INTRINSIC REGULAR HYPERSURFACES IN HEISENBERG GROUPS

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1. Introduction

In this paper we study the $\mathbb{H}$-regular surfaces, a class of *intrinsic regular* hypersurfaces in the setting of the Heisenberg group $\mathbb{H}^n = \mathbb{C}^n \times \mathbb{R} \equiv \mathbb{R}^{2n+1}$ endowed with a left-invariant metric d_∞ equivalent to its Carnot-Carathéodory (CC) metric. In particular we (locally) characterize them as *intrinsic regular* graphs inside $\mathbb{H}^n$ (see Theorems 1.2 and 1.3 below). Here hypersurface simply means topological codimension 1 surface and by the word "intrinsic" and "regular" we will mean of notions respectively involving the group structure of $\mathbb{H}^n$ and its differential structure as CC manifold in a sense we will precise below.

This notion of regular hypersurface has been introduced in the setting of Carnot groups, of which $\mathbb{H}^n$ is the simplest example, in order to study the classical problem of Geometric Measure Theory (GMT) of defining regural surfaces and different reasonable measures on them. Moreover this problem has been also carried out in the setting of Carnot groups and more generally in a metric space by many authors (see [50], [51], [10], [36], [38], [33], [14], [26], [32], [13], [3], [4], [27], [52], [48], [28], [40], [29], [42] and [6]). On the other hand the notion of intrinsic graph has been recently introduced and studied in [30] in the setting of a Carnot group even if it was already implicitly exploited in [27].

Throughout this paper, we shall denote the points of $\mathbb{H}^n$ by $P = [z,t] = [x+iy,t]$, $z \in \mathbb{C}^n$, $x, y \in \mathbb{R}^n$, $t \in \mathbb{R}$. If $P = [z,t]$, $Q = [\zeta,\tau] \in \mathbb{H}^n$ and $r > 0$, following the notations of [57], where the reader can find an exhaustive introduction to the Heisenberg group, we define the group operation

$$P \cdot Q := [z + \zeta, t + \tau + 2\Im m(z\bar{\zeta})] \tag{1.1}$$

and the family of non isotropic dilations

$$\delta_r(P) := [rz, r^2 t], \text{ for } r > 0. \tag{1.2}$$

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Moreover $\mathbb{H}^n$ can be endowed with the homogeneous norm

$$\|P\|_\infty := \max\{|z|, |t|^{1/2}\} \tag{1.3}$$

and the distance d_∞ we shall deal with is defined as

$$d_\infty(P,Q) := \|P^{-1} \cdot Q\|_\infty. \tag{1.4}$$

It is well-known that $\mathbb{H}^n$ is a Lie group of topological dimension $2n+1$, whereas the Hausdorff dimension of $(\mathbb{H}^n, d_\infty)$ is $Q := 2n+2$ (see Proposition 2.1).

$(\mathbb{H}^n, d_\infty)$ provides the simplest example of a metric space that is not Euclidean, even locally, but is still endowed with a sufficiently rich compatible underlying structure, due to the existence of intrinsic families of left translations and dilations respectively induced from the group law (1.1) and dilations (1.2). Indeed, the geometry of $\mathbb{H}^n$ is noneuclidean at every scale, since it was proved by S. Semmes ([56]) that there are no bilitschitz maps from $\mathbb{H}^n$ to any Euclidean space. Our interest can be viewed in the framework of the general project meant to develop GMT in the setting of metric spaces. Such a project, already embryonally contained in Federer's book ([22]), has been explicitly formulated and carried on in the last fewyears by De Giorgi ([18], [19], [17]), Preiss and Tisěr([54]), (Kirchheim [36]), (David & Semmes[14]), Ambrosio & Kirchheim ([3],[4]), Lorent ([39]) and Mattila ([45]). It is well-known that $\mathbb{H}^n$ is a Carnot group of step 2. Indeed its Lie algebra $\mathfrak{h}_n$ is (linearly) generated by

$$X_j = \frac{\partial}{\partial x_j} + 2y_j \frac{\partial}{\partial t}, \qquad Y_j = \frac{\partial}{\partial y_j} - 2x_j \frac{\partial}{\partial t}, \qquad \text{for } j = 1, \dots, n; \qquad T = \frac{\partial}{\partial t}, \tag{1.5}$$

and the only non-trivial commutator relations are

$$[X_j, Y_j] = -4T, \qquad \text{for } j = 1, \dots, n.$$

Throughout this paper, we shall identify vector fields and associated first order differential operators; thus the vector fields $X_1, \dots, X_n$, $Y_1, \dots, Y_n$ generate a vector bundle on $\mathbb{H}^n$, the so called horizontal vector bundle $\mathrm{H}\mathbb{H}^n$ according to the notation of Gromov (see [33] and [38]), that is a vector subbundle of $\mathrm{T}\mathbb{H}^n$, the tangent vector bundle of $\mathbb{H}^n$. Since each fiber of $\mathrm{H}\mathbb{H}^n$ can be canonically identified with a vector subspace of $\mathbb{R}^{2n+1}$, each section φ of $\mathrm{H}\mathbb{H}^n$ can be identified with a map $\varphi : \mathbb{H}^n \to \mathbb{R}^{2n+1}$. At each point $P \in \mathbb{H}$ the horizontal fiber is indicated as $\mathrm{H}\mathbb{H}^n_P$ and each fiber can be endowed with the scalar product $\langle \cdot, \cdot \rangle_P$ and the associated norm $|\cdot|_P$ that make the vector fields $X_1, \dots, X_n$, $Y_1, \dots, Y_n$ orthonormal. Hence we shall also identify a section of $\mathrm{H}\mathbb{H}^n$ with its canonical coordinates with respect to this moving frame. In this way, a section φ will be identified with the function $\varphi = (\varphi_1, \dots, \varphi_{2n}) : \mathbb{H}^n \to \mathbb{R}^{2n}$ such that $\varphi = \sum(\varphi_i X_i + \varphi_{n+i} Y_i)$. Analogously, if f is a real function defined in an open subset $\Omega \subset \mathbb{H}$, its $\mathbb{H}$-*gradient* is the section of $\mathrm{H}\mathbb{H}^n$ defined by $\nabla_{\mathbb{H}} f := (X_1 f, \dots, X_n f, Y_1 f, \dots, Y_n f)$. As it is common in Riemannian geometry, when dealing with two sections φ and φ' whose argument is not explicitely written, we shall drop the index P in the scalar product writing $\langle \varphi, \varphi' \rangle$ for $\langle \varphi(P), \varphi'(P) \rangle_P$. The same convention shall be adopted for the norm.

To introduce our results, let us start by recalling some related notions already existing in the literature.

The two key points we want to stress now are the notions of intrinsic regular hypersurface and graph in $\mathbb{H}^n$. A general and more complete discussion of these topics in Carnot groups can be found in [30].

Let us recall the notion of regular surface is related to a notion of rectifiability in a metric spaces which goes back to Federer (see [22] 3.2.14) and that has been used by Ambrosio & Kirchheim (see [3], [4]) in the framework of a theory of currents in metric spaces (as for the rectifiability in metric spaces see for instance [36], [54] and also the monograph [44] and the references therein). According to this notion, a "good" surface in a metric space should be the image of an open subset of an Euclidean space via a Lipschitz map. Unfortunately, such a notion does not fit the geometry of the Heisenberg group, that indeed would be, according with this definition, purely unrectifiable (see [3]). On the other hand, in the Euclidean setting $\mathbb{R}^n$, a $\mathbf{C}^1$-hypersurface can be equivalently viewed as the (local) set of zeros of a function $f : \mathbb{R}^n \to \mathbb{R}$ with non-vanishing gradient. Such a notion was easily transposed in [27] to the Heisenberg group, since there is an intrinsic notion of $\mathbf{C}^1_{\mathbb{H}}$-functions introduced by Folland and Stein (see [24]): we can say that a continuous real function f on $\mathbb{H}^n$ belongs to $\mathbf{C}^1_{\mathbb{H}}(\mathbb{H}^n)$ if $\nabla_{\mathbb{H}} f$ (in the sense of distributions) is a continuous vector-valued function. Thus, an $\mathbb{H}$-regular surface S will be locally defined as the set of points $P \in \mathbb{H}$ such that $f(P) = 0$, provided that $\nabla_{\mathbb{H}} f \neq 0$ on S (see Definition 2.13). A few comments are now in place to point out similar geometric properties (in the measure theoretical sense) of the $\mathbb{H}$-regular surfaces and classical (Euclidean) regular surfaces and to mention some of their applications.

First of all, we point out that the class of $\mathbb{H}$-regular surfaces is deeply different from the class of Euclidean regular surfaces, in the sense that there are $\mathbb{H}$-regular surfaces in $\mathbb{H}^1 \equiv \mathbb{R}^3$ that are (Euclidean) fractal sets (see [37]), and conversely there are continuously differentiable 2-submanifolds in $\mathbb{R}^3$ that are not $\mathbb{H}$-regular hypersurfaces (see [27], Remark 6.2 and Example 2). We notice that Euclidean continuously differentiable 2-manifolds are $\mathbb{H}$-regular surfaces provided they do not contain characteristic points, i.e. points P such that the Euclidean tangent space at P coincides with the horizontal fiber $\mathrm{H}\mathbb{H}^n_P$ at P. Frobenius theorem yields that, for a general smooth manifold, the set of characteristic points has empty interior; in fact there are few characteristic points ([13], [5], [42]).

The important point supporting the choice of the notion is the fact that this definition yields an Implicit Function Theorem, proved in [27] for the Heisenberg group and in [28] for a general Carnot group (see also [11]), so that a $\mathbb{H}$-regular surface locally is a X_i-*graph* (or a Y_i-*graph*) ($i = 1, \dots, n$), namely (see Definition 2.19) there is a continuous parameterization of S

$$\Phi : \omega \subset (\mathbb{V}_i, \rho) \to (S, d_\infty) \quad (\text{or } \Phi : \omega \subset (\mathbb{V}_{i+n}, |\cdot|) \to (S, d_\infty)) \tag{1.6}$$

$$\Phi(A) := A \cdot (\phi(A)e_i) \quad (\text{or } \Phi(A) := A \cdot (\phi(A)e_{i+n})) \tag{1.7}$$

where $\phi : \omega \to \mathbb{R}$ is continuous, $V_i := \{(x, y, t) \in \mathbb{H}^n : x_i = 0\}$ (or $V_{i+n} := \{(x, y, t) \in \mathbb{H}^n : y_i = 0\}$), $\omega \subset V_j$, $\{e_j : j = 1, \dots, 2n+1\}$ denotes the standard basis in $\mathbb{R}^{2n+1} \equiv \mathbb{H}^n$ and we consider $\rho \equiv |\cdot|$ the Euclidean distance on $V_j \equiv \mathbb{R}^{2n}$ ($j = 1, \dots, 2n$), (see Theorem 2.16). In general, such a parameterization is not continuosly differentiable or even Lipschitz continuous. Indeed it was proved in [37] that generally its best Hölder continuous regularity turns out to be of order 1/2 with respect to the distances given in (1.6). Nevertheless from this parameterization we infer that S is a topological submanifold of dimension $2n$. Besides, by using again the Implicit Function Theorem and the Blow-Up Theorem (see Theorem 2.17), an area type formula for the $(Q-1)$-dimensional spherical Hausdorff measure $\mathcal{S}_\infty^{Q-1}$ induced in $(\mathbb{H}^n, d_\infty)$ and the existence of the tangent group in the sense of GMT for $\mathbb{H}$-regular surfaces were established (see also [27] and [28]).

Based on this, also the notion of $\mathbb{H}$-rectifiability was introduced: a set $S \subset \mathbb{H}^n$ is said $(Q-1)$-*dimensional* $\mathbb{H}$-*rectifiable* if there exists a sequence of $\mathbb{H}$-regular surfaces $(S_i)_i$ in $\mathbb{H}^n$ such that $\mathcal{S}_\infty^{Q-1}(S \setminus \cup_{i\in\mathbb{N}} S_i) = 0$. This intrinsic notion of rectifiability has been proven particularly useful to obtain in [27] an analog of De Giorgi's structure theorem for sets of intrinsic finite perimeter in the setting of Heisenberg group, and later in the setting of a general Carnot group of step 2 ([29]). The notions of Euclidean and $\mathbb{H}$-rectifiability have been compared in [7]; generalizations of this notion of rectifiability have been studied by V. Magnani in [42] for general Carnot groups (see also [41] for a general account of GMT in Carnot groups).

One of the main aim of this paper is to find out necessary and sufficient (manageable) assumptions on $\phi : \omega \subset V_j \to \mathbb{R}$ ($j = 1, \dots, 2n$), besides the continuity, assuring that the intrinsic graph

$$S = G^j_{\mathbb{H},\phi} := \Phi(\omega) \tag{1.8}$$

is $\mathbb{H}$-regular if $\Phi : \omega \to \mathbb{H}^n$ is the map defined in (1.7). Namely which other (minimal) assumptions, more than the continuity of ϕ, need in order $G^j_{\mathbb{H},\phi}$ turns out to be $\mathbb{H}$-regular.

We will see that these additional assumptions will be characterized in terms of an intrinsic differential structure on the subgroup $V_j \equiv \mathbb{R}^{2n}$ ($j = 1, \dots, 2n$) induced by the *graph distance* defined on V_j in a classical way. More precisely whithout loss of generality we can assume $j = 1$. Then there is a natural identification between V_1 and $\mathbb{R}^{2n}$ given by the diffeomorphism

$$\iota : \mathbb{R}^{2n} \longrightarrow V_1 \subset \mathbb{H}^n \tag{1.9}$$

defined when $n = 1$ as

$$\iota(\eta, \tau) = (0, \eta, \tau), \tag{1.10}$$

while for $n \ge 2$ and $(\eta, v, \tau) \in \mathbb{R}^{2n} \equiv \mathbb{R}_\eta \times \mathbb{R}_v^{2n-2} \times \mathbb{R}_\tau$ ι is defined as

$$\iota((\eta, v, \tau)) = (0, v_2, \dots, v_n, \eta, v_{n+2}, \dots, v_{2n}, \tau), \tag{1.11}$$

where $v = (v_2, \dots, v_n, v_{n+2}, \dots, v_{2n})$. Thus, since V_1 is a subgroup of $\mathbb{H}^n$ closed with respect to the dilations in (1.2), $\mathbb{R}^{2n}$ can be endowed through this identification by a structure of homogeneous group in the sense of Folland and Stein (see [24]), i.e. we can define a group law in $\mathbb{R}^{2n}$

$$A \star B := \iota^{-1}(\iota(A) \cdot \iota(B)) \quad A, B \in \mathbb{R}^{2n} \tag{1.12}$$

and a family of intrinsic dilations $\delta^\star_\lambda : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ $(\lambda > 0)$

$$\delta^\star_\lambda(A) := \iota^{-1}(\delta_\lambda(\iota(A)) \quad \in \mathbb{R}^{2n} \tag{1.13}$$

such that $(\mathbb{R}^{2n}, \star, \delta^\star_\lambda)$ turns out to be a homogeneous group.

Then $(\mathbb{R}^{2n}, \star, \delta^\star_\lambda)$ can be endowed of a natural intrinsic linear structure and, inspired by Pansu's ideas (see [51] and also [27] section 5 and [41] section 3.1), we can naturally define a $\star$-linear functional $L : \mathbb{R}^{2n} \to \mathbb{R}$ as a homomorphism which is also positively homogeneous of degree 1 with respect to the dilations in (1.13).

Thus, fixed $\phi : \omega \subset \mathbb{R}^{2n} \to \mathbb{R}$, we can construct a map $\rho_\phi : \omega \times \omega \to [0, +\infty)$ defined as

$$\rho_\phi(A, B) := |\eta' - \eta| + |\tau' - \tau + 2(\phi(A) + \phi(B))(\eta' - \eta)|^{1/2} \tag{1.14}$$

when $n = 1$, $A = (\eta', \tau')$, $B = (\eta, \tau) \in \omega$ and

$$\rho_\phi(A, B) := |(\eta', v') - (\eta, v)|_{\mathbb{R}^{2n-1}} + |\tau' - \tau + 2(\phi(A) + \phi(B))(\eta' - \eta) + \sigma(v', v)|^{1/2} \tag{1.15}$$

when $n \ge 2$, $A = (\eta', v', \tau')$, $B = (\eta, v, \tau) \in \omega$ and

$$\sigma(v', v) := 2 \sum_{j=2}^{n} (v'_{n+j} v_j - v'_j v_{n+j})$$

if $v = (v_2, \dots, v_n, v_{n+2}, \dots, v_{2n})$, $v' = (v'_2, \dots, v'_n, v'_{n+2}, \dots, v'_{2n}) \in \mathbb{R}^{2n-2}$.

If there is $c_1 > 0$ such that

$$|\phi(A) - \phi(B)| \le c_1 \, \rho_\phi(A, B) \tag{1.16}$$

for all $A, B \in \omega$, then the quantity ρ_ϕ in (1.14) and (1.15) is a quasimetric on ω (see Proposition 3.1). We will call ρ_ϕ "graph distance" since in this case it is equivalent to the metric d_∞ restricted to the graph S in (1.8), i.e. there exists $c_2 > 0$ such that

$$\frac{1}{c_2} \rho_\phi(A, B) \le d_\infty(\Phi(A), \Phi(B)) \le c_2 \, \rho_\phi(A, B) \quad \forall A, B \in \omega \, . \tag{1.17}$$

Now we can state our notion of W^ϕ-differentiability.

Definition 1.1. *Let $\omega \subset \mathbb{R}^{2n}$ be an open set and let $\phi : \omega \to \mathbb{R}$ be a fixed continuous function. Let $A_0 \in \omega$ and $\psi : \omega \to \mathbb{R}$ be given. We say that ψ is W^ϕ-differentiable at A_0 if there is a $\star$-linear functional $L : \mathbb{R}^{2n} \to \mathbb{R}$ such that*

$$\lim_{A \to A_0} \frac{|\psi(A) - \psi(A_0) - L(A_0^{-1} \star A)|}{\rho_\phi(A, A_0)} = 0 \, . \tag{1.18}$$

We say that ψ is uniformly W^ϕ-differentiable at A_0 if there is a $\star$-linear functional $L:\mathbb{R}^{2n}\to\mathbb{R}$ such that, if we put

$$M_\phi(\psi,A_0,L,r):=\sup_{\substack{A\neq A'\\ A,A'\in B(A_0,r)}}\left\{\frac{|\psi(A')-\psi(A)-L(A^{-1}\star A')|}{\rho_\phi(A,A')}\right\} \tag{1.19}$$

where $B(A_0,r)$ denotes the Euclidean (open) ball centered at A_0 with radius r in $\mathbb{R}^{2n}$, then $\lim_{r\downarrow 0}M_\phi(\psi,A,L,r)=0$.

It is straightforward that the uniform W^ϕ-differentiability implies the W^ϕ-differentiability. Moreover it is a good definition since if ψ is W^ϕ-differentiable at $A_0\in\omega$, then there is an unique $\star$-linear functional $L:\mathbb{R}^{2n}\to\mathbb{R}$ verifying (1.18) and we will denote $L:=d_{W^\phi}\psi(A_0)$ and we will call it the W^ϕ-differential of ψ at A_0.

Let $\pi:\mathbb{R}^{2n}\to\mathbb{R}^{2n-1}$ the projection respectively defined as $\pi:\mathbb{R}^2\equiv\mathbb{R}_\eta\times\mathbb{R}_\tau\to\mathbb{R}_\eta$, $\pi((\eta,\tau)):=\eta$ when $n=1$ and $\pi:\mathbb{R}^{2n}\equiv\mathbb{R}_\eta\times\mathbb{R}_v^{2n-2}\times\mathbb{R}_\tau\to\mathbb{R}^{2n-1}\equiv\mathbb{R}_\eta\times\mathbb{R}_v^{2n-2}$, $\pi((\eta,v,\tau)):=(\eta,v)$ when $n\geq 2$. Let us denote by $\langle\cdot,\cdot\rangle$ the standard scalar product on $\mathbb{R}^{2n-1}$, i.e. $\langle\eta,\eta'\rangle:=\eta\,\eta'$ when $n=1$ and $\langle(\eta,v),(\eta',v')\rangle:=\eta\,\eta'+\sum_{j=2,j\neq n+1}^{2n}v_jv_j'$ when $n\geq 2$. Then we can simply characterize the $\star$-linear functionals on $(\mathbb{R}^{2n},\star,\delta_\lambda^\star)$. Indeed for every $\star$-linear function $L:\mathbb{R}^{2n}\to\mathbb{R}$ there is an unique $w_L\in\mathbb{R}^{2n-1}$ such that $L(A)=\langle w_L,\pi(A)\rangle$ for every $A\in\mathbb{R}^{2n}$ (see Proposition 2.15). In particular if $\psi:\omega\to\mathbb{R}$ is W^ϕ-differentiable at A_0 then we will denote $W^\phi\psi(A_0)$ the unique vector in $\mathbb{R}^{2n-1}$ for which $d_{W^\phi}\psi(A_0)(A)=\langle W^\phi\psi(A_0),\pi(A)\rangle$ for every $A\in\mathbb{R}^{2n}$.

The tangent space of V_1 is linearly generated by the vector fields which are the restrictions of $X_2,\dots,X_n,Y_1,\dots,Y_n,T$ to V_1, and so we can define the vector fields $\widetilde X_2\dots,\widetilde X_n,\widetilde Y_1,\dots,\widetilde Y_n$ and $\widetilde T$ on $\mathbb{R}^{2n}$ given by $\widetilde X_j:=(\iota^{-1})_*X_j$ and $\widetilde Y_j:=(\iota^{-1})_*Y_j,\widetilde T:=(\iota^{-1})_*T$, where $(\iota^{-1})_*$ is the usual push forward of vector fields after the diffeomorphism ι^{-1}. In coordinates, they can be written as

$$\begin{aligned}\widetilde Y_1(\eta,\tau)&=\frac{\partial}{\partial\eta}\\ \widetilde T(\eta,\tau)&=\frac{\partial}{\partial\tau}.\end{aligned}\tag{1.20}$$

if $n=1$ and as

$$\begin{aligned}\widetilde X_j(\eta,v,\tau)&=\frac{\partial}{\partial v_j}+2v_{j+n}\frac{\partial}{\partial\tau}\quad&&\text{for } j=2,\dots,n\\ \widetilde Y_1(\eta,v,\tau)&=\frac{\partial}{\partial\eta}\\ \widetilde Y_j(\eta,v,\tau)&=\frac{\partial}{\partial v_{j+n}}-2v_j\frac{\partial}{\partial\tau}\quad&&\text{for } j=2,\dots,n\\ \widetilde T(\eta,v,\tau)&=\frac{\partial}{\partial\tau}\end{aligned}\tag{1.21}$$

if $n \geq 2$. For $n+1 \leq j \leq 2n$ we will also use the notation $\widetilde{X}_j := \widetilde{Y}_{j-n}$; notice that the vector fields $\widetilde{X}_j, \widetilde{Y}_j, \widetilde{T}$ are $\star$-left invariant.

