On jump minimizing liftings for $\mathbb{S}^1$-valued maps and connections with Ambrosio-Tortorelli-type Γ-limits

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Abstract

This paper is concerned with the Γ-limits of Ambrosio-Tortorelli-type functionals, for maps u defined on an open bounded set $\Omega \subset \mathbb{R}^n$ and taking values in the unit circle $\mathbb{S}^1 \subset \mathbb{R}^2$. Depending on the domain of the functional, two different Γ-limits are possible, one of which is nonlocal, and related to the notion of jump minimizing lifting, i.e., a lifting of a map u whose measure of the jump set is minimal. The latter requires ad hoc compactness results for sequences of liftings which, besides being interesting by themselves, also allow to deduce existence of a jump minimizing lifting.

Key words: Jump minimizing liftings, Γ-convergence, $\mathbb{S}^1$-valued maps, free boundary problems.

AMS (MOS) subject classification: 49Q15, 49Q20, 49J45, 58E12.

1 Introduction

This paper is devoted to the asymptotic analysis, via Γ-convergence, of regularized free discontinuity functionals for maps defined on a connected bounded open set $\Omega \subset \mathbb{R}^n$ with Lipschitz boundary and taking values in the unit circle $\mathbb{S}^1 \subset \mathbb{R}^2$. More specifically, define the two functional domains

$$\begin{aligned} \widehat{\mathcal{A}}_{\mathbb{S}^1} &:= \big\{(u,v) \in W^{1,1}(\Omega;\mathbb{S}^1) \times W^{1,2}(\Omega) \colon v|\nabla u| \in L^2(\Omega),\ 0 \le v \le 1\big\} \subset L^1(\Omega;\mathbb{S}^1) \times L^1(\Omega), \\ \mathcal{A}_{\mathbb{S}^1} &:= \big\{(u,v) \in W^{1,2}(\Omega;\mathbb{S}^1) \times W^{1,2}(\Omega) \colon\ 0 \le v \le 1\big\} \subset L^1(\Omega;\mathbb{S}^1) \times L^1(\Omega). \end{aligned} \tag{1.1}$$

For $\varepsilon \in (0,1]$ let us consider the corresponding family of Ambrosio-Tortorelli-type functionals

$$\widehat{\mathrm{AT}}_\varepsilon^{\mathbb{S}^1}, \mathrm{AT}_\varepsilon^{\mathbb{S}^1} : L^1(\Omega;\mathbb{S}^1) \times L^1(\Omega) \to [0,+\infty]$$

given by

$$\widehat{\mathrm{AT}}_\varepsilon^{\mathbb{S}^1}(u,v) := \begin{cases} \mathrm{AT}_\varepsilon(u,v) & \text{if } (u,v) \in \widehat{\mathcal{A}}_{\mathbb{S}^1}, \\ +\infty & \text{otherwise in } L^1(\Omega;\mathbb{S}^1) \times L^1(\Omega), \end{cases} \tag{1.2}$$

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and

$$\mathrm{AT}^{\mathbb{S}^1}_\varepsilon(u,v) := \begin{cases} \mathrm{AT}_\varepsilon(u,v) & \text{if } (u,v)\in\mathcal{A}_{\mathbb{S}^1}, \\ +\infty & \text{otherwise in } L^1(\Omega;\mathbb{S}^1)\times L^1(\Omega), \end{cases} \tag{1.3}$$

where

$$\mathrm{AT}_\varepsilon(u,v) := \int_\Omega \left(v^2|\nabla u|^2 + \varepsilon|\nabla v|^2 + \frac{(v-1)^2}{4\varepsilon} \right) \mathrm{d}x, \tag{1.4}$$

with $|\nabla u|^2$ indicating the Frobenius norm of ∇u. It is clear that

$$\mathcal{A}_{\mathbb{S}^1} \subset \widehat{\mathcal{A}}_{\mathbb{S}^1}, \qquad \mathrm{AT}^{\mathbb{S}^1}_\varepsilon \geq \widehat{\mathrm{AT}}^{\mathbb{S}^1}_\varepsilon .$$

Our main results, connecting these functionals with Mumford-Shah-type functionals, read as follows.

Theorem 1.1 (Γ-convergence of $\widehat{\mathrm{AT}}^{\mathbb{S}^1}_\varepsilon$)**.** *Let $\Omega \subseteq \mathbb{R}^n$ be a connected bounded open set with Lipschitz boundary. We have*

$$\Gamma - L^1 \lim_{\epsilon\to 0^+} \widehat{\mathrm{AT}}^{\mathbb{S}^1}_\varepsilon = \mathrm{MS}_{\mathbb{S}^1},$$

where $\mathrm{MS}_{\mathbb{S}^1}\colon L^1(\Omega;\mathbb{S}^1)\times L^1(\Omega) \to [0,+\infty]$ *is given by*

$$\mathrm{MS}_{\mathbb{S}^1}(u,v) := \begin{cases} \displaystyle\int_\Omega |\nabla u|^2 \mathrm{d}x + \mathcal{H}^{n-1}(S_u) & \text{if } u\in SBV^2(\Omega;\mathbb{S}^1),\ v=1 \ a.e., \\ +\infty & \text{otherwise in } L^1(\Omega;\mathbb{S}^1)\times L^1(\Omega). \end{cases} \tag{1.5}$$

Theorem 1.2 (Γ-convergence of $\mathrm{AT}^{\mathbb{S}^1}_\varepsilon$)**.** *Let $\Omega \subseteq \mathbb{R}^n$ be a connected and simply-connected bounded open set with Lipschitz boundary. We have*

$$\Gamma - L^1 \lim_{\epsilon\to 0^+} \mathrm{AT}^{\mathbb{S}^1}_\varepsilon = \mathrm{MS}_{\mathrm{lift}},$$

where $\mathrm{MS}_{\mathrm{lift}}\colon L^1(\Omega;\mathbb{S}^1)\times L^1(\Omega) \to [0,+\infty]$ *is given by*

$$\mathrm{MS}_{\mathrm{lift}}(u,v) := \begin{cases} \displaystyle\int_\Omega |\nabla u|^2\, \mathrm{d}x + m_2[u] & \text{if } u\in SBV^2(\Omega;\mathbb{S}^1),\ v=1 \ a.e. \\ +\infty & \text{otherwise in } L^1(\Omega;\mathbb{S}^1)\times L^1(\Omega), \end{cases} \tag{1.6}$$

with

$$m_2[u] := \inf\{\mathcal{H}^{n-1}(S_\varphi)\colon \varphi\in GSBV^2(\Omega),\ e^{i\varphi}=u \ a.e.\ in\ \Omega\}. \tag{1.7}$$

In (1.5) and (1.7) the symbol $\mathcal{H}^{n-1}(S_\varphi)$ stands for the $(n-1)$-dimensional Hausdorff measure of the jump set S_φ of φ, a lifting[1] of u, and $SBV^2(\Omega;\mathbb{S}^1)$ (resp. $GSBV^2(\Omega)$) is the space of $\mathbb{S}^1$-valued maps with special bounded variation in Ω (resp. the space of generalized special bounded variation functions in Ω) whose absolutely continuous part of the gradient is square integrable and whose jump set has finite $(n-1)$-dimensional Hausdorff measure. To better understand the above results, some comments are in order.

(i) In Theorem 1.1 the Γ-limit in (1.5) is the classical Mumford-Shah functional for $\mathbb{S}^1$-valued maps, and indeed part of the proof is an adaptation of known results, together with some applications of the properties of liftings of nonsmooth maps;

[1] See Section 2.3.

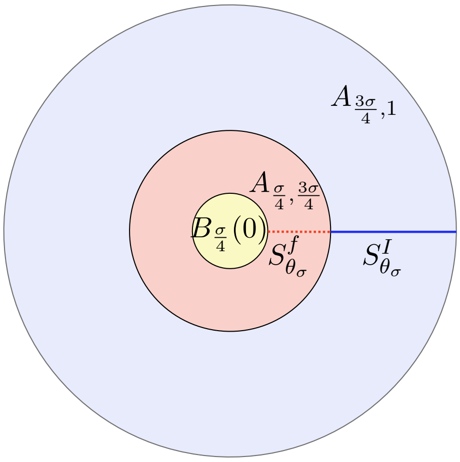

Figure 1: The dotted segment denotes the set $S^f_{\theta_\sigma}$ where $[\![\theta_\sigma]\!]$ varies between 0 and 2π, the continuous segment denotes the set $S^I_{\theta_\sigma}$ where $[\![\theta_\sigma]\!] = 2\pi$.

(ii) the second, more interesting, Γ-limit in Theorem 1.2 is nonlocal, it does not have an integral representation and depends on the minimization problem in (3.1). The singular term $m_2[u]$ is the penalization of a particular lifting of u, the one which minimizes the measure of the jump. It is clear that $\mathrm{MS}_{\mathrm{lift}} \geq \mathrm{MS}_{\mathbb{S}^1}$.

These two theorems show the huge difference made by the choice of the two domains (1.1) where defining the approximating functionals, in particular the requirements $u \in W^{1,1}(\Omega;\mathbb{S}^1)$, $v|\nabla u| \in L^2(\Omega)$, as opposite to the more standard requirement $u \in W^{1,2}(\Omega;\mathbb{S}^1)$. The reason of the corresponding different limit behaviours is topological in nature, and it is better explained by the following example, see also [26, pag. 30].

The example of the vortex map. Let $n = 2$, $\Omega = B_1(0) \subset \mathbb{R}^2$, and consider the vortex map $u_V(x) = \frac{x}{|x|}$ for any $x \in B_1(0) \setminus \{0\}$. Then $u_V \in W^{1,p}(B_1(0);\mathbb{S}^1)$ for any $p \in [1,2)$ and $u \notin W^{1,2}(B_1(0);\mathbb{S}^1)$. Thus in particular

$$\mathrm{MS}_{\mathbb{S}^1}(u_V, 1) = +\infty \qquad \text{since } u_V \notin SBV^2(\Omega;\mathbb{S}^1).$$

Also, any lifting of u_V jumps (in $B_1(0)$) at least on some curve connecting the origin to $\partial B_1(0)$. Now, let us modify u_V in a small neighbourhood of the origin to get a function which is in $SBV^2(\Omega;\mathbb{S}^1)$, by introducing a small jump. To this purpose consider the lifting of u_V given by the argument function[2] θ jumping on the positive real axis. Hence, in Ω, $S_\theta = (0,1) \times \{0\}$ with jump opening $[\![\theta]\!] = 2\pi$. Now, for $0 < r < R \leq 1$ define the annulus $A_{r,R} = B_R(0) \setminus \overline{B}_r(0)$, and let $\sigma \in (0,1)$. Consider the following perturbation of θ and u_V on $B_1(0) \setminus \{0\}$: let $\chi_\sigma \in C^\infty(B_\sigma(0),[0,1])$ be such that $\chi_\sigma \equiv 0$ in $B_{\sigma/4}(0)$, $\chi_\sigma \equiv 1$ in $A_{3/4\sigma,\sigma}$, $0 < \chi_\sigma < 1$ in $A_{\sigma/4,3\sigma/4}$, $|\nabla \chi_\sigma| \leq \frac{C}{\sigma}$ for some $C > 0$,

$$\theta_\sigma(x) := \begin{cases} \chi_\sigma(x)\theta(x) & \text{if } x \in B_\sigma(0) \setminus \{0\}, \\ \theta(x) & \text{if } x \in A_{\sigma,1}(0), \end{cases} \qquad u^{(\sigma)} := e^{i\theta_\sigma}.$$

Thus we have (see Figure 1)

[2] I.e., the imaginary part of the complex logarithm.

$$S_{\theta_\sigma} = S^f_{\theta_\sigma} \cup S^I_{\theta_\sigma} \quad \text{where} \quad S^f_{\theta_\sigma} = \Big(\frac{\sigma}{4}, \frac{3\sigma}{4}\Big) \times \{0\}\,, \quad S^I_{\theta_\sigma} = \Big(\frac{3\sigma}{4}, 1\Big) \times \{0\}\,,$$

moreover $[\![\theta_\sigma]\!]$ varies from 0 to 2π on $S^f_{\theta_\sigma}$ and $[\![\theta_\sigma]\!] \equiv 2\pi$ on $S^I_{\theta_\sigma}$. This in turn implies $u^{(\sigma)} = u_V$ on $A_{3\sigma/4,1}$ and $S_{u^{(\sigma)}} = S^f_{\theta_\sigma}$. We also have $u^{(\sigma)} \in SBV^2(B_1(0);\mathbb{S}^1)$, and $\mathrm{MS}_{\mathbb{S}^1}(u^{(\sigma)}, 1) < +\infty$, and $m_2[u^{(\sigma)}]$ turns out to be the length of the shortest segment joining the right extremum of $S_{u^{(\sigma)}}$ to the boundary of Ω. Indeed any lifting φ of $u^{(\sigma)}$ has the following properties: $S_\varphi \supset S^f_{\theta_\sigma}$, φ is a lifting of u_V on $A_{3\sigma/4,1}$ and φ is a lifting of $(1,0)$ on $B_\sigma(0)$. Hence, in order to be minimal it must be $S_\varphi = S_{\theta_\sigma}$ and therefore $m_2[u^{(\sigma)}] = \mathcal{H}^1(S_{\theta_\sigma}) = 1 - \sigma/4$.

The map $u^{(\sigma)}$ is one of the possible maps on which one may compute the two Γ-limits above. It turns out that the gradient integrability of the approximating maps is crucial to devise the two behaviours. In particular in Theorem 1.2, due to topological obstructions related to the fact that $u^{(\sigma)}$ has non-zero degree on any circle $\partial B_\rho(0)$ with $3\sigma/4 \le \rho < 1$ (as it coincides with the vortex map outside a small neighbourhood of the origin), the recovery sequence gives rise to the contribution $m_2[u^{(\sigma)}]$. Indeed, a naive approach to get the upper bound would be to adapt the classical construction given in [7, 8]. More precisely, the latter consists in regularising $u^{(\sigma)}$ in a neighbourhood with width $\ll \varepsilon$ of its jump set $S_{u^{(\sigma)}}$, being careful to keep values on the unit circle $\mathbb{S}^1$, and then define v_ε accordingly. However this leads to an approximating function $u_\varepsilon\colon B_1(0) \to \mathbb{S}^1$ which coincides with the vortex map far from the origin, and so necessarily u_ε has non-zero degree, which in turn implies[3] that $u_\varepsilon \notin W^{1,2}(B_1(0), \mathbb{S}^1)$. In particular, u_ε cannot be used to produce a recovery sequence in Theorem 1.2, but it provides a suitable construction for the upper bound in Theorem 1.1, being of class $W^{1,1}(B_1(0);\mathbb{S}^1)$ and satisfying $v_\varepsilon|\nabla u_\varepsilon| \in L^2(B_1(0))$. In order to construct a recovery sequence in the second case it is needed to consider approximating functions with zero degree. This is possible by modifying $u^{(\sigma)}$ in a neighbourhood of the jump set of a minimal lifting (for example θ_σ) and then define v_ε accordingly. Thus, passing to the limit one gets the contribution $\mathcal{H}^1(S_{\theta_\sigma}) = m_2[u^{(\sigma)}]$.

Main challenges of the proofs. From the previous example we learned that the nature of the Γ-limit (1.6) is related to topological obstructions when approximating a $\mathbb{S}^1$-valued map whose approximate gradient is square integrable. In particular, the non locality of $\mathrm{MS}_{\mathrm{lift}}$ leads to non trivial difficulties in the proof of the lower bound inequality. The basic idea consists in rewriting the approximating functionals in terms of liftings. Namely, given $u \in W^{1,2}(\Omega;\mathbb{S}^1)$ take a lifting $\varphi \in W^{1,2}(\Omega)$ of u, so that from the identity $|\nabla u| = |\nabla \varphi|$ it follows

$$\mathrm{AT}_\varepsilon(u, v) = \int_\Omega \Big(v^2|\nabla\varphi|^2 + \varepsilon|\nabla v|^2 + \frac{(v-1)^2}{4\varepsilon} \Big)\,\mathrm{d}x\,, \tag{1.8}$$

which is the Ambrosio-Tortorelli functional for the pair (φ, v). Now, if for any sequence $(u_{\varepsilon_k}, v_{\varepsilon_k}) \to (u, 1)$ in $L^1(\Omega;\mathbb{S}^1) \times L^1(\Omega)$ we would be able to prove that $\varphi_{\varepsilon_k} \to \varphi$ in $L^1(\Omega)$, with $\varphi_{\varepsilon_k} \in W^{1,2}(\Omega)$ lifting of u_{ε_k}, we could conclude by applying the classical results [7, 8], since the limit φ is a lifting of u. Unfortunately this is in general not true, as the energy provides a control of $\nabla\varphi_{\varepsilon_k}$ only in regions where v_{ε_k} is far from zero. Moreover a sequence of liftings might escape to infinity (since we can always add integer multiples of 2π). For this reason, the main issue concerning the lower bound in Theorem 1.2 is to get a compactness result on sequences of liftings. This leads us to another main result

[3]Indeed, if $u_\varepsilon \in W^{1,2}(B_1(0), \mathbb{S}^1)$, then by [15, Theorem 1.1] u_ε has a lifting in $W^{1,2}(B_1(0))$, which is not possible since u_ε has non zero degree (cf. [15, Theorem 1.2]).

contained in Theorems 3.1 and 3.3, which provides compactness and lower semi-continuity for a sequence of liftings associated to a single map u and more generally, to a converging sequence (u_{ε_k}). We stress that these compactness results have a local feature, in the sense that liftings converge up to locally subtract a suitable constant. Similar compactness results obtained by slight modifying the given sequence of functions can be found in [30] (see also [19] where the compactness result in $GSBD^p$ is used in the proof of our result). Theorems 3.1 and 3.3, besides being interesting by themselves, allow us, together with a refined local argument, to show the lower bound inequality in Theorem 1.2. Furthermore Theorem 3.1 provides existence of a solution to (1.7), that is, the infimum in (1.7) is actually a minimum. More precisely, in Corollary 3.2, we show that the following more general minimum problem has a solution. Let $p > 1$ and $u \in SBV^p(\Omega; \mathbb{S}^1)$; then there exists a lifting $\varphi_{\min} \in GSBV^p(\Omega)$ of u such that

$$\mathcal{H}^{n-1}(S_{\varphi_{\min}}) = \inf \left\{ \mathcal{H}^{n-1}(S_\varphi) : \varphi \in GSBV^p(\Omega),\ e^{i\varphi} = u \text{ a.e. in } \Omega \right\} . \tag{1.9}$$

It is worth to notice that, somehow surprisingly, the minimum in (1.7) is not attained in the class $SBV(\Omega)$, as a consequence of an example discussed in Section 3.1. This shows that the analysis of (1.7) is rather delicate.

The problem of finding a lifting minimizing the length of the jump set somehow resembles the related question of finding a lifting minimizing its BV-seminorm [15]. As for the latter, which is strongly linked with Plateau and optimal transport problems [16], the structure of a jump minimizing lifting might be related to optimal transport questions with different cost functions. We discuss in more details this issue in Subsection 3.2.

Further directions and open problems. Lifting theory is useful in several, apparently unrelated, contexts where *topological singularities* arise, such as screw dislocations in crystals, vortices in superconductors, the non-parametric Plateau problem in codimension-two and optimal transport problems. This shows similarities between different problems to which a suitable version of the Ambrosio-Tortorelli approximations for $\mathbb{S}^1$-valued maps might be addressed. In the following we briefly review some open problems which will be investigated in future works.

Dislocations and vortices. Dislocations are line-defects in metals that locally alter the crystalline structure and are the main source of plastic slips. From a mathematical point of view they can be identified with codimension-two singularities. In particular, in a simplified framework one can consider two-dimensional semi-discrete models for dislocations either of screw or edge type, which correspond to point singularities in a two-dimensional domain (see [4, 23, 25, 26, 31, 37] and references therein). It is well known that models for screw dislocations share similarities with Ginzburg Landau models for superconductors [2]. In particular, in recent works [23, 26] it was proposed a free-discontinuity model for screw dislocations. The latter is given by an energy of Mumford-Shah type for $\mathbb{S}^1$-valued maps which penalizes the measure of the jump set. From the analysis pursued in [23, 26] it turns out that such a model is equivalent (to the leading order) to the Ginzburg Landau one. This seems to suggest that an Ambrosio Tortorelli approximation for screw dislocations, as well as for vortices in superconductors, is possible. In addition, this offers the possibility to extend the results of [23, 26] to the three-dimensional setting (see [21, 22, 32] for models in dimension three).

