Superlinear free-discontinuity models: relaxation and phase field approximation

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In this paper we develop the Direct Method in the Calculus of Variations for free-discontinuity energies whose bulk and surface densities exhibit superlinear growth, respectively for large gradients and small jump amplitudes. A distinctive feature of this kind of models is that the functionals are defined on SBV functions whose jump sets may have infinite measure. Establishing general lower semicontinuity and relaxation results in this setting requires new analytical techniques. In addition, we propose a variational approximation of certain superlinear energies via phase field models.

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1 Introduction

Variational models involving free-discontinuity problems play a fundamental role in fields such as fracture mechanics, image processing, and materials science. These models describe systems in which the energy consists of competing bulk and surface terms. A prototypical example takes the form

$$\int_{\Omega\setminus K}\Psi(\nabla u)\mathrm{d}x+\int_K g([u],\nu)\mathrm{d}\mathcal{H}^{n-1}. \tag{1.1}$$

In the context of solid mechanics, the domain $\Omega\subset\mathbb{R}^n$ represents the reference configuration of a hyperelastic body, while $u:\Omega\to\mathbb{R}^m$ denotes its deformation. The set $K\subset\Omega$ is the unknown (sufficiently regular) set of discontinuity points of u, finally ν and $[u]$ denote, respectively, the normal vector to K and the difference of the traces of u across K. The first term in the energy (1.1) represents the stored elastic energy, while the second term describes the contribution concentrated on the fractured surface K, which may account for both energy and dissipation. External loads and Dirichlet boundary conditions may also be incorporated into the model. For a comprehensive overview of the subject, see [AB95, Bra98, FM98, AFP00, BFM08] and references therein. While most of the literature focuses on the case where the discontinuity set K has finite $\mathcal{H}^{n-1}$-dimensional measure, we aim to extend the framework to settings in which $\mathcal{H}^{n-1}(K)$ may be infinite. This generalization requires new analytical techniques to establish lower semicontinuity and relaxation results under weaker regularity assumptions.

It is nowadays clear that the correct space to relax and study problems of type (1.1) is that of functions of bounded variation $BV(\Omega;\mathbb{R}^m)$ (or related spaces SBV, GBV, $GSBV$, see [AFP00]). More precisely, the weak formulation of problem (1.1) is

$$\int_{\Omega}\Psi(\nabla u)\mathrm{d}x+\int_{J_u} g([u],\nu_u)\mathrm{d}\mathcal{H}^{n-1}, \tag{1.2}$$

having replaced K by the jump set J_u of u and having labeled the corresponding normal vector ν_u.

In several models the energy density $\Psi:\mathbb{R}^{m\times n}\to[0,\infty)$ is assumed to have q-growth at infinity, $q>1$, while the surface energy density $g:\mathbb{R}^m\times S^{n-1}\to[0,\infty)$ is commonly chosen either such that

$$g(z,\nu)\geq c>0,$$

for some $c>0$ and for all $z\in\mathbb{R}^m$ and $\nu\in S^{n-1}$, or such that $g(0,\nu)=0$ and

$$\lim_{|z|\to 0}\frac{g(z,\nu)}{|z|}\in(0,\infty], \tag{1.3}$$

for all $\nu\in S^{n-1}$. While many classical models assume surface densities that remain bounded away from zero for small jumps, we consider *superlinear growth*, where the surface energy density converges to zero as the jump amplitude vanishes. This assumption reflects physical situations where fracture toughness increases continuously at small scales and leads to new mathematical challenges. Specifically, we will assume for all $\nu \in S^{n-1}$

$$\lim_{|z|\to 0} \frac{g(z,\nu)}{|z|} = \infty. \tag{1.4}$$

Under the latter assumptions the functional setting of the problem is naturally given by the space of Special Functions with Bounded Variation SBV introduced by De Giorgi and Ambrosio [DGA88] (see also [AFP00, Chapters 4-8]). Models of this type appear for instance in image processing [BDS00, ABG98, AG00, Mor03], in fracture mechanics [FCO14, ACO24], in screw dislocations in single crystal plasticity [DLSVG24], in the theory of smectic thin films [BCS23], and in the variational derivation of the Read and Shockley formula for the energy of small angle grain boundaries in polycrystals [LL17, FGS23]. These models include the case of jump sets with infinite length, but the analysis of the corresponding function space is still missing.

In view of the closure and compactness properties of the underlying space $(G)SBV$ [AFP00, Theorems 4.7, 4.8 and 4.36], the existence of minimizers for a functional of form (1.2) is known using the Direct Method of the Calculus of Variations in the isotropic case. This refers to the setting where the bulk energy density $\Psi(\xi)$ depends only on the modulus of the gradient, i.e., $\Psi(\xi) = \hat{\Psi}(|\xi|)$, and the surface energy density $g(z,\nu)$ depends only on the jump magnitude, i.e., $g(z,\nu) = \hat{g}(|z|)$. Under the assumption that both Ψ and g satisfy the aforementioned growth conditions, Ψ is convex, and g is concave, existence can be established. See below for a more detailed discussion of the relevant literature.

In this paper we develop the theory of the Direct Method for this class of problems, focusing in particular on the properties of the appropriate functions spaces, as well as on the lower semicontinuity and relaxation of energies of the form (1.2), with more general bulk and surface densities. We also construct a variational approximation via phase field models, which is particularly relevant for numerical approximation.

The lower semicontinuity property for anisotropic surface energies has been studied first in [AB90] in the framework of minimal partitions, i.e. SBV functions which take finitely many values, obtaining as necessary condition the BV-ellipticity of g (see (2.4) for the precise definition), see [AFP00, Theorem 5.14].

More generally, the lower semicontinuity property for anisotropic surface integrals with BV-elliptic, bounded and continuous densities g has been first established in [Amb90, Theorem 3.3] along sequences $(u_j)_j \subset SBV \cap L^\infty(\Omega;\mathbb{R}^m)$ such that

(g-a) $u_j \to u$ in measure on Ω, $u \in SBV \cap L^\infty(\Omega;\mathbb{R}^m)$, $\sup_j \|u_j\|_{L^\infty(\Omega)} < \infty$, and $(\nabla u_j)_j$ is equiintegrable in $L^1(A;\mathbb{R}^{m\times n})$ for every $A \subset\subset \Omega$;

(g-b) $\sup_j \mathcal{H}^{n-1}(J_{u_j}) < \infty$.

We note that the statement of [Amb90, Theorem 3.3] is formulated by requiring g to be bounded from below by a strictly positive constant, nevertheless the conclusion holds in the more general framework as stated above with the same proof.

In what follows we extend [Amb90, Theorem 3.3] to a large class of energies compatible with Ambrosio's SBV compactness theorem. More precisely, we consider surface integrands $g : \mathbb{R}^m \times S^{n-1} \to [0, \infty)$ such that

$$g \text{ is } BV\text{-elliptic,} \tag{1.5}$$

$$g(z, \nu) = g(-z, -\nu), \tag{1.6}$$

$$\frac{1}{c} g_0(|z|) \le g(z, \nu) \le c g_0(|z|), \tag{1.7}$$

$$|g(z, \nu) - g(z', \nu)| \le c g_0(|z - z'|), \tag{1.8}$$

for some $c > 0$ and for all $(z, \nu) \in \mathbb{R}^m \times S^{n-1}$, where $g_0 \in C^0([0, \infty); [0, \infty))$ is subadditive, nondecreasing, with $g_0(0) = 0$, and such that for some $\gamma \in (0, 1)$

$$\lim_{s \to 0} \frac{g_0(s)}{s^\gamma} = \ell \in (0, \infty) \tag{1.9}$$

(extensive comments on the assumption in (1.9) will be given after the proof of Proposition 2.1). In view of (1.8), g is necessarily continuous with g_0 as modulus of continuity.

Proposition 1.1. *Let g_0 and g satisfy* (1.5)-(1.9)*, and let $u_j \in (GSBV(\Omega))^m$ satisfy*

(g-a') $u_j \to u$ in measure, $u \in (GSBV(\Omega))^m$, and $(\nabla u_j)_j$ is equiintegrable in $L^1(A; \mathbb{R}^{m \times n})$ for every $A \subset\subset \Omega$.

Then

$$\int_{J_u} g([u], \nu_u) \mathrm{d}\mathcal{H}^{n-1} \le \liminf_{j \to \infty} \int_{J_{u_j}} g([u_j], \nu_{u_j}) \mathrm{d}\mathcal{H}^{n-1}. \tag{1.10}$$

Our proof follows the key idea introduced in [Amb90, Theorem 3.3], replacing the sequence u_j by a sequence of piecewise constant functions via the coarea formula. However, since in our setting the length of the jump sets may be infinite, the estimates are more delicate and rely on monotonicity, subadditivity, and γ-growth of g_0 at the origin. All approximation errors are explicitly quantified in terms of g_0.

Instead, the lower semicontinuity property for volume energies under quasiconvexity and q-growth conditions, $q > 1$, on the integrand Ψ has been established first by Ambrosio in [Amb94] (see also [AFP00, Theorem 5.29]) along sequences $(u_j)_j \subset SBV(\Omega; \mathbb{R}^m)$ such that

(Ψ-a) $u_j \to u$ in $L^1(\Omega; \mathbb{R}^m)$, $u \in SBV(\Omega; \mathbb{R}^m)$, and $\sup_j \|\nabla u_j\|_{L^q(\Omega)} < \infty$, $q \in (1, \infty)$;

(Ψ-b) $\sup_j \mathcal{H}^{n-1}(J_{u_j}) < \infty$.

The result has then been extended by Kristensen in [Kri99] under the assumptions:

(Ψ-a') the conditions in (Ψ-a) hold for an exponent in the larger range $q \in [1, \infty]$;

(Ψ-b') there is $g_0 \in C^0([0,\infty);[0,\infty))$ concave, nondecreasing, with $g_0(0) = 0$, and such that

$$\lim_{s\to 0} \frac{g_0(s)}{s} = \infty\,, \tag{1.11}$$

for which

$$\sup_j \int_{J_{u_j}} g_0(|[u_j]|)\mathrm{d}\mathcal{H}^{n-1} < \infty\,; \tag{1.12}$$

(Ψ-c) $\Psi \in C^0(\mathbb{R}^{m\times n};\mathbb{R})$, $(\max\{-\Psi(\nabla u_j), 0\})_j$ is equiintegrable.

Kristensen's proof hinges upon the fact that Young measures generated by sequences $(\nabla u_j)_j$ satisfying (Ψ-a') and (Ψ-b') are actually gradient q-Young measures for $\mathcal{L}^n$-almost every point. The use of Young measures allows one to treat the problem for a class of normal integrands depending on the lower order variables (x, u). We present here a more elementary proof of the result for autonomous functionals, which still relies on some fundamental ideas introduced in [Kri99]. Key ingredients include an approximation argument for nonnegative, superlinear quasiconvex functions by quasiconvex ones with linear growth (see Proposition 2.3 below), and an extension of the quasiconvexity inequality for quasiconvex functions with linear growth to BV test functions (see Equation (2.78) below). For the former, rather than relying on the refined theory of Young measures mentioned above, we employ a nowadays elementary truncation argument based on the maximal function (cf. Lemma 2.4).

Proposition 1.2. *Let $\Psi : \mathbb{R}^{m\times n} \to \mathbb{R}$ be a quasiconvex integrand satisfying*

$$|\Psi(\xi)| \le C(1 + |\xi|^q) \tag{1.13}$$

for some $q \in [1,\infty)$ and $C > 0$. Let $u_j \in (GSBV(\Omega))^m$ be satisfying

(Ψ-a") $u_j \to u$ in measure, $u \in (GSBV(\Omega))^m$, and $\sup_j \|\nabla u_j\|_{L^q(\Omega)} < \infty$;

(Ψ-b") (1.12) holds for some $g_0 \in C^0([0,\infty);[0,\infty))$ subadditive, nondecreasing, with $g_0(0) = 0$, and satisfying (1.11);

and (Ψ-c). Then,

$$\int_\Omega \Psi(\nabla u)\mathrm{d}x \le \liminf_{j\to\infty} \int_\Omega \Psi(\nabla u_j)\mathrm{d}x. \tag{1.14}$$

As a consequence of the superlinearity assumptions of the energy densities, we combine Propositions 1.1 and 1.2 to deduce a lower semicontinuity result for the full energy in (1.2) (cf. Theorem 2.5). A L^1 lower semicontinuity result for free-discontinuity energies along sequences satisfying (Ψ-a) and (g-b), i.e. with gradients equi-bounded in some L^q, $q > 1$, and with equi-bounded measure of the jump sets, has been previously established in [BFLM02, Theorem 4].

A key challenge in the analysis of free-discontinuity energies is their relaxation under minimal regularity assumptions. Theorem 1.3 provides a relaxation result for functionals of type (1.2) without requiring Ψ to be quasiconvex or g to be BV-elliptic. The proof makes an essential use of the density result obtained in [CFI25, Corollary 2.3].

Theorem 1.3. *Let $\Psi : \mathbb{R}^{m\times n} \to [0, \infty)$ be continuous and satisfy*

$$\left(\frac{1}{c}|\xi|^q - c\right) \vee 0 \le \Psi(\xi) \le c(|\xi|^q + 1), \tag{1.15}$$

for all $\xi \in \mathbb{R}^{m\times n}$ and some $q > 1$, and let g_0 and g satisfy (1.6)-(1.9)*. Let $H : L^1(\Omega; \mathbb{R}^m) \to [0, \infty]$ be*

$$H(u) := \begin{cases} \displaystyle\int_\Omega \Psi(\nabla u)\mathrm{d}x + \int_{J_u} g([u], \nu)\mathrm{d}\mathcal{H}^{n-1}, & \text{if } u \in SBV(\Omega; \mathbb{R}^m), \\ \infty, & \text{otherwise.} \end{cases} \tag{1.16}$$

Then, the relaxation $\overline{H}$ with respect to the strong topology of $L^1(\Omega; \mathbb{R}^m)$ is the functional

$$\overline{H}(u) = \int_\Omega \Psi^{\mathrm{qc}}(\nabla u)\mathrm{d}x + \int_{J_u} g_{BV}([u], \nu_u)\mathrm{d}\mathcal{H}^{n-1}, \tag{1.17}$$

if $u \in (GSBV(\Omega))^m$ with $\nabla u \in L^q(\Omega; \mathbb{R}^{m\times n})$, and $\overline{H}(u) = \infty$ otherwise.

For the definition of the quasiconvex envelope Ψ^{qc} of Ψ and of the BV-elliptic envelope g_{BV} of g we refer to (2.50) and (3.1), respectively. A similar statement holds by replacing the strong L^1-convergence by the convergence in measure. We leave the details to the interested reader.

A powerful approach to approximating free-discontinuity problems is through *phase field methods*, which replace sharp discontinuities with diffuse interfaces. This approach is widely used in numerical simulations of fracture mechanics and offers many advantages in computational implementation. In the final part of this paper, we establish a Γ-convergence result for a class of anisotropic phase field functionals of Ambrosio-Tortorelli type, thereby rigorously justifying their use as approximations to functionals of the form (1.2), where Ψ and g exhibit superlinear growth in the sense described above.

Phase field approximations are especially popular in the numerical literature on fracture mechanics, since functionals defined on Sobolev spaces are significantly more tractable from a computational standpoint. In some contexts, they are also interpreted as genuine diffuse models and used in place of the corresponding singular limit models. Various phase field models have been proposed and studied in the mechanics literature on cohesive-zone models for fracture; see, for example, [LG11, VdB13, Wu17, WNN+20, CdB21, LCM23, FL23, LCM25, ACF25a, ACF25b] and references therein. For a broader overview of the literature on this topic, we refer to the introductions of [CFI16, CFI24].

Our phase field models are generalizations to the vector-valued anisotropic case of those considered in [CFI16, Section 7.2]. For all $\varepsilon > 0$, p and $q > 1$, $\ell > 0$, we consider the functionals $\mathcal{F}_{\varepsilon,p,q} \colon L^1(\Omega;\mathbb{R}^{m+1}) \to [0,\infty]$ given by

$$\mathcal{F}_{\varepsilon,p,q}(u,v) := \int_\Omega \left(f^q_{\varepsilon,p,q}(v)\Psi(\nabla u) + \frac{(1-v)^{q'}}{q' q^{q'/q}\varepsilon} + \varepsilon^{q-1}|\nabla v|^q \right) \mathrm{d}x \tag{1.18}$$

if $(u,v) \in W^{1,q}(\Omega;\mathbb{R}^m \times [0,1])$ and ∞ otherwise, where $q'= q/(q-1)$ denotes the conjugate exponent of q, and for some $\ell \in (0,\infty)$ and for every $t \in [0,1)$

$$f_p(t) := \frac{\ell t}{(1-t)^p}, \qquad f_{\varepsilon,p,q}(t) := 1 \wedge \varepsilon^{1-1/q} f_p(t), \qquad f_{\varepsilon,p,q}(1) := 1\,.$$

For the precise set of assumptions on Ψ we refer to Section 4.1. For the sake of simplicity, in this introduction we only consider the case in which the q-recession function Ψ_∞ of Ψ satisfies the so-called projection property (see (4.1) for the definition of Ψ_∞), namely

$$\Psi_\infty(\xi) \geq \Psi_\infty(\xi\nu \otimes \nu) \quad \text{for every } (\xi,\nu) \in \mathbb{R}^{m\times n} \times S^{n-1}. \tag{1.19}$$

For example, consider for $n = m = 2$ and $\alpha > 0$ the functions

$$\begin{aligned} \psi^{(\alpha)}(\xi) &:= (|\xi|^2 - 2)^2_+ + \alpha(\det \xi - 1)^2, \\ \hat\psi^{(\alpha)}(\xi) &:= (|\xi|^2 - 2\det \xi)^2 + \alpha(\det \xi - 1)^2, \end{aligned} \tag{1.20}$$

where $(f)^2_+ := (f \vee 0)^2$. These functions are polyconvex, obey the growth condition (1.15) with $q = 4$, and are minimized for $\xi \in \mathrm{SO}(2)$. The $q-$recession functions are $\psi^{(\alpha)}_\infty(\xi) = |\xi|^4 + \alpha(\det\xi)^2$, and $\hat\psi^{(\alpha)}_\infty(\xi) = (|\xi|^2 - 2\det\xi)^2 + \alpha(\det\xi)^2$. The first one clearly obeys (1.19). However, considering $\nu = e_1$ and $\xi = \mathrm{diag}(1,t)$ for $t \to 0$ shows that $\hat\psi^{(\alpha)}_\infty$ does not obey (1.19).

Under such an assumption we obtain the following result.

Theorem 1.4. *Let Ψ be satisfying* (1.15)*,* (1.19)*, and* (4.2)*. Then for all $(u,v) \in L^1(\Omega;\mathbb{R}^{m+1})$ it holds*

$$\Gamma(L^1)\text{-}\lim_{\varepsilon\to 0} \mathcal{F}_{\varepsilon,p,q}(u,v) = \mathcal{F}_{p,q}(u,v),$$

where

$$\mathcal{F}_{p,q}(u,v) := \int_\Omega \Psi^{\mathrm{qc}}(\nabla u)\mathrm{d}x + \int_{J_u} g([u],\nu_u)\mathrm{d}\mathcal{H}^{n-1},$$

if $u \in (GSBV \cap L^1(\Omega))^m$ with $\nabla u \in L^q(\Omega;\mathbb{R}^{m\times n})$ and $v = 1$ $\mathcal{L}^n$-a.e. on Ω, and $\mathcal{F}_{p,q}(u,v) := \infty$ otherwise, where

$$g(z,\nu) := \lim_{T\uparrow\infty} \inf_{\mathcal{V}^T_z} \int_{-T/2}^{T/2} \Big(f^q_p(\beta(t))\Psi_\infty\big(\alpha'(t)\otimes\nu\big) + \frac{(1-\beta(t))^{q'}}{q' q^{q'/q}} + |\beta'(t)|^q \Big)\mathrm{d}t$$

and

$$\mathcal{V}_z^T := \{(\alpha,\beta) \in W^{1,q}((-{}^T\!/2, {}^T\!/2);\mathbb{R}^{m+1}) : \\ 0 \le \beta \le 1,\ \beta(\pm {}^T\!/2) = 1,\ \alpha(-{}^T\!/2) = 0,\ \alpha({}^T\!/2) = z\}.$$

More generally, discarding the assumption in (1.19) we provide a lower and an upper bound for the asymptotic analysis of the functionals $\mathcal{F}_{\varepsilon,p,q}$ which differ in principle only for what the surface energy densities are concerned (see discussion after (4.7) and (4.8)). We do not know whether the latters coincide or not in general, except under the assumption in (1.19) (cf. Theorem 4.1).

The proof of the Γ-liminf of Theorem 4.1 follows the same strategy of that used to treat the linear case [CFI24]. However, two different scales appear when one looks at the local behavior of a sequence $1 - v_\varepsilon \to 0$ for which the energy is bounded, $\varepsilon^{1/2} \ll \varepsilon^{1/2p}$, coming from the competition of the first two terms of the energy (this will become clear in Section 4 below). This prevents us to eliminate the truncation by 1 (by truncating $1 - v_\varepsilon$ itself by $\varepsilon^{1/2}$) when looking at the blow-up around a jump point of u. We are then led to define a surface energy density $g_{\inf}$ (cf. (4.7)) involving a truncation of the coefficient $\varepsilon^{q-1} f_p^q(v)$. Notice that in the linear case $p = 1$ the two scales match and the truncation by 1 can be neglected in the blow-up around a jump point of u.

It suffices to prove the Γ-limsup inequality for functions $u \in SBV \cap L^\infty(\Omega;\mathbb{R}^m)$ for which there exists a locally finite decomposition of $\mathbb{R}^n$ in simplexes, such that u is affine in the interior of each of them, thanks to the density result [CFI25, Corollary 2.3] and to the relaxation Theorem 1.3. The rest of the proof is an explicit construction obtained rescaling properly and gluing together the optimal profile of a surface energy density $g_{\sup}(z,\nu)$ (with no truncation involved, at variance with $g_{\inf}$, cf. (4.8)) with the affine function defining u in each simplex. In particular we show that $g_{\inf} = g_{\sup}$ in case the projection property in (1.19) holds.

Finally, the compactness of families $\mathcal{F}_{\varepsilon,p,q}(u_\varepsilon, v_\varepsilon)$ under a L^1 bound on u_ε and the convergence of minimizers of functionals of the form

$$\mathcal{F}_{\varepsilon,p,q}(u,v) + \int_\Omega \Big(\eta_\varepsilon \Psi(\nabla u) + |u - w|^r\Big) \mathrm{d}x,$$

where ${}^{\eta_\varepsilon}\!/_{\varepsilon^{q-1}} \to 0$ and $w \in L^r(\Omega;\mathbb{R}^m)$ with $r > 1$, can be addressed in a standard way thanks to Corollary 4.8, see for example [CFI25, Section 6].

The paper is organized as follows: in Section 2 we prove Propositions 1.1 and 1.2, and in Section 3 we prove Theorem 1.3, respectively. In Section 4 we introduce in details the phase field model and prove Theorem 1.4 (see Theorem 4.2, and the more general results in Propositions 4.11 and 4.12).

1.1 Notation

In the entire paper $\Omega \subset \mathbb{R}^n$ is bounded, open, Lipschitz, $\mathcal{A}(\Omega)$ denotes the family of open subsets of Ω, $\mathcal{B}(\Omega)$ denotes the family of Borel subsets of Ω, and $|\cdot|$ denotes the Euclidean norm, $|\xi|^2 := \sum_{ij} \xi_{ij}^2 = \mathrm{Tr}\left(\xi^T \xi\right)$ for $\xi \in \mathbb{R}^{m\times n}$.