Let $\phi : \omega \to \mathbb{R}$ be a given continuous function and $n \geq 1$; we will denote with $W^\phi := (W_2^\phi, \dots W_{2n}^\phi)$ the family of $(2n-1)$ first-order differential operators defined by

$$W_j^\phi := \begin{cases} \widetilde{X}_j = \dfrac{\partial}{\partial v_j} + 2v_{j+n}\dfrac{\partial}{\partial \tau} & \text{if } 2 \leq j \leq n \\ \widetilde{Y}_1 - 4\phi\widetilde{T} = \dfrac{\partial}{\partial \eta} - 4\phi\dfrac{\partial}{\partial \tau} & \text{if } j = n+1 \\ \widetilde{Y}_{j-n} = \dfrac{\partial}{\partial v_j} - 2v_{j-n}\dfrac{\partial}{\partial \tau} & \text{if } n+2 \leq j \leq 2n, \end{cases}$$

while when $n=1$ we put $W^\phi = W_2^\phi := \widetilde{Y}_1 - 4\phi\widetilde{T} = \frac{\partial}{\partial \eta} - 4\phi\frac{\partial}{\partial \tau}$.

We will prove that if $\phi \in \mathbf{C}^1(\omega)$ then $\mathbf{C}^1(\omega)$ functions are uniformly W^ϕ-differentiable too. More precisely if ϕ, $\psi \in \mathbf{C}^1(\omega)$, then ψ is uniformly W^ϕ-differentiable at A for every $A \in \omega$ and

$$\begin{aligned} W_j^\phi\psi(A) &= \widetilde{X}_j\psi(A) \qquad \text{if } j \neq n+1; \\ W_{n+1}^\phi\psi(A) &= \frac{\partial\psi}{\partial\eta}(A) - 4\phi(A)\frac{\partial\psi}{\partial\tau}(A)\,. \end{aligned}$$

In particular let us notice the (nonlinear) differential operator

$$\mathbf{C}^1(\omega) \ni \phi \to \mathfrak{B}\phi := W_{n+1}^\phi\phi \tag{1.22}$$

is a Burgers' type operator which can be also represented in distributional form as

$$\mathfrak{B}\phi = \frac{\partial\phi}{\partial\eta} - 2\frac{\partial\phi^2}{\partial\tau}$$

(see also Remark 5.2).

Now we are in order to state the main results of this paper. The former is the characterization of $\mathbb{H}$-regular intrinsic graph $G^1_{\mathbb{H},\phi}$ (defined as in (1.8)) in terms of the uniform W^ϕ-differentiability of $\phi : \omega \subset \mathbb{R}^{2n} \to \mathbb{R}$ (see Theorem 4.1). Moreover also an area type formula for $G^1_{\mathbb{H},\phi}$ with respect to the $(Q-1)$-spherical Hausdorff measure $\mathcal{S}_\infty^{Q-1}$ is proved (see (4.2)). We will collect them in Theorem 1.2 below.

The latter is the characterization of the uniformly W^ϕ-differentiability of $\phi : \omega \subset \mathbb{R}^{2n} \to \mathbb{R}$ in terms of existence of the derivatives $W_j^\phi\phi$ $(j = 2, \dots, 2n)$ in ω in a suitable sense (Theorem 1.3 below).

Theorem 1.2. *Let $\omega \subset \mathbb{R}^{2n}$ be an open set and let $\phi : \omega \to \mathbb{R}$ be a continuous function. Let $\Phi : \omega \to \mathbb{H}^n$ be the function defined by $\Phi(A) := \iota(A) \cdot \phi(A)e_1$ and let $S := \Phi(\omega)$. Then the following conditions are equivalent:*

(i) *S is an $\mathbb{H}$-regular surface and $\nu_S^{(1)}(P) < 0$ for all $P \in S$, where we denote with $\nu_S(P) = (\nu_S^{(1)}(P), \dots, \nu_S^{(2n)}(P))$ the horizontal normal to S at a point $P \in S$;*

(ii) *ϕ is uniformly W^ϕ-differentiable at any $A \in \omega$, and the vector function $W^\phi \phi : \omega \to \mathbb{R}^{2n-1}$ is continuous.*

Moreover, for all $P \in S$ we have

$$\nu_S(P) = \left(-\frac{1}{\sqrt{1+|W^\phi\phi|^2}}, \frac{W^\phi\phi}{\sqrt{1+|W^\phi\phi|^2}} \right) (\Phi^{-1}(P)), \tag{1.23}$$

and

$$\mathcal{S}_\infty^{Q-1}(S) = c(n) \int_\omega \sqrt{1+|W^\phi\phi|^2}\, d\mathcal{L}^{2n} \tag{1.24}$$

where $\mathcal{L}^{2n}$ denotes the Lebesgue measure on $\mathbb{R}^{2n}$.

Theorem 1.3. *Let $\omega \subset \mathbb{R}^{2n}$ be an open set and let $\phi : \omega \to \mathbb{R}$ be a continuous function. Then the following conditions are equivalent:*

(i) *ϕ is uniformly W^ϕ-differentiable at A for each $A \in \omega$;*

(ii) *there exist $w \in \mathbf{C}^0(\omega, \mathbb{R}^{2n-1})$ such that*

$$w = \left(\widetilde{X}_2\phi, \dots, \widetilde{X}_n\phi, \mathfrak{B}\phi, \widetilde{Y}_2\phi, \dots, \widetilde{Y}_n\phi \right)$$

in distributional sense in ω, and a family $\{\phi_\epsilon\}_{\epsilon>0} \subset \mathbf{C}^1(\omega)$ such that, for any open set $\omega' \Subset \omega$, we have

$$\phi_\epsilon \to \phi \text{ and } W^{\phi_\epsilon}\phi_\epsilon \to w \text{ uniformly on } \omega'. \tag{1.25}$$

Moreover, $w = W^\phi\phi$ on ω and

$$\lim_{r\to 0^+} \sup \left\{ \frac{|\phi(A) - \phi(B)|}{|A-B|^{\frac{1}{2}}} : A, B \in \omega', 0 < |A-B| < r \right\} = 0. \tag{1.26}$$

for each open set $\omega' \Subset \omega$.

The proof of Theorem 1.2 relies on a Mean Value Theorem for functions in $\mathbf{C}^1_{\mathbb{H}}(\mathbb{H}^n)$ (see Lemma 4.2), the Implicit Function Theorem and Whitney Extension Theorem (see Theorem 2.18). In particular, by means of Whitney Extension Theorem and the definition of intrinsic graph (1.8), we can transfer the notion of $\mathbf{C}^1_{\mathbb{H}}$ intrinsic differentiablity from $\mathbb{H}^n$ in the one of uniform W^ϕ-differentiability on the subgroup $V_1 \equiv \mathbb{R}^{2n}$ and viceversa.

Theorems 1.2 and 1.3 give some partial answers to the problem of the *good* parameterization of $\mathbb{H}$-regular surfaces proposed in [27]: to find out a model metric space such that each $\mathbb{H}$-regular surface can be locally viewed as its image through a bi-lipschitz continuous map (see also [55], [56] for similar problems in a more general setting and [52], [12] for Carnot groups). Indeed from Theorem 1.2 we infer that, if $S = G^1_{\mathbb{H},\phi} := \Phi(\omega)$ is $\mathbb{H}$-regular, then $\phi : (\omega, \rho_\phi) \to \mathbb{R}$ is locally Lipischitz continuous, i.e. (1.16) locally holds as well as (1.17). On the other hand (1.17) means the parameterization Φ in (1.6) is locally bi-lipschitz continuous provided $\rho \equiv \rho_\phi$ (see Corollary 4.3). Moreover by Theorem 1.3 it can be proved it is no longer true that $\phi : (\omega, \rho) \to \mathbb{R}$ is locally Lipischitz continuous when ρ denotes the so- called *parabolic* metric on $\mathbb{R}^2 = \mathbb{R}_\eta \times \mathbb{R}_\tau$, i.e the metric $\rho \equiv \rho_\phi$ in (1.14) with $\phi \equiv 0$ (see Corollary5.10). Anyway it is still an interesting open problem to understand whether, for instance, for a given $\mathbb{H}$-regular $S = G^1_{\mathbb{H},\phi}$ there exist a metric ρ on V_1, independent of S, and a suitable locally bi-lipschitz continuous parameterization $\tilde{\Phi} : \tilde{\omega} \subset (V_1, \rho) \to (S, d_\infty)$.

Let us stress that (1.24) and (1.23) are the exact counterparts of the analogous formulas for the inward normal and the area of (Euclidean) regular $(n-1)$- graphs in $\mathbb{R}^n$, provided the replacement of $W^\phi\phi$ with the classical gradient $\nabla\phi$ of the paramete rization ϕ.

The proof of Theorem 1.3 relies on the construction of an exponential map for vector fields $W^\phi_j (j = 2, \dots, 2n)$ (see Lemma 5.6) and on *a priori* uniform Hölder continuous estimates for $C^1(\omega)$ solutions ϕ of the first-order nonlinear PDE's system

$$W^\phi_j \phi = w_j \quad j = 2, \dots, 2n$$

with given $w_j \in C^0(\omega)$ (see Theorem 5.9). Let us notice that the construction of an exponential map for the vector field W^ϕ_{n+1} is not trivial since its coefficients are only Hölder continuous and then it requires an *ad hoc* argument.

Theorem 1.3 allows also the construction of explicit simple examples of uniform W^ϕ-differentiable functions $\phi : \omega \subset \mathbb{R}^2 \to \mathbb{R}$ which are not Euclidean C^1-regular. For instance in the case of the first Heisenberg group $\mathbb{H}^1$ the following corollary holds (see Corollary 5.11).

Corollary 1.4. *Let $\omega := (a, b) \times (c, d) \subset \mathbb{R}^2 \equiv \mathbb{R}_\eta \times \mathbb{R}_\tau$ and let $\phi : \omega \to \mathbb{R}$ be a continuous function which depends only on τ, i.e. $\phi = \phi(\tau) : (c, d) \to \mathbb{R}$. Suppose that $\phi^2 : (c, d) \to \mathbb{R}$ is of class $\mathbf{C}^1$. Then ϕ is uniformly W^ϕ-differentiable at A for every $A \in \omega$ and*

$$W^\phi\phi(A) = (\mathfrak{B}\phi)(A) = -2(\phi^2)'(A).$$

In particular from Corollary 1.4, Theorems 1.3 and 1.2 it follows that, if $\phi : \mathbb{R} \to \mathbb{R}$, $\phi(\tau) := |\tau|^\alpha$ with $1/2 < \alpha < 1$, then the intrinsic graph $S = G^1_{\mathbb{H},\phi}$ is a $\mathbb{H}$-regular surface in $\mathbb{H}^1$ but no longer an Euclidean regular graph in any neighborhood of the origin (see Example 3.9 in [30]).

Let us point out that, by Theorem 1.2 and the example given in [37] of an $\mathbb{H}$-regular surface in $\mathbb{H}^1 \equiv \mathbb{R}^3$ not 2-Euclidean rectifiable, it follows that uniformly W^ϕ-differentiable functions can be much more irregular from the Euclidean point of view than the previous one.

Eventually, let us give a short abstract of the paper. In section 2 we introduce our notations and we recall more or less known results; in section 3 we study the graph distance and the notion of W^ϕ-differentiability in $\mathbb{R}^{2n}$; in sections 4 and 5 we essentially prove the results respectively collected in Theorems 1.2 and 1.3.

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2. Notations and preliminary results

Besides te group operation in $\mathbb{H}^n$ and the dilations defined in the introduction, it is also useful to consider the group translations $\tau_P : \mathbb{H}^n \to \mathbb{H}^n$ defined as

$$Q \mapsto \tau_P(Q) := P \cdot Q$$

for any fixed $P \in \mathbb{H}^n$. We denote as $P^{-1} := [-z, -t]$ the inverse of P and as 0 the origin of $\mathbb{R}^{2n+1}$. We shall endow $\mathbb{H}^n$ with the homogeneous norm $\|P\|_\infty := \max\{|z|, |t|^{1/2}\}$ and with the distance, associated to the norm,

$$d_\infty(P, Q) := \|P^{-1} \cdot Q\|_\infty. \tag{2.1}$$

We explicitely observe that

Proposition 2.1. *The function d_∞ defined by (2.1) is a distance in $\mathbb{H}^n$ and the usual properties related with translations and dilations hold, i.e. $\forall\, P, Q, Q' \in \mathbb{H}^n$ and $\forall r > 0$*

$$d_\infty(\tau_P Q, \tau_P Q') = d_\infty(Q, Q') \quad \textit{and} \quad d_\infty(\delta_r Q, \delta_r Q') = r\, d_\infty(Q, Q'). \tag{2.2}$$

In addition, for any bounded subset Ω of $\mathbb{H}^n$ there exist positive constants $c_1(\Omega)$, $c_2(\Omega)$ such that

$$c_1(\Omega)|P - Q|_{\mathbb{R}^{2n+1}} \le d_\infty(P, Q) \le c_2(\Omega)|P - Q|^{1/2}_{\mathbb{R}^{2n+1}} \tag{2.3}$$

for $P, Q \in \Omega$. In particular, the topologies defined by d_∞ and by the Euclidean distance coincide on $\mathbb{H}^n$.

Remark 2.2. We stress that, because the topologies defined by d_∞ and by the Euclidean distance coincide, the topological dimension of $\mathbb{H}^n$ is $2n + 1$. On the contrary the Hausdorff dimension of $(\mathbb{H}^n, d_\infty)$ is $Q = 2n + 2$.

From now on, $U(P, r)$ will be the open ball with centre P and radius r with respect to the distance d_∞. We notice that $U(P, r)$ is an Euclidean Lipschitz domain in $\mathbb{R}^{2n+1}$.

There is a natural measure dh on $\mathbb{H}^n$ which is given by the Lebesgue measure $d\mathcal{L}^{2n+1} = dz\, dt$ on $\mathbb{C}^n \times \mathbb{R}$. The measure dh is left (and right) invariant and it is the Haar measure of the group. If $E \subset \mathbb{H}^n$ then $|E|$ is its Lebesgue measure and ω_k will denote the k-dimensional Lebesgue measure of the unit Euclidean ball in $\mathbb{R}^k$.

Definition 2.3. *(see [22]) We shall denote by $\mathcal{H}^m$ the m-dimensional Hausdorff measure obtained from the Euclidean distance in $\mathbb{R}^{2n+1} \simeq \mathbb{H}^n$, and by $\mathcal{H}^m_\infty$ the m-dimensional Hausdorff measure obtained from the distance d_∞ in $\mathbb{H}^n$. Analogously, $\mathcal{S}^m$ and $\mathcal{S}^m_\infty$ will denote the corresponding spherical measures.*

Translation invariance and homogeneity under dilations of Hausdorff measures follow as usual from (2.2), more precisely we have

Proposition 2.4. *Let $\Omega \subseteq \mathbb{H}^n$, $P \in \mathbb{H}^n$ and $m,r \in [0,\infty)$. Then*

$$\mathcal{H}^m_\infty(\tau_P\Omega) = \mathcal{H}^m_\infty(\Omega) \quad \text{and} \quad \mathcal{H}^m_\infty(\delta_r(\Omega)) = r^m\mathcal{H}^m_\infty(\Omega).$$

To simplify the notations it will be sometimes useful to adopt the convention $X_j := Y_{j-n}$ for $n+1 \leq j \leq 2n$, where X_j and Y_j $(j = 1,\dots,n)$ are the generators of the Lie algebra $\mathfrak{h}_n$ defined in (1.5).

For sake of completness, let us recall here the definition of the Carnot–Carathéodory metric associated with $X_1,\dots,X_n,Y_1,\dots,Y_n$. In fact, this definition has been developed in a much larger setting (see, e.g., [23], [49]).

Definition 2.5. *We say that an absolutely continuous curve $\gamma : [0,T] \to \mathbb{H}^n$ is a subunit curve with respect to $X_1,\dots,X_n,Y_1,\dots,Y_n$ if there exist real measurable functions $a_1(s),\dots,a_{2n}(s)$, $s \in [0,T]$ such that $\sum_j a_j^2 \leq 1$ and*

$$\dot\gamma(s) = \sum_{j=1}^n a_j(s)X_j(\gamma(s)) + \sum_{j=1}^n a_{j+n}(s)Y_j(\gamma(s)), \qquad \text{for a.e. } s \in [0,T].$$

If $P_1,P_2 \in \mathbb{H}^n$, their Carnot–Carathéodory distance $d_C(P_1,P_2)$ is

$$d_C(P_1,P_2) = \inf\left\{T > 0 : \exists\gamma : [0,T] \to \mathbb{H}^n \text{ subunit}, \gamma(0) = P_1,\ \gamma(T) = P_2\right\}.$$

Notice that the above set of curves joining P_1 and P_2 is not empty, by Chow's Theorem, since the rank of the Lie algebra generated by $X_1,\dots,X_n,Y_1,\dots,Y_n$ is maximal, and hence d_C is a distance on $\mathbb{H}^n$. We shall denote by $U_C(P,r)$ the open balls for d_C.

The following result is well known: see, for instance, [8], [59].

Proposition 2.6. *The Carnot–Carathéodory distance d_C is (globally) equivalent to the distance d_∞ defined in (2.1).*

If Ω is an open subset of $\mathbb{H}^n$ and $k \geq 0$ is a non negative integer, the symbols $\mathbf{C}^k(\Omega)$, $\mathbf{C}^\infty(\Omega)$ indicate the usual (Euclidean) spaces of real valued continuously differentiable functions. We denote by $\mathbf{C}^k(\Omega, H\mathbb{H}^n)$ the set of all C^k-sections of $H\mathbb{H}^n$ where the C^k regularity is understood as regularity between smooth manifolds. The notions of $\mathbf{C}^k_c(\Omega, H\mathbb{H}^n)$, $\mathbf{C}^\infty(\Omega, H\mathbb{H}^n)$ and $\mathbf{C}^\infty_c(\Omega, H\mathbb{H}^n)$ are defined analogously.

Definition 2.7. *Let $[z,t], P_0 \in \mathbb{H}^n$ be given. We set*

$$\pi_{P_0}([z,t]) = \sum_{j=1}^n x_j X_j(P_0) + \sum_{j=1}^n y_j Y_j(P_0).$$

The map $P_0 \to \pi_{P_0}([z,t])$ is a smooth section of $\mathrm{H}\mathbb{H}^n$.

The similarity among some statements in $\mathbb{H}^n$ with others in $\mathbb{R}^{2n+1}$ is clear using intrinsic notions of gradient for functions $f : \mathbb{H}^n \to \mathbb{R}$ and of divergence for sections of $\mathrm{H}\mathbb{H}^n$.

Definition 2.8. *If Ω is an open subset of $\mathbb{H}^n$, $f \in \mathbf{C}^1(\Omega)$ and $\varphi = (\varphi_1, \dots, \varphi_{2n}) \in \mathbf{C}^1(\Omega, \mathrm{H}\mathbb{H}^n)$, define*

$$\nabla_{\mathbb{H}} f := (X_1 f, \dots, X_n f, Y_1 f, \dots, Y_n f) \tag{2.4}$$

and

$$div_{\mathbb{H}} \varphi := \sum_{j=1}^n X_j \varphi_j + Y_j \varphi_{n+j}. \tag{2.5}$$

Alternatively $\nabla_{\mathbb{H}} f$ can be defined as the section of $\mathrm{H}\mathbb{H}^n$

$$\nabla_{\mathbb{H}} f := \sum_{j=1}^n (X_j f) X_j + (Y_j f) Y_j$$

whose canonical coordinates are $(X_1 f, \dots, X_n f, Y_1 f, \dots, Y_n f)$ (observe that this is consistent with the identification we mentioned of sections and their coordinates).

Finally, we write

$$\widehat{\nabla_{\mathbb{H}}} f := (X_2 f, \dots, X_n f, Y_1 f, \dots, Y_n f).$$

We shall denote by $\mathbf{C}^k_{\mathbb{H}}(\Omega)$ the set of continuous real functions f in Ω such that $\nabla_{\mathbb{H}} f$ is of class $\mathbf{C}^{k-1}$ in Ω. Moreover, we shall denote by $\mathbf{C}^k_{\mathbb{H}}(\Omega, \mathrm{H}\mathbb{H}^n)$ the set of all sections φ of $\mathrm{H}\mathbb{H}^n$ whose canonical coordinates φ_j belong to $\mathbf{C}^k_{\mathbb{H}}(\Omega)$ for $j = 1, \dots, 2n$.

Remark 2.9. We stress that the inclusion $\mathbf{C}^1(\Omega) \subset \mathbf{C}^1_{\mathbb{H}}(\Omega)$ is strict; see for example [27], Remark 5.9.

In $\mathbb{H}^n$ there is a natural definition of bounded variation functions and of finite perimeter sets (see [25], [32], [9]).

Definition 2.10. *We say that $f : \Omega \to \mathbb{R}$ is of bounded $\mathbb{H}$-variation in an open set $\Omega \subset \mathbb{H}^n$, ($f \in BV_{\mathbb{H}}(\Omega)$), if $f \in L^1(\Omega)$ and if*

$$\int_\Omega d|\nabla_{\mathbb{H}} f| := \sup \left\{ \int_\Omega f \, div_{\mathbb{H}} \varphi \, dh \ : \varphi \in \mathbf{C}^1_c(\Omega, \mathrm{H}\mathbb{H}^n), \ |\varphi(P)|_P \le 1 \right\} < +\infty. \tag{2.6}$$

Analogously the space $BV_{\mathbb{H},\mathrm{loc}}(\Omega)$ is defined in the usual way.

Definition 2.11. *We say that $E \subset \mathbb{H}^n$ is a locally finite $\mathbb{H}$-perimeter set (or a $\mathbb{H}$-Caccioppoli set) if $\mathbf{1}_E \in BV_{\mathbb{H},\mathrm{loc}}(\mathbb{H}^n)$, where we indicate as $\mathbf{1}_E$ the characteristic function of the set E. In this case, the measure $|\nabla_{\mathbb{H}}\mathbf{1}_E|$ will be called $\mathbb{H}$-perimeter of E and will be denoted by $|\partial E|_{\mathbb{H}}$.*

For $\mathbb{H}$-Caccioppoli sets the following divergence-type theorem holds (see [27])

Theorem 2.12. *There exists a $|\partial E|_{\mathbb{H}}$-measurable section ν_E of $\mathrm{H}\mathbb{H}^n$ such that*

$$-\int_E div_{\mathbb{H}}\varphi \, dh = \int_{\mathbb{H}^n} \langle \nu_E, \varphi\rangle \, d|\partial E|_{\mathbb{H}} \qquad \forall \varphi \in \mathbf{C}_0^\infty(\Omega; \mathrm{H}\mathbb{H}^n);$$
$$|\nu_E(P)|_P = 1 \qquad for\ |\partial E|_{\mathbb{H}} - a.e.\ P \in \mathbb{H}^n.$$

Here, the measurability of ν_E is meant in the sense that its coordinates $\nu_1, \dots, \nu_{2n}$ are $|\partial E|_{\mathbb{H}}$-measurable functions.

The function ν_E can be interpreted $|\partial E|_{\mathbb{H}}$-almost everywhere as a generalized inward “horizontal” normal to the set E.

Definition 2.13. *We shall say that $S \subset \mathbb{H}^n$ is a $\mathbb{H}$-regular hypersurface if for every $P \in S$ there exist an open ball $U(P,r)$ and a function $f \in \mathbf{C}^1_{\mathbb{H}}(U(P,r))$ such that*

(i) $$S \cap U(P,r) = \{Q \in U(P,r) : f(Q) = 0\};$$

(ii) $$\nabla_{\mathbb{H}} f(P) \neq 0.$$

We will denote with $\nu_S(P)$ the horizontal normal to S at a point $P \in S$, i.e. the unit vector

$$\nu_S(P) := -\frac{\nabla_{\mathbb{H}} f(P)}{|\nabla_{\mathbb{H}} f(P)|_P},$$

and with $T^g_{\mathbb{H}} S(P)$ the tangent group to S at P, i.e. the proper subgroup of $\mathbb{H}^n$ defined by

$$T^g_{\mathbb{H}} S(P) := \{Q : \langle\, \nabla_{\mathbb{H}}(f \circ \tau_P)(0),\, \pi_0(Q)) \,\rangle_0 = 0\}.$$

Finally, we use the notation $T_{\mathbb{H}} S(P)$ for the tangent plane to S at P, i.e. the lateral $P \cdot T^g_{\mathbb{H}} S(P)$.

