\mathrm{M} = (\mu_k^i)_{1\le k,i\le 3}$ given by $\mu_k^i := \boldsymbol{\mu}_k \cdot \mathbf{e}_i$ and $\boldsymbol{\mu}_k = 2\sigma_3\lambda\langle r\rangle\mathbb{F}_k$.

Lemma 4.17. *Assume* $\mathrm{Ma}(\varepsilon)$ *obeys* (4.3)*. Then for every* $\boldsymbol{\phi} \in C_c^\infty(\Omega;\mathbb{R}^3)$ *with* $\mathrm{div}_x\boldsymbol{\phi} = 0$ *there exists a sequence* $(\boldsymbol{\phi}_\varepsilon)_{\varepsilon>0}$ *that satisfies* $\boldsymbol{\phi}_\varepsilon = 0$ *in* $\Omega\setminus\Omega_\varepsilon$, (2.13)–(2.16) *and* (2.21)*. Moreover, almost surely,*

$$\|\nabla_x(\boldsymbol{\phi}_\varepsilon - \boldsymbol{\phi})\|^p_{L^p(\Omega;\mathbb{R}^{3\times 3})} \lesssim \begin{cases} \varepsilon^{6(2-p)\delta} + \varepsilon^{(1+2\delta)(2-p)} + \varepsilon^{2-p+(p-1)\delta} & \text{for } 1 < p \le 2, \\ \varepsilon^{3(2-p)} + \varepsilon^{2-p+\delta} & \text{for } p > 2. \end{cases} \tag{4.55}$$

Remark 4.18. Note that (4.55) combined with (4.3) yields (2.16). Indeed, (2.16) requires $p = \frac{3\gamma}{2\gamma-3}$. Plugging into (4.55), we find $3(2-p) = 3\frac{\gamma-6}{2\gamma-3}$, which is the last term of the exponent given in (4.3).

Proof. The construction of $(\boldsymbol{\phi}_\varepsilon)_{\varepsilon>0}$ that satisfy $\boldsymbol{\phi}_\varepsilon = 0$ in $\Omega\setminus\Omega_\varepsilon$, (2.13), and

$$\boldsymbol{\phi}_\varepsilon \rightharpoonup \boldsymbol{\phi} \quad \text{weakly in } W^{1,2}(\Omega;\mathbb{R}^3),$$

is done in [20, Lemma 2.5.]. We repeat the construction to show that also (2.15), (2.21), and (4.55) hold. This in turn implies the validity of (2.14).

Step 1: Validity of (4.55)**.** Let $\theta > 1$ be fixed. Let $H_{\rm g}^\varepsilon, H_{\rm b}^\varepsilon, \Omega_{\rm b}^\varepsilon$ be given as in Lemma 4.4 and Lemma 4.5. We split our domain into two parts $\Omega = \Omega_{\rm b}^\varepsilon \cup (\Omega\setminus\Omega_{\rm b}^\varepsilon)$. Next, we define

$$\boldsymbol{\phi}_\varepsilon := \begin{cases} \boldsymbol{\phi}_{\rm b}^\varepsilon & \text{in } \Omega_{\rm b}^\varepsilon, \\ \boldsymbol{\phi}_{\rm g}^\varepsilon & \text{in } \Omega\setminus\Omega_{\rm b}^\varepsilon, \end{cases}$$

with $\boldsymbol{\phi}_{\rm b}^\varepsilon$ and $\boldsymbol{\phi}_{\rm g}^\varepsilon$ constructed as follows.

Construction of $\phi_{\mathrm{b}}^{\varepsilon}$ and control over $\|\nabla_x(\phi_{\mathrm{b}}^{\varepsilon}-\phi)\|^p_{L^p(\Omega;\mathbb{R}^{3\times 3})}$. We choose $\boldsymbol{\Psi}=\phi$ in Lemma 4.14 and define $\phi_{\mathrm{b}}^{\varepsilon} := \boldsymbol{\Psi}_{\mathrm{b}}^{\varepsilon}$.

Construction of $\phi_{\mathrm{g}}^{\varepsilon}$ and control over $\|\nabla_x(\phi_{g}^{\varepsilon}-\phi)\|^p_{L^p(\Omega;\mathbb{R}^{3\times 3})}$. For each $z_i\in n^\varepsilon$ let $(\phi^i_{1,\varepsilon},\pi^i_\varepsilon)$ and $(\phi^i_{2,\varepsilon},q^i_\varepsilon)$ be the unique weak solutions to

$$\begin{cases} \Delta\phi^i_{1,\varepsilon}-\nabla_x\pi^i_\varepsilon=0 & \text{in } \mathbb{R}^3\setminus B_1,\\ \operatorname{div}_x\phi^i_{1,\varepsilon}=0 & \text{in } \mathbb{R}^3\setminus B_1,\\ \phi^i_{1,\varepsilon}=\phi(a_{\varepsilon,i}\cdot+\varepsilon z_i) & \text{on } \partial B_1,\\ \phi^i_{1,\varepsilon}\to 0 & \text{for } |x|\to\infty, \end{cases} \quad\text{and}\quad \begin{cases} \Delta\phi^i_{2,\varepsilon}-\nabla_x q^i_\varepsilon=0 & \text{in } B_2\setminus B_1,\\ \operatorname{div}_x\phi^i_{2,\varepsilon}=0 & \text{in } B_2\setminus B_1,\\ \phi^i_{2,\varepsilon}=\phi^i_{1,\varepsilon}\left(\frac{d_{\varepsilon,i}\cdot}{2a_{\varepsilon,i}}\right) & \text{on } \partial B_1,\\ \phi^i_{2,\varepsilon}=0 & \text{on } \partial B_2, \end{cases}$$

respectively. Recalling the sets $T_i, C_i, D_i, B_{2,i}$ from (4.8), we now define

$$\phi_{\mathrm{g}}^{\varepsilon} := \begin{cases} 0 & \text{in } T_i,\\ \phi-\phi^i_{1,\varepsilon}\left(\frac{\cdot-\varepsilon z_i}{a_{\varepsilon,i}}\right) & \text{in } C_i,\\ \phi-\phi^i_{2,\varepsilon}\left(\frac{2(\cdot-\varepsilon z_i)}{d_{\varepsilon,i}}\right) & \text{in } D_i,\\ \phi & \text{in } \Omega\setminus\bigcup_{z_i\in n^\varepsilon} B_{2,i}. \end{cases}$$

