

On jump minimizing liftings for $\mathbb{S}^1$ -valued maps and connections with Ambrosio-Tortorelli-type Γ -limits

Giovanni Bellettini* Roberta Marziani† Riccardo Scala‡

August 12, 2026

Abstract

The main result of this paper is a compactness property for sequences of liftings of Sobolev maps defined on an open bounded set $\Omega \subset \mathbb{R}^n$ and taking values in the unit circle $\mathbb{S}^1 \subset \mathbb{R}^2$. As a consequence, we prove that the problem of finding a lifting of a map, having the shortest jump set, has a solution. Next, we obtain two Γ -limits for sequences of Ambrosio-Tortorelli-type functionals on $\mathbb{S}^1$ -valued maps, one of which turns out to be nonlocal, for topological reasons.

Résumé

Le résultat principal de cet article est une propriété de compacité pour des suites de relèvements d'applications de Sobolev définies sur un ouvert borné $\Omega \subset \mathbb{R}^n$ et à valeurs dans le cercle unité $\mathbb{S}^1 \subset \mathbb{R}^2$. Comme conséquence, nous montrons que le problème consistant à trouver un relèvement d'une application dont l'ensemble de saut est de longueur minimale admet une solution. Nous obtenons ensuite deux Γ -limites pour des suites de fonctionnelles de type Ambrosio-Tortorelli définies sur des applications à valeurs dans $\mathbb{S}^1$, dont l'une s'avère non locale pour des raisons topologiques.

Key words: $\mathbb{S}^1$ -valued maps, compactness of sequences of liftings, jump minimizing liftings, Γ -convergence, free boundary problems.

AMS (MOS) subject classification: 49Q15, 49Q20, 49J45, 58E12.

1 Introduction

In the last decades the analysis of $\mathbb{S}^1$ -valued maps has been extensively developed, due to its relevance in many physical models where topological singularities arise, such as, for example, Ginzburg-Landau vortices in superconductors and dislocations in crystals [2–4, 24, 26, 27, 32, 39]. An interesting link has been addressed also with the Plateau problem in higher codimension and in the Cartesian setting (cf. [1, 9–12, 40]). A central issue in

*Dipartimento di Scienze Matematiche, Informatiche e Fisiche, via delle Scienze 206, 33100 Udine, Italy and International Centre for Theoretical Physics ICTP, Mathematics Section, 34151 Trieste, Italy. E-mail: giovanni.bellettini@uniud.it

†Dipartimento di Ingegneria dell'Informazione e Scienze Matematiche, Università di Siena, 53100 Siena, Italy. E-mail: roberta.marziani@unisi.it

‡Dipartimento di Ingegneria dell'Informazione e Scienze Matematiche, Università di Siena, 53100 Siena, Italy. E-mail: riccardo.scala@unisi.it

these contexts is the existence, for a given $\mathbb{S}^1$ -valued map u , of suitably regular liftings, i.e., real valued maps φ such that

$$u = e^{i\varphi} = \cos \varphi + i \sin \varphi,$$

where we identify $\mathbb{R}^2$ with the complex plane. We refer to the monograph [16] and references therein for an exhaustive discussion on the topic.

In this paper we analyse compactness properties for sequences of liftings, which are essential for addressing the following question. Let $\Omega \subset \mathbb{R}^n$ be a suitable open set, let $p > 1$ and $u \in SBV^p(\Omega; \mathbb{S}^1)$. We seek a lifting $\varphi_{\min} \in GSBV^p(\Omega)$ of u such that

$$\mathcal{H}^{n-1}(S_{\varphi_{\min}}) = \inf \{ \mathcal{H}^{n-1}(S_{\varphi}) : \varphi \in GSBV^p(\Omega), e^{i\varphi} = u \text{ a.e. in } \Omega \}. \quad (1.1)$$

Here $\mathcal{H}^{n-1}(S_{\varphi})$ stands for the $(n-1)$ -dimensional Hausdorff measure of the jump set S_{φ} of φ . Moreover, $SBV^p(\Omega; \mathbb{S}^1)$ (resp. $GSBV^p(\Omega)$) denotes the space of $\mathbb{S}^1$ -valued maps with special bounded variation in Ω (resp. the space of generalized special bounded variation functions in Ω) whose absolutely continuous part of the gradient is in L^p and whose jump set has finite $(n-1)$ -dimensional Hausdorff measure. We call a lifting $\varphi_{\min}$ as above a *jump minimizing lifting*. As a byproduct of our analysis we also study Γ -convergence of sequences of Ambrosio-Tortorelli type functionals for $\mathbb{S}^1$ -valued maps, highlighting the occurrence of a Lavrentiev phenomenon depending on the choice of the functional domain.

The next two theorems can be considered as the main results of the present paper; they provide compactness and lower semicontinuity for a sequence of liftings associated to a single map u and more generally, to a converging sequence (u_k) . It is clear that Theorem 1.2 extends Theorem 1.1; however, for clarity of presentation and to make the demonstrations more readable, we prefer to give two separate statements. In what follows $\Omega \subset \mathbb{R}^n$ is a bounded open set with Lipschitz boundary.

Theorem 1.1 (Compactness and lower semicontinuity, I). *Let $p > 1$, $u \in SBV^p(\Omega; \mathbb{S}^1)$ and $(\varphi_k)_{k \geq 1} \subset GSBV^p(\Omega)$ be a sequence of liftings of u with*

$$\sup_{k \in \mathbb{N}} \mathcal{H}^{n-1}(S_{\varphi_k}) < +\infty.$$

Then there exist a Caccioppoli partition $(E_i)_{i \geq 1}$ of Ω and a not relabelled subsequence of indices k for which the following holds:

there exist a lifting $\varphi_{\infty} \in GSBV^p(\Omega)$ of u in Ω , and a sequence $(d_k^{(i)})_{k \geq 1} \subset \mathbb{Z}$ for any $i \in \mathbb{N}$, so that

$$\begin{aligned} \lim_{k \rightarrow +\infty} (\varphi_k(x) - 2\pi d_k^{(i)}) &= \varphi_{\infty}(x) \quad \forall i \in \mathbb{N}, \text{ for a.e. } x \in E_i, \\ \lim_{k \rightarrow +\infty} |\varphi_k(x) - 2\pi d_k^{(i)}| &= +\infty \quad \forall i \in \mathbb{N}, \text{ for a.e. } x \in \Omega \setminus E_i, \end{aligned} \quad (1.2)$$

and

$$\liminf_{k \rightarrow +\infty} \mathcal{H}^{n-1}(S_{\varphi_k}) \geq \mathcal{H}^{n-1}(S_{\varphi_{\infty}}). \quad (1.3)$$

We recall that a Caccioppoli partition of Ω is a countable family $(E_i)_{i \geq 1}$ of mutually disjoint finite perimeter sets whose union is Ω and whose series of perimeters is finite.

A direct consequence of Theorem 1.1 is the existence of a minimizer of (1.1), that is, the infimum in (1.1) is attained, for any $u \in SBV^p(\Omega; \mathbb{S}^1)$, and so a jump minimizing lifting exists (see Corollary 3.1). Theorems 1.1 (as well as Theorem 1.2) is relevant for

applications to various problems in the calculus of variations. For instance, finding a lifting that minimizes the length of the jump set is reminiscent of the related question of minimizing the BV -seminorm [16]. In turn, the latter is closely connected to Plateau-type problems and optimal transport [15]. Similarly, the structure of a jump minimizing lifting may be related to optimal transport problems with different cost functions. We discuss these connections in some detail in Subsection 3.3.

It is worth noticing that, somewhat surprisingly, the minimum in (1.1) is not attained within the class $SBV(\Omega)$, as shown by the example discussed in Section 3.2. This highlights the delicate nature of the problem (1.1). At the same time, in Theorem 3.2 we show that, for suitable Sobolev maps $u \in W^{1,p}(\Omega; \mathbb{S}^1)$, there exists a jump minimizing lifting belonging to $SBV^p(\Omega)$. We point out that Theorem 3.2 is currently established only in the two-dimensional setting. Its proof relies partly on an approximation argument for $\varphi_{\min}$ (see Proposition 3.3) which holds in arbitrary dimension, and partly on techniques from geometric measure theory, such as decomposition of integral 1-currents, that crucially exploit properties of two-dimensional setting. We believe that Theorem 3.2 could be extended to a more general framework, but we leave this analysis for future developments.

A generalization of Theorem 1.1 to a sequence of maps (u_k) is stated as follows.

Theorem 1.2 (Compactness and lower semicontinuity, II). *Let $p > 1$, $u, u_k \in SBV^p(\Omega; \mathbb{S}^1)$ be such that*

$$u_k \xrightarrow{*} u \quad \text{in } BV(\Omega; \mathbb{S}^1), \quad (1.4)$$

and let $\varphi_k \in GSBV^p(\Omega)$ be a lifting of u_k , for all $k \geq 1$. Suppose

$$\sup_{k \in \mathbb{N}} \mathcal{H}^{n-1}(S_{\varphi_k}) < +\infty.$$

Then there exist a Caccioppoli partition (E_i) of Ω and a not relabelled subsequence of indices k for which the following holds:

there exist a lifting $\varphi_\infty \in GSBV^p(\Omega)$ of u in Ω , and a sequence $(d_k^{(i)})_{k \geq 1} \subset \mathbb{Z}$ for any $i \in \mathbb{N}$, so that

$$\begin{aligned} \lim_{k \rightarrow +\infty} (\varphi_k(x) - 2\pi d_k^{(i)}) &= \varphi_\infty(x) \quad \forall i \in \mathbb{N}, \quad \text{for a.e. } x \in E_i, \\ \lim_{k \rightarrow +\infty} |\varphi_k(x) - 2\pi d_k^{(i)}| &= +\infty \quad \forall i \in \mathbb{N}, \quad \text{for a.e. } x \in \Omega \setminus E_i, \end{aligned} \quad (1.5)$$

and

$$\liminf_{k \rightarrow +\infty} \mathcal{H}^{n-1}(S_{\varphi_k}) \geq \mathcal{H}^{n-1}(S_{\varphi_\infty}). \quad (1.6)$$

We stress that these compactness results have a local feature, in the sense that liftings converge up to locally subtract a suitable constant. More general compactness results in the framework of functions of bounded deformations can be found in [20] (see also [19]) and, with different techniques, obtained by slightly modifying the given sequence of functions, in [31]. The proof of Theorems 1.1 and 1.2 is partially analogous to the arguments in [19, 20], which are here suitably modified to take advantage of the specific structure of liftings.

The above results may be applied to free discontinuity problems, in particular to the asymptotic analysis, via Γ -convergence, of Ambrosio-Tortorelli type functionals for maps defined on $\Omega \subset \mathbb{R}^n$ and taking values in $\mathbb{S}^1$. More specifically, let us define the two functional domains

$$\begin{aligned}\widehat{\mathcal{A}}_{\mathbb{S}^1} &:= \{(u, v) \in W^{1,1}(\Omega; \mathbb{S}^1) \times W^{1,2}(\Omega) : v|\nabla u| \in L^2(\Omega), 0 \leq v \leq 1\} \subset L^1(\Omega; \mathbb{S}^1) \times L^1(\Omega), \\ \mathcal{A}_{\mathbb{S}^1} &:= \{(u, v) \in W^{1,2}(\Omega; \mathbb{S}^1) \times W^{1,2}(\Omega) : 0 \leq v \leq 1\} \subset L^1(\Omega; \mathbb{S}^1) \times L^1(\Omega).\end{aligned}\tag{1.7}$$

For $\varepsilon \in (0, 1]$ let us consider the corresponding family of functionals

$$\widehat{\text{AT}}_\varepsilon^{\mathbb{S}^1}, \text{AT}_\varepsilon^{\mathbb{S}^1} : L^1(\Omega; \mathbb{S}^1) \times L^1(\Omega) \rightarrow [0, +\infty]$$

given by

$$\widehat{\text{AT}}_\varepsilon^{\mathbb{S}^1}(u, v) := \begin{cases} \text{AT}_\varepsilon(u, v) & \text{if } (u, v) \in \widehat{\mathcal{A}}_{\mathbb{S}^1}, \\ +\infty & \text{otherwise in } L^1(\Omega; \mathbb{S}^1) \times L^1(\Omega), \end{cases}\tag{1.8}$$

and

$$\text{AT}_\varepsilon^{\mathbb{S}^1}(u, v) := \begin{cases} \text{AT}_\varepsilon(u, v) & \text{if } (u, v) \in \mathcal{A}_{\mathbb{S}^1}, \\ +\infty & \text{otherwise in } L^1(\Omega; \mathbb{S}^1) \times L^1(\Omega), \end{cases}\tag{1.9}$$

where

$$\text{AT}_\varepsilon(u, v) := \int_{\Omega} \left(v^2 |\nabla u|^2 + \varepsilon |\nabla v|^2 + \frac{(v-1)^2}{4\varepsilon} \right) dx,\tag{1.10}$$

with $|\nabla u|^2$ indicating the Frobenius norm of ∇u . With this choice, if φ is a suitably regular lifting of the map u , i.e., $e^{i\varphi} = u$, then $|\nabla \varphi| = |\nabla u|$. It is clear that

$$\mathcal{A}_{\mathbb{S}^1} \subset \widehat{\mathcal{A}}_{\mathbb{S}^1}, \quad \text{AT}_\varepsilon^{\mathbb{S}^1} \geq \widehat{\text{AT}}_\varepsilon^{\mathbb{S}^1}.$$

The next two main results connect these functionals with Mumford-Shah-type functionals, and read as follows.

Theorem 1.3 (Γ -convergence of $\widehat{\text{AT}}_\varepsilon^{\mathbb{S}^1}$). *Let $\Omega \subset \mathbb{R}^n$ be a connected bounded open set with Lipschitz boundary. We have*

$$\Gamma\text{-}L^1 \lim_{\varepsilon \rightarrow 0^+} \widehat{\text{AT}}_\varepsilon^{\mathbb{S}^1} = \text{MS}_{\mathbb{S}^1},$$

where $\text{MS}_{\mathbb{S}^1} : L^1(\Omega; \mathbb{S}^1) \times L^1(\Omega) \rightarrow [0, +\infty]$ is given by

$$\text{MS}_{\mathbb{S}^1}(u, v) := \begin{cases} \int_{\Omega} |\nabla u|^2 dx + \mathcal{H}^{n-1}(S_u) & \text{if } u \in SBV^2(\Omega; \mathbb{S}^1), v = 1 \text{ a.e.}, \\ +\infty & \text{otherwise in } L^1(\Omega; \mathbb{S}^1) \times L^1(\Omega). \end{cases}\tag{1.11}$$

Theorem 1.4 (Γ -convergence of $\text{AT}_\varepsilon^{\mathbb{S}^1}$). *Let $\Omega \subset \mathbb{R}^n$ be a connected and simply-connected bounded open set with Lipschitz boundary. We have*

$$\Gamma\text{-}L^1 \lim_{\varepsilon \rightarrow 0^+} \text{AT}_\varepsilon^{\mathbb{S}^1} = \text{MS}_{\text{lift}},$$

where $\text{MS}_{\text{lift}} : L^1(\Omega; \mathbb{S}^1) \times L^1(\Omega) \rightarrow [0, +\infty]$ is given by

$$\text{MS}_{\text{lift}}(u, v) := \begin{cases} \int_{\Omega} |\nabla u|^2 dx + m_2[u] & \text{if } u \in SBV^2(\Omega; \mathbb{S}^1), v = 1 \text{ a.e.} \\ +\infty & \text{otherwise in } L^1(\Omega; \mathbb{S}^1) \times L^1(\Omega), \end{cases}\tag{1.12}$$

with

$$m_2[u] := \inf \{ \mathcal{H}^{n-1}(S_\varphi) : \varphi \in GSBV^2(\Omega), e^{i\varphi} = u \text{ a.e. in } \Omega \}.\tag{1.13}$$

We emphasize that, in Theorem 1.4, the assumption that Ω is simply connected is essential. Indeed, if Ω is not simply connected, the theorem fails. A simple counterexample is provided by the vortex map $u_V(x) = \frac{x}{|x|}$ defined for x in the annulus $\Omega := B_R(0) \setminus \overline{B_\rho(0)} \subset \mathbb{R}^2$, $0 < \rho < R$. The map u_V is smooth in Ω , and therefore its Γ -limit reduces to $\int_\Omega |\nabla u_V|^2 dx$, whereas $m_2[u_V] = R - \rho > 0$.

To better understand the two theorems above, some comments are in order.

- (i) The Γ -limit in (1.11) is the classical Mumford-Shah functional for $\mathbb{S}^1$ -valued maps, and indeed part of the proof of Theorem 1.3 is an adaptation of known results, together with some applications of the properties of liftings of nonsmooth maps;
- (ii) the second, more interesting, Γ -limit in Theorem 1.4 is nonlocal, it does not have an integral representation and depends on the minimization problem in (1.13). The singular term $m_2[u]$ is the penalization of a particular lifting of u , the one which minimizes the measure of the jump. It is clear that $\text{MS}_{\text{lift}} \geq \text{MS}_{\mathbb{S}^1}$.

These two theorems show the huge difference made by the choice of the two domains (1.7) where defining the approximating functionals, in particular the requirements $u \in W^{1,1}(\Omega; \mathbb{S}^1)$, $v|\nabla u| \in L^2(\Omega)$, as opposed to the more standard requirement $u \in W^{1,2}(\Omega; \mathbb{S}^1)$. The reason of the corresponding different limit behaviours is topological in nature, and it is better explained by the following example, see also [27, pag. 30].

An example of an SBV^2 map with vorticity. Let $n = 2$ and $\Omega = B_1(0) \subset \mathbb{R}^2$. We construct a map $u \in SBV^2(B_1(0); \mathbb{S}^1)$ with the following properties: it coincides with the vortex map $u_V(x) = \frac{x}{|x|}$ for $x \in B_1(0) \setminus B_{1/2}(0)$ and it jumps along the segment $(\frac{1}{4}, \frac{1}{2}) \times \{0\}$. To define u we modify u_V using a cut-off function to get a map in $SBV^2(\Omega; \mathbb{S}^1)$ with a small jump. For example, consider the lifting of u_V given by the determination of argument function $\theta : \Omega \rightarrow [0, 2\pi]$ jumping on the positive real axis (i.e., the choice of a determination of the imaginary part of the complex logarithm jumping on $\{(x, 0) : x > 0\}$). Hence, in $B_1(0)$, $S_\theta = (0, 1) \times \{0\}$ with constant jump opening $[\![\theta]\!] = 2\pi$. Now, let $h \in C^\infty(\overline{B_1(0)}; [0, 1])$ be such that $h \equiv 0$ in $B_{1/4}(0)$, $h \equiv 1$ in $B_1(0) \setminus B_{1/2}(0)$, $0 < h < 1$ in $B_{1/2}(0) \setminus B_{1/4}(0)$ (hence $|\nabla h| \leq C$ for some $C > 0$), and define

$$\theta_h := \begin{cases} h\theta & \text{in } B_{1/2}(0) \setminus \{0\}, \\ \theta & \text{in } B_1(0) \setminus B_{1/2}(0), \end{cases} \quad u := e^{i\theta_h}.$$

Thus we have (see Figure 1)

$$S_{\theta_h} = S_{\theta_h}^f \cup S_{\theta_h}^I \quad \text{where} \quad S_{\theta_h}^f = \left(\frac{1}{4}, \frac{1}{2}\right) \times \{0\}, \quad S_{\theta_h}^I = \left(\frac{1}{2}, 1\right) \times \{0\},$$

moreover $[\![\theta_h]\!]$ varies from 0 to 2π on $S_{\theta_h}^f$ and $[\![\theta_h]\!] \equiv 2\pi$ on $S_{\theta_h}^I$. This in turn implies $u = u_V$ on $B_1(0) \setminus B_{1/2}(0)$, and

$$S_u = S_{\theta_h}^f.$$

We also have $u \in SBV^2(B_1(0); \mathbb{S}^1)$, $\text{MS}_{\mathbb{S}^1}(u, 1) < +\infty$, and $m_2[u]$ turns out to be the length of the shortest segment joining the right endpoint of S_u to the boundary of Ω . Indeed any lifting φ of u has the following properties: $S_\varphi \supset S_{\theta_h}^f$, φ is a lifting of u_V on $B_1(0) \setminus B_{1/2}(0)$ and φ is a lifting of $(1, 0)$ on $B_{1/4}(0)$. Hence, in order to be minimal it must satisfy $S_\varphi = S_{\theta_h}$ and therefore $m_2[u] = \mathcal{H}^1(S_{\theta_h}) = 3/4$.

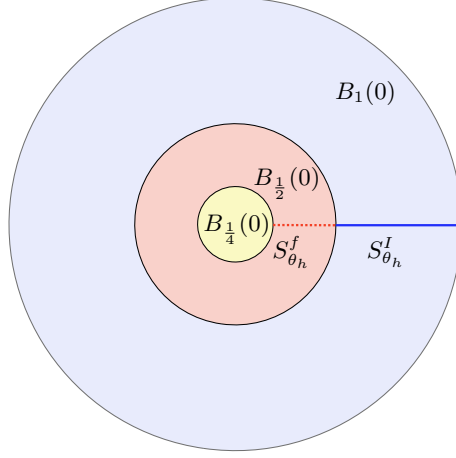


Figure 1: The dotted segment denotes the set $S_{\theta_h}^f$ where $\llbracket \theta_h(\cdot) \rrbracket$ varies between 0 and 2π , the solid segment denotes the set $S_{\theta_h}^I$ where $\llbracket \theta \rrbracket \equiv 2\pi$.

The map u provides an example for which one may compute the two Γ -limits above, and it turns out that the gradient integrability of the approximating maps is crucial in producing the two behaviours. In particular in Theorem 1.4, owing to topological obstructions related to the fact that u has non-zero degree on every circle $\partial B_\rho(0)$ with $1/2 \leq \rho < 1$ (since it coincides with the vortex map on these circles), the recovery sequence gives rise to the contribution $m_2[u]$. To see this, any construction consisting of a perturbation in a small ε -neighborhood of S_u as in [7, 8] would provide approximating functions $u_\varepsilon: B_1(0) \rightarrow \mathbb{S}^1$ which coincide with the vortex map far from the origin, and so necessarily u_ε has non-zero degree, which in turn implies that $u_\varepsilon \notin W^{1,2}(B_1(0); \mathbb{S}^1)$. Indeed, if $u_\varepsilon \in W^{1,2}(B_1(0); \mathbb{S}^1)$, then by [16, Theorem 1.1] u_ε has a lifting in $W^{1,2}(B_1(0))$, which is not possible since u_ε has non zero degree (cf. [16, Theorem 1.2]). In particular, u_ε cannot be used to produce a recovery sequence in Theorem 1.4, but it provides a suitable construction for the upper bound in Theorem 1.3, being of class $W^{1,1}(B_1(0); \mathbb{S}^1)$ and satisfying $v_\varepsilon |\nabla u_\varepsilon| \in L^2(B_1(0))$. In order to construct a recovery sequence in this case, we need to consider approximating functions with zero degree. This is possible by modifying u in a neighbourhood of the jump set of a minimal lifting (i.e., θ_h) and then define v_ε accordingly. Thus, passing to the limit one gets the contribution $\mathcal{H}^1(S_{\theta_h}) = m_2[u]$.