As pointed out in the introduction, Euclidean regular hypersurfaces and of $\mathbb{H}$-regular hypersurfaces are disjoint classes.

Let us introduce some useful subspaces of $\mathfrak{h}_n$ (here $\widehat{X}_j$ means that in an enumeration we omit X_j):

$$\begin{aligned}
&\mathfrak{o} := \mathrm{span}\{X_1, \dots, X_{2n}\};\\
&\mathfrak{v}_j := \mathrm{span}\{X_1, \dots, \widehat{X}_j \dots, X_{2n}, T\} \quad (1 \le j \le 2n);\\
&\mathfrak{o}_j := \mathrm{span}\{X_1, \dots, \widehat{X}_j \dots, X_{2n}\} \quad (1 \le j \le 2n);\\
&\mathfrak{l}_j := \mathrm{span}\{X_j\} \quad (1 \le j \le 2n);\\
&\mathfrak{z} := \mathrm{span}\{T\}
\end{aligned}$$

and let $\pi_{\mathfrak{o}}, \pi_{\mathfrak{v}_j}, \pi_{\mathfrak{o}_j}, \pi_{\mathfrak{l}_j}, \pi_{\mathfrak{z}}$ be the projections of $\mathfrak{h}_n$ onto $\mathfrak{o}, \mathfrak{v}_j, \mathfrak{o}_j, \mathfrak{l}_j$ and $\mathfrak{z}$ respectively. Define the following subsets of $\mathbb{H}^n$:

$$\begin{aligned}
&O := \exp(\mathfrak{o}) = \{P \in \mathbb{H}^n : p_{2n+1} = 0\};\\
&V_j := \exp(\mathfrak{v}_j) = \{P \in \mathbb{H}^n : p_j = 0\};\\
&O_j := \exp(\mathfrak{o}_j) = O \cap V_j = \{P \in \mathbb{H}^n : p_j = p_{2n+1} = 0\};\\
&L_j := \exp(\mathfrak{l}_j) = \{P \in \mathbb{H}^n : p_i = 0\ \forall i \neq j\};\\
&Z := \exp(\mathfrak{z}) = \{P \in \mathbb{H}^n : p_1 = \cdots = p_{2n} = 0\}.
\end{aligned}$$

and let $\pi_O, \pi_{V_j}, \pi_{O_j}, \pi_{L_j}$ and π_Z be the maps defined by $\exp \circ \pi_{\mathfrak{o}} \circ \exp^{-1}, \exp \circ \pi_{\mathfrak{v}_j} \circ \exp^{-1}$ and so on; we will refer to them as orthogonal projections of $\mathbb{H}^n$ on O, V_j, O_j, L_j and Z.

The following properties of these projections are straightforward:

Proposition 2.14. *For any $P, Q \in \mathbb{H}^n$ we have*

$$\begin{aligned}
&\pi_{O_1}(P) = \pi_O \circ \pi_{V_1}(P) = \pi_{V_1} \circ \pi_O(P)\\
&\pi_{O_1}(P \cdot Q) = \pi_{O_1}(\pi_{O_1}(P) \cdot \pi_{O_1}(Q))\\
&\pi_Z(P \cdot Q) = \pi_Z(P) \cdot \pi_Z(Q) \cdot \pi_Z(\pi_O(P) \cdot \pi_O(Q))\\
&\|\pi_M(P)\|_\infty \leq \|P\|_\infty \quad \forall M \in \{O, O_1, V_1, L_1, Z\}
\end{aligned}$$

Let us observe that Z is the center of the group, and that only Z, L_j and V_j are subgroups; O_j is a subgroup only if $n = 1$ (because in this case it coincides with L_j), while O is never a subgroup. We agree to indicate with αe_j the point $\exp(\alpha X_j) \in L_j$; then for each $P \in \mathbb{H}^n$ there is a unique way to write P in the form $P_{V_j} \cdot P_{L_j}$ for points $P_{V_j} \in V_j, P_{L_j} \in L_j$: it is sufficient to take $P_{L_j} = p_j e_j$ and $P_{V_j} = P \cdot P_{L_j}^{-1} \in V_j$.

Recalling the definitions of ι and of the product law $\star$ on $\mathbb{R}^{2n}$ given in the introduction (see (1.10), (1.11), and (1.12)), we will use $\tau_A^\star$ to indicate the left translation by A in $\mathbb{R}^{2n}$. Explicitely, if $n > 1$ and $A = (\eta, v, \tau), B = (\eta', v', \tau') \in \mathbb{R}^{2n}$ we have

$$A \star B = (\eta + \eta', v + v', \tau + \tau' + \sigma(v, v')) \tag{2.7}$$

where

$$\sigma(v, v') = 2 \sum_{j=2}^{n} (v_{n+j} v'_j - v_j v'_{n+j}) \tag{2.8}$$

if $v = (v_2, \ldots, v_n, v_{n+2}, \ldots v_{2n})$, $v' = (v'_2, \ldots, v'_n, v'_{n+2}, \ldots v'_{2n})$. Instead if $n = 1$ and $A = (\eta, \tau), B = (\eta', \tau') \in \mathbb{R}^2$ we simply have

$$A \star B = (\eta + \eta', \tau + \tau'). \tag{2.9}$$

Notice that in both cases the induced group structure is the one arising from direct product $\mathbb{R} \times \mathbb{R}$ if $n = 1$, and $\mathbb{R} \times \mathbb{H}^{n-1}$ if $n > 1$, via the identification $\mathbb{R}^{2n} = \mathbb{R}_\eta \times (\mathbb{R}_v^{2n-2} \times \mathbb{R}_\tau) = \mathbb{R} \times \mathbb{H}^{n-1}$.

As we did in the introduction, we can define via ι a family of intrinsic dilations $\delta_\lambda^\star$ ($\lambda > 0$) on $\mathbb{R}^{2n}$, which can be written explicitely as

$$\begin{array}{ll} \delta_\lambda^\star(\eta, v, \tau) = (\lambda\eta, \lambda v, \lambda^2\tau) & \text{for } n \geq 2 \\ \delta_\lambda^\star(\eta, \tau) = (\lambda\eta, \lambda^2\tau) & \text{for } n = 1. \end{array}$$

As we already said, we define a $\star$-linear functional $L : \mathbb{R}^{2n} \to \mathbb{R}$ as a homomorphism which is also positively homogeneous of degree 1 with respect to the dilations, i.e. $L \circ \delta_\lambda^\star = \lambda L$. The following Proposition comes from Proposition 5.4 in [27]:

Proposition 2.15. *Let $L : \mathbb{R}^{2n} \to \mathbb{R}$ be a $\star$-linear functional; then there is a unique vector $w_L \in \mathbb{R}^{2n-1}$ such that $L(A) = \langle A, w_L\rangle$, where we intend that*

$$\begin{array}{ll} \langle A, w_L\rangle = \eta w_{Ln+1} + \displaystyle\sum_{j=2, j\neq n+1}^{2n} v_j w_{Lj} & \text{if } n \geq 2, w_L = (w_{L2}, \dots, w_{L2n}) \text{ and } A = (\eta, v, \tau) \\ \langle A, w_L\rangle = \eta w_{L2} & \text{if } n = 1, w_L = w_{L2} \text{ and } A = (\eta, \tau). \end{array}$$

Conversely, through the previous formulas we can associate to each $w \in \mathbb{R}^{2n-1}$ a unique $\star$-linear functional L_w.

Observe that the choice of the enumeration of the components of w_L has been made in order to be coherent with the one made for the components of v and with the fact that η is the $(n+1)$-th coordinate of $\iota(A)$.

Finally, let us recall the following results, which will be crucial in our paper: their proofs can be found in [27].

Theorem 2.16. [Implicit Function Theorem] *Let Ω be an open set in $\mathbb{H}^n$, $0 \in \Omega$, and let $f \in \mathbf{C}^1_{\mathbb{H}}(\Omega)$ be such that $X_1 f(0) > 0$, $f(0) = 0$. Let*

$$\begin{array}{l} E := \{[z,t] \in \Omega : f([z,t]) < 0\} \\ S := \{[z,t] \in \Omega : f([z,t]) = 0\}; \end{array}$$

then there exist $\delta, h > 0$ such that, if we put $I := [-\delta, \delta] \times [-\delta, \delta]^{2n-2} \times [-\delta^2, \delta^2] \subset \mathbb{R}^{2n}$, $J := \{se_1 \in L_1 : s \in [-h, h]\}$ and $\mathcal{U} := \iota(I) \cdot J$, we have that

E has finite $\mathbb{H}$-perimeter in $\mathcal{U}$;
$\partial E \cap \mathcal{U} = S \cap \mathcal{U}$;
$\nu_E(P) = -\nabla_{\mathbb{H}} f(P)/|\nabla_{\mathbb{H}} f(P)|_P = \nu_S(P)$ for all $P \in S \cap \mathcal{U}$.

Moreover there exists a unique continuous function $\phi : I \to [-h, h]$ such that $S \cap \overline{\mathcal{U}} = \Phi(I)$, where Φ is the map $I \ni A \mapsto \iota(A) \cdot \phi(A) e_1 \in \mathbb{H}^n$, and the $\mathbb{H}$-perimeter has the integral representation

$$|\partial E|_{\mathbb{H}}(\mathcal{U}) = \int_I \frac{|\nabla_{\mathbb{H}} f|}{X_1 f}(\Phi(A))\, d\mathcal{L}^{2n}(A). \tag{2.10}$$

Theorem 2.17. *Let Ω be an open set in $\mathbb{H}^n$ and let $E \in \mathbb{H}^n$ be such that $\partial E \cap \Omega = S \cap \Omega$ where $S \subset \mathbb{H}^n$ is an $\mathbb{H}$-regular surface. If $P_0 \in S$ and $r > 0$ put*

$$E_{P_0,r} := \delta_{1/r}(P_0^{-1} \cdot E) = \{P \in \mathbb{H}^n : \delta_r(P_0^{-1} \cdot P) \in E\}.$$

Then there is a $c(n) > 0$ such that

(i) $\lim_{r\to 0} |\partial E_{P_0,r}|_{\mathbb{H}}(U(0,1)) = \lim_{r\to 0} \dfrac{|\partial E|_{\mathbb{H}}(U(P_0,r))}{r^{2n+1}} = \mathcal{H}^{2n}(T^g_{\mathbb{H}}S(P^0)\cap U(0,1)) = c(n)$;

(ii) $|\partial E|_{\mathbb{H}} \llcorner \Omega = c(n)\, \mathcal{S}^{Q-1}_{\infty} \llcorner (S\cap\Omega)$.

Theorem 2.18. [Whitney Extension Theorem] *Let $F \subset \mathbb{H}^n$ be a closed set, and let $f : F \to \mathbb{R}$, $k : F \to \mathrm{H}\mathbb{H}^n$ be two continuous functions. We set*

$$R(Q,P) := \frac{f(Q) - f(P) - \langle k(P), \pi_P(P^{-1}\cdot Q)\rangle_P}{d(P,Q)},$$

and, if $K \subset F$ is a compact set,

$$\rho_K(\delta) := \sup\{|R(Q,P)| : P,Q\in K,\ 0 < d_\infty(P,Q) < \delta\}.$$

If $\rho_K(\delta)\to 0$ as $\delta\to 0$ for every compact set $K\subset F$, then there exist $\tilde f : \mathbb{H}^n \to \mathbb{R}$, $\tilde f \in \mathbf{C}^1_{\mathbb{H}}(\mathbb{H}^n)$ such that $\tilde f_{|F} \equiv f$ and $\nabla_{\mathbb{H}} \tilde f_{|F} \equiv k$.

Taking the Implicit Function theorem into account we can give the following notion of intrinsic graph in $\mathbb{H}^n$.

Definition 2.19. *A set $S\subset\mathbb{H}^n$ is an X_1-graph if there is a function $\phi : \omega\subset\mathbb{R}^{2n}\to\mathbb{R}$ such that $S = \{\iota(A)\cdot\phi(A)e_1 : A\in\omega\}$.*

More generally, after fixing an identification $\iota_j : \mathbb{R}^{2n}\to V_j$, for $j = 2,\dots,2n$ we can define X_j-graphs as those subsets S of $\mathbb{H}^n$ for which there exists a function $\phi : \omega\subset\mathbb{R}^{2n}\to\mathbb{R}$ such that $S = \{\iota_j(A)\cdot\phi(A)e_j : A\in\omega\}$.

A general definition of intrinsic graph in $\mathbb{H}^n$, which applies also to surfaces with topological codimension bigger than 1, is given in [30]. In particular this notion is stable with respect to left translations of the group; more precisely, from Proposition 3.11 in [30] we infer

Proposition 2.20. *Let $S\subset\mathbb{H}^n$ be an X_j-graph, i.e. $S = \{\Phi(A) := \iota_j(A)\cdot\phi(A)e_j : A\in\omega\}$. Let $P = (p_1,\dots p_{2n+1})\in\mathbb{H}^n$, $P = P_{V_j}\cdot P_{L_j}$ with $P_{L_j} = p_je_j\in L_j$ and $P_{V_j}\in V_j$. Then the translated set τ_PS still is an X_j-graph; more precisely, if we define $\sigma_P : \mathbb{R}^{2n}\to\mathbb{R}^{2n}$ by $\sigma_P(A) := \iota_j^{-1}(P\cdot\iota_j(A)\cdot P_{L_j}^{-1})$, we have*

$$\tau_PS = \{\Phi'(A) := \iota_j(A)\cdot\phi'(A)e_j : A\in\omega'\},$$

where $\omega' := \sigma_P(\omega)$ and $\phi' : \omega'\to\mathbb{R}$ is defined by

$$\phi'(A) = p_j + \phi(\sigma_{P^{-1}}(A)).$$

In addition we have $\Phi' = \tau_P\circ\Phi\circ\sigma_{P^{-1}}$.

Remark 2.21. In Theorem 2.16, and more generally in the rest of the paper, we made a precise choice, i.e. to consider only regular hypersurfaces that are zero sets of functions $f\in\mathbf{C}^1_{\mathbb{H}}$ such that $X_1f > 0$. This fact, somehow, makes X_1 a "privileged" direction: for example, observe that such surfaces results X_1-graphs, i.e. functions on V_1, and that we translate points of V_1 by an element with all the coordinates null except the first one. One can prove that this is not restrictive; the key tool in this sense are the so-called "horizontal rotations", see [43], section 2.1.

We end the section with an improvement of Theorem 2.16:

Proposition 2.22. *Under the same assumptions of Theorem 2.16, let $\widetilde{X}_j, \widetilde{Y}_j$ be the vector fields defined in* (1.20) *and* (1.21)*, and let $\mathfrak{B}\phi$ be the distribution $\frac{\partial\phi}{\partial\eta} - 2\frac{\partial\phi^2}{\partial\tau}$ on I, where ϕ and I are given by Theorem 2.16). Then if $n > 1$ we have for $j = 2, \dots, n$*

$$\widetilde{X}_j\phi = -\frac{X_j f}{X_1 f}\circ\Phi, \quad \widetilde{Y}_j\phi = -\frac{Y_j f}{X_1 f}\circ\Phi, \quad \mathfrak{B}\phi = -\frac{Y_1 f}{X_1 f}\circ\Phi \tag{2.11}$$

where the equalities must be understood in distributional sense on I. Moreover, the $\mathbb{H}$-perimeter has the integral representation

$$|\partial E|_{\mathbb{H}}(\mathcal{U}) = c(n)\mathcal{S}_\infty^{Q-1}\llcorner(S\cap\mathcal{U}) = \int_I \sqrt{1+(\mathfrak{B}\phi)^2+\sum_{j=2}^n[|\widetilde{X}_j\phi|^2+|\widetilde{Y}_j\phi|^2]}\, d\mathcal{L}^{2n}. \tag{2.12}$$

If $n = 1$ we have simply

$$|\partial E|_{\mathbb{H}}(\mathcal{U}) = c(1)\mathcal{S}_\infty^{Q-1}\llcorner(S\cap\mathcal{U}) = \int_I \sqrt{1+|\mathfrak{B}\phi|^2}\, d\eta\, d\tau.$$

Proof. We will give the proof only for the case $n \geq 2$; the generalization to $n = 1$ should not present difficulties.

Arguing as in Step 1 of the proof of Theorem 2.16 (Theorem 6.5 in [27]) we can suppose that there exists a family of functions $f_\epsilon : \mathcal{U} \to \mathbb{R}$ such that $f_\epsilon \in \mathbf{C}^1(\overline{\mathcal{U}})$, $X_1 f_\epsilon > 0$ on $\mathcal{U}$ and

$$X_j f_\epsilon \to X_j f,\ Y_j f_\epsilon \to Y_j f_\epsilon \quad \text{uniformly on } \mathcal{U} \quad (j = 1, \dots, n).$$

Now, following Step 4 of the same proof, we obtain the existence (for ϵ_0 small enough and h as in Theorem 2.16) of functions $\phi_\epsilon : I \to [-h, h], 0 < \epsilon < \epsilon_0$ such that

$$\begin{aligned} &f_\epsilon(\iota(A)\cdot\phi_\epsilon(A)e_1) = 0 \quad \text{for all } A \in I \\ &\phi_\epsilon \to \phi \quad \text{uniformly on } I \text{ for } \epsilon \to 0. \end{aligned}$$

It is not difficult to prove that $\phi_\epsilon \in \mathbf{C}^1(I)$; indeed, following once again the proof of the Implicit Function Theorem, the fact that $f_\epsilon \in \mathbf{C}^1$ implies that also

$$\begin{aligned} g_\epsilon \quad : \quad & [-h, h]\times I \to \mathbb{R} \\ & (\xi, \eta, v, \tau) \longmapsto f_\epsilon(\iota(\eta, v, \tau)\cdot\xi e_1) \end{aligned}$$

is also $\mathbf{C}^1$. As ϕ_ϵ is obtained by means of the classical Implicit Function Theorem (so that $g_\epsilon(\phi_\epsilon(\eta, v, \tau), \eta, v, \tau) = 0$), we get that ϕ_ϵ is $\mathbf{C}^1$ too. This implies (it is sufficient to differentiate the equality $f_\epsilon(\iota(\eta,v,\tau)\cdot\phi_\epsilon(\eta,v,\tau)e_1)\equiv 0)$ that

$$\begin{aligned}
\frac{\partial\phi_\epsilon}{\partial\eta}(A) &= -\frac{Y_1 f_\epsilon + 4\phi_\epsilon T f_\epsilon}{X_1 f_\epsilon}(\Phi_\epsilon(A))\\
\frac{\partial\phi_\epsilon}{\partial v_j}(A) &= -\frac{\frac{\partial}{\partial x_j} f_\epsilon}{X_1 f_\epsilon}(\Phi_\epsilon(A))\\
\frac{\partial\phi_\epsilon}{\partial v_{j+n}}(A) &- \frac{\frac{\partial}{\partial y_j} f_\epsilon}{X_1 f_\epsilon}(\Phi_\epsilon(A))\\
\frac{\partial\phi_\epsilon}{\partial\tau}(A) &= -\frac{T f_\epsilon}{X_1 f_\epsilon}(\Phi_\epsilon(A))
\end{aligned}$$

for all $A=(\eta,v,\tau)\in\mathring{I}$ and all $j=2,\dots,n$; obviously, Φ_ϵ is the map $A\longmapsto\iota(A)\cdot\phi_\epsilon(A)e_1$.

Then for $j=2,\dots,n$ we get

$$\begin{aligned}
\widetilde{X}_j\phi_\epsilon &= -\frac{X_j f_\epsilon}{X_1 f_\epsilon}\circ\Phi_\epsilon\\
\widetilde{Y}_j\phi_\epsilon &= -\frac{Y_j f_\epsilon}{X_1 f_\epsilon}\circ\Phi_\epsilon\\
\mathfrak{B}\phi_\epsilon &= \frac{\partial\phi_\epsilon}{\partial\eta} - 2\frac{\partial\phi_\epsilon^2}{\partial\tau} = \frac{\partial\phi_\epsilon}{\partial\eta} - 4\phi_\epsilon\frac{\partial\phi_\epsilon}{\partial\tau} = -\frac{Y_1 f_\epsilon}{X_1 f_\epsilon}\circ\Phi_\epsilon
\end{aligned}$$

from which (2.11) follows.

The integral representation (2.12) follows from the area type formula (2.10), together with (2.11). □

Remark 2.23. Starting from Theorem 2.22, it is not difficult to prove the following fact: let Ω be an open subset of $\mathbb{H}^n$, and let $f\in\mathbf{C}^1_{\mathbb{H}}(\Omega)$ be such that $X_1 f>0$ on $S:=\{f=0\}$. Suppose that S is intrinsically parameterized by $\phi:\omega\subset\mathbb{R}^{2n}\to\mathbb{R}$ (i.e. $S:=\Phi(\omega)$, where as usual $\Phi(A):=\iota(A)\cdot\phi(A)e_1$) and let $E:=\{f<0\}$. Then for each Borel set $F\subset\Omega$ we have

$$|\partial E|_{\mathbb{H}}(F)=c(n)\mathcal{S}^{Q-1}_\infty(F\cap S)=\int_{\Phi^{-1}(F)}\sqrt{1+(\mathfrak{B}\phi)^2+\sum_{j=2}^n[(\widetilde{X}_j\phi)^2+(\widetilde{Y}_j\phi)^2]}\,d\mathcal{L}^{2n} \tag{2.13}$$

if $n\geq 2$, and

$$|\partial E|_{\mathbb{H}}(F)=c(1)\mathcal{S}^{Q-1}_\infty(F\cap S)=\int_{\Phi^{-1}(F)}\sqrt{1+(\mathfrak{B}\phi)^2}\;d\mathcal{L}^2 \tag{2.14}$$

if $n=1$.

Remark 2.24. The operator $\mathfrak{B}$ is known in the literature as Burger's operator: see for example [21], section 3.4.

3. Graph distance and W^ϕ-differentiability

Let ω be an open, connected and bounded subset of $\mathbb{R}^{2n} = \mathbb{R}_\eta \times \mathbb{R}^{2n-2}_v \times \mathbb{R}_\tau$ if $n \geq 1$, of $\mathbb{R}^2 = \mathbb{R}_\eta \times \mathbb{R}_\tau$ if $n = 1$. If $n \geq 2$ and $A = (\eta, v, \tau) \in \mathbb{R}^{2n}$ and $r > 0$ are given, we define

$$I_r(A) := \left\{(\eta', v', \tau') \in \mathbb{R}^{2n} : |(\eta', v') - (\eta, v)| < r, |\tau' - \tau| < r\right\},$$

while if $n = 1$ and $A = (\eta, \tau)$ we put

$$I_r(A) := \left\{(\eta', \tau') \in \mathbb{R}^2 : |\eta' - \eta| < r, |\tau' - \tau| < r\right\}.$$

Let $\phi : \omega \to \mathbb{R}$ be a given function; we will indicate with W^ϕ the family of first-order operators $(W^\phi_2, \ldots W^\phi_{2n})$ (the reasons of the enumeration from 2 will be clear later) defined for $n \geq 2$ by

$$W^\phi_j := \begin{cases} \widetilde{X}_j = \dfrac{\partial}{\partial v_j} + 2v_{j+n}\dfrac{\partial}{\partial \tau} & \text{if } 2 \leq j \leq n \\ \widetilde{Y}_1 - 4\phi\widetilde{T} = \dfrac{\partial}{\partial \eta} - 4\phi\dfrac{\partial}{\partial \tau} & \text{if } j = n+1 \\ \widetilde{Y}_{j-n} = \dfrac{\partial}{\partial v_j} - 2v_{j-n}\dfrac{\partial}{\partial \tau} & \text{if } n+2 \leq j \leq 2n, \end{cases} \tag{3.1}$$

while for $n = 1$ we put $W^\phi = W^\phi_2 := \widetilde{Y}_1 - 4\phi\widetilde{T} = \frac{\partial}{\partial \eta} - 4\phi\frac{\partial}{\partial \tau}$.