By [31, Theorem 7.1.] and [19, Theorem II.4.3.] we have

$$\begin{aligned} \|\nabla_x\phi^i_{1,\varepsilon}\|^p_{L^p(\mathbb{R}\setminus B_1;\mathbb{R}^{3\times 3})} &\lesssim \|\phi(a_{\varepsilon,i}\cdot-z_i)\|^p_{W^{1-\frac{1}{p},p}(\partial B_1;\mathbb{R}^3)} \lesssim \|\phi(a_{\varepsilon,i}\cdot-z_i)\|^p_{W^{1,p}(B_1;\mathbb{R}^3)}\\ &\lesssim a_{\varepsilon,i}^{-d}\|\phi\|^p_{L^p(T_i;\mathbb{R}^3)}+a_{\varepsilon,i}^{-(3-p)}\|\nabla_x\phi\|^p_{L^p(T_i;\mathbb{R}^{3\times 3})}. \end{aligned} \tag{4.56}$$

Hence, it follows

$$\begin{aligned} \|\nabla_x(\phi_{\mathrm{g}}^{\varepsilon}-\phi)\|^p_{L^p(C_i;\mathbb{R}^{3\times 3})} &\lesssim a_{\varepsilon,i}^{3-p}\|\nabla_x\phi^i_{1,\varepsilon}\|^p_{L^p(\mathbb{R}\setminus B_1;\mathbb{R}^{3\times 3})} \lesssim a_{\varepsilon,i}^{-p}\|\phi\|^p_{L^p(T_i;\mathbb{R}^3)}+\|\nabla_x\phi\|^p_{L^p(T_i;\mathbb{R}^{3\times 3})}\\ &\lesssim a_{\varepsilon,i}^{3-p}\|\phi\|^p_{C^1(T_i;\mathbb{R}^3)}+a_{\varepsilon,i}^{3}\|\phi\|^p_{C^1(T_i;\mathbb{R}^3)} \lesssim a_{\varepsilon,i}a_{\varepsilon,i}^{-(p-2)}. \end{aligned} \tag{4.57}$$

Let us now consider a cut-off function $\eta\in C_c^\infty(\mathbb{R}^3)$ such that $\eta=0$ in $\mathbb{R}^3\setminus B_{d_{\varepsilon,i}a_{\varepsilon,i}^{-1}}$ and $\eta=1$ in $B_{\frac{1}{2}d_{\varepsilon,i}a_{\varepsilon,i}^{-1}}$. Using [19, Theorem IV.6.1.], [19, Theorem II.4.3.], (4.56), and $|\nabla_x\eta|\lesssim\frac{a_{\varepsilon,i}}{d_{\varepsilon,i}}$, we deduce

$$\begin{aligned} \|\nabla_x(\phi_{\mathrm{g}}^{\varepsilon}-\phi)\|^p_{L^p(D_i;\mathbb{R}^{3\times 3})} &\lesssim d_{\varepsilon,i}^{3-p}\|\nabla_x\phi^i_{2,\varepsilon}\|^p_{L^p(B_2\setminus B_1;\mathbb{R}^{3\times 3})} \lesssim d_{\varepsilon,i}^{3-p}\left\|\nabla_x\phi^i_{2,\varepsilon}\right\|^p_{W^{1-\frac{1}{p},p}(\partial(B_2\setminus B_1);\mathbb{R}^{3\times 3})}\\ &\lesssim d_{\varepsilon,i}^{3-p}\left\|(\eta\phi^i_{1,\varepsilon})\left(\frac{d_{\varepsilon,i}\cdot-\varepsilon z_i}{2a_{\varepsilon,i}}\right)\right\|^p_{W^{1,p}(B_2\setminus B_1;\mathbb{R}^3)}\\ &\lesssim a_{\varepsilon,i}^3 d_{\varepsilon,i}^{-p}\left\|\eta\phi^i_{1,\varepsilon}\right\|^p_{L^p(E_i^0;\mathbb{R}^3)}+a_{\varepsilon,i}^{3-p}\left\|\nabla_x(\eta\phi^i_{1,\varepsilon})\right\|^p_{L^p(E_i^0;\mathbb{R}^{3\times 3})}\\ &\lesssim a_{\varepsilon,i}^3 d_{\varepsilon,i}^{-p}\left\|\phi^i_{1,\varepsilon}\right\|^p_{L^p(E_i^0;\mathbb{R}^3)}+a_{\varepsilon,i}^{3-p}\left\|\nabla_x\phi^i_{1,\varepsilon}\right\|^p_{L^p(E_i^0;\mathbb{R}^{3\times 3})}+a_{\varepsilon,i}^3 d_{\varepsilon,i}^{-p}\left\|\phi^i_{1,\varepsilon}\right\|^p_{L^p(E_i^0;\mathbb{R}^3)}. \end{aligned}$$

Since

$$\|\phi(a_{\varepsilon,i}\cdot+\varepsilon z_i)\|^2_{L^2(B_2;\mathbb{R}^3)}=a_{\varepsilon,i}^3\|\phi\|^2_{L^2(B_{2a_{\varepsilon,i}}(\varepsilon z_i);\mathbb{R}^3)}\lesssim a_{\varepsilon,i}^3 a_{\varepsilon,i}^{-3}\|\phi\|^2_{C^0(\Omega;\mathbb{R}^3)}\lesssim 1$$

and

$$\|\nabla_x(\phi(a_{\varepsilon,i}\cdot+\varepsilon z_i))\|^2_{L^2(B_2;\mathbb{R}^{3\times 3})}=a_{\varepsilon,i}^2 a_{\varepsilon,i}^3\|\nabla_x\phi\|^2_{L^2(B_{2a_{\varepsilon,i}}(\varepsilon z_i);\mathbb{R}^{3\times 3})}\lesssim a_{\varepsilon,i}^2 a_{\varepsilon,i}^3 a_{\varepsilon,i}^{-3}\|\phi\|^2_{C^1(\Omega;\mathbb{R}^3)}\lesssim a_{\varepsilon,i}^2,$$

we have $\|\phi(a_{\varepsilon,i}\cdot+\varepsilon z_i)\|_{W^{1,2}(B_2;\mathbb{R}^3)}\lesssim 1$.