Ideas of the proofs. Compactness properties for sequences of liftings are delicate, as liftings may escape to infinity. To see this, it is enough to observe that one can always modify a lifting by adding integer multiples of 2π ; moreover, in the $GSBV$ setting, such shifts can be performed locally with different multiples of 2π , provided they are carried out along a Caccioppoli partition. This explains why Theorems 1.1 and 1.2 are essentially optimal. As already pointed out, analogous compactness results have been obtained in [19, 20] for general sequences of functions in the $GSBD$ framework. In our setting we provide a different proof, which exploits the specific structure of liftings. In particular, the constants to be subtracted can be chosen as suitable integer multiples of 2π . In this sense, sequences of liftings naturally induce sequences of Caccioppoli partitions of the domain Ω . The proofs of Theorems 1.1 and 1.2 therefore rely on an iterative argument in which, at each step, one selects the largest component of the underlying Caccioppoli partition and suitably normalizes the corresponding liftings. The proof of Theorem 1.2 involves an

additional difficulty, as the k -th lifting is associated with a function u_k converging to u . To address this issue, we decompose each lifting as the sum of a lifting of u and a remainder term that vanishes in the limit. This decomposition allows us to apply Theorem 1.1.

We now explain how Theorem 1.2 is used in the Γ -convergence analysis. The previous example shows that the nature of the Γ -limit (1.12) is related to topological obstructions arising when approximating a $\mathbb{S}^1$ -valued map whose approximate gradient is square-integrable. In particular, the non locality of MS_{lift} leads to non trivial difficulties in the proof of the lower-bound inequality. The basic idea is to rewrite the approximating functionals in terms of liftings. More precisely, given $u \in W^{1,2}(\Omega; \mathbb{S}^1)$ take a lifting $\varphi \in W^{1,2}(\Omega)$ of u , so that from the identity $|\nabla u| = |\nabla \varphi|$ it follows

$$\text{AT}_\varepsilon(u, v) = \int_{\Omega} \left(v^2 |\nabla \varphi|^2 + \varepsilon |\nabla v|^2 + \frac{(v-1)^2}{4\varepsilon} \right) dx, \quad (1.14)$$

which is the Ambrosio-Tortorelli functional for the pair (φ, v) . Now, if for any sequence $(u_{\varepsilon_k}, v_{\varepsilon_k}) \rightarrow (u, 1)$ in $L^1(\Omega; \mathbb{S}^1) \times L^1(\Omega)$ we were able to prove that $\varphi_{\varepsilon_k} \rightarrow \varphi$ in $L^1(\Omega)$, with $\varphi_{\varepsilon_k} \in W^{1,2}(\Omega)$ lifting of u_{ε_k} , we could conclude by applying the classical results [7, 8], since the limit φ is a lifting of u . Unfortunately this convergence in general is not valid, as the energy provides a control of $\nabla \varphi_{\varepsilon_k}$ only in regions where v_{ε_k} is far from zero. Moreover, as already observed, sequences of liftings might escape to infinity (since we can always add integer multiples of 2π). Nevertheless, by means of a truncation argument based on the Modica-Mortola trick combined with the coarea formula, one can modify $(\varphi_{\varepsilon_k})$ on a vanishing set to construct a new sequence (ϕ_{ε_k}) satisfying the assumptions of Theorem 1.2 and hence the corresponding convergence conclusions. This and a Vitali covering argument allow one to deduce the lower bound inequality in Theorem 1.4.

Content of the paper. In Section 2 we collect some notation and recall some useful tools which will be employed to prove the main results. In Section 3 we prove (Theorem 3.2) the existence of an $SBV(\Omega)$ -jump minimizing lifting for a map whose Jacobian determinant has a finite number of singularities. We also provide an example where the jump minimizing lifting is not in $SBV(\Omega)$ but just in $GSBV(\Omega)$ (Section 3.2). Furthermore, we describe a connection with optimal transport (Section 3.3). In Section 4 we provide the proofs of the compactness Theorems 1.1 and 1.2. Finally, in Section 5, after establishing some density and approximation results in $SBV(\Omega)$, both for $\mathbb{S}^1$ -valued functions and for liftings, we prove Theorem 1.3. In its proof, we do not make use of the results of Section 5.1, but instead rely on those of Section 3. Section 5.3 is devoted to the proof of Theorem 1.4. We conclude the paper with a short section containing remarks and open problems (Section 6).

2 Notation and preliminaries

In this section we collect some notation, and recall some notions concerning SBV and $GSBV$ functions and related Γ -convergence results [6], currents theory in dimension two [28], liftings and Jacobian of liftings [16]. In what follows:

- $n \geq 1$ is a fixed integer and $p > 1$ is a fixed real number;
- $\partial^* A$ denotes the reduced boundary of finite perimeter set $A \subset \mathbb{R}^n$;
- $|\cdot|$ and $\mathcal{H}^{n-1}$ denote the Lebesgue measure and the $(n-1)$ -dimensional Hausdorff measure in $\mathbb{R}^n$, respectively;

- χ_A denotes the characteristic function of the set $A \subset \mathbb{R}^n$;
- for $a, b \in \mathbb{R}^n$ the symbol $a \otimes b$ denotes the tensor product between a and b ;
- $\mathbb{S}^1$ is the unit circle in $\mathbb{R}^2$ which is identified with the unit circle in $\mathbb{C}$, i.e.,

$$\mathbb{S}^1 := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\} = \{x + iy \in \mathbb{C} : x^2 + y^2 = 1\}.$$

2.1 SBV and GSBV functions

Let $U \subset \mathbb{R}^n$ be open and bounded, and $m \geq 1$ an integer. We denote by $BV(U; \mathbb{R}^m)$ the space of vector-valued *functions with bounded variation in U* , and with $|\cdot|_{BV}$ and $\|\cdot\|_{BV}$ the BV seminorm and norm, respectively, i.e.

$$|u|_{BV} := |Du|(U), \quad \|u\|_{BV} := \|u\|_{L^1} + |u|_{BV},$$

see [6]. We say that $u \in BV(U; \mathbb{R}^m)$ belongs to the space of *special functions with bounded variation in U* , i.e., $u \in SBV(U; \mathbb{R}^m)$, if its distributional gradient is a finite $\mathbb{R}^{m \times n}$ -valued Radon measure without Cantor part, that is,

$$Du = \nabla u \mathcal{L}^n + \llbracket u \rrbracket \otimes \nu_u \mathcal{H}^{n-1} \llcorner S_u,$$

where ∇u is the approximate gradient of u , S_u is the approximate jump set of u , $\llbracket u \rrbracket = u^+ - u^-$ is the jump opening and ν_u is the unit normal to S_u , see [6, Def. 3.67]. A measurable function $u : U \rightarrow \mathbb{R}^m$ belongs to the space of *generalised special functions with bounded variation in U* , that is, $u \in GSBV(U; \mathbb{R}^m)$, if $\phi \circ u \in SBV_{\text{loc}}(U)$ for any $\phi \in C^1(\mathbb{R}^m)$ with $\nabla \phi$ compactly supported. Recall that $f \in SBV_{\text{loc}}(U)$ if $f \in SBV(U')$ for every open set $U' \subset\subset U$. If $m = 1$ we write $BV(U) = BV(U; \mathbb{R})$, $SBV(U) = SBV(U; \mathbb{R})$ and $GSBV(U) = GSBV(U; \mathbb{R})$.

Remark 2.1 (Equivalent definition of $GSBV$ for $m = 1$). $GSBV(U)$ can be equivalently defined as the space of measurable functions $u : U \rightarrow \mathbb{R}$ such that $u \wedge M \vee (-M) \in SBV_{\text{loc}}(U)$ for any $M > 0$.

For $p > 1$ we set

$$SBV^p(U; \mathbb{R}^m) = \{u \in SBV(U; \mathbb{R}^m) : \nabla u \in L^p(U; \mathbb{R}^{m \times n}) \text{ and } \mathcal{H}^{n-1}(S_u) < +\infty\},$$

and

$$GSBV^p(U; \mathbb{R}^m) = \{u \in GSBV(U; \mathbb{R}^m) : \nabla u \in L^p(U; \mathbb{R}^{m \times n}) \text{ and } \mathcal{H}^{n-1}(S_u) < +\infty\}.$$

Again, when $m = 1$ we write $SBV^p(U) = SBV^p(U; \mathbb{R})$ and $GSBV^p(U) = GSBV^p(U; \mathbb{R})$. We set

$$\begin{aligned} BV(U; \mathbb{S}^1) &= \{u \in BV(U; \mathbb{R}^2) : |u| = 1 \text{ a.e. in } U\}, \\ SBV(U; \mathbb{S}^1) &= \{u \in SBV(U; \mathbb{R}^2) : |u| = 1 \text{ a.e. in } \Omega\}, \end{aligned}$$

and for $p > 1$

$$SBV^p(U; \mathbb{S}^1) = \{u \in SBV(U; \mathbb{S}^1) : \nabla u \in L^p(U; \mathbb{R}^{2 \times 2}), \mathcal{H}^{n-1}(S_u) < +\infty\}.$$

Eventually, a (finite or countable) family (E_i) of finite perimeter subsets of a finite perimeter set F is called a Caccioppoli partition of F if the sets E_i are pairwise disjoint, and their union is F . The next technical observation will be needed later (Section 5.2).

Remark 2.2 (Approximation of a BV function by smooth functions). Let $A \subset \mathbb{R}^n$ be a bounded open set with Lipschitz boundary and $\varphi \in BV(A)$. By [6, Theorem 3.9], for any $\delta > 0$, there exists a function $\varphi_\delta \in C^\infty(A)$ such that

$$\int_A |\varphi - \varphi_\delta| dx < \delta, \quad \int_A |\nabla \varphi_\delta| dx \leq |D\varphi|(A) + \delta.$$

Moreover, by inspecting the proof, φ_δ can be chosen so that

$$\varphi = \varphi_\delta \quad \mathcal{H}^{n-1} - \text{a.e. on } \partial A,$$

in the sense of BV -traces on ∂A (see e.g. [6, page 181]).

2.2 Currents

In some proofs we will deal with measures in $\mathbb{R}^2$ which can be identified with currents in the sense of Federer. We just recall here those definitions used to our purposes, and we refer to the monographs [28, 34] for a more detailed discussion on the subject.

Let $\Omega \subseteq \mathbb{R}^2$ be open with Lipschitz boundary, and $k \in \{0, 1, 2\}$; we denote by $\mathcal{D}^k(\Omega)$ the topological vector space of k -forms on Ω . Its dual space $\mathcal{D}_k(\Omega)$ is known as the space of k -currents in Ω . For a current $T \in \mathcal{D}_k(\Omega)$, we denote the mass of T in Ω by $|T|_\Omega$, defined by

$$|T|_\Omega := \sup\{T(\omega) : \omega \in \mathcal{D}^k(\Omega), |\omega| \leq 1\}. \quad (2.1)$$

Notice that 0-currents in $\mathcal{D}_0(\Omega)$ with finite mass can be identified with finite Radon measures on Ω with finite total variation.

Given a current $T \in \mathcal{D}_k(\Omega)$ with $k \in \{1, 2\}$, we introduce its boundary $\partial T \in \mathcal{D}_{k-1}(\Omega)$ as

$$\partial T(\omega) := T(d\omega) \quad \forall \omega \in \mathcal{D}^{k-1}(\Omega),$$

where $d\omega$ denotes the external differential of ω . By convention, in the case $k = 0$ it is $\partial T = 0$.

A $\mathcal{H}^1$ -rectifiable set $R \subset \Omega$ is a set for which there are (at most) countably many Lipschitz maps $\phi_h : \mathbb{R} \rightarrow \mathbb{R}^2$ such that

$$R \subseteq N \cup \bigcup_{h=0}^{+\infty} \phi_h(\mathbb{R}), \quad \mathcal{H}^1(N) = 0.$$

We say that $T \in \mathcal{D}_1(\Omega)$ is rectifiable if there exist a $\mathcal{H}^1$ -rectifiable set R , a unit simple 1-vector $\tau(x)$ for $\mathcal{H}^1$ -a.e. $x \in R$, and a measurable function $\theta : R \rightarrow \mathbb{R}$ such that

$$T(\omega) = \int_R \theta(x) \langle \omega(x), \tau(x) \rangle d\mathcal{H}^1(x), \quad \omega \in \mathcal{D}^1(\Omega). \quad (2.2)$$

Here $\langle \cdot, \cdot \rangle$ stands for the duality pairing between 1-covectors and 1-vectors. A rectifiable current $T \in \mathcal{D}_1(\Omega)$ is said to be an integral 1-current if θ takes integer values, τ is tangent to R , and $|T|_\Omega + |\partial T|_\Omega < +\infty$. Given an integral current $T \in \mathcal{D}_1(\Omega)$ we also denote it as

$$T = (R, \theta, \tau), \quad (2.3)$$

whenever R is the $\mathcal{H}^1$ -rectifiable set, θ the $\mathbb{Z}$ -valued function, τ the 1-vector tangent to R such that (2.2) takes place.

The structure of integral 1-currents is quite rigid. Specifically, from Federer decomposition theorem [28, Sections 4.2.25 and 4.5.9], if $T \in \mathcal{D}_1(\Omega)$ is a 1-integral current, then T can be written as a sum

$$T = \sum_{i=1}^{+\infty} T_i, \quad \text{with} \quad \sum_{i=1}^{+\infty} |T_i|_\Omega = |T|_\Omega, \quad \sum_{i=1}^{+\infty} |\partial T_i|_\Omega = |\partial T|_\Omega, \quad (2.4)$$

and where each T_i is the image of a Lipschitz curve, in the sense that there exists a Lipschitz curve $\alpha_i : [0, 1] \rightarrow \mathbb{R}^2$ injective on $[0, 1)$, and such that,

$$T_i(\omega) = \int_0^1 \langle \omega(\alpha_i(t)), \dot{\alpha}_i(t) \rangle dt.$$

In this case, we denote T_i also by $[\![\alpha_i]\!]$. Furthermore, $\partial T_i = 0$ in $\mathcal{D}_0(\Omega)$ if and only if either the curve α_i is closed, or it has endpoints in $\partial\Omega$. In general, $\partial T_i = \delta_{y_i} - \delta_{x_i}$ where $x_i = \alpha_i(0)$ and $y_i = \alpha_i(1)$. In particular, since $|\partial T|_\Omega$ is finite, it follows that ∂T is a finite sum of Dirac deltas, namely

$$\partial T = \sum_{i=1}^{\ell} z_i \delta_{p_i}, \quad \sum_{i=1}^{\ell} |z_i| = |\partial T|_\Omega,$$

where $\ell \in \mathbb{N}$, $z_i \in \mathbb{Z}$, and p_i are distinct points in Ω .

Assume now Ω is simply-connected. If $T_i = [\![\alpha_i]\!]$ and $\partial T_i = 0$, we distinguish two cases: if T_i is a loop, i.e. $\alpha_i(0) = \alpha_i(1)$, we have that $\alpha_i([0, 1])$ is a Jordan curve and we denote by $\widehat{G}_i$ its interior. If instead $\alpha_i(0) \neq \alpha_i(1)$, $\alpha_i(0), \alpha_i(1) \in \partial\Omega$ we can consider the curve in $\partial\Omega$ connecting $\alpha_i(0)$ and $\alpha_i(1)$ with smaller length, and the concatenation of this curve with α_i becomes a Jordan curve $\widehat{\alpha}_i$ whose interior we denote by $\widehat{G}_i$. In both cases $\Omega \cap \partial \widehat{G}_i = \alpha_i([0, 1])$. Now, by [36, Theorem 7.9.1], there exists an integral 2-current S_i in $\mathbb{R}^2$ such that $\partial S_i = [\![\alpha_i]\!]$ (if α_i is a loop) or $\partial S_i = [\![\widehat{\alpha}_i]\!]$ (otherwise). By constancy lemma [36, Theorem 7.3.1] it straightforwardly follows that S_i coincides, up to a sign, with the integration over the finite perimeter set $\widehat{G}_i \subseteq \Omega$ whose reduced boundary (up to $\mathcal{H}^1$ -negligible sets) agrees with $\alpha_i([0, 1])$ (in Ω); precisely, there is $\sigma_i \in \{-1, 1\}$ such that the current $S_i \in \mathcal{D}_2(\Omega)$ is

$$S_i(\omega) = \sigma_i \int_{\widehat{G}_i} \langle \omega, e_1 \wedge e_2 \rangle dx \quad (2.5)$$

(where $\langle \cdot, \cdot \rangle$ is the duality product between 2-covectors and 2-vectors) and $\partial S_i = T_i$. Moreover, by adding a constant in $\mathbb{Z}$ if necessary, we can always replace the set $\widehat{G}_i$ by $\Omega \setminus \widehat{G}_i$; in particular, we can assume that the set of integration in (2.5) is the one between $\widehat{G}_i$ and $\Omega \setminus \widehat{G}_i$ with smaller Lebesgue measure. As a consequence, denoting this set by G_i , by the isoperimetric inequality [36, Theorem 7.9.1] and by construction, since Ω is Lipschitz, there is a constant $c_\Omega > 0$, depending on Ω , such that

$$|G_i| \leq c_\Omega |T_i|_\Omega^2. \quad (2.6)$$

2.3 Liftings of $\mathbb{S}^1$ -valued maps

Let $\Omega \subset \mathbb{R}^n$ be a bounded connected open set with Lipschitz boundary, and $u : \Omega \rightarrow \mathbb{S}^1$ be a measurable function. A *lifting* of u is a measurable function $\varphi : \Omega \rightarrow \mathbb{R}$ such that

$$u(x) = e^{i\varphi(x)} \quad \text{for a.e. } x \in \Omega.$$

Given a Borel set $B \subseteq \Omega$, we say that $\varphi : B \rightarrow \mathbb{R}$ is a lifting of u on B if the previous equality holds a.e. on B . Clearly, if φ is a lifting of u , then so is $\varphi + 2\pi m$ for all $m \in \mathbb{Z}$.

If u has some regularity, a natural question is whether φ can be chosen with the same regularity. The answer is partially positive, see [16, 25] for more details. In particular, if Ω is simply connected we have

- (1) If $u \in C^k(\bar{\Omega}; \mathbb{S}^1)$ for some $k \geq 0$, then u has a lifting $\varphi \in C^k(\bar{\Omega})$, unique (mod 2π), [16, Lemma 1.1];
- (2) If $u \in C^k(\bar{\Omega}; \mathbb{S}^1) \cap W^{1,p}(\Omega; \mathbb{S}^1)$ for some $p \in [1, +\infty]$, then u has a lifting $\varphi \in C^k(\bar{\Omega}) \cap W^{1,p}(\Omega)$; If $n = 1$ and $u \in W^{1,p}(\Omega; \mathbb{S}^1)$ for some $p \in [1, +\infty)$, then u has a lifting $\varphi \in W^{1,p}(\Omega)$;
- (3) If $n \geq 2$ and $u \in W^{1,p}(\Omega; \mathbb{S}^1)$ for some $p \in [2, +\infty)$, then u has a lifting $\varphi \in W^{1,p}(\Omega)$. Moreover φ is unique (mod 2π);
- (4) If $n \geq 2$, then for every $p \in [1, 2)$ there exists $u \in W^{1,p}(\Omega; \mathbb{S}^1)$ having no liftings in $W^{1,p}(\Omega)$, see [16, Theorem 1.2, Remark 1.9].

A well-known example showing (4) when $n = 2$ is the vortex map u_V discussed in the introduction. Indeed it can be shown [16, Pages 17-19] that there are no liftings of u in $W^{1,1}(B_1)$ (and thus there are no liftings in $W^{1,p}(B_1)$ for all $p \in [1, 2)$).

Next we recall some regularity results on liftings of BV maps (see [25, Theorem 1.1], and also [16, Theorem 1.4]).

Theorem 2.3 (Davila-Ignat). *Let $u \in BV(\Omega; \mathbb{S}^1)$. Then there exists a lifting $\varphi \in BV(\Omega)$ such that $\|\varphi\|_{L^\infty} \leq 2\pi$ and $|\varphi|_{BV} \leq 2|u|_{BV}$.*

Remark 2.4. In the above theorem Ω does not have to be simply connected. If $u \in SBV(\Omega; \mathbb{S}^1)$ then φ of Theorem 2.3 can be chosen in $SBV(\Omega)$. If $u \in SBV^p(\Omega; \mathbb{S}^1)$ for some $p > 1$ then φ of Theorem 2.3 can be chosen in $SBV^p(\Omega) \cap L^\infty(\Omega; [0, 2\pi])$.

As usual, for any lifting $\varphi \in SBV(\Omega)$ of u we write $S_\varphi = S_\varphi^I \cup S_\varphi^f$ where S_φ^I stands for the “integer” part of the jump, and S_φ^f for the “fractional” part. More precisely,

$$S_\varphi^I := \{x \in S_\varphi : \llbracket \varphi \rrbracket(x) \in 2\pi\mathbb{Z}\}, \quad S_\varphi^f := S_\varphi \setminus S_\varphi^I.$$

Notice that in particular $S_\varphi^f = S_u$.

Now, let $u \in BV(\Omega; \mathbb{S}^1)$ and consider the minimum problem

$$\inf\{|\varphi|_{BV} : \varphi \in BV(\Omega), e^{i\varphi} = u \text{ a.e. in } \Omega\}.$$

We recall that there exists a minimizer $\varphi \in BV(\Omega)$ such that $|\varphi|_{BV} \leq 2|u|_{BV}$ and $0 \leq \int_\Omega \varphi \, dx \leq 2\pi|\Omega|$ (see [16, page 25]). As explained in the introduction, we are instead concerned with the existence of a lifting that minimizes the measure of the jump set. This will be the subject of Section 3.

2.4 Jacobians and liftings of $\mathbb{S}^1$ -valued maps from 2-dimensional domains

In this section we assume $\Omega \subseteq \mathbb{R}^2$, and consider a Sobolev map $u \in W^{1,1}(\Omega; \mathbb{S}^1)$. We recall the definition of distributional Jacobian determinant for Sobolev maps [16, Page 12, formulas (1.33) and (1.35)] that is

$$\text{Det}(\nabla u) = \frac{1}{2} \text{Div}(u_1 \nabla^\perp u_2 - u_2 \nabla^\perp u_1) =: \frac{1}{2} \text{Div}(j(u)), \quad (2.7)$$

where the divergence operator Div is considered in the sense of distributions, and we use the notation

$$j(u) := u_1 \nabla^\perp u_2 - u_2 \nabla^\perp u_1 \in L^1(\Omega; \mathbb{R}^2),$$

$\nabla^\perp = (-\partial_2, \partial_1)$. Following [16], for any two points $x, y \in \overline{\Omega}$ we denote by

$$d_\Omega(x, y) := \min\{|x - y|, \text{dist}(x, \partial\Omega) + \text{dist}(y, \partial\Omega)\}.$$

It is well-known [16] that if $u \in W^{1,1}(\Omega; \mathbb{S}^1)$ then there are two sequences of points $(y_i) \subset \overline{\Omega}$ and $(x_i) \subset \overline{\Omega}$ such that $\sum_{i=1}^{+\infty} d_\Omega(x_i, y_i) < +\infty$ and

$$\text{Det}(\nabla u) = \pi \sum_{i=1}^{+\infty} (\delta_{x_i} - \delta_{y_i}), \quad (2.8)$$

Of course, it might happen that, instead of sequences of points, there is a finite number $N \in \mathbb{N}$ of couples $(x_i, y_i) \in \overline{\Omega} \times \overline{\Omega}$ such that

$$\text{Det}(\nabla u) = \pi \sum_{i=1}^N (\delta_{x_i} - \delta_{y_i}). \quad (2.9)$$

Remark 2.5. Assume $\text{Det}(\nabla u)$ is a finite Radon measure. Since the points x_i, y_i might belong to $\partial\Omega$ and since they are not distinct in general, $\text{Det}(\nabla u)$ is, as a distribution in Ω , a sum of the form

$$\text{Det}(\nabla u) = \pi \sum_{i=1}^M z_i \delta_{p_i} \quad (2.10)$$

where $z_i \in \mathbb{Z}$, p_i are distinct points in Ω , $M \in \mathbb{N}$. As an example, the vortex map $u_V(x) = \frac{x}{|x|}$ for nonzero x in $\Omega = B_r(0)$, has

$$\text{Det}(\nabla u_V) = \pi(\delta_0 - \delta_q)$$

for any choice of $q \in \partial B_r(0)$. In such a case, as $q \in \partial B_r(0)$, the Dirac delta δ_q does not contribute to the measure $\text{Det}(\nabla u_V)$ in the open set $B_r(0)$, and so we can also write $\text{Det}(\nabla u_V) = \pi\delta_0$ in $B_r(0)$.