From now on $\phi : \omega \to \mathbb{R}$ will be a fixed continuous function, and Φ will indicate the function $\omega \ni A \mapsto \iota(A) \cdot \phi(A)e_1 \in \mathbb{H}^n$; explicitely

$$\begin{aligned} &\Phi(\eta, v, \tau) = (\phi(\eta, v, \tau), v_2, \ldots, v_n, \eta, v_{n+2}, \ldots, v_{2n}, \tau + 2\eta\phi(\eta, v, \tau)) && \text{if } n \geq 2 \\ &\Phi(\eta, \tau) = (\phi(\eta, \tau), \eta, \tau + 2\eta\phi(\eta, \tau)) && \text{if } n = 1. \end{aligned}$$

For $A, B \in \omega$ we define the graph distance

$$\rho_\phi(A, B) := \|\pi_{O_1}(\Phi(A)^{-1} \cdot \Phi(B))\|_\infty + \|\pi_Z(\Phi(A)^{-1} \cdot \Phi(B))\|_\infty \tag{3.2}$$

which is equivalent to $\|\pi_{V_1}(\Phi(A)^{-1} \cdot \Phi(B))\|_\infty$. Explicitely, for $n \geq 2$ and $A = (\eta, v, \tau)$, $B = (\eta', v', \tau')$ we have

$$\rho_\phi(A, B) = |(\eta', v') - (\eta, v)| + |\tau' - \tau + 2(\phi(B) + \phi(A))(\eta' - \eta) + \sigma(v', v)|^{1/2}$$

where $\sigma(v', v)$ has been defined in (2.8); if $n = 1$ and $A = (\eta, \tau), B' = (\eta', \tau')$ we have $\rho_\phi(A, B) = |\eta' - \eta| + |\tau' - \tau + 2(\phi(B) + \phi(A))(\eta' - \eta)|^{1/2}$.

With this definition we are able to prove the following

Proposition 3.1. *If there is an $L > 0$ such that*

$$|\phi(A) - \phi(B)| \leq L\,\rho_\phi(A, B) \tag{3.3}$$

for all $A, B \in I$, then the quantity ρ_ϕ in (3.2) is a quasimetric on I, id est

(i) *$\rho_\phi(A, B) = 0 \Leftrightarrow A = B$;*
(ii) *$\rho_\phi(A, B) = \rho_\phi(B, A)$;*
(iii) *there exists $q > 1$ such that $\rho_\phi(A, B) \leq q\,[\,\rho_\phi(A, C) + \rho_\phi(C, B)\,]$*

for all $A, B, C \in I$.

Proof. The assertions in (i) and (ii) are straightforward, while for (iii) we have

$$\begin{aligned}
\rho_\phi(A,B) &\le 2d_\infty(\Phi(A),\Phi(B)) \\
&\le 2[\, d_\infty(\Phi(A),\Phi(C)) + d_\infty(\Phi(C),\Phi(B))\,] \\
&\le 2[\, |\phi(A)-\phi(C)| + \rho_\phi(A,C) + |\phi(C)-\phi(B)| + \rho_\phi(C,B)\,] \\
&\le 2(L+1)[\, \rho_\phi(A,C) + \rho_\phi(C,B)\,].
\end{aligned}$$

□

Let us observe that if ϕ satisfies the condition (3.3), then it is locally $1/2$-Hölder continuous in the Euclidean sense, i.e. for all compact set $K \subset \omega$ there exist an $L' = L'(K) > 0$ such that

$$|\phi(B) - \phi(A)| \le L'|B-A|^{1/2} \tag{3.4}$$

for all $A, B \in K$. First, let us observe that for any $P \in \mathbb{H}^n, \alpha \in \mathbb{R}$

$$\begin{aligned}
\|\pi_Z(P\cdot \alpha e_1)\|_\infty &\le \|\pi_Z(P)\|_\infty + \sqrt{2}|\alpha|^{1/2}\|\pi_{V_1}(P)\|_\infty^{1/2} \\
\|\pi_Z(\alpha e_1\cdot P)\|_\infty &\le \|\pi_Z(P)\|_\infty + \sqrt{2}|\alpha|^{1/2}\|\pi_{V_1}(P)\|_\infty^{1/2}.
\end{aligned}$$

Now let $M := \sup_K |\phi|, \Delta := \sup_{A\in K}|A|$ and, as before, $\phi := \phi(A), \phi' := \phi(B)$; then

$$\begin{aligned}
&|\phi(B)-\phi(A)|\,/L \le \rho_\phi(B,A) \qquad (3.5)\\
&= \|\pi_{O_1}(-\phi e_1\cdot\iota(A)^{-1}\cdot\iota(B)\cdot\phi' e_1)\|_\infty + \|\pi_Z(-\phi e_1\cdot\iota(A)^{-1}\cdot\iota(B)\cdot\phi' e_1)\|_\infty \\
&\le |B-A| + \|\pi_Z(\iota(A)^{-1}\cdot\iota(B)\cdot\phi' e_1)\|_\infty + \sqrt{2M}\|\pi_{V_1}(\iota(A)^{-1}\cdot\iota(B)\cdot\phi' e_1)\|_\infty^{1/2} \\
&\le (2\sqrt{\Delta}+\sqrt{2M})|B-A|^{1/2} + \|\pi_Z(\iota(A)^{-1}\iota(B))\|_\infty + \sqrt{2M}\|\pi_{V_1}(\iota(A)^{-1}\iota(B))\|_\infty^{1/2} \\
&\le (2\sqrt{\Delta}+2\sqrt{2M}+C(K))|B-A|^{1/2}.
\end{aligned}$$

where in the last passage we used (2.3) (this is the reason of the constant $C(K)$).

Now we have all the tools to state our notion of W^ϕ-differentiability as given in Definition 1.1; let us remark that, if ψ is uniformly W^ϕ-differentiable at A, then it is also W^ϕ-differentiable at A, as (1.18) is satisfied with the same L as in (1.19).

Remark 3.2. If ψ is W^ϕ-differentiable at A, then it is continuous at A. Indeed, if $L \in \mathbb{R}^{2n-1}$ is such that (1.18) holds and w_L is as in Proposition 2.15, then for any $B \in \omega$

$$\psi(B)-\psi(A) = \frac{\psi(B)-\psi(A)-\langle w_L, A^{-1}\star B\rangle}{\rho_\phi(A,B)}\cdot\rho_\phi(A,B) + \langle w_L, A^{-1}\star B\rangle$$

and we deduce the continuity of ψ at A from the W^ϕ-differentiability at A together with the fact that $\rho_\phi(A,B)$ is bounded near A.

Remark 3.3. We stress the fact that if $\psi : \omega \to \mathbb{R}$ is uniformly W^ϕ-differentiable at $A \in \omega$, then ψ is Lipschitz continuous (between the spaces (ω, ρ_ϕ) and $(\mathbb{R}, d_{eucl})$) in a neighbourhood of A; in fact there exist $C, r > 0$ such that

$$\frac{|\psi(B) - \psi(A) - L(A^{-1} \star B)|}{\rho_\phi(A, B)} \le C$$

for all $B \in I_r(A)$, whence

$$|\psi(B) - \psi(A)| \le |\langle w_L, A^{-1} \star B\rangle| + C\rho_\phi(A, B) \le (|w_L| + C)\rho_\phi(A, B)$$

We will indicate the $\star$-linear functional L such that (1.18) holds with $d_{W^\phi}\psi(A)$; we will call the vector w_L the W^ϕ-differential of ψ at A, and we will indicate it with $W^\phi\psi(A)$, writing $[W^\phi\psi(A)]_j$ for w_{Lj}, $j = 2, \dots, 2n$. These definitions are well posed because of the following

Lemma 3.4. *Let $\phi, \psi : \omega \to \mathbb{R}$ be such that ψ is W^ϕ-differentiable at $A \in \omega$, and let L be a $\star$-linear functional such that* (1.18) *holds; then L is unique.*

Proof. We have to prove that, if $w := w_L, w' := w_{L'} \in \mathbb{R}^{2n-1}$ are given by Proposition 2.15, then $w = w'$. We will give the proof only for the case $n \ge 2$, as it can be easily adapted for $n = 1$. Therefore let $A = (\eta, v, \tau)$: it is easy to prove that

$$\lim_{B=(\eta',v',\tau')\to A} \frac{\langle w - w', (\eta' - \eta, v' - v)\rangle}{\rho_\phi(A, B)} = 0. \tag{3.6}$$

Let

$$\begin{aligned}\mathcal{A} &= \{(\eta', v', \tau') \in \omega : \rho_\phi(A, (\eta', v', \tau')) = |(\eta' - \eta, v' - v)|\}\\ &:= \{(\eta', v', \tau') \in \omega : \pi_Z(\Phi'^{-1} \cdot \Phi) = 0\}\\ &= \{(\eta', v', \tau') \in I : \tau' = \tau - 2(\phi' + \phi)(\eta' - \eta) - \sigma(v', v)\}\end{aligned}$$

where, here and in the following, we write Φ', Φ, ϕ' and ϕ instead of $\Phi(\eta', v', \tau')$, $\Phi(\eta, v, \tau)$, $\phi(\eta', v', \tau')$ and $\phi(\eta, v, \tau)$ respectively. Let $\delta_2 > 0$ be such that $I := \overline{I_{\delta_2}(A)} \subset \omega$; we want to prove that there exists a $\delta_1 > 0$ with the property that for all (η', v') with $|(\eta' - \eta, v' - v)| \le \delta_1$ there is a $\tau' \in [\tau - \delta_2, \tau + \delta_2]$ such that $\tau' = \tau - 2(\phi' + \phi)(\eta' - \eta) - \sigma(v', v)$, i.e. $(\eta', v', \tau') \in \mathcal{A}$. Being ϕ continuous we can suppose that $|\phi| \le M$ on I; then, for each (η', v') with $|(\eta' - \eta, v' - v)| \le \delta_1$, the functions $\gamma_{(\eta',v')}(\tau') := \tau - 2(\phi(\eta', v', \tau') + \phi(\eta, v, t))(\eta' - \eta) - \sigma(v', v)$ map the closed interval $[\tau - \delta_2, \tau + \delta_2]$ into itself provided δ_1 is sufficiently small. In fact

$$\begin{aligned}&|\gamma_{(\eta',v')}(\tau') - \tau|\\ =\ &\left|2(\phi' + \phi)(\eta' - \eta) + 2\textstyle\sum_{j=2}^n (v_j v'_{n+j} - v_{n+j} v'_j)\right|\\ =\ &\left|2(\phi' + \phi)(\eta' - \eta) + 2\textstyle\sum_{j=2}^n (v_j(v'_{n+j} - v_{n+j}) - v_{n+j}(v'_j - v_j))\right|\\ \le\ &2M\delta_1 + 2|v|\delta_1\end{aligned} \tag{3.7}$$

so it is sufficient to choose δ_1 such that $(2M + 2|v|)\delta_1 \le \delta_2$. Therefore the fixed point theorem guarantees that $\gamma_{(\eta',v')}$ has a fixed point $\tau'(\eta', v')$ if $|(\eta' - \eta, v' - v)| \le \delta_1$, so that $(\eta', v', \tau'(\eta', v')) \in \mathcal{A}$, i.e. $\rho_\phi((\eta', v', \tau'(\eta', v')), (\eta, v, \tau)) = |(\eta' - \eta, v' - v)|$; moreover, it is not difficult to prove that $\tau'(\eta', v') \to \tau$ if $(\eta', v') \to (\eta, v)$ (it's sufficient to use the very same estimate as in (3.7)). Now, for each $j = 2, \dots 2n$, we can easily construct a sequence $A^h = (\eta^h, v^h, \tau^h) \in \mathcal{A}$ such that

- $A^h \to A$;
- $\eta^h \equiv \eta$, $v_i^h \equiv v_i \, \forall \, i \neq j$ and $\rho_\phi(A^h, A) = v_j^h - v_j > 0$ if $j \neq n+1$;
- $v^h \equiv v$ and $\rho_\phi(A^h, A) = \eta^h - \eta > 0$ if $j = n+1$.

By (3.6) we obtain

$$0 = \lim_{h\to\infty} \frac{\langle w - w', (\eta^h - e, v^h - v)\rangle}{\rho_\phi(A^h, A)} \equiv w_j - w_j'$$

for all $j = 2, \dots 2n$, whence $w = w'$. □

Remark 3.5. Let $A \in \omega$ and $P := \Phi(A) = \iota(A) \cdot \phi(A) e_1$. With the same notations of Proposition 2.20, set $\sigma_{P^{-1}}(B) := \iota^{-1}(P^{-1} \cdot \iota(B) \cdot P_{L_1})$ and $\omega' := \sigma_{P^{-1}}(\omega)$. Let $\alpha\Theta$ denote the element $(0, \dots, 0, \alpha) \in \mathbb{R}^{2n}$ and define

$$\begin{aligned} \phi' :\ & \omega' \longrightarrow \mathbb{R} \\ & B = (\eta', v', \tau') \mapsto \phi(\sigma_P(B)) - \phi(A) = \phi(A \star B \star (-4\phi(A)(\eta' - \eta))\Theta) - \phi(A); \end{aligned}$$

then $\Phi'(\omega') = \tau_{P^{-1}}(\Phi(\omega))$, where as usual $\Phi'(B) = \iota(B) \cdot \phi(B) e_1$.

It is not difficult to show that a function $\psi \in \mathbf{C}^0(\omega)$ is W^ϕ-differentiable (resp. uniformly W^ϕ-differentiable) at $B \in \omega$ if and only if $\psi \circ \sigma_P \in \mathbf{C}^0(\omega')$ is $W^{\phi'}$-differentiable (resp. uniformly $W^{\phi'}$-differentiable) at $\sigma_{P^{-1}}(B) \in \omega'$: the key observation is that $\rho_\phi(B, B') = \rho_{\phi'}(\sigma_{P^{-1}}(B), \sigma_{P^{-1}}(B')))$.

The following Proposition shows that uniformly W^ϕ-differentiable functions have continuous W^ϕ-differentials:

Proposition 3.6. *Let $\phi, \psi : \omega \to \mathbb{R}$ be two continuous functions; suppose that there exists an $\overline{A} \in \omega$ such that ψ in uniformly W^ϕ-differentiable at $\overline{A}$ and that ψ is W^ϕ-differentiable in an open neighbourhood $\mathcal{U}$ of $\overline{A}$. Then $W^\phi : \mathcal{U} \to \mathbb{R}^{2n-1}$ is continuous at $\overline{A}$.*

Proof. As usual we give the proof only for $n \geq 2$.

Suppose that the thesis is not true; then there exist a $\delta > 0$ and a sequence $\{A^j\} \subset \mathcal{U}$ such that $A^j \to \overline{A}$ and

$$|W^\phi \psi(A^j) - W^\phi \psi(\overline{A})| \geq 3\delta.$$

By the uniform W^ϕ-differentiability of ψ at $\overline{A}$ we can find an open rectangle I centered at $\overline{A}$ such that

$$\sup_{\substack{A, B \in I \\ A = (\eta, v, \tau) \neq B = (\eta', v', \tau')}} \left\{ \frac{|\psi(B) - \psi(A) - \langle W^\phi \psi(\overline{A}), (\eta' - \eta, v' - v)\rangle|}{\rho_\phi(B, A)} \right\} \leq \delta. \tag{3.8}$$

There is no loss of generality if we suppose that $A^j = (\eta^j, v^j, \tau^j) \in I$ for all j; then, using the W^ϕ-differentiability of ψ at A^j and reasoning as in Lemma 3.4, we can find a sequence of points $B^j = (\eta'^j, v'^j, \tau'^j) \in I$ such that

$$\frac{|\psi(B^j) - \psi(A^j) - \langle W^\phi\psi(A^j), (\eta'^j - \eta^j, v'^j - v^j)\rangle|}{\rho_\phi(B^j, A^j)} \le \delta; \tag{3.9}$$

$$\rho_\phi(B^j, A^j) = |(\eta'^j - \eta^j, v'^j - v^j)|; \tag{3.10}$$

(3.11) the $(2n-1)$-vectors $(\eta'^j - \eta^j, v'^j - v^j)$ and $(W^\phi\psi(A^j) - W^\phi\psi(\overline{A}))$ are linearly dependent.

Observe that (3.10) and (3.11) imply that $|\langle W^\phi\psi(A^j) - W^\phi\psi(\overline{A}), (\eta'^j - \eta^j, v'^j - v^j)\rangle| = |W^\phi\psi(A^j) - W^\phi\psi(\overline{A})|\rho_\phi(B^j, A^j) \ge 3\delta\rho_\phi(B^j, A^j)$. Then, using also (3.9), we get

$$\begin{aligned}
&\frac{|\psi(B^j) - \psi(A^j) - \langle W^\phi\psi(\overline{A}), (\eta'^j - \eta^j, v'^j - v^j)\rangle|}{\rho_\phi(B^j, A^j)} \\
&\qquad\ge \frac{|\langle W^\phi\psi(A^j) - W^\phi\psi(\overline{A}), (\eta'^j - \eta^j, v'^j - v^j)\rangle|}{\rho_\phi(B^j, A^j)} + \\
&\qquad\quad - \frac{|\psi(B^j) - \psi(A^j) - \langle W^\phi\psi(A^j), (\eta'^j - \eta^j, v'^j - v^j)\rangle|}{\rho_\phi(B^j, A^j)} \\
&\qquad\ge \frac{3\delta\rho_\phi(B^j, A^j) - \delta\rho_\phi(B^j, A^j)}{\rho_\phi(B^j, A^j)} \ge 2\delta
\end{aligned}$$

which contradicts (3.8). □

It is not clear whether the converse is true, i.e. if W^ϕ-differentiability in an open neighbourhood and continuity of the W^ϕ-differential imply uniform W^ϕ-differentiability. Observe that this is true when we consider the classical notion of differentiability.

Recalling how we defined the family W^ϕ of the $2n$-1 first-order operators W_j^ϕ, the following Proposition explains why we call the vector w_L (with L as in (1.18)) the W^ϕ-differential of ψ: the fact is that the j-th component of this vector is (at least for regular maps) the derivative of ψ in the W_j^ϕ-direction (with the usual identification between vector fields and first-order operators):

Proposition 3.7. *Let $\phi, \psi : \omega \to \mathbb{R}$ be continuous functions such that ψ is W^ϕ-differentiable at a point $A = (\eta, v, \tau) \in \omega$ (respectively $A = (\eta, \tau)$ if $n = 1$). For $j = 2, \ldots, 2n$ let $\gamma^j : [-\delta, \delta] \to \omega$ be a $\mathbf{C}^1$-integral curve of the vector field W_j^ϕ with $\gamma^j(0) = A$ and such that the map*

$$[-\delta, \delta] \ni s \longmapsto \phi(\gamma^j(s)) \in \mathbb{R}$$

is of class $\mathbf{C}^1$. *Then we have*

$$\lim_{s\to 0}\frac{\psi(\gamma^j(s))-\psi(\gamma^j(0))}{s}=\left[W^\phi\psi(A)\right]_j. \tag{3.12}$$

Proof. Again we accomplish the proof only for $n\geq 2$.

Let us fix the following notation: if $\gamma^j(s)=(\eta(s),v(s),\tau(s))$ we set

$$\begin{array}{l}\gamma^j_i(s):=v_i(s)\quad\text{for } 2\leq i\leq 2n, i\neq n+1\\ \gamma^j_{n+1}(s):=\eta(s)\\ \gamma^j_{2n+1}(s):=\tau(s)\end{array}$$

For $j\neq n+1$ the thesis is obvious: indeed we must have $\gamma^j(s)=A\star\exp(s\widetilde{X}_j)$ i.e. $\iota(\gamma^j(s))=\iota(A)\cdot\exp(sX_j)$, and so

$$\begin{aligned}\rho_\phi(A,\gamma^j(s))=\|\pi_{O_1}(-\phi(A)e_1\cdot\exp(sX_j)\cdot\phi(\gamma^j(s))e_1)\|_\infty+\\ +\|\pi_Z(-\phi(A)e_1\cdot\exp(sX_j)\cdot\phi(\gamma^j(s))e_1)\|_\infty=|s|\end{aligned}$$

which gives immediately (3.12) as a consequence of the W^ϕ-differentiability and of the fact that $\gamma^j_i(s)\equiv v_i$ for $i\notin\{j,2n+1\}$ and $\gamma^j_j(s)=v_j+s$.

For $j=n+1$ we have

$$\begin{array}{l}\gamma^{n+1}_i(s)=v_i\quad\text{if } i\neq n+1,2n+1\\ \gamma^{n+1}_{n+1}(s)=\eta+s\\ \gamma^{n+1}_{2n}(s)=\tau-4\int_0^s\phi(\gamma^{n+1}(r))dr\end{array} \tag{3.13}$$

and so

$$\begin{aligned}&\rho_\phi(\gamma^{n+1}(s),\gamma^{n+1}(0))\\ =\ &|s|+\left|-4\int_0^s\phi(\gamma^{n+1}(r))dr+2[\phi(\gamma^{n+1}(s))+\phi(A)]s\right|^{1/2}\\ =\ &|s|\left(1+\frac{1}{|s|}\left|-4\int_0^s\phi(\gamma^{n+1}(r))dr+2[\phi(\gamma^{n+1}(s))+\phi(A)]s\right|^{1/2}\right)\\ =:\ &|s|\left(1+\frac{1}{|s|}|\Delta(s)|^{1/2}\right).\end{aligned} \tag{3.14}$$

We want to prove that $|\Delta(s)|\leq Cs^2$ for a certain $C>0$; indeed, we have that the map $s\mapsto\gamma^{n+1}_{2n+1}(s)$ is of class $\mathbf{C}^2$ (because of (3.13) and the hypothesis that $s\mapsto\phi(\gamma^{n+1}(s))$ is $\mathbf{C}^1$) and then

$$\begin{aligned}\Delta(s)&=-4\int_0^s\phi(\gamma^{n+1}(r))dr+2[\phi(\gamma^{n+1}(s))+\phi(A)]s\\ &=-4\int_0^s[\phi(\gamma^{n+1}(r))-\phi(A)]dr+2[\phi(\gamma^{n+1}(s))-\phi(A)]s\\ &=O(s^2).\end{aligned} \tag{3.15}$$

Then by (3.14) we get $\rho_\phi(\gamma^{n+1}(s), A) \le (1+\sqrt{C})|s|$ and so

$$\frac{|\,\psi(\gamma^{n+1}(s)) - \psi(\gamma^{n+1}(0)) - [W^\phi\psi(A)]_{n+1}s\,|}{|s|} \le$$
$$\le (1+\sqrt{C})\frac{|\,\psi(\gamma^{n+1}(s)) - \psi(A) - L_{W^\phi\psi(A)}(A^{-1}\star\gamma^{n+1}(s))\,|}{\rho_\phi(\gamma^{n+1}(s), A)}.$$

By letting $s \to 0$ and using the W^ϕ-differentiability of ψ at A we get the thesis (3.12). $\square$

The following result shows that the class of ϕ, ψ such that ψ is W^ϕ-differentiable (in fact, uniformly W^ϕ-differentiable) is not empty, and gives an explicit formula for $W^\phi\psi$.