The Plateau and optimal transport problems. The Cartesian Plateau problem consists in finding an area-minimizing surface among all Cartesian surfaces spanning an appropriate Jordan space curve. While the codimension-one setting has been exhaustively understood

[34] very few is known in codimension two. This has to do with the fact that the relaxed Cartesian area in codimension larger than one is not subadditive, and thus has no integral representation. A couple of examples in codimension two where the computation of the relaxed graph area is possible are the vortex map u_V and the triple junction function [1]. Both are $\mathbb{S}^1$-valued maps with singularities, and the corresponding relaxed area has a nonlocal feature involving liftings, similarly as in (1.6) (cf. [9, 10, 12, 38] and references therein). Also, the singular contribution appearing in the case of $\mathbb{S}^1$-valued maps is related with optimal transport problems with different cost functions. In the specific case of $\mathbb{S}^1$-valued Sobolev maps, the singularities should be optimally connected each other, in a similar fashion as for the minimal connection of dipoles in the minimization of the BV-seminorm of liftings [16] (see also [13, 39]). Connecting the singularities in the relaxed Cartesian area is related with finding an optimal Cartesian vertical current filling the holes of the graph of the given map u, and this often requires the study of a Plateau problem in codimension-one with partial free boundary [11].

In the simplified setting of $\mathbb{S}^1$-constrained relaxation [33, pg. 611 eq. 4], the optimal value of the graph area has a singular contribution characterized in terms of $\mathbb{S}^1$ vertical Cartesian currents [33, pg. 612, Theorem 1] and that is again related to an optimal transport question on how to connect the singularities of u. A future development of the present research is to approximate the relaxed area functional on $\mathbb{S}^1$-valued functions by an Ambrosio-Tortorelli energy where the bulk term has linear growth in the gradient (in the same spirit of [3]).

We emphasize that phase-field models based on the Ambrosio-Tortorelli functionals have been already employed to study problems related with optimal transport questions (see e.g. [27] and references therin, and [20] for the relation with the steiner tree problem). While the link of Corollary 3.2 with the optimal transport question involves the cost function $\psi \equiv 1$ (see Subsection 3.2), we believe that the analysis of the Ambrosio-Tortorelli functional for $\mathbb{S}^1$-valued maps with linear growth (instead of quadratic) would give rise to a minimization question similar to (1.9) involving different cost functions. Also this will be object of future development.

Content of the paper. In Section 2 we collect some notation and recall some useful tools which will be employed to prove the main results. In Section 3 we provide an example where the jump minimizing lifting is not in $SBV(\Omega)$ but just in $GSBV(\Omega)$. Furthermore, we describe a connection with optimal transport, we state the two compactness results Theorem 3.1, Theorem 3.3 and the existence of a minimizer to (1.7), Corollary 3.2. In Section 4 we provide the proofs of Theorems 3.1 and 3.3. Eventually in Section 5, after establishing some density and approximation results in $SBV(\Omega)$, both for $\mathbb{S}^1$-valued functions and for liftings, we prove Theorem 1.1; in the proof we do not make use of the results of Section 5.1, but we utilize the results of Section 3. Finally, Section 5.3 is devoted to the proof of Theorem 1.2.

2 Notation and preliminaries

In this section we collect some notation, and recall some notions concerning SBV and $GSBV$ functions [6] and lifting theory [15]. In what follows:

- $n \geq 1$ is a fixed integer and $p > 1$ is a fixed real number;

- $\partial^* A$ denotes the reduced boundary of finite perimeter set $A \subset \mathbb{R}^n$;

- $|\cdot|$ and $\mathcal{H}^{n-1}$ denote the Lebesgue measure and the $(n-1)$-dimensional Hausdorff measure in $\mathbb{R}^n$, respectively;
- χ_A denotes the characteristic function of the set $A \subset \mathbb{R}^n$;
- for $a, b \in \mathbb{R}^n$ the symbol $a \otimes b$ denotes the tensor product between a and b;
- $\mathbb{S}^1 := \{(x,y) \in \mathbb{R}^2 \colon x^2 + y^2 = 1\}$ is the unit circle in $\mathbb{R}^2$.

2.1 SBV and $GSBV$ functions

Let $U \subset \mathbb{R}^n$ be open and bounded, and $m \geq 1$ an integer. We denote by $BV(U;\mathbb{R}^m)$ the space of vector-valued *functions with bounded variation in* U, and with $|\cdot|_{BV}$ and $\|\cdot\|_{BV}$ the BV seminorm and norm, respectively, i.e.

$$|u|_{BV} := |Du|(U), \qquad \|u\|_{BV} := \|u\|_{L^1} + |u|_{BV},$$

see [6]. We say that $u \in BV(U;\mathbb{R}^m)$ belongs to the space of *special functions with bounded variation in* U , i.e., $u \in SBV(U;\mathbb{R}^m)$, if its distributional gradient is a finite $\mathbb{R}^{m\times n}$-valued Radon measure without Cantor part, that is,

$$Du = \nabla u \mathcal{L}^n + [\![u]\!] \otimes \nu_u \mathcal{H}^{n-1} \llcorner S_u \,,$$

where ∇u is the approximate gradient of u, S_u is the approximate jump set of u, $[\![u]\!] = u^+ - u^-$ is the jump opening and ν_u is the unit normal, see [6, Def. 3.67] to S_u. A measurable function $u : U \to \mathbb{R}^m$ belongs to the space of *generalised special functions with bounded variation in* U, that is, $u \in GSBV(U;\mathbb{R}^m)$, if $\phi \circ u \in SBV_{\mathrm{loc}}(U)$ for any $\phi \in C^1(\mathbb{R}^m)$ with $\nabla\phi$ compactly supported.[4] If $m = 1$ we write $BV(U) = BV(U;\mathbb{R})$, $SBV(U) = SBV(U;\mathbb{R})$ and $GSBV(U) = GSBV(U;\mathbb{R})$.

Remark 2.1 (Equivalent definition of $GSBV$ for $m = 1$). $GSBV(U)$ can be equivalently defined as the space of measurable functions $u \colon U \to \mathbb{R}$ such that $u \wedge M \vee (-M) \in SBV_{\mathrm{loc}}(U)$ for any $M > 0$.

For $p > 1$ we set

$$SBV^p(U;\mathbb{R}^m) = \{u \in SBV(U;\mathbb{R}^m) : \nabla u \in L^p(U;\mathbb{R}^{m\times n}) \text{ and } \mathcal{H}^{n-1}(S_u) < +\infty\},$$

and

$$GSBV^p(U;\mathbb{R}^m) = \{u \in GSBV(U;\mathbb{R}^m) : \nabla u \in L^p(U;\mathbb{R}^{m\times n}) \text{ and } \mathcal{H}^{n-1}(S_u) < +\infty\}.$$

Again, when $m = 1$ we write $SBV^p(U) = SBV^p(U;\mathbb{R}^m)$ and $GSBV^p(U) = GSBV^p(U;\mathbb{R}^m)$. We set

$$BV(U;\mathbb{S}^1) = \{u \in BV(U;\mathbb{R}^2) \colon |u| = 1 \text{ a.e. in } U\}\,,$$
$$SBV(U;\mathbb{S}^1) = \{u \in SBV(U;\mathbb{R}^2) \colon |u| = 1 \text{ a.e. in } \Omega\}\,,$$

and for $p > 1$

$$SBV^p(U;\mathbb{S}^1) = \{u \in SBV(U;\mathbb{S}^1) \colon \nabla u \in L^p(U;\mathbb{R}^{2\times 2}),\, \mathcal{H}^{n-1}(S_u) < +\infty\}\,.$$

Eventually, a (finite or countable) family (E_i) of finite perimeter subsets of a finite perimeter set F is called a Caccioppoli partition of F if the sets E_i are pairwise disjoint, and their union is F. The next technical observation will be needed later (Section 5.2).

[4]Recall that $f \in SBV_{\mathrm{loc}}(U)$ if $f \in SBV(K)$ for every $K \subset U$ compact.

Remark 2.2 (Approximation of a BV function by smooth functions). Let $A \subset \mathbb{R}^n$ be a bounded open set with Lipschitz boundary and $\varphi \in BV(A)$. By [6, Theorem 3.9], for any $\delta > 0$, there exists a function $\varphi_\delta \in C^\infty(A)$ such that

$$\int_A |\varphi - \varphi_\delta| \, \mathrm{d}x < \delta \,, \quad \int_A |\nabla \varphi_\delta| \, \mathrm{d}x \leq |D\varphi|(A) + \delta \,.$$

Moreover, by inspecting the proof,

$$\varphi = \varphi_\delta \quad \mathcal{H}^{n-1} - \text{a.e. on } \partial A \,,$$

in the sense of BV-traces on ∂A (see e.g. [6, page 181]). To see this last property we recall that

$$\varphi_\delta := \sum_{h \geq 1} (\varphi \psi_h) * \rho_h \,,$$

where

- $(\psi_h)_{h\geq 1}$ is a partition of unity relative to the covering $(A_h)_{h\geq 1}$ of A defined as

$$A_1 := \{x \in A \colon \text{ dist } (x, \partial A) > 2^{-1}\} \,,$$

$$A_h := \{x \in A \colon (h+1)^{-1} < \text{ dist } (x, \partial A) < (h-1)^{-1}\} \quad \text{ for } h \geq 2 \,;$$

- $(\rho_h)_{h\geq 1}$ is a family of mollifiers such that

$$\text{supp } ((\varphi \psi_h) * \rho_h) \subset A_h \,,$$

$$\int_A \Big[|(\varphi \psi_h) * \rho_h - \varphi \psi_h| + |(\varphi \nabla \psi_h) * \rho_h - \varphi \nabla \psi_h| \Big] \, \mathrm{d}x < 2^{-h} \delta \,.$$

Thus, for $\mathcal{H}^{n-1}$-a.e. $x_0 \in \partial A$, we have

$$\begin{aligned} |\varphi(x_0) - \varphi_\delta(x_0)| &\leq \lim_{r \searrow 0} \frac{2}{\omega_n} \frac{1}{r^n} \int_{A \cap B_r(x_0)} |\varphi(y) - \varphi_\delta(y)| \, \mathrm{d}y \\ &\leq \lim_{r \searrow 0} \frac{2}{\omega_n} \frac{1}{r^n} \int_{A \cap B_r(x_0)} \sum_{h \geq 1} \Big| \varphi(y) \psi_h(y) - (\varphi \psi_h) * \rho_h(y) \Big| \, \mathrm{d}y \,. \end{aligned} \tag{2.1}$$

Now we observe that $A_h \cap B_r(x_0) \neq \emptyset$ if and only if $\frac{1}{h+1} < r$, i.e., $h > \frac{1}{r} - 1$. Hence for $y \in B_r(x_0)$, the sum on the right hand side of (2.1) reduces to

$$\sum_{h \geq \frac{1}{r} - 1} \Big| \varphi(y) \psi_h(y) - (\varphi \psi_h) * \rho_h(y) \Big| \leq \sum_{h \geq \frac{1}{r} - 1} 2^{-h} \delta = 2^{-\frac{1}{r} + 2} \delta \,.$$

We conclude

$$|\varphi(x_0) - \varphi_\delta(x_0)| \leq \lim_{r \searrow 0} \frac{2}{\omega_n} \frac{1}{r^n} 2^{-\frac{1}{r} + 2} \delta r^n = 0 \,.$$

2.2 Γ-approximation and compactness for the Mumford-Shah functional

We recall two classical results [7,8] (see also [28] for the generalization to the vector case), and [6, Theorem 4.8].

Theorem 2.3 (**Ambrosio-Tortorelli**). *Let* $\Omega \subset \mathbb{R}^n$ *be an open set with Lipschitz boundary, and*

$$\mathcal{A} := \{(u,v) \in W^{1,2}(\Omega;\mathbb{R}^m) \times W^{1,2}(\Omega) \colon 0 \le v \le 1\}.$$

Then the functionals

$$\mathrm{AT}_\varepsilon(u,v) := \begin{cases} \displaystyle\int_\Omega \left(v^2 |\nabla u|^2 + \varepsilon |\nabla v|^2 + \frac{(v-1)^2}{4\varepsilon} \right) \mathrm{d}x & \text{if } (u,v) \in \mathcal{A}\,, \\ +\infty & \text{otherwise in } L^1(\Omega;\mathbb{R}^m) \times L^1(\Omega)\,, \end{cases}$$

$\Gamma - L^1$*-converge to the Mumford-Shah functional*

$$\mathrm{MS}(u,v) := \begin{cases} \displaystyle\int_\Omega |\nabla u|^2 \,\mathrm{d}x + \mathcal{H}^{n-1}(S_u) & \text{if } u \in GSBV^2(\Omega;\mathbb{R}^m),\ v = 1 \text{ a.e.} \\ +\infty & \text{otherwise in } L^1(\Omega;\mathbb{R}^m) \times L^1(\Omega)\,, \end{cases}$$

as $\varepsilon \to 0^+$.

Remark 2.4. By inspecting the proof of Theorem 2.3 one actually deduces the following properties:

(i) **Decoupling.** Let $\varepsilon_k \searrow 0$, and let $((u_k, v_k))_{k \ge 1} \subset \mathcal{A}$ be a sequence converging to $(u,1)$ in $L^1(\Omega;\mathbb{R}^m) \times L^1(\Omega)$ with $\sup_{k \in \mathbb{N}} \mathrm{AT}_{\varepsilon_k}(u_k, v_k) < +\infty$. Then

$$\begin{aligned} \liminf_{k \to +\infty} \int_\Omega v_k^2 |\nabla u_k|^2 \,\mathrm{d}x &\ge \int_\Omega |\nabla u|^2 \,\mathrm{d}x\,, \\ \liminf_{k \to +\infty} \int_\Omega \left(\varepsilon_k |\nabla v_k|^2 + \frac{(v_k - 1)^2}{4\varepsilon_k} \right) \mathrm{d}x &\ge \mathcal{H}^{n-1}(S_u)\,. \end{aligned} \tag{2.2}$$

(ii) **Γ-convergence on a larger domain.** The result still holds if one replaces $\mathcal{A}$ with the larger class

$$\widehat{\mathcal{A}} := \{(u,v) \in W^{1,1}(\Omega;\mathbb{R}^m) \times W^{1,2}(\Omega) \colon v|\nabla u| \in L^2(\Omega),\ 0 \le v \le 1\}.$$

Theorem 2.5 (**Compactness in** SBV). *Let* $p > 1$, $\Omega \subset \mathbb{R}^n$ *be a bounded open set and* $(\varphi_k)_{k \ge 1} \subset SBV(\Omega)$ *be a sequence satisfying*

$$\sup_{k \in \mathbb{N}} \Big(\int_\Omega |\nabla \varphi_k|^p \,\mathrm{d}x + \mathcal{H}^{n-1}(S_{\varphi_k}) \Big) < +\infty,$$

and

$$\sup_{k \in \mathbb{N}} \|\varphi_k\|_\infty < +\infty. \tag{2.3}$$

Then there exists a subsequence of (φ_k) *weakly-star converging in* $BV(\Omega)$ *to a function belonging to* $SBV(\Omega)$.

Removing assumption (2.3) leads to the following result, which is the $GSBV^p$ variant of [19, Theorem 1.1], and that will be applied in the proof of Lemma 4.1 to an appropriate sequence of liftings.

Theorem 2.6 (Compactness). *Let $p > 1$, $\Omega \subset \mathbb{R}^n$ be an open set and $(\phi_k)_{k\geq 1} \subset GSBV^p(\Omega)$ be a sequence satisfying*

$$\sup_{k\in\mathbb{N}} \Big(\int_\Omega |\nabla \phi_k|^p \, \mathrm{d}x + \mathcal{H}^{n-1}(S_{\phi_k}) \Big) < +\infty.$$

Then there exist a (not-relabelled) subsequence and a function $\varphi_\infty \in GSBV^p(\Omega)$ such that:

$$\begin{aligned}
&E := \{x \in \Omega : |\phi_k(x)| \to +\infty\} \text{ has finite perimeter},\\
&\varphi_\infty = 0 \text{ on } E,\\
&\phi_k \to \varphi_\infty \text{ a.e. on } \Omega \setminus E\,,\\
&\nabla\phi_k \to \nabla\varphi_\infty \text{ in } L^1(\Omega\setminus E;\mathbb{R}^n)\,,\\
&\mathcal{H}^{n-1}(S_{\varphi_\infty} \cup \partial^* E) \leq \liminf_{k\to+\infty} \mathcal{H}^{n-1}(S_{\phi_k}).
\end{aligned}$$

2.3 Liftings of $\mathbb{S}^1$-valued maps

Let $\Omega \subset \mathbb{R}^n$ be a bounded connected and simply connected open set with Lipschitz boundary, and $u\colon \Omega \to \mathbb{S}^1$ be a measurable function. A *lifting* of u is a measurable function $\varphi\colon \Omega \to \mathbb{R}$ such that

$$u(x) = e^{i\varphi(x)} \quad \text{for a.e. } x \in \Omega\,.$$

Given a Borel set $B \subseteq \Omega$, we say that $\varphi : B \to \mathbb{R}$ is a lifting of u on B if the previous equality holds a.e. on B. Clearly, if φ is a lifting of u, then so is $\varphi + 2\pi m$ for all $m \in \mathbb{Z}$.

If u has some regularity, a natural question is whether φ can be chosen with the same regularity. The answer is partially positive, see [15, 24] for more details:

(1) If $u \in C^k(\overline{\Omega};\mathbb{S}^1)$ for some $k \geq 0$, then u has a lifting $\varphi \in C^k(\overline{\Omega})$, unique (mod. 2π), [15, Lemma 1.1];

(2) If $u \in C^k(\overline{\Omega};\mathbb{S}^1) \cap W^{1,p}(\Omega;\mathbb{S}^1)$ for some $p \in [1,+\infty]$, then u has a lifting $\varphi \in C^k(\overline{\Omega}) \cap W^{1,p}(\Omega)$; If $n = 1$ and $u \in W^{1,p}(\Omega;\mathbb{S}^1)$ for some $p \in [1,+\infty)$, then u has a lifting $\varphi \in W^{1,p}(\Omega)$;

(3) If $n \geq 2$ and $u \in W^{1,p}(\Omega;\mathbb{S}^1)$ for some $p \in [2,+\infty)$, then u has a lifting $\varphi \in W^{1,p}(\Omega)$. Moreover φ is unique (mod 2π);

(4) If $n \geq 2$, then for every $p \in [1,2)$ there exists $u \in W^{1,p}(\Omega;\mathbb{S}^1)$ for which there are no liftings $\varphi \in W^{1,p}(\Omega)$, see [15, Theorem 1.2, Remark 1.9].

A well-known example of property (4) when $n = 2$ is the vortex map u_V discussed in the introduction. Indeed it can be shown [15, Pages 17-19] that there are no liftings of u in $W^{1,1}(B_1)$ (and thus there are no liftings in $W^{1,p}(B_1)$ for all $p \in [1,2)$).

Next we recall some regularity results on lifting of BV maps.

Theorem 2.7 (Davila-Ignat). *Let $u \in BV(\Omega;\mathbb{S}^1)$. Then there exists a lifting $\varphi \in BV(\Omega)$ such that $\|\varphi\|_{L^\infty} \leq 2\pi$ and $|\varphi|_{BV} \leq 2|u|_{BV}$.*

Proof. See [24, Theorem 1.1], and also [15, Theorem 1.4]. □

Remark 2.8. If $u \in SBV(\Omega; \mathbb{S}^1)$ then φ of Theorem 2.7 can be chosen in $SBV(\Omega)$. If $u \in SBV^p(\Omega; \mathbb{S}^1)$ for some $p > 1$ then φ of Theorem 2.7 can be chosen in $SBV^p(\Omega) \cap L^\infty(\Omega; [0, 2\pi])$.

As usual, for any lifting $\varphi \in SBV(\Omega)$ of u we write[5] $S_\varphi = S^I_\varphi \cup S^f_\varphi$ where

$$S^I_\varphi := \{x \in S_\varphi : [\![\varphi]\!](x) \in 2\pi\mathbb{Z}\}, \qquad S^f_\varphi := S_\varphi \setminus S^I_\varphi.$$

Notice that in particular $S^f_\varphi = S_u$.