Let now $u \in W^{1,1}(\Omega; \mathbb{S}^1)$ and let $\varphi \in SBV(\Omega)$ be a lifting of u ; then the distributional gradient of φ reads

$$D\varphi = \nabla\varphi \mathcal{L}^2 + D^j\varphi, \quad (2.11)$$

where $D^j\varphi = \llbracket\varphi\rrbracket n_{S_\varphi} \mathcal{H}^1 \llcorner S_\varphi$, n_{S_φ} being a unit normal to S_φ and $\llbracket\varphi\rrbracket$ the corresponding jump amplitude. We then consider the rotated, by the counterclockwise $\pi/2$ -rotation, of $D^j\varphi$, namely

$$(D^j\varphi)^\perp = \llbracket\varphi\rrbracket n_{S_\varphi}^\perp \mathcal{H}^1 \llcorner S_\varphi, \quad (2.12)$$

where $n_{S_\varphi}^\perp = (-(n_{S_\varphi})_2, (n_{S_\varphi})_1)$ is an oriented tangent vector to S_φ . The vector-valued measure in (2.12) can be identified with 2π times the integral 1-current

$$T_\varphi = (S_\varphi, \frac{\llbracket \varphi \rrbracket}{2\pi}, n_{S_\varphi}^\perp) \quad (2.13)$$

which indeed has $\frac{\llbracket \varphi \rrbracket}{2\pi} \in \mathbb{Z} \mathcal{H}^1$ -a.e.. If we also identify the elements of $\mathcal{D}_0(\Omega)$ with scalar measures, $\partial T_\varphi \in \mathcal{D}_0(\Omega)$ corresponds to minus the divergence of T_φ , and so

$$\partial T_\varphi = -\frac{1}{2\pi} \text{Div}((D^j \varphi)^\perp) = \frac{1}{2\pi} \text{Div}(\nabla^\perp \varphi) \quad (2.14)$$

where we have used that

$$\text{Div}((D\varphi)^\perp) = \text{Curl}(D\varphi) = 0.$$

Also here $\text{Curl}(D\varphi)$ is defined in a distributional sense. Therefore, by (2.7), it follows

$$\text{Det}(\nabla u) = \frac{1}{2} \text{Div}(\nabla^\perp \varphi) = -\frac{1}{2} \text{Curl}(\nabla \varphi), \quad (2.15)$$

where $\nabla \varphi$ is the approximate gradient of φ . We can then state the following:

Proposition 2.6. *Let $u \in W^{1,1}(\Omega; \mathbb{S}^1)$ and let $\varphi \in SBV(\Omega)$ be a lifting of u . Then (2.15) holds in the sense of distribution. Moreover, if T_φ is the current in (2.13), then*

$$\partial T_\varphi = \frac{1}{\pi} \text{Det}(\nabla u).$$

2.5 Γ -approximation and compactness for the Mumford-Shah functional

In view of the application to Theorems 1.3, 1.4, we recall two classical results [7, 8] (see also [30] for the generalization to the vector case), and [6, Theorem 4.8].

Theorem 2.7 (Ambrosio-Tortorelli). *Let $\Omega \subset \mathbb{R}^n$ be an open set with Lipschitz boundary, and*

$$\mathcal{A} := \{(u, v) \in W^{1,2}(\Omega; \mathbb{R}^m) \times W^{1,2}(\Omega): 0 \leq v \leq 1\}.$$

Then the functionals

$$\text{AT}_\varepsilon(u, v) := \begin{cases} \int_\Omega \left(v^2 |\nabla u|^2 + \varepsilon |\nabla v|^2 + \frac{(v-1)^2}{4\varepsilon} \right) dx & \text{if } (u, v) \in \mathcal{A}, \\ +\infty & \text{otherwise in } L^1(\Omega; \mathbb{R}^m) \times L^1(\Omega), \end{cases}$$

Γ - L^1 -converge to the Mumford-Shah functional

$$\text{MS}(u, v) := \begin{cases} \int_\Omega |\nabla u|^2 dx + \mathcal{H}^{n-1}(S_u) & \text{if } u \in GSBV^2(\Omega; \mathbb{R}^m), v = 1 \text{ a.e.} \\ +\infty & \text{otherwise in } L^1(\Omega; \mathbb{R}^m) \times L^1(\Omega), \end{cases}$$

as $\varepsilon \rightarrow 0^+$.

Remark 2.8. By inspecting the proof of Theorem 2.7 one actually deduces the following properties:

- (i) **Decoupling.** Let $\varepsilon_k \searrow 0$, and let $((u_k, v_k))_{k \geq 1} \subset \mathcal{A}$ be a sequence converging to $(u, 1)$ in $L^1(\Omega; \mathbb{R}^m) \times L^1(\Omega)$ with $\sup_{k \in \mathbb{N}} AT_{\varepsilon_k}(u_k, v_k) < +\infty$. Then

$$\begin{aligned} \liminf_{k \rightarrow +\infty} \int_{\Omega} v_k^2 |\nabla u_k|^2 dx &\geq \int_{\Omega} |\nabla u|^2 dx, \\ \liminf_{k \rightarrow +\infty} \int_{\Omega} \left(\varepsilon_k |\nabla v_k|^2 + \frac{(v_k - 1)^2}{4\varepsilon_k} \right) dx &\geq \mathcal{H}^{n-1}(S_u). \end{aligned} \tag{2.16}$$

- (ii) **Γ -convergence on a larger domain.** The result still holds if one replaces $\mathcal{A}$ with the larger class

$$\widehat{\mathcal{A}} := \{(u, v) \in W^{1,1}(\Omega; \mathbb{R}^m) \times W^{1,2}(\Omega) : v|\nabla u| \in L^2(\Omega), 0 \leq v \leq 1\}.$$

Property (ii) is a consequence of the inclusion $\mathcal{A} \subset \widehat{\mathcal{A}}$ and of property (i). Indeed, by $\mathcal{A} \subset \widehat{\mathcal{A}}$ it follows that a recovery sequence in $\mathcal{A}$ is also a recovery sequence in $\widehat{\mathcal{A}}$, and thus the limsup inequality holds. For the liminf inequality consider a sequence $(u_k, v_k) \subset \widehat{\mathcal{A}}$ converging to $(u, 1)$ in $L^1(\Omega; \mathbb{R}^m) \times L^1(\Omega)$. Since the integrability of $|\nabla u_k|$ does not interfere with the Modica-Mortola term, by property (i) it follows

$$\liminf_{k \rightarrow +\infty} \int_{\Omega} \left(\varepsilon_k |\nabla v_k|^2 + \frac{(v_k - 1)^2}{4\varepsilon_k} \right) dx \geq \mathcal{H}^{n-1}(S_u).$$

For the bulk term instead fix $0 < \eta < 1$. By letting $\hat{u}_k = u_k \chi_{\{v_k \geq \eta\}}$ we find

$$\begin{aligned} \liminf_{k \rightarrow +\infty} \int_{\Omega} v_k^2 |\nabla u_k|^2 dx &\geq \liminf_{k \rightarrow +\infty} \eta^2 \int_{\Omega} \chi_{\{v_k \geq \eta\}} |\nabla u_k|^2 dx \\ &= \liminf_{k \rightarrow +\infty} \eta^2 \int_{\Omega} |\nabla \hat{u}_k|^2 dx = \eta^2 \int_{\Omega} |\nabla u|^2 dx, \end{aligned}$$

where the last equality follows by observing that $\chi_{\{v_k \geq \eta\}}$ converges to 1 in $L^1(\Omega)$ and thus, up to subsequence, $|\nabla \hat{u}_k| \rightharpoonup |\nabla u|$ in $L^2(\Omega)$. Eventually, by the arbitrariness of η , we can take the limit as $\eta \rightarrow 1$ and conclude.

Theorem 2.9 (Compactness in SBV). *Let $p > 1$, $\Omega \subset \mathbb{R}^n$ be a bounded open set and $(\varphi_k)_{k \geq 1} \subset SBV(\Omega)$ be a sequence satisfying*

$$\sup_{k \in \mathbb{N}} \left(\int_{\Omega} |\nabla \varphi_k|^p dx + \mathcal{H}^{n-1}(S_{\varphi_k}) \right) < +\infty,$$

and

$$\sup_{k \in \mathbb{N}} \|\varphi_k\|_{\infty} < +\infty. \tag{2.17}$$

Then there exists a subsequence of (φ_k) weakly-star converging in $BV(\Omega)$ to a function belonging to $SBV(\Omega)$.

Removing assumption (2.17) leads to the following result, which is the $GSBV^p$ variant of [19, Theorem 1.1], and that will be applied in the proof of Lemma 4.1 to an appropriate sequence of liftings.

Theorem 2.10 (Compactness). *Let $p > 1$, $\Omega \subset \mathbb{R}^n$ be an open set and $(\phi_k)_{k \geq 1} \subset GSBV^p(\Omega)$ be a sequence satisfying*

$$\sup_{k \in \mathbb{N}} \left(\int_{\Omega} |\nabla \phi_k|^p dx + \mathcal{H}^{n-1}(S_{\phi_k}) \right) < +\infty.$$

Then there exist a (not relabelled) subsequence and a function $\varphi_{\infty} \in GSBV^p(\Omega)$ such that:

$$E := \{x \in \Omega : |\phi_k(x)| \rightarrow +\infty\} \text{ has finite perimeter,}$$

$$\varphi_{\infty} = 0 \text{ on } E,$$

$$\phi_k \rightarrow \varphi_{\infty} \text{ a.e. on } \Omega \setminus E,$$

$$\nabla \phi_k \rightarrow \nabla \varphi_{\infty} \text{ in } L^1(\Omega \setminus E; \mathbb{R}^n),$$

$$\mathcal{H}^{n-1}(S_{\varphi_{\infty}} \cup \partial^* E) \leq \liminf_{k \rightarrow +\infty} \mathcal{H}^{n-1}(S_{\phi_k}).$$

3 On jump minimizing liftings

In view of the applications in Section 5, we are concerned with the analysis of the following minimization problem: let $p > 1$, $u \in SBV^p(\Omega; \mathbb{S}^1)$ and define

$$m_p[u] := \inf \{ \mathcal{H}^{n-1}(S_{\varphi}) : \varphi \in GSBV^p(\Omega), e^{i\varphi} = u \text{ a.e. in } \Omega \}. \quad (3.1)$$

According to Remark 2.4, the set of competitors φ in (3.1) is non empty. In the next section we discuss some consequences of Theorems 1.1 and 1.2. The first one is the existence of a minimizer of (3.1). Such a minimizer in general is not a function of bounded variation, and for this reason is necessary to set the problem in the space $GSBV^p(\Omega)$ instead of $SBV^p(\Omega)$. In Section 3.2 we exhibit a map $u \in SBV^p(\Omega; \mathbb{S}^1)$ having minimizer of (3.1) which is not a BV -function. On the other hand, in Theorem 3.2 we give some sufficient condition for which a minimizer belongs to $SBV^p(\Omega)$.

3.1 Main consequences of compactness of sequences of liftings

The first consequence follows from Theorem 1.1:

Corollary 3.1 (Existence). *Let $p > 1$. Then there exists a minimizer $\varphi \in GSBV^p(\Omega)$ of (3.1).*

As already pointed out, in general a minimizer $\varphi_{\min}$ of (3.1) does not exist in $SBV^p(\Omega)$. However, at least in dimension $n = 2$, we can give some sufficient condition in order that $\varphi_{\min} \in SBV^p(\Omega)$. The following result holds:

Theorem 3.2. *Let $p > 1$, let $\Omega \subseteq \mathbb{R}^2$ be a bounded simply-connected Lipschitz open set and let $u \in W^{1,p}(\Omega; \mathbb{S}^1)$. Assume $\text{Det}(\nabla u) \in \mathcal{M}_b(\Omega)$. Then there exists $\bar{\varphi}_{\min} \in SBV^p(\Omega)$ minimizer of (3.1).*

Before proving Theorem 3.2, we show the following approximation property:

Proposition 3.3. *Let $p > 1$, let $\Omega \subseteq \mathbb{R}^n$ be a bounded Lipschitz open set, and let $u \in SBV^p(\Omega; \mathbb{S}^1)$. Let $\varphi_{\min}$ be a minimizer of (3.1); then there exists a sequence $(\varphi_k) \subset SBV^p(\Omega) \cap L^{\infty}(\Omega)$ liftings of u , such that*

$$\varphi_k \rightarrow \varphi_{\min} \quad \text{pointwise a.e. in } \Omega, \quad (3.2)$$

$$\mathcal{H}^{n-1}(S_{\varphi_k}) \rightarrow \mathcal{H}^{n-1}(S_{\varphi_{\min}}). \quad (3.3)$$

Proof. Following [25] for any $t \in \mathbb{S}^1$ we define $L(t) = \theta$, where θ is the unique number in $[0, 2\pi)$ such that $e^{i\theta} = t$ (we use here complex notation for a point $t \in \mathbb{R}^2$, where $\mathbb{R}^2$ is identified with $\mathbb{C}$). Hence, given $u \in SBV^p(\Omega; \mathbb{S}^1)$ and $\alpha \in [0, 2\pi)$ we consider the lifting $\varphi^\alpha := L(e^{i\alpha}u) - \alpha$ of u . Thanks to [25, Theorem 1.2] we have

$$\int_0^{2\pi} |\varphi^\alpha|_{BV} d\alpha \leq 4\pi |u|_{BV}. \quad (3.4)$$

In particular there exists $\alpha \in [0, 2\pi)$ such that $|\varphi^\alpha|_{BV} \leq 2|u|_{BV}$ (this is exactly the lifting for which [25, Theorem 1.1] holds). Observe that $\varphi^\alpha \in L^\infty(\Omega)$ with

$$\|\varphi^\alpha\|_{L^\infty} \leq 4\pi, \quad (3.5)$$

$S_u \subseteq S_{\varphi^\alpha}$, and

$$\llbracket \varphi^\alpha \rrbracket = 2\pi \quad \mathcal{H}^{n-1} - \text{a.e. on } S_{\varphi^\alpha} \setminus S_u.$$

From this we deduce that

$$\mathcal{H}^{n-1}(S_{\varphi^\alpha}) = \mathcal{H}^{n-1}(S_u) + \mathcal{H}^{n-1}(S_{\varphi^\alpha} \setminus S_u) \leq \mathcal{H}^{n-1}(S_u) + \frac{1}{\pi} |u|_{BV} < +\infty.$$

Now we consider the map

$$\psi := \varphi^\alpha - \varphi_{\min},$$

that satisfies $S_\psi \subseteq S_{\varphi^\alpha} \cup S_{\varphi_{\min}}$, and so $\mathcal{H}^{n-1}(S_\psi) < +\infty$. We notice that ψ takes values in $2\pi\mathbb{Z}$, as difference of two liftings of the same map u ; hence, for all $z \in \mathbb{Z}$ we define

$$E_z := \{x \in \Omega : \psi(x) = 2\pi z\},$$

which has finite perimeter since $\partial^* E_z \subseteq S_\psi$ (up to $\mathcal{H}^{n-1}$ -negligible sets). In particular the family $\mathcal{P} := \{E_z : z \in \mathbb{Z}\}$ forms a Caccioppoli partition of Ω , i.e., a partition of Ω with finite total perimeter, namely

$$P(\mathcal{P}, \Omega) := \frac{1}{2} \sum_{z \in \mathbb{Z}} \mathcal{H}^{n-1}(\partial^* E_z) \leq \mathcal{H}^{n-1}(\partial\Omega) + \mathcal{H}^{n-1}(S_\psi) < +\infty. \quad (3.6)$$

We are now ready to define the liftings φ_k . For all $k \in \mathbb{N}$, $k \geq 1$, we set

$$\varphi_k(x) := \begin{cases} \varphi_{\min}(x) & \text{if } x \in E_z \text{ for some } |z| \leq k, \\ \varphi^\alpha(x) & \text{if } x \in E_z \text{ for some } |z| > k \end{cases} \quad (3.7)$$

We denote

$$F_k := \bigcup_{\substack{z \in \mathbb{Z} \\ |z| \leq k}} E_z,$$

and so, since $\varphi_k(x) = \varphi^\alpha(x)$ for $x \in E_z$ with $|z| > k$, we notice that $|\varphi_k(x)| \leq 4\pi$ on $\Omega \setminus F_k$ by (3.5). On the other hand, if $x \in E_z$ with $|z| \leq k$, it holds

$$|\varphi_k(x)| = |\varphi_{\min}(x)| = |\varphi^\alpha(x) - \psi(x)| \leq 4\pi + 2\pi|z| \leq 2\pi(2+k).$$

Therefore $\varphi_k \in L^\infty(\Omega)$ and $\|\varphi_k\|_{L^\infty} \leq 2\pi(2+k)$. Moreover $\varphi_k \neq \varphi_{\min}$ on $\Omega \setminus F_k$ and since $F_k \nearrow \Omega$ as $k \rightarrow +\infty$ (up to negligible sets), we infer $\varphi_k \rightarrow \varphi_{\min}$ pointwise a.e. on Ω . Finally, since $\varphi_k = \varphi_{\min}$ on F_k , we observe that, up to $\mathcal{H}^{n-1}$ -negligible sets,

$$S_{\varphi_k} \subseteq S_{\varphi_{\min}} \cup \bigcup_{\substack{z \in \mathbb{Z} \\ |z| > k}} \partial^* E_z$$

and so

$$\mathcal{H}^{n-1}(S_{\varphi_k}) \leq \mathcal{H}^{n-1}(S_{\varphi_{\min}}) + \sum_{\substack{z \in \mathbb{Z} \\ |z| > k}} \mathcal{H}^{n-1}(\partial^* E_z).$$

The series on the right-hand side is convergent by (3.6) and so its tail vanishes as $k \rightarrow \infty$. Thus the second convergence in (3.2) follows by minimality of $S_{\varphi_{\min}}$. $\square$

A consequence of this result is that the infimum in (3.1) can be computed among SBV^p maps, namely, for $p > 1$ and any $u \in SBV^p(\Omega; \mathbb{S}^1)$,

$$m_p[u] = \inf \{ \mathcal{H}^{n-1}(S_\varphi) : \varphi \in SBV^p(\Omega), e^{i\varphi} = u \text{ a.e. in } \Omega \}.$$

We are now in a position to prove Theorem 3.2.

Proof of Theorem 3.2. We divide the proof into two steps. In the first one we decompose the jump set of the liftings φ_k given by Proposition 3.3. In the second one we modify these liftings in $\bar{\varphi}_k$ to ensure that a weak star limit of the new sequence is the lifting $\bar{\varphi}_{\min}$.

Step 1: By (2.10) the distributional determinant of u has the form

$$\text{Det}(\nabla u) = \pi \sum_{i=1}^M z_i \delta_{p_i},$$

for some $M \in \mathbb{N}$, $z_i \in \mathbb{Z}$, and distinct points $p_i \in \Omega$. Let $\varphi_{\min}$ be a solution of (3.1), let us consider the liftings φ_k provided by Proposition 3.3, and let $T_k := T_{\varphi_k} \in \mathcal{D}_1(\Omega)$ be the currents in (2.13) associate with the lifting φ_k . By Proposition 2.6, for all k it holds

$$\partial T_k = \frac{1}{\pi} \text{Det}(\nabla u) = \sum_{i=1}^M z_i \delta_{p_i} \quad |\partial T_k|_\Omega = \sum_{i=1}^M |z_i|.$$

Since by Proposition 3.3 the liftings φ_k are of bounded variation, we infer that

$$2\pi |T_k|_\Omega = |D^j \varphi_k|(\Omega) < +\infty,$$

and so T_k is an integral 1-current. We can then use decomposition (2.4) to write, for all $k \geq 1$,

$$\frac{1}{2\pi} D^j \varphi_k = T_k = \sum_{i=1}^{+\infty} T_k^i, \quad \text{with} \quad \sum_{i=1}^{+\infty} |T_k^i|_\Omega = |T_k|_\Omega, \quad \sum_{i=1}^{+\infty} |\partial T_k^i|_\Omega = |\partial T_k|_\Omega, \quad (3.8)$$

and we know that for all T_k^i , if $\partial T_k^i \neq 0$, there exists a Lipschitz arc $\alpha_k^i : [0, 1] \rightarrow \bar{\Omega}$ such that either $\alpha(0) \in \{p_i : i = 1, \dots, n\}$ or $\alpha(1) \in \{p_i : i = 1, \dots, n\}$ (or even both). In particular there are only a finite number of indices i such that this happens (since $n < +\infty$), and up to relabelling we can assume that there exists $n_k \leq N := \sum_{i=1}^M |z_i| = |\partial T_k|_\Omega$ such that

$$\partial T_k^i \neq 0 \quad \text{if and only if } i \leq n_k.$$

Up to passing to a subsequence, we can assume that $n_k = \bar{n}$ for all $k \geq 1$. In the case $i > n_k = \bar{n}$, the currents T_k^i satisfy $\partial T_k^i = 0$, and so there are Lipschitz curves α_k^i which

have both endpoints on $\partial\Omega$ or have coinciding endpoints. For each of these arcs, there exist a finite perimeter set $E_k^i \subseteq \Omega$ such that $\mathcal{H}^1(\Omega \cap (\partial^* E_k^i \Delta \alpha([0, 1])) = 0$ and, by (2.6),

$$|E_k^i| \leq c_\Omega \mathcal{H}^1(\Omega \cap \partial^* E_k^i)^2 = c_\Omega |T_k^i|_\Omega^2$$

This implies, by (3.8) and according to the fact that $|T_k|_\Omega < +\infty$,

$$\sum_{i > \bar{n}} |E_k^i| < +\infty. \quad (3.9)$$

Now we notice that, from the second condition in (3.8), we have that, up to $\mathcal{H}^1$ -negligible sets,

$$S_{\varphi_k} = \bigcup_{i \geq 1} \alpha_k^i([0, 1]). \quad (3.10)$$

Observing that $\alpha_k^i([0, 1]) \subseteq S_{\varphi_k}$ by (3.10), there is a constant $C > 0$, independent of i and k , such that

$$\mathcal{H}^1(\alpha_k^i([0, 1])) \leq \mathcal{H}^1(S_{\varphi_k}) \leq C.$$

Step 2: We modify the liftings φ_k as follows: we define

$$\widehat{\varphi}_k := \varphi_k - 2\pi \sum_{i=\bar{n}+1}^{\infty} \chi_{E_k^i}.$$

By (3.8) we deduce that

$$|D^j \widehat{\varphi}_k|(\Omega) = 2\pi \sum_{i=1}^{\bar{n}} |T_k^i|_\Omega$$

and since $|\nabla \widehat{\varphi}_k| = |\nabla u|$ and $\widehat{\varphi}_k \in L^1(\Omega)$ by (3.9), we conclude that $\widehat{\varphi}_k \in SBV^p(\Omega)$. We then subtract to $\widehat{\varphi}_k$ a constant multiple of 2π so that the mean value of the new function $\overline{\varphi}_k$ belongs to $[0, 2\pi)$; hence we can apply Poincarè inequality [6, Theorem 3.44] to infer that there is a constant $\overline{C} > 0$, independent of k , such that

$$\|\overline{\varphi}_k\|_{BV} \leq \overline{C}, \quad \forall k \geq 1.$$

To control the total variation of $\overline{\varphi}_k$ we have used here that

$$|D^j \overline{\varphi}_k|(\Omega) = 2\pi \sum_{i \leq \bar{n}} |T_k^i|_\Omega = 2\pi \sum_{i \leq \bar{n}} \mathcal{H}^1(\alpha_k^i([0, 1])) \leq 2\pi \bar{n} \mathcal{H}^1(S_{\varphi_k}) < C.$$

Moreover, since $\partial^* E_k^i \subseteq \alpha_k^i([0, 1]) \subseteq S_{\varphi_k}$, we also infer $S_{\overline{\varphi}_k} \subseteq S_{\varphi_k}$, and therefore

$$\mathcal{H}^1(S_{\overline{\varphi}_k}) \leq \mathcal{H}^1(S_{\varphi_k}).$$

We now pass to the limit as $k \rightarrow \infty$, and we deduce that there is a function $\overline{\varphi} \in SBV^p(\Omega)$ such that

$$\overline{\varphi}_k \rightharpoonup \overline{\varphi} \quad \text{weakly}^* \text{ in } BV(\Omega).$$

We can now invoke Theorem 1.1 to exploit formula (1.3) which implies

$$\mathcal{H}^1(S_{\overline{\varphi}}) \leq \liminf_{k \rightarrow \infty} \mathcal{H}^1(S_{\overline{\varphi}_k}) \leq \liminf_{k \rightarrow \infty} \mathcal{H}^1(S_{\varphi_k}) = \mathcal{H}^1(S_{\varphi_{\min}}).$$

This concludes the proof, by observing that $\overline{\varphi}_{\min} := \overline{\varphi}$ is a lifting of u belonging to $SBV^p(\Omega)$. $\square$

In the preceding proof, to control the total variation of $\bar{\varphi}_k$ we have used that the number of curves α_k^i with some endpoint in $\{p_i, i = 1, \dots, n\}$ is bounded by $\bar{n}$. This makes it difficult to extend the previous argument to the general case $u \in W^{1,p}(\Omega; \mathbb{S}^1)$, without assuming boundedness of the total variation of $\text{Det}(\nabla u)$. At the present stage we do not know whether Theorem 3.2 holds without this assumption.