Theorem 3.8. *Let $\phi, \psi \in \mathbf{C}^1(\omega)$; then ψ is uniformly W^ϕ-differentiable at A for all $A \in \omega$ and*

$$W^\phi\psi(A) = \left(\widetilde{X}_2\psi, \dots, \widetilde{X}_n\psi, \frac{\partial\psi}{\partial\eta} - 4\phi\frac{\partial\psi}{\partial\tau}, \widetilde{Y}_2\psi, \dots, \widetilde{Y}_n\psi\right)(A)$$

for all $A \in \omega$. In particular, $W^\phi\psi : \omega \to \mathbb{R}^{2n-1}$ is continuous.

Proof. Let us fix $A = (\overline{\eta}, \overline{v}, \overline{\tau}) \in \omega$ ($A = (\overline{\eta}, \overline{\tau})$ if $n = 1$) and set

$$w(A) := \left(\widetilde{X}_2\psi, \dots, \widetilde{X}_n\psi, \frac{\partial\psi}{\partial\eta} - 4\phi\frac{\partial\psi}{\partial\tau}, \widetilde{Y}_2\psi, \dots, \widetilde{Y}_n\psi\right)(A) \in \mathbb{R}^{2n-1}$$

if $n \ge 2$, while for $n = 1$ we set

$$w(A) := \frac{\partial\psi}{\partial\eta}(A) - 4\phi(A)\frac{\partial\psi}{\partial\tau}(A).$$

Following Definition 1.1, and (1.19) in particular, we have to prove that

$$\lim_{r\to 0} M_\phi(\psi, A, w(A), r) = 0. \tag{3.16}$$

Therefore let $B, B' \in \omega$ be sufficiently close to A (in a way we are going to specify), and for $n \ge 2$ let $\overline{X}, \overline{W}$ be the $\mathbf{C}^1$ vector fields given by

$$\overline{X} := \sum_{j=2, j\neq n+1}^{2n} (v_j' - v_j)\widetilde{X}_j, \qquad \overline{W} := \frac{\partial}{\partial\eta} - 4\phi\frac{\partial}{\partial\tau}.$$

Define

$$\begin{aligned}
B^* &:= \exp(\overline{X})(B) \\
&= B \star (0, (v_2' - v_2, \dots, v_n' - v_n, v_{n+2}' - v_{n+2}, \dots, v_{2n}' - v_{2n}), 0) \\
&= (\eta, v', \tau - \sigma(v', v)) \\
B'' &:= \exp((\eta' - \eta)\overline{W})(B^*) = (\eta', v', \tau'') \quad \text{(for a certain } \tau''\text{)};
\end{aligned}$$

observe that B^* and B'' are well defined if $B, B' \in I_{\delta_0}(A)$ for a sufficiently small δ_0. For $n = 1$, $\overline{X}$ is not defined and we set $B^* = B$ and $B'' := \exp((\eta' - \eta)\overline{W})(B) = (\eta', \tau'')$.

As ψ is of class $\mathbf{C}^1$ we have

$$
\begin{aligned}
& \psi(B') - \psi(B) \\
= \; & [\psi(B') - \psi(B'')] + [\psi(B'') - \psi(B^*)] + [\psi(B^*) - \psi(B)] \\
= \; & [\psi(B') - \psi(B'')] + \int_0^{\eta'-\eta} (\overline{W}\psi)\big(\exp(s\overline{W})(B^*)\big)\, ds + \\
& + \int_0^1 \sum_{\substack{j=2 \\ j \neq n+1}}^{2n} (v_j' - v_j)\widetilde{X}_j\psi(\exp(s\overline{X})(B)) \\
= \; & [\psi(B') - \psi(B'')] + \sum_{j=2, j\neq n+1}^{2n} (v_j' - v_j)\widetilde{X}_j\phi(A) + \\
& + (\eta' - \eta)\overline{W}\psi(A) + o(|(\eta' - \eta, v' - v)|) \\
= \; & [\psi(B') - \psi(B'')] + \langle w(A), (\eta' - \eta, v' - v)\rangle + o(\rho_\phi(B', B)).
\end{aligned}
\tag{3.17}
$$

For $n = 1$ the same calculation leads to

$$\psi(B') - \psi(B) = [\psi(B') - \psi(B'')] + w(A)(\eta' - \eta) + o(\rho_\phi(B', B)).$$

Therefore it is sufficient to prove that $\psi(B') - \psi(B'') = o(\rho_\phi(B', B))$. We have

$$\frac{|\psi(B') - \psi(B'')|}{\rho_\phi(B', B)} \le \omega_\psi(\delta_0) \cdot \frac{|\tau' - \tau''|^{1/2}}{\rho_\phi(B', B)} \tag{3.18}$$

where

$$\omega_\psi(\delta) := \sup\left\{ \frac{|\psi(A') - \psi(A'')|}{|A' - A''|^{1/2}} : A' \neq A'' \in I_\delta(A) \right\}, \tag{3.19}$$

and where we know that $\omega_\psi(\delta) \to 0$ as $\delta \downarrow 0$ because of the fact that ψ is $\mathbf{C}^1$. So we have to prove that $|\tau' - \tau''|^{1/2}/\rho_\phi(B', B)$ is bounded in a proper neighbourhood of

A. Let's observe that

$$
\begin{aligned}
& |\tau' - \tau''| \\
= \ & \left|\tau' - \tau + \sigma(v', v) + 4\int_0^{\eta'-\eta} \phi(\exp(s\overline{W})(B^*))\, ds\right| \\
\leq \ & |\tau' - \tau + 2(\phi(B') + \phi(B))(\eta' - \eta) + \sigma(v', v)| + \\
& +2\left|2\int_0^{\eta'-\eta} \phi(\exp(s\overline{W})(B^*))\, ds - (\phi(B') + \phi(B))(\eta' - \eta)\right| \\
\leq \ & \rho_\phi(B', B)^2 + 2|\phi(B') - \phi(B'')||\eta' - \eta| + 2|\phi(B) - \phi(B^*)||\eta' - \eta| + \\
& +2\left|2\int_0^{\eta'-\eta} \phi(\exp(s\overline{W})(B^*)\, ds - [\phi(B'') + \phi(B^*)](\eta' - \eta)\right| \\
=: \ & \rho_\phi(B', B)^2 + R_1(B', B) + R_2(B', B) + R_3(B', B).
\end{aligned}
\tag{3.20}
$$

For the case $n = 1$ we arrive to (3.20) with the same line (it is sufficient to follow the same steps "erasing" the term $\sigma(v', v)$).

Now we want to prove that there exist $C_1, C_2 > 0$ such that

$$R_3(B', B) \leq C_1|\eta' - \eta|^2 \tag{3.21}$$

$$R_2(B', B) \leq C_2\rho_\phi(B', B)^2 \tag{3.22}$$

for all $B', B \in I_{\delta_0}(A)$, and that for all $\epsilon > 0$ there is a $\delta_\epsilon \in]0, \delta_0]$ such that, for $\delta \in]0, \delta_\epsilon[$,

$$R_1(B', B) \leq |\eta' - \eta|^2 + \epsilon|\tau' - \tau''| \tag{3.23}$$

for all $B', B \in I_\delta(A)$. These estimates are sufficient to conclude: in fact, choosing $\epsilon := 1/2$ and using (3.20), (3.21), (3.23) and (3.22), we get

$$|\tau' - \tau''| \leq \rho_\phi(B', B)^2 + C_1|\eta' - \eta|^2 + |\eta' - \eta|^2 + |\tau' - \tau''|/2 + C_2\rho_\phi(B', B)^2$$

whence

$$|\tau' - \tau''|^{1/2} \leq C_3\rho_\phi(B, B')$$

which is the thesis.

For $s \in [-\delta_0, \delta_0]$ we can define

$$g(s) := 2\int_0^s \phi(\exp(r\overline{W})(B^*))\, dr - \left[\phi\big(\exp(s\overline{W})(B^*)\big) + \phi(B^*)\right]s; \tag{3.24}$$

as in (3.15) one can prove that there is a $C_1 > 0$ such that

$$|g(s)| \leq C_1 s^2 \quad \text{for all } s \in [-\delta_0, \delta_0], \tag{3.25}$$

so that (3.21) follows with $s = \eta' - \eta$.

Define ω_ϕ as in (3.19) (with ϕ instead of ψ), then

$$
\begin{aligned}
R_1(B, B') \ &\leq \ 2\omega_\phi(\delta)|\tau' - \tau''|^{1/2}|\eta' - \eta| \\
&\leq \ |\eta' - \eta|^2 + \omega_\phi(\delta)^2|\tau' - \tau''|.
\end{aligned}
$$

Since ϕ is $\mathbf{C}^1$, $\omega_\phi(\delta) \to 0$ for $\delta \downarrow 0$, and so for all $\epsilon > 0$ there is a $\delta_\epsilon > 0$ such that for all $\delta \in]0, \delta_\epsilon]$ we have $\omega_\phi(\delta)^2 \le \epsilon$, whence (3.23) follows.

Finally, observe that (3.22) follows from $R_2(B, B') = 0$ if $n = 1$, and from

$$\begin{aligned} R_2(B,B') &= |\eta' - \eta||\phi(B) - \phi(B^*)| \\ &= |\eta' - \eta| \left| \sum_{j=2,\, j\neq n+1}^{2n} (v'_j - v_j)(w_j + o(1)) \right| \\ &\le 2C_2|\eta' - \eta||v' - v| \le C_2|(\eta' - \eta, v' - v)|^2 \le C_2 \rho_\phi(B', B)^2 \end{aligned}$$

if $n \ge 2$. $\square$

4. $\mathbb{H}$-regular graphs and W^ϕ-differentiability.

In this section we are going to characterize the $\mathbb{H}$-regular graphs in terms of the uniform W^ϕ-differentiability of their parametrizations. The main theorem of the section is the following

Theorem 4.1. *Let $\phi : \omega \to \mathbb{R}$ be a continuous function and let $\Phi : \omega \to \mathbb{H}^n$ be the function defined by*

$$\Phi(A) := \iota(A) \cdot \phi(A)e_1.$$

Let $S := \Phi(\omega)$. Then the following conditions are equivalent:

(i) *S is an $\mathbb{H}$-regular surface and $\nu_S^{(1)}(P) < 0$ for all $P \in S$, where $\nu_S(P) = (\nu_S^{(1)}(P), \dots, \nu_S^{(2n)}(P))$ denotes the horizontal normal to S at a point $P \in S$;*

(ii) *ϕ is uniformly W^ϕ-differentiable at any $A \in \omega$.*

Moreover, for all $P \in S$ we have

$$\nu_S(P) = \left(-\frac{1}{\sqrt{1 + |W^\phi \phi|^2}}, \frac{W^\phi \phi}{\sqrt{1 + |W^\phi \phi|^2}} \right) (\Phi^{-1}(P)) \in \mathbb{R} \times \mathbb{R}^{2n-1}. \tag{4.1}$$

and

$$\mathcal{S}_\infty^{Q-1}(S) = c(n) \int_\omega \sqrt{1 + |W^\phi \phi(A)|^2}\, d\mathcal{L}^{2n}(A). \tag{4.2}$$

Proof. We will give the proof only for $n \ge 2$, since the generalization to $n = 1$ is immediate.

Let us begin with the proof of the implication (i)$\Rightarrow$(ii). Let $P = \Phi(A) \in S$, where $A = (\eta, v, \tau) \in \omega$; then there exist an $r_0 > 0$ and a function $f \in \mathbf{C}^1_{\mathbb{H}}(U(P, r_0))$ such that

$S \cap U(P, r_0) = \{Q \in U(P, r_0) : f(Q) = 0\}$

$\nabla_{\mathbb{H}} f(Q) = (X_1 f(Q), \dots, X_n f(Q), Y_1 f(Q), \dots, Y_n f(Q)) \neq 0$ for all $Q \in U(P, r_0)$.

As $\nu_S(Q) = -\nabla_{\mathbb{H}} f(Q)/|\nabla_{\mathbb{H}} f(Q)|$, by hypothesis we have that

$$X_1 f(P) > 0 \text{ for all } P \in S \cap U(P^0, r_0). \tag{4.3}$$

Moreover without loss of generality we can suppose that

$$A = (\eta, v, \tau) = (0, 0, 0) \text{ and } P = \Phi(0, 0, 0) = 0. \tag{4.4}$$

Indeed, if this is not the case, let us consider $S' := \tau_{P^{-1}}(S) = \Phi'(\omega')$, where we use the same notations of Remark 3.5. We have that $S' \cap U(0, r_0)$ is an $\mathbb{H}$-regular surface because it is the zero set of the function $f' : \mathbb{H}^n \ni Q \mapsto f(P \cdot Q) \in \mathbb{R}$, and by left invariance $X_1 f'(Q) = X_1 f(P \cdot Q) > 0$ for all $Q \in U(0, r_0)$. Finally (again by Remark 3.5), ϕ' (which is equal to $\phi \circ \sigma_P$ up to an additive constant) is uniformly $W^{\phi'}$-differentiable if and only if ϕ is uniformly W^{ϕ}-differentiable.

By the unicity of the parametrization provided by the Implicit Function Theorem we can assume that there is a $\overline{\delta} > 0$ such that $\overline{I_{\overline{\delta}}} = \overline{I_{\overline{\delta}}(0, 0, 0)} \Subset \omega$ and

$$f(\Phi(B)) = 0 \text{ for all } B \in \overline{I_{\overline{\delta}}}. \tag{4.5}$$

With the assumptions in (4.4), by the continuity of Φ for each $r \in]0, r_0/4[$ there is a $0 < \delta_r < r$ such that

$$\Phi(I_{\delta_r}(0, 0, 0)) \subset U(0, r). \tag{4.6}$$

Let us recall the following

Lemma 4.2. *Let $f \in \mathbf{C}^1_{\mathbb{H}}(U(P, r))$. Then there exists a $C = C(P, r_0)$ such that, for each $Q \in U(P, r_0/2)$, $r \in]0, r_0/4[$ and $Q' \in U(Q, r)$ we have*

$$\begin{aligned} &|f(Q') - f(Q) - \langle \nabla_{\mathbb{H}} f(Q), \pi(Q^{-1}Q')\rangle_Q| \le \\ &\qquad \le C\, d_\infty(Q, Q')\, \|\nabla_{\mathbb{H}} f(\cdot) - \nabla_{\mathbb{H}} f(Q)\|_{L^\infty(U(Q, 2d_\infty(Q,Q')))}. \end{aligned}$$

The proof of this fact can be found in [47], Theorem 2.3.3.

Therefore, for each $B = (\eta, v, \tau), B' = (\eta', v', \tau') \in I_{\delta_r}(0)$, with δ_r sufficiently small, we get, by applying Lemma 4.2 to our f with $P = 0$, $Q = \Phi(B)$, $Q' = \Phi(B')$, that

$$\begin{aligned} & |\langle \nabla_{\mathbb{H}} f(\Phi(B)),\, \pi(\Phi(B)^{-1}\Phi(B'))\rangle| \\ = \;& |f(\Phi(B')) - f(\Phi(B)) + \langle\, \nabla_{\mathbb{H}} f(\Phi(B)),\, \pi(\Phi(B)^{-1}\Phi(B'))\,\rangle| \\ \le \;& C_1\, R(\delta_r)\, d_\infty(\Phi(B'),\, \Phi(B)) \\ \le \;& C_2\, R(\delta_r)\, \big[\|\pi_{L_1}(\Phi(B)^{-1}\Phi(B'))\|_\infty + \|\pi_{O_1}(\Phi(B)^{-1}\Phi(B'))\|_\infty + \\ & + \|\pi_Z(\Phi(B)^{-1}\Phi(B'))\|_\infty\big] \\ \le \;& C_2\, R(\delta_r)\, [\, |\phi(B') - \phi(B)| + \rho_\phi(B, B')\,] \end{aligned} \tag{4.7}$$

where C_1 is given by Lemma 4.2 and

$$R(\delta) := \sup\big\{\|\nabla_{\mathbb{H}} f(\cdot) - \nabla_{\mathbb{H}} f(P')\|_{L^\infty(U(P', 2d_\infty(P', P'')))} : P', P'' \in \Phi(I_\delta(0, 0)).\big\}$$

By the uniform continuity of $\nabla_{\mathbb{H}} f : \overline{U(0, r_0/2)} \to H\mathbb{H}^n$ we have

$$\lim_{r \downarrow 0} R(\delta_r) = 0. \tag{4.8}$$

Setting $\widehat{\nabla_{\mathbb{H}}} f := (X_2 f, \dots, X_n f, Y_1 f, \dots, Y_n f)$, (4.7) and (4.3) imply

$$
\begin{aligned}
&\Big|\phi(B') - \phi(B) + \frac{\langle \widehat{\nabla_{\mathbb{H}}}(\Phi(B)),\, (\eta'-\eta, v'-v)\rangle}{X_1 f(\Phi(B))}\Big| \\
&= \frac{|\langle \nabla_{\mathbb{H}} f(\Phi(B)),\, \pi(\Phi(B)^{-1}\Phi(B'))\rangle|}{X_1 f(\Phi(B))} \\
&\le \Big[\inf_{B(0,r_0)} X_1 f\Big]^{-1} C_2 R(\delta_r)[|\phi(B') - \phi(B)| + \rho_\phi(B,B')]
\end{aligned} \tag{4.9}
$$

for any $B, B' \in I_{\delta_r}$. By (4.8) we can suppose

$$
\frac{C_2}{\inf_{B(0,r_0)} X_1 f} R(\delta_{\overline{r}}) \le \frac{1}{2}
$$

for a certain $\overline{r} \in]0, r_0/4[$, and so

$$
\begin{aligned}
|\phi(B') - \phi(B)| &\le \left|\phi(B') - \phi(B) + \frac{\langle \widehat{\nabla_{\mathbb{H}}}(\Phi(B)),\, (\eta'-\eta, v'-v)\rangle}{X_1 f(\Phi(B))}\right| + \\
&\quad + \left|\frac{\langle \widehat{\nabla_{\mathbb{H}}}(\Phi(B)),\, (\eta'-\eta, v'-v)\rangle}{X_1 f(\Phi(B))}\right| \\
&\le [|\phi(B') - \phi(B)| + \rho_\phi(B,B')]/2 + C_3|(\eta'-\eta, v'-v)|
\end{aligned}
$$

for each $B, B' \in I_{\delta_{\overline{r}}}$. Therefore there exists a constant $C_4 > 0$ such that

$$
|\phi(B') - \phi(B)| \le C_4 \rho_\phi(B,B'). \tag{4.10}
$$

Putting together (4.9) and (4.10) we get that there is a $C_5 > 0$ for which

$$
\left|\phi(B') - \phi(B) + \frac{\langle \widehat{\nabla_{\mathbb{H}}}(\Phi(B)),\, (\eta'-\eta, v'-v)\rangle}{X_1 f(\Phi(B))}\right| \le C_5 R(\delta_r)\rho_\phi(B,B') \tag{4.11}
$$

and so

$$
\begin{aligned}
&\frac{\left|\phi(B') - \phi(B) + \left\langle \dfrac{\widehat{\nabla_{\mathbb{H}}} f(0)}{X_1 f(0)},\, (\eta'-\eta, v'-v)\right\rangle\right|}{\rho_\phi(B,B')} \le \\
&\qquad\qquad \le C_5 R(\delta_r) + \sup_{I_{\delta_r}(0)} \left|\frac{\widehat{\nabla_{\mathbb{H}}} f(\Phi(\cdot))}{X_1 f(\Phi(\cdot))} - \frac{\widehat{\nabla_{\mathbb{H}}} f(0)}{X_1 f(0)}\right|
\end{aligned}
$$

for each $B, B' \in I_{\delta_r}(0)$ with $r \le \overline{r}$. Thanks to (4.8) and the fact that f is of class $\mathbf{C}^1_{\mathbb{H}}$ we get that $\lim_{r\downarrow 0} L_\phi\big(\phi, 0, \frac{\widehat{\nabla_{\mathbb{H}}} f(0)}{X_1 f(0)}, \delta_r\big) = 0$, i.e. ϕ is uniformly W^ϕ-differentiable at 0 and

$$
W^\phi \phi(0) = -\frac{\widehat{\nabla_{\mathbb{H}}} f}{X_1 f}(0). \tag{4.12}
$$

More generally, we can say that

$$W^\phi\phi(\Phi^{-1}(P)) = -\frac{\widehat{\nabla_{\mathbb{H}} f}}{X_1 f}(P)$$

from which (4.1) follows because

$$\begin{aligned}\nu_S(P) &= -\frac{\nabla_{\mathbb{H}} f(P)}{|\nabla_{\mathbb{H}} f(P)|} = \frac{\big(-X_1 f(P),\, X_1 f(P) W^\phi\phi(\Phi^{-1}(P))\big)}{\sqrt{X_1(fP)^2[1+|W^\phi\phi(\Phi^{-1}(P))|^2]}}\\ &= \left(-\frac{1}{\sqrt{1+|W^\phi\phi|^2}}, \frac{W^\phi\phi}{\sqrt{1+|W^\phi\phi|^2}}\right)(\Phi^{-1}(P)).\end{aligned}$$

So the implication (i)⇒(ii) is completely proved.