Furthermore, from [31, Theorem 6.1] we get

$$|\nabla_x \phi^i_{1,\varepsilon}(x)| \lesssim \|\phi(a_{\varepsilon,i}\cdot+\varepsilon z_i)\|_{W^{1,2}(B_2;\mathbb{R}^3)}|x|^{-2} \lesssim |x|^{-2} \quad \text{for every } |x|\geq 3.$$

As $d_{\varepsilon,i}a_{\varepsilon,i}^{-1} \geq \varepsilon^{-\delta}$ we have for $\varepsilon \ll 1$ that $|x|\geq 3$ for each $x\in E_i^0$. In this way

$$\|\nabla_x \phi^i_{1,\varepsilon}\|^p_{L^p(E_i^0;\mathbb{R}^{3\times 3})} \lesssim \int_{E_i^0}|x|^{-2p} \lesssim \left(\frac{a_{\varepsilon,i}}{d_{\varepsilon,i}}\right)^{3(p-1)-p}.$$

Finally, we get

$$\begin{aligned}\|\nabla_x(\phi^\varepsilon_{\mathrm{g}}-\phi)\|^p_{L^p(D_i;\mathbb{R}^{3\times 3})} &\lesssim a^3_{\varepsilon,i}d^{-p}_{\varepsilon,i}\,\|\phi^i_{1,\varepsilon}\|^p_{L^p(E_i^0;\mathbb{R}^3)} + a^{3-p}_{\varepsilon,i}\,\|\nabla_x\phi^i_{1,\varepsilon}\|^p_{L^p(E_i^0;\mathbb{R}^{3\times 3})}\\ &\lesssim a^{3-p}_{\varepsilon,i}\left(\frac{a_{\varepsilon,i}}{d_{\varepsilon,i}}\right)^{3(p-1)-p} \lesssim a_{\varepsilon,i}d^{(2-p)}_{\varepsilon,i}\left(\frac{a_{\varepsilon,i}}{d_{\varepsilon,i}}\right)^{p-1} \lesssim a_{\varepsilon,i}d^{2-p}_{\varepsilon,i}\varepsilon^{\delta(p-1)(d-2)}. \end{aligned} \tag{4.58}$$

On the bad holes, $\phi^\varepsilon_{\mathrm{g}}=\phi$. On the good holes we have $a_{\varepsilon,i}\lesssim \varepsilon^{1+2\delta}$ and thus

$$\|\nabla_x(\phi^\varepsilon_{\mathrm{g}}-\phi)\|^p_{L^p(T_i;\mathbb{R}^3)} \lesssim \|\phi\|_{C^1(\Omega,\mathbb{R}^3)}a^3_{\varepsilon,i} \lesssim a_{\varepsilon,i}\varepsilon^{2(1+2\delta)}. \tag{4.59}$$

Combining (4.57), (4.58), and (4.59), we find

$$\begin{aligned}\|\nabla_x(\phi^\varepsilon_{\mathrm{g}}-\phi)\|^p_{L^p(\Omega;\mathbb{R}^{3\times 3})} &\lesssim \sum_{z_i\in n^\varepsilon} a_{\varepsilon,i}(a^{2-p}_{\varepsilon,i}+d^{2-p}_{\varepsilon,i}\varepsilon^{(p-1)\delta}+\varepsilon^{2(1+2\delta)}),\\ &\lesssim \begin{cases} (\varepsilon^{(1+2\delta)(2-p)}+\varepsilon^{2-p}\varepsilon^{(p-1)\delta}+\varepsilon^{2(1+2\delta)})\sum_{z_i\in n^\varepsilon}a_{\varepsilon,i} & \text{for } 1<p<2\\ (1+\varepsilon^\delta+\varepsilon^{2(1+2\delta)})\sum_{z_i\in n^\varepsilon}a_{\varepsilon,i} & \text{for } p=2,\\ (\varepsilon^{3(2-p)}+\varepsilon^{(1+\delta)(2-p)}\varepsilon^{(p-1)\delta}+\varepsilon^{2(1+2\delta)})\sum_{z_i\in n^\varepsilon}a_{\varepsilon,i} & \text{for } p>2.\end{cases}\end{aligned}$$

Lastly, we use the Strong Law of Large Numbers Lemma 4.6 to conclude that almost surely

$$\|\nabla_x(\phi^\varepsilon_{\mathrm{g}}-\phi)\|^p_{L^p(\Omega;\mathbb{R}^{3\times 3})} \lesssim \begin{cases}\varepsilon^{(1+2\delta)(2-p)}+\varepsilon^{2-p+(p-1)\delta} & \text{for } 1<p\leq 2,\\ \varepsilon^{3(2-p)}+\varepsilon^{2-p+\delta} & \text{for } p>2.\end{cases} \tag{4.60}$$

Eventually, combining (4.49) and (4.60), we find (4.55).

Step 2: Validity of (2.21). By definition of $\phi^\varepsilon_{\mathrm{b}}$ and $\phi^\varepsilon_{\mathrm{g}}$ we have

$$\phi_\varepsilon = \phi^\varepsilon_{\mathrm{b}}+\phi^\varepsilon_{\mathrm{g}}-\phi,$$

therefore we can write

$$\int_0^T\int_\Omega \nabla_x\phi_\varepsilon : \nabla_x\mathbf{v}_\varepsilon \,\mathrm{d}x\,\mathrm{d}t = \int_0^T\int_\Omega \nabla_x(\phi^\varepsilon_{\mathrm{b}}-\phi) : \nabla_x\mathbf{v}_\varepsilon\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_\Omega \nabla_x\phi^\varepsilon_{\mathrm{g}} : \nabla_x\mathbf{v}_\varepsilon\,\mathrm{d}x\,\mathrm{d}t. \tag{4.61}$$