3.2 An example

In this section we will show that, in general, a lifting minimizing the right-hand side of (3.1) cannot be found in $SBV^p(\Omega)$, for any $p \geq 1$. To this purpose we will exhibit a map $u \in SBV^p(\Omega; \mathbb{S}^1)$, for all $p \geq 1$, and a lifting φ of it, such that the following hold:

- (a) $\varphi \in GSBV^p(\Omega) \setminus SBV(\Omega)$;
- (b) $S_\varphi = S_u$;
- (c) $\Omega \setminus S_\varphi$ is arcwise connected.

From (b) and (c) it follows that φ is the only minimizer (modulo addition of a constant in $2\pi\mathbb{Z}$) of (3.1) for u . Moreover, (a) tells us that φ is not a function of bounded variation, and so in particular it does not belong to $SBV^p(\Omega)$.

The idea behind this example is to consider in $\Omega := (0, 1)^2 \subset \mathbb{R}^2$ a function u that winds infinitely many times around $\mathbb{S}^1$ in a suitable subdomain Z of Ω (obtained by union of rectangles R_n defined below), but having bounded L^1 -norm of the approximate gradient. In this way the corresponding lifting φ becomes unbounded and has a very large jump on the boundary of Z in Ω . This makes φ of unbounded variation, whereas the function $u = e^{i\varphi}$ has bounded amplitude of the jump set; since ∂Z has finite length, the total variation of u will be controlled.

To begin with we consider $\Omega := (0, 1)^2$ and three sequences (R_n) , (T_n) , (B_n) of open rectangles contained in Ω defined, for an even integer $n \geq 2$, as follows:

$$\begin{aligned} R_n \text{ has vertices } & \left(\frac{1}{2^{n+1}}, 0\right), \left(\frac{1}{2^{n-1}}, 0\right), \left(\frac{1}{2^{n-1}}, \frac{1}{10^n} + \frac{1}{n^2}\right), \left(\frac{1}{2^{n+1}}, \frac{1}{10^n} + \frac{1}{n^2}\right); \\ T_n \text{ has vertices } & \left(\frac{1}{2^{n+1}}, \frac{1}{n^2}\right), \left(\frac{1}{2^{n-1}}, \frac{1}{n^2}\right), \left(\frac{1}{2^{n-1}}, \frac{1}{10^n} + \frac{1}{n^2}\right), \left(\frac{1}{2^{n+1}}, \frac{1}{10^n} + \frac{1}{n^2}\right); \\ B_{n+1} \text{ has vertices } & \left(\frac{1}{2^{n+2}}, 0\right), \left(\frac{1}{2^n}, 0\right), \left(\frac{1}{2^n}, \frac{1}{10^{n+1}}\right), \left(\frac{1}{2^{n+2}}, \frac{1}{10^{n+1}}\right). \end{aligned}$$

Observe that the R_n 's are pairwise disjoint and the closures $\bar{T}_n$ of T_n are pairwise disjoint. Also the B_n 's are pairwise disjoint, but their boundaries share some part of the lateral edges (see Fig. 2). Furthermore, for all $n, m \geq 2$ even,

$$T_n \subset R_n, \quad B_{n+1} \subset R_n \cup R_{n+2}, \quad \bar{T}_n \cap \bar{B}_m = \emptyset.$$

Now, we define a map $\varphi : \Omega \rightarrow \mathbb{R}$

$$\varphi := \begin{cases} n^2 & \text{in } T_n \text{ for } n \geq 2 \text{ even,} \\ (n+1)^2 & \text{in } B_{n+1} \text{ for } n \geq 2 \text{ even,} \\ 4 & \text{in } \Omega \setminus \bigcup_{\substack{n \geq 2 \\ n \text{ even}}} R_n. \end{cases} \quad (3.11)$$

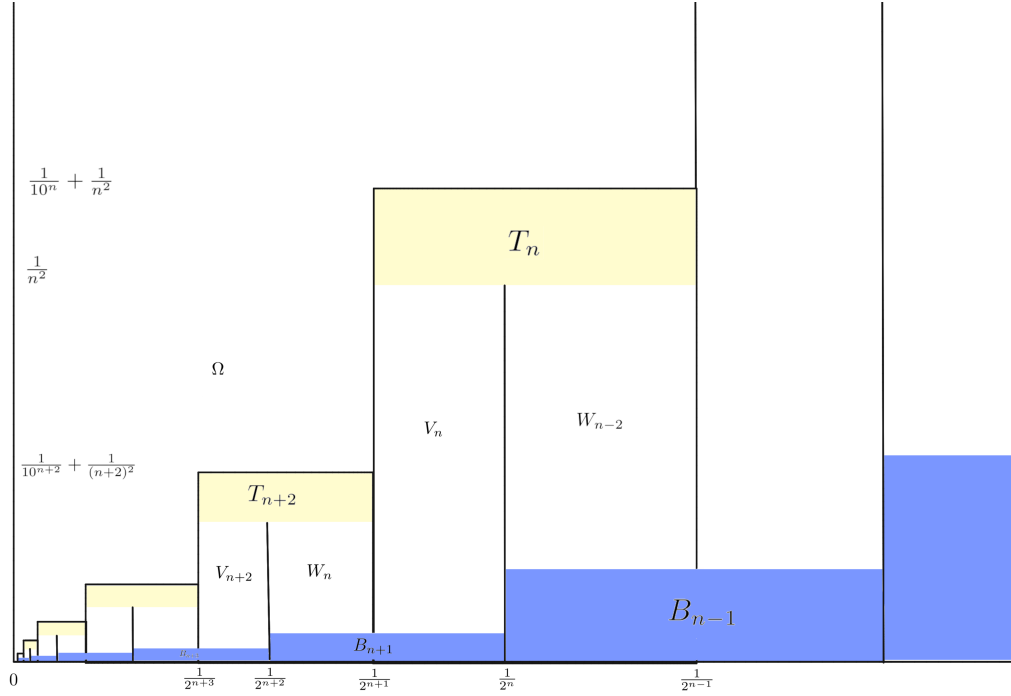


Figure 2: The rectangles T_n and B_n in a portion of $\Omega = (0, 1)^2$.

The value 4 has no particular meaning: any positive constant could be chosen as well. A similar comment applies to (3.12) below. It remains to define φ on

$$\bigcup_{\substack{n \geq 2 \\ n \text{ even}}} R_n \setminus \left(\bigcup_{\substack{n \geq 2 \\ n \text{ even}}} (T_n \cup B_{n+1}) \right).$$

To this purpose we set

$$\varphi := 4 \text{ in the open rectangle with vertices } \left(\frac{1}{4}, 0 \right), \left(\frac{1}{2}, 0 \right), \left(\frac{1}{2}, \frac{1}{4} + \frac{1}{10^2} \right), \left(\frac{1}{4}, \frac{1}{4} + \frac{1}{10^2} \right). \quad (3.12)$$

Further, we consider two sequences (V_n) , (W_n) of open rectangles, defined for any even $n \geq 2$, as follows:

$$\begin{aligned} V_n \text{ has vertices } & \left(\frac{1}{2^{n+1}}, \frac{1}{10^{n+1}} \right), \left(\frac{1}{2^n}, \frac{1}{10^{n+1}} \right), \left(\frac{1}{2^n}, \frac{1}{n^2} \right), \left(\frac{1}{2^{n+1}}, \frac{1}{n^2} \right), \\ W_n \text{ has vertices } & \left(\frac{1}{2^{n+2}}, \frac{1}{10^{n+1}} \right), \left(\frac{1}{2^{n+1}}, \frac{1}{10^{n+1}} \right), \left(\frac{1}{2^{n+1}}, \frac{1}{(n+2)^2} \right), \left(\frac{1}{2^{n+2}}, \frac{1}{(n+2)^2} \right), \end{aligned}$$

see Fig. 2.

By (3.11) the value of φ on the top edge of V_n is n^2 , whereas on the bottom edge is $(n+1)^2$, and we define φ on V_n as the affine interpolation of these two values. The value of φ on the top edge of W_n is $(n+2)^2$, whereas on the bottom edge is $(n+1)^2$, and we define φ on W_n as the affine interpolation of these two values. Observe that, looking from top to bottom, φ increases in V_n and decreases in W_n . Clearly,

$$\varphi \notin L^\infty(\Omega), \quad (3.13)$$

φ is piecewise constant in $\Omega \setminus (\bigcup_{\substack{n \geq 2 \\ n \text{ even}}} (V_n \cup W_n))$, piecewise affine in $\bigcup_{\substack{n \geq 2 \\ n \text{ even}}} V_n \cup W_n$, (3.14)

and

$$\begin{aligned} |\nabla \varphi| &= \left| \frac{\partial \varphi}{\partial x_2} \right| = \frac{(n+1)^2 - n^2}{\frac{1}{n^2} - \frac{1}{10^{n+1}}} \approx n^3 \quad \text{on } V_n, \\ |\nabla \varphi| &= \left| \frac{\partial \varphi}{\partial x_2} \right| = \frac{(n+2)^2 - (n+1)^2}{\frac{1}{(n+2)^2} - \frac{1}{10^{n+1}}} \approx (n+1)^3 \quad \text{on } W_n, \end{aligned}$$

being the height of V_n (resp. W_n) of order $\frac{1}{n^2}$ (resp. $\frac{1}{(n+1)^2}$). However, the bases of V_n and W_n have length $\frac{1}{2^{n+1}}$ and $\frac{1}{2^{n+2}}$ respectively, so

$$\|\nabla \varphi\|_{L^p(V_n)}^p \approx |V_n| n^{3p} = \frac{n^{3p-2}}{2^{n+1}} =: a_n, \quad \|\nabla \varphi\|_{L^p(W_n)}^p \approx |W_n| (n+1)^{3p} = \frac{(n+1)^{3p-2}}{2^{n+2}} =: b_n.$$

In particular

$$\nabla \varphi \in L^p(\Omega; \mathbb{R}^2) \quad \forall p \geq 1, \quad (3.15)$$

since $\sum_n a_n < +\infty$ and $\sum_n b_n < +\infty$. On the other hand, denoting Σ_T the union (for $n \geq 4$ even) of the top and left edges of the T_n 's, by Σ_B the union (for $n \geq 4$ even) of the lateral edges of the B_n 's, and by Σ_{VW} the union of the lateral edges of all V_n 's and W_n 's, we see that the jump set S_φ of φ is exactly

$$S_\varphi = \Sigma_T \cup \Sigma_B \cup \Sigma_{VW};$$

furthermore,

$$\Omega \setminus S_\varphi \text{ is arcwise connected,} \quad (3.16)$$

and it is easy to check that

$$\mathcal{H}^1(S_\varphi) < +\infty. \quad (3.17)$$

In particular, from (3.14), (3.17) and (3.15) we deduce $\varphi \in GSBV^p(\Omega)$, for any $p \geq 1$.

However,

$$\varphi \notin SBV(\Omega).$$

Indeed, although $D\varphi$ has no Cantor part, the total variation $|D^j \varphi|(\Omega)$ of the jump part $D^j \varphi$ of $D\varphi$ is infinite. This can be seen by observing that, on each lateral edge of V_n , the jump opening $[\![\varphi]\!]$ is of order n on a segment of length of order n^{-2} . Thus

$$|D^j \varphi|(\Omega) \geq \sum_{\substack{n \geq 2 \\ n \text{ even}}} n^{-1} = +\infty.$$

We claim that φ is the unique (up to addition of a constant) solution of (3.1) for

$$u := e^{i\varphi}.$$

We have $u \in BV(\Omega; \mathbb{S}^1)$ and $|\nabla u| = |\nabla \varphi|$. Now, we observe that

$$S_u = S_\varphi, \quad (3.18)$$

namely $\mathcal{H}^1(S_u \Delta S_\varphi) = 0$, and therefore

$$u \in SBV^p(\Omega; \mathbb{S}^1).$$

Equality (3.18) is due to the fact that the subset of S_φ where $\llbracket \varphi \rrbracket \in 2\pi\mathbb{Z}$ is $\mathcal{H}^1$ -negligible. This additionally implies that any lifting ψ of u must satisfy $S_\psi \supseteq S_u$, and so

$$\mathcal{H}^1(S_\psi) \geq \mathcal{H}^1(S_\varphi). \quad (3.19)$$

We conclude from (3.18) and (3.19) that φ is a minimizer of (3.1), and $\varphi \notin SBV^p(\Omega)$, for any $p \geq 1$. Finally, from (3.16) it follows that φ is unique, up to an additive constant in $2\pi\mathbb{Z}$.

3.3 A connection with optimal transport: an example

We observe in this subsection that the minimization problem (3.1) is closely related to a question arising in optimal transport (see [37, 38] for the setting and similar formulations). To illustrate this, we consider an example in a special case: for a connected and simply-connected bounded Lipschitz open set $\Omega \subset \mathbb{R}^2$, we fix $1 < p < 2$, $N \in \mathbb{N}$, and a Sobolev map $u \in W^{1,p}(\Omega; \mathbb{S}^1)$ with distributional Jacobian determinant

$$\text{Det}(\nabla u) = \pi \sum_{i=1}^N (\delta_{x_i} - \delta_{y_i}). \quad (3.20)$$

We set $\mu := \sum_{i=1}^N \delta_{x_i}$ and $\nu := \sum_{i=1}^N \delta_{y_i}$, where the points x_i and y_i belong to Ω and are not necessarily distinct, and we consider the class of all integer-multiplicity 1-currents whose boundary is $\mu - \nu$, namely

$$\mathcal{T}(\mu, \nu) := \{T \in \mathcal{D}_1(\Omega) : T = (R, \theta, \tau) \text{ is an i.m.c. such that } \partial T = \mu - \nu\}.$$

Here we use the notation $T = (R, \theta, \tau)$ introduced in (2.3).

A classical optimal transport problem can be formulated in the class $\mathcal{T}(\mu, \nu)$ in the following way: one studies the minimization problem

$$\Psi(\mu, \nu) := \inf \left\{ \mathcal{H}^1(R) : (R, \theta, \tau) \in \mathcal{T}(\mu, \nu) \right\}. \quad (3.21)$$

We claim that

$$m_p[u] = \Psi(\mu, \nu), \quad (3.22)$$

which shows the connection between our minimization problem (3.1) and optimal transport.

By Theorem 3.2, there is $\varphi \in SBV^p(\Omega)$, lifting of u , that is a minimizer of (3.1). Let T_φ be the 1-current as in (2.13); by Proposition 2.6

$$\partial T_\varphi = \mu - \nu \quad \text{in } \mathcal{D}_0(\Omega).$$

In particular T_φ is admissible for (3.21), i.e., $T_\varphi \in \mathcal{T}(\mu, \nu)$, and we obtain

$$m_p[u] \geq \inf \{ \mathcal{H}^1(R) : (R, \theta, \tau) \in \mathcal{T}(\mu, \nu) \}. \quad (3.23)$$

The opposite inequality is more delicate, and we only sketch the arguments to obtain it; let $T = (R, \theta, \tau)$ be admissible for (3.21), and let us decompose $T = \sum_{i=1}^{+\infty} T_i$ as in (2.4), so

T_i is either a loop with multiplicity ± 1 or a Lipschitz parametrized path from one point y_j to some x_k . Up to erasing the loops (operation that does not increase the energy in (3.21)) we may suppose that there are exactly N such Lipschitz paths, $T = \sum_{i=1}^N T_i$. In particular $T_i = (R_i, 1, \tau)$ is such that R_i is a closed set image of $[0, 1]$ under the Lipschitz injective map α_i , and so $R = \cup_{i=1}^N R_i$ is closed. Consider then the open set $\Omega \setminus R$ and let φ be a lifting of u in $\Omega \setminus R$ with jump set of minimal length. We claim that $\mathcal{H}^1(S_\varphi \cap (\Omega \setminus R)) = 0$; this is equivalent to say that there exists a lifting of u on $\Omega \setminus R$ with no jumps. The latter can be shown by the following observation: we connect the components of R with a curve σ in such a way $\Omega \setminus (R \cup \sigma)$ is simply-connected. Then by [16, Lemma 1.8] the lifting φ has no jumps on $\Omega \setminus (R \cup \sigma)$. To see that there is no jump of φ on σ , it is enough to observe that, given any closed loop Γ in $\Omega \setminus R$, the topological degree of u on Γ must be null (by construction of R).

From the claim, since $\varphi \in SBV^p(\Omega)$ satisfies $S_\varphi \subset R$, we easily infer

$$m_p[u] \leq \inf\{\mathcal{H}^1(R) : (R, \theta, \tau) \in \mathcal{T}(\mu, \nu)\} \quad (3.24)$$

by the arbitrariness of T .

In a similar manner, we see how the problem in (3.1) and the aforementioned transport problem can be used to solve Steiner-type problems. Again, assume for simplicity that the singularities of $u \in W^{1,p}(\Omega; \mathbb{S}^1)$ satisfy (3.20). Assume also that $x_i = \bar{x}$ for all $i = 1, \dots, N$, and that the points y_i are all distinct and different from $\bar{x}$. We look for the connected set of minimal length containing the $N + 1$ points in the family $C = \{\bar{x}, y_i : i = 1, \dots, N\}$. To do this, consider a compact connected set K with $\mathcal{H}^1(K) < +\infty$, containing C and let us suppose that $\text{dist}(C, \partial\Omega) > \mathcal{H}^1(K)$; then the Steiner problem for C can be proven to be

$$m_p[u] = \inf\{\mathcal{H}^1(L) : L \text{ is connected, } L \supset C\}, \quad (3.25)$$

and the jump set of a jump minimizing lifting for u is a minimizer of the right-hand side in the above expression.

4 Proofs of Theorems 1.1 and 1.2

In this section, we prove the compactness theorems. We point out that these results can be deduced from the general theorems valid in GSD proved in [20]. However, for our purposes it is crucial that the limit be a lifting, a property that is ensured by locally subtracting an integer multiple of 2π . Moreover, in our specialized setting, the compactness proofs can be simplified, and we therefore present them in detail.

Next Lemmas 4.1 and 4.2, independent one each other, are the building blocks of an iterative argument needed for the proof of Theorem 1.1. Some arguments in the proof of Lemma 4.2 will be also used in the proof of Theorem 1.4.