Now we have to prove the converse, i.e. (ii)⇒(i). Let $A = (\eta, v, \tau) \in \omega$ and $P = \Phi(A) \in S$. We have to find an $r_0 > 0$ and a $f \in \mathbf{C}^1_{\mathbb{H}}(U(P, r_0))$ such that

$$S \cap U(P, r_0) = \{Q \in U(0, r_0) : f(Q) = 0\} \tag{4.13}$$

$$X_1 f(Q) > 0 \text{ for all } Q \in U(P, r_0). \tag{4.14}$$

Let δ_1 be such that $I_{\delta_1}(A) \Subset \omega$; as $\Phi : \omega \to S$ is an homeomorphism we can suppose that

$$S \cap \overline{\mathcal{U}} = \Phi\big(\overline{I_{\delta_1}(A)}\big)$$

for a certain open neighbourhood $\mathcal{U} \Subset \mathbb{H}^n$ of P. Let $C := S \cap \overline{\mathcal{U}}$ and $g : C \to \mathbb{R}$ defined by $g(z,t) := 0$. Define

$$\begin{aligned} k \ &: \ C \longrightarrow H\mathbb{H}^n \equiv \mathbb{R}^{2n}\\ &\quad (z,t) \longmapsto (1, -W^\phi\phi(\Phi^{-1}(z,t)))\end{aligned}$$

We start by proving, thanks to Whitney's extention Theorem 2.18, that there is an $f \in \mathbf{C}^1_{\mathbb{H}}(\mathbb{H}^n, \mathbb{R})$ such that

$$f \equiv g \equiv 0 \quad \text{on C} \tag{4.15}$$

$$\nabla_{\mathbb{H}} f(z,t) = (z,t) = (1, -W^\phi\phi(\Phi^{-1}(z,t))) \quad \text{for all } (z,t) \in C. \tag{4.16}$$

Consider a compact subset K of C; for $Q, Q' \in K$ and $\delta > 0$ let

$$\begin{aligned} R(Q,Q') &:= \frac{g(Q') - g(Q) - \langle\, k(Q), \pi_Q(Q^{-1}Q')\,\rangle_Q}{d_\infty(Q,Q')} = -\frac{\langle\, k(Q), \pi_Q(Q^{-1}Q')\,\rangle_Q}{d_\infty(Q,Q')}\\ \rho_K(\delta) &:= \sup\{|R(Q,Q')| : Q, Q' \in K, 0 < d_\infty(Q,Q') < \delta\}.\end{aligned}$$

In order to apply Whitney's Theorem (which will provide the desired f) we have only to prove that

$$\lim_{\delta \downarrow 0} \rho_K(\delta) = 0. \tag{4.17}$$

Let us suppose that the converse is true, id est there is an $\epsilon_0 > 0$ such that for all $h \in \mathbb{N}$ there are $Q^h, Q^{h\prime} \in K, Q^h = \Phi(B^h), Q^{h\prime} = \Phi(B^{h\prime}), B^h = (\eta^h, v^h, \tau^h), B^{h\prime} = (\eta^{h\prime}, v^{h\prime}, \tau^{h\prime})$ for which

$$0 < d_\infty(Q^h, Q^{h\prime}) < 1/h \tag{4.18}$$

$$\epsilon_0 < |R(Q^h, Q^{h\prime})| \leq \frac{|\phi^{h\prime} - \phi^h - \langle W^\phi\phi(B^h), (\eta^{h\prime} - \eta^h, v^{h\prime} - v^h) \rangle|}{\rho_\phi(B^h, B^{h\prime})} \tag{4.19}$$

where as usual we indicated with $\phi^{h\prime}, \phi^h$ the quantities $\phi(B^{h\prime})$ and $\phi(B^h)$ respectively. In (4.19) we used the fact that $d_\infty(\Phi(B), \Phi(B')) \geq \rho_\phi(B, B')$; this estimate, together with (4.18), implies that $\rho_\phi(B^h, B^{h\prime})) \leq 1/h$ and so

$$|(\eta^{h\prime} - \eta^h, v^{h\prime} - v^h)| \leq 1/h \tag{4.20}$$

$$|\tau^{h\prime} - \tau^h + 2(\phi^{h\prime} + \phi^h)(\eta^{h\prime} - \eta^h) + \sigma(v^{h\prime}, v^h)| \leq 1/h^2. \tag{4.21}$$

If we set $M := \sup_K |\phi|$ and $\alpha := \sup_K |(\eta, v)|$ we get

$$\begin{aligned} |\tau^{h\prime} - \tau^h| &\leq 1/h^2 + 2|\phi^{h\prime} + \phi^h||\eta^{h\prime} - \eta^h| + 2|\sigma(v'^h, v^h)| \quad (*) \\ &\leq 1/h^2 + 4M|\eta^{h\prime} - \eta^h| + 2\alpha|v^{h\prime} - v^h| \quad (**) \\ &\leq C/h \end{aligned} \tag{4.22}$$

where $C := 1 + 4M + 2\alpha > 0$ depends only on K. In $(*)$ we used that $\sigma(v^{h\prime}, v^h) = 2\sum_{j=2}^n [v_{n+j}^{h\prime}(v_j^h - v_j^{h\prime}) - v_j^{h\prime}(v_{n+j}^h - v_{n+j}^{h\prime})]$, while (4.20) justifies $(**)$. But since K is compact there is a $B = (\eta', v', \tau') \in \overline{I_{\delta_1}(A)} \supset K$ such that

$$\lim_{h\to\infty} B^h = \lim_{h\to\infty} B^{h\prime} = B.$$

In particular $B^h, B^{h\prime} \in \overline{I_{C/h}(B)}$, and by (4.19) and the continuity of the W^ϕ-differential we get that for any h

$$0 < \epsilon_0 \leq M_\phi(\phi, B, W^\phi\phi(B), C/h)$$

which contradicts the fact that ϕ is uniformly W^ϕ-differentiable at $B \in I_{\delta_1}(A)$. This is sufficient to apply Whitney' Extention Theorem, and so we get the existence of an $f \in \mathbf{C}^1_{\mathbb{H}}(\mathbb{H}^n, \mathbb{R})$ for which (4.15) and (4.16) hold.

The proof of the implication (ii)⇒(i) will be complete if we prove the validity of (4.13) and (4.14) for a certain r_0. Let $S' := \{Q \in \mathbb{H}^n : f(Q) = 0, \nabla_{\mathbb{H}} f(Q) \neq 0\}$; as we have already shown, we can suppose that $P = 0$ and $A = 0$. As $0 \in S \cap \mathcal{U} \subset S'$ we have

$$f(0) = 0 \quad \text{and} \quad \nabla_{\mathbb{H}} f(0) = (1, -W^\phi\phi(0))$$

and by the Implicit Function Theorem there are an open neighbourhood $\mathcal{U}'$ of 0 and a continuous function $\phi' : \overline{I_{\delta'}(0)}$ such that

$$\begin{aligned} \Phi' \;:\; &\overline{I_{\delta'}(0)} \to S' \cap \overline{\mathcal{U}'} \\ &B \longmapsto \iota(B) \cdot \phi'(B) e_1 \end{aligned}$$

is an homeomorphism. Therefore $\Phi'^{-1}(S' \cap \mathcal{U}')$ is an open subset of $\overline{I_{\delta'}(0)}$ which contains 0, and so there exists a $\delta'' \in]0, \delta'[$ for which $I_{\delta''}(0) \subset \Phi'^{-1}(S' \cap \mathcal{U}')$; by the uniqueness of the parametrization we get that $\Phi' \equiv \Phi$ on $I_{\delta''}(0)$.

Now, let $\mathcal{U}''$ and $\mathcal{U}'''$ be open neighbourhoods of 0 in $\mathbb{H}^n$ such that

$$S \cap \mathcal{U}'' = \Phi(I_{\delta''}(0)) = \Phi'(I_{\delta''}(0)) = S' \cap \mathcal{U}''' \tag{4.23}$$

and let $r_0 > 0$ be such that $U(0, r_0) \subset \mathcal{U}'' \cap \mathcal{U}'''$. Then by (4.23) we get $U(0, r_0) \cap S = U(0, r_0) \cap S'$, from which (4.13) and (4.14) follow.

Finally, the area type formula (4.2) follows from Corollary 2.23 after finding a global f (that is given only locally), which can be done by a standard argument involving a partition of the unity. This completes the proof of the Theorem. □

Corollary 4.3. *With the same notations of Theorem 4.1, suppose that $S := \Phi(\omega)$ is $\mathbb{H}$-regular; then $\phi : (\omega, \rho_\phi) \to \mathbb{R}$ is locally Lipschitz continuous.*

Proof. The thesis follows from Theorem 4.1 and Remark 3.3. □

Now we want to establish some Hölder continuity properties for uniformly W^ϕ-differentiable functions on ω and therefore for parametrizations of $\mathbb{H}$-regular graphs; in particular we want to improve the Hölder continuity obtained in (3.4). More precisely we have the following

Proposition 4.4. *Let $\phi : \omega \to \mathbb{R}$ be uniformly W^ϕ-differentiable at $A \in \omega$. Then there is an $r_0 > 0$ such that $I_{r_0}(A) \Subset \omega$ and*

$$\lim_{r \downarrow 0} \ \sup \left\{ \frac{|\phi(B') - \phi(B)|}{|B' - B|^{1/2}} : B, B' \in I_{r_0}(A), 0 < |B - B'| < r \right\} = 0.$$

Proof. Again we treat only the case $n \geq 2$.

If $B = (\eta, v, \tau)$ and $B' = (\eta', v', \tau')$ let us set

$$R(\delta) := \sup \left\{ \frac{|\phi(B') - \phi(B) - \langle W^\phi \phi(A), (\eta' - \eta, v' - v) \rangle}{\rho_\phi(B', B)} : B' \neq B \in I_\delta(A) \right\};$$

by the uniform W^ϕ-differentiability of ϕ at A we now that $\lim_{\delta \downarrow 0} R(\delta) = 0$. In particular there is an $r_0 > 0$ such that ϕ is Lipschitz continuous between the (quasi) metric spaces $(I_{r_0}(A), \rho_\phi)$ and $\mathbb{R}$ (equipped with the standard euclidean distance), i.e. (3.3) holds. Then by (3.4) (see the passages that lead to (3.5)) there is a $C_1 > 0$ such that

$$\rho_\phi(B', B) \leq C_1 |B' - B|^{1/2} \quad \text{for all } B', B \in I_{r_0}(A). \tag{4.24}$$

But if $B' \neq B \in I_r(A)$, $0 < r < r_0$, we have

$$\begin{aligned}\frac{|\phi(B') - \phi(B)|}{|B' - B|^{1/2}} &\leq \frac{|\phi(B') - \phi(B) - \langle W^\phi \phi(A), (\eta' - \eta, v' - v)\rangle|}{\rho_\phi(B', B)} \cdot \frac{\rho_\phi(B', B)}{|B' - B|^{1/2}} + \\ &\quad + |W^\phi \phi(A)| \frac{|(\eta' - \eta, v' - v)|}{|B' - B|^{1/2}} \\ &\leq C_1 R(r) + C_2 |W^\phi \phi(A)| r^{1/2} \longrightarrow 0 \qquad \text{for } r \downarrow 0.\end{aligned}$$

This completes the proof. □

From Proposition 4.4 and a standard compactness argument we get the following

Corollary 4.5. *Let $\phi : \omega \to \mathbb{R}$ be a continuous function, and let $\Phi : \omega \to \mathbb{H}^n$ be defined as usual: $\Phi(A) = \iota(A) \cdot \phi(A) e_1$. Let $S := \Phi(\omega)$, and suppose that S is an $\mathbb{H}$-regular surface with $\nu_S^{(1)}(P) < 0$ for all $P \in S$; then for each $\omega' \Subset I$ we have*

$$\lim_{r \downarrow 0} \quad \sup \left\{ \frac{|\phi(A) - \phi(B)|}{|A - B|^{1/2}} : A, B \in \omega', 0 < |A - B| < r \right\} = 0.$$

Finally, we stess an interesting approximation property for the parametrizations of $\mathbb{H}$-regular graphs:

Proposition 4.6. *Let $\phi : \omega \to \mathbb{R}$ be a continuous function which is uniformly W^ϕ-differentiable at any $A \in \omega$; then for any $A \in \omega$ there is a $\delta = \delta(A) > 0$, with $I_\delta(A) \Subset \omega$, and a family $\{\phi_\epsilon\}_{\epsilon > 0} \subset \mathbf{C}^1(\overline{I_\delta(A)}, \mathbb{R})$ such that*

$$\phi_\epsilon \to \phi \text{ and } W^{\phi_\epsilon} \phi_\epsilon \to W^\phi \phi \qquad \text{uniformly on } \overline{I_\delta(A)}.$$

Proof. Arguing as in the proof of Theorem 4.1 we can suppose that $A = 0$, $\Phi(0) = 0$ and

$$S \cap U(0, r) = \{P \in U(0, r) : f(P) = 0\}$$

for proper $r > 0$ and $f \in \mathbf{C}^1_{\mathbb{H}} U(0, r)$ such that $f \circ \Phi \equiv 0$ on $I_\delta(A)$, with δ sufficiently small. Moreover, arguing as in the proof of the Implicit Function Theorem 2.16 (see [27]), we can suppose that, for a certain $0 < r' < r$ (and considering possibly a smaller δ), there are two families $\{f_\epsilon\}_{\epsilon > 0} \subset \mathbf{C}^1(\overline{U(0, r')})$ and $\{\phi_\epsilon\}_{\epsilon > 0} \subset \mathbf{C}^1(\overline{I_\delta(A)})$ such that

$$\begin{aligned}&f_\epsilon \to f \text{ and } \nabla_{\mathbb{H}} f_\epsilon \to \nabla_{\mathbb{H}} f && \text{uniformly on } \overline{U(0, r')} \\ &\phi_\epsilon \to \phi \text{ and } -\frac{\widehat{\nabla_{\mathbb{H}}} f_\epsilon}{X_1 f_\epsilon} \circ \Phi_\epsilon \to -\frac{\widehat{\nabla_{\mathbb{H}}} f}{X_1 f} \circ \Phi = W^\phi \phi && \text{uniformly on } \overline{I_\delta(A)}\end{aligned}$$

where $\Phi_\epsilon(A) := \iota(A) \cdot \phi_\epsilon(A) e_1$ is such that $f_\epsilon \circ \Phi_\epsilon = 0$; indeed the set $S_\epsilon := \{P \in \overline{U(0, r')} : f_\epsilon(P) = 0\} \supset \Phi_\epsilon(\overline{I_\delta(A)})$ is an (Euclidean) $\mathbf{C}^1$-surface, and then its parametrization ϕ_ϵ is uniformly W_{ϕ_ϵ}-differentiable and

$$W_{\phi_\epsilon} \phi_\epsilon = -\frac{\widehat{\nabla_{\mathbb{H}}} f_\epsilon}{X_1 f_\epsilon} \circ \Phi_\epsilon$$

from which the thesis follows. □

5. Characterization of the uniform W^ϕ-differentiability and some applications.

The main result we are going to prove in this section is the following

Theorem 5.1. *Let $\phi : \omega \to \mathbb{R}$ be a continuous function. Then the following conditions are equivalent:*

(i) *ϕ is uniformly W^ϕ-differentiable at A for each $A \in \omega$;*
(ii) *there exist a $w \in \mathbf{C}^0(\omega, \mathbb{R}^{2n-1})$ such that, in distributional sense,*

$$\begin{array}{ll} w = (\widetilde{X}_2\phi, \dots, \widetilde{X}_n\phi, \mathfrak{B}\phi, \widetilde{Y}_2\phi, \dots, \widetilde{Y}_n\phi) & \text{if } n \geq 2 \\ w = \mathfrak{B}\phi & \text{if } n = 1 \end{array}$$

and there is a family $\{\phi_\epsilon\}_{\epsilon>0} \subset \mathbf{C}^1(\omega)$ such that, for any open $\omega' \Subset \omega$, we have

$$\phi_\epsilon \to \phi \text{ and } W^{\phi_\epsilon}\phi_\epsilon \to w \text{ uniformly on } \omega'. \tag{5.1}$$

Moreover, $w = W^\phi\phi$ on ω and

$$\lim_{r\to 0^+} \sup\left\{\frac{|\phi(A) - \phi(B)|}{|A - B|^{1/2}} : A, B \in \omega', 0 < |A - B| < r\right\} = 0. \tag{5.2}$$

for each $\omega' \Subset \omega$.

Remark 5.2. Let $n = 1$ and $w \equiv 0$ then the functions $\phi : \omega \to \mathbb{R}$ satisfying condition (ii) of Theorem 5.1 are entropy solutions of Burgers' scalar conservation law in classical sense. Indeed by performing the change of variables $\mathbb{R}^2 = \mathbb{R}_x \times R_t \to \mathbb{R}^2 = \mathbb{R}_\eta \times R_\tau$, $(x, t) \to (t, -4x)$, Burgers' operator $\mathfrak{B}$ can be represented in classical way with respect to the variables (x, t) as

$$\mathfrak{B}u = \frac{\partial u}{\partial t} + \frac{1}{2}\frac{\partial u^2}{\partial x}$$

if $u = u(x, t) \in C^1(\omega^*)$ and $\omega^* \subset \mathbb{R}^2$ is a fixed open set (see [21], chapter III, section 3). In this case condition (ii) of Theorem 5.1 reads as the existence of a function $u : \omega^* \to \mathbb{R}$ and of a family $\{u_\varepsilon\}_\varepsilon \subset C^1(\omega^*)$ such that

$$u_\epsilon \to u \text{ and } \mathfrak{B}u_\epsilon \to 0 \text{ uniformly on } \omega' \tag{5.3}$$

for any open $\omega' \Subset \omega^*$. Let us assume now $\omega^* = (a, b) \times (-\delta, \delta)$ and let $g(x) := u(x, 0)$ if $x \in (a, b)$. Then we claim u is an *entropy solution* of the initial-value problem

$$\begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{1}{2}\dfrac{\partial u^2}{\partial x} = 0 & \text{in } (a, b) \times (0, \delta) \\ u = g & \text{on } (a, b) \times \{t = 0\}, \end{cases}$$

More precisely, by definition (see [21], chapter XI, section 11.4.3), we have to prove that

$$u \in C^0([0,\delta), L^1_{loc}(a,b)) \cap L^\infty_{loc}(\omega^*)\,; \tag{5.4}$$

$$u(\cdot,t) \to g \text{ in } L^1_{loc}(a,b) \text{ as } t \to 0^+\,; \tag{5.5}$$

$$\int_{\omega^*}\left[e(u)\frac{\partial v}{\partial t} + d(u)\frac{\partial v}{\partial x}\right] dxdt \geq 0 \tag{5.6}$$

for each $v \in C^1_c(\omega^*)$, $v \geq 0$ and for each entropy/entropy flux pair (e,d), i.e. two smooth functions $e, d : \mathbb{R} \to \mathbb{R}$ such that e is convex and $e'(u)u = d'(u)\,\forall u \in \mathbb{R}$. Then (5.4) and (5.5) follow at once because $u \in C^0(\omega^*)$. As $u_\varepsilon \in C^1(\omega^*)$

$$\frac{\partial(e(u_\epsilon))}{\partial t} + \frac{\partial(d(u_\epsilon))}{\partial x} = w_\epsilon\, e'(u_\epsilon) \text{ in } \omega^* \tag{5.7}$$

in pointwise sense with $w_\epsilon = \mathfrak{B}u_\epsilon$ and, by (5.3), $w_\epsilon \to 0$ uniformly in ω' for any open $\omega' \Subset \omega^*$. Therefore multiplying both sides of (5.7) for a given $v \in C^1_c(\omega^*)$, integrating by parts and taking the limit as $\epsilon \to 0^+$ we get (5.6) too (actually with an equality, so with no entropy production).

Remark 5.3. Let $n \geq 2$ and let assume that $\phi : \omega \to \mathbb{R}$ satisfies condition (ii) of Theorem 5.1 with $w \equiv 0$ in an open connected set $\omega \subset \mathbb{R}^{2n}$, then ϕ is constant in ω. Indeed for a fixed $A_0 \in \omega$ let $B = B(A_0, r_0) \subset \omega$ an Euclidean ball centered at A_0 with radius $r_0 > 0$,and, for a fixed $\eta \in \mathbb{R}$, let $B_\eta := \{(v,\tau) \in \mathbb{R}^{2n-2}_v \times \mathbb{R}_\tau : (\eta, v, \tau) \in B\}$, $\phi_\eta(v,\tau) := \phi(\eta, v, \tau)$ if $(v,\tau) \in B_\eta$. Since ϕ is continuous in ω, B_η is an open connected set in $R^{2n-2}_v \times \mathbb{R}_\tau \equiv \mathbb{H}^{n-1}$ and

$$\widetilde{X}_j \phi_\eta = \widetilde{Y}_j \phi_\eta = 0 \text{ in } B_\eta \quad (j = 2, \dots, n)\,,$$

in distributional sense we get

$$\phi(\eta, v, \tau) = \phi(\eta) \quad \forall (\eta, v, \tau) \in B\,. \tag{5.8}$$

In fact a Poincaré inequality holds in $(\mathbb{H}^{n-1}, d_C)$ with respect to the horizontal gradient $\nabla_{\mathbb{H}} := (\widetilde{X}_2, \dots, \widetilde{X}_n, \widetilde{Y}_2, \dots, \widetilde{Y}_n)$ (see, for instance,[34] , Proposition 11.17) and then there exists a constant $c > 0$ such

$$\int_{U_C(P,r)} |\phi_\eta - \phi_{\eta, U_C}|\, d\mathcal{L}^{2n-1} \leq c\, r \int_{U_C(P,r)} |\nabla_{\mathbb{H}} \phi_\eta|\, d\mathcal{L}^{2n-1}$$

for every $P \in \mathbb{H}^{n-1}$, $r > 0$ such that $U_C(P,r) := \{Q \in \mathbb{H}^{n-1} : d_C(P,Q) < r\} \subset B_\eta$ and

$$\phi_{\eta, U_C} := \frac{1}{\mathcal{L}^{2n-1}(U_C(P,r))} \int_{U_C(P,r)} \phi_\eta\, d\mathcal{L}^{2n-1}\,.$$

On the other hand by (5.8) we infer

$$\mathfrak{B}\phi = \frac{\partial \phi}{\partial \eta} = 0 \text{ in } B$$

in distributional sense. Thus ϕ is constant in $B = B(A_0, r_0)$ for all $A_0 \in \omega$ for suitable $r_0 > 0$. As ϕ is continuous in ω and ω is connected we can conclude that ϕ actually is constant in the whole ω.

In order to prove Theorem 5.1 we will need some further notation and preliminary results.

Let $\phi : \omega \to \mathbb{R}$ be a continuous function, and suppose that for all $A \in \omega$ there are $0 < \delta_2 < \delta_1$ such that, for each $j \in \{2, \dots, 2n\}$ there exists a map

$$\begin{aligned} \gamma_j \quad &: \quad [-\delta_2, \delta_2] \times \overline{I_{\delta_2}(A)} \to \overline{I_{\delta_1}(A)} \Subset \omega \\ &\quad (s, B) \longmapsto \gamma_j^B(s) \end{aligned}$$

such that $\gamma_j^B \in \mathbf{C}^1([-\delta_2, \delta_2], \mathbb{R}^{2n})$ for each $B \in \overline{I_{\delta_2}(A)}$ and, with the usual identification between vector fields and differential operators,

$$\text{(E.1)} \quad \begin{cases} \dot{\gamma}_j^B = W_j^\phi \circ \gamma_j^B = \begin{cases} \widetilde{X}_j \circ \gamma_j^B & \text{if } j \neq n+1 \\ \dfrac{\partial}{\partial v_\eta} - 4(\phi \circ \gamma_{n+1}^B) \dfrac{\partial}{\partial \tau} & \text{if } j = n+1 \end{cases} \\ \gamma_j^B(0) = B; \end{cases}$$

(E.2) there is a suitable continuous function $w_j : \omega \to \mathbb{R}$ (depending only on ϕ) such that, for each $s \in [-\delta_2, \delta_2]$, $\phi(\gamma_j^B(s)) - \phi(\gamma_j^B(0)) = \int_0^s w_j(\gamma_j^B(r))\, dr$

We will call the $\{\gamma_j\}$ a family of exponential maps of W^ϕ at A; we will write $\exp_A(sW_j^\phi)(B) := \gamma_j^B(s)$.

Remark 5.4. Notice that if the exponential maps of W^ϕ at A exist, then the map

$$[-\delta_2, \delta_2] \ni s \longmapsto \phi(\exp_A(sW_j^\phi)(B))$$

is of class $\mathbf{C}^1$ for each $j \in \{2, \dots, 2n\}$ and each $B \in I_{\delta_2}(A)$.