Using that $\|\mathbf{v}_\varepsilon\|_{L^2(0,T;W^{1,2}(\Omega;\mathbb{R}^3))}\lesssim 1$ and (4.50) we deduce

$$\int_0^T\int_\Omega \nabla_x(\phi^\varepsilon_{\mathrm{b}}-\phi):\nabla_x\mathbf{v}_\varepsilon\,\mathrm{d}x\,\mathrm{d}t \lesssim \|\nabla_x(\phi^\varepsilon_{\mathrm{b}}-\phi)\|_{L^2(\Omega;\mathbb{R}^{3\times 3})}\|\mathbf{v}_\varepsilon\|_{L^2(0,T;W^{1,2}(\Omega;\mathbb{R}^3))} \xrightarrow{\varepsilon\to 0} 0.$$

Hence, it is left to show that

$$\lim_{\varepsilon\to 0}\int_0^T\int_\Omega \nabla_x\boldsymbol{\phi}_{\mathrm{g}}^\varepsilon : \nabla_x\mathbf{v}_\varepsilon \,\mathrm{d}x\,\mathrm{d}t = \int_0^T\int_\Omega \nabla_x\boldsymbol{\phi} : \nabla_x\mathbf{v}\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_\Omega (\mathrm{M}\boldsymbol{\phi})\cdot\mathbf{v}\,\mathrm{d}x\,\mathrm{d}t. \tag{4.62}$$

To this end we write

$$\begin{aligned}\int_0^T\int_\Omega \nabla_x\boldsymbol{\phi}_{\mathrm{g}}^\varepsilon : \nabla_x\mathbf{v}_\varepsilon\,\mathrm{d}x\,\mathrm{d}t = \int_0^T\int_\Omega \nabla_x\boldsymbol{\phi} : \nabla_x\mathbf{v}_\varepsilon\,\mathrm{d}x\,\mathrm{d}t &+ \int_0^T\int_{\bigcup_{z_i\in n^\varepsilon} D_i} \nabla_x(\boldsymbol{\phi}_{\mathrm{g}}^\varepsilon-\boldsymbol{\phi}) : \nabla_x\mathbf{v}_\varepsilon\,\mathrm{d}x\,\mathrm{d}t \\ &+ \int_0^T\int_{\bigcup_{z_i\in n^\varepsilon} C_i} \nabla_x(\boldsymbol{\phi}_{\mathrm{g}}^\varepsilon-\boldsymbol{\phi}) : \nabla_x\mathbf{v}_\varepsilon\,\mathrm{d}x\,\mathrm{d}t.\end{aligned} \tag{4.63}$$

Since $\mathbf{v}_\varepsilon \rightharpoonup \mathbf{v}$ weakly in $L^2(0,T;W^{1,2}(\Omega;\mathbb{R}^3))$, the first term on the right-hand side converges to $\int_0^T\int_\Omega \nabla_x\boldsymbol{\phi} : \nabla_x\mathbf{v}\,\mathrm{d}x\,\mathrm{d}t$. To control the second term on the right-hand side we use (4.58) and the boundedness of $\mathbf{v}_\varepsilon$ in $L^2(0,T;W^{1,2}(\Omega;\mathbb{R}^3))$ to see that

$$\begin{aligned}\left|\int_0^T\int_{\bigcup_{z_i\in n^\varepsilon} D_i} \nabla_x(\boldsymbol{\phi}_{\mathrm{g}}^\varepsilon-\boldsymbol{\phi}) : \nabla_x\mathbf{v}_\varepsilon\,\mathrm{d}x\,\mathrm{d}t\right| &\lesssim \|\nabla_x(\boldsymbol{\phi}_{\mathrm{g}}^\varepsilon-\boldsymbol{\phi})\|_{L^2(\bigcup_{z_i\in n^\varepsilon} D_i;\mathbb{R}^{3\times 3})}\|\mathbf{v}_\varepsilon\|_{L^2(0,T;W^{1,2}(\Omega;\mathbb{R}^3))} \\ &\lesssim \left(\sum_{z_i\in n^\varepsilon} a_{\varepsilon,i}\varepsilon^\delta\right)^{\frac12}\|\mathbf{v}_\varepsilon\|_{L^2(0,T;W^{1,2}(\Omega;\mathbb{R}^3))} \lesssim \varepsilon^{\frac{\delta}{2}}\left(\varepsilon^3\sum_{z_i\in n^\varepsilon} r_i\right)^{\frac12}.\end{aligned}$$

Due to the Lemma 4.6 the right-hand side vanishes almost surely for $\varepsilon\to 0$. Setting $\boldsymbol{\phi}=(\phi_1,\phi_2,\phi_3)$ and letting $\boldsymbol{\omega}_k^\varepsilon$ be the function given by Lemma 4.8 we define a new function $\overline{\boldsymbol{\phi}}_\varepsilon$ by

$$\overline{\boldsymbol{\phi}}_\varepsilon := \begin{cases} \sum_{k=1}^3 \phi_k(\varepsilon z_i)(\boldsymbol{\omega}_k^\varepsilon((\cdot-\varepsilon z_i)a_{\varepsilon,i}^{-1})-\mathbf{e}_k) & \text{in } C_i \text{ for } z_i\in n^\varepsilon, \\ 0 & \text{in } \Omega\setminus(\bigcup_{z_i\in n^\varepsilon} C_i).\end{cases}$$

We write the last term of (4.63) as

$$\int_0^T\int_{\bigcup_{z_i\in n^\varepsilon} C_i} \nabla_x(\boldsymbol{\phi}_{\mathrm{g}}^\varepsilon-\boldsymbol{\phi}) : \nabla_x\mathbf{v}_\varepsilon\,\mathrm{d}x\,\mathrm{d}t = \int_0^T\int_{\bigcup_{z_i\in n^\varepsilon} C_i} \nabla_x(\boldsymbol{\phi}_{\mathrm{g}}^\varepsilon-\boldsymbol{\phi}-\overline{\boldsymbol{\phi}}_\varepsilon) : \nabla_x\mathbf{v}_\varepsilon\,\mathrm{d}x\,\mathrm{d}t \tag{4.64}$$

$$+\int_0^T\int_{\bigcup_{z_i\in n^\varepsilon} C_i} \nabla_x\overline{\boldsymbol{\phi}}_\varepsilon : \nabla_x\mathbf{v}_\varepsilon\,\mathrm{d}x\,\mathrm{d}t. \tag{4.65}$$