Lemma 4.1 (Localized compactness). *Let $p > 1$, let $\Omega \subset \mathbb{R}^n$ be a bounded open set with Lipschitz boundary, and $u \in SBV^p(\Omega; \mathbb{S}^1)$. Let $(\varphi_k)_{k \geq 1} \subset GSBV^p(\Omega)$ be a sequence of liftings of u , and suppose*

$$C := \sup_{k \geq 1} \mathcal{H}^{n-1}(S_{\varphi_k}) < +\infty. \quad (4.1)$$

Let $F \subset \Omega$ be a nonempty finite perimeter set in Ω . Then, for a not relabelled subsequence, there exist a sequence of integers $(d_k)_{k \geq 1} \subset \mathbb{Z}$, a finite perimeter set

$$E \subseteq F$$

in Ω , and a function $\varphi_\infty \in GSBV^p(\Omega)$, such that:

$$\varphi_\infty \text{ is a lifting of } u \text{ in } E, \quad \varphi_\infty = 0 \quad \text{in } \Omega \setminus E, \quad (4.2)$$

$$\begin{aligned} \lim_{k \rightarrow +\infty} (\varphi_k(x) - 2\pi d_k) &= \varphi_\infty(x) \quad \text{for a.e. } x \in E, \\ \lim_{k \rightarrow +\infty} |\varphi_k(x) - 2\pi d_k| &= +\infty \quad \text{for a.e. } x \in F \setminus E, \end{aligned} \quad (4.3)$$

$$|E| \geq \frac{n^n \omega_n |F|^n}{2^n (2C + \mathcal{H}^{n-1}(\partial^* F))^n} > 0, \quad (4.4)$$

If furthermore

$$|\varphi_k(x) - 2\pi d_k| \rightarrow +\infty \quad \text{for a.e. } x \in \Omega \setminus E, \quad (4.5)$$

then

$$\liminf_{k \rightarrow +\infty} \mathcal{H}^{n-1}(S_{\varphi_k} \cap A) \geq \mathcal{H}^{n-1}((S_{\varphi_\infty} \cup \partial^* E) \cap A) \quad \text{for any open set } A \subseteq \Omega, \quad (4.6)$$

and

$$\mathcal{H}^{n-1}(F \cap \partial^* E) = \mathcal{H}^{n-1}(F \cap \partial^*(F \setminus E)) \leq C. \quad (4.7)$$

Proof. Define $v_k := \varphi_k - \varphi_1 \in GSBV^p(\Omega)$ which, since φ_k and φ_1 are liftings of u , is piecewise constant (i.e., the absolutely continuous and Cantor parts of Dv_k vanish) and takes values in $2\pi\mathbb{Z}$. Therefore, it induces a Caccioppoli partition $\{V_m^k : m \in \mathbb{Z}\}$ of F , where

for any $m \in \mathbb{Z}$ $V_m^k := \{x \in F : v_k(x) = 2\pi m\}$ has finite perimeter,

$$v_k = \sum_{m \in \mathbb{Z}} 2\pi m \chi_{V_m^k}, \quad \sum_{m \in \mathbb{Z}} |V_m^k| = |F|.$$

Moreover, since $\partial^* V_m^k \subset S_{\varphi_k} \cup S_{\varphi_1} \cup \partial^* F$, the perimeter of the partition satisfies

$$\begin{aligned} \frac{1}{2} \sum_{m \in \mathbb{Z}} P(V_m^k) &\leq \mathcal{H}^{n-1}(S_{\varphi_k} \cup S_{\varphi_1}) + \mathcal{H}^{n-1}(\partial^* F) \\ &\leq \mathcal{H}^{n-1}(S_{\varphi_k}) + \mathcal{H}^{n-1}(S_{\varphi_1}) + \mathcal{H}^{n-1}(\partial^* F) \\ &\leq 2C + \mathcal{H}^{n-1}(\partial^* F), \end{aligned} \quad (4.8)$$

where C is the constant in (4.1). Now, for any $k \in \mathbb{N}$, $k \geq 1$, we select $d_k \in \mathbb{Z}$ for which

$$|V_{d_k}^k| = \max \left\{ |V_m^k| : m \in \mathbb{Z} \right\}.$$

In particular, from (4.8),

$$P(V_{d_k}^k) \leq 2(2C + \mathcal{H}^{n-1}(\partial^* F)). \quad (4.9)$$

Using also the isoperimetric inequality, we find

$$\begin{aligned} |F| &= \sum_{m \in \mathbb{Z}} |V_m^k| \leq |V_{d_k}^k|^{\frac{1}{n}} \sum_{m \in \mathbb{Z}} |V_m^k|^{1-\frac{1}{n}} \leq \frac{|V_{d_k}^k|^{\frac{1}{n}}}{n\omega_n^{1/n}} \sum_{m \in \mathbb{Z}} P(V_m^k) \\ &\leq \frac{|V_{d_k}^k|^{1/n}}{n\omega_n^{1/n}} 2(2C + \mathcal{H}^{n-1}(\partial^* F)). \end{aligned}$$

Thus passing to the limit

$$\liminf_{k \rightarrow +\infty} |V_{d_k}^k| \geq \frac{n^n \omega_n |F|^n}{2^n (2C + \mathcal{H}^{n-1}(\partial^* F))^n}. \quad (4.10)$$

Hence, from (4.9) and (4.10), there are a not relabelled subsequence and a finite perimeter set

$$G \subset F$$

such that

$$(\chi_{V_{d_k}^k}) \text{ converges to } \chi_G \text{ a.e. in } F \text{ and weakly star in } BV(\Omega; \{0, 1\}) \text{ as } k \rightarrow +\infty. \quad (4.11)$$

Consequently,

$$|G| \geq \frac{n^n \omega_n |F|^n}{2^n (2C + \mathcal{H}^{n-1}(\partial^* F))^n}. \quad (4.12)$$

Next, we define

$$\widehat{\varphi}_k := \varphi_k - 2\pi d_k.$$

Notice that $\mathcal{H}^{n-1}(S_{\widehat{\varphi}_k}) \leq \mathcal{H}^{n-1}(S_{\varphi_k}) \leq C$; moreover by construction

$$\widehat{\varphi}_k = \varphi_1 \text{ on } V_{d_k}^k \quad \forall k \in \mathbb{N},$$

and, from (4.11),

$$\widehat{\varphi}_k \rightarrow \varphi_1 \quad \text{pointwise a.e. in } G.$$

However, there could be other regions of F out of G , where the sequence $(\widehat{\varphi}_k)$ converges pointwise, that we need to identify. To this aim, let us consider, for all integers $N \geq 1$, the sequence of truncated functions $(\widehat{\varphi}_k \wedge N) \vee (-N)$, which is precompact in $SBV^p(\Omega)$. Using a diagonal argument, we select a further (not relabelled) subsequence such that $(\widehat{\varphi}_k \wedge N) \vee (-N)$ converge in $L^1(\Omega)$ and pointwise a.e. in F for all $N \in \mathbb{N}$. Define

$$F_N := \left\{ x \in F : \lim_{k \rightarrow \infty} |(\widehat{\varphi}_k(x) \wedge N) \vee (-N)| = N \right\}.$$

Then $F_N \supset F_{N+1}$, and

$$\bigcap_{N \in \mathbb{N}} F_N := \{x \in F : |\widehat{\varphi}_k(x)| \rightarrow +\infty\}.$$

As a consequence, setting

$$E := F \setminus \bigcap_{N \in \mathbb{N}} F_N, \quad (4.13)$$

we have

$$G \subseteq E \subseteq F, \quad (4.14)$$

E has finite perimeter in Ω (indeed E has finite perimeter in F , and F has finite perimeter in Ω , therefore E has finite perimeter in Ω), and setting

$$\varphi_\infty(x) = \begin{cases} \lim_k \widehat{\varphi}_k(x) & \text{in } E \\ 0 & \text{in } \Omega \setminus E \end{cases} \quad \text{for a. e. } x \in \Omega,$$

we have $\varphi_\infty \in GSBV^p(\Omega)$, $\varphi_\infty = \varphi_1$ a.e. in G , and φ_∞ is a lifting of u in E . At the same time

$$|\widehat{\varphi}_k(x)| \rightarrow +\infty \quad \text{for a.e. } x \in F \setminus E.$$

Thus, using also (4.12) and (4.14), which imply $|E| \geq |G| \geq \frac{n^n \omega_n |F|^n}{2^n (2C + \mathcal{H}^{n-1}(\partial^* F))^n}$, statements (4.2)-(4.4) are proven.

It remains to prove (4.6). Assume $|\widehat{\varphi}_k| \rightarrow +\infty$ a.e. in $\Omega \setminus E$. Then, by Theorem 2.10, for any subsequence $(\widehat{\varphi}_{k_h})$ we can extract a further subsequence $(\widehat{\varphi}_{k_{h_j}})$ such that

$$\mathcal{H}^{n-1}((S_{\varphi_\infty} \cup \partial^* E) \cap A) \leq \liminf_{j \rightarrow +\infty} \mathcal{H}^{n-1}(S_{\varphi_{k_{h_j}}} \cap A) \text{ for any open set } A \subseteq \Omega.$$

Hence, the same holds for the original sequence $(\widehat{\varphi}_k)$,

$$\mathcal{H}^{n-1}((S_{\varphi_\infty} \cup \partial^* E) \cap A) \leq \liminf_{k \rightarrow +\infty} \mathcal{H}^{n-1}(S_{\varphi_k} \cap A) \text{ for any open set } A \subseteq \Omega,$$

which shows (4.6). This additionally implies

$$P(E, F) \leq C = \sup_{k \in \mathbb{N}} \mathcal{H}^{n-1}(S_{\varphi_k}),$$

which shows (4.7). □

Lemma 4.2. *Let $p > 1$ and $(\varphi_k)_{k \geq 1} \subset GSBV^p(\Omega)$ be a sequence of functions with*

$$C := \sup_{k \in \mathbb{N}} \mathcal{H}^{n-1}(S_{\varphi_k}) < +\infty. \quad (4.15)$$

Let $N \geq 1$ be an integer, $E_1, \dots, E_N \subset \Omega$ be pairwise disjoint nonempty finite perimeter sets and $\varphi_\infty^1, \dots, \varphi_\infty^N$ be functions in $GSBV^p(\Omega)$ with the following properties: for any $i = 1, \dots, N$,

$$\begin{aligned} \varphi_\infty^{(i)} &= 0 \quad \text{a.e. in } \Omega \setminus E_i, \\ \liminf_{k \rightarrow +\infty} \mathcal{H}^{n-1}(S_{\varphi_k} \cap B) &\geq \mathcal{H}^{n-1}\left((S_{\varphi_\infty^{(i)}} \cup \partial^* E_i) \cap B\right) \text{ for any open ball } B \subset \Omega. \end{aligned} \quad (4.16)$$

Define $\Phi_N \in GSBV^p(\Omega)$ as

$$\Phi_N(x) := \begin{cases} \varphi_\infty^{(i)}(x) & \text{if } x \in E_i \text{ for some } i = 1, \dots, N, \\ 0 & \text{if } x \in \Omega \setminus (\cup_{i=1}^N E_i). \end{cases}$$

Then

$$\liminf_{k \rightarrow +\infty} \mathcal{H}^{n-1}(S_{\varphi_k}) \geq \mathcal{H}^{n-1}\left(S_{\Phi_N} \cup (\Omega \cap \partial^*(\cup_{i=1}^N E_i))\right), \quad (4.17)$$

and

$$\mathcal{H}^{n-1}\left(\Omega \cap \partial^*(\cup_{i=1}^N E_i)\right) = \mathcal{H}^{n-1}\left(\Omega \cap \partial^*(\Omega \setminus \cup_{i=1}^N E_i)\right) \leq C. \quad (4.18)$$

Proof. The equality in (4.18) follows from $\Omega \cap \partial^* E = \Omega \cap \partial^*(\Omega \setminus E)$, and the inequality is a consequence of (4.17) and (4.15). So, let us prove (4.17).

To shortcut the notation we set, for all $i = 1, \dots, N$,

$$\Sigma_i := \Omega \cap \partial^* E_i, \quad S_i := S_{\varphi_\infty^{(i)}} \setminus \partial^* E_i,$$

and

$$\Sigma := \bigcup_{i=1}^N \Sigma_i, \quad S := \bigcup_{i=1}^N S_i,$$

are $(n-1)$ -rectifiable with $\mathcal{H}^{n-1}(\Sigma) < +\infty$, $\mathcal{H}^{n-1}(S) < +\infty$. We fix $\delta \in (0, 1)$ and, for any $i = 1, \dots, N$, for $\mathcal{H}^{n-1}$ -a.e. $x \in S_i$ we choose a radius $r(x)$ such that

$$\begin{aligned} B_\rho(x) &\subseteq \Omega, \\ \mathcal{H}^{n-1}(B_\rho(x) \cap S_i) &\geq (1 - \delta)\omega_{n-1}\rho^{n-1}, \\ \mathcal{H}^{n-1}(B_\rho(x) \cap \Sigma) + \sum_{\substack{m \neq i \\ m=1}}^N \mathcal{H}^{n-1}(B_\rho(x) \cap S_m) &\leq \delta\omega_{n-1}\rho^{n-1} \end{aligned} \quad \forall \rho \in (0, r(x)). \quad (4.19)$$

We collect all such balls $B_\rho(x)$ satisfying (4.19) in a family denoted $\mathcal{B}_i$. Furthermore, possibly reducing the value of $r(x)$, for any $i = 1, \dots, N$ and for $\mathcal{H}^{n-1}$ -a.e. $x \in \Sigma_i$ we may suppose that

$$\begin{aligned} B_\rho(x) &\subseteq \Omega, \\ \mathcal{H}^{n-1}(B_\rho(x) \cap \Sigma_i) &\geq (1 - \delta)\omega_{n-1}\rho^{n-1}, \\ \mathcal{H}^{n-1}(B_\rho(x) \cap S) + \sum_{\substack{m \neq i \\ m=1}}^N \mathcal{H}^{n-1}(B_\rho(x) \cap \Sigma_m) &\leq \delta\omega_{n-1}\rho^{n-1}, \end{aligned} \quad \rho \in (0, r(x)), \quad (4.20)$$

and we collect such balls in a family $\mathcal{B}_{i+N}$. The family $\bigcup_{n=1}^{2N} \mathcal{B}_n$, forms a Vitali covering of $\Sigma \cup S$, and so by Vitali covering theorem we can choose countable many points $x_k \in \Omega$ and radii $\rho_k \in (0, r(x_k))$ such that the family

$$\mathcal{B} := \left\{ B \in \bigcup_{j=1}^{2N} \mathcal{B}_j : B = B_{\rho_k}(x_k) \text{ for some } k \in \mathbb{N} \right\}$$

consists of mutually disjoint balls and covers $\Sigma \cup S$ up to a $\mathcal{H}^{n-1}$ -negligible set, with

$$\sum_{k=1}^{+\infty} \rho_k^{n-1} < +\infty.$$

In addition

$$\mathcal{H}^{n-1}(S \cup \Sigma) = \sum_{B \in \mathcal{B}} \mathcal{H}^{n-1}(B \cap (S \cup \Sigma)) \geq (1 - \delta)\omega_{n-1} \sum_{k=1}^{+\infty} \rho_k^{n-1}, \quad (4.21)$$

the inequality following from (4.19) and (4.20). From the same formulas, for any $n = 1, \dots, N$, and $B = B_\rho(x) \in \mathcal{B}_i$, $i \neq j$, it holds

$$\mathcal{H}^{n-1}(B \cap S_j) \leq \sum_{\substack{m \neq i \\ m=1}}^N \mathcal{H}^{n-1}(B \cap S_m) \leq \delta\omega_{n-1}\rho^{n-1}.$$

This, together with (4.21), implies

$$\sum_{B \in \mathcal{B} \setminus \mathcal{B}_j} \mathcal{H}^{n-1}(B \cap S_j) \leq \delta\omega_{n-1} \sum_{k=1}^{+\infty} \rho_k^{n-1} \leq \frac{\delta}{1 - \delta} \mathcal{H}^{n-1}(S \cup \Sigma). \quad (4.22)$$

Similarly, for all $h = 1, \dots, N$,

$$\sum_{B \in \mathcal{B} \setminus \mathcal{B}_{N+h}} \mathcal{H}^{n-1}(B \cap \Sigma_h) \leq \frac{\delta}{1-\delta} \mathcal{H}^{n-1}(S \cup \Sigma). \quad (4.23)$$

From (4.22) and (4.23) it follows

$$\begin{aligned} \sum_{B \in \mathcal{B} \cap \mathcal{B}_j} \mathcal{H}^{n-1}(B \cap S_j) &= \sum_{B \in \mathcal{B}} \mathcal{H}^{n-1}(B \cap S_j) - \sum_{B \in \mathcal{B} \setminus \mathcal{B}_j} \mathcal{H}^{n-1}(B \cap S_j) \\ &\geq \mathcal{H}^{n-1}(S_j) - \frac{\delta \mathcal{H}^{n-1}(S \cup \Sigma)}{1-\delta}, \end{aligned} \quad (4.24)$$

for all $j = 1, \dots, N$, and analogously for all $h = 1, \dots, N$,

$$\sum_{B \in \mathcal{B} \cap \mathcal{B}_{N+h}} \mathcal{H}^{n-1}(B \cap \Sigma_h) \geq \mathcal{H}^{n-1}(\Sigma_h) - \frac{\delta \mathcal{H}^{n-1}(S \cup \Sigma)}{1-\delta}. \quad (4.25)$$

From (4.16), for any $j = 1, \dots, N$ we obtain

$$\liminf_{k \rightarrow +\infty} \mathcal{H}^{n-1}(S_{\varphi_k} \cap B) \geq \mathcal{H}^{n-1}(S_{\varphi_\infty^j} \cap B) \geq \mathcal{H}^{n-1}(S_j \cap B), \quad (4.26)$$

and for all $h = 1, \dots, N$

$$\liminf_{k \rightarrow +\infty} \mathcal{H}^{n-1}(S_{\varphi_k} \cap B) \geq \mathcal{H}^{n-1}(B \cap \partial^* E_h) \geq \mathcal{H}^{n-1}(\Sigma_h \cap B). \quad (4.27)$$

Now, summing (4.26) over all $B \in \mathcal{B} \cap \mathcal{B}_j$ and (4.27) over all $B \in \mathcal{B} \cap \mathcal{B}_{N+h}$, and then over n, h respectively, we infer

$$\begin{aligned} &\sum_{j=1}^N \sum_{B \in \mathcal{B} \cap \mathcal{B}_j} \mathcal{H}^{n-1}(S_j \cap B) + \sum_{h=1}^N \sum_{B \in \mathcal{B} \cap \mathcal{B}_{N+h}} \mathcal{H}^{n-1}(\Sigma_h \cap B) \\ &\leq \liminf_{k \rightarrow +\infty} \sum_{j=1}^{2N} \sum_{B \in \mathcal{B} \cap \mathcal{B}_j} \mathcal{H}^{n-1}(S_{\varphi_k} \cap B) \leq \liminf_{k \rightarrow +\infty} \mathcal{H}^{n-1}(S_{\varphi_k}), \end{aligned} \quad (4.28)$$

where the first inequality is a consequence of Fatou's Lemma. Combining (4.28) with (4.24) and (4.25) we get

$$\sum_{j=1}^N \mathcal{H}^{n-1}(S_j) + \sum_{h=1}^N \mathcal{H}^{n-1}(\Sigma_h) - \frac{2\delta N \mathcal{H}^{n-1}(S \cup \Sigma)}{1-\delta} \leq \liminf_{k \rightarrow +\infty} \mathcal{H}^{n-1}(S_{\varphi_k}).$$

By the arbitrariness of $\delta > 0$ we conclude

$$\mathcal{H}^{n-1}(S \cup \Sigma) \leq \liminf_{k \rightarrow +\infty} \mathcal{H}^{n-1}(S_{\varphi_k}),$$

that implies (4.17). □

4.1 Proof of Theorem 1.1

Let $M := \sup_{k \in \mathbb{N}} \mathcal{H}^{n-1}(S_{\varphi_k}) < +\infty$. To show compactness of (φ_k) , the idea is to first apply Lemma 4.1 and find a subset E_1 of Ω where pointwise convergence of $\varphi_k - 2\pi d_k^{(1)}$ takes place. Then we utilize an iterative argument, constructing a second set E_2 in $\Omega \setminus E_1$, then a third one $E_3 \subset \Omega \setminus (E_1 \cup E_2)$, and so on, in such a way that, for suitable constants $d_k^{(i)}$, the sequence $\varphi_k - 2\pi d_k^{(i)}$ converges pointwise in E_i .

Base case $N = 1$. We set

$$F_1 := \Omega. \quad (4.29)$$

From Lemma 4.1 we find, for a not relabelled subsequence, a finite perimeter set $E_1 \subseteq F_1$, a sequence $(d_k^{(1)})_{k \geq 1} \subset \mathbb{Z}$ and a function $\varphi_\infty^{(1)} \in GSBV^p(\Omega)$, such that

$$\begin{aligned} \varphi_\infty^{(1)} \text{ is a lifting of } u \text{ in } E_1, \quad \varphi_\infty^{(1)} &= 0 \quad \text{in } F_1 \setminus E_1, \\ |E_1| &\geq \frac{n^n \omega_n |F_1|^n}{2^n (2M + \mathcal{H}^{n-1}(\partial\Omega))^n}, \end{aligned}$$

and

$$\begin{aligned} (\varphi_k(x) - 2\pi d_k^{(1)}) &\rightarrow \varphi_\infty^{(1)}(x) \quad \text{for a.e. } x \in E_1, \\ |\varphi_k(x) - 2\pi d_k^{(1)}| &\rightarrow +\infty \quad \text{for a.e. } x \in F_1 \setminus E_1. \end{aligned} \quad (4.30)$$

Moreover, since $F_1 = \Omega$, from the final part of the statement of Lemma 4.1, we have

$$\liminf_{k \rightarrow +\infty} \mathcal{H}^{n-1}(S_{\varphi_k} \cap A) \geq \mathcal{H}^{n-1}((S_{\varphi_\infty^{(1)}} \cup \partial^* E_1) \cap A) \text{ for any open set } A \subseteq F_1, \quad (4.31)$$

see (4.6).

Iterative case $N \rightsquigarrow N+1$. Let $N \geq 2$ be an integer, $E_1, \dots, E_N \subset \Omega$ be pairwise disjoint nonempty finite perimeter sets, and define, together with (4.29),

$$F_i := \Omega \setminus \bigcup_{j=1}^{i-1} E_j \quad \text{for } i = 2, \dots, N,$$

so that

$$E_1 \subseteq F_1, \quad E_2 \subseteq F_2, \dots, \quad E_N \subseteq F_N.$$

Suppose that:

- (i) For all $i = 1, \dots, N$, there exists a function $\varphi_\infty^{(i)} \in GSBV^p(\Omega)$ which is a lifting of u in E_i , and $\varphi_\infty^{(i)} = 0$ in $\Omega \setminus E_i$;
- (ii) For all $i = 1, \dots, N$ we have

$$|E_i| \geq \frac{n^n \omega_n |F_i|^n}{2^n (2M + \mathcal{H}^{n-1}(\partial^* F_i))^n} \geq \frac{n^n \omega_n |F_i|^n}{2^n (3M + \mathcal{H}^{n-1}(\partial\Omega))^n};$$

- (iii) For all $i = 1, \dots, N$, there exist sequences $(d_k^{(i)})_k \subset \mathbb{Z}$, such that

$$\begin{aligned} (\varphi_k(x) - 2\pi d_k^{(i)}) &\rightarrow \varphi_\infty^{(i)}(x) \quad \text{for a.e. } x \in E_i, \\ |\varphi_k(x) - 2\pi d_k^{(i)}| &\rightarrow +\infty \quad \text{for a.e. } x \in \Omega \setminus E_i, \\ \liminf_{k \rightarrow +\infty} \mathcal{H}^{n-1}(S_{\varphi_k} \cap A) &\geq \mathcal{H}^{n-1}((S_{\varphi_\infty^{(i)}} \cup \partial^* E_i) \cap A) \text{ for any open set } A \subseteq \Omega. \end{aligned} \quad (4.32)$$

We now want to find a finite perimeter set $E_{N+1} \subseteq \Omega$ disjoint from $\cup_{i=1}^N E_i$ and a function $\varphi_\infty^{(N+1)} \in GSBV^p(\Omega)$, such that, for a not relabelled subsequence, $E_1, \dots, E_{N+1}$, (φ_k) , and $\varphi_\infty^{(N+1)}$ satisfy properties (i)-(iii) above, with N replaced by $N+1$.

To this purpose we set, for $N \geq 1$,

$$F_{N+1} := \Omega \setminus \left(\bigcup_{i=1}^N E_i \right),$$

and we let $\Phi_N \in GSBV^p(\Omega)$ be defined as

$$\Phi_N(x) := \begin{cases} \varphi_\infty^{(i)}(x) & \text{if } x \in E_i \text{ for } i = 1, \dots, N, \\ 0 & \text{if } x \in \Omega \setminus (\cup_{i=1}^N E_i). \end{cases} \quad (4.33)$$

If $F_{N+1} = \emptyset$ there is nothing to prove. Assume then $F_{N+1} \neq \emptyset$. From Lemma 4.2 we have

$$\liminf_{k \rightarrow +\infty} \mathcal{H}^{n-1}(S_{\varphi_k}) \geq \mathcal{H}^{n-1}(S_{\Phi_N} \cup (\Omega \cap \partial^*(\cup_{i=1}^N E_i))), \quad (4.34)$$

and

$$\mathcal{H}^{n-1}(\Omega \cap \partial^*(\cup_{i=1}^N E_i)) = \mathcal{H}^{n-1}(\Omega \cap \partial^* F_{N+1}) \leq M. \quad (4.35)$$

Next, applying Lemma 4.1 to $F = F_{N+1}$, there exist a sequence $(d_k^{(N+1)})_{k \geq 1} \subset \mathbb{Z}$, a finite perimeter set $E_{N+1} \subseteq F_{N+1}$ (and thus $E_{N+1} \cap (\cup_{i=1}^N E_i) = \emptyset$) and a function $\varphi_\infty^{(N+1)} \in GSBV^p(\Omega)$ such that

$$\begin{aligned} \varphi_\infty^{(N+1)} &= 0 \quad \text{in } \Omega \setminus E_{N+1}, \quad \varphi_\infty^{(N+1)} \text{ is a lifting of } u \text{ in } E_{N+1}, \\ |E_{N+1}| &\geq \frac{n^n \omega_n |F_{N+1}|^n}{2^n (2M + \mathcal{H}^{n-1}(\partial^* F_{N+1}))^n}, \end{aligned} \quad (4.36)$$

and

$$\begin{aligned} (\varphi_k(x) - 2\pi d_k^{(N+1)}) &\rightarrow \varphi_\infty^{(N+1)}(x) \quad \text{for a.e. } x \in E_{N+1}, \\ |\varphi_k(x) - 2\pi d_k^{(N+1)}| &\rightarrow +\infty \quad \text{for a.e. } x \in F_{N+1} \setminus E_{N+1}, \\ \liminf_{k \rightarrow +\infty} \mathcal{H}^{n-1}(S_{\varphi_k} \cap A) &\geq \mathcal{H}^{n-1}((S_{\varphi_\infty^{(N+1)}} \cup \partial^* E_{N+1}) \cap A) \text{ for any open set } A \subseteq F_{N+1}. \end{aligned} \quad (4.37)$$

Combining (4.35) and (4.36) we readily get

$$|E_{N+1}| \geq \frac{n^n \omega_n |F_{N+1}|^n}{2^n (2M + \mathcal{H}^{n-1}(\partial^* F_{N+1}))^n} \geq \frac{n^n \omega_n |F_{N+1}|^n}{2^n (3M + \mathcal{H}^{n-1}(\partial\Omega))^n} > 0.$$

Moreover by (4.32), also

$$|\varphi_k(x) - 2\pi d_k^{(N+1)}| \rightarrow +\infty \quad \text{for a.e. } x \in \Omega \setminus E_{N+1}. \quad (4.38)$$

Thus the sets $E_1, \dots, E_{N+1}$ satisfy (i)-(iii) with $(d_k^{(i)})_{k \geq 1} \subset \mathbb{Z}$ and $\varphi_\infty^{(i)} \in GSBV^p(\Omega)$ for $i \in \{1, \dots, N+1\}$.