Remark 5.5. Observe that, because of the left invariance of the fields $\widetilde{X}_j$, for $j \neq n$ one must have

$$\exp_A(sW_j^\phi)(B) = B \star \iota^{-1}(\exp sX_j) = B \star \iota^{-1}(s\, e_j). \tag{5.9}$$

Moreover, if there are the exponential maps of W^ϕ at A (in particular there are w_j as in $(E.2)$), then for any $\lambda = (\lambda_2, \dots, \lambda_n, \lambda_{n+2}, \dots, \lambda_{2n}) \in \mathbb{R}^{2n-2}$ there exists also an exponential map for the field $\sum \lambda_j W_j^\phi$, i.e. there are two continuous maps $\gamma_\lambda : [-\delta_2, \delta_2] \times \overline{I_{\delta_2}(A)} \to \overline{I_{\delta_1}(A)} \Subset \omega$ (with, possibly, a $\delta_2 > 0$ smaller than the one in (E.1), depending on λ) and $w_\lambda : \omega \to \mathbb{R}$ such that

$$\begin{aligned} &\dot{\gamma_\lambda}(s, B) = \sum \lambda_j W_j^\phi(\gamma_\lambda(s, B)) \\ &\gamma_\lambda(0, B) = B \\ &\phi(\gamma_\lambda(s, B)) - \phi(\gamma_\lambda(0, B)) = \int_0^s w_\lambda(\gamma(r, B))\, dr \end{aligned}$$

In fact, it is sufficient to take $\gamma_\lambda(s,B) := B \star (0, s\lambda, 0)$ and $w_\lambda := \sum \lambda_j w_j$.

The following Lemma provides sufficient conditions to guarantee the existence of exponential maps of W^ϕ.

Lemma 5.6. *Let $\phi : \omega \to \mathbb{R}$ be continuous, and suppose that*

(i) *there exists $w \in \mathbf{C}^0(\omega)$ such that, in distributional sense,*

$$\begin{array}{ll} w = (w_2, \dots, w_{2n}) = (\widetilde{X}_2\phi, \dots, \widetilde{X}_n\phi, \mathfrak{B}\phi, \widetilde{X}_{n+2}\phi, \dots, \widetilde{X}_{2n}\phi) & \text{if } n \geq 2 \\ w = \mathfrak{B}\phi & \text{if } n = 1 \end{array}$$

(ii) *there is a family of functions $\{\phi_\epsilon\}_{\epsilon>0} \subset \mathbf{C}^1(\omega, \mathbb{R})$ such that for each $\omega' \Subset \omega$ we have*

$$\phi_\epsilon \to \phi,\ W^{\phi_\epsilon}\phi_\epsilon \to w \qquad \text{uniformly on } \overline{\omega'}.$$

Then for each $A \in \omega$ there are $0 < \delta_2 < \delta_1$ such that, for each $j = 2, \dots, 2n$, there exists $\exp_A(sW_j^\phi)(B) \in \overline{I_{\delta_1}(A)} \Subset \omega$ for all $(s,B) \in [-\delta_2, \delta_2] \times \overline{I_{\delta_2}(A)}$; moreover,

$$w_j(B) = \frac{d}{ds}\phi\big(\exp_A(sW_j^\phi)(B)\big)_{|s=0}$$

for each $B \in I_{\delta_2}(A)$.

Proof. Again we can suppose $n \geq 2$, as for $n = 1$ the proof can easily be derived.

There is no problem if $j \neq n+1$; in fact by (5.9) it is sufficient to set

$$\exp_A(sW_j^\phi)(B) := B \star \exp(s\widetilde{X}_j)$$

which is defined on $[-\delta_2, \delta_2] \times \overline{I_{\delta_2}(A)}$ for a sufficiently small δ_2 with values in $\overline{I_{\delta_1(A^0)}} \Subset \omega$. Then (E.1) is fulfilled by construction and (E.2) comes from the hypothesis that $w_j = \widetilde{X}_j\phi$ in distributional sense.

For $j = n+1$ and $\epsilon > 0$ consider the Cauchy problem

$$\begin{cases} \dot\gamma_\epsilon(s,B) = \dfrac{\partial}{\partial\eta} - 4\phi_\epsilon(\gamma_\epsilon(s,B))\dfrac{\partial}{\partial\tau} = W_{n+1}^{\phi_\epsilon}(\gamma_\epsilon(s,B)) \\ \gamma_\epsilon(0,B) = B \end{cases}$$

which has a solution $\gamma_\epsilon : [-\delta_2(\epsilon), \delta_2(\epsilon)] \times \overline{I_{\delta_2(\epsilon)}(A)} \to \overline{I_{\delta_1}(A)}$. By Peano's estimate on the existence time for solutions of ordinary differential equations we obtain that $\delta_2(\epsilon)$ can be taken greater than $C/\|\phi_\epsilon\|_{L^\infty(I_{\delta_1}(A))}$ (where the constant C depends only on δ_1), and so we get a $\delta_2 > 0$ such that $\delta_2(\epsilon) \geq \delta_2$ for all ϵ. Now, on the compact $[-\delta_2, \delta_2] \times \overline{I_{\delta_2}(A)}$ the functions γ_ϵ are uniformly continuous by hypothesis, and by Ascoli-Arzelá's Theorem we get a sequence $\{\epsilon_h\}_h$ such that $\epsilon_h \to 0$ as $h \to \infty$ and $\gamma_{\epsilon_h} \to \gamma$ uniformly on $[-\delta_2, \delta_2] \times \overline{I_{\delta_2}(A)}$. Remembering that

$$\gamma_{\epsilon_h}(s,B) = B + \int_0^s \left[\frac{\partial}{\partial\eta} - 4\phi_{\epsilon_h}(\gamma_{\epsilon_h}(s,B))\frac{\partial}{\partial\tau}\right] ds$$

$$\phi_{\epsilon_h}(\gamma_{\epsilon_h}(s,A)) - \phi_{\epsilon_h}(\gamma_{\epsilon_h}(0,B)) = \int_0^s W_{n+1}^{\phi_{\epsilon_h}}\phi_{\epsilon_h}(\gamma_{\epsilon_h}(s,B))\, ds$$

and for $j \to \infty$ we get (all the involved convergences are uniform)

$$\gamma(s,B) = B + \int_0^s \left[\frac{\partial}{\partial\eta} - 4\phi(\gamma(s,B))\frac{\partial}{\partial\tau}\right] ds$$
$$\phi(\gamma(s,B)) - \phi(\gamma(0,B)) = \int_0^s w_{n+1}(\gamma(s,B))\, ds$$

i.e. (E.1) and (E.2). □

As in Euclidean spaces the gradient of a function is the vector composed by the derivatives along the exponentials of the vectors of the canonical basis, we will prove, in the following theorem, that the W^ϕ-differential is the vector made by the derivatives along the exponentials of W^ϕ.

Theorem 5.7. *Let $\phi : \omega \to \mathbb{R}$ be a continuous function such that, for a certain $A \in \omega$, the following conditions are fulfilled:*

(i) *there are $0 < \delta_2 < \delta_1$ such that, for each $j = 2, \dots, 2n$ there exist a family of exponential maps*

$$\exp_A(sW_j^\phi) : [-\delta_2, \delta_2] \times \overline{I_{\delta_2}(A)} \to \overline{I_{\delta_1}(A)}.$$

(ii) *for each $\omega' \Subset \omega$*

$$\lim_{r\to 0^+} \sup\left\{\frac{|\phi(B') - \phi(B)|}{|B' - B|^{1/2}} : B', B \in \overline{\omega'}, 0 < |B' - B| \le r\right\} = 0;$$

Then ϕ is uniformly W^ϕ-differentiable at A and

$$[(W^\phi\phi)(A)]_j = \frac{d}{ds}\phi\big(\exp_A(sW_j^\phi)(A)\big)_{|s=0}.$$

Proof. For $n \ge 2$ let $A = (\overline{\eta}, \overline{v}, \overline{\tau}), B = (\eta, v, \tau), B' = (\eta', v', \tau') \in \omega$, while for $n = 1$ $A = (\overline{\eta}, \overline{\tau}), B = (\eta, \tau), B' = (\eta', \tau') \in \omega$, and let $w = (w_2, \dots, w_{2n})$ be as in (E.2). We have to prove that

$$\lim_{\delta\to 0} M_\phi(\phi, A, w(A), \delta) = 0 \tag{5.10}$$

where M_ϕ is defined as in (1.19).

The proof is exactly the same as in Theorem 3.8: at first, for $n > 1$, we define the vector field $\overline{X} := \sum_{j=2, j\neq n+1}^{2n}(v'_j - v_j)W_j^\phi = \sum_{j=2, j\neq n+1}^{2n}(v'_j - v_j)\widetilde{X}_j$, and then we set

$$\begin{aligned} B^* &:= \exp_A(\overline{X})(B) \\ &= B \star (0, (v'_2 - v_2, \dots, v'_n - v_n, v'_{n+2} - v_{n+2}, \dots, v'_{2n} - v_{2n}), 0) \\ &= (\eta, v', \tau - \sigma(v', v)). \end{aligned}$$

If $n = 1$, $\overline{X}$ has no meaning and we simply define $B^* := B$.

Now the big obstacle is that in general we cannot integrate along the vector field W_{n+1}^ϕ, i.e. we cannot define $B'' := \exp\big((\eta' - \eta)(\frac{\partial}{\partial\eta} - 4\phi\frac{\partial}{\partial\tau})\big)(B^*)$; however, this problem can be solved using the existence of exponential maps, more precisely by posing

$$B'' := \exp_A((\eta'-\eta)W^\phi_{n+1})(B^*) = \begin{array}{l}(\eta', v', \tau'') \text{ if } n \geq 2 \\ (\eta', \tau'') \text{ if } n = 1\end{array} \quad \text{(for a certain } \tau'').$$

Therefore, we can rewrite (3.17) as

$$\begin{aligned}
\phi(B') - \phi(B) &= [\phi(B') - \phi(B'')] + [\phi(B'') - \phi(B^*)] + [\phi(B^*) - \phi(B)] \\
&= [\phi(B') - \phi(B'')] + \int_0^{\eta'-\eta} w_{n+1}(\exp_A(sW^\phi_{n+1})(B^*))\, ds + \\
&\quad + \int_0^1 \sum_{\substack{j=2 \\ j\neq n+1}}^{2n} (v'_j - v_j) w_j(\exp_A(s\overline{X})(B)) \quad (*) \\
&= [\phi(B') - \phi(B'')] + \sum_{j=2, j\neq n+1}^{2n} (v'_j - v_j) w_j(A) + \\
&\quad + (\eta'-\eta) w_{n+1}(A) + o(|(\eta'-\eta, v'-v)|) \\
&= [\phi(B') - \phi(B'')] + \langle w(A), (\eta'-\eta, v'-v)\rangle + o(\rho_\phi(B', B))
\end{aligned}$$

if $n \geq 2$, and as

$$\phi(B') - \phi(B) = [\phi(B') - \phi(B'')] + w(A)(\eta'-\eta) + o(\rho_\phi(B', B))$$

if $n = 1$. In the passage signed with $(*)$ we have used the continuity of the w_j at A.

Reasoning as in (3.18) and (3.19), the keypoint is again to prove that $|\tau' - \tau''|^{1/2}/\rho_\phi(B', B'')$ is bounded in a neighbourhood of A, and rewriting (3.20) we obtain

$$\begin{aligned}
&|\tau' - \tau''| \\
&= \left|\tau' - \tau + \sigma(v', v) + 4\int_0^{\eta'-\eta} \phi(\exp_A(sW^\phi_{n+1})(B^*))\, ds\right| \\
&\leq |\tau' - \tau + 2(\phi(B') + \phi(B))(\eta'-\eta) + \sigma(v', v)| + \\
&\quad +2\left|2\int_0^{\eta'-\eta} \phi(\exp_A(sW^\phi_{n+1})(B^*))\, ds - (\phi(B') + \phi(B))(\eta'-\eta)\right| \\
&\leq \rho_\phi(B', B)^2 + 2|\phi(B') - \phi(B'')||\eta'-\eta| + 2|\phi(B) - \phi(B^*)||\eta'-\eta| + \\
&\quad +2\left|2\int_0^{\eta'-\eta} \phi(\exp_A(sW^\phi_{n+1})(B^*)\, ds - [\phi(B'') + \phi(B^*)](\eta'-\eta)\right| \\
&=: \rho_\phi(B', B)^2 + R_1(B', B) + R_2(B', B) + R_3(B', B)
\end{aligned} \tag{5.11}$$

for $n \geq 2$; for $n = 1$ simply don't consider the term $\sigma(v', v)$. Therefore we have once again to prove (3.21), (3.22), (3.23); this can be done following exactly the same line as in the proof of Theorem 3.8 and using (E.1) and (E.2): the only thing one must pay attention to is to write $\exp_A(\cdot\, W^{\phi}_{n+1})$ instead of $\exp(\cdot\, \overline{W})$ in (3.24). □

Now the proof of Theorem 5.1 is in order.

Proof (of Theorem 5.1). We will accomplish the proof only for $n \geq 2$, because as usual the generalization to $n = 1$ is immediate. Let us begin with the proof of the implication $(i) \Rightarrow (ii)$.

The statement in (5.2) follows from Theorem 4.1 and Corollary 4.5. By Proposition 4.6 we get that for each $B \in \omega$ there is a $\delta(B) > 0$ (such that $I_{\delta(B)}(B) \Subset \omega$) and a family of $\mathbf{C}^1$ functions $\{\phi_{\epsilon,B} : \overline{I_{\delta(B)}(B)} \to \mathbb{R}\}_{0<\epsilon<1}$ such that

$$\phi_{\epsilon,B} \to \phi \text{ and } W_{\phi_{\epsilon,B}}\phi_{\epsilon,B} \to W^{\phi}\phi \quad \text{uniformly on } \overline{I_{\delta(B)}(B)}. \tag{5.12}$$

As $\mathcal{F} := \{I_{\delta(B)}(B) : B \in \omega\}$ is an open covering of ω we can associate a partition of the unity $\{\theta_i : i \in \mathbb{N}\}$ which is subordinate to it, i.e.

$$\theta_i \in \mathbf{C}^{\infty}_c(\omega), 0 \leq \theta_i \leq 1 \text{ on } \omega \text{ for all } i \tag{5.13}$$

$$\begin{aligned}&\{\text{spt } \theta_i\}_{i\in\mathbb{N}} \text{ form a locally finite covering of } \omega, \text{ and for all } i \in \mathbb{N}\\ &\text{there is an } I_i := I_{\delta(B(i))}(B(i)) \in \mathcal{F} \text{ such that spt } \theta_i \subset I_i\end{aligned} \tag{5.14}$$

$$\sum_{i=1}^{\infty} \theta_i \equiv 1 \text{ on } \omega. \tag{5.15}$$

Let $\phi_{\epsilon,i} := \phi_{\epsilon,B(i)} : \mathbb{R}^{2n} \to \mathbb{R}$, where from now on, if necessary, we use the convention of extending functions by letting them vanish outside their domain. Let $\phi_\epsilon := \sum_{i=1}^{\infty} \theta_i \phi_{\epsilon,i}$; by construction $\phi_\epsilon \in \mathbf{C}^1(\omega)$ and

$$\begin{aligned}
\frac{\partial \phi_\epsilon}{\partial \eta} &= \sum_{i=1}^{\infty}\left(\frac{\partial \theta_i}{\partial \eta}\phi_{\epsilon,i} + \theta_i \frac{\partial \phi_{\epsilon,i}}{\partial \eta}\right) && (\forall\, n)\\
\frac{\partial \phi_\epsilon}{\partial v_j} &= \sum_{i=1}^{\infty}\left(\frac{\partial \theta_i}{\partial v_j}\phi_{\epsilon,i} + \theta_i \frac{\partial \phi_{\epsilon,i}}{\partial v_j}\right) && (n \geq 2)\\
\frac{\partial \phi_\epsilon}{\partial \tau} &= \sum_{i=1}^{\infty}\left(\frac{\partial \theta_i}{\partial \tau}\phi_{\epsilon,i} + \theta_i \frac{\partial \phi_{\epsilon,i}}{\partial \tau}\right) && (\forall\, n).
\end{aligned}$$

In particular

$$W^{\phi_\epsilon}\phi_\epsilon = \sum_{i=1}^{\infty}\left(\phi_{\epsilon,i} W^{\phi_\epsilon}\theta_i + \theta_i W^{\phi_\epsilon}\phi_{\epsilon,i}\right) \text{ on } \omega.$$

We have to show that (5.1) holds for any fixed $\omega' \Subset \omega$; by (5.14) there is only a finite number of index $i_1, \ldots, i_k$ such that $\overline{\omega'} \cap \text{ spt } \theta_{i_h} \neq \emptyset \;\forall\; h = 1, \ldots, k$, and

$\overline{\omega'} \subset \cup_{h=1}^k \text{ spt } \theta_{i_h}$. Then

$$\phi_\epsilon = \sum_{h=1}^k \theta_{i_h}\phi_{\epsilon,i_h} \quad \text{and} \quad \phi = \sum_{h=1}^k \theta_{i_h}\phi \quad \text{on } \overline{\omega'} \tag{5.16}$$

$$W^{\phi_\epsilon}\phi_\epsilon = \sum_{h=1}^k (\phi_{\epsilon,i_h} W^{\phi_\epsilon}\theta_{i_h} + \theta_{i_h} W^{\phi_\epsilon}\phi_{\epsilon,i_h}) \quad \text{on } \overline{\omega'}. \tag{5.17}$$

Equations (5.16) and (5.17), together with (5.12), give

$$\phi_\epsilon \to \phi \tag{5.18}$$

$$W^{\phi_\epsilon}\phi_\epsilon \to \sum_{h=1}^k (\phi W^\phi_{i_h} + \theta_{i_h} W^\phi \phi) =: w \tag{5.19}$$

uniformly on $\overline{\omega'}$, where we put

$$W^\phi_{i_h} := \left(\widetilde{X}_2\theta_{i_h}, \dots, \widetilde{X}_n\theta_{i_h}, \frac{\partial\theta_{i_h}}{\partial\eta} - 4\phi\frac{\partial\theta_{i_h}}{\partial\tau}, \widetilde{Y}_2\theta_{i_h}, \dots, \widetilde{Y}_n\theta_{i_h}\right).$$

Observing that $\sum_{h=1}^k \phi W^\phi_{i_h} = 0$ we get that $w = W^\phi\phi \in \mathbf{C}^0(\omega, \mathbb{R}^{2n-1})$ and

$$w = (\widetilde{X}_2\phi, \dots, \widetilde{X}_n\phi, \mathfrak{B}\phi, \widetilde{Y}_2\phi, \dots, \widetilde{Y}_n\phi)$$

in distributional sense.

The reverse implication $(ii) \Rightarrow (i)$ follows from Lemma 5.6 and Theorem 5.7. The hypothesis (ii) of Theorem 5.7 (i.e. the assertion in (5.2)) is satisfied because of the following Theorem 5.9: the key observation is that, thanks to the uniform convergence of ϕ_ϵ and $W^{\phi_\epsilon}\phi_\epsilon$, we can estimate $\|\phi_\epsilon\|_{L^\infty(\omega'')}$ and $\|W^{\phi_\epsilon}\phi_\epsilon\|_{L^\infty(\omega'')}$ uniformly in ϵ for any $\omega'' \Subset \omega$. Moreover, the uniform convergence of $W^{\phi_\epsilon}\phi_\epsilon$ allows us to choose a modulus of continuity for $W^{\phi_\epsilon}\phi_\epsilon$ which is independeent of ϵ. Therefore there is a function $\alpha :]0, +\infty[\to \mathbb{R}$, which does not depend on ϵ, such that $\lim_{r\to 0}\alpha(r) = 0$ and

$$\sup\left\{\frac{|\phi_\epsilon(B') - \phi_\epsilon(B)|}{|B' - B|^{1/2}} : B', B \in \omega', 0 < |B' - B| \le r\right\} \le \alpha(r).$$

which implies (5.2). $\square$

Theorem 5.8. *Let $I \subset \mathbb{R}^{2n}$ be a rectangle and let $\phi \in \mathbf{C}^1(I)$ be such that $W^\phi\phi = (w_2, \dots, w_{2n}) \in \mathbf{C}^0(I, \mathbb{R}^{2n-1})$, i.e.*

$$\begin{cases} \widetilde{X}_j\phi = w_j, \ \widetilde{Y}_j\phi = w_{j+n} \quad \textit{for all } j = 2, \dots, n \\ \dfrac{\partial\phi}{\partial\eta} - 4\phi\dfrac{\partial\phi}{\partial\tau} = w_{n+1}. \end{cases}$$

Then for all rectangle $I' \Subset I$ there exists a function $\alpha :]0, +\infty[\to [0, +\infty[$, which depends only on I'', $\|\phi\|_{L^\infty(I'')}$ (where I'' is any open rectangle such that $I' \Subset I'' \Subset$

I), on $\|W^\phi\phi\|_{L^\infty(I'')}$ and on the modulus of continuity of w_{n+1} on I'', such that $\lim_{r\to 0}\alpha(r) = 0$ *and*

$$\sup\left\{\frac{|\phi(A)-\phi(B)|}{|A-B|^{1/2}} : A, B \in I', 0 < |A-B| \le r\right\} \le \alpha(r). \tag{5.20}$$

Proof. As usual we can suppose $n \ge 2$; the proof can be easily adapted to the case $n = 1$.

Let's begin by setting $K := \sup_{A\in I''}|A|$, $M := \|\phi\|_{L^\infty(I'')}$, $N := \|W^\phi\phi\|_{L^\infty(I'')}$; let β be the modulus of continuity of w_{n+1} on I'', i.e. an increasing function $]0,+\infty[\ni r \to \beta(r) \in [0,+\infty[$ such that $|w_{n+1}(A) - w_{n+1}(B)| \le \beta(|A-B|)$ for all $A, B \in I''$ and $\lim_{r\to 0}\beta(r) = 0$. We divide the proof in several steps.

STEP 1. Let us fix another rectangle $J \subset \mathbb{R}^{2n}$ such that $I' \Subset J \Subset I''$, and let us introduce the following notation: for $A = (\eta, v, \tau) \in J$ we define γ_A as the curve solution of the Cauchy problem

$$\begin{cases} \dot{\gamma_A}(t) = \dfrac{\partial}{\partial\eta} - 4\phi(\gamma_A(t))\dfrac{\partial}{\partial\tau} \\ \gamma_A(\eta) = A \end{cases}$$

By standard considerations on ordinary differential equations, we have $\gamma_A \in \mathbf{C}^1([\eta-\epsilon, \eta+\epsilon], I'')$ for a certain $\epsilon > 0$ which does not depend on A; moreover, we can choose ϵ so that $\gamma_A([\eta-\epsilon, \eta+\epsilon]) \subset J$ for all $A \in I'$. Let $\gamma_A(t) = (\eta + t, v, \tau_A(t))$; then

$$\frac{d^2}{dt^2}\tau_{A_0}(t) = \frac{d}{dt}[-4\phi(\gamma_{A_0}(t))] = -4w_{n+1}(\gamma_{A_0}(t)). \tag{5.21}$$

STEP 2. Set $\delta(r) := \max\{r^{1/4}, \beta(Er^{1/4})^{1/2}\}$, where $E > 0$ is a constant which will be specified later; we start by proving that $\alpha'(r) \le \delta(r) + 2N^{1/2}\delta(r) + Nr^{1/2}$ for r "sufficiently small" (in a way we are going to specify, but depending on K, M, N and β only), where

$$\alpha'(r) := \sup\left\{\frac{|\phi(A)-\phi(B)|}{|A-B|^{1/2}} : A = (\eta, v, \tau), B = (\eta', v, \tau') \in I', 0 < |A-B| \le r\right\}.$$

Suppose on the contrary that there exist $A = (\eta, v, \tau), B = (\eta', v, \tau') \in I'$ such that $|A-B|$ is "sufficiently small" and

$$\frac{|\phi(A)-\phi(B)|}{|A-B|^{1/2}} > \delta + 2N^{1/2}\delta + Nr^{1/2},$$

where from now on we will write δ instead of $\delta(|A-B|)$. We observe explicitely that by definition of $\delta(r)$ we have $\delta' := \delta(|\tau-\tau'|) \le \delta$ and so

$$\begin{aligned}\frac{\beta\big(|\tau-\tau'|+8M|\tau-\tau'|^{1/2}/\delta\big)}{\delta^2} &\le \frac{\beta\big(|\tau-\tau'|+8M|\tau-\tau'|^{1/2}/\delta'\big)}{\delta'^2}\\ &\le \frac{\beta\big(|\tau-\tau'|+8M|\tau-\tau'|^{1/4}\big)}{\delta'^2}\\ &\le 1\end{aligned} \tag{5.22}$$

provided $E>0$ is such that $|\tau-\tau'|+8M|\tau-\tau'|^{1/4} \le E|\tau-\tau'|^{1/4}$.