Now observe that $-\boldsymbol{\phi}_{1,\varepsilon}^i - \sum_{k=1}^3\phi_k(\varepsilon z_i)\,(\boldsymbol{\omega}_k-\mathbf{e}_k)$ solves the Stokes problem in the exterior domain $\mathbb{R}^3\setminus B_1$ with boundary datum $-\boldsymbol{\phi}(a_{\varepsilon,i}\cdot -\varepsilon z_i)+\boldsymbol{\phi}(\varepsilon z_i)$. As a consequence

$$\begin{aligned}\|\nabla_x(\boldsymbol{\phi}_{\mathrm{g}}^\varepsilon-\boldsymbol{\phi}-\overline{\boldsymbol{\phi}}_\varepsilon)\|_{L^2(C_i;\mathbb{R}^{3\times3})}^2 &\lesssim a_{\varepsilon,i}^{-2}\|\nabla_x(-\boldsymbol{\phi}_{1,\varepsilon}^i-\sum_{k=1}^3\phi_k(\varepsilon z_i)\,(\boldsymbol{\omega}_k-\mathbf{e}_k))\left((\cdot-\varepsilon z_i)a_{\varepsilon,i}^{-1}\right)\|_{L^2(C_i;\mathbb{R}^{3\times3})}^2 \\ &\lesssim a_{\varepsilon,i}\|\nabla_x(-\boldsymbol{\phi}_{1,\varepsilon}^i-\sum_{k=1}^3\phi_k(\varepsilon z_i)\,(\boldsymbol{\omega}_k-\mathbf{e}_k))\|_{L^2(\mathbb{R}\setminus B_1;\mathbb{R}^{3\times3})}^2 \\ &\lesssim a_{\varepsilon,i}\|\boldsymbol{\phi}(a_{\varepsilon,i}\cdot+\varepsilon z_i)-\boldsymbol{\phi}(\varepsilon z_i)\|_{W^{1-\frac12,2}(\partial B_1;\mathbb{R}^3)}^2 \\ &\lesssim a_{\varepsilon,i}\|(\eta(\boldsymbol{\phi}-\boldsymbol{\phi}(\varepsilon z_i)))(a_{\varepsilon,i}\cdot+\varepsilon z_i)\|_{W^{1-\frac12,2}(\partial B_1;\mathbb{R}^3)}^2,\end{aligned}$$

where η is a cut-off function in $B_{2a_{\varepsilon,i}}(\varepsilon z_i)\setminus T_i$ with $\eta=0$ on $\partial B_{2a_{\varepsilon,i}}(\varepsilon z_i)$ and $\eta=1$ on ∂T_i. Now, we use [19, Theorem II.4.3], $|\nabla_x\eta|\lesssim a_{\varepsilon,i}^{-1}$, and a Lipschitz estimate to conclude

$$\begin{aligned}a_{\varepsilon,i}\|(\eta(\boldsymbol{\phi}-\boldsymbol{\phi}(\varepsilon z_i)))(a_{\varepsilon,i}\cdot+\varepsilon z_i)\|_{W^{1-\frac12,2}(\partial B_1;\mathbb{R}^3)}^2 &\lesssim a_{\varepsilon,i}\|(\eta(\boldsymbol{\phi}-\boldsymbol{\phi}(\varepsilon z_i)))(a_{\varepsilon,i}\cdot+\varepsilon z_i)\|_{W^{1,2}(B_2\setminus B_1;\mathbb{R}^3)}^2 \\ &\lesssim a_{\varepsilon,i}^{-2}\|(\boldsymbol{\phi}-\boldsymbol{\phi}(\varepsilon z_i))\|_{L^2(B_{2a_{\varepsilon,i}}(\varepsilon z_i)\setminus T_i;\mathbb{R}^3)}^2 + \|\eta\nabla_x(\boldsymbol{\phi}-\boldsymbol{\phi}(\varepsilon z_i))\|_{L^2(B_{2a_{\varepsilon,i}}(\varepsilon z_i)\setminus T_i;\mathbb{R}^{3\times3})}^2\end{aligned}$$

$$\lesssim a_{\varepsilon,i}^{-2}\|\boldsymbol{\phi}-\boldsymbol{\phi}(\varepsilon z_i)\|^2_{L^2(B_{2a_{\varepsilon,i}}(\varepsilon z_i)\setminus T_i;\mathbb{R}^3)}+\|\nabla_x\boldsymbol{\phi}\|^2_{L^2(B_{2a_{\varepsilon,i}}(\varepsilon z_i)\setminus T_i;\mathbb{R}^{3\times 3})}$$
$$\lesssim a_{\varepsilon,i}^{-2}\|\boldsymbol{\phi}\|^2_{C^1(\Omega;\mathbb{R}^3)}\int_{B_{2a_{\varepsilon,i}}(\varepsilon z_i)\setminus T_i}|x-\varepsilon z_i|^2\,\mathrm{d}x+a_{\varepsilon,i}^d\|\boldsymbol{\phi}\|^2_{C^1(\Omega;\mathbb{R}^3)}$$
$$\lesssim a_{\varepsilon,i}^{-2}a_{\varepsilon,i}^5\|\boldsymbol{\phi}\|^2_{C^1(\Omega;\mathbb{R}^3)}+a_{\varepsilon,i}^3\|\boldsymbol{\phi}\|^2_{C^1(\Omega;\mathbb{R}^3)}.$$