Conclusion. We now combine the base case and the iterative case to conclude the proof. First we note that each time we apply the iterative case we have to extract a subsequence.

Hence, taking a diagonal (not relabelled) subsequence of $(\varphi_k)_{k \geq 1}$, this yields a sequence $(E_i)_{i \geq 1}$ of mutually disjoint finite perimeter sets in Ω such that for every $m \geq 1$, $E_1, \dots, E_m$ satisfy properties (i)-(iii) (with m in place of N) with the corresponding $(d_k^{(i)})_{k \geq 1} \subset \mathbb{Z}$ and $\varphi_\infty^{(i)} \in GSBV^p(\Omega)$ for $i \in \{1, \dots, m\}$. In particular, from (4.34),

$$\liminf_{k \rightarrow +\infty} \mathcal{H}^{n-1}(S_{\varphi_k}) \geq \mathcal{H}^{n-1}(S_{\Phi_m} \cup (\Omega \cap \partial^*(\cup_{i=1}^m E_i))), \quad \forall m \geq 1 \quad (4.39)$$

with Φ_m as in (4.33) for $N = m$. We next show that

$$|\Omega \setminus (\cup_{i=1}^{+\infty} E_i)| = 0.$$

To this aim, since $\sum_{m=1}^{+\infty} |E_m| < +\infty$, the sequence $(|E_m|)$ tends to zero as $m \rightarrow +\infty$ (note that it might be $E_i = \emptyset$ for $i \geq \bar{i}$, and this case is simpler to treat), and by the inequality $|E_m| \geq \frac{n^n \omega_n |F_m|^2}{2^n (3M + \mathcal{H}^{n-1}(\partial\Omega))^2}$ we also infer that $|F_m| \rightarrow 0$. In particular,

$$|\Omega \setminus (\cup_{i=1}^{+\infty} E_i)| = \lim_{m \rightarrow +\infty} |\Omega \setminus (\cup_{i=1}^{m-1} E_i)| = \lim_{m \rightarrow +\infty} |F_m| = 0.$$

Finally we define

$$\varphi_\infty(x) := \varphi_\infty^{(i)}(x) \quad \text{if } x \in E_i \text{ for some } i \in \mathbb{N}.$$

We now show that $\varphi_\infty \in GSBV^p(\Omega)$ and that (1.3) holds true. By definition of φ_∞ and Φ_m it follows that

$$\mathcal{H}^{n-1}(S_{\varphi_\infty}) \leq \lim_{m \rightarrow +\infty} \mathcal{H}^{n-1}(S_{\Phi_m}) \leq \lim_{m \rightarrow +\infty} \mathcal{H}^{n-1}(S_{\Phi_m} \cup (\Omega \cap \partial^*(\cup_{i=1}^m E_i))).$$

This together with (4.39) yields (1.3). Eventually, being φ_∞ limit of liftings of u , it is itself a lifting of u in Ω and therefore from the identity $|\nabla u| = |\nabla \varphi_\infty|$ we infer $\varphi_\infty \in GSBV^p(\Omega)$. $\square$

4.2 Proof of Theorem 1.2

Let $\varphi \in SBV^p(\Omega) \cap L^\infty(\Omega)$ be a lifting of u with

$$|\varphi|_{BV} \leq 2|u|_{BV} \leq C, \quad (4.40)$$

whose existence is ensured by Theorem 2.3 and Remark 2.4. The idea of the proof is to construct a sequence $(\tilde{\psi}_k)_{k \geq 1} \subset SBV^p(\Omega) \cap L^\infty(\Omega)$ (see (4.49)) having the following properties:

$$\begin{aligned} \tilde{\psi}_k &\rightarrow 0 \quad \text{in } L^1(\Omega) \text{ and weakly}^* \text{ in } BV(\Omega), \\ \tilde{\varphi}_k &:= \varphi_k - \tilde{\psi}_k = \varphi + 2\pi \sum_{z \in \mathbb{Z}} z \chi_{F_k^z}, \\ \sup_{k \in \mathbb{N}} \mathcal{H}^{n-1}(\tilde{\varphi}_k) &< +\infty, \end{aligned} \quad (4.41)$$

with $(F_k^z)_{z \in \mathbb{Z}}$ a Caccioppoli partition of Ω for all $k \geq 1$, see also (4.51). Note that each $\tilde{\varphi}_k$ is a lifting of the limit map u . Hence we can apply Theorem 1.1 to the sequence $(\tilde{\varphi}_k)$, and eventually from the convergence of (a not relabelled subsequence of) $(\tilde{\varphi}_k)_{k \geq 1}$ and Theorem 2.10 deduce compactness and lower semicontinuity for the original sequence $(\varphi_k)_{k \geq 1}$. We now give the details of the proof.

Step 1: construction of $\widetilde{\psi}_k$. For $k \geq 1$ define first

$$\psi_k := \varphi_k - \varphi \in GSBV^p(\Omega).$$

Since φ is a lifting of u , we have $\llbracket \varphi \rrbracket \in 2\pi\mathbb{Z}$ on $S_\varphi \setminus S_u$, thus, using also (4.40), $\mathcal{H}^{n-1}(S_\varphi) = \mathcal{H}^{n-1}(S_u) + \mathcal{H}^{n-1}(S_\varphi \setminus S_u) < +\infty$, and $|\nabla u| = |\nabla \varphi|$ a.e. on Ω . Moreover, since $S_{\psi_k} \subseteq S_{\varphi_k} \cup S_\varphi$ and $|\nabla \psi_k| \leq |\nabla \varphi_k| + |\nabla \varphi| = |\nabla u_k| + |\nabla u|$ a.e. on Ω , we have, using also (1.4),

$$\sup_{k \geq 1} \left(\int_{\Omega} |\nabla \psi_k| \, dx + \mathcal{H}^{n-1}(S_{\psi_k}) \right) \leq C < +\infty. \quad (4.42)$$

For every $k \in \mathbb{N}$ and $z \in \mathbb{Z}$ we define

$$\psi_k^z := (\psi_k \wedge 2\pi z) \vee 2\pi(z-1) \in SBV^p(\Omega),$$

whose jump set decomposes as

$$S_{\psi_k^z} = S_{\psi_k^z}^1 \cup S_{\psi_k^z}^2 \cup S_{\psi_k^z}^3 \cup S_{\psi_k^z}^4,$$

with

$$\begin{aligned} S_{\psi_k^z}^1 &:= \{x \in S_{\psi_k^z} : 2\pi(z-1) < (\psi_k^z)^-(x) < (\psi_k^z)^+(x) < 2\pi z\}, \\ S_{\psi_k^z}^2 &:= \{x \in S_{\psi_k^z} : 2\pi(z-1) = (\psi_k^z)^-(x) < (\psi_k^z)^+(x) < 2\pi z\}, \\ S_{\psi_k^z}^3 &:= \{x \in S_{\psi_k^z} : 2\pi(z-1) < (\psi_k^z)^-(x) < (\psi_k^z)^+(x) = 2\pi z\}, \\ S_{\psi_k^z}^4 &:= \{x \in S_{\psi_k^z} : 2\pi(z-1) = (\psi_k^z)^-(x) < (\psi_k^z)^+(x) = 2\pi z\}, \end{aligned}$$

where $(\psi_k^z)^\pm$ are the two traces of ψ_k^z on $S_{\psi_k^z}$. As $\psi_k \in GSBV^p(\Omega)$ we have, up to a $\mathcal{H}^{n-1}$ -negligible set,

$$\bigcup_{z \in \mathbb{Z}} S_{\psi_k^z}^1 \cup \bigcup_{z \in \mathbb{Z}} S_{\psi_k^z}^2 \cup \bigcup_{z \in \mathbb{Z}} S_{\psi_k^z}^3 \subseteq S_{\psi_k}. \quad (4.43)$$

Notice that, for $\mathcal{H}^{n-1}$ -a.e. $x \in S_{\psi_k}$, x belongs to at most 2 sets appearing in the union on the left-hand side of (4.43); hence in particular from (4.42), for all $k \geq 1$

$$\sum_{z \in \mathbb{Z}} \left(\mathcal{H}^{n-1}(S_{\psi_k^z}^1) + \mathcal{H}^{n-1}(S_{\psi_k^z}^2) + \mathcal{H}^{n-1}(S_{\psi_k^z}^3) \right) \leq 2\mathcal{H}^{n-1}(S_{\psi_k}) \leq C, \quad (4.44)$$

for some constant $C > 0$. Furthermore by definition of ψ_k^z ,

$$\int_{\Omega} |\nabla \psi_k| \, dx = \sum_{z \in \mathbb{Z}} \int_{\Omega} |\nabla \psi_k^z| \, dx \leq C \quad \forall k \in \mathbb{N}. \quad (4.45)$$

Consider now the function $\tau_k^z \in SBV^p(\Omega)$, defined by

$$\tau_k^z := \left| \psi_k^z - 2\pi \left(z - \frac{1}{2} \right) \right|,$$

which satisfies $0 \leq \tau_k^z \leq \pi$,

$$\int_{\Omega} |\nabla \psi_k^z| \, dx = \int_{\Omega} |\nabla \tau_k^z| \, dx, \quad |\llbracket \tau_k^z \rrbracket| \leq |\llbracket \psi_k^z \rrbracket| \quad \mathcal{H}^{n-1} - \text{a.e. on } S_{\tau_k^z},$$

$$S_{\tau_k^z} \subseteq S_{\psi_k^z}^1 \cup S_{\psi_k^z}^2 \cup S_{\psi_k^z}^3 ,$$

owing to the fact that τ_k^z has null jump on $S_{\psi_k^z}^4$. In particular, we infer from (4.44) and (4.45) that

$$\sum_{z \in \mathbb{Z}} |D\tau_k^z|(\Omega) \leq C \quad \forall k \in \mathbb{N}, \quad (4.46)$$

for some positive constant C .

Now, for $t \geq 0$, let $(E_k^z)^t := \{x \in \Omega : \tau_k^z(x) < t\}$; by the coarea formula,

$$\int_0^\pi \mathcal{H}^{n-1}(\partial^*(E_k^z)^t) dt = |D\tau_k^z|(\Omega),$$

hence, summing over z and using (4.46),

$$\int_0^\pi \sum_{z \in \mathbb{Z}} (\mathcal{H}^{n-1}(\partial^*(E_k^z)^t)) dt = \sum_{z \in \mathbb{Z}} \int_0^\pi \mathcal{H}^{n-1}(\partial^*(E_k^z)^t) dt \leq C.$$

Whence, for any $k \in \mathbb{Z}$ we can find a number $t_k \in (0, \pi/2)$ such that

$$\sum_{z \in \mathbb{Z}} \mathcal{H}^{n-1}(\partial^*(E_k^z)^{t_k}) \leq C, \quad (4.47)$$

with $C > 0$ independent of k . We observe that

$$(E_k^z)^{t_k} = \left\{ x \in \Omega : 2\pi\left(z - \frac{1}{2}\right) - t_k < \psi_k^z(x) = \psi_k(x) < 2\pi\left(z + \frac{1}{2}\right) + t_k \right\}.$$

For every k let $(F_k^z)_{z \in \mathbb{Z}}$ be the Caccioppoli partition of Ω defined as

$$F_k^z := \left\{ x \in \Omega : \psi_k(x) \in \left(2\pi\left(z - \frac{1}{2}\right) - t_k, 2\pi\left(z + \frac{1}{2}\right) - t_k \right) \right\}.$$

Thus $\partial^* F_k^z \subset \partial^*(E_k^z)^{t_k} \cup \partial^*(E_k^{z+1})^{t_k}$ and

$$\sum_{z \in \mathbb{Z}} \mathcal{H}^{n-1}(\partial^* F_k^z) \leq 2 \sum_{z \in \mathbb{Z}} \mathcal{H}^{n-1}(\partial^*(E_k^z)^{t_k}) \leq C, \quad (4.48)$$

and so for each $k \in \mathbb{N}$ the family $(F_k^z)_{z \in \mathbb{Z}}$ is a Caccioppoli partition of Ω , with equibounded (in k) total perimeter.

We finally introduce the map $\tilde{\psi}_k \in SBV^p(\Omega) \cap L^\infty(\Omega)$ as

$$\tilde{\psi}_k := \sum_{z \in \mathbb{Z}} \left(\psi_k - \left(2\pi\left(z - \frac{1}{2}\right) - t_k \right) \right) \chi_{F_k^z} - \pi - t_k = \psi_k - 2\pi \sum_{z \in \mathbb{Z}} z \chi_{F_k^z}, \quad (4.49)$$

which satisfies $-\pi \leq \tilde{\psi}_k \leq \pi$. Moreover, since $S_{\tilde{\psi}_k} \subseteq S_{\psi_k} \cup \bigcup_{z \in \mathbb{Z}} \partial^* F_k^z$, by (4.48) we deduce that

$$\mathcal{H}^{n-1}(S_{\tilde{\psi}_k}) + \|\tilde{\psi}_k\|_{BV} \leq C, \quad (4.50)$$

for all $k \geq 1$. Therefore, up to a not relabelled subsequence, we can also suppose

$$\tilde{\psi}_k \rightarrow \tilde{\psi} \quad \text{in } L^1(\Omega) \text{ and weakly* in } BV(\Omega)$$

for some $\tilde{\psi} \in SBV^p(\Omega) \cap L^\infty(\Omega)$. As asserted in (4.41), we now want to show that $\tilde{\psi} = 0$ a.e. on Ω . Let us go back to the functions u_k that, up to subsequences, are converging pointwise a.e. to u . Hence for a.e. x we have

$$u_k(x) - u(x) = e^{i\varphi(x)}(e^{i(\varphi_k(x) - \varphi(x))} - 1) \rightarrow 0.$$

This in turn implies that the map $x \mapsto \text{dist}(\varphi_k(x) - \varphi(x), 2\pi\mathbb{Z}) = \text{dist}(\psi_k(x), 2\pi\mathbb{Z})$ is converging pointwise a.e. to 0, and by the dominated convergence theorem, in $L^1(\Omega)$. Then there is $\bar{k} \geq 0$ such that for a.e. $x \in \Omega$ and for all $k \geq \bar{k}$ there is $K = K(x, k) \in \mathbb{Z}$ such that

$$x \in F_k^K \quad \text{and} \quad |\tilde{\psi}_k(x)| = |\psi_k(x) - 2\pi K| \rightarrow 0 \quad \text{as } k \rightarrow +\infty,$$

and so by the dominated convergence theorem we conclude $\tilde{\psi} = 0$.

Step 2: compactness and lower semicontinuity. We first observe that $\psi_k - \tilde{\psi}_k \in GSBV^p(\Omega)$ takes values in $2\pi\mathbb{Z}$. As a consequence, the maps $\tilde{\varphi}_k = \varphi_k - \tilde{\psi}_k = \psi_k - \tilde{\psi}_k + \varphi$ are all liftings of u and from (4.50) we also have

$$\sup_{k \geq 1} \mathcal{H}^{n-1}(S_{\tilde{\varphi}_k}) < +\infty. \quad (4.51)$$

We now apply Theorem 1.1 to the sequence $(\tilde{\varphi}_k)_{k \geq 1}$, and deduce the existence of a Caccioppoli partition $(E_i)_{i \geq 1}$ of Ω , of sequences $(d_k^{(i)})_{k \geq 1} \subset 2\pi\mathbb{Z}$ for all $i \geq 1$, and of a lifting $\varphi_\infty \in GSBV^p(\Omega)$ of u , such that

$$\begin{aligned} \lim_{k \rightarrow +\infty} (\tilde{\varphi}_k(x) - 2\pi d_k^{(i)}) &= \varphi_\infty(x) \quad \text{for a.e. } x \in E_i, \\ \lim_{k \rightarrow +\infty} |\tilde{\varphi}_k(x) - 2\pi d_k^{(i)}| &= +\infty \quad \text{for a.e. } x \in \Omega \setminus E_i, \end{aligned}$$

for all $i \geq 1$. As a consequence, by Step 2,

$$\begin{aligned} \lim_{k \rightarrow +\infty} (\varphi_k(x) - 2\pi d_k^{(i)}) &= \varphi_\infty(x) \quad \text{for a.e. } x \in E_i, \\ \lim_{k \rightarrow +\infty} |\varphi_k(x) - 2\pi d_k^{(i)}| &= +\infty \quad \text{for a.e. } x \in \Omega \setminus E_i, \end{aligned}$$

for all $i \geq 1$.

It remains to show (1.6). For all $i \geq 1$ let $\varphi_\infty^{(i)} := \varphi_\infty$ in E_i and $\varphi_\infty^{(i)} := 0$ in $\Omega \setminus E_i$. By Theorem 2.10 and the fact that $S_{\varphi_k} = S_{\varphi_k - 2\pi d_k^{(i)}}$ it follows

$$\liminf_{k \rightarrow +\infty} \mathcal{H}^{n-1}(S_{\varphi_k} \cap A) \geq \mathcal{H}^{n-1}((S_{\varphi_\infty^{(i)}} \cup \partial^* E_i) \cap A),$$

for any open set $A \subset \Omega$. Then by Lemma 4.2 for all $N \geq 1$ we get

$$\liminf_{k \rightarrow +\infty} \mathcal{H}^{n-1}(S_{\varphi_k}) \geq \mathcal{H}^{n-1}(S_{\bar{\varphi}_\infty^{(N)}} \cup (\partial^*(\cup_{i=1}^N E_i))),$$

where

$$\bar{\varphi}_\infty^{(N)} := \begin{cases} \varphi_\infty^{(i)} & \text{if } x \in E_i \text{ for some } i = 1, \dots, N, \\ 0 & \text{if } x \in \Omega \setminus (\cup_{i=1}^N E_i). \end{cases}$$

Hence, observing that $\varphi_\infty = \bar{\varphi}_\infty^{(N)}$ in $\cup_{i=1}^N E_i$, and letting $N \rightarrow +\infty$ we get

$$\liminf_{k \rightarrow +\infty} \mathcal{H}^{n-1}(S_{\varphi_k}) \geq \mathcal{H}^{n-1}(S_{\varphi_\infty} \cup (\partial^*(\cup_{i=1}^{+\infty} E_i))) \geq \mathcal{H}^{n-1}(S_{\varphi_\infty}).$$

□

5 Γ -convergence of functionals on $\mathbb{S}^1$ -valued maps

In this section we prove Theorems 1.3 and 1.4. The proof of Theorem 1.3 follows by suitably adapting the arguments of [7, 8, 30]. Instead, the proof of Theorem 1.4 (and more specifically the lower bound inequality) requires some new ideas which rely on the compactness result for liftings (Theorem 1.2). For convenience we introduce the localised Modica-Mortola-type (or Allen-Cahn type) functionals

$$\text{MM}_\varepsilon(v, A) := \int_A \left(\varepsilon |\nabla v|^2 + \frac{(v-1)^2}{4\varepsilon} \right) dx \quad \forall v \in W^{1,2}(\Omega),$$

for every open set $A \subseteq \Omega$.

5.1 Some density and approximation results for $\mathbb{S}^1$ -valued maps

For $E \subset \mathbb{R}^n$ we denote by $\mathcal{M}^{n-1}(E)$ its $(n-1)$ -dimensional Minkowski content, i.e.,

$$\mathcal{M}^{n-1}(E) = \lim_{\rho \searrow 0} \frac{|\{x \in \mathbb{R}^n : \text{dist}(x, E) < \rho\}|}{2\rho},$$

provided the limit exists.

Proposition 5.1 (Density in $SBV^2(\Omega; \mathbb{S}^1)$). *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with Lipschitz boundary, $u \in SBV^2(\Omega; \mathbb{S}^1)$ and let $\hat{\Omega} \subset \mathbb{R}^n$ be a bounded open set with $\Omega \subset\subset \hat{\Omega}$. Then u has an extension $\hat{u} \in SBV^2(\hat{\Omega}; \mathbb{S}^1)$ with*

$$\mathcal{H}^{n-1}(S_{\hat{u}} \cap \partial\Omega) = 0.$$

Moreover, there exists a sequence $(\hat{u}_k) \subset SBV^2(\hat{\Omega}; \mathbb{S}^1)$ such that:

- (i) $\hat{u}_k$ converges to $\hat{u}$ in $L^1(\hat{\Omega}; \mathbb{S}^1)$;
- (ii) $\nabla \hat{u}_k$ converges to $\nabla \hat{u}$ in $L^2(\hat{\Omega}; \mathbb{R}^{n \times n})$;
- (iii) $\lim_{k \rightarrow \infty} \mathcal{H}^{n-1}(S_{\hat{u}_k}) = \mathcal{H}^{n-1}(S_{\hat{u}})$;
- (iv) $\lim_{k \rightarrow \infty} \mathcal{H}^{n-1}(S_{\hat{u}_k} \cap \overline{\Omega}) = \mathcal{H}^{n-1}(S_{\hat{u}} \cap \Omega) = \mathcal{H}^{n-1}(S_u)$;
- (v) $\mathcal{H}^{n-1}(\overline{S_{\hat{u}_k}} \setminus S_{\hat{u}_k}) = 0$ and $\mathcal{H}^{n-1}(S_{\hat{u}_k} \cap K) = \mathcal{M}^{n-1}(S_{\hat{u}_k} \cap K)$ for every compact set $K \subset \hat{\Omega}$.

Note that (ii)-(iii) guarantee the convergence of $\text{MS}_{\mathbb{S}^1}(\hat{u}_k, 1)$ to $\text{MS}_{\mathbb{S}^1}(\hat{u}, 1)$ in $\hat{\Omega}$.