Let $u \in BV(\Omega; \mathbb{S}^1)$ and consider the minimum problem

$$\inf\{|\varphi|_{BV} : \varphi \in BV(\Omega),\ e^{i\varphi} = u \text{ a.e. in } \Omega\}.$$

Then there exists a minimizer $\varphi \in BV(\Omega)$ such that $|\varphi|_{BV} \leq 2|u|_{BV}$ and $0 \leq \int_\Omega \varphi \, dx \leq 2\pi|\Omega|$ [15, page 25]. As explained in the introduction, we are instead concerned with the existence of a lifting which minimizes the measure of the jump set. This will be the argument of the next section.

3 On jump minimizing liftings

In view of the applications in Section 5, we are concerned with the analysis of the following minimization problem, which has an independent interest: let $p > 1$, let $u \in SBV^p(\Omega; \mathbb{S}^1)$ and define

$$m_p[u] := \inf\left\{\mathcal{H}^{n-1}(S_\varphi) : \varphi \in GSBV^p(\Omega),\ e^{i\varphi} = u \text{ a.e. in } \Omega\right\}. \tag{3.1}$$

We observe that, in (3.1), the set of φ between parentheses is non empty, due to Remark 2.8. The reason of utilizing the space $GSBV^p(\Omega)$ instead that $SBV^p(\Omega)$ can be understood from the following example.

3.1 An example

In this section we will show that, in general, a lifting minimizing the right-hand side of (3.1) cannot be found in $SBV^p(\Omega)$, for any $p \geq 1$. The strategy consists in constructing a real-valued function φ such that, letting $u := e^{i\varphi}$, the following hold:

- $\varphi \in GSBV^p(\Omega) \setminus SBV(\Omega)$;
- $S_\varphi = S_u$;
- $\Omega \setminus S_\varphi$ is arcwise connected.

In this way φ is the only minimizer (modulo addition of a constant in $2\pi\mathbb{Z}$) of (3.1) for u.

[5] S^I_φ stands for the "integer" part of the jump, and S^f_φ for the "fractional" part.

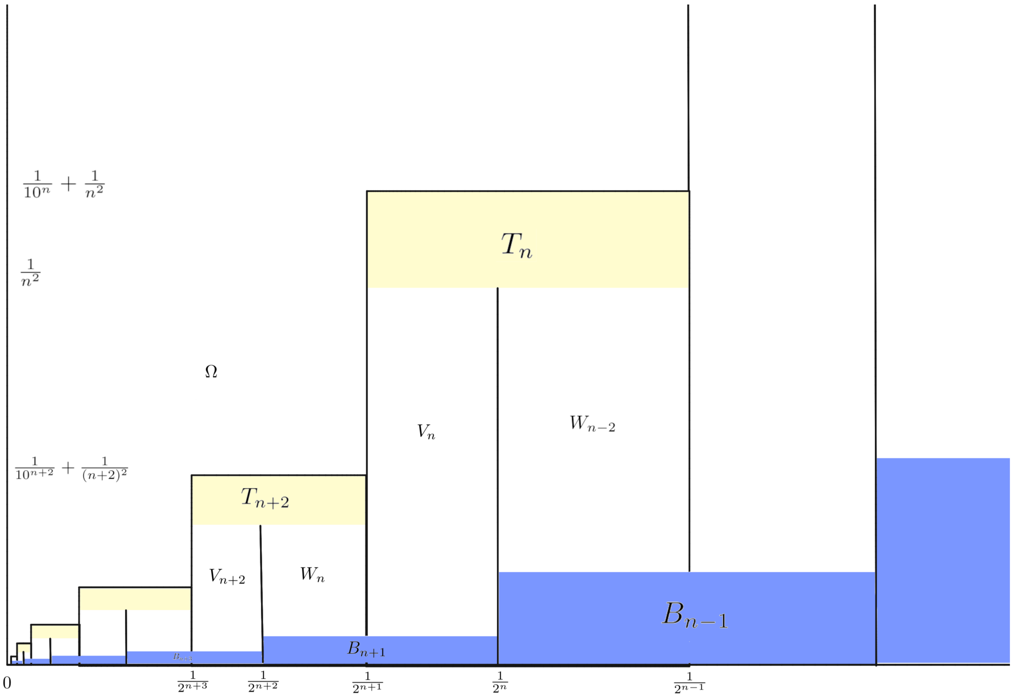

Figure 2: The rectangles T_n and B_n in a portion of $\Omega = (0,1)^2$.

We consider $\Omega := (0,1)^2 \subset \mathbb{R}^2$, and three sequences (R_n), (T_n), (B_n) of open rectangles contained in Ω defined, for an even integer $n \geq 2$, as follows:

$$R_n \text{ has vertices } \left(\frac{1}{2^{n+1}}, 0\right), \left(\frac{1}{2^{n-1}}, 0\right), \left(\frac{1}{2^{n-1}}, \frac{1}{10^n} + \frac{1}{n^2}\right), \left(\frac{1}{2^{n+1}}, \frac{1}{10^n} + \frac{1}{n^2}\right);$$

$$T_n \text{ has vertices } \left(\frac{1}{2^{n+1}}, \frac{1}{n^2}\right), \left(\frac{1}{2^{n-1}}, \frac{1}{n^2}\right), \left(\frac{1}{2^{n-1}}, \frac{1}{10^n} + \frac{1}{n^2}\right), \left(\frac{1}{2^{n+1}}, \frac{1}{10^n} + \frac{1}{n^2}\right);$$

$$B_{n+1} \text{ has vertices } \left(\frac{1}{2^{n+2}}, 0\right), \left(\frac{1}{2^n}, 0\right), \left(\frac{1}{2^n}, \frac{1}{10^{n+1}}\right), \left(\frac{1}{2^{n+2}}, \frac{1}{10^{n+1}}\right).$$

Observe that the R_n's are pairwise disjoint and the closures $\overline{T}_n$ of T_n are pairwise disjoint. Also the B_n's are pairwise disjoint, but their boundaries share some part of the lateral edges (see Fig. 2). Furthermore, for all $n, m \geq 2$ even,

$$T_n \subset R_n, \quad B_{n+1} \subset R_n \cup R_{n+2}, \quad \overline{T}_n \cap \overline{B}_m = \varnothing.$$

Now, we define a map[6] $\varphi : \Omega \to \mathbb{R}$

$$\varphi := \begin{cases} n^2 & \text{in } T_n \text{ for } n \geq 2 \text{ even}, \\ (n+1)^2 & \text{in } B_{n+1} \text{ for } n \geq 2 \text{ even}, \\ 4 & \text{in } \Omega \setminus \bigcup_{\substack{n \geq 2 \\ n \text{ even}}} R_n. \end{cases} \tag{3.2}$$

It remains to define φ on

$$\bigcup_{\substack{n \geq 2 \\ n \text{ even}}} R_n \setminus \Big(\bigcup_{\substack{n \geq 2 \\ n \text{ even}}} (T_n \cup B_{n+1}) \Big).$$

[6] The value 4 has not particular meaning: any positive constant could be chosen as well. A similar comment applies to (3.3)

To this purpose we set

$$\varphi := 4 \text{ in the open rectangle with vertices } \Big(\frac{1}{4}, 0\Big), \ \Big(\frac{1}{2}, 0\Big), \ \Big(\frac{1}{2}, \frac{1}{4} + \frac{1}{10^2}\Big), \ \Big(\frac{1}{4}, \frac{1}{4} + \frac{1}{10^2}\Big). \tag{3.3}$$

Further, we consider two sequences (V_n), (W_n) of open rectangles, defined for any even $n \geq 2$, as follows:

$$V_n \text{ has vertices } \Big(\frac{1}{2^{n+1}}, \frac{1}{10^{n+1}}\Big), \ \Big(\frac{1}{2^n}, \frac{1}{10^{n+1}}\Big), \ \Big(\frac{1}{2^n}, \frac{1}{n^2}\Big), \ \Big(\frac{1}{2^{n+1}}, \frac{1}{n^2}\Big),$$

$$W_n \text{ has vertices } \Big(\frac{1}{2^{n+2}}, \frac{1}{10^{n+1}}\Big), \ \Big(\frac{1}{2^{n+1}}, \frac{1}{10^{n+1}}\Big), \ \Big(\frac{1}{2^{n+1}}, \frac{1}{(n+2)^2}\Big), \ \Big(\frac{1}{2^{n+2}}, \frac{1}{(n+2)^2}\Big),$$

see Fig. 2.

By (3.2) the value of φ on the top edge of V_n is n^2, whereas on the bottom edge is $(n+1)^2$, and we define φ on V_n as the affine interpolation of these two values. The value of φ on the top edge of W_n is $(n+2)^2$, whereas on the bottom edge is $(n+1)^2$, and we define φ on W_n as the affine interpolation of these two values. Observe that, looking from top to bottom, φ increases in V_n and decreases in W_n. Clearly,

$$\varphi \notin L^\infty(\Omega), \tag{3.4}$$

$$\varphi \text{ is piecewise constant in } \Omega \setminus \Big(\bigcup_{\substack{n \geq 2 \\ n \text{ even}}} (V_n \cup W_n)\Big), \text{ piecewise affine in } \bigcup_{\substack{n \geq 2 \\ n \text{ even}}} V_n \cup W_n, \tag{3.5}$$

and

$$|\nabla \varphi| = \Big|\frac{\partial \varphi}{\partial x_2}\Big| = \frac{(n+1)^2 - n^2}{\frac{1}{n^2} - \frac{1}{10^{n+1}}} \approx n^3 \qquad \text{on } V_n,$$

$$|\nabla \varphi| = \Big|\frac{\partial \varphi}{\partial x_2}\Big| = \frac{(n+2)^2 - (n+1)^2}{\frac{1}{(n+2)^2} - \frac{1}{10^{n+1}}} \approx (n+1)^3 \qquad \text{on } W_n,$$

being the height of V_n (resp. W_n) of order $\frac{1}{n^2}$ (resp. $\frac{1}{(n+1)^2}$). However, the bases of V_n and W_n have length $\frac{1}{2^{n+1}}$ and $\frac{1}{2^{n+2}}$ respectively, so

$$\|\nabla \varphi\|^p_{L^p(V_n)} \approx |v_n| n^{3p} = \frac{n^{3p-2}}{2^{n+1}} =: a_n, \quad \|\nabla \varphi\|^p_{L^p(W_n)} \approx |w_n|(n+1)^{3p} = \frac{(n+1)^{3p-2}}{2^{n+2}} =: b_n.$$

In particular

$$\nabla \varphi \in L^p(\Omega; \mathbb{R}^2) \qquad \forall p \geq 1, \tag{3.6}$$

since $\sum_n a_n < +\infty$ and $\sum_n b_n < +\infty$. On the other hand, denoting Σ_T the union (for $n \geq 4$ even) of the top and left edges of the T_n's, by Σ_B the union (for $n \geq 4$ even) of the lateral edges of the B_n's, and by Σ_{VW} the union of the lateral edges of all V_n's and W_n's, we see that the jump set S_φ of φ is exactly

$$S_\varphi = \Sigma_T \cup \Sigma_B \cup \Sigma_{VW};$$

furthermore,

$$\Omega \setminus S_\varphi \text{ is arcwise connected,} \tag{3.7}$$

and it is easy to check that

$$\mathcal{H}^1(S_\varphi) < +\infty. \tag{3.8}$$

In particular, from (3.5), (3.8) and (3.6) we deduce $\varphi \in GSBV^p(\Omega)$, for any $p \geq 1$.

However,

$$\varphi \notin SBV(\Omega).$$

Indeed, although $D\varphi$ has no Cantor part, the total variation $|D^j\varphi|(\Omega)$ of the jump part $D^j\varphi$ of $D\varphi$ is infinite. This can be seen by observing that, on each lateral edge of V_n, the jump opening $[\![\varphi]\!]$ is of order n on a segment of length of order n^{-2}. Thus

$$|D^j\varphi|(\Omega) \geq \sum_{\substack{n\geq 2\\ n \text{ even}}} n^{-1} = +\infty.$$

We claim that φ is the unique (up to addition of a constant) solution of (3.1) for

$$u := e^{i\varphi}.$$

We have $u \in BV(\Omega; \mathbb{S}^1)$ and $|\nabla u| = |\nabla \varphi|$. Now, we observe that

$$S_u = S_\varphi, \tag{3.9}$$

namely $\mathcal{H}^1(S_u \Delta S_\varphi) = 0$, and therefore

$$u \in SBV^p(\Omega; \mathbb{S}^1).$$

Equality (3.9) is due to the fact that the subset of S_φ where $[\![\varphi]\!] \in 2\pi\mathbb{Z}$ is $\mathcal{H}^1$-negligible. This additionally implies that any lifting ψ of u must satisfy $S_\psi \supseteq S_u$, and so

$$\mathcal{H}^1(S_\psi) \geq \mathcal{H}^1(S_\varphi). \tag{3.10}$$

We conclude from (3.9) and (3.10) that φ is a minimizer of (3.1), and $\varphi \notin SBV^p(\Omega)$, for any $p \geq 1$. Finally, from (3.7) it follows that φ is unique, modulo addition of a constant in $2\pi\mathbb{Z}$.

3.2 A connection with optimal transport: an example

We observe in this subsection that the minimization problem (3.1) has a strict relation with a question arising from optimal transport (see [35, 36] for the setting and similar formulations). To do so, we give an example in a special case, leaving the more general ones to future treatments: for a connected and simply-connected bounded open set $\Omega \subset \mathbb{R}^2$, we fix $1 < p < 2$, $N \in \mathbb{N}$, and a Sobolev map $u \in W^{1,p}(\Omega; \mathbb{S}^1)$ with distributional Jacobian determinant

$$\mathrm{Det}(\nabla u) = \pi \sum_{i=1}^{N} (\delta_{x_i} - \delta_{y_i}), \tag{3.11}$$

(see [15]). We set $\mu := \sum_{i=1}^N \delta_{x_i}$ and $\nu := \sum_{i=1}^N \delta_{y_i}$, where the points x_i's and $y_i' s$ belong to Ω are not necessarily distinct, and we consider the class of all integer multiplicity 1-currents whose boundary is $\mu - \nu$, namely

$$\mathcal{T}(\mu, \nu) := \{T \in \mathcal{D}_1(\Omega) : T = (R, \theta, \tau) \text{ is an i.m.c. such that } \partial T = \mu - \nu\}.$$

Here $T = (R, \theta, \tau)$ is the integer multiplicity current given by

$$T(\omega) = \int_R \theta(x)\langle \omega(x), \tau(x)\rangle d\mathcal{H}^1 \qquad\qquad \forall \omega \in \mathcal{D}^1(\Omega),$$

with $R \subset \Omega$ a 1-rectifiable set, τ a tangent 1-vector to it, and $\theta : R \to \mathbb{Z}$ a $\mathcal{H}^1$-measurable function. A classical optimal transport problem can be formulated on the class $\mathcal{T}(\mu, \nu)$ in the following way: one fixes a cost function $\psi : \mathbb{Z} \times \mathbb{S}^1 \to \mathbb{R}^+$ and study the minimization problem

$$\Psi(\mu, \nu) := \inf \Big\{ \int_R \psi(\theta, \tau) d\mathcal{H}^1 : (R, \theta, \tau) \in \mathcal{T}(\mu, \nu)\Big\}. \tag{3.12}$$

We refer to In the special case $\psi \equiv 1$ we have $\Psi(\mu, \nu) = \int_R \psi(\theta, \tau) d\mathcal{H}^1 = \mathcal{H}^1(R)$ and we claim that

$$m_p[u] = \inf\{\mathcal{H}^1(R) : (R, \theta, \tau) \in \mathcal{T}(\mu, \nu)\}, \tag{3.13}$$

which shows the connection between our minimization problem (3.1) and optimal transport. We sketch the main steps to prove this, leaving details to future developments. Let $\varphi \in SBV^p(\Omega)$ be a lifting[7] of u; then

$$D\varphi = \nabla\varphi\, \mathcal{L}^2 + D^j\varphi, \tag{3.14}$$

and since $D^j\varphi = [\![\varphi]\!] n_{S_\varphi} \mathcal{H}^1 \llcorner S_\varphi$, its rotated[8] $(D^j\varphi)^\perp = [\![\varphi]\!] n^\perp_{S_\varphi} \mathcal{H}^1 \llcorner S_\varphi$ can be identified with 2π times the 1-current

$$T_\varphi = (S_\varphi, \frac{[\![\varphi]\!]}{2\pi}, n^\perp_{S_\varphi})$$

which has integer multiplicity. We claim that

$$\partial T_\varphi = \mu - \nu \quad \text{ in } \mathcal{D}_0(\Omega);$$

indeed, if we identify $T_\varphi \in \mathcal{D}_1(\Omega)$ with a vector-valued measure and the elements of $\mathcal{D}_0(\Omega)$ with scalar measures, $\partial T_\varphi \in \mathcal{D}_0(\Omega)$ corresponds to minus the divergence of T_φ, and so

$$\partial T_\varphi = -\frac{1}{2\pi}\mathrm{Div}((D^j\varphi)^\perp) = \frac{1}{2\pi}\mathrm{Div}(\nabla^\perp\varphi) \tag{3.15}$$

where we have used that

$$\mathrm{Div}(D^\perp\varphi) = \mathrm{Curl}(D\varphi) = 0.$$

Here $\mathrm{Curl}(D\varphi)$ is defined in a distributional sense. Recalling the definition of distributional Jacobian determinant for Sobolev maps [15, Page 12, formula (1.33)] one has

$$\mathrm{Det}(\nabla u) = \frac{1}{2}\mathrm{Div}(u_1\nabla^\perp u_2 - u_2\nabla^\perp u_1) = \frac{1}{2}\mathrm{Div}(\nabla^\perp\varphi) \tag{3.16}$$

so we conclude $\partial T_\varphi = \mu - \nu$.
In particular T_φ is admissible for (3.12), i.e., $T_\varphi \in \mathcal{T}(\mu, \nu)$, and we obtain

$$m_p[u] \geq \inf\{\mathcal{H}^1(R) : (R, \theta, \tau) \in \mathcal{T}(\mu, \nu)\}. \tag{3.17}$$

[7] We here suppose, for simplicity, that the infimum in (3.1) can be obtained restricting on the space $SBV^p(\Omega)$.

[8] We take the conterclockwise $\pi/2$-rotation.

To see that also the opposite inequality holds, let $T = (R, \theta, \tau)$ be admissible for (3.12), and let us decompose $T = \sum_{i=1}^{+\infty} T_i$ in indecomposable components, so that by Federer decomposition theorem [29, Sections 4.2.25 and 4.5.9] T_i is either a loop with multiplicity ± 1 or a Lipschitz parametrized path from one point y_j to some x_k. Up to erasing the loops (operation that does not increase the energy in (3.12)) we may suppose that there are exactly N such Lipschitz paths, $T = \sum_{i=1}^{N} T_i$. In particular $T_i = (R_i, 1, \tau)$ is such that R_i is a closed set image of $[0, 1]$ under the Lipschitz injective map γ_i, and so $R = \cup_{i=1}^{N} R_i$ is closed. Consider then the open set $\Omega \setminus R$ and let φ be a lifting of u in $\Omega \setminus R$ with jump set of minimal length. We claim that $\mathcal{H}^1(S_\varphi \cap (\Omega \setminus R)) = 0$; this is equivalent to say that there exists a lifting of u on $\Omega \setminus R$ with no jumps. The latter can be shown by the following observation: we connect the components of R with a curve σ in such a way $\Omega \setminus (R \cup \sigma)$ is simply-connected. Then by [15, Lemma 1.8] the lifting φ has no jumps[9] on $\Omega \setminus (R \cup \sigma)$. To see that there is no jump of φ on σ, it is enough to observe that, given any closed loop Γ in $\Omega \setminus R$, the topological degree of u on Γ must be null (by construction of R).

From the claim, since $\varphi \in SBV^p(\Omega)$ satisfies $S_\varphi \subset R$, we easily infer

$$m_p[u] \leq \inf\{\mathcal{H}^1(R) : (R, \theta, \tau) \in \mathcal{T}(\mu, \nu)\} \tag{3.18}$$

by the arbitrariness of T.