Thus

$$\Big|\int_0^T\int_{\bigcup_{z_i\in n^\varepsilon}C_i}\nabla_x(\boldsymbol{\phi}_{\mathrm{g}}^\varepsilon-\boldsymbol{\phi}-\overline{\boldsymbol{\phi}}_\varepsilon):\nabla_x\mathbf{v}_\varepsilon\,\mathrm{d}x\,\mathrm{d}t\Big|\le\|\nabla_x\mathbf{v}_\varepsilon\|_{L^2(0,T;L^2(\bigcup_{z_i\in n^\varepsilon}C_i;\mathbb{R}^{3\times3}))}\|\nabla_x(\boldsymbol{\phi}_{\mathrm{g}}^\varepsilon-\boldsymbol{\phi}-\overline{\boldsymbol{\phi}}_\varepsilon)\|_{L^2(\bigcup_{z_i\in n^\varepsilon}C_i;\mathbb{R}^{3\times3})}$$
$$\lesssim\|\mathbf{v}_\varepsilon\|_{L^2(0,T;W^{1,2}(\Omega;\mathbb{R}^3))}\left(\sum_{z_i\in n^\varepsilon}\|\nabla_x(\boldsymbol{\phi}_{\mathrm{g}}^\varepsilon-\boldsymbol{\phi}-\overline{\boldsymbol{\phi}}_\varepsilon)\|^2_{L^2(C_i;\mathbb{R}^{3\times3})}\right)^{\frac12}$$
$$\lesssim\|\mathbf{v}_\varepsilon\|_{L^2(0,T;W^{1,2}(\Omega;\mathbb{R}^3))}\left(\sum_{z_i\in n^\varepsilon}a_{\varepsilon,i}^3\right)^{\frac12}\lesssim\varepsilon^{1+2\delta}\|\mathbf{v}_\varepsilon\|_{L^2(0,T;W^{1,2}(\Omega;\mathbb{R}^3))}\left(\sum_{z_i\in n^\varepsilon}a_{\varepsilon,i}\right)^{\frac12}.$$

From $\|\mathbf{v}_\varepsilon\|_{L^2(0,T;W^{1,2}(\Omega;\mathbb{R}^3))}\lesssim 1$ and Lemma 4.6, the right-hand side in the above estimate vanishes almost surely for $\varepsilon\to 0$ and so does the first term on the right-hand side of (4.64). In addition, the last term of (4.64) yields

$$\lim_{\varepsilon\to0}\int_0^T\int_{\bigcup_{z_i\in n^\varepsilon}C_i}\nabla_x\overline{\boldsymbol{\phi}}_\varepsilon:\nabla_x\mathbf{v}_\varepsilon\,\mathrm{d}x\,\mathrm{d}t=\int_0^T\int_\Omega \mathrm{M}\boldsymbol{\phi}\cdot\mathbf{v}\,\mathrm{d}x\,\mathrm{d}t.\tag{4.66}$$

Indeed, by (H5) we have

$$\lim_{\varepsilon\to0}\sum_{k=1}^3\int_0^T\int_\Omega\nabla_x\boldsymbol{\omega}_k^\varepsilon:\nabla_x(\phi_k\mathbf{v}_\varepsilon)\,\mathrm{d}x\,\mathrm{d}t=\sum_{k=1}^3\int_0^T\int_\Omega\boldsymbol{\mu}_k\cdot(\phi_k\mathbf{v})\,\mathrm{d}x\,\mathrm{d}t=\int_0^T\int_\Omega(\mathrm{M}\boldsymbol{\phi})\cdot\mathbf{v}\,\mathrm{d}x\,\mathrm{d}t.\tag{4.67}$$

Hence

$$\begin{aligned}&\int_0^T\int_{\bigcup_{z_i\in n^\varepsilon}C_i}\nabla_x\overline{\boldsymbol{\phi}}_\varepsilon:\nabla_x\mathbf{v}_\varepsilon\,\mathrm{d}x\,\mathrm{d}t-\sum_{k=1}^3\int_0^T\int_\Omega\nabla_x\boldsymbol{\omega}_k^\varepsilon:\nabla_x(\phi_k\mathbf{v}_\varepsilon)\,\mathrm{d}x\,\mathrm{d}t\\&\quad=\sum_{k=1}^3\int_0^T\sum_{z_i\in n^\varepsilon}\int_{C_i}(\phi_k(\varepsilon z_i)-\phi_k)\nabla_x\boldsymbol{\omega}_k^\varepsilon:\nabla_x\mathbf{v}_\varepsilon\,\mathrm{d}x\,\mathrm{d}t-\sum_{k=1}^3\int_0^T\int_\Omega\nabla_x\boldsymbol{\omega}_k^\varepsilon:(\mathbf{v}_\varepsilon\otimes\nabla_x\phi_k)\,\mathrm{d}x\,\mathrm{d}t\\&\quad-\sum_{k=1}^3\int_0^T\int_{\bigcup_{z_i\in n^\varepsilon}D_i}\phi_k\nabla_x\boldsymbol{\omega}_k^\varepsilon:\nabla_x\mathbf{v}_\varepsilon\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}\tag{4.68}$$

As $\boldsymbol{\phi}$ is smooth, we use a Lipschitz estimate, Lemma 4.8, $\|\mathbf{v}_\varepsilon\|_{L^2(0,T;W^{1,2}(\Omega,\mathbb{R}^3))}\lesssim 1$, and $d_{\varepsilon,i}\le\varepsilon$ to conclude

$$\begin{aligned}&\Big|\sum_{k=1}^3\int_0^T\sum_{z_i\in n^\varepsilon}\int_{C_i}(\phi_k(\varepsilon z_i)-\boldsymbol{\phi})\nabla_x\boldsymbol{\omega}_k^\varepsilon:\nabla_x\mathbf{v}_\varepsilon\,\mathrm{d}x\,\mathrm{d}t\Big|\\&\le\sum_{k=1}^3\int_0^T\sum_{z_i\in n^\varepsilon}\int_{C_i}\|\phi_k(\varepsilon z_i)-\boldsymbol{\phi}\|_{C^0(C_i;\mathbb{R}^3)}|\nabla_x\boldsymbol{\omega}_k^\varepsilon:\nabla_x\mathbf{v}_\varepsilon|\,\mathrm{d}x\,\mathrm{d}t\\&\lesssim d_{\varepsilon,i}\sum_{k=1}^3\int_0^T\int_{\bigcup_{z_i\in n^\varepsilon}C_i}|\nabla_x\boldsymbol{\omega}_k^\varepsilon:\nabla_x\mathbf{v}_\varepsilon|\,\mathrm{d}x\,\mathrm{d}t\lesssim d_{\varepsilon,i}\sum_{k=1}^3\|\nabla_x\boldsymbol{\omega}_k^\varepsilon\|_{L^2(\Omega;\mathbb{R}^{3\times3})}\|\mathbf{v}_\varepsilon\|_{L^2(0,T;W^{1,2}(\Omega;\mathbb{R}^3))}\lesssim\varepsilon.\end{aligned}$$