We will set $Q_1 := (-\frac{1}{2}, \frac{1}{2})^n$ and we will denote by Q^ν, $\nu \in \mathcal{S}^{n-1}$, a unit cube with one face orthogonal to ν centered in the origin; for $t > 0$ we set $Q_t^\nu := tQ^\nu$.

We use standard notation for Sobolev and BV functions, referring to [AFP00] when needed.

We shall indicate with $\mathcal{F}_{\varepsilon,p,q}(u, v; A)$ the functional with integration restricted to A; if $A = \Omega$, the dependence on the set of integration will be dropped. This convention will be adopted for all the functionals that will be introduced in what follows.

Many arguments below are based on a truncation procedure. We recall briefly the definition of the truncation operator acting on vector-valued functions and its main properties. We fix a sequence $(a_k)_k \subset (0, \infty)$ such that $2a_k < a_{k+1}$, $a_k \uparrow \infty$, and such that there are functions $\mathcal{T}_k \in C_c^1(\mathbb{R}^m; \mathbb{R}^m)$ satisfying

$$\mathcal{T}_k(z) := \begin{cases} z, & \text{if } |z| \le a_k, \\ 0, & \text{if } |z| \ge a_{k+1} \end{cases} \tag{1.21}$$

and $\mathrm{Lip}\,(\mathcal{T}_k) \le 1$.

Let $u \in (GSBV(\Omega))^m$ obey $\nabla u \in L^1(\Omega; \mathbb{R}^{m\times n})$ and $\int_{J_u} g_0(|[u]|)\mathrm{d}\mathcal{H}^{n-1} < \infty$ with g_0 as in (1.9). Setting $u_k := \mathcal{T}_k(u) \in SBV(\Omega; \mathbb{R}^m)$, we have that u_k converges to u in measure, $\nabla u_k = \nabla u$ $\mathcal{L}^n$-a.e. in $\Omega_k := \{x \in \Omega : |u(x)| \le a_k\}$, $J_{u_k} \subseteq J_u$, $\nu_{u_k} = \nu_u$, $|[u_k]| \le |[u]|$ $\mathcal{H}^{n-1}$-a.e. in J_{u_k}, and $u_k^\pm(x) = u^\pm(x)$ $\mathcal{H}^{n-1}$-a.e. in $J_{u_k} \cap \Omega_k$. Since

$$\mathcal{H}^{n-1}(\{x \in J_u : |u^\pm(x)| = \infty\}) = 0 \tag{1.22}$$

(see [AF02, Proposition 2.12, Remark 2.13]), we get $\chi_{J_{u_k}} \to \chi_{J_u}$ and $u_k^\pm \to u^\pm$ $\mathcal{H}^{n-1}$-a.e. in J_u.

Following De Giorgi's averaging/slicing procedure on the codomain, the family $\mathcal{T}_k$ will be used in several instances along the paper to obtain from a sequence converging in measure to a limit belonging to L^∞, a sequence with the same L^1 limit which is in addition equi-bounded in L^∞. Moreover, this substitution can be done up to paying an error in energy which can be made arbitrarily small.

2 Lower semicontinuity

The aim of this section is to prove a lower semicontinuity result for free-discontinuity energies with autonomous bulk densities superlinear for large gradients, and translation independent surface densities superlinear for small amplitudes of the jump.

More precisely, define H_g, $H_\Psi : L^1(\Omega; \mathbb{R}^m) \times \mathcal{B}(\Omega) \to [0, \infty]$ by

$$H_g(u; A) := \int_{A\cap J_u} g([u], \nu_u)\mathrm{d}\mathcal{H}^{n-1} \tag{2.1}$$

and

$$H_\Psi(u; A) := \int_A \Psi(\nabla u)\mathrm{d}x \tag{2.2}$$

for $u \in (GSBV(\Omega))^m$, and ∞ otherwise in L^1. In case $A = \Omega$ we drop the dependence on the set in H_g and H_Ψ.

Our aim is to prove the following: If $u_j \to u$ strongly in L^1, u_j, $u \in (GSBV(\Omega))^m$, g and Ψ superlinear and regular enough (see below for the precise hypotheses), then

$$H_g(u) + H_\Psi(u) \leq \liminf_{j\to\infty}(H_g(u_j) + H_\Psi(u_j)). \tag{2.3}$$

Necessary conditions for L^1-lower semicontinuity require g to be BV-elliptic and Ψ to be quasiconvex (cf. [AFP00, Theorem 5.14 and Theorem 5.26]). We record the two definitions in what follows for the sake of convenience.

A Borel-measurable function $g : \mathbb{R}^m \times S^{n-1} \to [0,\infty)$ is said to be BV-elliptic (see [AFP00, Definition 5.13]) if for every $z \in \mathbb{R}^m$, $\nu \in S^{n-1}$, and every $u \in SBV(Q^\nu;\mathbb{R}^m)$ piecewise constant, with $\{u \neq z\chi_{\{x\cdot\nu>0\}}\} \subset\subset Q^\nu$ one has

$$g(z,\nu) \leq \int_{Q^\nu\cap J_u} g([u],\nu_u)\mathrm{d}\mathcal{H}^{n-1}. \tag{2.4}$$

We recall that a function $u \in SBV(\Omega;\mathbb{R}^m)$ is called piecewise constant if it is constant on each element of a Caccioppoli partition of Ω, in the sense that there are countably many sets of finite perimeter E_j such that $\sum_j \mathcal{H}^{n-1}(\Omega \cap \partial^* E_j) < \infty$, $\mathcal{L}^n(\Omega \setminus \cup_j E_j) = 0$, and u is constant on each of them. Equivalently, $\mathcal{H}^{n-1}(J_u) < \infty$ and $\nabla u = 0$ $\mathcal{L}^n$-a.e., see [AFP00, Theorem 4.23]. We shall show below (see Lemma 3.1) that there are equivalent variants of the definition of BV-ellipticity, using classes of test functions which are either smaller (costant on finitely many polyhedra) or larger (SBV with $\nabla u = 0$ $\mathcal{L}^n$-a.e.).

A locally bounded Borel-measurable function $\Psi : \mathbb{R}^{m\times n} \to [0,\infty)$ is said to be quasiconvex (see [AFP00, Definition 5.25]) if for every $\xi \in \mathbb{R}^{m\times n}$, and every $\varphi \in C_c^\infty(Q_1;\mathbb{R}^m)$ one has

$$\Psi(\xi) \leq \int_{Q_1} \Psi(\xi + \nabla\varphi)\mathrm{d}x. \tag{2.5}$$

In case of superlinear growth for g and Ψ the problem decouples, and one can prove two corresponding separate lower semicontinuity inequalities (cp. [AFP00, Section 5.4]). Therefore, in the next two sections we will address separately semicontinuity for surface integrals and for bulk integrals, respectively, specifying explicitely the precise set of assumptions on the corresponding energy densities, and finally deduce semicontinuity for free-discontinuity energies in Theorem 2.5 in the end of the section.

2.1 Lower semicontinuity for surface energies

In this section we prove Proposition 1.1. To this aim we first modify to our setting an approximation argument with piecewise constant functions introduced in [Amb90, Theorem 3.3] for the case $\mathcal{H}^{n-1}(J_u) < \infty$. For the ensuing result to hold, g does not need to be BV-elliptic, but the quantitative superlinear growth in (1.9) is important to treat the part where the amplitude of the jump is small.

Proposition 2.1. *Assume that* g_0 *and* g *are as in* (1.6)-(1.9). *Let* $u \in SBV(\Omega;\mathbb{R}^m)$, $\varepsilon > 0$, $\delta > 4\varepsilon$, *and* $\eta \in (0,\varepsilon)$. *Then there is* $u_\varepsilon \in SBV(\Omega;\mathbb{R}^m)$ *piecewise constant such that* $\|u-u_\varepsilon\|_{L^\infty(\Omega)} \le \varepsilon$, $|Du_\varepsilon|(\Omega) \le \sqrt{m}|Du|(\Omega)$,

$$\begin{aligned} H_g(u_\varepsilon) \le &\left(1 + C\left(\frac{\varepsilon}{\delta}\right)^\gamma + C\left(\frac{\eta}{\varepsilon}\right)^{1-\gamma}\right) H_g(u) \\ &+ C\left(\frac{\varepsilon}{\eta}\right)^\gamma H_g(u,\{|[u]| \in [\eta,\delta)\}) + C\varepsilon^{\gamma-1}\|\nabla u\|_{L^1(\Omega)}. \end{aligned} \tag{2.6}$$

Proof. **Step 1. Construction.** The construction is done componentwise. Assume for a moment that $m=1$. For $\varepsilon > 0$ and $\rho \in [0,1)$ we set

$$u_{\varepsilon,\rho}(x) := \varepsilon\left\lfloor \frac{u(x)}{\varepsilon} + \rho \right\rfloor = \sum_{k\in\mathbb{N}\setminus\{0\}} \varepsilon\chi_{\{u\ge(k-\rho)\varepsilon\}} - \sum_{k\in\mathbb{N}} \varepsilon\chi_{\{u<(-k-\rho)\varepsilon\}} \tag{2.7}$$

where we used that for $t \in \mathbb{R}$

$$\lfloor t \rfloor := \max\{k\in\mathbb{Z} : k \le t\} = \#\{k \ge 1 : k \le t\} - \#\{k \le 0 : t < k\}.$$

For $\mathcal{L}^1$-almost all $\rho \in (0,1)$, all functions $\chi_{\{u\ge(k-\rho)\varepsilon\}}$ and $\chi_{\{u<(-k-\rho)\varepsilon\}}$ are in $BV(\Omega)$, with pure-jump distributional gradient. Therefore, for $\mathcal{L}^1$-almost all ρ we have $u_{\varepsilon,\rho} \in GBV(\Omega)$ with $\nabla u_{\varepsilon,\rho} = 0$ $\mathcal{L}^n$-a.e. on Ω. It is also clear that $|u_{\varepsilon,\rho} - u| < \varepsilon$ pointwise.

We estimate, using $D\chi_{\{u<(-k-\rho)\varepsilon\}} = -D\chi_{\{u\ge(-k-\rho)\varepsilon\}}$,

$$|Du_{\varepsilon,\rho}|(\Omega) \le \varepsilon \sum_{k\in\mathbb{Z}} |D\chi_{\{u\ge(k-\rho)\varepsilon\}}|(\Omega) \tag{2.8}$$

so that, averaging over ρ,

$$\begin{aligned} \int_{[0,1)} |Du_{\varepsilon,\rho}|(\Omega)\mathrm{d}\rho &\le \varepsilon \sum_{k\in\mathbb{Z}} \int_{[0,1)} |D\chi_{\{u\ge(k-\rho)\varepsilon\}}|(\Omega)\mathrm{d}\rho \\ &= \int_{\mathbb{R}} |D\chi_{\{u\ge t\}}|(\Omega)\mathrm{d}t = |Du|(\Omega). \end{aligned}$$

We can therefore pick $\rho \in [0,1)$ such that $|Du_{\varepsilon,\rho}|(\Omega) \le |Du|(\Omega)$.

We repeat this for each component, using $\varepsilon/\sqrt{m}$ in place of ε, and obtain $\rho \in [0,1)^m$ such that the resulting function u_ε obeys

$$u_\varepsilon \in BV(\Omega;\mathbb{R}^m), \quad \nabla u_\varepsilon = 0 \ \mathcal{L}^n\text{-a.e. on } \Omega, \quad \|u - u_\varepsilon\|_{L^\infty(\Omega)} \le \varepsilon, \tag{2.9}$$

and such that each component $i \in \{1,\dots,m\}$ obeys

$$|Du_\varepsilon^i|(\Omega) \le |Du^i|(\Omega) \tag{2.10}$$

which in particular implies

$$|Du_\varepsilon|(\Omega) \le \sqrt{m}|Du|(\Omega). \tag{2.11}$$

Step 2. Estimates. From the pointwise bound in (2.9) we obtain

$$|[u]-[u_\varepsilon]| \le 2\varepsilon \qquad \mathcal{H}^{n-1}\text{-a.e. on } J_u \cup J_{u_\varepsilon}, \tag{2.12}$$

where as usual we set $[u] = 0$ on $\Omega \setminus J_u$, and the same for u_ε. Necessarily, $\nu_u = \pm\nu_{u_\varepsilon}$ $\mathcal{H}^{n-1}$-a.e. on the set where at least one of the jumps in (2.12) has modulus larger than 2ε. For $s > 0$ we define

$$A_s := \{x \in J_u : |[u](x)| \ge s\}. \tag{2.13}$$

In the following we consider separately the four sets

$$A_\delta, \quad A_\eta \setminus A_\delta, \quad J_u \setminus A_\eta, \quad \text{and} \quad \Omega \setminus J_u. \tag{2.14}$$

As $\delta > 2\varepsilon$, by (2.12) one has $\mathcal{H}^{n-1}(A_\delta \setminus J_{u_\varepsilon}) = 0$. At the same time, by monotonicity of g_0 and $g_0(|z|) \le Cg(z,\nu)$ we have

$$\mathcal{H}^{n-1}(A_\delta)g_0(\delta) \le CH_g(u). \tag{2.15}$$

Therefore, again by monotonicity and subadditivity of g_0

$$\int_{A_\delta} g_0(|[u]-[u_\varepsilon]|)\mathrm{d}\mathcal{H}^{n-1} \le g_0(2\varepsilon)\mathcal{H}^{n-1}(A_\delta) \le C\frac{g_0(\varepsilon)}{g_0(\delta)}H_g(u) \tag{2.16}$$

and analogously

$$|Du - Du_\varepsilon|(A_\delta) \le 2\varepsilon\mathcal{H}^{n-1}(A_\delta) \le C\frac{\varepsilon}{g_0(\delta)}H_g(u). \tag{2.17}$$

Here and in what follows we denote by $C > 0$ a constant independent from ε, δ, η and u, which may vary from line to line.

We next turn to the second set. Since g_0 is nondecreasing, similarly

$$\int_{A_\eta \setminus A_\delta} g_0(|[u]-[u_\varepsilon]|)\mathrm{d}\mathcal{H}^{n-1} \le g_0(2\varepsilon)\mathcal{H}^{n-1}(A_\eta \setminus A_\delta) \le C\frac{g_0(\varepsilon)}{g_0(\eta)}H_g(u; A_\eta \setminus A_\delta). \tag{2.18}$$

Finally, on the rest $|[u]| \le \eta < \varepsilon$, and therefore $|[u_\varepsilon]| \le 3\varepsilon$. We have, using that $|[u_\varepsilon]|$ is either 0 or at least $\varepsilon/\sqrt{m}$,

$$\int_{\Omega \setminus A_\eta} g_0(|[u_\varepsilon]|)\mathrm{d}\mathcal{H}^{n-1} \le g_0(3\varepsilon)\mathcal{H}^{n-1}(J_{u_\varepsilon} \setminus A_\eta) \le C\frac{g_0(\varepsilon)}{\varepsilon}|Du_\varepsilon|(\Omega \setminus A_\eta). \tag{2.19}$$

The last term is critical, as the first factor diverges, and thus the second factor requires a careful estimate. To this aim using (2.10) and (2.12) we estimate as follows for any component $i \in \{1, \dots, m\}$:

$$\begin{aligned} |Du_\varepsilon^i|(\Omega \setminus A_\eta) &= |Du_\varepsilon^i|(\Omega) - |Du_\varepsilon^i|(A_\eta) \\ &\le |Du^i|(\Omega) - |Du^i|(A_\eta) + |Du_\varepsilon^i - Du^i|(A_\eta) \\ &\le \|\nabla u^i\|_{L^1(\Omega)} + |Du^i|(J_u) - |Du^i|(A_\eta) + 2\varepsilon\mathcal{H}^{n-1}(A_\eta) \\ &= \|\nabla u^i\|_{L^1(\Omega)} + |Du^i|(J_u \setminus A_\eta) + 2\varepsilon\left[\mathcal{H}^{n-1}(A_\delta) + \mathcal{H}^{n-1}(A_\eta \setminus A_\delta)\right]. \end{aligned} \tag{2.20}$$

Summing over i leads to

$$\begin{aligned}|Du_\varepsilon|(\Omega\setminus A_\eta) \le & m\|\nabla u\|_{L^1(\Omega)} + m|Du|(J_u\setminus A_\eta)\\ & + 2\varepsilon m\left[\mathcal{H}^{n-1}(A_\delta)+\mathcal{H}^{n-1}(A_\eta\setminus A_\delta)\right].\end{aligned} \tag{2.21}$$

In order to estimate the contribution in $J_u\setminus A_\eta$ we define $\psi(s) := \min_{t\in(0,s]} g_0(t)/t$, and observe that subadditivity of g_0 implies that

$$\frac{g_0(2t)}{2t} \le \frac{2g_0(t)}{2t} = \frac{g_0(t)}{t} \qquad \text{for all } t\in(0,s]. \tag{2.22}$$

Therefore $\psi(2s) \ge \min_{t\in[s,2s]} g_0(t)/t \ge g_0(s)/(2s)$. In $J_u\setminus A_\eta$ we have $0 < |[u]| < \eta$, so that by subadditivity of g_0

$$\begin{aligned}|Du|(J_u\setminus A_\eta) = & \int_{J_u\setminus A_\eta} \frac{|[u]|}{g_0(|[u]|)} g_0(|[u]|)\mathrm{d}\mathcal{H}^{n-1}\\ \le & \frac{1}{\psi(2\eta)}\int_{J_u\setminus A_\eta} g_0(|[u]|)\mathrm{d}\mathcal{H}^{n-1} \le C\frac{\eta}{g_0(\eta)} H_g(u;\Omega\setminus A_\eta).\end{aligned} \tag{2.23}$$

At the same time, $|[u]| \ge \eta$ in A_η, and therefore being g_0 nondecreasing

$$\mathcal{H}^{n-1}(A_\eta\setminus A_\delta) \le \frac{1}{g_0(\eta)}\int_{A_\eta\setminus A_\delta} g_0(|[u]|)\mathrm{d}\mathcal{H}^{n-1} \le \frac{C}{g_0(\eta)} H_g(u, A_\eta\setminus A_\delta). \tag{2.24}$$

Finally, A_δ had been estimated in (2.15). Inserting the estimates (2.23), (2.24) and (2.15) in (2.21), the last term in (2.19) can be estimated by

$$\begin{aligned}&\frac{g_0(\varepsilon)}{\varepsilon}|Du_\varepsilon|(\Omega\setminus A_\eta)\\ &\quad\le C\frac{g_0(\varepsilon)}{\varepsilon}\left[\|\nabla u\|_{L^1(\Omega)} + |Du|(J_u\setminus A_\eta) + 2\varepsilon\mathcal{H}^{n-1}(A_\delta) + 2\varepsilon\mathcal{H}^{n-1}(A_\eta\setminus A_\delta)\right]\\ &\quad\le C\frac{g_0(\varepsilon)}{\varepsilon}\|\nabla u\|_{L^1(\Omega)} + C\frac{g_0(\varepsilon)}{\varepsilon}\frac{\eta}{g_0(\eta)} H_g(u;\Omega\setminus A_\eta)\\ &\qquad + C\frac{g_0(\varepsilon)}{g_0(\delta)} H_g(u) + C\frac{g_0(\varepsilon)}{g_0(\eta)} H_g(u; A_\eta\setminus A_\delta)\\ &\quad\le C\varepsilon^{\gamma-1}\|\nabla u\|_{L^1(\Omega)} + C\left(\frac{\eta}{\varepsilon}\right)^{1-\gamma} H_g(u;\Omega\setminus A_\eta)\\ &\qquad + C\left(\frac{\varepsilon}{\delta}\right)^{\gamma} H_g(u) + C\left(\frac{\varepsilon}{\eta}\right)^{\gamma} H_g(u; A_\eta\setminus A_\delta),\end{aligned} \tag{2.25}$$

where in the last line we used $g_0(s)\sim s^\gamma$ as $s\to 0$ (cf. (1.9)). Collecting the estimates in (2.16), (2.18), (2.19), and (2.25) and recalling that $A_\delta \subseteq A_\eta$ we finally get using (1.8),

$$\begin{aligned}H_g(u_\varepsilon) \le& H_g(u) + c\int_{A_\eta} g_0(|[u]-[u_\varepsilon]|)\mathrm{d}\mathcal{H}^{n-1} + c\int_{\Omega\setminus A_\eta} g_0(|[u_\varepsilon]|)\mathrm{d}\mathcal{H}^{n-1}\\ \le& \left[1 + C\left(\frac{\varepsilon}{\delta}\right)^\gamma + C\left(\frac{\eta}{\varepsilon}\right)^{1-\gamma}\right] H_g(u)\\ &+ C\varepsilon^{\gamma-1}\|\nabla u\|_{L^1(\Omega)} + C\left(\frac{\varepsilon}{\eta}\right)^\gamma H_g(u; A_\eta\setminus A_\delta).\end{aligned} \tag{2.26}$$

The conclusion in (2.6) then follows at once. □

We stress that the specific superlinear behaviour of power type of g_0 at 0 (cf. (1.9)) has been used only in the last inequality of (2.25), and it is exploited in what follows to control the prefactor $\frac{g_0(\varepsilon)}{\varepsilon}\frac{\eta}{g_0(\eta)}$ in the third line of (2.25).

Indeed, we will apply Proposition 2.1 in Step 2 of the proof of Proposition 1.1 below to a sequence u_ε with $\|\nabla u_\varepsilon\|_{L^1}$ vanishing, with the choice $\varepsilon = M_\varepsilon\eta_\varepsilon$, M_ε diverging. On the one hand, choosing ε suitably with respect to $\|\nabla u_\varepsilon\|_{L^1}$, the first summand in (2.25) vanishes. The third summand is no problem. Moreover, the diverging prefactor $\frac{g_0(\varepsilon)}{g_0(\eta)}$ in the fourth summand can be compensated by selecting appropriately the slice $A_{\eta_\varepsilon}\setminus A_{\delta_\varepsilon}$. On the other hand, the subadditivity of g_0 implies only that $\frac{g_0(\varepsilon)}{\varepsilon}\frac{\eta_\varepsilon}{g_0(\eta_\varepsilon)} = \frac{g_0(M_\varepsilon\eta_\varepsilon)}{M_\varepsilon g_0(\eta_\varepsilon)}$ is bounded, so that the second summand becomes a nonvanishing error term in the estimate. The power-type behaviour of g_0 in the origin is needed exactly to render infinitesimal the latter term. Of course, more general behaviours are allowed for g_0.

Note that in case $g_0(t)\sim t|\ln t|$ the previous subadditivity estimate is sharp, and thus with Proposition 1.1 below one cannot establish the lower semicontinuity of energies having densities with almost linear growth. For instance with our method we cannot cope with the Read and Shockley model for low-angle grain boundaries in polycrystals. If one assumes isotropy, however, also in this situation lower semicontinuity follows from either a slicing argument or the biconvexity of the integrand (see [Amb89, Theorem 2.1], [Amb90, Theorem 3.7], see also for another proof [AFP00, Theorem 4.7]).

By truncation we can obtain a similar approximation result for functions in $GSBV$.