Proof. The proof follows by suitably adapting [14, Lemma 4.11] and [7, Proposition 5.3] to the $\mathbb{S}^1$ -constrained case. For this reason we give here only the main steps. According to Remark 2.4 there exists a lifting $\varphi \in SBV^2(\Omega)$ of u . Since $\partial\Omega$ is Lipschitz, by a standard reflection argument we can construct an extension $\hat{\varphi} \in SBV^2(\hat{\Omega})$ of φ such that

$$\mathcal{H}^{n-1}(S_{\hat{\varphi}} \cap \partial\Omega) = 0.$$

By defining

$$\hat{u}: \hat{\Omega} \rightarrow \mathbb{S}^1, \quad \hat{u} := e^{i\hat{\varphi}},$$

we immediately get that $\hat{u}$ is an extension of u , and

$$\int_{\hat{\Omega}} |\nabla \hat{u}|^2 dx = \int_{\hat{\Omega}} |\nabla \hat{\varphi}|^2 dx, \quad \mathcal{H}^{n-1}(S_{\hat{u}}) \leq \mathcal{H}^{n-1}(S_{\hat{\varphi}}),$$

and thus $\widehat{u} \in SBV^2(\widehat{\Omega}; \mathbb{S}^1)$. Moreover

$$\mathcal{H}^{n-1}(S_{\widehat{u}} \cap \partial\Omega) \leq \mathcal{H}^{n-1}(S_{\widehat{\varphi}} \cap \partial\Omega) = 0. \quad (5.1)$$

By [17, Lemma 4.3] we know that, for every integer $k > 0$, there exists $\widehat{u}_k \in SBV^2(\widehat{\Omega}; \mathbb{S}^1)$ such that

$$\begin{aligned} & \int_{\widehat{\Omega}} |\nabla \widehat{u}_k|^2 dx + \mathcal{H}^{n-1}(S_{\widehat{u}_k}) + k \int_{\widehat{\Omega}} |\widehat{u}_k - \widehat{u}| dx \\ &= \min_{w \in SBV^2(\widehat{\Omega}; \mathbb{S}^1)} \left(\int_{\widehat{\Omega}} |\nabla w|^2 dx + \mathcal{H}^{n-1}(S_w) + k \int_{\widehat{\Omega}} |w - \widehat{u}| dx \right). \end{aligned}$$

Clearly $\|\widehat{u}_k\|_{\infty} = 1$ and

$$\int_{\widehat{\Omega}} |\nabla \widehat{u}_k|^2 dx + \mathcal{H}^{n-1}(S_{\widehat{u}_k}) + k \int_{\widehat{\Omega}} |\widehat{u}_k - \widehat{u}| dx \leq \int_{\widehat{\Omega}} |\nabla \widehat{u}|^2 dx + \mathcal{H}^{n-1}(S_{\widehat{u}}) < +\infty. \quad (5.2)$$

Thus, in particular, $(\widehat{u}_k)$ converges to $\widehat{u}$ in $L^1(\widehat{\Omega}; \mathbb{S}^1)$ as $k \rightarrow +\infty$, and (i) follows. By [5] we also get, up to a not relabelled subsequence,

$$\nabla \widehat{u}_k \rightharpoonup \nabla \widehat{u} \quad \text{in } L^2(\widehat{\Omega}; \mathbb{R}^{n \times n}), \quad \liminf_{k \rightarrow +\infty} \mathcal{H}^{n-1}(S_{\widehat{u}_k}) \geq \mathcal{H}^{n-1}(S_{\widehat{u}}).$$

This and (5.2) imply

$$\int_{\widehat{\Omega}} |\nabla \widehat{u}|^2 dx + \mathcal{H}^{n-1}(S_{\widehat{u}}) = \lim_{k \rightarrow +\infty} \left(\int_{\widehat{\Omega}} |\nabla \widehat{u}_k|^2 dx + \mathcal{H}^{n-1}(S_{\widehat{u}_k}) \right),$$

from which we readily deduce (ii) and (iii). Again by [5] and (5.1) we have

$$\mathcal{H}^{n-1}(S_{\widehat{u}} \cap \Omega) \leq \liminf_{k \rightarrow +\infty} \mathcal{H}^{n-1}(S_{\widehat{u}_k} \cap \Omega) \leq \liminf_{k \rightarrow +\infty} \mathcal{H}^{n-1}(S_{\widehat{u}_k} \cap \overline{\Omega}),$$

$$\mathcal{H}^{n-1}(S_{\widehat{u}} \setminus \Omega) = \mathcal{H}^{n-1}(S_{\widehat{u}} \setminus \overline{\Omega}) \leq \liminf_{k \rightarrow +\infty} \mathcal{H}^{n-1}(S_{\widehat{u}_k} \setminus \overline{\Omega}),$$

which in turn imply (iv). Moreover, invoking [17, Lemma 4.5] we have $\mathcal{H}^{n-1}(\overline{S_{\widehat{u}_k}} \setminus S_{\widehat{u}_k}) = 0$. Finally, thanks to the density estimate in [17, Lemma 4.9] and arguing exactly as in [7, Proposition 5.3] it can be shown that

$$\mathcal{H}^{n-1}(S_{\widehat{u}_k} \cap K) = \mathcal{M}^{n-1}(S_{\widehat{u}_k} \cap K),$$

for every compact set $K \subset \widehat{\Omega}$ and (v) is proven. $\square$

A similar result holds also for liftings:

Proposition 5.2 (Density result for liftings). *Let Ω and $\widehat{\Omega} \subset \mathbb{R}^n$ be as in Proposition 5.1. Let $u \in SBV^2(\Omega; \mathbb{S}^1)$ and $\varphi \in SBV^2(\Omega) \cap L^\infty(\Omega)$ be a lifting of u . Then φ has an extension $\widehat{\varphi} \in SBV^2(\widehat{\Omega}) \cap L^\infty(\widehat{\Omega})$ satisfying $\mathcal{H}^{n-1}(S_{\widehat{\varphi}} \cap \partial\Omega) = 0$. Moreover, there exists a sequence $(\widehat{\varphi}_k) \subset SBV^2(\widehat{\Omega})$ such that:*

- (i) $\widehat{\varphi}_k$ converges to $\widehat{\varphi}$ in $L^1(\widehat{\Omega})$;
- (ii) $\nabla \widehat{\varphi}_k$ converges to $\nabla \widehat{\varphi}$ in $L^2(\widehat{\Omega}; \mathbb{R}^n)$;
- (iii) $\lim_{k \rightarrow \infty} \mathcal{H}^{n-1}(S_{\widehat{\varphi}_k}) = \mathcal{H}^{n-1}(S_{\widehat{\varphi}})$;

- (iv) $\lim_{k \rightarrow \infty} \mathcal{H}^{n-1}(S_{\widehat{\varphi}_k} \cap \overline{\Omega}) = \mathcal{H}^{n-1}(S_{\widehat{\varphi}} \cap \Omega) = \mathcal{H}^{n-1}(S_{\varphi});$
- (v) $\mathcal{H}^{n-1}(\overline{S}_{\widehat{\varphi}_k} \setminus S_{\widehat{\varphi}_k}) = 0$ and $\mathcal{H}^{n-1}(S_{\widehat{\varphi}_k} \cap K) = \mathcal{M}^{n-1}(S_{\widehat{\varphi}_k} \cap K)$ for every compact set $K \subset \widehat{\Omega};$
- (vi) $\widehat{u}_k := e^{i\widehat{\varphi}_k}$ converges to u in $L^1(\Omega; \mathbb{S}^1).$

Note that (ii)-(iii) guarantee the convergence of $\text{MS}(\widehat{\varphi}_k, 1)$ to $\text{MS}(\widehat{\varphi}, 1)$ in $\widehat{\Omega}.$

Proof. From [14, Lemma 4.11] we can find $\widehat{\varphi}$ as in the statement and $(\widehat{\varphi}_k) \subset SBV^2(\widehat{\Omega})$ that satisfy (i)–(v). This in turn implies also (vi). $\square$

5.2 Proof of Theorem 1.3

If $\varepsilon_k \searrow 0$ and $((u_k, v_k))_{k \geq 1} \subset L^1(\Omega; \mathbb{S}^1) \times L^1(\Omega, [0, 1])$ is a sequence converging to (u, v) in $L^1(\Omega; \mathbb{S}^1) \times L^1(\Omega),$ arguing exactly as in [30], one gets

$$\liminf_{k \rightarrow +\infty} \text{AT}_{\varepsilon_k}^{\mathbb{S}^1}(u_k, v_k) \geq \text{MS}_{\mathbb{S}^1}(u, v).$$

Therefore, we only need to prove the upper bound. Let $\varepsilon_k \searrow 0$ and $u \in SBV^2(\Omega; \mathbb{S}^1).$ We have to find a sequence $((u_k, v_k)) \subset \widehat{\mathcal{A}}_{\mathbb{S}^1}$ converging to $(u, 1)$ in $L^1(\Omega; \mathbb{S}^1) \times L^1(\Omega)$ for which

$$\limsup_{k \rightarrow +\infty} \text{AT}_{\varepsilon_k}^{\mathbb{S}^1}(u_k, v_k) \leq \text{MS}_{\mathbb{S}^1}(u, 1). \quad (5.3)$$

We note that, in general, we cannot take $((u_k, v_k)) \subset \mathcal{A}_{\mathbb{S}^1},$ as it happens for the example described in the introduction.

We fix a bounded open set $\widehat{\Omega} \subset \subset \mathbb{R}^n,$ with $\Omega \subset \subset \widehat{\Omega}.$ By Proposition 5.1 it suffices to show (5.3) for $u \in SBV^2(\widehat{\Omega}; \mathbb{S}^1)$ with

$$\mathcal{H}^{n-1}(S_u \cap \partial\Omega) = 0, \quad \mathcal{H}^{n-1}(\overline{S}_u \setminus S_u) = 0, \quad \mathcal{H}^{n-1}(S_u \cap K) = \mathcal{M}^{n-1}(S_u \cap K), \quad (5.4)$$

for every compact set $K \subset \widehat{\Omega}.$

By Theorem 2.3 and Remark 2.4 there exists a lifting $\varphi \in SBV^2(\widehat{\Omega})$ of $u.$ For any $0 < \rho \ll 1$ define

$$(S_u)_\rho := \{x \in \widehat{\Omega} : \text{dist}(x, S_u) < \rho\}.$$

Let $0 < \xi_k$ with $\lim_{k \rightarrow +\infty} \xi_k/\varepsilon_k = 0$ be such that $(S_u)_{\xi_k}$ has Lipschitz boundary. By Remark 2.2 applied to $A = (S_u)_{\xi_k}$ and $\delta = \varepsilon_k,$ there exists $\varphi_k \in C^\infty((S_u)_{\xi_k})$ such that

$$\int_{(S_u)_{\xi_k}} |\varphi - \varphi_k| dx < \varepsilon_k, \quad \int_{(S_u)_{\xi_k}} |\nabla \varphi_k| dx \leq |D\varphi|((S_u)_{\xi_k}) + \varepsilon_k, \quad (5.5)$$

and, in the sense of BV -traces,

$$\varphi_k = \varphi \quad \text{a.e. on} \quad \partial(S_u)_{\xi_k}. \quad (5.6)$$

We define $\omega_k \in SBV^2(\widehat{\Omega})$ as follows:

$$\omega_k := \begin{cases} \varphi_k & \text{in } (S_u)_{\xi_k} \\ \varphi & \text{in } \widehat{\Omega} \setminus (S_u)_{\xi_k} \end{cases}.$$

Then by (5.5) the sequence (ω_k) converges to φ strictly in $BV(\widehat{\Omega})$ and

$$S_{\omega_k} \subseteq S_\varphi.$$

We set

$$u_k := e^{i\omega_k} = \begin{cases} e^{i\varphi_k} & \text{in } (S_u)_{\xi_k} \\ u & \text{in } \widehat{\Omega} \setminus (S_u)_{\xi_k} \end{cases}$$

so that

$$u_k \in W^{1,1}(\widehat{\Omega}; \mathbb{S}^1) \quad (5.7)$$

and (u_k) converges to u in $L^1(\Omega; \mathbb{S}^1)$. Since we modified u only in a neighbourhood of S_u in general $u_k \notin W^{1,2}(\widehat{\Omega}; \mathbb{S}^1)$, as it happens in the example in the introduction.

The construction of (v_k) is done as in [18]. For completeness here we sketch the proof and refer to [18] for details. Precisely, we define $(v_k) \subset W^{1,2}(\widehat{\Omega})$ as

$$v_k(x) := \begin{cases} 0 & x \in (S_u)_{\xi_k} \\ 1 - \exp\left(-\frac{\text{dist}(x, S_u) - \xi_k}{2\varepsilon_k}\right) & x \in \widehat{\Omega} \setminus (S_u)_{\xi_k} \end{cases},$$

Then (v_k) converges to 1 in $L^1(\widehat{\Omega})$ as $k \rightarrow +\infty$. Moreover

$$v_k |\nabla u_k| \in L^2(\widehat{\Omega})$$

and hence, using (5.7), we get the crucial inclusion

$$(u_k, v_k) \subset \widehat{\mathcal{A}}_{\mathbb{S}^1}.$$

It remains to show (5.3). To this aim we have

$$\limsup_{k \rightarrow +\infty} \int_{\Omega} v_k^2 |\nabla u_k|^2 dx \leq \limsup_{k \rightarrow +\infty} \int_{\Omega \setminus (S_u)_{\xi_k}} |\nabla u|^2 dx \leq \int_{\Omega} |\nabla u|^2 dx. \quad (5.8)$$

Moreover

$$\text{MM}_{\varepsilon_k}(v_k, \Omega) = \frac{|(S_u)_{\xi_k} \cap \overline{\Omega}|}{\varepsilon_k} + \text{MM}_{\varepsilon_k}(v_k, \Omega \setminus (S_u)_{\xi_k}). \quad (5.9)$$

From (5.4) it follows

$$\limsup_{k \rightarrow +\infty} \frac{|(S_u)_{\xi_k} \cap \overline{\Omega}|}{2\varepsilon_k} = \limsup_{k \rightarrow +\infty} \frac{|(S_u)_{\xi_k} \cap \overline{\Omega}|}{2\xi_k} \frac{\xi_k}{\varepsilon_k} = 0. \quad (5.10)$$

Moreover, following the lines of [18] it can be shown that

$$\text{MM}_{\varepsilon_k}(v_k, \Omega \setminus (S_u)_{\xi_k}) = -\frac{1}{2\varepsilon_k} |(S_u)_{\xi_k}| + \int_0^{+\infty} (s + \varepsilon_k) e^{-s} \frac{|(S_u)_{s\varepsilon_k + \xi_k} \cap \overline{\Omega}|}{2(s\varepsilon_k + \xi_k)} ds. \quad (5.11)$$

Gathering together (5.9)–(5.11) we find

$$\text{MM}_{\varepsilon_k}(v_k, \Omega) = \frac{1}{2\varepsilon_k} |(S_u)_{\xi_k} \cap \overline{\Omega}| + \int_0^{+\infty} (s + \varepsilon_k) e^{-s} \frac{|(S_u)_{s\varepsilon_k + \xi_k} \cap \overline{\Omega}|}{2(s\varepsilon_k + \xi_k)} ds. \quad (5.12)$$

From (5.4) it follows

$$\limsup_{k \rightarrow +\infty} \frac{|(S_u)_{s\varepsilon_k + \xi_k} \cap \overline{\Omega}|}{2(s\varepsilon_k + \xi_k)} = \mathcal{H}^{n-1}(S_u \cap \overline{\Omega}).$$

This, together with the fact that $\int_0^{+\infty} (s + \varepsilon_k) e^{-s} ds \rightarrow 1$ as $k \rightarrow +\infty$, imply

$$\limsup_{k \rightarrow +\infty} \text{MM}_{\varepsilon_k}(v_k, \Omega) \leq \mathcal{H}^{n-1}(S_u \cap \overline{\Omega}). \quad (5.13)$$

Finally, combining (5.8), (5.10) and (5.13) we obtain (5.3). $\square$

5.3 Proof of Theorem 1.4

Step 1: Lower bound. Let $\varepsilon_k \searrow 0$ as $k \rightarrow +\infty$. We have to show that, for every sequence $((u_k, v_k))_{k \geq 1} \subset L^1(\Omega; \mathbb{S}^1) \times L^1(\Omega)$ converging to (u, v) in $L^1(\Omega; \mathbb{S}^1) \times L^1(\Omega)$,

$$\liminf_{k \rightarrow +\infty} \text{AT}_{\varepsilon_k}^{\mathbb{S}^1}(u_k, v_k) \geq \text{MS}_{\text{lift}}(u, v). \quad (5.14)$$

We may assume

$$\sup_{k \in \mathbb{N}} \text{AT}_{\varepsilon_k}^{\mathbb{S}^1}(u_k, v_k) \leq C < +\infty \quad (5.15)$$

so that $(u_k, v_k) \in \mathcal{A}_{\mathbb{S}^1}$, $v = 1$ a.e. in Ω and, up to a not relabelled subsequence,

$$\begin{aligned} \liminf_{k \rightarrow +\infty} \text{AT}_{\varepsilon_k}^{\mathbb{S}^1}(u_k, v_k) &= \lim_{k \rightarrow +\infty} \text{AT}_{\varepsilon_k}^{\mathbb{S}^1}(u_k, v_k), \\ \liminf_{k \rightarrow +\infty} \text{MM}_{\varepsilon_k}(v_k, \Omega) &= \lim_{k \rightarrow +\infty} \text{MM}_{\varepsilon_k}(v_k, \Omega). \end{aligned}$$

From Theorem 2.7 and the decoupling property (2.16), it follows that

$$\begin{aligned} \liminf_{k \rightarrow +\infty} \text{AT}_{\varepsilon_k}^{\mathbb{S}^1}(u_k, v_k) &\geq \int_{\Omega} |\nabla u|^2 \, dx + \mathcal{H}^{n-1}(S_u), \\ \liminf_{k \rightarrow +\infty} \int_{\Omega} v_k^2 |\nabla u_k|^2 \, dx &\geq \int_{\Omega} |\nabla u|^2 \, dx, \\ \liminf_{k \rightarrow +\infty} \text{MM}_{\varepsilon_k}(v_k, \Omega) &\geq \mathcal{H}^{n-1}(S_u). \end{aligned}$$

In particular, from (5.15), $u \in SBV^2(\Omega; \mathbb{S}^1)$. Hence it remains to show

$$\lim_{k \rightarrow +\infty} \text{MM}_{\varepsilon_k}(v_k, \Omega) \geq m_2[u]. \quad (5.16)$$

For every $k \geq 1$ we start by selecting a lifting $\varphi_k \in W^{1,2}(\Omega)$ of u_k . Since $|\nabla u_k| = |\nabla \varphi_k|$ we have

$$+\infty > C \geq \text{AT}_{\varepsilon_k}^{\mathbb{S}^1}(u_k, v_k) = \int_{\Omega} \left(v_k^2 |\nabla \varphi_k|^2 + \varepsilon_k |\nabla v_k|^2 + \frac{(v_k - 1)^2}{4\varepsilon_k} \right) \, dx.$$

Thus, using the coarea formula,

$$C \geq \text{MM}_{\varepsilon_k}(v_k, \Omega) \geq \int_{\Omega} (1 - v_k) |\nabla v_k| \, dx = \int_0^1 (1 - t) \mathcal{H}^{n-1}(\partial^* F_k^t) \, dt \quad (5.17)$$

for any $k \geq 1$, where $F_k^t := \{x \in \Omega : v_k(x) \leq t\}$. Let $\eta', \eta'' \in (0, 1)$, $\eta' < \eta''$ be fixed. By (5.17) and the mean value theorem there exists $t(k) \in (\eta', \eta'')$ such that

$$C \geq (1 - \eta'')(\eta'' - \eta') \mathcal{H}^{n-1}(\partial^* F_k^{t(k)}). \quad (5.18)$$

Moreover

$$C \geq \int_{\Omega} v_k^2 |\nabla \varphi_k|^2 \, dx \geq (\eta')^2 \int_{\Omega} \chi_{\Omega \setminus F_k^{t(k)}} |\nabla \varphi_k|^2 \, dx. \quad (5.19)$$

Then, using a standard argument (see for instance [18, Theorem 4]), we set

$$\phi_k := \varphi_k \chi_{\Omega \setminus F_k^{t(k)}} \in SBV^2(\Omega).$$

In our present situation, such an argument will allow us to use the compactness Theorem 1.2. We have $S_{\phi_k} \subset \partial^* F_k^{t(k)}$ and, for the absolutely continuous parts of the gradients, $\nabla \phi_k = \nabla \varphi_k \chi_{\Omega \setminus F_k^{t(k)}}$ and so, from (5.19) and (5.18),

$$\int_{\Omega} |\nabla \phi_k|^2 dx + \mathcal{H}^{n-1}(S_{\phi_k}) \leq C(\eta', \eta'') \quad (5.20)$$

for some $C(\eta', \eta'') > 0$ depending on η', η'' and independent of k . Let also

$$\bar{u}_k := e^{i\phi_k} = u_k \chi_{\Omega \setminus F_k^{t(k)}} + (1, 0) \chi_{F_k^{t(k)}},$$

which, by (5.18), are uniformly bounded in $BV(\Omega; \mathbb{S}^1)$. Since

$$|F_k^{t(k)}| \rightarrow 0, \quad (5.21)$$

the sequence $(\bar{u}_k)$ weakly star converges to u in $BV(\Omega; \mathbb{S}^1)$. Hence, using (5.20), we can apply Theorem 1.2 to the sequence $(\phi_k)_{k \geq 1}$ (with $p = 2$) and get, for a not relabelled subsequence, a Caccioppoli partition $(E_i)_{i \in \mathbb{N}}$ of Ω , sequences $(d_k^{(i)})_{k \geq 1} \subset \mathbb{Z}$ for any integer $i \geq 1$ and a lifting $\varphi_{\infty} \in GSBV^2(\Omega)$ of u , such that

$$\begin{aligned} \lim_{k \rightarrow +\infty} (\phi_k(x) - 2\pi d_k^{(i)}) &= \varphi_{\infty}(x) \quad \forall i \in \mathbb{N}, \text{ for a.e. } x \in E_i, \\ \lim_{k \rightarrow +\infty} |\phi_k(x) - 2\pi d_k^{(i)}| &= +\infty \quad \forall i \in \mathbb{N}, \text{ for a.e. } x \in \Omega \setminus E_i. \end{aligned}$$

Again, using (5.21), the same holds for φ_k , i.e.,

$$\begin{aligned} \lim_{k \rightarrow +\infty} (\varphi_k(x) - 2\pi d_k^{(i)}) &= \varphi_{\infty}(x) \quad \forall i \in \mathbb{N}, \text{ for a.e. } x \in E_i, \\ \lim_{k \rightarrow +\infty} |\varphi_k(x) - 2\pi d_k^{(i)}| &= +\infty \quad \forall i \in \mathbb{N}, \text{ for a.e. } x \in \Omega \setminus E_i. \end{aligned} \quad (5.22)$$

Then (5.16) follows if we show

$$\liminf_{k \rightarrow +\infty} \text{MM}(v_k, \Omega) \geq \mathcal{H}^{n-1}(S_{\varphi_{\infty}}), \quad (5.23)$$

since, being φ_{∞} a lifting of u , we have

$$\mathcal{H}^{n-1}(S_{\varphi_{\infty}}) \geq m_2[u].$$

For any integer $K \geq 1$ we consider the truncated function

$$\varphi_k^{i,K} := ((\varphi_k - 2\pi d_k^{(i)}) \wedge K) \vee (-K) \in W^{1,2}(\Omega), \quad (5.24)$$

and

$$\phi_k^{i,K} := \varphi_k^{i,K} \chi_{\Omega \setminus F_k^{t(k)}}.$$

Since $S_{\phi_k^{i,K}} \subset \partial^* F_k^{t(k)}$ and $\llbracket \phi_k^{i,K} \rrbracket \leq K$, we have

$$\begin{aligned} |D\phi_k^{i,K}|(\Omega) &= \int_{\Omega} |\nabla \phi_k| \chi_{\{|\varphi_k - 2\pi d_k^{(i)}| < K\}} dx + \int_{S_{\phi_k^{i,K}}} \llbracket \phi_k^{i,K} \rrbracket d\mathcal{H}^{n-1} \\ &\leq \int_{\Omega} |\nabla \phi_k| dx + K \mathcal{H}^{n-1}(\partial^* F_k^{t(k)}) \leq C. \end{aligned}$$