Consequently, the first term on the right-hand side of (4.68) vanishes as $\varepsilon\to0$. Since $\mathbf{v}_\varepsilon\rightharpoonup\mathbf{v}$ weakly in $L^2(0,T;W^{1,2}(\Omega;\mathbb{R}^3))$ and $\boldsymbol{\omega}_k^\varepsilon\to\mathbf{e}_k$ strongly in $L^2(\Omega;\mathbb{R}^3)$ by (H3), together with $\mathrm{div}_x\mathbf{v}=0$, we have by partial integration

$$\lim_{\varepsilon\to 0}\sum_{k=1}^{3}\int_0^T\int_\Omega \nabla_x \boldsymbol{\omega}_k^\varepsilon : (\mathbf{v}_\varepsilon\otimes\nabla_x\phi_k)\,\mathrm{d}x\,\mathrm{d}t$$
$$= -\lim_{\varepsilon\to 0}\sum_{k=1}^{3}\int_0^T\int_\Omega \mathrm{div}_x(\mathbf{v}_\varepsilon)(\boldsymbol{\omega}_k^\varepsilon\cdot\nabla_x\phi_k)+\Delta_x\phi_k((\boldsymbol{\omega}_k^\varepsilon-\mathbf{e}_k)\cdot\mathbf{v}_\varepsilon)\,\mathrm{d}x\,\mathrm{d}t = 0,$$

and also the second term on the right-hand side of (4.68) vanishes. Eventually, from Lemma 4.8 and the fact that $\|\mathbf{v}_\varepsilon\|_{L^2(0,T;W^{1,2}(\Omega;\mathbb{R}^3))}\lesssim 1$ we get

$$\Big|\sum_{k=1}^{3}\int_0^T\int_{\bigcup_{z_i\in n^\varepsilon}D_i}\phi_k\nabla_x\boldsymbol{\omega}_k^\varepsilon:\nabla_x\mathbf{v}_\varepsilon\,\mathrm{d}x\,\mathrm{d}t\Big|\lesssim\|\phi_k\|_{C^0(\Omega;\mathbb{R}^3)}\|\nabla_x\boldsymbol{\omega}_k^\varepsilon\|_{L^2(\bigcup_{z_i\in n^\varepsilon}A_i;\mathbb{R}^{3\times3})}\|\mathbf{v}_\varepsilon\|_{L^2(0,T;W^{1,2}(\Omega;\mathbb{R}^3))}\lesssim\varepsilon^{\frac{\delta}{2}},$$

hence also the last term of (4.68) vanishes for $\varepsilon\to 0$. This together with (4.67) proves (4.66). □

Acknowledgements

R. M. has received funding from the European Union's Horizon research and innovation program under the Marie Skłodowska-Curie agreement No 101150549. The research that led to the present paper was partially supported by a grant from the GNAMPA group of INdAM. F. O. has been supported by the Czech Science Foundation (GAČR) project 22-01591S, and the Czech Academy of Sciences project L100192351. The Institute of Mathematics, CAS is supported by RVO:67985840.

References

[1] Anna Abbatiello and Eduard Feireisl. On a class of generalized solutions to equations describing incompressible viscous fluids. *Annali di Matematica Pura ed Applicata (1923-)*, 199(3):1183–1195, 2020.

[2] Grégoire Allaire. *Homogénéisation des équations de Stokes et de Navier-Stokes*. PhD thesis, Paris 6, 1989.

[3] Grégoire Allaire. Homogenization of the Stokes flow in a connected porous medium. *Asymptotic Anal.*, 2(3):203–222, 1989.

[4] Grégoire Allaire. Homogenization of the Navier–Stokes equations in open sets perforated with tiny holes. I. Abstract framework, a volume distribution of holes. *Arch. Rational Mech. Anal.*, 113(3):209–259, 1990.

[5] Grégoire Allaire. Homogenization of the Navier–Stokes equations in open sets perforated with tiny holes. II. Noncritical sizes of the holes for a volume distribution and a surface distribution of holes. *Arch. Rational Mech. Anal.*, 113(3):261–298, 1990.

[6] Annika Bach, Roberta Marziani, and Caterina Ida Zeppieri. Γ-convergence and stochastic homogenisation of singularly-perturbed elliptic functionals. *Calc. Var.* **62***, 199*, 2023.

[7] Danica Basarić and Nilasis Chaudhuri. Low Mach number limit on perforated domains for the evolutionary Navier–Stokes–Fourier system. *Nonlinearity*, 37(6):065008, 2024.

[8] Danica Basarić, Florian Oschmann, and Jiaojiao Pan. Qualitative derivation of a density dependent incompressible Darcy law. *arXiv preprint arXiv:2502.14602*, 2025.

[9] Peter Bella, Eduard Feireisl, and Florian Oschmann. Γ–convergence for nearly incompressible fluids. *Journal of Mathematical Physics*, 64(9), 2023.

[10] Peter Bella and Florian Oschmann. Homogenization and low Mach number limit of compressible Navier-Stokes equations in critically perforated domains. *Journal of Mathematical Fluid Mechanics*, 24(3):1–11, 2022.

[11] Peter Bella and Florian Oschmann. Inverse of divergence and homogenization of compressible Navier–Stokes equations in randomly perforated domains. *Arch. Ration. Mech. Anal.*, 247(2):14, 2023.

[12] Doïna Cioranescu and François Murat. Un terme étrange venu d'ailleurs. I. In *Nonlinear partial differential equations and their applications. Collège de France Seminar, Vol. III*, volume 70 of *Res. Notes in Math.*, pages 154–178, 425–426. Pitman, Boston, Mass.-London, 1982.

[13] Henry Darcy. *Les fontaines publiques de la ville de Dijon.* Victor Dalmont, 1856.

[14] Francesco De Anna, Anja Schlömerkemper, and Arghir Dani Zarnescu. Colloidal homogenization for the hydrodynamics of nematic liquid crystals. In *Proceedings A*, volume 481, page 20240192. The Royal Society, 2025.