In a similar manner, we see how the problem in (3.1) and the aforementioned transport problem can be used to solve Steiner-type problems. Again, assume for simplicity that the singularities of $u \in W^{1,p}(\Omega; \mathbb{S}^1)$ satisfy (3.11). Assume also that $x_i = \overline{x}$ for all $i = 1, \dots, N$, and that the points y_i are all distinct and different from $\overline{x}$. We look for the connected set of minimal length containing the $N + 1$ points in the family $C = \{\overline{x}, y_i : i = 1, \dots, N\}$. To do this, consider a compact connected set K with $\mathcal{H}^1(K) < +\infty$, containing C and let us suppose that $\mathrm{dist}(C, \partial\Omega) > \mathcal{H}^1(K)$; then the Steiner problem for C can be proven to be

$$m_p[u] = \inf\{\mathcal{H}^1(L) : L \text{ is connected, } L \supset C\}, \tag{3.19}$$

and the jump set of a jump minimizing lifting for u is a minimizer of the right-hand side in the above expression.

3.3 Main results on sequences of liftings

The main results of the first part of the paper are the following.

Theorem 3.1 (Compactness and lower semi-continuity). *Let $p > 1$, $u \in SBV^p(\Omega; \mathbb{S}^1)$ and $(\varphi_k)_{k \geq 1} \subset GSBV^p(\Omega)$ be a sequence of liftings of u with*

$$\sup_{k \in \mathbb{N}} \mathcal{H}^{n-1}(S_{\varphi_k}) < +\infty.$$

Then there exist a Caccioppoli partition $(E_i)_{i \geq 1}$ of Ω and a not-relabelled subsequence of indices k for which the following holds:

(a) there exists a lifting $\varphi_\infty \in GSBV^p(\Omega)$ of u in Ω,

(b) there exists a sequence $(d_k^{(i)})_{k \geq 1} \subset \mathbb{Z}$ for any $i \in \mathbb{N}$,

[9] Although the domain $\Omega \setminus (R \cup \sigma)$ is not Lipschitz, the same result can be obtained by approximating it with suitable Lipschitz subdomains.

so that

$$\begin{aligned} &\lim_{k\to+\infty} (\varphi_k(x) - 2\pi d_k^{(i)}) = \varphi_\infty(x) \qquad \forall i\in\mathbb{N}, \quad \text{for a.e. } x\in E_i\,, \\ &\lim_{k\to+\infty} |\varphi_k(x) - 2\pi d_k^{(i)}| = +\infty \qquad \forall i\in\mathbb{N}, \quad \text{for a.e. } x\in \Omega\setminus E_i\,, \end{aligned} \tag{3.20}$$

and

$$\liminf_{k\to+\infty} \mathcal{H}^{n-1}(S_{\varphi_k}) \geq \mathcal{H}^{n-1}(S_{\varphi_\infty})\,. \tag{3.21}$$

Corollary 3.2 (**Existence**). *Let $p>1$. Then there exists a minimizer $\varphi\in GSBV^p(\Omega)$ of* (3.1).

As pointed out in the introduction, in general a minimizer of (3.1) does not exist in $SBV^p(\Omega)$.

The generalization of Theorem 3.1 to a sequence (u_k), needed in the proof of Theorem 1.2, reads as follows.

Theorem 3.3. *Let $p>1$, $u, u_k \in SBV^p(\Omega;\mathbb{S}^1)$ be such that*

$$u_k \overset{*}{\rightharpoonup} u \quad \text{in } BV(\Omega;\mathbb{S}^1), \tag{3.22}$$

and let $\varphi_k\in GSBV^p(\Omega)$ be a lifting of u_k, for all $k\geq 1$. Suppose

$$\sup_{k\in\mathbb{N}} \mathcal{H}^{n-1}(S_{\varphi_k}) < +\infty\,.$$

Then there exist a Caccioppoli partition (E_i) of Ω and a not-relabelled subsequence of indices k for which the following holds:

(a) there exists a lifting $\varphi_\infty\in GSBV^p(\Omega)$ of u in Ω,

(b) there exist a sequence $(d_k^{(i)})_{k\geq 1}\subset\mathbb{Z}$ for any $i\in\mathbb{N}$,

so that

$$\begin{aligned} &\lim_{k\to+\infty} (\varphi_k(x) - 2\pi d_k^{(i)}) = \varphi_\infty(x) \qquad \forall i\in\mathbb{N}, \quad \text{for a.e. } x\in E_i\,, \\ &\lim_{k\to+\infty} |\varphi_k(x) - 2\pi d_k^{(i)}| = +\infty \qquad \forall i\in\mathbb{N}, \quad \text{for a.e. } x\in \Omega\setminus E_i\,, \end{aligned} \tag{3.23}$$

and

$$\liminf_{k\to+\infty} \mathcal{H}^{n-1}(S_{\varphi_k}) \geq \mathcal{H}^{n-1}(S_{\varphi_\infty})\,. \tag{3.24}$$

4 Proofs of Theorems 3.1 and 3.3

Next Lemmas 4.1 and 4.2, independent one each other, are the building blocks of an iterative argument needed for the proof of Theorem 3.1. Some arguments in the proof of Lemma 4.2 will be also used in the proof of Theorem 1.2.

Lemma 4.1 (**Localized compactness**). *Let $p>1$, let $\Omega\subset\mathbb{R}^n$ be a bounded open set with Lipschitz boundary, and $u\in SBV^p(\Omega;\mathbb{S}^1)$. Let $(\varphi_k)_{k\geq 1}\subset GSBV^p(\Omega)$ be a sequence of liftings of u, and suppose*

$$C := \sup_{k\geq 1} \mathcal{H}^{n-1}(S_{\varphi_k}) < +\infty. \tag{4.1}$$

Let $F \subset \Omega$ be a nonempty finite perimeter set in Ω. Then, for a not-relabelled subsequence, there exist a sequence of integers $(d_k)_{k\geq 1} \subset \mathbb{Z}$, a finite perimeter set

$$E \subseteq F$$

in Ω, and a function $\varphi_\infty \in GSBV^p(\Omega)$, such that:

$$\varphi_\infty \textit{ is a lifting of } u \textit{ in } E\,, \quad \varphi_\infty = 0 \quad \textit{in } \Omega \setminus E\,, \tag{4.2}$$

$$\begin{aligned} &\lim_{k\to+\infty} (\varphi_k(x) - 2\pi d_k) = \varphi_\infty(x) \qquad \textit{for a.e. } x \in E\,, \\ &\lim_{k\to+\infty} |\varphi_k(x) - 2\pi d_k| = +\infty \qquad \textit{for a.e. } x \in F \setminus E, \end{aligned} \tag{4.3}$$

$$\begin{aligned} &|E| \geq \frac{n^n \omega_n |F|^n}{2^n(2C + \mathcal{H}^{n-1}(\partial^* F))^n} > 0, \\ &\mathcal{H}^{n-1}(F \cap \partial^* E) = \mathcal{H}^{n-1}(F \cap \partial^*(F \setminus E)) \leq C. \end{aligned} \tag{4.4}$$

If furthermore

$$|\varphi_k(x) - 2\pi d_k| \to +\infty \qquad \text{for a.e. } x \in \Omega \setminus E, \tag{4.5}$$

then

$$\liminf_{k\to+\infty} \mathcal{H}^{n-1}\big(S_{\varphi_k} \cap A\big) \geq \mathcal{H}^{n-1}\big((S_{\varphi_\infty} \cup \partial^* E) \cap A\big) \quad \textit{for any open set } A \subseteq \Omega. \tag{4.6}$$

Proof. Define[10] $v_k := \varphi_k - \varphi_1 \in GSBV^p(\Omega)$ which, since φ_k and φ_1 are liftings of u, is piecewise constant (i.e., the absolutely continuous and Cantor parts of Dv_k vanish) and takes values in $2\pi\mathbb{Z}$. Therefore, it induces a Caccioppoli partition $\{V_m^k : m \in \mathbb{Z}\}$ of F, where

$$\text{for any } m \in \mathbb{Z} \qquad V_m^k := \{x \in F : v_k(x) = 2\pi m\} \text{ has finite perimeter,}$$

$$v_k = \sum_{m\in\mathbb{Z}} 2\pi m \chi_{V_m^k}, \qquad \sum_{m\in\mathbb{Z}} |V_m^k| = |F|.$$

Moreover, since $\partial^* V_m^k \subset S_{\varphi_k} \cup S_{\varphi_1} \cup \partial^* F$, the perimeter of the partition satisfies

$$\begin{aligned} \frac{1}{2}\sum_{m\in\mathbb{Z}} P(V_m^k) &\leq \mathcal{H}^{n-1}(S_{\varphi_k} \cup S_{\varphi_1}) + \mathcal{H}^{n-1}(\partial^* F) \\ &\leq \mathcal{H}^{n-1}(S_{\varphi_k}) + \mathcal{H}^{n-1}(S_{\varphi_1}) + \mathcal{H}^{n-1}(\partial^* F) \\ &\leq 2C + \mathcal{H}^{n-1}(\partial^* F)\,, \end{aligned} \tag{4.7}$$

where C is the constant in (4.1). Now, for any $k \in \mathbb{N}$, $k \geq 1$, we select $d_k \in \mathbb{Z}$ for which

$$|V_{d_k}^k| = \max\Big\{|V_m^k| : m \in \mathbb{Z}\Big\}.$$

In particular, from (4.7),

$$P(V_{d_k}^k) \leq 2(2C + \mathcal{H}^{n-1}(\partial^* F)). \tag{4.8}$$

[10] There is no special reason in the choice of φ_1; any element of the sequence (φ_k) could be chosen as well.

Using also the isoperimetric inequality, we find

$$
\begin{aligned}
|F| = \sum_{m\in\mathbb{Z}} |V_m^k| \le |V_{d_k}^k|^{\frac{1}{n}} \sum_{m\in\mathbb{Z}} |V_m^k|^{1-\frac{1}{n}} \le \frac{|V_{d_k}^k|^{\frac{1}{n}}}{n\omega_n^{1/n}} \sum_{m\in\mathbb{Z}} P(V_m^k) \\
\le \frac{|V_{d_k}^k|^{1/n}}{n\omega_n^{1/n}} 2(2C + \mathcal{H}^{n-1}(\partial^* F)).
\end{aligned}
$$

Thus passing to the limit

$$
\liminf_{k\to+\infty} |V_{d_k}^k| \ge \frac{n^n \omega_n |F|^n}{2^n (2C + \mathcal{H}^{n-1}(\partial^* F))^n}. \tag{4.9}
$$

Hence, from (4.8) and (4.9), there are a not-relabelled subsequence and a finite perimeter set

$$
G \subset F
$$

such that

$$
(\chi_{V_{d_k}^k}) \text{ converges to } \chi_G \ \text{ a.e. in } F \text{ and weakly star in } BV(\Omega;\{0,1\}) \text{ as } k \to +\infty. \tag{4.10}
$$

Consequently,

$$
|G| \ge \frac{n^n \omega_n |F|^n}{2^n (2C + \mathcal{H}^{n-1}(\partial^* F))^n}. \tag{4.11}
$$

Next, we define

$$
\widehat{\varphi}_k := \varphi_k - 2\pi d_k.
$$

Notice that $\mathcal{H}^{n-1}(S_{\widehat{\varphi}_k}) \le \mathcal{H}^{n-1}(S_{\varphi_k}) \le C$; moreover by construction

$$
\widehat{\varphi}_k = \varphi_1 \text{ on } V_{d_k}^k \qquad \forall k \in \mathbb{N},
$$

and, from (4.10),

$$
\widehat{\varphi}_k \to \varphi_1 \qquad \text{pointwise a.e. in } G\,.
$$

However, there could be other regions of F out of G, where the sequence $(\widehat{\varphi}_k)$ converges pointwise, that we need to identify. To this aim, let us consider, for all integers $N \ge 1$, the sequence of truncated functions $(\widehat{\varphi}_k \wedge N) \vee (-N)$, which is precompact in $SBV^p(\Omega)$. Using a diagonal argument, we select a further (not-relabelled) subsequence such that $(\widehat{\varphi}_k \wedge N) \vee (-N)$ converge in $L^1(\Omega)$ and pointwise a.e. in F for all $N \in \mathbb{N}$. Define

$$
F_N := \left\{ x \in F : \lim_{k\to\infty} |(\widehat{\varphi}_k(x) \wedge N) \vee (-N)| = N \right\}.
$$

Then $F_N \supset F_{N+1}$, and

$$
\bigcap_{N\in\mathbb{N}} F_N := \{x \in F : |\widehat{\varphi}_k(x)| \to +\infty\}.
$$

As a consequence, setting

$$
E := F \setminus \bigcap_N F_N, \tag{4.12}
$$

we have

$$G \subseteq E \subseteq F, \tag{4.13}$$

E has finite perimeter in Ω[11], and setting

$$\varphi_\infty(x) = \begin{cases} \lim_k \widehat{\varphi}_k(x) & \text{in } E \\ 0 & \text{in } \Omega \setminus E \end{cases} \qquad \text{for a. e. } x \in \Omega,$$

we have $\varphi_\infty \in GSBV^p(\Omega)$, $\varphi_\infty = \varphi_1$ a.e. in G, and φ_∞ is a lifting of u in E. At the same time

$$|\widehat{\varphi}_k(x)| \to +\infty \qquad \text{for a.e. } x \in F \setminus E.$$

Thus, using also (4.11) and (4.13), which imply $|E| \geq |G| \geq \frac{n^n \omega_n |F|^n}{2^n(2C+\mathcal{H}^{n-1}(\partial^* F))^n}$, statements (4.2)-(4.4) are proven.

It remains to prove (4.6). Assume $|\widehat{\varphi}_k| \to +\infty$ a.e. in $\Omega \setminus E$. Then, by Theorem 2.6, for any subsequence $(\widehat{\varphi}_{k_h})$ we can extract a further subsequence $(\widehat{\varphi}_{k_{h_j}})$ such that

$$\mathcal{H}^{n-1}((S_{\varphi_\infty} \cup \partial^* E) \cap A) \leq \liminf_{j \to +\infty} \mathcal{H}^{n-1}(S_{\varphi_{k_{h_j}}} \cap A) \text{ for any open set } A \subseteq \Omega.$$

Hence, the same holds for the original sequence $(\widehat{\varphi}_k)$,

$$\mathcal{H}^{n-1}((S_{\varphi_\infty} \cup \partial^* E) \cap A) \leq \liminf_{k \to +\infty} \mathcal{H}^{n-1}(S_{\varphi_k} \cap A) \text{ for any open set } A \subseteq \Omega,$$

which shows (4.6). This additionally implies

$$P(E, F) \leq C = \sup_{k \in \mathbb{N}} \mathcal{H}^{n-1}(S_{\varphi_k}),$$

which shows the last equality in (4.4). $\square$

Lemma 4.2. *Let $p > 1$ and $(\varphi_k)_{k \geq 1} \subset GSBV^p(\Omega)$ be a sequence of functions with*

$$C := \sup_{k \in \mathbb{N}} \mathcal{H}^{n-1}(S_{\varphi_k}) < +\infty. \tag{4.14}$$

Let $N \geq 1$ be an integer, $E_1, \dots, E_N \subset \Omega$ be pairwise disjoint nonempty finite perimeter sets and $\varphi^1_\infty, \dots, \varphi^N_\infty$ be functions in $GSBV^p(\Omega)$ with the following properties: for any $i = 1, \dots, N$,

$$\begin{aligned} &\varphi^{(i)}_\infty = 0 \qquad \text{a.e. in } \Omega \setminus E_i\,, \\ &\liminf_{k \to +\infty} \mathcal{H}^{n-1}(S_{\varphi_k} \cap B) \geq \mathcal{H}^{n-1}\Big((S_{\varphi^{(i)}_\infty} \cup \partial^* E_i) \cap B\Big) \text{ for any open ball } B \subset \Omega. \end{aligned} \tag{4.15}$$

Define $\Phi_N \in GSBV^p(\Omega)$ as

$$\Phi_N(x) := \begin{cases} \varphi^{(i)}_\infty(x) & \text{if } x \in E_i \text{ for some } i = 1, \dots, N\,, \\ 0 & \text{if } x \in \Omega \setminus \big(\cup_{i=1}^N E_i\big). \end{cases}$$

Then

$$\liminf_{k \to +\infty} \mathcal{H}^{n-1}(S_{\varphi_k}) \geq \mathcal{H}^{n-1}\Big(S_{\Phi_N} \cup \big(\Omega \cap \partial^*\big(\cup_{i=1}^N E_i\big)\big)\Big), \tag{4.16}$$

and

$$\mathcal{H}^{n-1}\Big(\Omega \cap \partial^*\big(\cup_{i=1}^N E_i\big)\Big) = \mathcal{H}^{n-1}\Big(\Omega \cap \partial^*\big(\Omega \setminus \cup_{i=1}^N E_i\big)\Big) \leq C. \tag{4.17}$$

[11] E has finite perimeter in F, and F has finite perimeter in Ω, therefore E has finite perimeter in Ω.

Proof. The equality in (4.17) follows from $\Omega \cap \partial^* E = \Omega \cap \partial^*(\Omega \setminus E)$, and the inequality is a consequence of (4.16) and (4.14). So, let us prove (4.16).

To shortcut the notation we set, for all $i = 1, \dots, N$,

$$\Sigma_i := \Omega \cap \partial^* E_i, \qquad S_i := S_{\varphi_\infty^{(i)}} \setminus \partial^* E_i\,,$$

and

$$\Sigma := \bigcup_{i=1}^N \Sigma_i\,, \qquad S := \bigcup_{i=1}^N S_i,$$

are $(n-1)$-rectifiable with $\mathcal{H}^{n-1}(\Sigma) < +\infty$, $\mathcal{H}^{n-1}(S) < +\infty$. We fix $\delta \in (0,1)$ and, for any $i = 1, \dots, N$, for $\mathcal{H}^{n-1}$-a.e. $x \in S_i$ we choose a radius $r(x)$ such that

$$\begin{aligned} & B_\rho(x) \subseteq \Omega\,, \\ & \mathcal{H}^{n-1}(B_\rho(x) \cap S_i) \geq (1-\delta)\omega_{n-1}\rho^{n-1}, \\ & \mathcal{H}^{n-1}(B_\rho(x) \cap \Sigma) + \sum_{\substack{m \neq i \\ m=1}}^N \mathcal{H}^{n-1}(B_\rho(x) \cap S_m) \leq \delta\omega_{n-1}\rho^{n-1} \end{aligned} \qquad \forall \rho \in (0, r(x)). \quad (4.18)$$

We collect all such balls $B_\rho(x)$ satisfying (4.18) in a family denoted $\mathcal{B}_i$. Furthermore, possibly reducing the value of $r(x)$, for any $i = 1, \dots, N$ and for $\mathcal{H}^{n-1}$-a.e. $x \in \Sigma_i$ we may suppose that

$$\begin{aligned} & B_\rho(x) \subseteq \Omega\,, \\ & \mathcal{H}^{n-1}(B_\rho(x) \cap \Sigma_i) \geq (1-\delta)\omega_{n-1}\rho^{n-1}, \\ & \mathcal{H}^{n-1}(B_\rho(x) \cap S) + \sum_{\substack{m \neq i \\ m=1}}^N \mathcal{H}^{n-1}(B_\rho(x) \cap \Sigma_m) \leq \delta\omega_{n-1}\rho^{n-1}, \end{aligned} \qquad \rho \in (0, r(x)), \quad (4.19)$$

and we collect such balls in a family $\mathcal{B}_{i+N}$. The family $\cup_{n=1}^{2N} \mathcal{B}_n$, forms a Vitali covering of $\Sigma \cup S$, and so by Vitali covering theorem we can choose countable many points $x_k \in \Omega$ and radii $\rho_k \in (0, r(x_k))$ such that the family

$$\mathcal{B} := \left\{ B \in \bigcup_{j=1}^{2N} \mathcal{B}_j : B = B_{\rho_k}(x_k) \text{ for some } k \in \mathbb{N} \right\}$$

consists of mutually disjoint balls and covers $\Sigma \cup S$ up to a $\mathcal{H}^{n-1}$-negligible set, with

$$\sum_{k=1}^{+\infty} \rho_k^{n-1} < +\infty.$$

In addition

$$\mathcal{H}^{n-1}(S \cup \Sigma) = \sum_{B \in \mathcal{B}} \mathcal{H}^{n-1}(B \cap (S \cup \Sigma)) \geq (1-\delta)\omega_{n-1} \sum_{k=1}^{+\infty} \rho_k^{n-1}, \qquad (4.20)$$

the inequality following from (4.18) and (4.19). From the same formulas, for any $n = 1, \dots, N$, and $B = B_\rho(x) \in \mathcal{B}_i$, $i \neq j$, it holds

$$\mathcal{H}^{n-1}(B \cap S_j) \leq \sum_{\substack{m \neq i \\ m=1}}^N \mathcal{H}^{n-1}(B \cap S_m) \leq \delta\omega_{n-1}\rho^{n-1}.$$