Corollary 2.2. *Assume that g_0 and g are as in* (1.6)-(1.9). *Let $u\in(GSBV(\Omega))^m$ with $|\nabla u|\in L^1(\Omega)$, $\varepsilon>0$, $\delta>4\varepsilon$, and $\eta\in(0,\varepsilon)$. Then there is $u_\varepsilon\in SBV(\Omega;\mathbb{R}^m)$ which takes (away from a $\mathcal{L}^n$-null set) finitely many values with*

$$\mathcal{L}^n(\{|u-u_\varepsilon|>\varepsilon\}) < \varepsilon \tag{2.27}$$

and

$$\begin{aligned}H_g(u_\varepsilon) \le& C\varepsilon + \left(1 + C\left(\frac{\varepsilon}{\delta}\right)^\gamma + C\left(\frac{\eta}{\varepsilon}\right)^{1-\gamma}\right) H_g(u)\\ &+ C\left(\frac{\varepsilon}{\eta}\right)^\gamma H_g(u;\{|[u]|\in[\eta,\delta)\}) + C\varepsilon^{\gamma-1}\|\nabla u\|_{L^1(\Omega)}.\end{aligned} \tag{2.28}$$

If $u \in L^1(\Omega;\mathbb{R}^m)$, *then* $\|u_\varepsilon - u\|_{L^1(\Omega)} \le C\varepsilon$.

The fact that the set $u_\varepsilon(\Omega\setminus N)$, for a $\mathcal{L}^n$-null set N, is finite, implies that $\nabla u_\varepsilon = 0$ $\mathcal{L}^n$-a.e. on Ω and that $u_\varepsilon \in L^\infty(\Omega;\mathbb{R}^m)$.

Proof. It suffices to reduce to a function in L^∞ and then to apply Proposition 2.1. We can assume $H_g(u) < \infty$. Then, by (1.22)

$$\lim_{M\to\infty}\int_{\{|u^+|\ge M\}\cap J_u} g([u],\nu_u)\mathrm{d}\mathcal{H}^{n-1} = \lim_{M\to\infty}\int_{\{|u^-|\ge M\}\cap J_u} g([u],\nu_u)\mathrm{d}\mathcal{H}^{n-1} = 0. \tag{2.29}$$

Recalling the definition of the sequence a_k before (1.21), there is k such that

$$\mathcal{L}^n(\{|u| > a_k\}) \le \varepsilon \tag{2.30}$$

and

$$\int_{A_k} g([u],\nu_u)\mathrm{d}\mathcal{H}^{n-1} \le \eta^\gamma\varepsilon^{1-\gamma}, \tag{2.31}$$

where $A_k := (\{|u^+| > a_k\}\cup\{|u^-| > a_k\})\cap J_u$. If $u \in L^1(\Omega;\mathbb{R}^m)$, we can additionally have $\int_{\{|u|>a_k\}}|u|\mathrm{d}x < \varepsilon$.

Let $\hat{u} := \mathcal{T}_k(u)$. Then $\hat{u} \in L^\infty \cap SBV(\Omega;\mathbb{R}^m)$ and $\|\nabla\hat{u}\|_{L^1(\Omega)} \le \|\nabla u\|_{L^1(\Omega)}$. We estimate, using that (up to $\mathcal{H}^{n-1}$-null sets) $J_{\hat{u}} \subseteq J_u$, $|[\hat{u}]| \le |[u]|$ on A_k, (1.7), the monotonicity of g_0, and finally (2.31),

$$\begin{aligned} H_g(\hat{u};A_k) =& \int_{A_k} g([\hat{u}],\nu_{\hat{u}})\mathrm{d}\mathcal{H}^{n-1} \le C\int_{A_k} g_0(|[\hat{u}]|)\mathrm{d}\mathcal{H}^{n-1} \\ \le& C\int_{A_k} g_0(|[u]|)\mathrm{d}\mathcal{H}^{n-1} \le C\eta^\gamma\varepsilon^{1-\gamma}\,. \end{aligned} \tag{2.32}$$

Recalling that $\eta \le \varepsilon$ and $[\hat{u}] = [u]$ on $J_{\hat{u}}\setminus A_k$,

$$H_g(\hat{u}) \le H_g(u) + H_g(\hat{u};A_k) \le H_g(u) + C\varepsilon \tag{2.33}$$

and

$$\begin{aligned} H_g(\hat{u};\{|[\hat{u}]|\in[\eta,\delta)\}) \le& H_g(u;\{|[u]|\in[\eta,\delta)\}) + H_g(\hat{u};A_k) \\ \le& H_g(u;\{|[u]|\in[\eta,\delta)\}) + C\eta^\gamma\varepsilon^{1-\gamma}\,. \end{aligned} \tag{2.34}$$

The estimates (2.27) and (2.28) follow using Proposition 2.1 for $\hat{u}$ and recalling that $\eta \le \varepsilon$. Indeed, by $\|\hat{u} - u_\varepsilon\|_{L^\infty(\Omega)} \le \varepsilon$ and (2.30) we get

$$\mathcal{L}^n(\{|u - u_\varepsilon| > \varepsilon\}) \le \mathcal{L}^n(\{|\hat{u} - u| > 0\}) \le \mathcal{L}^n(\{|u| > a_k\}) \le \varepsilon.$$

The L^1 estimate follows from

$$\begin{aligned} \|u - u_\varepsilon\|_{L^1(\Omega)} \le& \|\hat{u} - u_\varepsilon\|_{L^1(\Omega)} + \|u - \hat{u}\|_{L^1(\Omega)} \\ \le& \mathcal{L}^n(\Omega)\varepsilon + \int_{\{|u|>a_k\}}(|u| + |\mathcal{T}_k(u)|)\mathrm{d}x \\ \le& \mathcal{L}^n(\Omega)\varepsilon + 2\int_{\{|u|>a_k\}}|u|\mathrm{d}x \le C\varepsilon. \end{aligned}$$

□

We are now ready to establish the claimed lower semicontinuity property for surface integrands.

Proof of Proposition 1.1. We divide the proof in several steps.

Step 1. Blow-up. Without loss of generality we can assume that $(H_g(u_j))_j$ is bounded and that the measures

$$\mu_g^j := g([u_j], \nu_j)\mathcal{H}^{n-1} \llcorner J_{u_j}$$

converge weak-$*$ in $\mathcal{M}(\Omega)$ to some positive finite Radon measure μ_g. We shall show that $H_g(u) \le \mu_g(\Omega)$, and for this it suffices to show that

$$g([u](x_0), \nu_u(x_0)) \le \frac{\mathrm{d}\mu_g}{\mathrm{d}\mathcal{H}^{n-1} \llcorner J_u}(x_0),$$

for $\mathcal{H}^{n-1}$-almost every $x_0 \in J_u$. By [AFP00, Theorem 4.34] it suffices to prove the last inequality for points $x_0 \in J_u$ such that

$$\frac{\mathrm{d}\mu_g}{\mathrm{d}\mathcal{H}^{n-1} \llcorner J_u}(x_0) = \lim_{\rho\to 0} \frac{\mu_g(Q_\rho^\nu(x_0))}{\mathcal{H}^{n-1}(J_u \cap Q_\rho^\nu(x_0))} \quad \text{exists finite}$$

and

$$\lim_{\rho\to 0} \frac{\mathcal{H}^{n-1}(J_u \cap Q_\rho^\nu(x_0))}{\rho^{n-1}} = 1,$$

where $\nu := \nu_u(x_0)$ and $Q_\rho^\nu(x_0) := x_0 + \rho Q^\nu$ is the cube centred in x_0, with side length ρ, and one face orthogonal to ν. We remark that such conditions define a set of full measure in J_u. First note that

$$\frac{\mathrm{d}\mu_g}{\mathrm{d}\mathcal{H}^{n-1} \llcorner J_u}(x_0) = \lim_{\rho\to 0} \frac{\mu_g(Q_\rho^\nu(x_0))}{\rho^{n-1}} = \lim_{\substack{\rho\in I\\ \rho\to 0}} \lim_{j\to\infty} \frac{\mu_g^j(Q_\rho^\nu(x_0))}{\rho^{n-1}},$$

where we have used that $\mu_g^j \rightharpoonup \mu$ w^*-$\mathcal{M}(\Omega)$ and we have set $I := \{\rho \in (0, \frac{2}{\sqrt{n}}\operatorname{dist}(x_0, \partial\Omega)) : \mu_g(\partial Q_\rho^\nu(x_0)) = 0\}$. Therefore, a change of variables yields that

$$\frac{\mathrm{d}\mu_g}{\mathrm{d}\mathcal{H}^{n-1} \llcorner J_u}(x_0) = \lim_{\substack{\rho\in I\\ \rho\to 0}} \lim_{j\to\infty} H_g(u_j^\rho; Q^\nu), \tag{2.35}$$

where we have denoted by $u_j^\rho(y) := u_j(x_0 + \rho y)$ the rescaling of u_j.

Since $|\nabla u_j|$ is equiintegrable, and therefore in particular bounded in L^1, by the Dunford-Pettis Theorem [AFP00, Theorem 1.38] it is weakly pre-compact in $L^1(\Omega)$, hence there exists $h \in L^1(\Omega)$ such that a subsequence satisfies $|\nabla u_j| \rightharpoonup h$ w-$L^1(\Omega)$ as $j \to \infty$. Hence,

$$\lim_{\rho\to 0} \lim_{j\to\infty} \frac{1}{\rho^{n-1}} \int_{Q_\rho^\nu(x_0)} |\nabla u_j| \mathrm{d}x = \lim_{\rho\to 0} \frac{1}{\rho^{n-1}} \int_{Q_\rho^\nu(x_0)} h \, \mathrm{d}x = 0, \tag{2.36}$$

for $\mathcal{H}^{n-1}$-almost every $x_0 \in J_u$ by [AFP00, Theorem 2.56 and equation (2.41)]. Note that by scaling one has

$$\lim_{\rho\to 0}\lim_{j\to\infty}\frac{1}{\rho^{n-1}}\int_{Q^\nu_\rho(x_0)}|\nabla u_j|\mathrm{d}x = \lim_{\rho\to 0}\lim_{j\to\infty}\int_{Q^\nu}|\nabla u^\rho_j|\mathrm{d}y\,. \tag{2.37}$$

Recalling that $u_j \to u$ in measure as $j\to\infty$, using (2.35), (2.36), and (2.37) we can find by diagonalization two subsequences $\{\rho_k\}_k$ infinitesimal and $\{j_k\}_k$ diverging such that as $k\to\infty$

$$\begin{aligned} u^{\rho_k}_{j_k} \to [u](x_0)\chi_{\{y\cdot\nu>0\}} + u^-(x_0) \quad \text{in measure,}\\ \nabla u^{\rho_k}_{j_k} \to 0 \quad \text{in} \quad L^1(Q^\nu;\mathbb{R}^{m\times n}),\\ \frac{\mathrm{d}\mu_g}{\mathrm{d}\mathcal{H}^{n-1}\llcorner J_u}(x_0) = \lim_{k\to\infty} H_g(u^{\rho_k}_{j_k};Q^\nu)\,. \end{aligned}$$

Step 2. Reduction to a piecewise constant sequence. By Step 1 it is enough to prove that for $z\in\mathbb{R}^m$ and $\nu\in S^{n-1}$ we have

$$g(z,\nu) \le \liminf_{j\to\infty} H_g(u_j;Q^\nu),$$

whenever $u_j \to u_0 := z\chi_{\{x\cdot\nu>0\}}$ in measure and $\nabla u_j \to 0$ in $L^1(Q^\nu;\mathbb{R}^{m\times n})$, as $j\to\infty$. We can also assume that the sequence $H_g(u_j;Q^\nu)$ is bounded.

Consider any sequence $r_j\to 0$ such that $r_j^{\gamma-1}\|\nabla u_j\|_{L^1(Q^\nu)}\to 0$, and then $M_j\to\infty$, $K_j\to\infty$ such that $r_jM_j^{2K_j+1}\to 0$ and $M_j^\gamma K_j^{-1}\to 0$. Select by averaging $k_j\in\{1,\dots,K_j\}$ such that

$$H_g(u_j;\{|[u_j]|\in[r_jM_j^{2k_j-1}, r_jM_j^{2k_j+1})\}) \le \frac{2}{K_j}H_g(u_j;Q^\nu). \tag{2.38}$$

By Corollary 2.2, applied for all $j\in\mathbb{N}$ with $\varepsilon_j := r_jM_j^{2k_j}$, $\eta_j := \varepsilon_j/M_j \to 0$, $\delta_j := \varepsilon_jM_j\to 0$, we find $v_j\in SBV(Q^\nu;\mathbb{R}^m)$ such that $\nabla v_j = 0$ $\mathcal{L}^n$-a.e. on Q^ν taking finitely many values away from a $\mathcal{L}^n$-null set,

$$\mathcal{L}^n(\{|u_j-v_j|>\varepsilon_j\}) \le \varepsilon_j, \tag{2.39}$$

$$\begin{aligned} H_g(v_j;Q^\nu) \le & C\varepsilon_j + \left(1 + C\left(\frac{\varepsilon_j}{\delta_j}\right)^\gamma + C\left(\frac{\eta_j}{\varepsilon_j}\right)^{1-\gamma}\right)H_g(u_j;Q^\nu)\\ & + C\left(\frac{\varepsilon_j}{\eta_j}\right)^\gamma H_g(u_j;Q^\nu\cap\{|[u_j]|\in[\eta_j,\delta_j)\}) + C\varepsilon_j^{\gamma-1}\|\nabla u_j\|_{L^1(Q^\nu)}. \end{aligned} \tag{2.40}$$

By definition $\frac{\varepsilon_j}{\delta_j} = \frac{\eta_j}{\varepsilon_j} = M_j^{-1}\to 0$ as $j\to\infty$, and moreover recalling the definition of r_j and k_j (cf. (2.38))

$$\begin{aligned} &\liminf_{j\to\infty}\left(\left(\frac{\varepsilon_j}{\eta_j}\right)^\gamma H_g(u_j;Q^\nu\cap\{|[u_j]|\in[\eta_j,\delta_j)\}) + \frac{1}{\varepsilon_j^{1-\gamma}}\|\nabla u_j\|_{L^1(Q^\nu)}\right)\\ &\quad\le 2\limsup_{j\to\infty}\frac{M_j^\gamma}{K_j}H_g(u_j;Q^\nu) + \limsup_{j\to\infty}\frac{1}{r_j^{1-\gamma}}\|\nabla u_j\|_{L^1(Q^\nu)} = 0\,. \end{aligned}$$

In conclusion, from (2.40) we infer

$$\liminf_{j\to\infty} H_g(v_j;Q^\nu) \le \liminf_{j\to\infty} H_g(u_j;Q^\nu)\,. \tag{2.41}$$

From $u_j \to u_0$ in measure and (2.39) we deduce $v_j \to u_0$ in measure. If $u_j \to u_0$ in L^1, then $v_j \to u_0$ in L^1.

Step 3. Conclusion. The rest of the proof is similar to [AFP00, Theorem 5.14]. By Steps 1 and 2, it is sufficient to prove (1.10) for a sequence of piecewise constant functions, each taking finitely many values. Thus, in order to apply the definition of BV-ellipticity of g, we need to modify the boundary datum of v_j.

Define $\phi : \mathbb{R}^m \to [0,\infty)$ by $\phi(z) := g_0(|z|)$. For any function $w \in SBV(Q^\nu;\mathbb{R}^m)$ with $w(Q^\nu)$ a finite set, we have that $\phi\circ w \in SBV \cap L^\infty(Q^\nu)$, with $J_{\phi\circ w} \subseteq J_w$ (up to null sets) and $\nabla(\phi\circ w) = 0$ $\mathcal{L}^n$-a.e. on Q^ν. Further, using subadditivity and monotonicity of g_0

$$[\phi\circ w] = \phi(w^+) - \phi(w^-) \le g_0(|w^-| + |[w]|) - g_0(|w^-|) \le g_0(|[w]|) \tag{2.42}$$

so that, repeating the computation with the two traces swapped,

$$|[\phi\circ w]| \le g_0(|[w]|). \tag{2.43}$$

Therefore for every j we have

$$|D(\phi\circ v_j)|(Q^\nu) \le \int_{J_{v_j}} g_0(|[v_j]|)\mathrm{d}\mathcal{H}^{n-1} \le CH_g(v_j;Q^\nu) \tag{2.44}$$

and we obtain that $\phi\circ v_j$ is compact in $L^1(Q^\nu)$ (possibly after subtracting the average). Since g_0 is continuous, from convergence in measure of v_j to u_0 we obtain, possibly passing to a subsequence and dropping a null set, pointwise convergence of v_j to u_0 and of $\phi\circ v_j$ to $\phi\circ u_0$. Therefore $\phi\circ v_j \to \phi\circ u_0$ in $L^1(Q^\nu)$. By Fubini's theorem, up to subsequences for $\mathcal{L}^1$-a.e. $t\in(0,1)$ it holds

$$\lim_{j\to\infty}\int_{\partial Q^\nu_t} |\phi\circ v_j - \phi\circ u_0|\mathrm{d}\mathcal{H}^{n-1} = 0. \tag{2.45}$$

In addition, up to subsequences, by convergence in measure and another application of Fubini's theorem we may assume that $v_j \to u_0$ $\mathcal{H}^{n-1}\llcorner\partial Q^\nu_t$-a.e. on ∂Q^ν_t and that $\mathcal{H}^{n-1}(\partial Q^\nu_t \cap J_{v_j}) = 0$ for all j.

Fix M with $\|u_0\|_{L^\infty(Q^\nu)} < M$. By pointwise convergence and dominated convergence,

$$\lim_{j\to\infty}\int_{\partial Q^\nu_t} g_0(|v_j - u_0|)\chi_{\{|v_j|<M\}}\mathrm{d}\mathcal{H}^{n-1} = 0 \tag{2.46}$$

and

$$\lim_{j\to\infty}\int_{\partial Q^\nu_t} \chi_{\{|v_j|\ge M\}}\mathrm{d}\mathcal{H}^{n-1} = 0. \tag{2.47}$$

At the same time, using $g_0(|v_j - u_0|) \le g_0(|u_0|) + g_0(|v_j|) = 2g_0(|u_0|) + g_0(|v_j|) - g_0(|u_0|)$,

$$\begin{aligned}&\int_{\partial Q_t^\nu} g_0(|v_j - u_0|)\chi_{\{|v_j|\ge M\}}\mathrm{d}\mathcal{H}^{n-1} \le \\ &\int_{\partial Q_t^\nu} 2g_0(|u_0|)\chi_{\{|v_j|\ge M\}}\mathrm{d}\mathcal{H}^{n-1} + \int_{\partial Q_t^\nu} |\phi\circ v_j - \phi\circ u_0|\mathrm{d}\mathcal{H}^{n-1}.\end{aligned} \tag{2.48}$$

Combining these estimates leads to

$$\lim_{j\to\infty}\int_{\partial Q_t^\nu} g_0(|v_j - u_0|)\mathrm{d}\mathcal{H}^{n-1} = 0 \tag{2.49}$$

for almost every $t \in (0,1)$.

Also, by [AFP00, Lemma 5.15], for $\mathcal{L}^1$-a.e. $t \in (0,1)$ the function $w_j := v_j\chi_{Q_t^\nu} + u_0\chi_{Q^\nu\setminus Q_t^\nu}$ satisfies $(w_j^+, w_j^-, \nu_w) = (v_j, u_0, \nu_t)$ for $\mathcal{H}^{n-1}$-a.e. $x \in \partial Q_t^\nu$, where ν_t denotes the inner normal vector to Q_t^ν. Hence,

$$\begin{aligned}&\liminf_{j\to\infty} H_g(v_j;Q^\nu) \ge \liminf_{j\to\infty} H_g(v_j;Q_t^\nu)\\ &\ge \liminf_{j\to\infty}\Big(H_g(w_j;Q^\nu) - \int_{\partial Q_t^\nu} g(v_j - u_0,\nu_t)\mathrm{d}\mathcal{H}^{n-1}\Big) - H_g(u_0;Q^\nu\setminus\overline{Q_t^\nu})\\ &\qquad\qquad \ge g(z,\nu) - g(z,\nu)(1-t^{n-1}),\end{aligned}$$

where in the last inequality we have used the BV-ellipticity of g and (2.49). Hence, as $t\to 1^-$, we conclude from Step 2

$$\liminf_{j\to\infty} H_g(u_j;Q^\nu) \ge \liminf_{j\to\infty} H_g(v_j;Q^\nu) \ge g(z,\nu) = H_g(u_0;Q^\nu). \qquad \square$$

2.2 Lower semicontinuity for bulk energies

The aim of this section is to prove Proposition 1.2. We first prove an approximation argument for nonnegative superlinear quasiconvex functions with linear quasiconvex ones. Actually, for later purposes (cf. Section 4.4) we do not assume quasiconvexity of Ψ, and only suppose that $\Psi : \mathbb{R}^{m\times n} \to [0,\infty)$ is continuous, and for some $q > 1$ satisfies (1.15), namely

$$\Big(\frac{1}{c}|\xi|^q - c\Big)\vee 0 \le \Psi(\xi) \le c(|\xi|^q + 1) \qquad \text{for all } \xi\in\mathbb{R}^{m\times n}.$$

We are then led to introduce the quasiconvex envelope h^{qc} of a continuous function $h : \mathbb{R}^{m\times n} \to [0,\infty)$, which is defined as

$$h^{\mathrm{qc}}(\xi) := \sup\{\Phi(\xi) : \Phi \le h,\ \Phi \text{ quasiconvex}\}, \tag{2.50}$$

and characterized by (see [Dac08, Theorem 6.9])

$$h^{\mathrm{qc}}(\xi) = \inf\left\{\int_{(0,1)^n} h(\xi + \nabla\varphi)\mathrm{d}x : \varphi \in W_0^{1,\infty}((0,1)^n;\mathbb{R}^m)\right\}. \tag{2.51}$$

Proposition 2.3. *Let $\Psi : \mathbb{R}^{m\times n} \to [0,\infty)$ be continuous and satisfy* (1.15). *For $\delta \in (0,1)$ $\ell > 0$, and $p > 1$, let*

$$h_\delta(\xi) := \Psi(\xi) \wedge \frac{\ell}{(1-\delta^{q'})^{p-1}} \Psi^{1/q}(\xi), \tag{2.52}$$

where as usual $q' := q/(q-1)$. Then

$$\sup_{\delta\in(0,1)} h_\delta^{\mathrm{qc}}(\xi) = \lim_{\delta\uparrow 1} h_\delta^{\mathrm{qc}}(\xi) = \Psi^{\mathrm{qc}}(\xi)\,.$$

The proof is based on the following variant of Zhang's truncation result [Zha92] (see also [EG92, Sect. 6.6.2]), the version on a bounded domain stated below appears for instance in [FJM02, Proposition A.1] The treatment of boundary data is discussed for example in [DHM00, Lemma 4.1].

Lemma 2.4. *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz set, $s \in [1,\infty)$. There is $c_* = c_*(s,\Omega,m) > 0$ such that for any $u \in W^{1,s}(\Omega;\mathbb{R}^m)$ and $\lambda > 0$ there is $w_\lambda \in W^{1,\infty}(\Omega;\mathbb{R}^m)$ such that $\|\nabla w_\lambda\|_{L^\infty(\Omega)} \le c_*\lambda$ and*

$$\lambda^s \mathcal{L}^n(\{w_\lambda \ne u\}) \le c_* \int_{\{|\nabla u|>\lambda\}} |\nabla u|^s \mathrm{d}x\,.$$

If additionally $u \in W_0^{1,s}(\Omega;\mathbb{R}^m)$ then $w_\lambda \in W_0^{1,\infty}(\Omega;\mathbb{R}^m)$.

Proof of Proposition 2.3. The inequality $\sup_{\delta\in(0,1)} h_\delta^{\mathrm{qc}}(\xi) \le \Psi^{\mathrm{qc}}(\xi)$ follows immediately from $h_\delta \le \Psi$.