Hence, up to a subsequence (depending on K), $(\phi_k^{i,K})$ converges to some $\phi_\infty^{i,K}$ in $L^1(\Omega)$. Moreover from (5.22) it holds $\phi_\infty^{i,K} := (\varphi_\infty \wedge K) \vee (-K)$ in E_i and $\Omega \setminus E_i = F_i^+ \cup F_i^-$ are such that $\phi_\infty^{i,K} = \pm K$ in $F_i^\pm$. As $|F_k^{t(k)}| \rightarrow 0$ it follows that $\varphi_k^{i,K}$ converges to $\phi_\infty^{i,K}$ in $L^1(\Omega)$. Hence the sequence $((\varphi_k^{i,K}, v_k))$ converges to $(\phi_\infty^{i,K}, 1)$ in $L^1(\Omega) \times L^1(\Omega)$ and

$$\int_A \left(v_k^2 |\nabla \varphi_k^{i,K}|^2 + \varepsilon_k |\nabla v_k|^2 + \frac{(v_k - 1)^2}{4\varepsilon_k} \right) dx \leq C,$$

for any open set $A \subset \Omega$, for some $C > 0$ independent of k . Thus, from the decoupling property (2.16),

$$\begin{aligned} \liminf_{k \rightarrow +\infty} \int_A v_k^2 |\nabla \varphi_k^{i,K}|^2 dx &\geq \int_A |\nabla \phi_\infty^{i,K}|^2 dx, \\ \liminf_{k \rightarrow +\infty} \text{MM}_{\varepsilon_k}(v_k, A) &\geq \mathcal{H}^{n-1}(S_{\phi_\infty^{i,K}} \cap A). \end{aligned} \tag{5.25}$$

In particular $\phi_\infty^{i,K} \in SBV^2(\Omega)$. We fix an integer $N \geq 1$ and set, for all $i = 1, \dots, N$,

$$\Sigma_i^K := (S_{\phi_\infty^{i,K}} \cap \partial^* E_i) \cap \Omega, \quad S_i^K := S_{\phi_\infty^{i,K}} \setminus \partial^* E_i,$$

$$\Sigma^{N,K} := \bigcup_{i=1}^N \Sigma_i^K, \quad S^{N,K} := \bigcup_{i=1}^N S_i^K.$$

We now argue exactly as in the proof of Lemma 4.2, so, given $\delta \in (0, 1)$, after defining the families of balls $\mathcal{B}$ and $\mathcal{B}_j$ ($j = 1, \dots, 2N$), we arrive at

$$\sum_{B \in \mathcal{B} \cap \mathcal{B}_j} \mathcal{H}^{n-1}(B \cap S_j^K) \geq \mathcal{H}^{n-1}(S_j^K) - \frac{\delta \mathcal{H}^{n-1}(S^{N,K} \cup \Sigma^{N,K})}{1 - \delta}, \tag{5.26}$$

for all $j = 1, \dots, N$, and

$$\sum_{B \in \mathcal{B} \cap \mathcal{B}_{N+h}} \mathcal{H}^{n-1}(B \cap \Sigma_h^K) \geq \mathcal{H}^{n-1}(\Sigma_h^K) - \frac{\delta \mathcal{H}^{n-1}(S^{N,K} \cup \Sigma^{N,K})}{1 - \delta}, \tag{5.27}$$

for all $h = 1, \dots, N$. From (5.25) we know that, for all $j = 1, \dots, N$,

$$\liminf_{k \rightarrow +\infty} \text{MM}_{\varepsilon_k}(v_k, B) \geq \mathcal{H}^{n-1}(S_j^K \cap B), \tag{5.28}$$

for any $B \in \mathcal{B} \cap \mathcal{B}_j$, and for all $h = 1, \dots, N$

$$\liminf_{k \rightarrow +\infty} \text{MM}_{\varepsilon_k}(v_k, B) \geq \mathcal{H}^{n-1}(\Sigma_h^K \cap B), \tag{5.29}$$

for any ball $B \in \mathcal{B} \cap \mathcal{B}_{N+h}$. So, summing (5.28) over all $B \in \mathcal{B} \cap \mathcal{B}_j$ and (5.29) over all $B \in \mathcal{B} \cap \mathcal{B}_{N+h}$, and then over j, h respectively, we conclude again, using (5.26) and (5.27),

$$\begin{aligned} \liminf_{k \rightarrow +\infty} \text{MM}_{\varepsilon_k}(v_k, \Omega) &= \lim_{k \rightarrow +\infty} \text{MM}_{\varepsilon_k}(v_k, \Omega) \\ &\geq \sum_{j=1}^N \mathcal{H}^{n-1}(S_j^K) + \sum_{h=1}^N \mathcal{H}^{n-1}(\Sigma_h^K) - \frac{2\delta N \mathcal{H}^{n-1}(S^{N,K} \cup \Sigma^{N,K})}{1 - \delta} \end{aligned}$$

and by the arbitrariness of $\delta > 0$,

$$\lim_{k \rightarrow +\infty} \text{MM}_{\varepsilon_k}(v_k, \Omega) \geq \mathcal{H}^{n-1}(S^{N,K} \cup \Sigma^{N,K}).$$

The left hand-side in the above inequality is a limit, hence it does not depend on the subsequence, and so in particular on K and N . Letting first $N \rightarrow +\infty$ and then $K \rightarrow +\infty$ we finally get

$$\liminf_{k \rightarrow +\infty} \text{MM}_{\varepsilon_k}(v_k, \Omega) \geq \mathcal{H}^{n-1}(S_{\varphi_\infty}),$$

and (5.16) follows.

Step 2: Upper bound. Let $\varepsilon_k \searrow 0$ and $u \in SBV^2(\Omega; \mathbb{S}^1)$. We have to find a sequence $((u_k, v_k)) \subset \mathcal{A}_{\mathbb{S}^1}$ converging to $(u, 1)$ in $L^1(\Omega; \mathbb{S}^1) \times L^1(\Omega)$ and

$$\limsup_{k \rightarrow +\infty} \text{AT}_{\varepsilon_k}^{\mathbb{S}^1}(u_k, v_k) \leq \text{MS}_{\text{lift}}(u, 1). \quad (5.30)$$

We note that we cannot follow the strategy used in the proof of the upper bound in Theorem 1.3, since that construction does not ensure that the functions u_k belong to $W^{1,2}(\Omega; \mathbb{S}^1)$.

By Corollary 3.1 we can select a jump minimizing lifting $\varphi \in GSBV^2(\Omega)$ of u , $u = e^{i\varphi}$ and

$$\mathcal{H}^{n-1}(S_\varphi) = m_2[u]. \quad (5.31)$$

The main difference from the construction used in Theorem 1.3 is that we modify u in a neighbourhood of S_φ rather than of S_u . We divide the proof into two cases.

Case 1: $\varphi \in L^\infty(\Omega)$. Let $\widehat{\Omega}$, with $\Omega \subset \subset \widehat{\Omega} \subset \subset \mathbb{R}^2$, be open. Since $\varphi \in SBV^2(\Omega) \cap L^\infty(\Omega)$, by Proposition 5.2 we can assume, without loss of generality, that $\varphi \in SBV^2(\widehat{\Omega})$ with

$$\mathcal{H}^{n-1}(S_\varphi \cap \partial\Omega) = 0, \quad \mathcal{H}^{n-1}(\overline{S_\varphi} \setminus S_\varphi) = 0, \quad \mathcal{H}^{n-1}(S_\varphi \cap K) = \mathcal{M}^{n-1}(S_\varphi \cap K), \quad (5.32)$$

for every compact set $K \subset \widehat{\Omega}$. In this way, arguing as in [7, 8] we can construct $(\varphi_k, v_k) \in W^{1,2}(\widehat{\Omega}) \times W^{1,2}(\widehat{\Omega})$ such that $(\varphi_k, v_k) \rightarrow (\varphi, 1)$ in $L^1(\widehat{\Omega}) \times L^1(\widehat{\Omega})$ and

$$\limsup_{k \rightarrow +\infty} \text{AT}_{\varepsilon_k}(\varphi_k, v_k) \leq \text{MS}(\varphi, 1). \quad (5.33)$$

Next we let $u_k := e^{i\varphi_k} \in W^{1,2}(\widehat{\Omega}; \mathbb{S}^1)$, so that $(u_k, v_k) \rightarrow (u, 1)$ in $L^1(\widehat{\Omega}; \mathbb{S}^1) \times L^1(\widehat{\Omega})$. Moreover, from (5.31), (5.33), $|\nabla u_k| = |\nabla \varphi_k|$ and $|\nabla u| = |\nabla \varphi|$ we get

$$\text{AT}_{\varepsilon_k}(\varphi_k, v_k) = \text{AT}_{\varepsilon_k}^{\mathbb{S}^1}(u_k, v_k), \quad \text{MS}(\varphi, 1) = \text{MS}_{\text{lift}}(u, 1)$$

and so

$$\limsup_{k \rightarrow +\infty} \text{AT}_{\varepsilon_k}^{\mathbb{S}^1}(u_k, v_k) \leq \text{MS}_{\text{lift}}(u, 1).$$

Case 2: $\varphi \notin L^\infty(\Omega)$. For $N \in \mathbb{N}$, $N \geq 1$ we let $\varphi^{(N)} := \varphi \wedge N \vee (-N) \in L^\infty(\Omega)$. Then, as $N \rightarrow +\infty$, we have

$$\varphi^{(N)} \rightarrow \varphi \quad \text{in } L^1(\Omega), \quad \nabla \varphi^{(N)} \rightarrow \nabla \varphi \quad \text{in } L^2(\Omega; \mathbb{R}^n),$$

so that

$$u^{(N)} := e^{i\varphi^N} \rightarrow e^{i\varphi} = u \quad \text{in } L^1(\Omega; \mathbb{S}^1), \quad \nabla u^{(N)} \rightarrow \nabla u \quad \text{in } L^2(\Omega; \mathbb{R}^{n \times n}).$$

In addition,

$$\mathcal{H}^{n-1}(S_{\varphi^{(N)}}) \leq \mathcal{H}^{n-1}(S_\varphi).$$

This in turn implies

$$\begin{aligned} \limsup_{N \rightarrow +\infty} \left(\int_{\Omega} |\nabla \varphi^{(N)}|^2 dx + \mathcal{H}^{n-1}(S_{\varphi^{(N)}}) \right) &\leq \int_{\Omega} |\nabla \varphi|^2 dx + \mathcal{H}^{n-1}(S_\varphi) \\ &\leq \int_{\Omega} |\nabla u|^2 dx + m_2[u]. \end{aligned} \quad (5.34)$$

For each $N \geq 1$ we can argue as in case 1 and find a sequence $(u_k^{(N)}, v_k^{(N)}) \rightarrow (u^{(N)}, 1)$ in $L^1(\Omega; \mathbb{S}^1) \times L^1(\Omega)$ satisfying

$$\limsup_{k \rightarrow +\infty} \text{AT}_{\varepsilon_k}^{\mathbb{S}^1}(u_k^{(N)}, v_k^{(N)}) \leq \int_{\Omega} |\nabla \varphi^{(N)}|^2 dx + \mathcal{H}^{n-1}(S_\varphi^{(N)}). \quad (5.35)$$

Combining (5.34) with (5.35) we have

$$\limsup_{N \rightarrow +\infty} \limsup_{k \rightarrow +\infty} \text{AT}_{\varepsilon_k}^{\mathbb{S}^1}(u_k^{(N)}, v_k^{(N)}) \leq \int_{\Omega} |\nabla u|^2 dx + m_2[u].$$

Now we conclude using a diagonal argument. Namely, for any fixed $N \geq 1$ there exists $k_N \geq 1$ such that for $k \geq k_N$:

- $\|u_k^{(N)} - u^{(N)}\|_{L^1(\Omega; \mathbb{S}^1)} + \|v_k^{(N)} - 1\|_{L^1(\Omega)} \leq \frac{1}{N}$;
- $\text{AT}_{\varepsilon_k}^{\mathbb{S}^1}(u_k^{(N)}, v_k^{(N)}) \leq \int_{\Omega} |\nabla \varphi^{(N)}|^2 dx + \mathcal{H}^{n-1}(S_\varphi^{(N)}) + \frac{1}{N}$.

Without loss of generality we may assume $k_{N+1} \geq k_N$. Now, for every $k \geq 1$, we take $(u_k, v_k) := (u_k^{(N)}, v_k^{(N)})$ if $k \in [k_N, k_{N+1}) \cap \mathbb{N}$ and the proof is concluded. $\square$

6 Further directions and open problems

Suitable variants of the Ambrosio-Tortorelli approximation for $\mathbb{S}^1$ -valued maps could be investigated in order to characterise topological singularities in several settings. For instance, in the recent works [24, 27] a free-discontinuity model for screw dislocations was proposed. The model is given by a Mumford-Shah type energy for $\mathbb{S}^1$ -valued maps that penalizes the measure of the jump set. The analysis carried out in [24, 27] show that, to leading order, this model is equivalent to the Ginzburg-Landau one. This seems to suggest that an Ambrosio-Tortorelli approximation for screw dislocations offers the possibility to extend the aforementioned results to the three-dimensional setting [22, 23, 33].

In the context of Plateau problem and optimal transport, a close connection with lifting theory has been observed [15]. In the specific case of $\mathbb{S}^1$ -valued Sobolev maps, the singularities should be optimally connected to one another, in a fashion similar to the minimal connection of dipoles arising in the minimization of the BV -seminorm of liftings [15] (see also [13, 41]). In the relaxed Cartesian-area setting, connecting the singularities is related to finding an optimal Cartesian vertical current that fills the holes in the

graph of the given map u ; this, in turn, requires the study of a Plateau problem in codimension one with partial free boundary [11]. In the simplified setting of $\mathbb{S}^1$ -constrained relaxation [35, pg. 611 eq. 4], the optimal value of the graph area contains a singular contribution characterized in terms of $\mathbb{S}^1$ vertical Cartesian currents [35, pg. 612, Theorem 1]. This contribution is again related to an optimal transport problem concerning how the singularities of u should be connected. A possible direction for further research is to approximate the relaxed area functional on $\mathbb{S}^1$ -valued maps by an Ambrosio-Tortorelli energy whose bulk term has linear growth in the gradient, in the spirit of [2].

We also emphasize that phase-field models based on the Ambrosio-Tortorelli functionals have already been used in optimal transport (see e.g. [29] and references therein, and [21] for the relation with the Steiner tree problem). While the link of Corollary 3.1 with optimal transport involves the cost function $\psi \equiv 1$ (see Subsection 3.3), we expect that the analysis of the Ambrosio-Tortorelli functionals for $\mathbb{S}^1$ -valued maps with linear growth (rather than quadratic) would lead to a minimization problem similar to (1.1), but involving different cost functions.

Data availability statement

No new data was created or analysed in this work.

Conflict of interest statement

The authors have no conflict of interest to declare that are relevant to the content of this article.

Acknowledgements

R. Marziani has received funding from the European Union's Horizon research and innovation program under the Marie Skłodowska-Curie agreement No 101150549. The authors are members of INdAM and join projects from GNAMPA. The research that led to the present paper was partially supported by the INdAM-GNAMPA project 2025 CUP E5324001950001. The first and third authors also acknowledge the partial financial support of the PRIN project 2022PJ9EFL "Geometric Measure Theory: Structure of Singular Measures, Regularity Theory and Applications in the Calculus of Variations", PNRR Italia Domani, funded by the European Union via the program NextGenerationEU (Missione 4 Componente 2) CUP:E53D23005860006. Views and opinions expressed are however those of the authors only and do not necessarily reflect those of the European Union or the European Research Executive Agency. Neither the European Union nor the granting authority can be held responsible for them. Finally, we thank the referees for their careful reading and valuable suggestions which allow us to improve the paper presentation and content.

References

- [1] E. Acerbi and G. Dal Maso. New lower semicontinuity results for polyconvex integrals. *Calc. Var. Partial Differential Equations*, 2:329–371, 1994.
- [2] R. Alicandro, A. Braides, and J. Shah. Free-discontinuity problems via functionals involving the L^1 -norm of the gradient and their approximation. *Interfaces Free Bound.*, 1:17–37, 1999.

- [3] R. Alicandro, M. Cicalese, and M. Ponsiglione. Variational equivalence between ginzburg–landau, XY spin systems and screw dislocations energies. *Indiana Univ. Math. J.*, 60:171–208, 2011.
- [4] R. Alicandro, L. De Luca, A. Garroni, and M. Ponsiglione. Metastability and dynamics of discrete topological singularities in two dimensions: a Γ -convergence approach. *Arch. Rational Mech. Anal.*, 214:269–330, 2014.
- [5] L. Ambrosio. On the lower semicontinuity of quasi-convex integrals in $SBV(\Omega; \mathbb{R}^k)$. *Nonlinear Anal.*, 23:405–425, 1994.
- [6] L. Ambrosio, N. Fusco, and D. Pallara. *Functions of Bounded Variation and Free Discontinuity Problems*. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, 2000.
- [7] L. Ambrosio and V. M. Tortorelli. Approximation of functionals depending on jumps by elliptic functionals via Γ -convergence. *Comm. Pure Appl. Math.*, 43(8):999–1036, 1990.
- [8] L. Ambrosio and V. M. Tortorelli. On the approximation of free-discontinuity problems. *Boll. Un. Mat. Ital.*, 6-B(1):105–123, 1992. (7).
- [9] G. Bellettini, A. Elshorbagy, M. Paolini, and R. Scala. On the relaxed area of the graph of discontinuous maps from the plane to the plane taking three values with no symmetry assumptions. *Ann. Mat. Pura Appl.*, 199:445–477, 2020.
- [10] G. Bellettini, A. Elshorbagy, and R. Scala. The L^1 -relaxed area of the graph of the vortex map: optimal lower bound. *Nonlinear Anal.*, 256:113803, 2025. Paper No. 113803.
- [11] G. Bellettini, R. Marziani, and R. Scala. A non-parametric plateau problem with partial free boundary. *J. École Polytechnique – Mathématiques*, 11:1035–1098, 2024.
- [12] G. Bellettini and M. Paolini. On the area of the graph of a singular map from the plane to the plane taking three values. *Adv. Calc. Var.*, 3:371–386, 2010.
- [13] G. Bellettini, R. Scala, and G. Scianna. L^1 -relaxed area of graphs of $\mathbb{S}^1$ -valued sobolev maps and its countably subadditive envelope. *Rev. Mat. Iberoam.*, 40:2135–2178, 2024.
- [14] A. Braides. *Approximation of Free-Discontinuity Problems*, volume 1694 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1998.
- [15] H. Brezis and P. Mironescu. The plateau problem from the perspective of optimal transport. *Comptes Rendus. Mathématique*, 357(7):597–612, 2019.
- [16] H. Brezis and P. Mironescu. *Sobolev Maps to the Circle*, volume 96 of *Progress in Nonlinear Differential Equations and their Applications*. Birkhäuser/Springer, New York, 2021.
- [17] M. Carriero and A. Leaci. S^k -valued maps minimizing the L^p norm of the gradient with free discontinuities. *Ann. Sc. Norm. Sup. Pisa Cl. Sci.*, 18:321–352, 1991.
- [18] A. Chambolle. An approximation result for special functions with bounded deformation. *J. Math. Pures Appl.*, 83:929–954, 2004.

- [19] A. Chambolle and V. Crismale. Compactness and lower semicontinuity in GSBD. *J. Eur. Math. Soc. (JEMS)*, 23:701–719, 2021.
- [20] A. Chambolle and V. Crismale. A general compactness theorem in G(S)BD. *Indiana Univ. Math. J.*, 74:233–249, 2025.
- [21] A. Chambolle, L. A. D. Ferrari, and B. Merlet. A phase-field approximation of the steiner problem in dimension two. *Adv. Calc. Var.*, 12(2):157–179, 2019.
- [22] S. Conti, A. Garroni, and R. Marziani. Line-tension limits for line singularities and application to the mixed-growth case. *Calc. Var. Partial Differential Equations*, 62:228, 2023.
- [23] S. Conti, A. Garroni, and M. Ortiz. The line-tension approximation as the dilute limit of linear-elastic dislocations. *Arch. Rational Mech. Anal.*, 218:699–755, 2015.
- [24] V. Crismale, L. De Luca, and R. Scala. Approximation of topological singularities through free discontinuity functionals: the critical and super-critical regimes. *Calc. Var.*, 65.
- [25] J. Dávila and R. Ignat. Lifting of BV functions with values in S^1 . *C. R. Math. Acad. Sci. Paris*, 337:159–164, 2003.
- [26] L. De Luca, A. Garroni, and M. Ponsiglione. Γ -convergence analysis of systems of edge dislocations: the self energy regime. *Arch. Rational Mech. Anal.*, 206:885–910, 2012.
- [27] L. De Luca, R. Scala, and N. Van Goethem. A new approach to topological singularities via a weak notion of jacobian for functions of bounded variation. *Indiana Univ. Math. J.*, 73(2):723–779, 2024.
- [28] H. Federer. *Geometric Measure Theory*, volume 153 of *Die Grundlehren der mathematischen Wissenschaften*. Springer-Verlag, New York, 1969.
- [29] L. Ferrari, C. Dirks, and B. Wirth. Phase field approximations of branched transportation problems. *Calc. Var. Partial Differential Equations*, 59:37, 2020.
- [30] M. Focardi. On the variational approximation of free-discontinuity problems in the vectorial case. *Math. Models Methods Appl. Sci.*, 11:663–684, 2001.
- [31] M. Friedrich. A compactness result in $GSBV^p$ and applications to Γ -convergence for free discontinuity problems. *Calc. Var. Partial Differential Equations*, 58(86), 2019.
- [32] A. Garroni, G. Leoni, and M. Ponsiglione. Gradient theory for plasticity via homogenization of discrete dislocations. *J. Eur. Math. Soc. (JEMS)*, 12:1231–1266, 2010.
- [33] A. Garroni, R. Marziani, and R. Scala. Derivation of a line-tension model for dislocations from a nonlinear three-dimensional energy: The case of quadratic growth. *SIAM J. Math. Anal.*, 53(4):4252–4302, 2021.
- [34] M. Giaquinta, G. Modica, and J. Souček. *Cartesian Currents in the Calculus of Variations I*, volume 37 of *Ergebnisse der Mathematik und ihrer Grenzgebiete*. Springer-Verlag, Berlin-Heidelberg, 1998.

- [35] M. Giaquinta, G. Modica, and J. Souček. *Cartesian Currents in the Calculus of Variations II. Variational Integrals*, volume 38 of *Ergebnisse der Mathematik und ihrer Grenzgebiete*. Springer-Verlag, Berlin-Heidelberg, 1998.
- [36] G. Krantz and R. Parks. *Geometric Integration Theory*. Cornerstones. Birkhäuser Boston, Inc., Boston, MA, 2008.
- [37] A. Marchese, A. Massaccesi, S. Stuvard, and R. Tione. A multi-material transport problem with arbitrary marginals. *Calc. Var. Partial Differential Equations*, 60:88, 2021.
- [38] A. Marchese, A. Massaccesi, and R. Tione. A multimaterial transport problem and its convex relaxation via rectifiable G -currents. *SIAM J. Math. Anal.*, 51(3), 2019.
- [39] M. Ponsiglione. Elastic energy stored in a crystal induced by screw dislocations: from discrete to continuous. *SIAM J. Math. Anal.*, 39:449–469, 2007.
- [40] R. Scala. Optimal estimates for the triple junction function and other surprising aspects of the area functional. *Ann. Sc. Norm. Super. Pisa Cl. Sci.*, 20(2):491–564, 2020.
- [41] R. Scala and G. Scianna. On the L^1 -relaxed area of graphs of BV piecewise constant maps taking three values. *Adv. Calc. Var.*, 18(2):339–365, 2025.