This, together with (4.20), imply

$$\sum_{B\in\mathcal{B}\setminus\mathcal{B}_j} \mathcal{H}^{n-1}(B\cap S_j) \le \delta\omega_{n-1}\sum_{k=1}^{+\infty}\rho_k^{n-1} \le \frac{\delta}{1-\delta}\mathcal{H}^{n-1}(S\cup\Sigma). \tag{4.21}$$

Similarly, for all $h=1,\dots,N$,

$$\sum_{B\in\mathcal{B}\setminus\mathcal{B}_{N+h}} \mathcal{H}^{n-1}(B\cap\Sigma_h) \le \frac{\delta}{1-\delta}\mathcal{H}^{n-1}(S\cup\Sigma). \tag{4.22}$$

From (4.21) and (4.22) it follows

$$\begin{aligned}\sum_{B\in\mathcal{B}\cap\mathcal{B}_j} \mathcal{H}^{n-1}(B\cap S_j) &= \sum_{B\in\mathcal{B}} \mathcal{H}^{n-1}(B\cap S_j) - \sum_{B\in\mathcal{B}\setminus\mathcal{B}_j} \mathcal{H}^{n-1}(B\cap S_j)\\ &\ge \mathcal{H}^{n-1}(S_j) - \frac{\delta\mathcal{H}^{n-1}(S\cup\Sigma)}{1-\delta}\,,\end{aligned} \tag{4.23}$$

for all $j=1,\dots,N$, and analogously for all $h=1,\dots,N$,

$$\sum_{B\in\mathcal{B}\cap\mathcal{B}_{N+h}} \mathcal{H}^{n-1}(B\cap\Sigma_h) \ge \mathcal{H}^{n-1}(\Sigma_h) - \frac{\delta\mathcal{H}^{n-1}(S\cup\Sigma)}{1-\delta}. \tag{4.24}$$

From (4.15), for any $j=1,\dots,N$ we obtain

$$\liminf_{k\to+\infty}\mathcal{H}^{n-1}(S_{\varphi_k}\cap B) \ge \mathcal{H}^{n-1}(S_{\varphi_\infty^j}\cap B) \ge \mathcal{H}^{n-1}(S_j\cap B)\,, \tag{4.25}$$

and for all $h=1,\dots,N$

$$\liminf_{k\to+\infty}\mathcal{H}^{n-1}(S_{\varphi_k}\cap B) \ge \mathcal{H}^{n-1}(B\cap\partial^* E_h) \ge \mathcal{H}^{n-1}(\Sigma_h\cap B)\,. \tag{4.26}$$

Now, summing (4.25) over all $B\in\mathcal{B}\cap\mathcal{B}_j$ and (4.26) over all $B\in\mathcal{B}\cap\mathcal{B}_{N+h}$, and then over n,h respectively, we infer

$$\begin{aligned}&\sum_{j=1}^{N}\sum_{B\in\mathcal{B}\cap\mathcal{B}_j}\mathcal{H}^{n-1}(S_j\cap B) + \sum_{h=1}^{N}\sum_{B\in\mathcal{B}\cap\mathcal{B}_{N+h}}\mathcal{H}^{n-1}(\Sigma_h\cap B)\\ &\le \liminf_{k\to+\infty}\sum_{j=1}^{2N}\sum_{B\in\mathcal{B}\cap\mathcal{B}_j}\mathcal{H}^{n-1}(S_{\varphi_k}\cap B) \le \liminf_{k\to+\infty}\mathcal{H}^{n-1}(S_{\varphi_k}),\end{aligned} \tag{4.27}$$

where the penultimate inequality is a consequence of Fatou's Lemma. Combining (4.27) with (4.23) and (4.24) we get

$$\sum_{j=1}^{N}\mathcal{H}^{n-1}(S_j) + \sum_{h=1}^{N}\mathcal{H}^{n-1}(\Sigma_h) - \frac{2\delta N\mathcal{H}^{n-1}(S\cup\Sigma)}{1-\delta} \le \liminf_{k\to+\infty}\mathcal{H}^{n-1}(S_{\varphi_k})\,.$$

By the arbitrariness of $\delta>0$ we conclude

$$\mathcal{H}^{n-1}(S\cup\Sigma) \le \liminf_{k\to+\infty}\mathcal{H}^{n-1}(S_{\varphi_k})\,,$$

that implies (4.16). □

4.1 Proof of Theorem 3.1

Let $M := \sup_{k\in\mathbb{N}} \mathcal{H}^{n-1}(S_{\varphi_k}) < +\infty$. To show compactness of (φ_k), we utilize an iterative argument.

Base case $N = 1$. We set

$$F_1 := \Omega. \tag{4.28}$$

From Lemma 4.1 we find, for a not-relabelled subsequence, a finite perimeter set $E_1 \subseteq F_1$, a sequence $(d_k^{(1)})_{k\geq 1} \subset \mathbb{Z}$ and a function $\varphi_\infty^{(1)} \in GSBV^p(\Omega)$, such that

$$\begin{aligned}
&\varphi_\infty^{(1)} \text{ is a lifting of } u \text{ in } E_1, \qquad \varphi_\infty^{(1)} = 0 \quad \text{in } F_1 \setminus E_1\,,\\
&|E_1| \geq \frac{n^n \omega_n |F_1|^n}{2^n(2M + \mathcal{H}^{n-1}(\partial\Omega))^n}\,,
\end{aligned}$$

and

$$\begin{aligned}
&(\varphi_k(x) - 2\pi d_k^{(1)}) \to \varphi_\infty^{(1)}(x) \qquad \text{for a.e. } x \in E_1\,,\\
&|\varphi_k(x) - 2\pi d_k^{(1)}| \to +\infty \qquad \text{for a.e. } x \in F_1 \setminus E_1.
\end{aligned} \tag{4.29}$$

Moreover, since $F_1 = \Omega$, from the final part of the statement of Lemma 4.1, we have

$$\liminf_{k\to+\infty} \mathcal{H}^{n-1}(S_{\varphi_k} \cap A) \geq \mathcal{H}^{n-1}((S_{\varphi_\infty^{(1)}} \cup \partial^* E_1) \cap A) \text{ for any open set } A \subseteq F_1, \tag{4.30}$$

see (4.6).

Iterative case $N \rightsquigarrow N+1$. Let $N \geq 2$ be an integer, $E_1, \dots, E_N \subset \Omega$ be pairwise disjoint nonempty finite perimeter sets, and define, together with (4.28),

$$F_i := \Omega \setminus \bigcup_{j=1}^{i-1} E_j \quad \text{for } i = 2, \dots, N,$$

so that

$$E_1 \subseteq F_1,\ E_2 \subseteq F_2, \dots,\ E_N \subseteq F_N.$$

Suppose that:

(i) For all $i = 1, \dots, N$, there exists a function $\varphi_\infty^{(i)} \in GSBV^p(\Omega)$ which is a lifting of u in E_i, and $\varphi_\infty^{(i)} = 0$ in $\Omega \setminus E_i$;

(ii) For all $i = 1, \dots, N$ we have

$$|E_i| \geq \frac{n^n \omega_n |F_i|^n}{2^n(2M + \mathcal{H}^{n-1}(\partial^* F_i))^n} \geq \frac{n^n \omega_n |F_i|^n}{2^n(3M + \mathcal{H}^{n-1}(\partial\Omega))^n};$$

(iii) For all $i = 1, \dots, N$, there exist sequences $(d_k^{(i)})_k \subset \mathbb{Z}$, such that

$$\begin{aligned}
&(\varphi_k(x) - 2\pi d_k^{(i)}) \to \varphi_\infty^{(i)}(x) \qquad \text{for a.e. } x \in E_i\,,\\
&|\varphi_k(x) - 2\pi d_k^{(i)}| \to +\infty \qquad \text{for a.e. } x \in \Omega \setminus E_i\,,\\
&\liminf_{k\to+\infty} \mathcal{H}^{n-1}(S_{\varphi_k} \cap A) \geq \mathcal{H}^{n-1}((S_{\varphi_\infty^{(i)}} \cup \partial^* E_i) \cap A) \text{ for any open set } A \subseteq \Omega.
\end{aligned} \tag{4.31}$$

We now want to find a finite perimeter set $E_{N+1} \subseteq \Omega$ disjoint from $\cup_{i=1}^N E_i$ and a function $\varphi_\infty^{(N+1)} \in GSBV^p(\Omega)$, such that, for a not-relabelled subsequence, $E_1, \dots, E_{N+1}$, (φ_k), and $\varphi_\infty^{(N+1)}$ satisfy properties (i)-(iii) above, with N replaced by $N+1$.

To this purpose we set, for $N \geq 1$,

$$F_{N+1} := \Omega \setminus \Big(\bigcup_{i=1}^N E_i \Big),$$

and we let $\Phi_N \in GSBV^p(\Omega)$ be defined as

$$\Phi_N(x) := \begin{cases} \varphi_\infty^{(i)}(x) & \text{if } x \in E_i \text{ for } i = 1, \dots, N\,, \\ 0 & \text{if } x \in \Omega. \end{cases} \tag{4.32}$$

If $F_{N+1} = \emptyset$ there is nothing to prove. Assume then $F_{N+1} \neq \emptyset$. From Lemma 4.2 we have

$$\liminf_{k \to +\infty} \mathcal{H}^{n-1}(S_{\varphi_k}) \geq \mathcal{H}^{n-1}(S_{\Phi_N} \cup (\Omega \cap \partial^*(\cup_{i=1}^N E_i)))\,, \tag{4.33}$$

and

$$\mathcal{H}^{n-1}\Big(\Omega \cap \partial^*(\cup_{i=1}^N E_i)\Big) = \mathcal{H}^{n-1}(\Omega \cap \partial^* F_{N+1}) \leq M\,. \tag{4.34}$$

Next, applying Lemma 4.1 to $F = F_{N+1}$, there exist a sequence $(d_k^{(N+1)})_{k \geq 1} \subset \mathbb{Z}$, a finite perimeter set $E_{N+1} \subseteq F_{N+1}$ (and thus $E_{N+1} \cap (\cap_{i=1}^N E_i) = \emptyset$) and a function $\varphi_\infty^{(N+1)} \in GSBV^p(\Omega)$ such that

$$\varphi_\infty^{(N+1)} = 0 \quad \text{in } \Omega \setminus E_{N+1}\,, \quad \varphi_\infty^{(N+1)} \text{ is a lifting of } u \text{ in } E_{N+1}\,,$$

$$|E_{N+1}| \geq \frac{n^n \omega_n |F_{N+1}|^2}{2^n (2M + \mathcal{H}^{n-1}(\partial^* F_{N+1}))^2}\,, \tag{4.35}$$

and

$$\begin{aligned} &(\varphi_k(x) - 2\pi d_k^{(N+1)}) \to \varphi_\infty^{(N+1)}(x) && \text{for a.e. } x \in E_{N+1}\,, \\ &|\varphi_k(x) - 2\pi d_k^{(N+1)}| \to +\infty && \text{for a.e. } x \in F_{N+1} \setminus E_{N+1}\,, \\ &\liminf_{k \to +\infty} \mathcal{H}^1(S_{\varphi_k} \cap A) \geq \mathcal{H}^1((S_{\varphi_\infty^{(N+1)}} \cup \partial^* E_{N+1}) \cap A) && \text{for any open set } A \subseteq F_{N+1}. \end{aligned} \tag{4.36}$$

Combining (4.34) and (4.35) we readily get

$$|E_{N+1}| \geq \frac{n^n \omega_n |F_{N+1}|^2}{2^n (2M + \mathcal{H}^{n-1}(\partial^* F_{N+1}))^2} \geq \frac{n^n \omega_n |F_{N+1}|^2}{2^n (3M + \mathcal{H}^{n-1}(\partial \Omega))^2} > 0\,.$$

Moreover by (4.31), also

$$|\varphi_k(x) - 2\pi d_k^{(N+1)}| \to +\infty \qquad \text{for a.e. } x \in \Omega \setminus E_{N+1}\,. \tag{4.37}$$

Thus the sets $E_1, \dots, E_{N+1}$ satisfy (i)-(iii) with $(d_k^{(i)})_{k \geq 1} \subset \mathbb{Z}$ and $\varphi_\infty^{(i)} \in GSBV^p(\Omega)$ for $i \in \{1, \dots, N+1\}$.

Conclusion. We now combine the base case and the iterative case to conclude the proof. First we note that each time we apply the iterative case we have to extract a subsequence. Hence, taking a diagonal (not-relabelled) subsequence of $(\varphi_k)_{k \geq 1}$, this yields a sequence

$(E_i)_{i\geq 1}$ of mutually disjoint finite perimeter sets in Ω such that for every $m\geq 1$, $E_1,\dots,E_m$ satisfy properties (i)-(iii) (with m in place of N) with the corresponding $(d_k^{(i)})_{k\geq 1}\subset\mathbb{Z}$ and $\varphi_\infty^{(i)}\in GSBV^p(\Omega)$ for $i\in\{1,\dots,m\}$. In particular, from (4.33),

$$\liminf_{k\to+\infty}\mathcal{H}^{n-1}(S_{\varphi_k})\geq\mathcal{H}^{n-1}(S_{\Phi_m}\cup(\Omega\cap\partial^*(\cup_{i=1}^m E_i)))\,,\quad\forall m\geq 1 \tag{4.38}$$

with Φ_m as in (4.32) for $N=m$. We next show that

$$|\Omega\setminus(\cup_{i=1}^{+\infty}E_i)|=0.$$

To this aim, since $\sum_{m=1}^{+\infty}|E_m|<+\infty$, the sequence $(|E_m|)$ tends to zero as $m\to+\infty$[12], and by the inequality $|E_m|\geq\frac{n^n\omega_n|F_m|^2}{2^n(3M+\mathcal{H}^{n-1}(\partial\Omega))^2}$ we also infer that $|F_m|\to 0$. In particular,

$$|\Omega\setminus(\cup_{i=1}^{+\infty}E_i)|=\lim_{m\to+\infty}|\Omega\setminus(\cup_{i=1}^{m-1}E_i)|=\lim_{m\to+\infty}|F_m|=0\,.$$

Finally we define

$$\varphi_\infty(x):=\varphi_\infty^{(i)}(x)\qquad\text{if } x\in E_i \text{ for some } i\in\mathbb{N}.$$

We now show that $\varphi_\infty\in GSBV^p(\Omega)$ and that (3.21) holds true. By definition of φ_∞ and Φ_m it follows that

$$\mathcal{H}^{n-1}(S_{\varphi_\infty})\leq\lim_{m\to+\infty}\mathcal{H}^{n-1}(S_{\Phi_m})\leq\lim_{m\to+\infty}\mathcal{H}^{n-1}(S_{\Phi_m}\cup(\Omega\cap\partial^*(\cup_{i=1}^m E_i))).$$

This together with (4.38) yields (3.21). Eventually, being φ_∞ limit of liftings of u, it is itself a lifting of u in Ω and therefore from the identity $|\nabla u|=|\nabla\varphi_\infty|$ we infer $\varphi_\infty\in GSBV^p(\Omega)$. □

4.2 Proof of Theorem 3.3

Let $\varphi\in SBV^p(\Omega)\cap L^\infty(\Omega)$ be a lifting of u with

$$|\varphi|_{BV}\leq 2|u|_{BV}\leq C, \tag{4.39}$$

whose existence is ensured by Theorem 2.7 and Remark 2.8. The idea of the proof is to construct a sequence $(\widetilde{\psi}_k)_{k\geq 1}\subset SBV^p(\Omega)\cap L^\infty(\Omega)$ (see (4.48)) having the following properties:

$$\begin{aligned}&\widetilde{\psi}_k\to 0\quad\text{in } L^1(\Omega)\text{ and weakly}^*\text{ in } BV(\Omega),\\ &\widetilde{\varphi}_k:=\varphi_k-\widetilde{\psi}_k=\varphi+2\pi\sum_{z\in\mathbb{Z}}z\chi_{F_k^z}\,,\\ &\sup_{k\in\mathbb{N}}\mathcal{H}^{n-1}(\widetilde{\varphi}_k)<+\infty,\end{aligned} \tag{4.40}$$

with $(F_k^z)_{z\in\mathbb{Z}}$ a Caccioppoli partition of Ω for all $k\geq 1$, see also (4.50). Note that each $\widetilde{\varphi}_k$ is a lifting of the limit map u. Hence we can apply Theorem 3.1 to the sequence $(\widetilde{\varphi}_k)$, and eventually from the convergence of (a not relabelled subsequence of) $(\widetilde{\varphi}_k)_{k\geq 1}$ and Theorem 2.6 deduce compactness and lower semicontinuity for the original sequence $(\varphi_k)_{k\geq 1}$. We now give the details of the proof.

[12]Note that it might be $E_i=\emptyset$ for $i\geq\bar{i}$, this case is simpler to treat.

Step 1: construction of $\widetilde{\psi}_k$. For $k \geq 1$ define first

$$\psi_k := \varphi_k - \varphi \in GSBV^p(\Omega).$$

Since φ is a lifting of u, we have $[\![\varphi]\!] \in 2\pi\mathbb{Z}$ on $S_\varphi \setminus S_u$, thus, using also (4.39), $\mathcal{H}^{n-1}(S_\varphi) = \mathcal{H}^{n-1}(S_u) + \mathcal{H}^{n-1}(S_\varphi \setminus S_u) < +\infty$, and $|\nabla u| = |\nabla \varphi|$ a.e. on Ω. Moreover, since $S_{\psi_k} \subseteq S_{\varphi_k} \cup S_\varphi$ and $|\nabla \psi_k| \leq |\nabla \varphi_k| + |\nabla \varphi| = |\nabla u_k| + |\nabla u|$ a.e. on Ω, we have, using also (3.22),

$$\sup_{k\geq 1} \Big(\int_\Omega |\nabla \psi_k| \,\mathrm{d}x + \mathcal{H}^{n-1}(S_{\widetilde{\psi}_k}) \Big) \leq C < +\infty\,. \tag{4.41}$$

For every $k \in \mathbb{N}$ and $z \in \mathbb{Z}$ we define

$$\psi_k^z := (\psi_k \wedge 2\pi z) \vee 2\pi(z-1) \in SBV^p(\Omega)\,,$$

whose jump set decomposes as

$$S_{\psi_k^z} = S^1_{\psi_k^z} \cup S^2_{\psi_k^z} \cup S^3_{\psi_k^z} \cup S^4_{\psi_k^z}\,,$$

with

$$\begin{aligned}
S^1_{\psi_k^z} &:= \{x \in S_{\psi_k^z} : 2\pi(z-1) < (\psi_k^z)^-(x) < (\psi_k^z)^+(x) < 2\pi z\}\,,\\
S^2_{\psi_k^z} &:= \{x \in S_{\psi_k^z} : 2\pi(z-1) = (\psi_k^z)^-(x) < (\psi_k^z)^+(x) < 2\pi z\}\,,\\
S^3_{\psi_k^z} &:= \{x \in S_{\psi_k^z} : 2\pi(z-1) < (\psi_k^z)^-(x) < (\psi_k^z)^+(x) = 2\pi z\}\,,\\
S^4_{\psi_k^z} &:= \{x \in S_{\psi_k^z} : 2\pi(z-1) = (\psi_k^z)^-(x) < (\psi_k^z)^+(x) = 2\pi z\},
\end{aligned}$$

where $(\psi_k^z)^\pm$ are the two traces of ψ_k^z on $S_{\psi_k^z}$. As $\psi_k \in GSBV^p(\Omega)$ we have, up to a $\mathcal{H}^{n-1}$-negligible set,

$$\bigcup_{z\in\mathbb{Z}} S^1_{v_k^z} \cup \bigcup_{z\in\mathbb{Z}} S^2_{v_k^z} \cup \bigcup_{z\in\mathbb{Z}} S^3_{v_k^z} \subseteq S_{v_k}\,. \tag{4.42}$$

Notice that, for $\mathcal{H}^{n-1}$-a.e. $x \in S_{\psi_k}$, x belongs to at most 2 sets appearing in the union on the left-hand side of (4.42); hence in particular from (4.41), for all $k \geq 1$

$$\sum_{z\in\mathbb{Z}} \Big(\mathcal{H}^{n-1}(S^1_{\psi_k^z}) + \mathcal{H}^{n-1}(S^2_{\psi_k^z}) + \mathcal{H}^{n-1}(S^3_{\psi_k^z}) \Big) \leq 2\mathcal{H}^{n-1}(S_{\psi_k}) \leq C\,, \tag{4.43}$$

for some constant $C > 0$. Furthermore by definition of ψ_k^z,

$$\int_\Omega |\nabla \psi_k| \,\mathrm{d}x = \sum_{z\in\mathbb{Z}} \int_\Omega |\nabla \psi_k^z| \,\mathrm{d}x \leq C \qquad \forall k \in \mathbb{N}. \tag{4.44}$$

Consider now the function $\tau_k^z \in SBV^p(\Omega)$, defined by

$$\tau_k^z := \Big| \psi_k^z - 2\pi\Big(z - \frac{1}{2}\Big) \Big|,$$

which satisfies $0 \leq \tau_k^z \leq \pi$,

$$\int_\Omega |\nabla \psi_k^z| \,\mathrm{d}x = \int_\Omega |\nabla \tau_k^z| \,\mathrm{d}x, \qquad |[\![\tau_k^z]\!]| \leq |[\![\psi_k^z]\!]| \quad \mathcal{H}^{n-1} - \text{a.e. on } S_{\tau_k^z}\,,$$

$$S_{\tau_k^z} \subseteq S^1_{\psi_k^z} \cup S^2_{\psi_k^z} \cup S^3_{\psi_k^z}\,,$$

owing to the fact that τ_k^z has null jump on $S^4_{\psi_k^z}$. In particular, we infer from (4.43) and (4.44) that

$$\sum_{z\in\mathbb{Z}} |D\tau_k^z|(\Omega) \le C \qquad \forall k \in \mathbb{N}, \tag{4.45}$$

for some positive constant C.