To prove the opposite inequality we discuss the case $\ell = 1$ for the sake of notational simplicity. Then, define $A_\delta := (1-\delta^{q'})^{\frac{(1-p)q}{q-1}}$, so that $A_\delta \to \infty$ for $\delta \uparrow 1$ and

$$h_\delta(\eta) = \Psi(\eta) \wedge A_\delta^{1-1/q} \Psi^{1/q}(\eta) = \Psi^{1/q}(\eta)\,[\Psi(\eta) \wedge A_\delta]^{1-1/q}\,. \tag{2.53}$$

We fix $\xi \in \mathbb{R}^{m\times n}$ and for any $\delta \in (0,1)$ choose $\varphi_\delta \in W_0^{1,\infty}(Q_1;\mathbb{R}^m)$ such that

$$\int_{Q_1} h_\delta(\xi + \nabla\varphi_\delta)\mathrm{d}x \le h_\delta^{\mathrm{qc}}(\xi) + 1 - \delta. \tag{2.54}$$

Let $N > 1$, chosen below, and set

$$K_{N,\delta} := \max\{k \in \mathbb{N} : N^{k+2} \le A_\delta^{1/q}\}. \tag{2.55}$$

Obviously $\lim_{\delta\uparrow 1} K_{N,\delta} = \infty$. We select $k_\delta \in \mathbb{N} \cap [1, K_{N,\delta}]$ such that

$$\int_{\{N^{k_\delta} \le |\nabla\varphi_\delta| < N^{k_\delta+1}\}} h_\delta(\xi + \nabla\varphi_\delta)\mathrm{d}x \le \frac{h_\delta^{\mathrm{qc}}(\xi) + 1}{K_{N,\delta}} \le \frac{\Psi^{\mathrm{qc}}(\xi) + 1}{K_{N,\delta}}. \tag{2.56}$$

By Lemma 2.4 applied with $s = 1$ and $\lambda := N^{k_\delta}$ there is $\theta_\delta \in W_0^{1,\infty}(Q_1;\mathbb{R}^m)$ such that $|\nabla\theta_\delta| \le c_*\lambda$ and

$$\mathcal{L}^n(\{\theta_\delta \ne \varphi_\delta\}) \le \frac{c_*}{\lambda} \int_{\{|\nabla\varphi_\delta|>\lambda\}} |\nabla\varphi_\delta|\mathrm{d}x.$$

The constant c_* depends only on n and m.

From the definition of h_δ we prove that there are $\delta_0 \in (0,1)$, $N_0 \in \mathbb{N}$, $c' > 0$, all depending only on ξ, n, m, and the parameters c and q from (1.15), such that for any $\delta \in (\delta_0, 1)$ and $N \geq N_0$ the following holds:

$$\text{if } |\eta| \leq N\lambda \text{ then } h_\delta(\xi+\eta) = \Psi(\xi+\eta), \tag{2.57}$$

$$\text{if } \lambda \leq |\eta| \leq N\lambda \text{ then } h_\delta(\xi+\eta) \geq c'|\eta|\lambda^{q-1}, \tag{2.58}$$

and

$$\text{if } |\eta| \geq N\lambda \text{ then } h_\delta(\xi+\eta) \geq c'|\eta|(N\lambda)^{q-1}. \tag{2.59}$$

To prove the first one, by (2.53) it suffices to show that in the relevant regime one has $\Psi(\xi+\eta) \leq A_\delta$. We compute from (1.15)

$$\begin{aligned}\Psi(\xi+\eta) \leq & c(|\xi+\eta|^q + 1) \leq c(2^{q-1}|\xi|^q + 2^{q-1}|\eta|^q + 1) \\ \leq & c2^{q-1}(|\xi|^q+1) + c2^{q-1}N^{q(K_{N,\delta}+1)}.\end{aligned}$$

For a suitable choice of δ_0 the first term is bounded by $A_\delta/2$. By (2.55), the second one is bounded by $c2^{q-1}A_\delta/N^q$, so that for a suitable choice of N_0 this is also bounded by $A_\delta/2$. This proves (2.57).

We next observe that if $|\eta| \geq \lambda$ then for a suitable choice of N_0 and $\tilde{c} > 0$, depending on $|\xi|$, q and c, we have $|\xi+\eta| \geq |\eta|/2$ and by (1.15)

$$\Psi(\xi+\eta) \geq \frac{1}{c}|\xi+\eta|^q - c \geq \frac{1}{\tilde{c}}|\eta|^q. \tag{2.60}$$

This and (2.57) prove (2.58).

Finally, we turn to (2.59). If $\Psi(\xi+\eta) \leq A_\delta$, then it follows from (2.60). In order to obtain the other case, using again (2.55) and then (2.60),

$$A_\delta^{1-1/q}\Psi^{1/q}(\xi+\eta) \geq N^{(k_\delta+2)(q-1)}\frac{1}{\tilde{c}^{1/q}}|\eta| \geq (N\lambda)^{q-1}\frac{1}{\tilde{c}^{1/q}}|\eta|.$$

This concludes the proof of (2.59).

Recalling the choice of θ_δ, and using (2.57) and $N_0 \geq c_*$ to obtain $h_\delta(\xi + \nabla\theta_\delta) = \Psi(\xi+\nabla\theta_\delta)$,

$$\begin{aligned}\Psi^{\mathrm{qc}}(\xi) \leq \int_{Q_1} \Psi(\xi+\nabla\theta_\delta)\mathrm{d}x &= \int_{\{\theta_\delta=\varphi_\delta\}} \Psi(\xi+\nabla\theta_\delta)\mathrm{d}x + \int_{\{\theta_\delta\neq\varphi_\delta\}} \Psi(\xi+\nabla\theta_\delta)\mathrm{d}x \\ \leq &\int_{Q_1} h_\delta(\xi+\nabla\varphi_\delta)\mathrm{d}x + c(c_*^q\lambda^q+1)\mathcal{L}^n(\{\theta_\delta\neq\varphi_\delta\}) \\ \leq & h_\delta^{\mathrm{qc}}(\xi) + (1-\delta) + C\lambda^{q-1}\int_{\{|\nabla\varphi_\delta|>\lambda\}} |\nabla\varphi_\delta|\mathrm{d}x.\end{aligned} \tag{2.61}$$

We split the last term in two parts. The integral on $\{\lambda < |\nabla\varphi_\delta| < N\lambda\}$ is estimated using (2.56) and (2.58), which lead to

$$\lambda^{q-1}\int_{\{\lambda<|\nabla\varphi_\delta|<N\lambda\}} |\nabla\varphi_\delta|\mathrm{d}x \leq C\frac{\Psi^{\mathrm{qc}}(\xi)+1}{K_{N,\delta}}. \tag{2.62}$$

For the contribution on the set $\{|\nabla\varphi_\delta| \geq N\lambda\}$ instead we use (2.59) and (2.54), to obtain

$$\lambda^{q-1}\int_{\{N\lambda\leq|\nabla\varphi_\delta|\}}|\nabla\varphi_\delta|\mathrm{d}x \leq CN^{1-q}(\Psi^{\mathrm{qc}}(\xi)+1). \tag{2.63}$$

Inserting the last two inequalities in (2.61) leads to

$$\Psi^{\mathrm{qc}}(\xi) \leq h_\delta^{\mathrm{qc}}(\xi) + (1-\delta) + C\left(\frac{1}{N^{q-1}} + \frac{1}{K_{N,\delta}}\right)(\Psi^{\mathrm{qc}}(\xi)+1).$$

Taking first $\delta \to 1$ and then $N \to \infty$ gives the result (recall that $\delta \mapsto h_\delta(\xi)$ is nondecreasing). □

We are now ready to prove the claimed lower semicontinuity property for bulk integrands.

Proof of Proposition 1.2. We divide the proof in several steps.

Step 1. Reduction to coercive integrands. Assume that (1.14) holds for coercive integrands Ψ, namely satisfying for every $\xi \in \mathbb{R}^{m\times n}$

$$\frac{1}{c}|\xi|^q - c \leq \Psi(\xi) \leq c(|\xi|^q + 1)\,. \tag{2.64}$$

We show next how to deduce lower semicontinuity for integrands satisfying more generally (1.13). Indeed, recalling (Ψ-a”), assumption (Ψ-c) can be equivalently stated as follows: for every $\varepsilon > 0$ there is $k_\varepsilon \geq 1$ such that

$$\sup_j \int_{\{\Psi(\nabla u_j) < -k_\varepsilon\}} |\Psi(\nabla u_j)|\mathrm{d}x < \varepsilon\,. \tag{2.65}$$

Clearly, we may suppose that $k_\varepsilon \to \infty$ as $\varepsilon \to 0$.

Let $q \geq 1$, and define $\Psi_\varepsilon(\xi) := \max\{\Psi(\xi), -k_\varepsilon\} + \varepsilon|\xi|^q$. Then,

$$\begin{aligned}
&\left|\int_\Omega \Psi_\varepsilon(\nabla u_j)\mathrm{d}x - \int_\Omega \Psi(\nabla u_j)\mathrm{d}x\right| \\
&\leq \int_{\{\Psi(\nabla u_j)<-k_\varepsilon\}} |\Psi(\nabla u_j)|\mathrm{d}x + k_\varepsilon \mathcal{L}^n(\{\Psi(\nabla u_j) < -k_\varepsilon\}) + \varepsilon\int_\Omega |\nabla u_j|^q \mathrm{d}x \\
&\leq C\varepsilon
\end{aligned}$$

for some $C > 0$, thanks to assumption (Ψ-a”) and (2.65). Clearly, Ψ_ε is coercive and quasiconvex, thus the corresponding integrand is lower semicontinuous so that

$$\liminf_{j\to\infty} \int_\Omega \Psi(\nabla u_j)\mathrm{d}x \geq \liminf_{j\to\infty}\int_\Omega \Psi_\varepsilon(\nabla u_j)\mathrm{d}x - C\varepsilon \geq \int_\Omega \Psi_\varepsilon(\nabla u)\mathrm{d}x - C\varepsilon\,,$$

and the conclusion follows since $\Psi_\varepsilon \geq \Psi$ letting $\varepsilon \to 0$.

We are thus left with proving inequality (1.14) for integrands satisfying (2.64). Since (2.64) implies that Ψ is bounded from below, adding a constant one easily reduces to the case of nonnegative integrands.

Step 2. Reduction to a truncated sequence converging in L^1. Let $\mathcal{T}_k$ be the truncation defined in (1.21). Having fixed $M \in \mathbb{N}$, by taking into account assumption (Ψ-a") and averaging, we choose, for every j, an integer $k_j \in \{M+1, \dots, 2M\}$ such that

$$\int_{\{a_{k_j} < |u_j| < a_{k_j+1}\}} |\nabla u_j|^q \mathrm{d}x \le \frac{C}{M} \tag{2.66}$$

for some $C > 0$, which implies that $\hat{u}_j := \mathcal{T}_{k_j}(u_j) \in SBV(\Omega; \mathbb{R}^m)$ obeys

$$\int_\Omega \Psi(\nabla \hat{u}_j)\mathrm{d}x \le \int_\Omega \Psi(\nabla u_j)\mathrm{d}x + \frac{C}{M} + C\mathcal{L}^n(\{|u_j| \ge a_M\}). \tag{2.67}$$

Indeed, we have

$$\begin{aligned}\int_\Omega \Psi(\nabla \hat{u}_j)\mathrm{d}x \le & \int_{\{|u_j| \le a_{k_j}\}} \Psi(\nabla u_j)\mathrm{d}x \\ & + \int_{\{a_{k_j} < |u_j| < a_{k_j+1}\}} \Psi(\nabla \hat{u}_j)\mathrm{d}x + \Psi(0)\mathcal{L}^n(\{|u_j| \ge a_{k_j+1}\}) \\ & \le \int_\Omega \Psi(\nabla u_j)\mathrm{d}x + C\int_{\{a_{k_j} < |u_j| < a_{k_j+1}\}} |\nabla u_j|^q \mathrm{d}x + C\mathcal{L}^n(\{|u_j| \ge a_{k_j}\}),\end{aligned}$$

where in the last inequality we have used that $\mathrm{Lip}(\mathcal{T}_{k_j}) \le 1$ together with (2.64). The inequality in (2.67) then follows from (2.66).

Passing to a further subsequence we can assume $k_j = k$ independent of j (but still depending on M). Notice that $\hat{u}_j$ and $\hat{u} := \mathcal{T}_k(u)$ satisfy (Ψ-a"), (Ψ-b"), and (Ψ-c) by definition of $\mathcal{T}_k$. Moreover, up to subsequences, $\hat{u}_j$ converges pointwisely to $\hat{u}$ and then in L^1 by dominated convergence.

Let us give for granted that (1.14) holds for $\hat{u}_j$ and $\hat{u} := \mathcal{T}_k(u)$. Then, (2.67) yields

$$\begin{aligned}\int_\Omega \Psi(\nabla \hat{u})\mathrm{d}x \le & \liminf_{j\to\infty} \int_\Omega \Psi(\nabla \hat{u}_j)\mathrm{d}x \\ & \le \liminf_{j\to\infty} \int_\Omega \Psi(\nabla u_j)\mathrm{d}x + \frac{C}{M} + C\mathcal{L}^n(\{|u| \ge a_M\}).\end{aligned}$$

By this and

$$\Big|\int_\Omega \Psi(\nabla \hat{u})\mathrm{d}x - \int_\Omega \Psi(\nabla u)\mathrm{d}x\Big| \le C\int_{\{|u| > a_M\}} (|\nabla u|^q + 1)\mathrm{d}x,$$

by sending $M \to \infty$ we conclude that (1.14) holds. In the rest of the proof we will assume u, $u_j \in L^\infty \cap SBV(\Omega; \mathbb{R}^m)$, and $u_j \to u$ in $L^1(\Omega; \mathbb{R}^m)$.

Step 3. Blow-up. For every $j \in \mathbb{N}$ consider the measures defined by

$$\mu_j := \Psi(\nabla u_j)\mathcal{L}^n \llcorner \Omega, \qquad \nu_j := g_0(|[u_j]|)\mathcal{H}^{n-1} \llcorner J_{u_j}.$$

Assumptions (2.64), (Ψ-a") and (Ψ-b") imply that $(\mu_j)_j$ and $(\nu_j)_j$ are equi-bounded in mass. Passing to a subsequence we may assume that $\mu_j \rightharpoonup \mu_\Psi$ and $\nu_j \rightharpoonup \nu_\Psi$ weakly* in the sense of measures on Ω as $j \to \infty$, for some $\mu_\Psi, \nu_\Psi \in \mathcal{M}_b^+(\Omega)$. In addition, we may also assume that the right-hand side of (1.14) is a limit, which is finite in view of (2.64) and assumption (Ψ-a"). Notice that if $H_\Psi(u) \leq \mu_\Psi(\Omega)$ then necessarily the conclusion in (1.14) holds. Clearly, to establish the former inequality it suffices to show that

$$\Psi(\nabla u(x_0))) \leq \frac{\mathrm{d}\mu_\Psi}{\mathrm{d}\mathcal{L}^n}(x_0) \tag{2.68}$$

for $\mathcal{L}^n$-a.e. $x_0 \in \Omega$. By Besicovitch derivation theorem [AFP00, Theorem 2.22] we have

$$\frac{\mathrm{d}\mu_\Psi}{\mathrm{d}\mathcal{L}^n}(x_0) + \frac{\mathrm{d}\nu_\Psi}{\mathrm{d}\mathcal{L}^n}(x_0) < \infty \tag{2.69}$$

for $\mathcal{L}^n$-a.e. $x_0 \in \Omega$. Next we observe that for $\mathcal{L}^n$-a.e. $x_0 \in \Omega$ one has

$$\frac{\mathrm{d}\mu_\Psi}{\mathrm{d}\mathcal{L}^n}(x_0) = \lim_{\rho \to 0} \frac{\mu_\Psi(Q_\rho(x_0))}{\rho^n} = \lim_{\substack{\rho \to 0 \\ \rho \in I}} \lim_{j \to \infty} \frac{\mu_j(Q_\rho(x_0))}{\rho^n}$$

and

$$\frac{\mathrm{d}\nu_\Psi}{\mathrm{d}\mathcal{L}^n}(x_0) = \lim_{\rho \to 0} \frac{\nu_\Psi(Q_\rho(x_0))}{\rho^n} = \lim_{\substack{\rho \to 0 \\ \rho \in I}} \lim_{j \to \infty} \frac{\nu_j(Q_\rho(x_0))}{\rho^n}$$

where $Q_\rho(x_0) := x_0 + (-\frac{1}{2}\rho, \frac{1}{2}\rho)^n$ and $I := \{\rho \in (0, \frac{2}{\sqrt{n}}\mathrm{dist}(x_0, \partial A)) : \mu_\Psi(\partial Q_\rho(x_0)) = \nu(\partial Q_\rho(x_0)) = 0\}$. We define $u^\rho : Q_1 \to \mathbb{R}^m$ by

$$u^\rho(y) := \frac{u(x_0 + \rho y) - u(x_0)}{\rho}.$$

By Calderón-Zygmund theorem [AFP00, Theorem 3.83], for $\mathcal{L}^n$-a.e. $x_0 \in \Omega$, after possibly extracting a further subsequence, $u^\rho(y) \to \nabla u(x_0)y$ in $L^1(Q_1;\mathbb{R}^m)$ as $\rho \to 0$. We shall establish (2.68) for all points x_0 satisfying (2.69), and for which $u^\rho \to \nabla u(x_0)y$ in $L^1(Q_1;\mathbb{R}^m)$.

We further define

$$u_j^\rho(y) := \frac{u_j(x_0 + \rho y) - u(x_0)}{\rho}$$

so that $u_j^\rho \to u^\rho$ in $L^1(Q_1;\mathbb{R}^m)$ as $j \to \infty$ for any fixed $\rho > 0$. We take a diagonal subsequence so that $w_i(y) := u_{j_i}^{\rho_i}(y) \to \nabla u(x_0)y$ in $L^1(Q_1;\mathbb{R}^m)$,

$$\frac{\mathrm{d}\mu_\Psi}{\mathrm{d}\mathcal{L}^n}(x_0) = \lim_{i \to \infty} \int_{Q_1} \Psi(\nabla w_i)\mathrm{d}x\,, \tag{2.70}$$

and

$$\sup_i \frac{1}{\rho_i} \int_{J_{w_i} \cap Q_1} g_0(\rho_i |[w_i]|)\mathrm{d}\mathcal{H}^{n-1} < \infty\,. \tag{2.71}$$

Step 3. Reduction to blow-ups bounded in L^∞ and with vanishing singular total variations. By a truncation argument analogous to that used in Step 2, having fixed $M \in \mathbb{N}$, we find $k_i \in \{M+1, \dots, 2M\}$ such that $\hat{w}_i := \mathcal{T}_{k_i}(w_i) \in SBV(Q_1;\mathbb{R}^m)$, with $\mathcal{T}_{k_i}$ defined in (1.21), obeys

$$\int_{Q_1} \Psi(\nabla \hat{w}_i)\mathrm{d}x \leq \int_{Q_1} \Psi(\nabla w_i)\mathrm{d}x + \frac{C}{M} + C\mathcal{L}^n(\{|w_i| \geq a_M\}). \tag{2.72}$$

Moreover, note that if $a_M > \|\nabla u(x_0)y\|_{L^\infty(Q_1)}$ then $w_i \to \nabla u(x_0)y$ implies $\hat{w}_i \to \nabla u(x_0)y$ in $L^1(Q_1;\mathbb{R}^m)$. From $\mathcal{T}_{k_i} \in C^1$ we deduce $J_{\hat{w}_i} \subset J_{w_i}$; from the fact that $\mathcal{T}_{k_i}$ is 1-Lipschitz and g_0 monotone that

$$\sup_i \frac{1}{\rho_i} \int_{J_{\hat{w}_i} \cap Q_1} g_0(\rho_i |[\hat{w}_i]|)\mathrm{d}\mathcal{H}^{n-1} \leq \sup_i \frac{1}{\rho_i} \int_{J_{w_i} \cap Q_1} g_0(\rho_i |[w_i]|)\mathrm{d}\mathcal{H}^{n-1} < \infty\,. \tag{2.73}$$

We observe that subadditivity and monotonicity of g_0 imply

$$\frac{g_0(\rho)}{\rho} \leq 2\frac{g_0(t\rho)}{t\rho} \tag{2.74}$$

for every $\rho > 0$ and $t \in (0,1]$. Define the set $J^1_{\hat{w}_i} := \{x \in J_{\hat{w}_i} : |[\hat{w}_i]| \geq 1\}$, then

$$\begin{aligned} |D^s\hat{w}_i|(Q_1) &= \int_{J_{\hat{w}_i} \cap Q_1} |[\hat{w}_i]|\mathrm{d}\mathcal{H}^{n-1} \\ &\leq 2\int_{(J_{\hat{w}_i} \setminus J^1_{\hat{w}_i}) \cap Q_1} \frac{g_0(\rho_i|[\hat{w}_i]|)}{g_0(\rho_i)}\mathrm{d}\mathcal{H}^{n-1} + 2a_{2M}\mathcal{H}^{n-1}(J^1_{\hat{w}_i} \cap Q_1) \\ &\leq 2\int_{(J_{\hat{w}_i} \setminus J^1_{\hat{w}_i}) \cap Q_1} \frac{g_0(\rho_i|[\hat{w}_i]|)}{g_0(\rho_i)}\mathrm{d}\mathcal{H}^{n-1} + 2a_{2M}\int_{J^1_{\hat{w}_i} \cap Q_1} \frac{g_0(\rho_i|[\hat{w}_i]|)}{g_0(\rho_i)}\mathrm{d}\mathcal{H}^{n-1} \\ &\leq C(1 + a_{2M})\frac{\rho_i}{g_0(\rho_i)}\,, \end{aligned} \tag{2.75}$$

where to deduce the first inequality we have used (2.74) and the L^∞ bound on $\hat{w}_i$; to deduce the second inequality we have used again the monotonicity of g_0, and to deduce the last we have used (2.73). Therefore, recalling that g_0 is superlinear in the origin, we infer that

$$|D^s\hat{w}_i|(Q_1) \to 0\,. \tag{2.76}$$

Step 4. Lower semicontinuity for coercive integrands. We are now ready to conclude the proof. Indeed, thanks to (2.64), if $q > 1$ we can find an increasing sequence of quasiconvex functions with linear growth $(\Psi_k)_k$ such that $\sup_{k\in\mathbb{N}} \Psi_k(\xi) = \Psi(\xi)$ by applying Proposition 2.3 with $\delta = \delta_k \to 1^-$ and setting $\Psi_k := h^{\mathrm{qc}}_{\delta_k}$. We claim that for every $k \in \mathbb{N}$

$$\liminf_{i\to\infty} \int_{Q_1} \Psi_k(\nabla \hat{w}_i)\mathrm{d}x \geq \Psi_k(\nabla u(x_0))\,. \tag{2.77}$$

Given the claim for granted, by (2.70), (2.72), and (2.77) we deduce that

$$\begin{aligned}\frac{\mathrm{d}\mu_\Psi}{\mathrm{d}\mathcal{L}^n}(x_0) &= \lim_{i\to\infty}\int_{Q_1}\Psi(\nabla w_i)\mathrm{d}x \geq \liminf_{i\to\infty}\int_{Q_1}\Psi(\nabla\hat{w}_i)\mathrm{d}x - \frac{C}{M} \\ &\geq \liminf_{i\to\infty}\int_{Q_1}\Psi_k(\nabla\hat{w}_i)\mathrm{d}x - \frac{C}{M} \geq \Psi_k(\nabla u(x_0)) - \frac{C}{M}\,.\end{aligned}$$

The inequality in (2.68) then follows at once by passing to the supremum on $k\in\mathbb{N}$ and letting $M\to\infty$.