[15] Lars Diening, Eduard Feireisl, and Yong Lu. The inverse of the divergence operator on perforated domains with applications to homogenization problems for the compressible Navier–Stokes system. *ESAIM: Control, Optimisation and Calculus of Variations*, 23(3):851–868, 2017.

[16] Eduard Feireisl and Yong Lu. Homogenization of stationary Navier–Stokes equations in domains with tiny holes. *Journal of Mathematical Fluid Mechanics*, 17(2):381–392, 2015.

[17] Eduard Feireisl, Yuliya Namlyeyeva, and Šárka Nečasová. Homogenization of the evolutionary Navier–Stokes system. *Manuscripta Math.*, 149(1-2):251–274, 2016.

[18] Eduard Feireisl and Antonín Novotný. *Singular limits in thermodynamics of viscous fluids*, volume 2. Springer, 2009.

[19] Giovanni Paolo Galdi. *An introduction to the mathematical theory of the Navier–Stokes equations.* Springer Monographs in Mathematics. Springer, New York, second edition, 2011. Steady-state problems.

[20] Arianna Giunti and Richard Matthias Höfer. Homogenisation for the Stokes equations in randomly perforated domains under almost minimal assumptions on the size of the holes. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 36(7):1829–1868, 2019.

[21] Arianna Giunti, Richard Matthias Höfer, and Juan J. L. Velázquez. Homogenization for the Poisson equation in randomly perforated domains under minimal assumptions on the size of the holes. *Comm. Partial Differential Equations*, 43(9):1377–1412, 2018.

[22] Richard Matthias Höfer, Karina Kowalczyk, and Sebastian Schwarzacher. Darcy's law as low Mach and homogenization limit of a compressible fluid in perforated domains. *Mathematical Models and Methods in Applied Sciences*, 31(09):1787–1819, 2021.

[23] Richard Matthias Höfer, Šárka Nečasová, and Florian Oschmann. Quantitative homogenization of the compressible Navier-Stokes equations towards Darcy's law. *arXiv preprint arXiv:2403.12616*, 2024.

[24] Andrii Khrabustovskyi and Michael Plum. Operator estimates for homogenization of the Robin Laplacian in a perforated domain. *Journal of Differential Equations*, 338:474–517, 2022.

[25] Günter Last and Mathew Penrose. *Lectures on the Poisson process*, volume 7. Cambridge University Press, 2017.

[26] Yong Lu and Florian Oschmann. Qualitative/quantitative homogenization of some non-Newtonian flows in perforated domains. *arXiv preprint arXiv:2406.17406*, 2024.

[27] Yong Lu, Jiaojiao Pan, and Peikang Yang. Homogenization of Inhomogeneous Incompressible Navier-Stokes Equations in Domains with Very Tiny Holes. *arXiv preprint arXiv:2501.05734*, 2025.

[28] Yong Lu and Milan Pokorný. Homogenization of stationary Navier–Stokes–Fourier system in domains with tiny holes. *J. Differential Equations*, 278:463–492, 2021.

[29] Yong Lu and Zhengmao Qian. Homogenization of some evolutionary non-Newtonian flows in porous media. *Journal of Differential Equations*, 411:619–639, 2024.

[30] Yong Lu and Sebastian Schwarzacher. Homogenization of the compressible Navier—Stokes equations in domains with very tiny holes. *Journal of Differential Equations*, 265(4):1371 – 1406, 2018.

[31] Paolo Maremonti, Remigio Russo, and Giulio Starita. On the Stokes equations: the boundary value problem. In *Advances in fluid dynamics*, volume 4, pages 69–140. Aracne, 1999.

[32] Roberta Marziani. Γ-convergence and stochastic homogenization of phase-transition functionals. *ESAIM: Control Optim. Calc. Var. 29* **44**, 2023.

[33] Roberta Marziani and Francesco Solombrino. Non-local approximation of free-discontinuity problems in linear elasticity and application to stochastic homogenization. *Proceedings of the Royal Society Edinburgh, 1-35*, 2023.

[34] Nader Masmoudi. Homogenization of the compressible Navier–Stokes equations in a porous medium. *ESAIM: Control, Optimisation and Calculus of Variations*, 8:885–906, 2002.

[35] Andro Mikelić. Homogenization of nonstationary Navier–Stokes equations in a domain with a grained boundary. *Ann. Mat. Pura Appl. (4)*, 158:167–179, 1991.

[36] Šárka Nečasová and Jiaojiao Pan. Homogenization problems for the compressible Navier–Stokes system in 2D perforated domains. *Mathematical Methods in the Applied Sciences*, 45(12):7859–7873, 2022.

[37] Florian Oschmann. Homogenization of the full compressible Navier-Stokes-Fourier system in randomly perforated domains. *Journal of Mathematical Fluid Mechanics*, 24(2):1–20, 2022.

[38] Florian Oschmann and Milan Pokorný. Homogenization of the unsteady compressible Navier-Stokes equations for adiabatic exponent $\gamma > 3$. *Journal of Differential Equations*, 377:271–296, 2023.

[39] Jiaojiao Pan. Homogenization of Non-Homogeneous Incompressible Navier–Stokes System in Critically Perforated Domains. *Journal of Mathematical Fluid Mechanics*, 27(2):1–16, 2025.

[40] Milan Pokorný and Emil Skříšovský. Homogenization of the evolutionary compressible Navier–Stokes–Fourier system in domains with tiny holes. *Journal of Elliptic and Parabolic Equations*, pages 1–31, 2021.

[41] Lucia Scardia, Konstantinos Zemas, and Caterina Ida Zeppieri. Homogenization of nonlinear Dirichlet problems in randomly perforated domains under minimal assumptions on the size of the perforations. *Probab. Theory Relat. Fields*, 2024.

[42] Luc Tartar. Incompressible fluid flow in a porous medium: convergence of the homogenization process. *Appendix of Non-homogeneous media and vibration theory*, pages 368–377, 1980.

[43] David Wiedemann and Malte A. Peter. A Darcy law with memory by homogenisation for evolving microstructure. *Journal of Mathematical Analysis and Applications*, 546(2):129222, 2025.