Now, for $t \ge 0$, let $(E_k^z)^t := \{x \in \Omega : \tau_k^z(x) < t\}$; by the coarea formula,

$$\int_0^\pi \mathcal{H}^{n-1}(\partial^*(E_k^z)^t)\,\mathrm{d}t = |D\tau_k^z|(\Omega)\,,$$

hence, summing over z and using (4.45),

$$\int_0^\pi \sum_{z\in\mathbb{Z}} (\mathcal{H}^{n-1}(\partial^*(E_k^z)^t))\,\mathrm{d}t = \sum_{z\in\mathbb{Z}} \int_0^\pi \mathcal{H}^{n-1}(\partial^*(E_k^z)^t)\,\mathrm{d}t \le C.$$

Whence, for any $k \in \mathbb{Z}$ we can find a number $t_k \in (0, \pi/2)$ such that

$$\sum_{z\in\mathbb{Z}} \mathcal{H}^{n-1}(\partial^*(E_k^z)^{t_k}) \le C\,, \tag{4.46}$$

with $C > 0$ independent of k. We observe that

$$(E_k^z)^{t_k} = \left\{ x \in \Omega : 2\pi\Big(z - \frac{1}{2}\Big) - t_k < \psi_k^z(x) = \psi_k(x) < 2\pi\Big(z - \frac{1}{2}\Big) + t_k \right\}.$$

For every k let $(F_k^z)_{z\in\mathbb{Z}}$ be the Caccioppoli partition of Ω defined as

$$F_k^z := \left\{ x \in \Omega : \psi_k(x) \in \Big(2\pi\Big(z - \frac{1}{2}\Big) - t_k, 2\pi\Big(z + 1 - \frac{1}{2}\Big) - t_k \Big) \right\}.$$

Thus $\partial^* F_k^z \subset \partial^*(E_k^z)^{t_k} \cup \partial^*(E_k^{z+1})^{t_k}$ and

$$\sum_{z\in\mathbb{Z}} \mathcal{H}^{n-1}(\partial^* F_k^z) \le 2 \sum_{z\in\mathbb{Z}} \mathcal{H}^{n-1}(\partial^*(E_k^z)^{t_k}) \le C\,, \tag{4.47}$$

and so for each $k \in \mathbb{N}$ the family $(F_k^z)_{z\in\mathbb{Z}}$ is a Caccioppoli partition of Ω, with equibounded (in k) total perimeter.

We finally introduce the map $\widetilde{\psi}_k \in SBV^p(\Omega) \cap L^\infty(\Omega)$ as

$$\widetilde{\psi}_k := \sum_{z\in\mathbb{Z}} \Big(\psi_k - (2\pi(z - \frac{1}{2}) - t_k) \Big) \chi_{F_k^z} - \pi - t_k = \psi_k - 2\pi \sum_{z\in\mathbb{Z}} z \chi_{F_k^z}, \tag{4.48}$$

which satisfies $-\pi \le \widetilde{\psi}_k \le \pi$. Moreover, since $S_{\widetilde{\psi}_k} \subseteq S_{\psi_k} \cup \bigcup_{z\in\mathbb{Z}} \partial^* F_k^z$, by (4.47) we deduce that

$$\mathcal{H}^{n-1}(S_{\widetilde{\psi}_k}) + \|\widetilde{\psi}_k\|_{BV} \le C, \tag{4.49}$$

for all $k \ge 1$. Therefore, up to a not relabelled subsequence, we can also suppose

$$\widetilde{\psi}_k \to \widetilde{\psi} \qquad \text{in } L^1(\Omega) \text{ and weakly}^* \text{ in } BV(\Omega)$$

for some $\widetilde{\psi} \in SBV^p(\Omega) \cap L^\infty(\Omega)$. As asserted in (4.40), we now want to show that $\widetilde{\psi} = 0$ a.e. on Ω. Let us go back to the functions u_k that, up to subsequences, are converging pointwise a.e. to u. Hence for a.e. x we have

$$u_k(x) - u(x) = e^{i\varphi(x)}(e^{i(\varphi_k(x)-\varphi(x))} - 1) \to 0\,.$$

This in turn implies that the map $x \mapsto \mathrm{dist}(\varphi_k(x) - \varphi(x), 2\pi\mathbb{Z}) = \mathrm{dist}(\psi_k(x), 2\pi\mathbb{Z})$ is converging pointwise a.e. to 0, and by the dominated convergence theorem, in $L^1(\Omega)$. Then there is $\bar{k} \geq 0$ such that for a.e. $x \in \Omega$ and for all $k \geq \bar{k}$ there is $K = K(x,k) \in \mathbb{Z}$ such that

$$x \in F_k^K \quad \text{and} \quad |\widetilde{\psi}_k(x)| = |\psi_k(x) - 2\pi K| \to 0 \qquad \text{as } k \to +\infty,$$

and so by the dominated convergence theorem we conclude $\widetilde{\psi} = 0$.

Step 2: compactness and lower semicontinuity. We first observe that $\psi_k - \widetilde{\psi}_k \in GSBV^p(\Omega)$ takes values in $2\pi\mathbb{Z}$. As a consequence, the maps $\widetilde{\varphi}_k = \varphi_k - \widetilde{\psi}_k = \psi_k - \widetilde{\psi}_k + \varphi$ are all liftings of u and from (4.49) we also have

$$\sup_{k\geq 1} \mathcal{H}^{n-1}(S_{\widetilde{\varphi}_k}) < +\infty. \tag{4.50}$$

We now apply Theorem 3.1 to the sequence $(\widetilde{\varphi}_k)_{k\geq 1}$, and deduce the existence of a Caccioppoli partition $(E_i)_{i\geq 1}$ of Ω, of sequences $(d_k^{(i)})_{k\geq 1} \subset 2\pi\mathbb{Z}$ for all $i \geq 1$, and of a lifting $\varphi_\infty \in GSBV^p(\Omega)$ of u, such that

$$\lim_{k\to+\infty} (\widetilde{\varphi}_k(x) - 2\pi d_k^{(i)}) = \varphi_\infty(x) \quad \text{for a.e. } x \in E_i\,,$$
$$\lim_{k\to+\infty} |\widetilde{\varphi}_k(x) - 2\pi d_k^{(i)}| = +\infty \quad \text{for a.e. } x \in \Omega \setminus E_i\,,$$

for all $i \geq 1$. As a consequence, by Step 2,

$$\lim_{k\to+\infty} (\varphi_k(x) - 2\pi d_k^{(i)}) = \varphi_\infty(x) \quad \text{for a.e. } x \in E_i\,,$$
$$\lim_{k\to+\infty} |\varphi_k(x) - 2\pi d_k^{(i)}| = +\infty \quad \text{for a.e. } x \in \Omega \setminus E_i\,,$$

for all $i \geq 1$.

It remains to show (3.24). For all $i \geq 1$ let $\varphi_\infty^{(i)} := \varphi_\infty$ in E_i and $\varphi_\infty^{(i)} := 0$ in $\Omega \setminus E_i$. By Theorem 2.6 and the fact that $S_{\varphi_k} = S_{\varphi_k - 2\pi d_k^{(i)}}$ it follows

$$\liminf_{k\to+\infty} \mathcal{H}^{n-1}(S_{\varphi_k} \cap A) \geq \mathcal{H}^{n-1}((S_{\varphi_\infty^{(i)}} \cup \partial^* E_i) \cap A)\,,$$

for any open set $A \subset \Omega$. Then by Lemma 4.2 for all $N \geq 1$ we get

$$\liminf_{k\to+\infty} \mathcal{H}^{n-1}(S_{\varphi_k}) \geq \mathcal{H}^{n-1}(S_{\overline{\varphi}_\infty^{(N)}} \cup (\partial^*(\cup_{i=1}^N E_i)))\,,$$

where

$$\overline{\varphi}_\infty^{(N)} := \begin{cases} \varphi_\infty^{(i)} & \text{if } x \in E_i \text{ for some } i = 1, \ldots, N\,, \\ 0 & \text{if } x \in \Omega \setminus (\cup_{i=1}^N E_i)\,. \end{cases}$$

Hence, observing that $\varphi_\infty = \overline{\varphi}_\infty^{(N)}$ in $\cup_{i=1}^N E_i$, and letting $N \to +\infty$ we get

$$\liminf_{k\to+\infty} \mathcal{H}^{n-1}(S_{\varphi_k}) \geq \mathcal{H}^{n-1}(S_{\varphi_\infty} \cup (\partial^*(\cup_{i=1}^{+\infty} E_i))) \geq \mathcal{H}^{n-1}(S_{\varphi_\infty})\,.$$

□

5 Γ-convergence of functionals on $\mathbb{S}^1$- valued maps

In this section we prove Theorems 1.1 and 1.2. The proof of Theorem 1.1 follows by suitably adapting the arguments of [7, 8, 28]. Instead, the proof of Theorem 1.2 (and more specifically the lower bound inequality) requires some new ideas which rely on the compactness result for liftings (Theorem 3.3). For convenience we introduce the localised Modica-Mortola-type (or Allen-Cahn type) functionals

$$\mathrm{MM}_\varepsilon(v, A) := \int_A \left(\varepsilon |\nabla v|^2 + \frac{(v-1)^2}{4\varepsilon} \right) \mathrm{d}x \qquad \forall v \in W^{1,2}(\Omega),$$

for every open set $A \subseteq \Omega$.

5.1 Some density and approximation results for $\mathbb{S}^1$-valued maps

For $E \subset \mathbb{R}^n$ we denote by $\mathcal{M}^{n-1}(E)$ its $(n-1)$-dimensional Minkowski content, i.e.,

$$\mathcal{M}^{n-1}(E) = \lim_{\rho \searrow 0} \frac{|\{x \in \mathbb{R}^n \colon \operatorname{dist}(x, E) < \rho\}|}{2\rho},$$

provided the limit exists.

Proposition 5.1 (Density in $SBV^2(\Omega;\mathbb{S}^1)$). *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with Lipschitz boundary, $u \in SBV^2(\Omega;\mathbb{S}^1)$ and let $\widehat{\Omega} \subset \mathbb{R}^n$ be a bounded open set with $\Omega \subset\subset \widehat{\Omega}$. Then u has an extension $\widehat{u} \in SBV^2(\widehat{\Omega};\mathbb{S}^1)$ with*

$$\mathcal{H}^{n-1}(S_{\widehat{u}} \cap \partial\Omega) = 0.$$

Moreover, there exists a sequence $(\widehat{u}_k) \subset SBV^2(\widehat{\Omega};\mathbb{S}^1)$ such that:

(i) $\widehat{u}_k$ *converges to* $\widehat{u}$ *in* $L^1(\widehat{\Omega};\mathbb{S}^1)$*;*

(ii) $\nabla\widehat{u}_k$ *converges to* $\nabla\widehat{u}$ *in* $L^2(\widehat{\Omega};\mathbb{R}^{n\times n})$*;*

(iii) $\lim_{k\to\infty} \mathcal{H}^{n-1}(S_{\widehat{u}_k}) = \mathcal{H}^{n-1}(S_{\widehat{u}})$*;*

(iv) $\lim_{k\to\infty} \mathcal{H}^{n-1}(S_{\widehat{u}_k} \cap \overline{\Omega}) = \mathcal{H}^{n-1}(S_{\widehat{u}} \cap \Omega) = \mathcal{H}^{n-1}(S_u)$*;*

(v) $\mathcal{H}^{n-1}(\overline{S}_{\widehat{u}_k} \setminus S_{\widehat{u}_k}) = 0$ *and* $\mathcal{H}^{n-1}(S_{\widehat{u}_k} \cap K) = \mathcal{M}^{n-1}(S_{\widehat{u}_k} \cap K)$ *for every compact set* $K \subset \widehat{\Omega}$*.*

Note that (ii)-(iii) guarantee the convergence of $\mathrm{MS}_{\mathbb{S}^1}(\widehat{u}_k, 1)$ to $\mathrm{MS}_{\mathbb{S}^1}(\widehat{u}, 1)$ in $\widehat{\Omega}$.

Proof. The proof follows by suitably adapting [14, Lemma 4.11] and [7, Proposition 5.3] to the $\mathbb{S}^1$-constrained case. For this reason we give here only the main steps. According to Remark 2.8 there exists a lifting $\varphi \in SBV^2(\Omega)$ of u. Since $\partial\Omega$ is Lipschitz, by a standard reflection argument we can construct an extension $\widehat{\varphi} \in SBV^2(\widehat{\Omega})$ of φ such that

$$\mathcal{H}^{n-1}(S_{\widehat{\varphi}} \cap \partial\Omega) = 0.$$

By defining

$$\widehat{u} \colon \widehat{\Omega} \to \mathbb{S}^1, \qquad \widehat{u} := e^{i\widehat{\varphi}},$$

we immediately get that $\widehat{u}$ is an extension of u, and

$$\int_{\widehat{\Omega}} |\nabla\widehat{u}|^2 \,\mathrm{d}x = \int_{\widehat{\Omega}} |\nabla\widehat{\varphi}|^2 \,\mathrm{d}x, \quad \mathcal{H}^{n-1}(S_{\widehat{u}}) \le \mathcal{H}^{n-1}(S_{\widehat{\varphi}}),$$

and thus $\widehat{u} \in SBV^2(\widehat{\Omega};\mathbb{S}^1)$. Moreover

$$\mathcal{H}^{n-1}(S_{\widehat{u}} \cap \partial\Omega) \le \mathcal{H}^{n-1}(S_{\widehat{\varphi}} \cap \partial\Omega) = 0. \tag{5.1}$$

By [17, Lemma 4.3] we know that, for every integer $k > 0$, there exists $\widehat{u}_k \in SBV^2(\widehat{\Omega};\mathbb{S}^1)$ such that

$$\begin{aligned} &\int_{\widehat{\Omega}} |\nabla \widehat{u}_k|^2 \,\mathrm{d}x + \mathcal{H}^{n-1}(S_{\widehat{u}_k}) + k \int_{\widehat{\Omega}} |\widehat{u}_k - \widehat{u}| \,\mathrm{d}x \\ &= \min_{w \in SBV^2(\widehat{\Omega};\mathbb{S}^1)} \left(\int_{\widehat{\Omega}} |\nabla w|^2 \,\mathrm{d}x + \mathcal{H}^{n-1}(S_w) + k \int_{\widehat{\Omega}} |w - \widehat{u}| \,\mathrm{d}x \right). \end{aligned}$$

Clearly $\|\widehat{u}_k\|_\infty = 1$ and

$$\int_{\widehat{\Omega}} |\nabla \widehat{u}_k|^2 \,\mathrm{d}x + \mathcal{H}^{n-1}(S_{\widehat{u}_k}) + k \int_{\widehat{\Omega}} |\widehat{u}_k - \widehat{u}| \,\mathrm{d}x \le \int_{\widehat{\Omega}} |\nabla \widehat{u}|^2 \,\mathrm{d}x + \mathcal{H}^{n-1}(S_{\widehat{u}}) < +\infty\,. \tag{5.2}$$

Thus, in particular, $(\widehat{u}_k)$ converges to $\widehat{u}$ in $L^1(\widehat{\Omega};\mathbb{S}^1)$ as $k \to +\infty$, and (i) follows. By [5] we also get, up to a not relabelled subsequence,

$$\nabla \widehat{u}_k \rightharpoonup \nabla \widehat{u} \quad \text{in } L^2(\widehat{\Omega};\mathbb{R}^{n\times n})\,, \qquad \liminf_{k\to+\infty} \mathcal{H}^{n-1}(S_{\widehat{u}_k}) \ge \mathcal{H}^{n-1}(S_{\widehat{u}})\,.$$

This and (5.2) imply

$$\int_{\widehat{\Omega}} |\nabla \widehat{u}|^2 \,\mathrm{d}x + \mathcal{H}^{n-1}(S_{\widehat{u}}) = \lim_{k\to+\infty} \left(\int_{\widehat{\Omega}} |\nabla \widehat{u}_k|^2 \,\mathrm{d}x + \mathcal{H}^{n-1}(S_{\widehat{u}_k}) \right),$$

from which we readily deduce (ii) and (iii). Again by [5] and (5.1) we have

$$\mathcal{H}^{n-1}(S_{\widehat{u}} \cap \Omega) \le \liminf_{k\to+\infty} \mathcal{H}^{n-1}(S_{\widehat{u}_k} \cap \Omega) \le \liminf_{k\to+\infty} \mathcal{H}^{n-1}(S_{\widehat{u}_k} \cap \overline{\Omega})\,,$$

$$\mathcal{H}^{n-1}(S_{\widehat{u}} \setminus \Omega) = \mathcal{H}^{n-1}(S_{\widehat{u}} \setminus \overline{\Omega}) \le \liminf_{k\to+\infty} \mathcal{H}^{n-1}(S_{\widehat{u}_k} \setminus \overline{\Omega})\,,$$

which in turn imply (iv). Moreover, invoking [17, Lemma 4.5] we have $\mathcal{H}^{n-1}(\overline{S}_{\widehat{u}_k} \setminus S_{\widehat{u}_k}) = 0$. Finally, thanks to the density estimate in [17, Lemma 4.9] and arguing exactly as in [7, Proposition 5.3] it can be shown that

$$\mathcal{H}^{n-1}(S_{\widehat{u}_k} \cap K) = \mathcal{M}^{n-1}(S_{\widehat{u}_k} \cap K)\,,$$

for every compact set $K \subset \widehat{\Omega}$ and (v) is proven. □

A similar result holds also for liftings:

Proposition 5.2 (**Density result for liftings**). *Let Ω and $\widehat{\Omega} \subset \mathbb{R}^n$ be as in Proposition 5.1. Let $u \in SBV^2(\Omega;\mathbb{S}^1)$ and $\varphi \in SBV^2(\Omega) \cap L^\infty(\Omega)$ be a lifting of u. Then φ has an extension $\widehat{\varphi} \in SBV^2(\widehat{\Omega}) \cap L^\infty(\widehat{\Omega})$ satisfying $\mathcal{H}^{n-1}(S_{\widehat{\varphi}} \cap \partial\Omega) = 0$. Moreover, there exists a sequence $(\widehat{\varphi}_k) \subset SBV^2(\widehat{\Omega})$ such that:*

(i) $\widehat{\varphi}_k$ *converges to* $\widehat{\varphi}$ *in* $L^1(\widehat{\Omega})$*;*

(ii) $\nabla \widehat{\varphi}_k$ *converges to* $\nabla \widehat{\varphi}$ *in* $L^2(\widehat{\Omega};\mathbb{R}^n)$*;*

(iii) $\lim_{k\to\infty} \mathcal{H}^{n-1}(S_{\widehat{\varphi}_k}) = \mathcal{H}^{n-1}(S_{\widehat{\varphi}})$*;*

(iv) $\lim_{k\to\infty} \mathcal{H}^{n-1}(S_{\widehat{\varphi}_k} \cap \overline{\Omega}) = \mathcal{H}^{n-1}(S_{\widehat{\varphi}} \cap \Omega) = \mathcal{H}^{n-1}(S_\varphi)$;

(v) $\mathcal{H}^{n-1}(\overline{S}_{\widehat{\varphi}_k} \setminus S_{\widehat{\varphi}_k}) = 0$ *and* $\mathcal{H}^{n-1}(S_{\widehat{\varphi}_k} \cap K) = \mathcal{M}^{n-1}(S_{\widehat{\varphi}_k} \cap K)$ *for every compact set* $K \subset \widehat{\Omega}$;

(vi) $\widehat{u}_k := e^{i\widehat{\varphi}_k}$ *converges to* u *in* $L^1(\Omega; \mathbb{S}^1)$.