To conclude we are left with establishing (2.77). From the definition in (2.52) we check that $0\leq\Psi_k(\xi)\leq C_k(|\xi|+1)$, for some $C_k>0$ and all $\xi\in\mathbb{R}^{m\times n}$. Therefore we may argue as in [Kri99, Lemma 1.6] to infer that for every $r\in(0,1)$

$$\begin{aligned}\int_{Q_1}\Psi_k(\nabla\hat{w}_i)\mathrm{d}x &\geq r^n\Psi_k(\nabla u(x_0)) \\ &- C_k|D^s\hat{w}_i|(Q_1) - \frac{C_k}{1-r}\int_{Q_1\setminus Q_r}|\hat{w}_i-\nabla u(x_0)y|\mathrm{d}y\,. \qquad (2.78)\end{aligned}$$

Hence, (2.76) and the convergence $\hat{w}_i\to\nabla u(x_0)y$ in $L^1(Q_1;\mathbb{R}^m)$ imply (2.77) by letting $i\to\infty$ and then $r\to 1^-$.

If $q=1$ we may argue as above in order to deduce inequality (2.78) directly for Ψ. The conclusion then follows at once. ☐

2.3 Lower semicontinuity for free-discontinuity energies

As a consequence of Propositions 1.1 and 1.2 we obtain the ensuing lower semicontinuity statement for a large class of superlinear free-discontinuity energies generalizing [Amb94, Theorem 4.5, Remark 4.6], and the result in [Kri99] (see the comments after [Kri99, Theorem 1.2] and Remark 1 after [Kri99, Theorem 6.2]).

Theorem 2.5. *Let $g:\mathbb{R}^m\times S^{n-1}\to[0,\infty)$ be satisfying* (1.5)-(1.9), *and let $\Psi:\mathbb{R}^{m\times n}\to[0,\infty)$ be continuous, quasiconvex and satisfying* (1.13) *with $q\in(1,\infty)$. Then*

$$H_g(u)+H_\Psi(u)\leq\liminf_{j\to\infty}(H_g(u_j)+H_\Psi(u_j))\,,$$

if u, $u_j\in(GSBV(\Omega))^m$ satisfy (Ψ-a”), (Ψ-b”), and (Ψ-c).

Proof. Proposition 1.2 directly implies the lower semicontinuity inequality for the bulk part. Furthermore, $(\nabla u_j)_j$ is equiintegrable in $L^1(\Omega;\mathbb{R}^m)$ by assumption (Ψ-a”) as $q>1$, so that (g-a’) is satisfied. The conclusion then follows at once thanks to Proposition 1.1 and the superadditivity of the inferior limit operator. ☐

3 Relaxation

In this section we dispense with the quasiconvexity assumption on Ψ and the BV-ellipticity assumption on g and find the lower semicontinuos envelope of the corresponding energy $F := H_\Psi + H_g$ with respect to the strong L^1 topology. We recall that, given a metric space (X, d) endowed with the topology induced by d, and given a functional $F : X \to [0, \infty]$, the relaxation $\overline{F} : X \to [0, \infty]$ is defined by

$$\overline{F}(u) := \inf\{\liminf_{k\to\infty} F(u_k) : \ u_k \xrightarrow{d} u\}.$$

The relaxation of H_Ψ, which is the restriction of F to Sobolev functions, is an integral functional with energy density given by the quasiconvex envelope of Ψ, which was defined in (2.50), see [Dac08]. If one assumes that $g \geq c > 0$ then, both in the case of partition problems (where u takes values in a finite or discrete set) and for functionals defined on SBV^p, which in particular implies finiteness of the measure of the jump, the relaxation is an integral functional with g replaced by its BV-elliptic envelope, see [BFLM02]. The BV-elliptic envelope of $g : \mathbb{R}^m \times S^{n-1} \to [0, \infty)$ is defined by

$$g_{BV}(z, \nu) := \sup\{h(z, \nu) : \ h \leq g, \ h \ BV\text{-elliptic}\} . \tag{3.1}$$

We recall (see (2.4)) that $g : \mathbb{R}^m \times S^{n-1} \to [0, \infty)$ is BV-elliptic if for every $z \in \mathbb{R}^m$, every $\nu \in S^{n-1}$, every piecewise constant $u \in SBV(Q^\nu; \mathbb{R}^m)$ such that $u - z\chi_{\{x\cdot\nu>0\}}$ has compact support in the cube Q^ν one has

$$g(z, \nu) \leq \int_{Q^\nu\cap J_u} g([u], \nu_u)\mathrm{d}\mathcal{H}^{n-1}. \tag{3.2}$$

Since we are dealing with test functions u which take values in an infinite set, there are several variants of this definition, with different sets of test functions, which are not obviously equivalent. We adopted the one from [AFP00] which uses piecewise constant functions, in the sense of functions which are constant on a Caccioppoli partition. Such functions have jump set of finite $\mathcal{H}^{n-1}$-measure, which is a stronger property than finiteness of the integral in (3.2). In [BFLM02, Sect. 3.1] the even smaller class of piecewise constant functions which take finitely many values is used. Alternatively, one could consider all SBV functions with $\nabla u = 0$ almost everywhere, or require the level sets to be polygonals. Based on the density result in [CFI25], we show in Lemma 3.1 that these definitions are all equivalent. To do this it is useful to introduce some notation.

We denote by $PA(\mathbb{R}^n; \mathbb{R}^m)$ the space of functions for which there exists a locally finite decomposition of $\mathbb{R}^n$ in simplexes, such that u is affine in the interior of each of them. By restriction to Ω one obtains that any $u \in PA(\mathbb{R}^n; \mathbb{R}^m)$ satisfies $u \in SBV \cap L^\infty(\Omega; \mathbb{R}^m)$.

For $h:\mathbb{R}^m\times S^{n-1}\to[0,\infty)$ we define $Th:\mathbb{R}^m\times S^{n-1}\to[0,\infty)$ by

$$Th(z,\nu):=\inf\left\{\int_{Q^\nu\cap J_u} h([u],\nu_u)\mathrm{d}\mathcal{H}^{n-1} : u\in PA(\mathbb{R}^n;\mathbb{R}^m); \nabla u=0 \;\; \mathcal{L}^n\text{-a.e.}; \right.$$
$$\left. \operatorname{supp}(u-z\chi_{\{x\cdot\nu>0\}})\subset\subset Q^\nu\right\} \tag{3.3}$$

and, provided h is Borel measurable, $T_*h:\mathbb{R}^m\times S^{n-1}\to[0,\infty)$ by

$$T_*h(z,\nu):=\inf\left\{\int_{Q^\nu\cap J_u} h([u],\nu_u)\mathrm{d}\mathcal{H}^{n-1} : u\in SBV(Q^\nu;\mathbb{R}^m); \nabla u=0 \;\; \mathcal{L}^n\text{-a.e.}; \right.$$
$$\left. \operatorname{supp}(u-z\chi_{\{x\cdot\nu>0\}})\subset\subset Q^\nu\right\}. \tag{3.4}$$

The set of test functions in (3.3) is smaller than in (3.2), since test functions take finitely many values, and the level sets are polygonals. In (3.4) the set of test functions is instead larger than in (3.2), as it includes functions with $\mathcal{H}^{n-1}(J_u\cap Q^\nu)=\infty$.

Lemma 3.1. *Let $g:\mathbb{R}^m\times S^{n-1}\to[0,\infty)$ satisfy* (1.6)-(1.8)*, for some nondecreasing, subadditive function $g_0\in C^0([0,\infty);[0,\infty))$ with $g_0^{-1}(0)=\{0\}$. Then g_{BV} is BV-elliptic, satisfies* (1.6)-(1.8)*, with the same g_0, and*

$$g_{BV}=Tg=T_*g. \tag{3.5}$$

Proof. The fact that g_{BV} is BV-elliptic follows from the fact that for any function h as in (3.1) and any test function u as in (3.2) one has

$$h(z,\nu)\le\int_{Q^\nu\cap J_u} h([u],\nu_u)\mathrm{d}\mathcal{H}^{n-1}\le\int_{Q^\nu\cap J_u} g_{BV}([u],\nu_u)\mathrm{d}\mathcal{H}^{n-1}. \tag{3.6}$$

One important ingredient of the proof is that for any locally bounded $h:\mathbb{R}^m\times S^{n-1}\to[0,\infty)$ one has

$$TTh=Th. \tag{3.7}$$

The inequality $TTh\le Th$ is obvious, the other one follows by a standard covering argument. Indeed, fix $z\in\mathbb{R}^m$ and $\nu\in S^{n-1}$. For $\varepsilon>0$, by definition of TTh there is u as in (3.3) such that

$$\int_{Q^\nu\cap J_u} Th([u],\nu_u)\mathrm{d}\mathcal{H}^{n-1}\le TTh(z,\nu)+\varepsilon.$$

Since u is piecewise constant on a triangulation of Q^ν, we can cover J_u (up to a null set) by countably many pairwise disjoint cubes $Q_j=x_j+Q_{r_j}^{\nu_j}$ with

$x_j \in J_u$, ν_j the normal to J_u in x_j, and $u(x) = u^\pm(x_j)$ almost everywhere on $Q_j \cap \{\pm(x - x_j) \cdot \nu_j > 0\}$. Further, since $u \in L^\infty(Q^\nu)$ for some M large

$$\sum_{j>M} \int_{J_u \cap Q_j} h([u], \nu_u) \mathrm{d}\mathcal{H}^{n-1} \leq \varepsilon.$$

If $j = 1, \ldots, M$, by definition of Th there is $w_j \in PA(\mathbb{R}^n; \mathbb{R}^m)$ such that $w_j = u$ in a neighbourhood of ∂Q_j and

$$\int_{Q_j \cap J_{w_j}} h([w_j], \nu_{w_j}) \mathrm{d}\mathcal{H}^{n-1} \leq r_j^{n-1} Th([u], \nu_u) + \frac{\varepsilon}{M}.$$

Defining $w := w_j$ in Q_j and $w := u$ otherwise in Q^ν, we get

$$\begin{aligned} Th(z, \nu) \leq \int_{J_w} h([w], \nu_w) \mathrm{d}\mathcal{H}^{n-1} &\leq \sum_{j=1}^{M} \int_{J_{w_j} \cap Q_j} h([w_j], \nu_{w_j}) \mathrm{d}\mathcal{H}^{n-1} + \varepsilon \\ &\leq \sum_{j=1}^{M} \mathcal{H}^{n-1}(J_u \cap Q_j) Th([u], \nu_u) + 2\varepsilon \leq TTh(z, \nu) + 3\varepsilon. \end{aligned}$$

As $\varepsilon \to 0$ we get $Th \leq TTh$. This concludes the proof of (3.7).

We next check that if h obeys (1.6)–(1.8), then so does Th. Indeed, property (1.6) is immediate. The upper bound in (1.7) follows from $Th \leq h$; the lower bound from the fact that since g_0 is subadditive the function $(z, \nu) \mapsto g_0(|z|)$ is BV-elliptic, see [AB90, Example 2.8]. Further, (3.7) implies (again by rescaling and gluing competitors) that Th is subadditive in the first argument. Hence $Th(z, \nu) \leq Th(z', \nu) + h(z - z', \nu) \leq Th(z', \nu) + cg_0(|z - z'|)$, which proves property (1.8).

In order to prove (3.5), it suffices to show that

$$T_* Tg = Tg. \tag{3.8}$$

Indeed, combining (3.8) with the immediate inequalities $T_* Tg \leq T_* g \leq Tg$ we deduce $T_* g = Tg$. Similarly since the set of test functions in (3.4) contains the one in (3.2), (3.8) implies in particular that Tg is BV-elliptic, hence it is one of the functions entering (3.1), and $Tg \leq g_{BV}$. Conversely, from the definition of Tg and the fact that g_{BV} is BV-elliptic one obtains $g_{BV} \leq Tg$. Therefore $Tg = g_{BV}$, which concludes the proof.

It remains to prove (3.8). The inequality $T_* Tg \leq Tg$ is obvious. Fix $z \in \mathbb{R}^m$, $\nu \in S^{n-1}$, and a test function u as in (3.4), that is, a function $u \in SBV(Q^\nu; \mathbb{R}^m)$ such that $\nabla u = 0$ $\mathcal{L}^n$-a.e. on Q^ν and $\operatorname{supp}(u - z\chi_{\{x \cdot \nu > 0\}}) \subset\subset Q^\nu$. We need to show that

$$Tg(z, \nu) \leq \int_{Q^\nu \cap J_u} Tg([u], \nu_u) \mathrm{d}\mathcal{H}^{n-1}. \tag{3.9}$$

The key point is to apply the density result in [CFI25]. One must be careful to ensure that the condition $\operatorname{supp}(u - z\chi_{\{x \cdot \nu > 0\}}) \subset\subset Q^\nu$ is preserved. To do this, we observe that if one applies the construction in [CFI25, Theorem 1.1] to a function $\tilde u$ with $\operatorname{supp}\tilde u \subset\subset \Omega$, then (for sufficiently large j) one also has $\operatorname{supp}\tilde u_j \subset\subset \Omega$. To check this it suffices to follow the proof of that theorem in [CFI25, Section 4.3]. One first extends $\tilde u$ by zero to a function defined on $\mathbb{R}^n$, then remarks that, in the notation of that proof, the cubes Q_z^γ with $z \in A_\delta$ obey $\operatorname{dist}(Q_z^\gamma, \partial\Omega) \geq \delta(\sqrt{n}/2 - 1/4) \geq \delta/3$, so that the large-jump part of the reconstructed sequence vanishes around the boundary. Secondly, for the small-jump and diffuse parts, by [CFI25, Theorem 1.1 Step 3 and Proposition 4.3(iv)] the construction vanishes outside $B_{\varepsilon\sqrt{n}}(\operatorname{supp}\tilde u)$.

Therefore an application of [CFI25, Theorem 1.1] to $\hat u := u - z\chi_{\{x\cdot\nu>0\}}$ (with $p=1$ and g_0) leads to a piecewise constant sequence $\hat u_j \in PA(\mathbb{R}^n;\mathbb{R}^m)$ which is compactly supported in Q^ν. For large j the function $u_j := \hat u_j + z\chi_{\{x\cdot\nu>0\}}$ is an admissible test function in (3.3) with $h = Tg$. Further, by [CFI25, Corollary 2.1], one has

$$TTg(z,\nu) \leq \lim_{j\to\infty} \int_{Q^\nu\cap J_{u_j}} Tg([u_j],\nu_{u_j})\mathrm{d}\mathcal{H}^{n-1} = \int_{Q^\nu\cap J_u} Tg([u],\nu_u)\mathrm{d}\mathcal{H}^{n-1}. \tag{3.10}$$

Recalling that $TTg = Tg$, this proves (3.9) and concludes the proof of (3.8). □

By Lemma 3.1 one easily obtains that for any $z\in\mathbb{R}^m$, $\nu\in S^{n-1}$, and $\varepsilon>0$ there is a function $u\in SBV(Q^\nu;\mathbb{R}^m)$ with $\nabla u = 0$ $\mathcal{L}^n$-a.e. such that

$$\|u - z\chi_{\{x\cdot\nu>0\}}\|_{L^1(Q^\nu)} \leq \varepsilon \text{ and } \int_{Q^\nu\cap J_u} g([u],\nu_u)\mathrm{d}\mathcal{H}^{n-1} \leq g_{BV}(z,\nu)+\varepsilon. \tag{3.11}$$

This is obtained by covering the mid-plane of Q^ν by small cubes $x_i + Q^\nu_\rho$ and by gluing together the corresponding translations of the function u_ρ defined on Q^ν_ρ, obtained by suitably rescaling a good competitor u for (3.5). The details are left to the reader.

Using this remark we will prove in fact the following stronger version of Theorem 1.3 that will be used in Proposition 4.12.

Theorem 3.2. *Under the same hypotheses as in Theorem 1.3, let $H_0 : L^1(\Omega;\mathbb{R}^m) \to [0,\infty]$ be*

$$H_0(u) := \begin{cases} \displaystyle\int_\Omega \Psi(\nabla u)\mathrm{d}x + \int_{J_u\cap\Omega} g([u],\nu_u)\mathrm{d}\mathcal{H}^{n-1}, & \text{if } u\in PA(\mathbb{R}^n;\mathbb{R}^m),\\ & \mathcal{H}^{n-1}(J_u\cap\partial\Omega)=0,\\ \infty, & \text{otherwise.} \end{cases} \tag{3.12}$$

Then, the relaxation $\overline{H}_0$ with respect to the strong topology of $L^1(\Omega;\mathbb{R}^m)$ is the functional

$$\overline{H}_0(u) = \int_\Omega \Psi^{\mathrm{qc}}(\nabla u)\mathrm{d}x + \int_{J_u} g_{BV}([u],\nu_u)\mathrm{d}\mathcal{H}^{n-1}, \tag{3.13}$$

if $u\in (GSBV(\Omega))^m$ with $\nabla u\in L^q(\Omega;\mathbb{R}^{m\times n})$, and $\overline{H}_0(u)=\infty$ otherwise.

Proof. Let $\mathcal{F}$ be the functional in the right-hand side of (3.13), we first show that $\mathcal{F} \le \overline{H}_0$. In view of the definition of H_0, we immediately have $\mathcal{F} \le H_0$. Since the functional $\mathcal{F}$ is lower semicontinuous with respect to the strong topology of $L^1(\Omega;\mathbb{R}^m)$ thanks to Lemma 3.1, Propositions 1.1, and 1.2, we conclude that $\mathcal{F} \le \overline{H}_0$.

Conversely, let us prove that $\mathcal{F} \ge \overline{H}_0$. By standard diagonal arguments, it suffices to prove the following: Given $u \in (GSBV \cap L^1(\Omega))^m$ such that $\mathcal{F}(u) < \infty$ and $\delta > 0$, there is $U_\delta \in PA(\mathbb{R}^n;\mathbb{R}^m)$ with $\mathcal{H}^{n-1}(J_{U_\delta} \cap \partial\Omega) = 0$ such that $\|u - U_\delta\|_{L^1(\Omega)} \le C\delta$ and $H_0(U_\delta;\Omega) \le \mathcal{F}(u;\Omega) + C\delta$.

A careful inspection of the proof of [CFI25, Corollary 2.4] applied with Ψ^{qc} (in place of Ψ), g_0, and g, gives that there exist an open set Ω', $\overline{\Omega} \subset \Omega'$, and a function $u_1 \in PA(\mathbb{R}^n;\mathbb{R}^m)$ such that $\|u_1 - u\|_{L^1(\Omega')} \le \delta$, $\mathcal{H}^{n-1}(J_{u_1} \cap \partial\Omega) = 0$ and

$$\mathcal{F}(u_1;\Omega') \le \mathcal{F}(u;\Omega) + \delta. \tag{3.14}$$

Indeed, [CFI25, Corollary 2.4] is based on [CFI25, Theorem 1.1]. An extension of u to $\mathbb{R}^n$, not relabelled, satisfying good energy estimates and such that $\mathcal{H}^{n-1}(J_u \cap \partial\Omega) = 0$, is first provided in [CFI25, Theorem 1.1, Step 1]. Then, in [CFI25, Theorem 1.1, end of Step 3] one can check that near $\partial\Omega$ our function u_1, that is w_ζ in the notation there, is defined as the piecewise affine function $\Pi_{\varepsilon,\zeta}u$, with $\zeta \in B_\varepsilon$. The choice of ζ is performed in [CFI25, Theorem 1.1, Step 7] and can be further refined excluding a $\mathcal{L}^n$-negligible set without affecting the conclusions of [CFI25, Theorem 1.1]. More precisely, using that $\mathcal{H}^{n-1} \llcorner \partial\Omega$ is finite, we choose $\zeta \in B_\varepsilon$ satisfying $\mathcal{H}^{n-1}(\partial\Omega \cap J_{\Pi_{\varepsilon,\zeta}u}) = 0$, in addition to the conditions imposed in [CFI25, Theorem 1.1, Step 7]. Indeed, the latter condition holds for every $\zeta \in B_\varepsilon$ up to a $\mathcal{L}^n$-negligible subset. The properties of $\Pi_{\varepsilon,\zeta}$ in [CFI25, Proposition 4.3] finally give (3.14) when Ψ and g are replaced by $|\cdot|^q$ and g_0. The analogous estimate with the given Ψ and g follows by the proof of [CFI25, Corollary 2.4].

In particular, we can assume that there are finitely many simplexes E_k such that $\Omega \cap E_k$ has positive measure for each k and u_1 is affine on each of them, and that $\Omega \cap E_k$ cover Ω up to a null set.

Let

$$M_\delta := 1 + \|\Psi(\nabla u_1)\|_{L^\infty(\Omega)} + \|\nabla u_1\|_{L^\infty(\Omega)} + \|[u_1]\|_{L^\infty(\Omega\cap J_{u_1};\mathcal{H}^{n-1})} + \mathcal{H}^{n-1}(J_{u_1}). \tag{3.15}$$

This constant is finite, since ∇u_1 takes finitely many values, and $[u_1]$ can be estimated with finitely many affine functions. It depends on u and δ via u_1, we keep only the dependence on δ explicit.