Let $C := (\eta, v, \tau') \in I'$; as $|A-C|^{1/2} = |\tau-\tau'|^{1/2}$ and $|C-B|^{1/2} = |\eta-\eta'|^{1/2}$ we have

$$\begin{aligned}\delta + 2N^{1/2}\delta + Nr^{1/2} &\le \frac{|\phi(A)-\phi(B)|}{|\eta-\eta'|^{1/2}+|\tau-\tau'|^{1/2}} \le\\ &\le \frac{|\phi(A)-\phi(C)|}{|\tau-\tau'|^{1/2}} + \frac{|\phi(C)-\phi(B)|}{|\eta-\eta'|^{1/2}} =: R_1+R_2.\end{aligned} \tag{5.23}$$

Therefore, up to subsequences, we can suppose that we have always $R_1 \ge \delta$ or $R_2 \ge +2N^{1/2}\delta + Nr^{1/2}$.

STEP 3. We want to prove that the first case cannot occur; in fact, we will prove that

$$\frac{|\phi(A)-\phi(C)|}{|\tau-\tau'|^{1/2}} \le \delta$$

for $A, B \in J$ (not for I' only!). We can suppose that $\tau > \tau'$ (for the other case it is sufficient to exchange the roles of A and C). Consider γ_A and γ_C; thanks to (5.21) we have, for $t \in [\eta-\epsilon, \eta+\epsilon]$

(5.24)

$$\begin{aligned}&\tau_A(t)-\tau_C(t)\\ &= \tau-\tau' + \int_\eta^t \left[\dot\tau_A(\eta)-\dot\tau_C(\eta) + \int_\eta^s \big[\ddot\tau_A(r)-\ddot\tau_C(r)\big]dr\right] ds\\ &\le \tau-\tau' - 4(t-\eta)\big[\phi(A)-\phi(C)\big] - 4\int_\eta^t\int_\eta^s \big[w_{n+1}(\gamma_A(r))-w_{n+1}(\gamma_C(r))\big]dr\,ds\\ &\le \tau-\tau'-4(t-\eta)\big[\phi(A)-\phi(C)\big] + (t-\eta)^2\beta\big(|\tau-\tau'|+8M|t-\eta|\big),\end{aligned}$$

where in the last inequality we used the fact that

$$\begin{aligned}|\gamma_A(r)-\gamma_C(r)| &\le |\gamma_A(\eta)-\gamma_C(\eta)| + |r-\eta|\,(\|\dot\tau_A\|_\infty + \|\dot\tau_C\|_\infty)\\ &\le |\tau-\tau'| + 8M|t-\eta|.\end{aligned}$$

If $\phi(A)-\phi(C)>0$ put $t := \eta + (\tau-\tau')^{1/2}/\delta$ in (5.24), and $t := \eta - (\tau-\tau')^{1/2}/\delta$ otherwise; if $|\tau-\tau'|$ is "sufficiently small" $\gamma_A(t)$ and $\gamma_C(t) \in I''$ are well defined (it is sufficient to take $\epsilon \ge (\tau-\tau')^{1/4} \ge (\tau-\tau')^{1/2}/\delta = |t-\eta|$) and from (5.22), (5.24) and $R_1 \geq \delta$ we get (in both cases)

$$\begin{aligned} & \tau_A(t) - \tau_C(t) \\ (5.25) \qquad & \leq \tau - \tau' - 4(\tau - \tau') + (\tau - \tau')\beta\big(|\tau - \tau'| + 8M|\tau - \tau'|^{1/2}/\delta\big)/\delta^2 \\ & \leq -2(\tau - \tau') \ < \ 0. \end{aligned}$$

This leads to a contradiction: in fact τ_A and τ_C are solutions of the same Cauchy problem

$$\dot{\tau}(s) = -4\phi(s, v, \tau(s))$$

with initial data $\tau(\eta) = \tau$ and τ' respectively. The contradiction is given by the fact that two such solutions cannot meet, while $\tau_A(\eta) - \tau_C(\eta) > 0$ and $\tau_A(t) - \tau_C(t) < 0$ for h sufficiently large.

STEP 4. Now let's examine the second case $R_2 \geq 2N^{1/2}\delta + Nr^{1/2}$; we can suppose that $\eta' < \eta$ (otherwise it is sufficient to exchange the roles of B and C). Consider γ_B; again, for $\eta - \eta'$ "sufficiently small" $D := \gamma_B(\eta) = (\eta, v, \tau'') \in J$ is well defined, and

$$(5.26) \qquad |\phi(B) - \phi(D)| = \left| \int_{\eta'}^{\eta} w_{n+1}(\gamma_B(t))dt \right| \leq N|\eta - \eta'|;$$

moreover

$$(5.27) \qquad |\tau'' - \tau'| = |4 \int_{\eta'}^{\eta} \phi(\gamma_B(t))\, dt| \leq 4N|\eta - \eta'|.$$

Then for $|\eta' - \eta|$ "sufficiently small" (and precisely when $N|\eta - \eta'|^{1/2} \leq |\eta - \eta'|^{1/4} \leq \delta$) we obtain

$$(5.28) \qquad \begin{aligned} |\phi(C) - \phi(D)| \ & \geq \ |\phi(C) - \phi(B)| - |\phi(B) - \phi(D)| \\ & \geq \ \big[\, 2N^{1/2}\delta + Nr^{1/2} - N|\eta - \eta'|^{1/2} \,\big]\, |\eta - \eta'|^{1/2} \\ & \geq \ 2N^{1/2}\delta|\eta - \eta'|^{1/2} \ \geq \ \delta|\tau'' - \tau'|^{1/2} \end{aligned}$$

so that we are in the first case again (with the couple $C, D \in J$ instead of A, C) which we have seen is not possible.

This proves that $\lim_{r\to 0} \alpha'(r) = 0$, and that we are able to control α' with only K, M, N and β. Observe that what we said up to now, properly translated in the notation we use when $n = 1$, gives directly the thesis for the case $n = 1$.

STEP 5. For the general case, let $A = (\eta, v, \tau), B = (\eta', v', \tau') \in I$, and set

$$A^* := A \star (0, v' - v, 0) = (\eta, v', \tau + \sigma(v, v')).$$

We can see A^* also as $\exp(\sum_{j=2, j\neq n+1}^{2n} (v'_j - v_j)W_j^\phi)(A)$ and so

$$\begin{aligned} |\phi(A) - \phi(A^*)| \ & \leq \ \left| \sum_{\substack{j=2 \\ j\neq n+1}}^{2n} \int_0^1 (v'_j - v_j)W_j^\phi \phi\big(\exp(t \textstyle\sum_{j=2, j\neq n+1}^{2n} (v'_j - v_j)W_j^\phi)(A)\big)dt \right| \\ & \leq \ N|v' - v| \leq N|A - B|. \end{aligned}$$

As $\left|\sigma(v,v')\right| = \left|2\sum_{j=2}^{n}[v_{n+j}(v'_j - v_j) - v_j(v'_{n+j} - v_{n+j})]\right| \le 2K|A-B|$ we get

$$\begin{aligned} |A^* - B| &\le |\eta' - \eta| + |\tau' - \tau| + |\sigma(v,v')| \\ &\le (2K+2)|A-B| \end{aligned}$$

and so

$$\begin{aligned} \frac{|\phi(A)-\phi(B)|}{|A-B|^{1/2}} &\le \frac{|\phi(A)-\phi(A^*)|}{|A-B|^{1/2}} + \frac{|\phi(A^*)-\phi(B)|}{|A-B|^{1/2}} \\ &\le N|A-B|^{1/2} + (2K+2)\frac{|\phi(A^*)-\phi(B)|}{|A^*-B|^{1/2}} \\ &\le N|A-B|^{1/2} + (2K+2)\alpha'(|A^*-B|^{1/2}) \\ &\le N|A-B|^{1/2} + (2K+2)\alpha'([(K+2)|A-B|]^{1/2}). \end{aligned}$$

STEP 6. The proof is accomplished for r "sufficiently small" only; however, this is sufficient to conclude. □

By a standard compactness argument we get the following

Theorem 5.9. *Let $\phi \in \mathbf{C}^1(\omega)$ such that $W^\phi\phi = (w_2,\dots,w_{2n}) \in \mathbf{C}^0(\omega,\mathbb{R}^{2n-1})$, i.e.*

$$\begin{cases} \widetilde{X}_j\phi = w_j,\ \widetilde{Y}_j\phi = w_{j+n} \quad \text{for all } j = 2,\dots,n \\ \dfrac{\partial\phi}{\partial\eta} - 4\phi\dfrac{\partial\phi}{\partial\tau} = w_{n+1}. \end{cases}$$

Then for all $\omega' \Subset \omega$ there exists a function $\alpha :]0,+\infty[\to [0,+\infty[$, which depends only on ω', $\|\phi\|_{L^\infty(\omega'')}$ (where ω'' is any open set such that $\omega' \Subset \omega'' \Subset \omega$), $\|W^\phi\phi\|_{L^\infty(\omega'')}$ and on the modulus of continuity of w_{n+1} on ω'', such that $\lim_{r\to 0}\alpha(r) = 0$ and

$$\sup\left\{\frac{|\phi(A)-\phi(B)|}{|A-B|^{1/2}} : A,B \in \omega', 0 < |A-B| \le r\right\} \le \alpha(r). \tag{5.29}$$

Let us conclude this section with two applications of Theorem 5.1 .

A first application is a negative answer to the problem of a good parameterization of $\mathbb{H}$- regular hypersurfaces. Indeed a natural question arising is the (local) Lipschitz continuity of $\phi : \omega \subset (\mathbb{R}^{2n},\rho) \to \mathbb{R}$ when ρ denotes the restriction distance of d_∞ to $V_1 \equiv \mathbb{R}^{2n}$. More precisely when $\rho \equiv \rho_{\mathcal{P}}$ being $\rho_{\mathcal{P}}$ the so- called *parabolic* distance on $\mathbb{R}^{2n} = \mathbb{R}_\eta \times \mathbb{R}^{2n-2}_v \times \mathbb{R}_\tau$ ($\mathbb{R}_\eta \times \mathbb{R}_\tau$ if $n=1$), i.e. the distance defined by

$$\begin{aligned} &\rho_{\mathcal{P}}((\eta,v,\tau),(\eta',v',\tau')) = |(\eta',v')-(\eta,v)| + |\tau'-\tau|^{1/2} \\ &\rho_{\mathcal{P}}((\eta,\tau),(\eta',\tau')) = |\eta'-\eta| + |\tau'-\tau|^{1/2} \qquad \text{if } n=1. \end{aligned}$$

.

Corollary 5.10. *There exist $\mathbb{H}$-regular surfaces $S = \Phi(\omega) \subset \mathbb{H}^1$ for which there is no constant $L > 0$ such that*

$$|\phi(\eta',\tau') - \phi(\eta,\tau)| \le L(|\eta-\eta'| + |\tau-\tau'|^{1/2}) \quad \textit{for all } (\eta,\tau),(\eta',\tau') \in \omega$$

for suitable continuous functions $\phi : \omega \to \mathbb{R}$ when $\Phi : \omega \to \mathbb{H}^1$ is the function

$$\Phi(A) := \iota(A) \cdot \phi(A) e_1.$$

Proof. By contradiction. Whithout loss of generality we can assume that $\omega = (a, b) \times (c, d)$, then for each $\tau \in (c, d)$ the function $\phi(\cdot, \tau)$ is Lipschitz continuous in (a, b). Therefore, for all $\tau \in (c, d)$ there exists the distributionial derivative $\frac{\partial \phi}{\partial \eta}(\cdot, \tau) \in L^\infty(a, b)$ in (a, b) and $||\frac{\partial \phi}{\partial \eta}(\cdot, \tau)||_{L^\infty(a,b)} \leq L$ for all $\tau \in (c, d)$. In particular there exists the distributional derivative $\frac{\partial \phi}{\partial \eta} \in L^\infty(\omega)$ in ω too. By Theorem 5.1 we know that

$$\mathfrak{B}\phi = \frac{\partial \phi}{\partial \eta} - 2\frac{\partial \phi^2}{\partial \tau} \in \mathbf{C}^0(\omega)$$

in distributional sense, thus $2\frac{\partial \phi^2}{\partial \tau} \in L^\infty_{loc}(\omega)$. Then $\phi^2 \in$Lip$_{loc}(\omega)$.

We claim that $S := \Phi(\omega)$ is Euclidean 2-rectifiable. Indeed there is no loss of generality in supposing that actually $\phi^2 \in$Lip(ω), i.e. $|\phi^2(A) - \phi^2(B)| \leq M|B - A|$ for some $M > 0$ and all $A, B \in \omega$. Then for $h \in \mathbb{N}$ set

$$\begin{aligned}
&\omega_h^+ := \{A \in \omega : \phi(A) > 1/h\}\\
&\omega_h^- := \{A \in \omega : \phi(A) < -1/h\}\\
&\omega_0 := \{A \in \omega : \phi(A) = 0\}
\end{aligned}$$

and observe that, when $A, B \in \omega_h^+$ or $A, B \in \omega_h^-$, we have

$$\begin{aligned}
2|\phi(A) - \phi(B)|/h &\leq |\phi(A) - \phi(B)| \cdot |\phi(A) + \phi(B)|\\
&= |\phi^2(A) - \phi^2(B)| \leq M|B - A|,
\end{aligned}$$

i.e. $\phi_{|\omega_h^\pm}$ is Lipschitz continuous; extending it to $\phi_h^\pm : \omega \to \mathbb{R}$ and defining $\Phi_h^\pm$ in the usual way, we get that $\Phi(\omega_h^\pm) \subset \Phi_h^\pm(\omega)$ is Euclidean 2-rectifiable. Observing that $\Phi(\omega_0) \subset V_1$, we get that also

$$\Phi(\omega) \subset \Phi(\omega_0) \cup \bigcup_h \Phi(\omega_h^+) \cup \bigcup_h \Phi(\omega_h^+)$$

is Euclidean 2-rectifiable. On the other hand there are $\mathbb{H}$-regular surfaces $S = \Phi(\omega) \subset \mathbb{H}^1$ which are not Euclidean 2-rectifiable (see [37], Theorem 3.1) and then a contradiction. □

A second interesting corollary of Theorem 5.1 provides a simple way to exihibit $\mathbb{H}$-regular surfaces in $\mathbb{H}^1$ not Euclidean regular:

Corollary 5.11. *Let $\phi : \omega \subset \mathbb{R}^2 \to \mathbb{R}$ be a continuous function which depends only on τ, i.e. $\phi = \phi(\tau) : I \to \mathbb{R}$ for a certain open (and possibly unbounded) interval $I \subset \mathbb{R}$, and suppose that $\phi^2 : I \to \mathbb{R}_+$ is of class $\mathbf{C}^1$. Then ϕ is uniformly W^ϕ-differentiable at A for every $A \in \omega$ and*

$$W^\phi \phi(A) = -2(\phi^2)'(A).$$

In particular, $W^\phi\phi$ is continuous and ϕ parameterizes an $\mathbb{H}$-regular surface in $\mathbb{H}^1$.

Proof. Thanks to Theorem 5.1, it is sufficient to find a family $\{\phi_\epsilon\}_\epsilon$ such that (5.1) holds. The family we are going to consider is of the form $\phi_\epsilon = \phi_\epsilon(\tau) := (\phi^2+\delta_\epsilon^2)^{1/2}\cdot g_\epsilon$, where δ_ϵ and g_ϵ are to be found; the key idea is to construct g_ϵ such that $g_\epsilon \to$ sign ϕ and g'_ϵ is "controlled", in a way we are going to specify; then our thesis becomes

$$\phi_\epsilon \to \phi \quad \text{and} \quad (\phi_\epsilon^2)' \to (\phi^2)' \quad \text{uniformly on } J \tag{5.30}$$

for each $J \Subset I$.

We recall the following general fact: let D, E two closed subsets of I such that $d(D,E) := \inf\{|a-b| : a \in D, b \in E\} \geq C > 0$; then there exists a $g \in \mathbf{C}^\infty(I,[-1,1])$ such that $g_{|D} \equiv 1, g_{|E} \equiv -1$ and $\|g'\|_\infty \leq 4/C$.

Now let us set

$$\alpha(r) := \sup\left\{\frac{|\phi(\tau')-\phi(\tau)|}{|\tau'-\tau|^{1/2}} : \tau',\tau \in J,\ 0<|\tau'-\tau|\leq r\right\},$$

and suppose that $\alpha(r) \to 0$ as $r \to 0^+$: then if we set $\delta_\epsilon := \alpha(\epsilon)\epsilon^{1/2}/2$ we have $\lim_{\epsilon\to 0}\delta_\epsilon = 0$. For each ϵ let $D_\epsilon := \{\tau : \phi(\tau) \geq \delta_\epsilon\} \cap J$ and $E_\epsilon := \{\tau : \phi(\tau) \leq -\delta_\epsilon\} \cap J$; by construction $d(D_\epsilon, E_\epsilon) \geq \epsilon$ and so there exists a $g_\epsilon \in \mathbf{C}^\infty(I,[-1,1])$ with $g_\epsilon \equiv 1$ on D_ϵ, $g_\epsilon \equiv -1$ on E_ϵ and $\|g'_\epsilon\|_\infty \leq 4/\epsilon = \alpha(\epsilon)^2/\delta_\epsilon^2$. As we said earlier, set $\phi_\epsilon := (\phi^2+\delta_\epsilon^2)^{1/2}g_\epsilon$; it is easy to prove that $\phi_\epsilon \to \phi$ uniformly on J and

$$\begin{aligned}
2\|(\phi_\epsilon^2)'-(\phi^2)'\|_{L^\infty(J)} &\leq 4\|g_\epsilon g'_\epsilon(\phi^2+\delta_\epsilon^2)\|_{L^\infty(J)} + 2\|(g_\epsilon^2-1)(\phi^2)'\|_{L^\infty(J)}\\
&\leq 4\|g_\epsilon g'_\epsilon(\phi^2+\delta_\epsilon^2)\|_{L^\infty(J\setminus(D_\epsilon\cup E_\epsilon))} + 4\|(\phi^2)'\|_{L^\infty(J\setminus(D_\epsilon\cup E_\epsilon))}\\
&\leq 8\frac{\alpha(\epsilon)^2}{\delta_\epsilon^2}\delta_\epsilon^2 + 4\|(\phi^2)'\|_{L^\infty(J\cap\{|\phi|\leq\delta_\epsilon\})} \longrightarrow 0
\end{aligned}$$

for $\epsilon \to 0^+$; in the last passage we used the implication $\phi(\tau)=0 \Rightarrow (\phi^2)'(\tau)=0$, and so $\|(\phi^2)'\|_{L^\infty(J\cap\{|\phi|\leq\delta_\epsilon\})} \to 0$ because of the continuity of $(\phi^2)'$.

Let us remark that ϕ_ϵ actually depends on J; however, if we consider a sequence $\{J^n\}_{n\in\mathbb{N}}$ of closed intervals such that $J^n \subset J^{n+1}$ and $J^n \uparrow]\alpha,\beta[$, we get sequences $\{\phi_\epsilon^n\}_\epsilon$ for each n, so that we can conclude with a diagonal argument.

Finally, we have to prove that $\alpha(r) \to 0$ as $r \to 0$. Suppose that the converse is true; then there exist $\sigma > 0$ and $a_h, b_h \in J$ such that

$$|\phi(a_h)-\phi(b_h)| > 2\sigma|a_h-b_h|^{1/2} \quad \text{and} \quad |a_h-b_h| \to 0. \tag{5.31}$$

We can suppose that $\phi(a_h)$ and $\phi(b_h)$ have the same sign (i.e. $\phi(a_h)\phi(b_h) \geq 0$); in fact, if this is not the case, by the continuity of ϕ there is a $c_h \in]a_h, b_h[$ such that $\phi(c_h) = 0$, and we can suppose that $c_h \in J$ (because there is no loss of generality supposing that J is an interval). As

$$2\sigma < \frac{|\phi(a_h)-\phi(b_h)|}{|a_h-b_h|^{1/2}} \leq \frac{|\phi(a_h)-\phi(c_h)|}{|a_h-c_h|^{1/2}} + \frac{|\phi(c_h)-\phi(b_h)|}{|c_h-b_h|^{1/2}}$$

there exists a $d_h \in \{a_h, b_h\}$ such that $|\phi(c_h) - \phi(d_h)| > \sigma|c_h - d_h|^{1/2}$. Therefore (possibly considering c_h and d_h instead of a_h and b_h) we can assume that a_h and b_h satisfy (5.31) (possibly with σ instead of 2σ) and that $\phi(a_h)$ and $\phi(b_h)$ have the same sign.

As J is compact, we can suppose (up to subsequences) that there is a $\overline{\tau} \in J$ such that $a_h \to \overline{\tau}$ and $b_h \to \overline{\tau}$. It is not possible that $\phi(\overline{\tau}) \neq 0$: in fact, ϕ is of class $\mathbf{C}^1$ in the open set $\{\tau : \phi(\tau) \neq 0\}$ (it is easy to show that here $\phi' = (\phi^2)'/2\phi$) that would imply the boundedness of the quantities $|\phi(a_h) - \phi(b_h)|/|a_h - b_h|$ for h sufficiently large, which is in contradiction with (5.31). Therefore $\phi(\overline{\tau}) = 0$ and so one must have $(\phi^2)'(\overline{\tau}) = 0$. As $\phi(a_h)$ and $\phi(b_h)$ have the same sign, we have $|\phi(a_h) - \phi(b_h)| \leq |\phi(a_h) + \phi(b_h)|$ and so

$$\begin{aligned}
\sigma^2 &< \left(\frac{|\phi(a_h) - \phi(b_h)|}{|a_h - b_h|^{1/2}}\right)^2 \\
&\leq \left(\frac{|\phi(a_h) - \phi(b_h)|}{|a_h - b_h|^{1/2}}\right)\left(\frac{|\phi(a_h) + \phi(b_h)|}{|a_h - b_h|^{1/2}}\right) \\
&= \frac{|\phi(a_h)^2 - \phi(b_h)^2|}{|a_h - b_h|} = (\phi^2)'(\tau_h)
\end{aligned}$$

for a certain τ_h contained in the interval between a_h and b_h. Therefore $\tau_h \to \overline{\tau}$ and so $(\phi^2)'(\overline{\tau}) \geq \sigma$ by the continuity of $(\phi^2)'$, which is a contradiction. $\square$

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