Note that (ii)-(iii) guarantee the convergence of $\mathrm{MS}(\widehat{\varphi}_k, 1)$ to $\mathrm{MS}(\widehat{\varphi}, 1)$ in $\widehat{\Omega}$.

Proof. From [14, Lemma 4.11] we can find $\widehat{\varphi}$ as in the statement and $(\widehat{\varphi}_k) \subset SBV^2(\widehat{\Omega})$ that satisfy (i)–(v). This in turn implies also (vi). □

5.2 Proof of Theorem 1.1

If $\varepsilon_k \searrow 0$ and $((u_k, v_k))_{k\geq 1} \subset L^1(\Omega; \mathbb{S}^1) \times L^1(\Omega, [0, 1])$ is a sequence converging to (u, v) in $L^1(\Omega; \mathbb{S}^1) \times L^1(\Omega)$, arguing exactly as in [28], one gets

$$\liminf_{k\to+\infty} \mathrm{AT}^{\mathbb{S}^1}_{\varepsilon_k}(u_k, v_k) \geq \mathrm{MS}_{\mathbb{S}^1}(u, v).$$

Therefore, we only need to prove the upper bound. Let $\varepsilon_k \searrow 0$ and $u \in SBV^2(\Omega; \mathbb{S}^1)$. We have to find a sequence $((u_k, v_k)) \subset \widehat{\mathcal{A}}_{\mathbb{S}^1}$ converging to $(u, 1)$ in $L^1(\Omega; \mathbb{S}^1) \times L^1(\Omega)$ for which

$$\limsup_{k\to+\infty} \mathrm{AT}^{\mathbb{S}^1}_{\varepsilon_k}(u_k, v_k) \leq \mathrm{MS}_{\mathbb{S}^1}(u, 1)\,. \tag{5.3}$$

We note that, in general, we cannot take $((u_k, v_k)) \subset \mathcal{A}_{\mathbb{S}^1}$, as it happens for the example described in the introduction.

We fix a bounded open set $\widehat{\Omega} \subset\subset \mathbb{R}^n$, with $\Omega \subset\subset \widehat{\Omega}$. By Proposition 5.1 it suffices to show (5.3) for $u \in SBV^2(\widehat{\Omega}; \mathbb{S}^1)$ with

$$\mathcal{H}^{n-1}(S_u \cap \partial\Omega) = 0\,, \quad \mathcal{H}^{n-1}(\overline{S}_u \setminus S_u) = 0\,, \quad \mathcal{H}^{n-1}(S_u \cap K) = \mathcal{M}^{n-1}(S_u \cap K)\,, \tag{5.4}$$

for every compact set $K \subset \widehat{\Omega}$.

By Theorem 2.7 and Remark 2.8 there exists a lifting $\varphi \in SBV^2(\widehat{\Omega})$ of u. For any $0 < \rho \ll 1$ define

$$(S_u)_\rho := \{x \in \widehat{\Omega} \colon \mathrm{dist}(x, S_u) < \rho\}.$$

Let $0 < \xi_k$ with $\lim_{k\to+\infty} \xi_k/\varepsilon_k = 0$ be such that $(S_u)_{\xi_k}$ has Lipschitz boundary. By Remark 2.2 applied to $A = (S_u)_{\xi_k}$ and $\delta = \varepsilon_k$, there exists $\varphi_k \in C^\infty((S_u)_{\xi_k})$ such that

$$\int_{(S_u)_{\xi_k}} |\varphi - \varphi_k| \, \mathrm{d}x < \varepsilon_k\,, \quad \int_{(S_u)_{\xi_k}} |\nabla \varphi_k| \, \mathrm{d}x \leq |D\varphi|\big((S_u)_{\xi_k}\big) + \varepsilon_k\,, \tag{5.5}$$

and, in the sense of BV-traces,

$$\varphi_k = \varphi \quad \text{a.e. on} \quad \partial(S_u)_{\xi_k}\,. \tag{5.6}$$

We define $\omega_k \in SBV^2(\widehat{\Omega})$ as follows:

$$\omega_k := \begin{cases} \varphi_k & \text{in } (S_u)_{\xi_k} \\ \varphi & \text{in } \widehat{\Omega} \setminus (S_u)_{\xi_k} \end{cases}.$$

Then by (5.5) the sequence (ω_k) converges to φ strictly in $BV(\widehat{\Omega})$ and

$$S_{\omega_k} \subseteq S_\varphi.$$

We set

$$u_k := e^{i\omega_k} = \begin{cases} e^{i\varphi_k} & \text{in } (S_u)_{\xi_k} \\ u & \text{in } \widehat{\Omega} \setminus (S_u)_{\xi_k} \end{cases}$$

so that

$$u_k \in W^{1,1}(\widehat{\Omega}; \mathbb{S}^1) \tag{5.7}$$

and (u_k) converges to u in $L^1(\Omega; \mathbb{S}^1)$. Since we modified u only in a neighbourhood of S_u in general $u_k \notin W^{1,2}(\widehat{\Omega}; \mathbb{S}^1)$, as it happens in the example in the introduction.

The construction of (v_k) is done as in [18]. Precisely, we define $(v_k) \subset W^{1,2}(\widehat{\Omega})$ as

$$v_k(x) := \begin{cases} 0 & x \in (S_u)_{\xi_k} \\ 1 - \exp\left(-\frac{\text{dist }(x, S_u) - \xi_k}{2\varepsilon_k}\right) & x \in \widehat{\Omega} \setminus (S_u)_{\xi_k} \end{cases},$$

Then (v_k) converges to 1 in $L^1(\widehat{\Omega})$ as $k \to +\infty$. Moreover

$$v_k |\nabla u_k| \in L^2(\widehat{\Omega})$$

and hence, using (5.7), we get the crucial inclusion

$$(u_k, v_k) \subset \widehat{\mathcal{A}}_{\mathbb{S}^1}.$$

It remains to show (5.3). To this aim we have

$$\limsup_{k\to+\infty} \int_\Omega v_k^2 |\nabla u_k|^2 \,\mathrm{d}x \le \limsup_{k\to+\infty} \int_{\Omega\setminus(S_u)_{\xi_k}} |\nabla u|^2 \,\mathrm{d}x \le \int_\Omega |\nabla u|^2 \,\mathrm{d}x\,. \tag{5.8}$$

Moreover

$$\mathrm{MM}_{\varepsilon_k}(v_k, \Omega) = \frac{|(S_u)_{\xi_k} \cap \overline{\Omega}|}{\varepsilon_k} + \mathrm{MM}_{\varepsilon_k}(v_k, \Omega \setminus (S_u)_{\xi_k}). \tag{5.9}$$

From (5.4) it follows

$$\limsup_{k\to+\infty} \frac{|(S_u)_{\xi_k} \cap \overline{\Omega}|}{2\varepsilon_k} = \limsup_{k\to+\infty} \frac{|(S_u)_{\xi_k} \cap \overline{\Omega}|}{2\xi_k} \frac{\xi_k}{\varepsilon_k} = 0. \tag{5.10}$$

Now, we estimate the second term on the right hand-side of (5.9). At almost all points in $\Omega \setminus (S_u)_{\xi_k}$ there holds

$$|\nabla v_k| = \frac{1}{2\varepsilon_k} \exp\left(-\frac{\text{dist }(x, S_u) - \xi_k}{2\varepsilon_k}\right) |\nabla \text{dist }(x, S_u)| = \frac{1}{2\varepsilon_k} \exp\left(-\frac{\text{dist }(x, S_u) - \xi_k}{2\varepsilon_k}\right),$$

from which, using also the coarea formula we deduce

$$\begin{aligned} \mathrm{MM}_{\varepsilon_k}(v_k, \Omega \setminus (S_u)_{\xi_k}) &= \frac{1}{2\varepsilon_k} \int_{\Omega\setminus(S_u)_{\xi_k}} \exp\left(-\frac{\text{dist }(x, S_u) - \xi_k}{\varepsilon_k}\right) \mathrm{d}x \\ &= \frac{1}{2\varepsilon_k} \int_{\xi_k}^{+\infty} e^{-\frac{t-\xi_k}{\varepsilon_k}} \mathcal{H}^{n-1}(\partial E_t) \,\mathrm{d}t\,, \end{aligned} \tag{5.11}$$

with $E_t := \{x \in \Omega \colon \mathrm{dist}(x, S_u) > t\}$. Next, having that

$$\mathcal{H}^{n-1}(\partial E_t) = \frac{\mathrm{d}}{\mathrm{d}t}\bigg(\int_0^t \mathcal{H}^{n-1}(\partial E_s)\,\mathrm{d}s\bigg) = \frac{\mathrm{d}}{\mathrm{d}t}|(S_u)_t \cap \overline{\Omega}|,$$

integrating by parts first, and using the change of variable $s = \frac{t-\xi_k}{\varepsilon_k}$ we obtain

$$\begin{aligned}
&\frac{1}{2\varepsilon_k}\int_{\xi_k}^{+\infty} e^{-\frac{t-\xi_k}{\varepsilon_k}}\mathcal{H}^{n-1}(\partial E_t)\,\mathrm{d}t \\
&= -\frac{1}{2\varepsilon_k}|(S_u)_{\xi_k}| + \frac{1}{2\varepsilon_k^2}\int_{\xi_k}^{+\infty} e^{-\frac{t-\xi_k}{\varepsilon_k}}|(S_u)_t \cap \overline{\Omega}|\,\mathrm{d}t \\
&= -\frac{1}{2\varepsilon_k}|(S_u)_{\xi_k}| + \int_0^{+\infty}(s+\varepsilon_k)e^{-s}\frac{|(S_u)_{s\varepsilon_k+\xi_k} \cap \overline{\Omega}|}{2(s\varepsilon_k+\xi_k)}\,\mathrm{d}s.
\end{aligned} \tag{5.12}$$

Gathering together (5.9)–(5.12) we find

$$\mathrm{MM}_{\varepsilon_k}(v_k, \Omega) = \frac{1}{2\varepsilon_k}|(S_u)_{\xi_k} \cap \overline{\Omega}| + \int_0^{+\infty}(s+\varepsilon_k)e^{-s}\frac{|(S_u)_{s\varepsilon_k+\xi_k} \cap \overline{\Omega}|}{2(s\varepsilon_k+\xi_k)}\,\mathrm{d}s\,. \tag{5.13}$$

From (5.4) it follows

$$\limsup_{k\to+\infty}\frac{|(S_u)_{s\varepsilon_k+\xi_k} \cap \overline{\Omega}|}{2(s\varepsilon_k+\xi_k)} = \mathcal{H}^{n-1}(S_u \cap \overline{\Omega})\,.$$

This, together with the fact that $\int_0^{+\infty}(s+\varepsilon_k)e^{-s}\,\mathrm{d}s \to 1$ as $k \to +\infty$, imply

$$\limsup_{k\to+\infty}\mathrm{MM}_{\varepsilon_k}(v_k, \Omega) \le \mathcal{H}^{n-1}(S_u \cap \overline{\Omega})\,. \tag{5.14}$$

Eventually, combining (5.8), (5.10) and (5.14) we infer (5.3). □

5.3 Proof of Theorem 1.2

Step 1: Lower bound. Let $\varepsilon_k \searrow 0$ as $k \to +\infty$. We have to show that, for every sequence $((u_k, v_k))_{k\ge1} \subset L^1(\Omega; \mathbb{S}^1) \times L^1(\Omega)$ converging to (u, v) in $L^1(\Omega; \mathbb{S}^1) \times L^1(\Omega)$,

$$\liminf_{k\to+\infty}\mathrm{AT}^{\mathbb{S}^1}_{\varepsilon_k}(u_k, v_k) \ge \mathrm{MS}_{\mathrm{lift}}(u, v)\,. \tag{5.15}$$

We may assume

$$\sup_{k\in\mathbb{N}}\mathrm{AT}^{\mathbb{S}^1}_{\varepsilon_k}(u_k, v_k) \le C < +\infty \tag{5.16}$$

so that $(u_k, v_k) \in \mathcal{A}_{\mathbb{S}^1}$, $v = 1$ a.e. in Ω and, up to a not relabelled subsequence,

$$\begin{aligned}
\liminf_{k\to+\infty}\mathrm{AT}^{\mathbb{S}^1}_{\varepsilon_k}(u_k, v_k) &= \lim_{k\to+\infty}\mathrm{AT}^{\mathbb{S}^1}_{\varepsilon_k}(u_k, v_k), \\
\liminf_{k\to+\infty}\mathrm{MM}_{\varepsilon_k}(v_k, \Omega) &= \lim_{k\to+\infty}\mathrm{MM}_{\varepsilon_k}(v_k, \Omega).
\end{aligned}$$

From Theorem 2.3 and the decoupling property (2.2), it follows that

$$\begin{aligned}
&\liminf_{k\to+\infty}\mathrm{AT}^{\mathbb{S}^1}_{\varepsilon_k}(u_k, v_k) \ge \int_\Omega |\nabla u|^2\,\mathrm{d}x + \mathcal{H}^{n-1}(S_u)\,, \\
&\liminf_{k\to+\infty}\int_\Omega v_k^2|\nabla u_k|^2\,\mathrm{d}x \ge \int_\Omega |\nabla u|^2\,\mathrm{d}x\,, \\
&\liminf_{k\to+\infty}\mathrm{MM}_{\varepsilon_k}(v_k, \Omega) \ge \mathcal{H}^{n-1}(S_u)\,.
\end{aligned}$$

In particular, from (5.16), $u \in SBV^2(\Omega;\mathbb{S}^1)$. Hence it remains to show

$$\lim_{k\to+\infty} \mathrm{MM}_{\varepsilon_k}(v_k,\Omega) \geq m_2[u]. \tag{5.17}$$

For every $k \geq 1$ we start by selecting a lifting $\varphi_k \in W^{1,2}(\Omega)$ of u_k. Since $|\nabla u_k| = |\nabla \varphi_k|$ we have

$$+\infty > C \geq \mathrm{AT}^{\mathbb{S}^1}_{\varepsilon_k}(u_k,v_k) = \int_\Omega \left(v_k^2|\nabla\varphi_k|^2 + \varepsilon_k|\nabla v_k|^2 + \frac{(v_k-1)^2}{4\varepsilon_k}\right) \mathrm{d}x\,.$$

Thus, using the coarea formula,

$$C \geq \mathrm{MM}_{\varepsilon_k}(v_k,\Omega) \geq \int_\Omega (1-v_k)|\nabla v_k|\,\mathrm{d}x = \int_0^1 (1-t)\mathcal{H}^{n-1}(\partial^* F_k^t)\,\mathrm{d}t \tag{5.18}$$

for any $k \geq 1$, where $F_k^t := \{x \in \Omega : v_k(x) \leq t\}$. Let $\eta', \eta'' \in (0,1)$, $\eta' < \eta''$ be fixed. By (5.18) and the mean value theorem there exists $t(k) \in (\eta',\eta'')$ such that

$$C \geq (1-\eta'')(\eta''-\eta')\mathcal{H}^{n-1}(\partial^* F_k^{t(k)})\,. \tag{5.19}$$

Moreover

$$C \geq \int_\Omega v_k^2 |\nabla\varphi_k|^2\,\mathrm{d}x \geq (\eta')^2 \int_\Omega \chi_{\Omega\setminus F_k^{t(k)}} |\nabla\varphi_k|^2\,\mathrm{d}x\,. \tag{5.20}$$

Then, setting

$$\phi_k := \varphi_k \chi_{\Omega\setminus F_k^{t(k)}} \in SBV^2(\Omega),$$

we have $S_{\phi_k} \subset \partial^* F_k^{t(k)}$ and, for the absolutely continuous parts of the gradients, $\nabla\phi_k = \nabla\varphi_k\chi_{\Omega\setminus F_k^{t(k)}}$ and so, from (5.20) and (5.19),

$$\int_\Omega |\nabla\phi_k|^2\,\mathrm{d}x + \mathcal{H}^{n-1}(S_{\phi_k}) \leq C(\eta',\eta'') \tag{5.21}$$

for some $C(\eta',\eta'') > 0$ depending on η',η'' and independent of k. Let also

$$\overline{u}_k := e^{i\phi_k} = u_k\chi_{\Omega\setminus F_k^{t(k)}} + (1,0)\chi_{F_k^{t(k)}},$$

which, by (5.19), are uniformly bounded in $BV(\Omega;\mathbb{S}^1)$. Since

$$|F_k^{t(k)}| \to 0, \tag{5.22}$$

the sequence $(\overline{u}_k)$ weakly star converges to u in $BV(\Omega;\mathbb{S}^1)$. Hence, using (5.21), we can apply Theorem 3.3 to the sequence $(\phi_k)_{k\geq1}$ (with $p = 2$) and get, for a not-relabelled subsequence, a Caccioppoli partition $(E_i)_{i\in\mathbb{N}}$ of Ω, sequences $(d_k^{(i)})_{k\geq1} \subset \mathbb{Z}$ for any integer $i \geq 1$ and a lifting $\varphi_\infty \in GSBV^2(\Omega)$ of u, such that

$$\lim_{k\to+\infty} (\phi_k(x) - 2\pi d_k^{(i)}) = \varphi_\infty(x) \qquad \forall i \in \mathbb{N},\ \text{for a.e. } x \in E_i\,,$$
$$\lim_{k\to+\infty} |\phi_k(x) - 2\pi d_k^{(i)}| = +\infty \qquad \forall i \in \mathbb{N},\ \text{for a.e. } x \in \Omega\setminus E_i.$$

Again, using (5.22), the same holds for φ_k, i.e.,

$$\begin{aligned} &\lim_{k\to+\infty} (\varphi_k(x) - 2\pi d_k^{(i)}) = \varphi_\infty(x) \qquad \forall i \in \mathbb{N},\ \text{for a.e. } x \in E_i\,,\\ &\lim_{k\to+\infty} |\varphi_k(x) - 2\pi d_k^{(i)}| = +\infty \qquad \forall i \in \mathbb{N},\ \text{for a.e. } x \in \Omega\setminus E_i. \end{aligned} \tag{5.23}$$

Then (5.17) follows if we show

$$\liminf_{k\to+\infty} \mathrm{MM}(v_k,\Omega) \geq \mathcal{H}^{n-1}(S_{\varphi_\infty}), \tag{5.24}$$

since, being φ_∞ a lifting of u, we have

$$\mathcal{H}^{n-1}(S_{\varphi_\infty}) \geq m_2[u].$$

For any integer $K \geq 1$ we consider the truncated function

$$\varphi_k^{i,K} := ((\varphi_k - 2\pi d_k^{(i)}) \wedge K) \vee (-K) \in W^{1,2}(\Omega)\,, \tag{5.25}$$

and

$$\phi_k^{i,K} := \varphi_k^{i,K} \chi_{\Omega\setminus F_k^{t(k)}}.$$

Since $S_{\phi_k^{i,K}} \subset \partial^* F_k^{t(k)}$ and $[\![\phi_k^{i,K}]\!] \leq K$, we have

$$\begin{aligned} |D\phi_k^{i,K}|(\Omega) &= \int_\Omega |\nabla\phi_k| \chi_{\{|\varphi_k - 2\pi d_k^{(i)}|<K\}}\,\mathrm{d}x + \int_{S_{\phi_k^{i,K}}} [\![\phi_k^{i,K}]\!]\,\mathrm{d}\mathcal{H}^{n-1} \\ &\leq \int_\Omega |\nabla\phi_k|\,\mathrm{d}x + K\mathcal{H}^{n-1}(\partial^* F_k^{t(k)}) \leq C\,. \end{aligned}$$