For each k we select a Lipschitz set $E_k' \subset \Omega \cap E_k$ such that $\mathrm{dist}(E_k', \partial E_k) > 0$ and

$$\mathcal{L}^n(\Omega \cap E_k \setminus E_k') \le \frac{\delta}{M_\delta} \mathcal{L}^n(\Omega \cap E_k). \tag{3.16}$$

We refer to Figure 1 for a sketch of the decomposition of the domain. Then, by the definition of quasiconvexity and a standard covering argument (see for example [Dac08, Theorem 9.1] and [ET99, Proposition 2.1 in Chapter X]), there

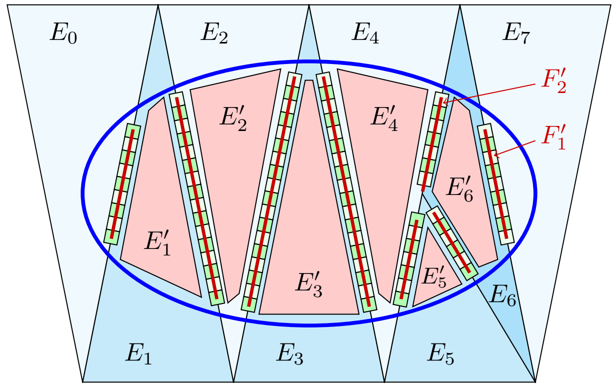

Figure 1: Sketch of the construction in the proof of Theorem 1.3. The set Ω (blue boundary) is covered by finitely many simplexes E_k (light blue); a large part of the interior of each of them is contained in the sets E_k'. The codimension-1 sets F_j' (red) cover most of the part of the faces inside Ω, and are in turn covered by the union of small cubes of size ρ (green).

exists $\varphi_k \in W_0^{1,q}(E_k';\mathbb{R}^m)$ such that $\|\varphi_k\|_{L^\infty(E_k')} \le \delta$ and

$$\int_{E_k'} \Psi(\nabla u_1 + \nabla\varphi_k)\mathrm{d}x \le \delta\mathcal{L}^n(E_k') + \int_{E_k'} \Psi^{\mathrm{qc}}(\nabla u_1)\mathrm{d}x. \tag{3.17}$$

Extending φ_k to 0 to the rest of Ω, we then define $u_2 : \Omega \to \mathbb{R}^m$ by

$$u_2 := u_1 + \sum_k \varphi_k. \tag{3.18}$$

We observe that

$$\|u_2 - u_1\|_{L^\infty(\Omega)} \le \delta, \tag{3.19}$$

with $u_2 = u_1$ in a neighbourhood of $\cup_k \partial E_k$. Further, from (3.17) we obtain

$$\int_{\Omega\cap E_k} \Psi(\nabla u_2)\mathrm{d}x \le \Psi(\nabla u_1|_{E_k})\mathcal{L}^n(\Omega\cap E_k\setminus E_k') + \delta\mathcal{L}^n(E_k') + \int_{\Omega\cap E_k} \Psi^{\mathrm{qc}}(\nabla u_1)\mathrm{d}x. \tag{3.20}$$

Summing over k, recalling (3.16), the fact that the E_k are disjoint and cover Ω, and the definition of M_δ, leads to

$$\int_\Omega \Psi(\nabla u_2)\mathrm{d}x \le 2\delta\mathcal{L}^n(\Omega) + \int_\Omega \Psi^{\mathrm{qc}}(\nabla u_1)\mathrm{d}x. \tag{3.21}$$

We next turn to the interfaces between the different sets E_k, which contain the set on which u_1 jumps. Consider the sets $\Omega\cap\partial E_k\cap\partial E_{k'}$, $k \ne k'$. We denote by F_j, $j = 1,\dots,J$, those which have positive $\mathcal{H}^{n-1}$ measure, and on which $[u_1]$

is not identically zero. Then

$$\mathcal{F}(u_1) = \sum_k \int_{E_k\cap\Omega} \Psi^{\mathrm{qc}}(\nabla u_1)\mathrm{d}x + \sum_j \int_{F_j} g_{BV}(u_1^+ - u_1^-, \nu_j)\mathrm{d}\mathcal{H}^{n-1}, \qquad (3.22)$$

where ν_j is the normal to F_j. We remark that for each j the function u_1 is affine on each side of F_j. As above, we fix relatively open sets $F_j' \subset F_j$ such that $\mathrm{dist}(F_j', \partial F_j) > 0$ (the boundary here is taken in the $n-1$-dimensional sense, each of them is contained in an $n-1$-dimensional plane) and

$$\sum_j \mathcal{H}^{n-1}(F_j \setminus F_j') \le \frac{\delta}{g_0(M_\delta)}. \qquad (3.23)$$

Fix $\rho > 0$, chosen below in dependence of u_1, E_k', and δ, and cover a large part of each F_j by cubes on a scale ρ. To do this, let $A_j : \mathbb{R}^n \to \mathbb{R}^n$ be an affine isometry that maps $\mathbb{R}^{n-1}$ (which we identify with $\mathbb{R}^{n-1} \times \{0\} \subset \mathbb{R}^n$) to the $n-1$-dimensional affine space containing F_j. For $y \in \rho\mathbb{Z}^{n-1}$ we let $Q_{j,y} := A_j(y + \rho(-\frac{1}{2}, \frac{1}{2})^n)$. Let I_j denote the set of $y \in \rho\mathbb{Z}^{n-1}$ such that $Q_{j,y} \cap F_j' \neq \emptyset$. Since $\mathrm{dist}(E_k', \partial E_k) > 0$ for all k and $\mathrm{dist}(F_j', F_l) > 0$ for $j \neq l$, for ρ sufficiently small we have that for all j and all $y \in I_j$ one has $u_1 = u_2$ on $Q_{j,y}$ and that, for all $l \neq j$, $Q_{j,y} \cap F_l = \emptyset$, and $Q_{j,y} \cap Q_{l,y'} = \emptyset$ for $y' \in I_l$.

For each j and $y \in I_j$, let $v_{j,y}^\pm := u_1^\pm(A_j(y))$ denote the two traces of u_1 at the center of $Q_{j,y}$ (the traces have point values, since u_1 is piecewise affine). From these two values we construct a piecewise constant function by

$$V_{j,y}(x) := \begin{cases} v_{j,y}^+, & \text{if } x \in A_j(y + \mathbb{R}^{n-1} \times [0,\infty)), \\ v_{j,y}^-, & \text{if } x \in A_j(y + \mathbb{R}^{n-1} \times (-\infty, 0)). \end{cases} \qquad (3.24)$$

We remark that $\|V_{j,y} - u_1\|_{L^\infty(Q_{j,y})} \le \rho\sqrt{n}M_\delta$.

By (3.11) there is $u_{j,y} \in SBV(Q_{j,y};\mathbb{R}^m)$ with $\nabla u_{j,y} = 0$ $\mathcal{L}^n$-a.e. on $Q_{j,y}$ such that

$$\int_{Q_{j,y}\cap J_{u_{j,y}}} g([u_{j,y}], \nu_{u_{j,y}})\mathrm{d}\mathcal{H}^{n-1} \le \rho^{n-1}(\frac{\delta}{M_\delta} + g_{BV}([V_{j,y}], \nu_j)), \qquad (3.25)$$

with

$$\mathrm{supp}\,(u_{j,y} - V_{j,y}) \subset\subset Q_{j,y}, \quad \text{and} \quad \|u_{j,y} - V_{j,y}\|_{L^1(Q_{j,y})} \le \delta\rho^n. \qquad (3.26)$$

We set

$$u^{(\rho)} := u_2 + \sum_j \sum_{y\in I_j} \chi_{Q_{j,y}}(u_{j,y} - V_{j,y}). \qquad (3.27)$$

By (3.26), no additional jump is inserted on $\partial Q_{j,y}$. In particular, $u^{(\rho)} \in SBV(\Omega;\mathbb{R}^m)$ with

$$\nabla u^{(\rho)} = \nabla u_2, \qquad (3.28)$$

so that (3.21) holds with $u^{(\rho)}$ in place of u_2, for any $\rho > 0$. Further,

$$\begin{aligned}
&\mathcal{H}^{n-1}(J_{u^{(\rho)}} \cap \partial\Omega) = 0,\\
&[u^{(\rho)}] = [u_{j,y}] + [u_2 - V_{j,y}] \text{ in } Q_{j,y},\\
&[u^{(\rho)}] = [u_2] \text{ outside } \bigcup_j \bigcup_{y\in I_j} Q_{j,y}.
\end{aligned} \tag{3.29}$$

We recall that $u_1 = u_2$ in each $Q_{j,y}$, so that

$$\|u_2 - V_{j,y}\|_{L^\infty(Q_{j,y})} = \|u_1 - V_{j,y}\|_{L^\infty(Q_{j,y})} \le \rho\sqrt{n}M_\delta. \tag{3.30}$$

Choosing ρ sufficiently small, we can have

$$g_0(2\rho\sqrt{n}M_\delta) \le \frac{\delta}{M_\delta}. \tag{3.31}$$

Since both u_2 and $V_{j,y}$ only jump on the midplane of $Q_{j,y}$, the second of (3.29), (1.8), (3.25), (3.30), and (3.31) lead to

$$\begin{aligned}
&\int_{Q_{j,y}\cap J_{u^{(\rho)}}} g([u^{(\rho)}], \nu_{u^{(\rho)}}) \mathrm{d}\mathcal{H}^{n-1}\\
&\qquad \le \int_{Q_{j,y}\cap J_{u_{j,y}}} g([u_{j,y}], \nu_{u_{j,y}}) \mathrm{d}\mathcal{H}^{n-1} + cg_0(2\rho\sqrt{n}M_\delta)\rho^{n-1}\\
&\qquad \le \int_{Q_{j,y}\cap F_j} \left[\frac{\delta}{M_\delta} + g_{BV}([V_{j,y}], \nu_j)\right] \mathrm{d}\mathcal{H}^{n-1} + c\frac{\delta}{M_\delta}\rho^{n-1}.
\end{aligned} \tag{3.32}$$

In turn, since g_{BV} obeys (1.8), we obtain by (3.30)

$$g_{BV}([V_{j,y}], \nu_j) \le g_{BV}([u_1], \nu_j) + cg_0(2\rho\sqrt{n}M_\delta) \tag{3.33}$$

$\mathcal{H}^{n-1}$-a.e. on $Q_{j,y} \cap F_j$, so that with (3.31) the above estimate reduces to

$$\int_{Q_{j,y}\cap J_{u^{(\rho)}}} g([u^{(\rho)}], \nu_{u^{(\rho)}}) \mathrm{d}\mathcal{H}^{n-1} \le 3c\frac{\delta}{M_\delta}\rho^{n-1} + \int_{Q_{j,y}\cap F_j} g_{BV}([u_1], \nu_j) \mathrm{d}\mathcal{H}^{n-1} \tag{3.34}$$

and summing over $y \in I_j$

$$\sum_{y\in I_j} \int_{Q_{j,y}\cap J_{u^{(\rho)}}} g([u^{(\rho)}], \nu_{u^{(\rho)}}) \mathrm{d}\mathcal{H}^{n-1} \le 3c\frac{\delta}{M_\delta}\mathcal{H}^{n-1}(F_j) + \int_{\Omega\cap F_j} g_{BV}([u_1], \nu_j) \mathrm{d}\mathcal{H}^{n-1}. \tag{3.35}$$

At the same time, the last of the (3.29) implies

$$\begin{aligned}
\int_{F_j\setminus\cup Q_{j,y}} g([u^{(\rho)}], \nu_{u^{(\rho)}}) \mathrm{d}\mathcal{H}^{n-1} &= \int_{F_j\setminus\cup Q_{j,y}} g([u_1], \nu_j) \mathrm{d}\mathcal{H}^{n-1}\\
&\le cg_0(M_\delta)\mathcal{H}^{n-1}(F_j \setminus F'_j),
\end{aligned} \tag{3.36}$$

where we used (3.15) and monotonicity of g_0. Since these sets cover the jump set of $u^{(\rho)}$, we conclude by (3.34) and (3.36)

$$\int_{\Omega\cap J_{u^{(\rho)}}} g([u^{(\rho)}],\nu_{u^{(\rho)}})d\mathcal{H}^{n-1} \le c g_0(M_\delta)\sum_j \mathcal{H}^{n-1}(F_j\setminus F_j') + 3c\frac{\delta}{M_\delta}\sum_j \mathcal{H}^{n-1}(F_j) + \int_{\Omega\cap J_{u_1}} g_{BV}([u_1],\nu_{u_1})d\mathcal{H}^{n-1}. \tag{3.37}$$

For ρ sufficiently small we conclude, with (3.23) and (3.15), that

$$\int_{\Omega\cap J_{u^{(\rho)}}} g([u^{(\rho)}],\nu_{u^{(\rho)}})d\mathcal{H}^{n-1} \le C\delta + \int_{\Omega\cap J_{u_1}} g_{BV}([u_1],\nu_{u_1})d\mathcal{H}^{n-1} \tag{3.38}$$

and, with (3.27), (3.19) and (3.26), that

$$\|u^{(\rho)}-u_1\|_{L^1(\Omega)} \le \mathcal{L}^n\big(\Omega\big)\|u_2-u_1\|_{L^\infty(\Omega)} + \sum_{j,y}\|u_{j,y}-V_{j,y}\|_{L^1(Q_{j,y})} \le C\delta\mathcal{L}^n(\Omega). \tag{3.39}$$

Combining (3.21), (3.28), and (3.38) we obtain $H_0(u^{(\rho)}) \le \mathcal{F}(u_1)+C\delta$, and with (3.14) the proof is concluded. □

4 Phase field approximation

In this section we establish a variational approximation of lower semicontinuous energies as in (1.2) by phase field models.

4.1 Model

We assume that $\Psi:\mathbb{R}^{m\times n}\to[0,\infty)$ is continuous, and for some $q>1$ satisfies (1.15), namely

$$\Big(\frac{1}{c}|\xi|^q - c\Big)\vee 0 \le \Psi(\xi) \le c(|\xi|^q+1) \qquad \text{for all } \xi\in\mathbb{R}^{m\times n}.$$

Moreover, we assume the ensuing limit to exist

$$\Psi_\infty(\xi) := \lim_{t\to\infty}\frac{\Psi(t\xi)}{t^q}, \tag{4.1}$$

and to be uniform on the set of ξ with $|\xi|=1$. This means that for every $\delta>0$ there is t_δ such that $|\Psi(t\xi)/t^q - \Psi_\infty(\xi)|\le\delta$ for all $t\ge t_\delta$ and all ξ with $|\xi|=1$, which is the same as

$$|\Psi(\xi)-\Psi_\infty(\xi)| \le \delta|\xi|^q \qquad \text{for all } |\xi|\ge t_\delta. \tag{4.2}$$

By scaling, $\Psi_\infty(t\xi) = t^q \Psi_\infty(\xi)$ and in particular $\Psi_\infty(0) = 0$. Uniform convergence also implies $\Psi_\infty \in C^0(\mathbb{R}^{m\times n})$ and (1.15) yields

$$\frac{1}{c}|\xi|^q \leq \Psi_\infty(\xi) \leq c|\xi|^q \qquad \text{for all } \xi \in \mathbb{R}^{m\times n}. \tag{4.3}$$

Following the analysis in the scalar case in [CFI16, Section 7.2] we provide an approximation of a model with power-law growth at small openings (cf. [CFI24] for the analogous model with linear growth). For all $\varepsilon > 0$, $q > 1$ and $p > 0$ we consider the functional $\mathcal{F}_{\varepsilon,p,q} : L^1(\Omega;\mathbb{R}^{m+1}) \times \mathcal{B}(\Omega) \to [0,\infty]$ given by

$$\mathcal{F}_{\varepsilon,p,q}(u,v;A) := \int_A \left(f^q_{\varepsilon,p,q}(v)\Psi(\nabla u) + \frac{(1-v)^{q'}}{q' q^{q'/q}\varepsilon} + \varepsilon^{q-1}|\nabla v|^q \right) \mathrm{d}x \tag{4.4}$$

if $(u,v) \in W^{1,q}(\Omega;\mathbb{R}^m \times [0,1])$ and ∞ otherwise, where q' denotes the conjugate exponent of q, and for every $t \in [0,1)$

$$f_p(t) := \frac{\ell t}{(1-t)^p}, \qquad f_{\varepsilon,p,q}(t) := 1 \wedge \varepsilon^{1-1/q} f_p(t), \qquad f_{\varepsilon,p,q}(1) := 1\,.$$

Here and below, $\ell > 0$ is a fixed parameter. Let us remark that $(0,\infty) \ni p \mapsto f_{\varepsilon,p,q}(t)$ is increasing for all $t \in [0,1]$. Therefore, we deduce that $\mathcal{F}_{\varepsilon,p,q} \geq \mathcal{F}_{\varepsilon,1,q}$ on $L^1(\Omega;\mathbb{R}^{m+1})$ if $p \geq 1$. In particular, for $q = 2$, the asymptotic analysis of the family $(\mathcal{F}_{\varepsilon,1,2})_{\varepsilon>0}$ has been performed in [CFI24], where its convergence to a functional of cohesive type with surface energy densities with linear growth for small jump amplitudes has been established. For every $q > 1$ and $p = 1$ one can prove a completely analogous result following the same arguments. Instead, here we will focus on the supercritical case $p > 1$ leading to surface energy densities with superlinear growth for small jump amplitudes, as investigated in [CFI16, Theorem 7.4] in the scalar isotropic case. In the subcritical setting, i.e. $p \in (0,1)$, a comparison argument yields that the Γ-limit is then trivial, i.e. identically null (cf. [ACF25a]).

To state our result we need to introduce some notation. For all Borel subsets $A \in \mathcal{B}(\Omega)$, and $(u,v) \in W^{1,q}(\Omega;\mathbb{R}^m \times [0,1])$, it is convenient to define for $M \in (0,\infty]$

$$\mathcal{F}^M_{\varepsilon,p,q}(u,v;A) := \int_A \Big((M^{q-1} \wedge \varepsilon^{q-1} f^q_p(v))\Psi_\infty(\nabla u) + \frac{(1-v)^{q'}}{q' q^{q'/q}\varepsilon} + \varepsilon^{q-1}|\nabla v|^q \Big) \mathrm{d}x. \tag{4.5}$$

In particular, we have

$$\mathcal{F}^\infty_{\varepsilon,p,q}(u,v;A) = \int_A \Big(\varepsilon^{q-1} f^q_p(v)\Psi_\infty(\nabla u) + \frac{(1-v)^{q'}}{q' q^{q'/q}\varepsilon} + \varepsilon^{q-1}|\nabla v|^q \Big) \mathrm{d}x. \tag{4.6}$$

Lower and upper bounds for $\Gamma(L^1)$-limits of $(\mathcal{F}_{\varepsilon,p,q})_\varepsilon$ will be expressed in terms of two different surface energies, defined respectively as

$$\begin{aligned} g_{\inf}(z,\nu) := \inf\{ \liminf_{j\to\infty} \mathcal{F}^{M_j}_{\varepsilon_j,p,q}(u_j,v_j;Q^\nu) :& \\ \|u_j - z\chi_{\{x\cdot\nu>0\}}\|_{L^1(Q^\nu)} \to 0,\ \varepsilon_j \to 0,\ M_j \to \infty\}& \end{aligned} \tag{4.7}$$

and

$$g_{\sup}(z,\nu) := \inf\{\liminf_{j\to\infty} \mathcal{F}^{\infty}_{\varepsilon_j,p,q}(u_j,v_j;Q^\nu) : \\ \|u_j - z\chi_{\{x\cdot\nu>0\}}\|_{L^1(Q^\nu)} \to 0,\, \varepsilon_j \to 0\}\,, \tag{4.8}$$

with $(z,\nu)\in\mathbb{R}^m\times S^{n-1}$. By Lemma 4.3, these expressions depend on ν and not on the choice of Q^ν. Obviously one can restrict to sequences $v_j\to 1$ in $L^1(Q^\nu)$. It is immediate to see that

$$g_{\inf}(z,\nu) \le g_{\sup}(z,\nu) \quad \text{for all } (z,\nu)\in\mathbb{R}^m\times S^{n-1}. \tag{4.9}$$

In general, we are not able to prove the equality of the above functions. As a consequence, we carry out the full Γ-convergence analysis under the conditional assumption

$$g_{\inf}(z,\nu) = g_{\sup}(z,\nu) \quad \text{for all } (z,\nu)\in\mathbb{R}^m\times S^{n-1}, \tag{4.10}$$

which we actually show to be verified in case the q-recession function Ψ_∞ of Ψ satisfies the projection property, namely

$$\Psi_\infty(\xi) \ge \Psi_\infty(\xi\nu\otimes\nu) \quad \text{for every } (\xi,\nu)\in\mathbb{R}^{m\times n}\times S^{n-1}. \tag{4.11}$$

In general, we provide lower and upper bounds for the Γ-inferior and superior limits which differ only for what the surface energy density is concerned. Indeed, a major difficulty in the asymptotic analysis is to prove that the limit does not depend on the chosen infinitesimal sequence, a fact related to the lack of rescaling property of the functional $\mathcal{F}_{\varepsilon,p,q}$. This issue was solved in the linear case $p=1$ (and $q=2$) in any dimension by an elementary truncation argument in the v-variable (see [CFI24, Proposition 4.1]). The latter has no immediate analogue in the current superlinear setting due to the presence of the two exponents $p>1$ (in f_p) and q (in the dissipation potential), which yield two different powers of ε as truncation thresholds. On the other hand, the one-dimensional superlinear setting has been dealt with in [CFI16, Theorem 7.4] with an ad hoc argument, exploiting an ε-independent alternative characterization of the surface energy density (cf. (4.50)). A different argument is used in what follows to settle the case in which Ψ_∞ satisfies the projection property.

Theorem 4.1. *Let Ψ be satisfying* (1.15), (4.2)*, and let $\mathcal{F}_{\varepsilon,p,q}$ be the functional defined in* (4.4)*, with p, $q>1$. Assume that* (4.10) *holds, and denote by $g(z,\nu)$ the common value.*

Then for all $(u,v)\in L^1(\Omega;\mathbb{R}^{m+1})$ it holds

$$\Gamma(L^1)\text{-}\lim_{\varepsilon\to 0} \mathcal{F}_{\varepsilon,p,q}(u,v) = \mathcal{F}_{p,q}(u,v),$$

where

$$\mathcal{F}_{p,q}(u,v) := \int_\Omega \Psi^{\mathrm{qc}}(\nabla u)\mathrm{d}x + \int_{J_u} g([u],\nu_u)\mathrm{d}\mathcal{H}^{n-1}, \tag{4.12}$$

if $u\in(GSBV\cap L^1(\Omega))^m$ with $\nabla u\in L^q(\Omega;\mathbb{R}^{m\times n})$ and $v=1$ $\mathcal{L}^n$-a.e. on Ω, and $\mathcal{F}_{p,q}(u,v):=\infty$ otherwise. In particular, g is BV-elliptic.

In case the projection property holds for Ψ_∞ we are able to prove the equality in (4.10) and then deduce Theorem 1.4 in the introduction (cf. [CFI24, Remark 3.6]). We recall the statement for the sake of convenience.

Theorem 4.2. *Let Ψ be satisfying* (1.15)*,* (4.2)*, and* (4.11)*. Then $g_{\inf}(z,\nu) = g_{\sup}(z,\nu) = g(z,\nu)$ for all $(z,\nu) \in \mathbb{R}^m \times S^{n-1}$, where*

$$g(z,\nu) := \lim_{T\uparrow\infty} \inf_{\mathcal{V}_z^T} \int_{-T/2}^{T/2} \Big(f_p^q(\beta(t))\Psi_\infty\big(\alpha'(t)\otimes\nu\big) + \frac{(1-\beta(t))^{q'}}{q' q^{q'/q}} + |\beta'(t)|^q \Big) \mathrm{d}t \tag{4.13}$$

and

$$\begin{aligned}\mathcal{V}_z^T := \{(\alpha,\beta) \in & W^{1,q}((-T/2, T/2);\mathbb{R}^{m+1}): \\ & 0 \le \beta \le 1,\ \beta(\pm T/2) = 1,\ \alpha(-T/2) = 0,\ \alpha(T/2) = z\}.\end{aligned} \tag{4.14}$$

In particular, the $\Gamma(L^1)$-convergence result stated in Theorem 4.1 holds in this case.

The proof uses in a crucial way the density result obtained in [CFI25]. In particular, this completes the proof of the scalar case [CFI16, Theorem 7.4] for which a density theorem was in fact missing.

Having fixed $p, q > 1$, throughout the rest of the paper to simplify the notation we omit to indicate the subscripts p and q in the approximating functionals $\mathcal{F}_{\varepsilon,p,q}$, in the limiting functional $\mathcal{F}_{p,q}$, and also in $\mathcal{F}^M_{\varepsilon,p,q}$ for every $M \in (0,\infty]$. Thus, we will simply write $\mathcal{F}_\varepsilon$, $\mathcal{F}$ and $\mathcal{F}^M_\varepsilon$, respectively. Similarly, we will write f_ε for $f_{\varepsilon,p,q}$.

4.2 Characterizations of the surface energy densities

In this section we prove several structural properties of the surface energy densities $g_{\inf}$ and $g_{\sup}$.

We first show that they are well-defined, in the sense that they do not depend on the choice of Q^ν. This is a standard argument, well known to experts. We briefly sketch it here to show that the interaction of the rescaling with the sequence M_j does not cause any problem.

Lemma 4.3. *The definitions of $g_{\sup}$ and $g_{\inf}$ do not depend on the choice of the cube Q^ν.*

Proof. We deal explicitly with $g_{\inf}$, the case of $g_{\sup}$ is similar but simpler, as one does not have to scale M_j. Fix $\nu \in S^{n-1}$, $z \in \mathbb{R}^m$, and two possible choices of the cube, call them Q^ν and Q^ν_* for definiteness. They differ by a rotation with axis ν. For clarity, in this proof we denote the two corresponding expressions for $g_{\inf}$ by $g_{\inf}(z,\nu;Q^\nu)$ and $g_{\inf}(z,\nu;Q^\nu_*)$, respectively. We intend to show that $g_{\inf}(z,\nu;Q^\nu_*) \le g_{\inf}(z,\nu;Q^\nu)$.

By definition of $g_{\inf}$ there are sequences u_j, v_j, $\varepsilon_j \to 0$, $M_j \to \infty$ such that

$$\lim_{j\to\infty} \mathcal{F}^{M_j}_{\varepsilon_j}(u_j, v_j; Q^\nu) = g_{\inf}(z,\nu;Q^\nu), \qquad \lim_{j\to\infty} \|u_j - z\chi_{\{x\cdot\nu>0\}}\|_{L^1(Q^\nu)} = 0. \tag{4.15}$$

Let $\lambda \in (0,1)$. Let $\{x_k\}_{k=1,\dots,K_\lambda}$ be points with $x_k \cdot \nu = 0$ and such that the cubes $q_k := x_k + \lambda Q_*^\nu$ are pairwise disjoint and contained in Q^ν. If one chooses the points on a regular grid aligned with Q_*^ν, one can ensure that

$$\lim_{\lambda\to 0} \lambda^{n-1} K_\lambda = 1. \tag{4.16}$$

From (4.15) we obtain that for any fixed λ

$$\limsup_{j\to\infty} \sum_{k=1}^{K_\lambda} \left[\mathcal{F}_{\varepsilon_j}^{M_j}(u_j, v_j; q_k) - \frac{g_{\inf}(z,\nu;Q^\nu)}{K_\lambda} \right] \leq 0, \tag{4.17}$$

therefore we can choose $k_* \in \{1,\dots,K_\lambda\}$ such that

$$\limsup_{j\to\infty} \left[\mathcal{F}_{\varepsilon_j}^{M_j}(u_j, v_j; q_{k_*}) - \frac{g_{\inf}(z,\nu;Q^\nu)}{K_\lambda} \right] \leq 0. \tag{4.18}$$

We also obtain that $\lim_{j\to\infty} \|u_j - z\chi_{\{x\cdot\nu>0\}}\|_{L^1(q_{k_*})} = 0$ by the second of (4.15) and $q_{k_*} \subset Q^\nu$. We define $u_j^{(\lambda)}(x) := u_j(x_{k_*} + \lambda x)$, $v_j^{(\lambda)}(x) := v_j(x_{k_*} + \lambda x)$, $\varepsilon_j^{(\lambda)} := \varepsilon_j/\lambda$, $M_j^{(\lambda)} := M_j/\lambda$. Then for $j \to \infty$ one has $\varepsilon_j^{(\lambda)} \to 0$, $M_j^{(\lambda)} \to \infty$, $u_j^{(\lambda)} \to z\chi_{\{x\cdot\nu>0\}}$ in $L^1(Q_*^\nu;\mathbb{R}^m)$, and a simple scaling, using that Ψ_∞ is positively q-homogeneous, gives

$$\begin{aligned}\limsup_{j\to\infty} \mathcal{F}_{\varepsilon_j^{(\lambda)}}^{M_j^{(\lambda)}}(u_j^{(\lambda)}, v_j^{(\lambda)}; Q_*^\nu) =& \limsup_{j\to\infty} \lambda^{1-n} \mathcal{F}_{\varepsilon_j}^{M_j}(u_j, v_j; q_{k_*}) \\ \leq& \frac{1}{\lambda^{n-1}K_\lambda} g_{\inf}(z,\nu;Q^\nu),\end{aligned}$$

implying

$$g_{\inf}(z,\nu;Q_*^\nu) \leq \frac{1}{\lambda^{n-1}K_\lambda} g_{\inf}(z,\nu;Q^\nu).$$

Since λ was arbitrary, with (4.16) the proof is concluded. □

We start showing that test sequences in the definitions of $g_{\inf}$ in (4.7) and of $g_{\sup}$ in (4.8) may be taken converging in L^q and satisfying periodic boundary conditions in $(n-1)$ directions orthogonal to ν and mutually orthogonal to each other, analogously to what established in [CFI24, Section 3.1] for $p = 1$. This is the content of the next two propositions. We fix a mollifier $\varphi_1 \in C_c^\infty(B_1;[0,\infty))$, with $\int_{B_1} \varphi_1 \mathrm{d}x = 1$ and $\varphi(-x) = \varphi(x)$, and set $\varphi_\varepsilon(x) := \varepsilon^{-n}\varphi_1(x/\varepsilon)$.

Proposition 4.4. *There are $\varepsilon_0 > 0$, $\gamma_1 > 0$, $\gamma_2 > 0$, depending only on p and q, such that the following holds. Let $(u,v) \in L^q(Q^\nu;\mathbb{R}^{m+1})$, $\varepsilon \in (0,\varepsilon_0)$, and let $(z,\nu) \in \mathbb{R}^m \times S^{n-1}$. Then, there are (u^*,v^*) such that*

$$\mathcal{F}_\varepsilon^\infty(u^*,v^*;Q^\nu) \leq \mathcal{F}_\varepsilon^\infty(u,v;Q^\nu) + Cd_q^{\gamma_1} + C\varepsilon^{\gamma_2},$$

where $d_q := \|u - z\chi_{\{x\cdot\nu>0\}}\|_{L^q(Q^\nu)}$, and

$$u^* = (z\chi_{\{x\cdot\nu>0\}}) * \varphi_\varepsilon, \qquad v^* = \chi_{\{|x\cdot\nu|\geq 2\varepsilon\}} * \varphi_\varepsilon \qquad \text{on } \partial Q^\nu. \tag{4.19}$$

The constant C may depend on q, p, ℓ, z, ν, and Ψ; but not on u, v, and ε.

Proof. **Step 1. Construction of u^* and v^*.** We write

$$U := (z\chi_{\{x\cdot\nu>0\}}) * \varphi_\varepsilon, \qquad V := \chi_{\{|x\cdot\nu|\geq 2\varepsilon\}} * \varphi_\varepsilon. \tag{4.20}$$

Obviously, $\|U - z\chi_{\{x\cdot\nu>0\}}\|^q_{L^q(Q^\nu)} \leq C\varepsilon$, with $C > 0$ depending on z, so that $\|u - U\|^q_{L^q(Q^\nu)} \leq C(\varepsilon + d_q^q)$. Moreover, by construction $\nabla U = 0$ if $|x \cdot \nu| \geq \varepsilon$, $V = 0$ if $|x \cdot \nu| \leq \varepsilon$, and $V = 1$ if $|x \cdot \nu| \geq 3\varepsilon$. Therefore, by $\Psi_\infty(0) = 0$ and $f_p(0) = 0$, we have

$$\mathcal{F}^\infty_\varepsilon(U, V; Q^\nu) = \mathcal{F}^\infty_\varepsilon(0, V; Q^\nu) \leq C + \varepsilon^{q-1} \int_{\{x\in Q^\nu : \varepsilon<|x\cdot\nu|<3\varepsilon\}} |\nabla V|^q \mathrm{d}x \leq C\,,$$

as $\|\nabla V\|_{L^\infty(\mathbb{R}^m)} \leq C/\varepsilon$. This implies that there is $c_0 > 0$ such that if $d_q \vee \mathcal{F}^\infty_\varepsilon(u, v; Q^\nu) \geq c_0$, then the pair (U, V) will give the thesis. Therefore, in the rest of the proof we can assume that

$$d_q \leq c_0 \text{ and } \mathcal{F}^\infty_\varepsilon(u, v; Q^\nu) \leq c_0.$$

Next, we fix $\eta \in [\varepsilon^{1/q}, 1/10]$ and set $K := \lfloor 4\eta^q/\varepsilon \rfloor \geq 4$, so that $(2K + 1)\varepsilon < 1$ and $K\varepsilon \geq \eta^q$. We let $R_k := Q^\nu_{1-k\varepsilon} \setminus Q^\nu_{1-(k+1)\varepsilon}$, where as usual $Q^\nu_r = rQ^\nu$ is the scaled cube. We select $k \in \{K + 1, \ldots, 2K\}$ such that

$$\|u - U\|^q_{L^q(R_k)} \leq \frac{C}{K} \|u - U\|^q_{L^q(Q^\nu)} \leq C \frac{\varepsilon + d_q^q}{K}\,, \tag{4.21}$$

and

$$\mathcal{F}^\infty_\varepsilon(u, v; R_k) + \mathcal{F}^\infty_\varepsilon(U, V; R_k) \leq \frac{C}{K}. \tag{4.22}$$

We fix $\theta \in C^1_c(Q^\nu_{1-k\varepsilon}; [0, 1])$ with $\theta = 1$ on $Q^\nu_{1-(k+1)\varepsilon}$ and $|\nabla\theta| \leq 3/\varepsilon$, and define

$$u^* := \theta u + (1 - \theta)U.$$

The construction of v^* is more complex. In the interior part, it should match v. In the exterior, V. In the interpolation region R_k, which has volume of order ε, it should be not larger than v and V, but also not larger than $1 - \eta$. The parameter $\eta \in (0, 1/10)$ will be chosen so that one can obtain a good estimate for the integral of $\varepsilon^{q-1} f_p^q(v)\Psi_\infty(\nabla u^*)$ over R_k. Therefore we first define

$$v_{\mathrm{mid}}(x) := \min\{1, 1 - \eta + \frac{1}{\varepsilon}\mathrm{dist}(x, R_k)\}, \tag{4.23}$$

which coincides with $1-\eta$ in the interpolation region R_k, and with 1 at distance larger than $\eta\varepsilon$ from it, in particular inside $Q^\nu_{1-(k+2)\varepsilon}$ and outside $Q^\nu_{1-(k-1)\varepsilon}$. Then, we set

$$v_{\mathrm{out}}(x) := \min\{1, V(x) + \frac{2}{\varepsilon}\mathrm{dist}(x, Q^\nu \setminus Q^\nu_{1-(k+1)\varepsilon})\} \tag{4.24}$$

which coincides with V outside $Q^\nu_{1-(k+1)\varepsilon}$, and with 1 inside $Q^\nu_{1-(k+2)\varepsilon}$ as well as for $|x\cdot\nu|\ge 3\varepsilon$, and finally

$$v_{\rm in}(x) := \min\{1, v(x) + \frac{2}{k\varepsilon}\mathrm{dist}(x, Q^\nu_{1-k\varepsilon})\}\,, \tag{4.25}$$

which is equal to v on $Q^\nu_{1-k\varepsilon}$ and to 1 at distance larger than $\frac{k\varepsilon}{2}$ from it. We then combine these three ingredients to obtain

$$v^* := \min\{v_{\rm in}, v_{\rm mid}, v_{\rm out}\}.$$

On ∂Q^ν, both $v_{\rm in}$ and $v_{\rm mid}$ equal 1, hence $v^* = v_{\rm out} = V$ on this set.

Step 2. Estimate of the elastic energy. By the definition of u^*,

$$|\nabla u^*| \le |\nabla u| + |\nabla U| + \frac{3}{\varepsilon}|u-U|$$

therefore by (4.3) in R_k we have

$$\Psi_\infty(\nabla u^*) \le C\Psi_\infty(\nabla u) + C\Psi_\infty(\nabla U) + \frac{C}{\varepsilon^q}|u-U|^q\,.$$

We observe that $v^* = \min\{v, 1-\eta, V\}$ in R_k. Set $F(t) := \varepsilon^{q-1}f_p^q(t)$ for brevity, one easily checks that F is increasing, with $F(0) = 0$. Since by construction $v^* = V = 0$ on $R_k\cap\{|x\cdot\nu|\le\varepsilon\}$, and $\nabla U = 0$ on $R_k\cap\{|x\cdot\nu|\ge\varepsilon\}$, the term $F(v^*)\Psi_\infty(\nabla U)$ vanishes in R_k. Therefore, using $F(v^*)\le F(v_{\rm mid})\le \varepsilon^{q-1}\ell^q/\eta^{pq}$ in R_k we have

$$F(v^*)\Psi_\infty(\nabla u^*) \le CF(v)\Psi_\infty(\nabla u) + C\frac{|u-U|^q}{\varepsilon\eta^{pq}}.$$

Integrating over R_k and using (4.22) in the first term, (4.21) in the second one,

$$\int_{R_k} F(v^*)\Psi_\infty(\nabla u^*)\mathrm{d}x \le \frac{C}{K} + C\frac{\varepsilon + d_q^q}{K\varepsilon\eta^{pq}}.$$

Recalling that the definition of K implies $K\varepsilon\ge\eta^q$,

$$\int_{R_k} F(v^*)\Psi_\infty(\nabla u^*)\mathrm{d}x \le \frac{C}{K} + C\frac{\varepsilon + d_q^q}{\eta^{q(p+1)}}.$$

Using that $u^* = U$ and $v^*\le v_{\rm out} = V$ in $Q^\nu\setminus Q^\nu_{1-k\varepsilon}$, that $V = 0$ on $Q^\nu\cap\{|x\cdot\nu|\le\varepsilon\}$, and $\nabla U = 0$ on $Q^\nu\cap\{|x\cdot\nu|\ge\varepsilon\}$, we have

$$\int_{Q^\nu\setminus Q^\nu_{1-k\varepsilon}} F(v^*)\Psi_\infty(\nabla u^*)\mathrm{d}x = 0. \tag{4.26}$$

In $Q^\nu_{1-(k+1)\varepsilon}$ instead we have $u^* = u$ and $v^*\le v_{\rm in} = v$, so that

$$\int_{Q^\nu_{1-(k+1)\varepsilon}} F(v^*)\Psi_\infty(\nabla u^*)\mathrm{d}x \le \int_{Q^\nu_{1-(k+1)\varepsilon}} F(v)\Psi_\infty(\nabla u)\mathrm{d}x.$$

Adding terms,

$$\int_{Q^\nu} F(v^*)\Psi_\infty(\nabla u^*)\mathrm{d}x \le \int_{Q^\nu} F(v)\Psi_\infty(\nabla u)\mathrm{d}x + \frac{C}{K} + C\frac{\varepsilon + d_q^q}{\eta^{q(p+1)}}. \tag{4.27}$$

Step 3. Estimate of the energy of the phase field. By the definition of v^*,

$$\mathcal{F}_\varepsilon^\infty(0, v^*; Q^\nu) \le \mathcal{F}_\varepsilon^\infty(0, v_{\text{in}}; Q^\nu) + \mathcal{F}_\varepsilon^\infty(0, v_{\text{mid}}; Q^\nu) + \mathcal{F}_\varepsilon^\infty(0, v_{\text{out}}; Q^\nu). \tag{4.28}$$

From (4.23) we have $|1 - v_{\text{mid}}| \le \eta$ with $|\{v_{\text{mid}} \ne 1\}| \le C\varepsilon$ and $|\nabla v_{\text{mid}}| \le 1/\varepsilon$ with $|\{\nabla v_{\text{mid}} \ne 0\}| \le C\varepsilon\eta$, so that

$$\mathcal{F}_\varepsilon^\infty(0, v_{\text{mid}}; Q^\nu) = \int_{Q^\nu} \Big(\frac{(1 - v_{\text{mid}})^{q'}}{q' q^{q'/q}\varepsilon} + \varepsilon^{q-1}|\nabla v_{\text{mid}}|^q\Big)\mathrm{d}x \le C\eta.$$

From the definition of V and v_{out}, we see that $|\{v_{\text{out}} \ne 1\}| \le C\varepsilon \cdot K\varepsilon$ and $\varepsilon|\nabla v_{\text{out}}| \le C$, so that

$$\mathcal{F}_\varepsilon^\infty(0, v_{\text{out}}; Q^\nu) \le CK\varepsilon.$$

Let $c_q > 0$ be such that for every $\delta \in (0, 1]$ and every a, $b \in \mathbb{R}^n$ one has $|a+b|^q \le (1+\delta)|a|^q + c_q\delta^{1-q}|b|^q$. As $v_{\text{in}} = v$ in $Q^\nu_{1-k\varepsilon}$, $v_{\text{in}} \ge v$, and $|\nabla v_{\text{in}}| \le |\nabla v| + 2/(k\varepsilon)$ in $Q^\nu \setminus Q^\nu_{1-k\varepsilon}$

$$\begin{aligned}\mathcal{F}_\varepsilon^\infty(0, v_{\text{in}}; Q^\nu) &\le (1 + \delta)\mathcal{F}_\varepsilon^\infty(0, v; Q^\nu) + \frac{c_q\varepsilon^{q-1}}{\delta^{q-1}(k\varepsilon)^q}\mathcal{L}^n(Q^\nu \setminus Q^\nu_{1-k\varepsilon}) \\ &\le (1 + \delta)\mathcal{F}_\varepsilon^\infty(0, v; Q^\nu) + \frac{c_q}{\delta^{q-1}k^{q-1}}.\end{aligned}$$

Recalling that $k \ge K + 1$ and $K\varepsilon \le 4\eta^q \le 4\eta$, (4.28) leads to

$$\mathcal{F}_\varepsilon^\infty(0, v^*; Q^\nu) \le (1 + \delta)\mathcal{F}_\varepsilon^\infty(0, v; Q^\nu) + C\eta + \frac{c_q}{\delta^{q-1}K^{q-1}}$$

so that choosing $\delta = K^{-1/q'}$

$$\mathcal{F}_\varepsilon^\infty(0, v^*; Q^\nu) \le \mathcal{F}_\varepsilon^\infty(0, v; Q^\nu) + C\eta + \frac{C}{K^{1/q'}}. \tag{4.29}$$

Step 4. Conclusion of the proof. Combining (4.29) with (4.27), using $1/K \le 1/K^{1/q'}$ and $K \ge \eta^q/\varepsilon$ to eliminate irrelevant terms leads to

$$\mathcal{F}_\varepsilon^\infty(u^*, v^*; Q^\nu) \le \mathcal{F}_\varepsilon^\infty(u, v; Q^\nu) + C\frac{\varepsilon + d_q^q}{\eta^{q(p+1)}} + C\eta + C\left(\frac{\varepsilon}{\eta^q}\right)^{1/q'}$$

for any $\eta \in [\varepsilon^q, 1/10]$. It remains to choose η. If $d_q \le \varepsilon^{1/q}$, using $\eta^q \le \eta$, the error term is controlled by

$$\frac{\varepsilon}{\eta^{q(p+1)}} + \eta + \left(\frac{\varepsilon}{\eta^q}\right)^{1/q'} \le \eta + 2\left(\frac{\varepsilon}{\eta^{q(p+1)}}\right)^{1/q'};$$

we pick $\eta := \varepsilon^{1/(q'+q(p+1))} \geq \varepsilon^{1/q}$ and conclude the proof. Similarly, if instead $d_q > \varepsilon^{1/q}$,

$$\left(\frac{d_q}{\eta^{p+1}}\right)^q + \eta + \left(\frac{d_q}{\eta}\right)^{q/q'} \leq \eta + 2\left(\frac{d_q}{\eta^{p+1}}\right)^{q/q'},$$

we conclude with $\eta := d_q^{q/(q'+q(p+1))} \wedge 1/10$. □

We are now ready to show equivalent characterizations of $g_{\inf}$ and $g_{\sup}$. We follow the arguments developed in [CFI24, Sections 3.1 and 3.2].

Proposition 4.5. *For every* $(z,\nu) \in \mathbb{R}^m \times S^{n-1}$

$$\begin{aligned} g_{\inf}(z,\nu) = \inf\{\liminf_{j\to\infty} \mathcal{F}_{\varepsilon_j}^{M_j}(u_j, v_j; Q^\nu) : \ & \|u_j - z\chi_{\{x\cdot\nu>0\}}\|_{L^q(Q^\nu)} \to 0, \\ & \varepsilon_j \to 0,\ M_j \to \infty,\ (u_j, v_j) \text{ satisfy } (4.19)\} \end{aligned} \tag{4.30}$$

and

$$\begin{aligned} g_{\sup}(z,\nu) = \inf\{\liminf_{j\to\infty} \mathcal{F}_{\varepsilon_j}^{\infty}(u_j, v_j; Q^\nu) : \ & \|u_j - z\chi_{\{x\cdot\nu>0\}}\|_{L^q(Q^\nu)} \to 0, \\ & \varepsilon_j \to 0,\ (u_j, v_j) \text{ satisfy } (4.19)\}. \end{aligned} \tag{4.31}$$

Proof. We give the proof only for $g_{\inf}$, that for $g_{\sup}$ being completely analogous.

Step 1. Reduction to an optimal sequence in (4.7) **converging in** $L^q(Q^\nu;\mathbb{R}^{m+1})$**.** Let $\varepsilon_j \to 0$, $M_j \to \infty$, and $(u_j, v_j) \to (z\chi_{\{x\cdot\nu>0\}}, 1)$ in $L^1(Q^\nu;\mathbb{R}^{m+1})$ be such that

$$g_{\inf}(z,\nu) = \lim_{j\to\infty} \mathcal{F}_{\varepsilon_j}^{M_j}(u_j, v_j; Q^\nu).$$

Recall that $v_j \in [0,1]$ $\mathcal{L}^n$-a.e. in Ω, therefore $v_j \to 1$ in $L^q(Q^\nu)$. By the definition of $\mathcal{T}_k$ in (1.21), we claim that for all j, $N \in \mathbb{N}$ there is $k_{N,j} \in \{N+1, \dots, 2N\}$ such that

$$\mathcal{F}_{\varepsilon_j}^{M_j}(\mathcal{T}_{k_{N,j}}(u_j), v_j; Q^\nu) \leq \Big(1 + \frac{C}{N}\Big)\mathcal{F}_{\varepsilon_j}^{M_j}(u_j, v_j; Q^\nu), \tag{4.32}$$

where $C > 0$ is a constant independent of N and j. If $a_N > |z| = \|z\chi_{\{x\cdot\nu>0\}}\|_{L^\infty(Q^\nu)}$ then $\mathcal{T}_{k_N,j}(u_j) \to z\chi_{\{x\cdot\nu>0\}}$ in $L^q(Q^\nu;\mathbb{R}^m)$, and (4.32) yields

$$\limsup_{j\to\infty} \mathcal{F}_{\varepsilon_j}^{M_j}(\mathcal{T}_{k_N,j}(u_j), v_j; Q^\nu) \leq \Big(1 + \frac{C}{N}\Big) g_{\inf}(z,\nu),$$

in turn implying by the arbitrariness of $N \in \mathbb{N}$

$$\begin{aligned} g_{\inf}(z,\nu) = \inf\{ & \liminf_{j\to\infty} \mathcal{F}_{\varepsilon_j}^{M_j}(u_j, v_j; Q^\nu) : \\ & \|u_j - z\chi_{\{x\cdot\nu>0\}}\|_{L^q(Q^\nu;\mathbb{R}^m)} \to 0, \varepsilon_j \to 0, M_j \to \infty\}. \end{aligned}$$

We are left with establishing (4.32). To this aim consider $\mathcal{T}_k(u_j)$ and note that

$$\begin{aligned} \mathcal{F}_{\varepsilon_j}^{M_j}(\mathcal{T}_k(u_j), v_j; Q^\nu) &= \mathcal{F}_{\varepsilon_j}^{M_j}(u_j, v_j; \{|u_j| \leq a_k\}) \\ &+ \mathcal{F}_{\varepsilon_j}^{M_j}(\mathcal{T}_k(u_j), v_j; \{a_k < |u_j| < a_{k+1}\}) + \mathcal{F}_{\varepsilon_j}^{M_j}(0, v_j; \{|u_j| \geq a_{k+1}\}). \end{aligned} \tag{4.33}$$

We estimate the second term in (4.33). The growth conditions on Ψ_∞ (cf. (4.3)) and $\mathrm{Lip}\,(\mathcal{T}_k) \leq 1$ yield for a constant $C > 1$

$$\begin{aligned}\mathcal{F}_{\varepsilon_j}^{M_j}(\mathcal{T}_k(u_j), v_j; \{a_k < |u_j| < a_{k+1}\}) \\ \leq C \int_{\{a_k<|u_j|<a_{k+1}\}} (M_j^q \wedge \varepsilon_j^{q-1} f_p^q(v_j))\Psi_\infty(\nabla u_j)\mathrm{d}x \\ + \mathcal{F}_{\varepsilon_j}^{M_j}(0, v_j; \{a_k < |u_j| < a_{k+1}\}) \,. \end{aligned} \tag{4.34}$$

Collecting (4.33) and (4.34) and using that for A and B disjoint $\mathcal{F}_{\varepsilon_j}^{M_j}(u_j, v_j; A)+ \mathcal{F}_{\varepsilon_j}^{M_j}(0, v_j; B) \leq \mathcal{F}_{\varepsilon_j}^{M_j}(u_j, v_j; A \cup B)$, we conclude that

$$\mathcal{F}_{\varepsilon_j}^{M_j}(\mathcal{T}_k(u_j), v_j; Q^\nu) \leq \mathcal{F}_{\varepsilon_j}^{M_j}(u_j, v_j; Q^\nu) + C\mathcal{F}_{\varepsilon_j}^{M_j}(u_j, v_j; \{a_k < |u_j| < a_{k+1}\}) \,.$$

Let now $N \in \mathbb{N}$, by averaging there exists $k_{N,j} \in \{N + 1, \ldots, 2N\}$ such that

$$\begin{aligned}\mathcal{F}_{\varepsilon_j}^{M_j}(\mathcal{T}_{k_{N,j}}(u_j), v_j; Q^\nu) &\leq \frac{1}{N} \sum_{k=N+1}^{2N} \mathcal{F}_{\varepsilon_j}^{M_j}(\mathcal{T}_k(u_j), v_j; Q^\nu) \\ &\leq \Big(1 + \frac{C}{N}\Big)\mathcal{F}_{\varepsilon_j}^{M_j}(u_j, v_j; Q^\nu) \,, \end{aligned} \tag{4.35}$$

i.e. (4.32).