Hence, up to a subsequence (depending on K), $(\phi_k^{i,K})$ converges to some $\phi_\infty^{i,K}$ in $L^1(\Omega)$. Moreover from (5.23) it holds $\phi_\infty^{i,K} := (\varphi_\infty \wedge K) \vee (-K)$ in E_i and $\Omega \setminus E_i = F_i^+ \cup F_i^-$ are such that $\phi_\infty^{i,K} = \pm K$ in $F_i^\pm$. As $|F_k^{t(k)}| \to 0$ it follows that $\varphi_k^{i,K}$ converges to $\phi_\infty^{i,K}$ in $L^1(\Omega)$. Hence the sequence $((\varphi_k^{i,K}, v_k))$ converges to $(\phi_\infty^{i,K}, 1)$ in $L^1(\Omega)\times L^1(\Omega)$ and

$$\int_A \left(v_k^2 |\nabla\varphi_k^{i,K}|^2 + \varepsilon_k |\nabla v_k|^2 + \frac{(v_k-1)^2}{4\varepsilon_k}\right) \mathrm{d}x \leq C\,,$$

for any open set $A \subset \Omega$, for some $C > 0$ independent of k. Thus, from the decoupling property (2.2),

$$\begin{aligned} &\liminf_{k\to+\infty} \int_A v_k^2 |\nabla\varphi_k^{i,K}|^2\,\mathrm{d}x \geq \int_A |\nabla\phi_\infty^{i,K}|^2\,\mathrm{d}x\,, \\ &\liminf_{k\to+\infty} \mathrm{MM}_{\varepsilon_k}(v_k, A) \geq \mathcal{H}^{n-1}(S_{\phi_\infty^{i,K}} \cap A). \end{aligned} \tag{5.26}$$

In particular $\phi_\infty^{i,K} \in SBV^2(\Omega)$. We fix an integer $N \geq 1$ and set, for all $i = 1, \ldots, N$,

$$\Sigma_i^K := (S_{\phi_\infty^{i,K}} \cap \partial^* E_i) \cap \Omega\,, \qquad S_i^K := S_{\phi_\infty^{i,K}} \setminus \partial^* E_i\,,$$

$$\Sigma^{N,K} := \bigcup_{i=1}^N \Sigma_i^K\,, \qquad S^{N,K} := \bigcup_{i=1}^N S_i^K\,.$$

We now argue exactly as in the proof of Lemma 4.2, so, given $\delta \in (0,1)$, after defining the families of balls $\mathcal{B}$ and $\mathcal{B}_j$ $(j = 1, \ldots, 2N)$, we arrive at

$$\sum_{B\in\mathcal{B}\cap\mathcal{B}_j} \mathcal{H}^{n-1}(B \cap S_j^K) \geq \mathcal{H}^{n-1}(S_j^K) - \frac{\delta\mathcal{H}^{n-1}(S^{N,K}\cup\Sigma^{N,K})}{1-\delta}\,, \tag{5.27}$$

for all $j = 1, \dots, N$, and

$$\sum_{B \in \mathcal{B} \cap \mathcal{B}_{N+h}} \mathcal{H}^{n-1}(B \cap \Sigma_h^K) \geq \mathcal{H}^{n-1}(\Sigma_h^K) - \frac{\delta \mathcal{H}^{n-1}(S^{N,K} \cup \Sigma^{N,K})}{1-\delta}, \tag{5.28}$$

for all $h = 1, \dots, N$. From (5.26) we know that, for all $j = 1, \dots, N$,

$$\liminf_{k \to +\infty} \mathrm{MM}_{\varepsilon_k}(v_k, B) \geq \mathcal{H}^{n-1}(S_j^K \cap B), \tag{5.29}$$

for any $B \in \mathcal{B} \cap \mathcal{B}_j$, and for all $h = 1, \dots, N$

$$\liminf_{k \to +\infty} \mathrm{MM}_{\varepsilon_k}(v_k, B) \geq \mathcal{H}^{n-1}(\Sigma_h^K \cap B), \tag{5.30}$$

for any ball $B \in \mathcal{B} \cap \mathcal{B}_{N+h}$. So, summing (5.29) over all $B \in \mathcal{B} \cap \mathcal{B}_j$ and (5.30) over all $B \in \mathcal{B} \cap \mathcal{B}_{N+h}$, and then over j, h respectively, we conclude again, using (5.27) and (5.28),

$$\begin{aligned}\liminf_{k \to +\infty} \mathrm{MM}_{\varepsilon_k}(v_k, \Omega) &= \lim_{k \to +\infty} \mathrm{MM}_{\varepsilon_k}(v_k, \Omega) \\ &\geq \sum_{j=1}^{N} \mathcal{H}^{n-1}(S_j^K) + \sum_{h=1}^{N} \mathcal{H}^{n-1}(\Sigma_h^K) - \frac{2\delta N \mathcal{H}^{n-1}(S^{N,K} \cup \Sigma^{N,K})}{1-\delta}\end{aligned}$$

and by the arbitrariness of $\delta > 0$,

$$\lim_{k \to +\infty} \mathrm{MM}_{\varepsilon_k}(v_k, \Omega) \geq \mathcal{H}^{n-1}(S^{N,K} \cup \Sigma^{N,K}).$$

The left hand-side in the above inequality is a limit, hence it does not depend on the subsequence, and so in particular on K and N. Letting first $N \to +\infty$ and then $K \to +\infty$ we finally get

$$\liminf_{k \to +\infty} \mathrm{MM}_{\varepsilon_k}(v_k, \Omega) \geq \mathcal{H}^{n-1}(S_{\varphi_\infty}),$$

and (5.17) follows.

Step 2: Upper bound. Let $\varepsilon_k \searrow 0$ and $u \in SBV^2(\Omega; \mathbb{S}^1)$. We have to find a sequence $((u_k, v_k)) \subset \mathcal{A}_{\mathbb{S}^1}$ converging to $(u, 1)$ in $L^1(\Omega; \mathbb{S}^1) \times L^1(\Omega)$ and

$$\limsup_{k \to +\infty} \mathrm{AT}_{\varepsilon_k}^{\mathbb{S}^1}(u_k, v_k) \leq \mathrm{MS}_{\mathrm{lift}}(u, 1). \tag{5.31}$$

We notice that we cannot follow the strategy used in the proof of the upper bound in Theorem 1.1, since that construction cannot ensure the functions u_k to belong to $W^{1,2}(\Omega; \mathbb{S}^1)$.

By Corollary 3.2 we can select a jump minimizing lifting $\varphi \in GSBV^2(\Omega)$ of u, $u = e^{i\varphi}$ and

$$\mathcal{H}^{n-1}(S_\varphi) = m_2[u]. \tag{5.32}$$

The main difference with the construction in Theorem 1.1 is that we will modify u in a neighbourhood of S_φ instead of S_u. We divide the proof into two cases.

Case 1: $\varphi \in L^\infty(\Omega)$. Let $\widehat{\Omega}$, with $\Omega \subset\subset \widehat{\Omega} \subset\subset \mathbb{R}^2$, be open. Since $\varphi \in SBV^2(\Omega) \cap L^\infty(\Omega)$, by Proposition 5.2 we can assume, without loss of generality, that $\varphi \in SBV^2(\widehat{\Omega})$ with

$$\mathcal{H}^{n-1}(S_\varphi \cap \partial\Omega) = 0, \quad \mathcal{H}^{n-1}(\overline{S}_\varphi \setminus S_\varphi) = 0, \quad \mathcal{H}^{n-1}(S_\varphi \cap K) = \mathcal{M}^{n-1}(S_\varphi \cap K), \tag{5.33}$$

for every compact set $K \subset \widehat{\Omega}$. In this way, arguing as in [7,8] we can construct $(\varphi_k, v_k) \in W^{1,2}(\widehat{\Omega}) \times W^{1,2}(\widehat{\Omega})$ such that $(\varphi_k, v_k) \to (\varphi, 1)$ in $L^1(\widehat{\Omega}) \times L^1(\widehat{\Omega})$ and

$$\limsup_{k\to+\infty} \mathrm{AT}_{\varepsilon_k}(\varphi_k, v_k) \le \mathrm{MS}(\varphi, 1)\,. \tag{5.34}$$

Next we let $u_k := e^{i\varphi_k} \in W^{1,2}(\widehat{\Omega}; \mathbb{S}^1)$, so that $(u_k, v_k) \to (u, 1)$ in $L^1(\widehat{\Omega}; \mathbb{S}^1) \times L^1(\widehat{\Omega})$. Moreover, from (5.32), (5.34), $|\nabla u_k| = |\nabla \varphi_k|$ and $|\nabla u| = |\nabla \varphi|$ we get

$$\mathrm{AT}_{\varepsilon_k}(\varphi_k, v_k) = \mathrm{AT}^{\mathbb{S}^1}_{\varepsilon_k}(u_k, v_k)\,, \quad \mathrm{MS}(\varphi, 1) = \mathrm{MS}_{\mathrm{lift}}(u, 1)$$

and so

$$\limsup_{k\to+\infty} \mathrm{AT}^{\mathbb{S}^1}_{\varepsilon_k}(u_k, v_k) \le \mathrm{MS}_{\mathrm{lift}}(u, 1)\,.$$

Case 2: $\varphi \notin L^\infty(\Omega)$. For $N \in \mathbb{N}$, $N \ge 1$ we let $\varphi^{(N)} := \varphi \wedge N \vee (-N) \in L^\infty(\Omega)$. Then, as $N \to +\infty$, we have

$$\varphi^{(N)} \to \varphi \quad \text{in } L^1(\Omega)\,, \quad \nabla\varphi^{(N)} \to \nabla\varphi \quad \text{in } L^2(\Omega; \mathbb{R}^n)\,,$$

so that

$$u^{(N)} := e^{i\varphi^N} \to e^{i\varphi} = u \quad \text{in } L^1(\Omega; \mathbb{S}^1)\,, \quad \nabla u^{(N)} \to \nabla u \quad \text{in } L^2(\Omega; \mathbb{R}^{n\times n})\,.$$

In addition,

$$\mathcal{H}^{n-1}(S_{\varphi^{(N)}}) \le \mathcal{H}^{n-1}(S_\varphi)\,.$$

This in turn implies

$$\begin{aligned} \limsup_{N\to+\infty} \left(\int_\Omega |\nabla\varphi^{(N)}|^2 \,\mathrm{d}x + \mathcal{H}^{n-1}(S_{\varphi^{(N)}}) \right) &\le \int_\Omega |\nabla\varphi|^2 \,\mathrm{d}x + \mathcal{H}^{n-1}(S_\varphi) \\ &\le \int_\Omega |\nabla u|^2 \,\mathrm{d}x + m_2[u]. \end{aligned} \tag{5.35}$$

For each $N \ge 1$ we can argue as in case 1 and find a sequence $(u_k^{(N)}, v_k^{(N)}) \to (u^{(N)}, 1)$ in $L^1(\Omega; \mathbb{S}^1) \times L^1(\Omega)$ satisfying

$$\limsup_{k\to+\infty} \mathrm{AT}^{\mathbb{S}^1}_{\varepsilon_k}(u_k^{(N)}, v_k^{(N)}) \le \int_\Omega |\nabla\varphi^{(N)}|^2 \,\mathrm{d}x + \mathcal{H}^{n-1}(S_\varphi^{(N)})\,. \tag{5.36}$$

Combining (5.35) with (5.36) we have

$$\limsup_{N\to+\infty} \limsup_{k\to+\infty} \mathrm{AT}^{\mathbb{S}^1}_{\varepsilon_k}(u_k^{(N)}, v_k^{(N)}) \le \int_\Omega |\nabla u|^2 \,\mathrm{d}x + m_2[u].$$

Now we conclude using a diagonal argument. Namely, for any fixed $N \ge 1$ there exists $k_N \ge 1$ such that for $k \ge k_N$:

- $\|u_k^{(N)} - u^{(N)}\|_{L^1(\Omega;\mathbb{S}^1)} + \|v_k^{(N)} - 1\|_{L^1(\Omega)} \le \frac{1}{N}$;
- $\mathrm{AT}^{\mathbb{S}^1}_{\varepsilon_k}(u_k^{(N)}, v_k^{(N)}) \le \int_\Omega |\nabla\varphi^{(N)}|^2 \,\mathrm{d}x + \mathcal{H}^{n-1}(S_\varphi^{(N)}) + \dfrac{1}{N}$.

Without loss of generality we may assume $k_{N+1} \ge k_N$. Now, for every $k \ge 1$, we take $(u_k, v_k) := (u_k^{(N)}, v_k^{(N)})$ if $k \in [k_N, k_{N+1}) \cap \mathbb{N}$ and the proof is concluded. □

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References

[1] E. Acerbi, and G. Dal Maso, New lower semicontinuity results for polyconvex integrals, *Calc. Var. Partial Differential Equations* **2** (1994), 329-371.

[2] R. Alicandro, M. Cicalese, and M. Ponsiglione, Variational equivalence between GinzburgLandau, XY spin systems and screw dislocations energies, *Indiana Univ. Math. J.* **60** (2011), 171—208.

[3] R. Alicandro, A. Braides, and J. Shah, Free-discontinuity problems via functionals involving the L^1-norm of the gradient and their approximation, *Interfaces Free Bound.* **1** (1999), 17–37

[4] R. Alicandro, L. De Luca, A. Garroni, and M. Ponsiglione, Metastability and dynamics of discrete topological singularities in two dimensions: a Γ-convergence approach, *Arch. Rational Mech. Anal.* **214** (2014), 269–330.

[5] L. Ambrosio, On the lower semicontinuity of quasi-convex integrals in $SBV(\Omega;\mathbb{R}^k)$, *Nonlinear Anal.* **23** (1994), 405–425.

[6] L. Ambrosio, N. Fusco, D. Pallara, Functions of Bounded Variation and Free Discontinuity Problems, Oxford Mathematical Monographs, *The Clarendon Press Oxford University Press, New York, 2000.*

[7] L. Ambrosio, V. M. Tortorelli, Approximation of functionals depending on jumps by elliptic functionals via Γ-convergence, *Comm. Pure Appl. Math.* **43** (1990) no.8, 999–1036.

[8] L. Ambrosio, V. M. Tortorelli, On the approximation of free-discontinuity problems, *Boll. Un. Mat. Ital.* (7) **6-B** (1992) no.1, 105–123.

[9] G. Bellettini, A. Elshorbagy, M. Paolini, and R. Scala, On the relaxed area of the graph of discontinuous maps from the plane to the plane taking three values with no symmetry assumptions, *Ann. Mat. Pura Appl.* **199**, 445-477 (2020).

[10] G. Bellettini, A. Elshorbagy and R. Scala, The L^1-relaxed area of the graph of the vortex map: optimal lower bound, *Nonlinear Anal.* **256** (2025), Paper No. 113803.

[11] G. Bellettini, R. Marziani, R. Scala, *A non-parametric Plateau problem with partial free boundary*, J. Echole Polytechnique – Mathématiques **11** (2024), pp. 1035-1098.

[12] G. Bellettini, and M. Paolini On the area of the graph of a singular map from the plane to the plane taking three values, *Adv. Calc. Var.* **3** (2010), 371-386.

[13] G. Bellettini, R. Scala and G. Scianna, L^1-relaxed area of graphs of $\mathbb{S}^1$-valued Sobolev maps and its countably subadditive envelope, *Rev. Mat. Iberoam.* **40** (2024), 2135-2178

[14] A. Braides, Approximation of Free-Discontinuity Problems, *Lecture Notes in Math.* **1694**, *Springer Verlag, Berlin, 1998.*

[15] H. Brezis, P. Mironescu, Sobolev Maps to the Circle, *Progress in Nonlinear Differential Equations and their Applications* **96**, *Birkhäuser/Springer, New York, 2021.*

[16] H. Brezis, P. Mironescu, The Plateau problem from the perspective of optimal transport, *Comptes Rendus. Mathématique,* **357**(7) (2019), 597–612.

[17] M. Carriero, A. Leaci, S^k-valued maps minimizing the L^p norm of the gradient with free discontinuities, *Ann. Sc. Norm. Sup. Pisa Cl. Sci.* **18** (1991), 321–352.

[18] A. Chambolle, An approximation result for special functions with bounded deformation, *J. Math. Pures Appl.* **83** (2004), 929–954.

[19] A. Chambolle, V. Crismale Compactness and lower semicontinuity in GSBD, *J. Eur. Math. Soc.* **23**, 701–719 (2021).

[20] Chambolle A., Ferrari L. A. D., Merlet B., *A phase-field approximation of the Steiner problem in dimension two*, Advances in Calculus of Variations, vol. 12, no. 2, (2019), 157-179.

[21] S. Conti, A. Garroni, and R. Marziani, Line-tension limits for line singularities and application to the mixed-growth case, *Calc. Var.* **62**, 228 (2023).

[22] S. Conti, A. Garroni, and M. Ortiz, The line-tension approximation as the dilute limit of linear-elastic dislocations, *Arch. Rational Mech. Anal.* **218** (2015) 699–755.

[23] V. Crismale, L. De Luca, R. Scala, Approximation of topological singularities through free discontinuity functionals: the critical and super-critical regimes, *arXiv:2407.08285* (2024).

[24] J. Dávila, R. Ignat, Lifting of BV functions with values in S^1, *C. R. Math. Acad. Sci. Paris* **337** (2003), 159–164.

[25] L. De Luca, A. Garroni, and M. Ponsiglione, Γ-convergence analysis of systems of edge dislocations: the self energy regime, *Arch. Rational Mech. Anal.* **206** (2012), 885–910.

[26] L. De Luca, R. Scala, N. Van Goethem, A new approach to topological singularities via a weak notion of Jacobian for functions of bounded variation, *Indiana Univ. Math. J.* **73**(2) (2024), 723–779.

[27] Ferrari L., Dirks C., Wirth B., *Phase field approximations of branched transportation problems*, Calc. Var. 59, 37 (2020).

[28] M. Focardi, On the variational approximation of free-discontinuity problems in the vectorial case, *Math. Models Methods Appl. Sci.* **11** (2001), 663–684.

[29] H. Federer, "Geometric Measure Theory", *Die Grundlehren der mathematischen Wissenschaften, Vol. 153, Springer-Verlag, New York Inc., New York,* 1969.

[30] M. Friedrich, A compactness result in $GSBV^p$ and applications to Γ-convergence for free discontinuity problems, *Calc. Var. PDE* **58**(86) (2019).

[31] A. Garroni, G. Leoni, and M. Ponsiglione, Gradient theory for plasticity via homogenization of discrete dislocations, *J. Eur. Math. Soc.* **12** (2010), 1231–1266.

[32] A. Garroni, R. Marziani, and R. Scala, Derivation of a Line-Tension Model for Dislocations from a Nonlinear Three-Dimensional Energy: The Case of Quadratic Growth, *SIAM J. Math. Anal.* **53**(4) (2021), 4252–4302.

[33] M. Giaquinta, G. Modica, and J. Souček, "Cartesian Currents in the Calculus of Variations II. Variational Integrals", *Ergebnisse der Mathematik und ihrer Grenzgebiete, Vol. 38, Springer-Verlag, Berlin-Heidelberg,* 1998.

[34] E. Giusti, "Minimal Surfaces and Functions of Bounded Variation", *Birkhäuser, Boston* 1984.

[35] A. Marchese, A. Massaccesi, S. Stuvard, and R. Tione, A multi-material transport problem with arbitrary marginals Calc. Var. **60**, 88 (2021).

[36] A. Marchese, A. Massaccesi, and R. Tione, A Multimaterial Transport Problem and its Convex Relaxation via Rectifiable G-currents, SIAM j. math. anal. **51**(3), (2019).

[37] M. Ponsiglione, Elastic energy stored in a crystal induced by screw dislocations: from discrete to continuous, *SIAM J. Math. Anal.* **39** (2007), 449–469.

[38] R. Scala, Optimal estimates for the triple junction function and other surprising aspects of the area functional, *Ann. Sc. Norm. Super. Pisa Cl. Sci.* (5) **20**(2), 491–564. (2020).

[39] R. Scala, G. Scianna, On the L^1-relaxed area of graphs of BV piecewise constant maps taking three values, *Adv. Calc. Var.* **18**(2), 339–365 (2025).