Step 2. Construction of an optimal sequence converging in $L^q(Q^\nu; \mathbb{R}^{m+1})$ **and satisfying** (4.19)**.** In view of Step 1 there is an optimal sequence for $g_{\inf}(z, \nu)$ in (4.7) converging in $L^q(Q^\nu; \mathbb{R}^{m+1})$. Therefore, we may apply Proposition 4.4 and conclude. □

We show next that the infimum in the definition of $g_{\sup}$ is achieved whatever infinitesimal sequence ε_j is chosen (cf. [CFI24, Proposition 3.2]).

Proposition 4.6. *For any* $(z, \nu) \in \mathbb{R}^m \times S^{n-1}$ *and any* $\varepsilon_j^* \downarrow 0$ *there is* $(u_j^*, v_j^*) \to (z\chi_{\{x\cdot\nu>0\}}, 1)$ *in* $L^q(Q^\nu; \mathbb{R}^{m+1})$*, with* $v_j^* \in [0, 1]$ $\mathcal{L}^n$*-a.e. in* Ω*, and satisfying* (4.19)*, such that*

$$\lim_{j\to\infty} \mathcal{F}_{\varepsilon_j^*}^\infty(u_j^*, v_j^*; Q^\nu) = g_{\sup}(z, \nu) \,. \tag{4.36}$$

Proof. We apply first Proposition 4.5 to infer that for some $\varepsilon_j \to 0$ and $(u_j, v_j) \to (z\chi_{\{x\cdot\nu>0\}}, 1)$ in $L^q(Q^\nu; \mathbb{R}^{m+1})$ satisfying (4.19) it holds

$$g_{\sup}(z, \nu) = \lim_{j\to\infty} \mathcal{F}_{\varepsilon_j}^\infty(u_j, v_j; Q^\nu) \tag{4.37}$$

(cf. (4.31)). Since $\lim_{k\to\infty} \lim_{j\to 0} \varepsilon_j^*/\varepsilon_k = 0$, we can select a nondecreasing sequence $k(j) \to \infty$ such that $\lambda_j := \varepsilon_j^*/\varepsilon_{k(j)} \to 0$. We let $\tilde{Q}^\nu := (\mathrm{Id}_n - \nu \otimes \nu)Q^\nu \subset \nu^\perp \subset \mathbb{R}^n$ and select $x_1, \ldots, x_{I_j} \in \tilde{Q}^\nu$, with $I_j := \lfloor 1/\lambda_j \rfloor^{n-1}$, such that $x_i + \tilde{Q}^\nu_{\lambda_j}$ are pairwise disjoint subsets of $\tilde{Q}^\nu$. We set

$$u_j^*(x) := \begin{cases} u_{k(j)}(\frac{x-x_i}{\lambda_j}), & \text{if } x - x_i \in Q^\nu_{\lambda_j} \text{ for some } i, \\ U_j^*(x), & \text{otherwise in } Q^\nu \end{cases}$$

and

$$v_j^*(x) := \begin{cases} v_{k(j)}(\frac{x-x_i}{\lambda_j}), & \text{if } x - x_i \in Q^\nu_{\lambda_j} \text{ for some } i, \\ V_j^*(x), & \text{otherwise in } Q^\nu, \end{cases}$$

where U_j^* and V_j^* are defined as in (4.20) using ε_j^*. One easily verifies that $U_j^*(x) = U_{k(j)}(\frac{x-y}{\lambda_j})$ for all $y \in \nu^\perp$, and the same for V_j^*. By the boundary conditions (4.19), these functions are continuous and therefore in $W^{1,q}(Q^\nu;\mathbb{R}^{m+1})$. We further estimate

$$\mathcal{F}^\infty_{\varepsilon_j^*}(u_j^*, v_j^*; Q^\nu) \le I_j \lambda_j^{n-1} \mathcal{F}^\infty_{\varepsilon_{k(j)}}(u_{k(j)}, v_{k(j)}; Q^\nu) + C\mathcal{H}^{n-1}(\tilde{Q}^\nu \setminus \cup_i (x_i + \tilde{Q}^\nu_{\lambda_j})).$$

Taking $j \to \infty$, and recalling (4.37), concludes the proof. □

In what follows we provide a further equivalent characterization for the surface energy density $g_{\sup}$ in the spirit of [CFI16, Proposition 4.3] (see also [CFI24, Proposition 3.3]).

Proposition 4.7. *For any $(z,\nu) \in \mathbb{R}^m \times S^{n-1}$ one has*

$$g_{\sup}(z,\nu) = \lim_{T\to\infty} \inf_{(u,v)\in\mathcal{U}^T_{z,\nu}} \frac{1}{T^{n-1}} \mathcal{F}^\infty_1(u,v;Q^\nu_T)\,, \tag{4.38}$$

where

$$\mathcal{U}^T_{z,\nu} := \Big\{(u,v) \in W^{1,q}(Q^\nu_T;\mathbb{R}^{m+1})\colon 0 \le v \le 1,\ v = \chi_{\{|x\cdot\nu|\ge 2\}} * \varphi_1 \ \text{and}\ u = (z\chi_{\{x\cdot\nu>0\}}) * \varphi_1 \ \text{on } \partial Q^\nu_T\Big\}. \tag{4.39}$$

Proof. For every $(z,\nu) \in \mathbb{R}^m \times S^{n-1}$ and $T > 0$ set

$$g_T(z,\nu) := \inf_{(u,v)\in\mathcal{U}^T_{z,\nu}} \frac{1}{T^{n-1}} \mathcal{F}^\infty_1(u,v;Q^\nu_T)\,. \tag{4.40}$$

We first prove that

$$\limsup_{T\to\infty} g_T(z,\nu) \le g_{\sup}(z,\nu)\,. \tag{4.41}$$

Indeed, if $T_j \uparrow \infty$ is a sequence achieving the superior limit on the left-hand side above, thanks to Proposition 4.6 we may consider $(u_j, v_j) \in W^{1,q}(Q^\nu;\mathbb{R}^{m+1})$ with $0 \le v_j \le 1$, $(u_j,v_j) \to (z\chi_{\{x\cdot\nu>0\}}, 1)$ in $L^q(Q^\nu;\mathbb{R}^{m+1})$,

$$u_j = (z\chi_{\{x\cdot\nu>0\}}) * \varphi_{\frac{1}{T_j}}\,, \qquad v_j = \chi_{\{|x\cdot\nu|\ge\frac{2}{T_j}\}} * \varphi_{\frac{1}{T_j}} \ \text{on } \partial Q^\nu, \tag{4.42}$$

and

$$\lim_{j\to\infty} \mathcal{F}^\infty_{1/T_j}(u_j,v_j;Q^\nu) = g_{\sup}(z,\nu). \tag{4.43}$$

Then, define $(\tilde{u}_j(y), \tilde{v}_j(y)) := \big(u_j(\frac{y}{T_j}), v_j(\frac{y}{T_j})\big)$ for $y \in Q^\nu_{T_j}$, and note that by a change of variable it is true that

$$\frac{1}{T_j^{n-1}} \mathcal{F}^\infty_1(\tilde{u}_j,\tilde{v}_j;Q^\nu_{T_j}) = \mathcal{F}^\infty_{1/T_j}(u_j,v_j;Q^\nu)\,,$$

and that $(\tilde u_j, \tilde v_j) \in \mathcal{U}^{T_j}_{z,\nu}$ in view of (4.42). Then, by (4.43), the choice of T_j and the definition of $g_T(z,\nu)$ we conclude straightforwardly (4.41).

In order to prove the converse inequality

$$\liminf_{T\to\infty} g_T(z,\nu) \geq g_{\sup}(z,\nu)\,, \tag{4.44}$$

we assume for the sake of notational simplicity $\nu = e_n$. We then fix $\rho > 0$ and take $T > 6$, depending on ρ, and $(u_T, v_T) \in \mathcal{U}^T_{z,e_n}$ such that

$$\frac{1}{T^{n-1}} \mathcal{F}^\infty_1(u_T, v_T; Q^{e_n}_T) \leq \liminf_{T\to\infty} g_T(z, e_n) + \rho\,. \tag{4.45}$$

Let $\varepsilon_j \to 0$ and set

$$u_j(y) := \begin{cases} u_T\Big(\dfrac{y}{\varepsilon_j} - d\Big), & \text{if } y \in \varepsilon_j(Q^{e_n}_T + d) \subset\subset Q^{e_n}, \\ (z\chi_{\{x\cdot e_n>0\}} * \varphi_1)(\frac{y}{\varepsilon_j}), & \text{otherwise in } Q^{e_n}, \end{cases} \tag{4.46}$$

$$v_j(y) := \begin{cases} v_T\Big(\dfrac{y}{\varepsilon_j} - d\Big), & \text{if } y \in \varepsilon_j(Q^{e_n}_T + d) \subset\subset Q^{e_n}, \\ (\chi_{\{|x\cdot e_n|>2\}} * \varphi_1)(\frac{y}{\varepsilon_j}), & \text{otherwise in } Q^{e_n}, \end{cases} \tag{4.47}$$

with $d \in \mathbb{Z}^{n-1} \times \{0\}$. Then, $(u_j, v_j) \to (z\chi_{\{x\cdot e_n>0\}}, 1)$ in $L^1(Q^{e_n}; \mathbb{R}^{m+1})$, and letting $I_{\varepsilon_j} := \{d \in \mathbb{Z}^{n-1} \times \{0\} : \varepsilon_j(Q^{e_n}_T + d) \subset\subset Q^{e_n}\}$, a change of variable yields (cf. also the discussion after (4.20))

$$\begin{aligned} g_{\sup}(z, e_n) &\leq \limsup_{j\to\infty} \mathcal{F}^\infty_{\varepsilon_j}(u_j, v_j; Q^{e_n}) \\ &\leq \limsup_{j\to\infty} \Big(\sum_{d\in I_{\varepsilon_j}} \mathcal{F}^\infty_{\varepsilon_j}(u_j, v_j; \varepsilon_j(Q^{e_n}_T + d)) \\ &\qquad + \frac{c}{\varepsilon_j} \mathcal{L}^n\Big(Q^{e_n} \cap \{|x_n| \leq 3\varepsilon_j\} \setminus \bigcup_{d\in I_{\varepsilon_j}} \varepsilon_j(Q^{e_n}_T + d)\Big)\Big) \\ &= \limsup_{j\to\infty} \varepsilon_j^{n-1} \# I_{\varepsilon_j}\, \mathcal{F}^\infty_1(u_T, v_T; Q^{e_n}_T) \\ &\leq \frac{1}{T^{n-1}} \mathcal{F}^\infty_1(u_T, v_T; Q^{e_n}_T) \leq \liminf_{T\to\infty} g_T(z, e_n) + \rho\,, \end{aligned}$$

by the choice of (u_T, v_T) and T (cf. (4.45)). As $\rho \to 0$ we get (4.44).

Estimates (4.41) and (4.44) yield the existence of the limit of $g_T(z,\nu)$ as $T \uparrow \infty$ and equality (4.38), as well. □

With this representation of $g_{\sup}$ at hand we can obtain a version of Proposition 4.6 which also accounts for a regularization term of the form $\eta_\varepsilon \int_\Omega |\nabla u|^q \mathrm{d}x$, which is useful to prove the convergence of minimizers stated in the introduction.

Corollary 4.8. *For any* $\varepsilon_j \downarrow 0$ *and* $\eta_j \downarrow 0$ *with* $\eta_j/\varepsilon_j^{q-1} \to 0$*, and any* $(z,\nu) \in \mathbb{R}^m \times S^{n-1}$ *there is* $(u_j, v_j) \to (z\chi_{\{x\cdot\nu>0\}}, 1)$ *in* $L^q(Q^\nu;\mathbb{R}^{m+1})$*, with* $v_j \in [0,1]$ $\mathcal{L}^n$*-a.e. in* Q^ν*, such that*

$$\lim_{j\to\infty} \mathcal{F}^\infty_{\varepsilon_j}(u_j, v_j; Q^\nu) = g_{\sup}(z,\nu)\,,$$

$$\lim_{j\to\infty} \eta_j \int_{Q^\nu} |\nabla u_j|^q \mathrm{d}x = 0\,,$$

and

$$u_j = (z\chi_{\{x\cdot\nu>0\}}) * \varphi_{\varepsilon_j}\,, \qquad v_j = \chi_{\{|x\cdot\nu|\ge 2\varepsilon_j\}} * \varphi_{\varepsilon_j} \qquad \text{on } \partial Q^\nu.$$

Proof. We use the same construction as in (4.46)-(4.47) (without loss of generality, explicitly written only for $\nu = e_n$), and compute similarly

$$\begin{aligned}\|\nabla u_j\|^q_{L^q(Q^{e_n})} &\le \sum_{d\in I_{\varepsilon_j}} \|\nabla u_j\|^q_{L^q(\varepsilon_j(Q_T^{e_n}+d))} + \frac{C}{\varepsilon_j^q}\mathcal{L}^n\left(Q^{e_n}\cap\{|x_n|\le\varepsilon_j\}\right)\\ &= \varepsilon_j^{n-q}\#I_{\varepsilon_j}\|\nabla u_T\|^q_{L^q(Q_T)} + \frac{C}{\varepsilon_j^{q-1}} \le \frac{C_T}{\varepsilon_j^{q-1}}.\end{aligned}$$

To conclude the proof it suffices to choose $T_j \to \infty$ so slow that $\eta_j C_{T_j}/\varepsilon_j^{q-1} \to 0$. □

We finally prove that $g_{\inf}$ and $g_{\sup}$ coincide, in case Ψ_∞ satisfies the projection property in (4.11). Before starting, we observe that continuity of Ψ_∞ implies that for any $\delta > 0$ there is $C(\delta) > 0$ such that

$$\Psi_\infty(\xi+\eta) \le (1+\delta)\Psi_\infty(\xi) + C(\delta)|\eta|^q \qquad \text{for all } \xi,\eta \in \mathbb{R}^{m\times n}, \tag{4.48}$$

which by q-homogeneity and coercivity immediately implies, for $L > 0$,

$$\Psi_\infty(\xi+\eta) \le (1+\delta+C(\delta)L^q)\Psi_\infty(\xi) \qquad \text{for all } \xi,\eta\in\mathbb{R}^{m\times n} \text{ with } |\eta|\le L|\xi|. \tag{4.49}$$

By scaling, it suffices to prove (4.48) in the case $|\xi+\eta| = 1$. If this were false, there would be $\delta > 0$ and sequences with

$$\Psi_\infty(\xi_k+\eta_k) > (1+\delta)\Psi_\infty(\xi_k) + k|\eta_k|^q.$$

As $\xi_k+\eta_k$ is bounded, we obtain $\eta_k \to 0$, and therefore, passing to a subsequence, $\xi_k \to \xi_*$ with $|\xi_*| = 1$. Continuity of Ψ_∞ in ξ_* then implies

$$\Psi_\infty(\xi_*) \ge (1+\delta)\Psi_\infty(\xi_*),$$

a contradiction. This proves (4.48).

Proposition 4.9. *Let Ψ_∞ satisfy the projection property in* (4.11)*. Then*

$$g_{\inf}(z,\nu)=g_{\sup}(z,\nu)=g(z,\nu) \quad \text{for every } (z,\nu)\in\mathbb{R}^m\times S^{n-1},$$

where g is defined in (4.13)*.*

In particular, if $\Psi_\infty(\xi)=|\xi|^q$, then $g(z,\nu)=g_{\rm scal}(|z|)$ for all $(z,\nu)\in\mathbb{R}^m\times S^{n-1}$, where for every $s\geq 0$

$$g_{\rm scal}(s):=\inf_{\mathcal{U}_s}\int_0^1 |1-\beta|(f_p^q(\beta)|\alpha'|^q+|\beta'|^q)^{1/q}\,\mathrm{d}\tau \tag{4.50}$$

and

$$\mathcal{U}_s:=\{(\alpha,\,\beta)\in W^{1,q}((0,1);\mathbb{R}^2):\,0\leq\beta\leq 1,\,\beta(0)=\beta(1)=1,\,\alpha(0)=0,\,\alpha(1)=s\}\,.$$

Proof. Let $(z,\nu)\in\mathbb{R}^m\times S^{n-1}$, $z\neq 0$, and $T>0$. It is convenient to recall first some notation used in the slicing technique and some results for which we refer to [AFP00]: if $(u,v)\in\mathcal{U}_z^T$, then for $\mathcal{H}^{n-1}$-a.e. $y\in\tilde{Q}_T^\nu:=(\mathrm{Id}_n-\nu\otimes\nu)Q_T^\nu\subset\nu^\perp$ the slices

$$u_y^\nu(t):=u(y+t\nu),\quad v_y^\nu(t):=v(y+t\nu)$$

belong to $\mathcal{V}_z^T$ defined in (4.14). Moreover, for every $(\alpha,\beta)\in W^{1,q}_{\rm loc}(\mathbb{R};\mathbb{R}^{m+1})$, $\varepsilon>0$, $A\in\mathcal{B}(\mathbb{R})$, and $M\in(0,\infty]$ set

$$\begin{aligned}\widetilde{\mathcal{F}}_\varepsilon^M(\alpha,\beta;A):=\int_A\Big((&M^{q-1}\wedge\varepsilon^{q-1}f_p^q(\beta(t)))\Psi_\infty(\alpha'(t)\otimes\nu)\\&+\frac{(1-\beta(t))^{q'}}{q'q^{q'/q}\varepsilon}+\varepsilon^{q-1}|\beta'(t)|^q\Big)\mathrm{d}t\,.\end{aligned}$$

Step 1. $g_{\sup}(z,\nu)=g(z,\nu)$**.** Set

$$\widetilde{g}_T(z,\nu):=\inf_{\mathcal{V}_z^T}\widetilde{\mathcal{F}}_1^\infty(\alpha,\beta;(-{}^T\!/_2,{}^T\!/_2))\,,$$

and note that (4.13) rewrites as $g(z,\nu)=\lim\limits_{T\to\infty}\widetilde{g}_T(z,\nu)$. On the one hand, to prove that

$$\liminf_{T\to\infty}\widetilde{g}_T(z,\nu)\geq g_{\sup}(z,\nu)\,, \tag{4.51}$$

we assume for the sake of notational simplicity $\nu=e_n$. We then fix $\rho>0$ and take $T>0$, depending on ρ, and $(\alpha_T,\beta_T)\in\mathcal{V}_z^T$ such that

$$\widetilde{\mathcal{F}}_1^\infty(\alpha_T,\beta_T;(-{}^T\!/_2,{}^T\!/_2))\leq\liminf_{T\to\infty}\widetilde{g}_T(z,e_n)+\rho\,. \tag{4.52}$$

Let $\varepsilon_j\to 0$ and set

$$u_j(y):=\begin{cases}\alpha_T\left(\dfrac{y_n}{\varepsilon_j}\right), & \text{if } |y_n|\leq\frac{T\varepsilon_j}{2},\\ z\chi_{\{x\cdot e_n>0\}}, & \text{otherwise in } Q^{e_n},\end{cases}$$

$$v_j(y) := \begin{cases} \beta_T\left(\dfrac{y_n}{\varepsilon_j}\right), & \text{if } |y_n|\le\frac{T\varepsilon_j}{2}, \\ 1, & \text{otherwise in } Q^{e_n}. \end{cases}$$

Then, $(u_j, v_j) \to (z\chi_{\{x\cdot e_n>0\}}, 1)$ in $L^1(Q^{e_n};\mathbb{R}^{m+1})$, and a change of variable yields

$$\begin{aligned}\limsup_{j\to\infty}\mathcal{F}^\infty_{\varepsilon_j}(u_j, v_j; Q^{e_n}) &= \limsup_{j\to\infty}\mathcal{F}^\infty_{\varepsilon_j}(u_j, v_j; \{|y_n| \le T\varepsilon_j/2\}) \\ &= \widetilde{\mathcal{F}}^\infty_1(\alpha_T, \beta_T; (-T/2, T/2)) \le \liminf_{T\to\infty} \widetilde{g}_T(z, e_n) + \rho\,,\end{aligned}$$

by the choice of (u_T, v_T) and T (cf. (4.52)). As $\rho \to 0$ we conclude (4.51) recalling the definition of $g_{\sup}$ in (4.8).

On the other hand, for every $(u, v) \in \mathcal{U}^T_z$, $T > 0$, we use the projection property in (4.11) and Fubini's theorem to deduce that

$$\mathcal{F}^\infty_1(u, v; Q^\nu_T) \ge \int_{\tilde{Q}^\nu_T} \widetilde{\mathcal{F}}^\infty_1(u^\nu_y, v^\nu_y; (-T/2, T/2))\mathrm{d}\mathcal{H}^{n-1}(y)\,.$$

In turn, as $(u^\nu_y, v^\nu_y) \in \mathcal{V}^T_z$ for $\mathcal{H}^{n-1}$-a.e. $y \in \tilde{Q}^\nu_T$, the latter inequality implies $g_T(z, \nu) \ge \widetilde{g}_T(z, \nu)$. Therefore, Proposition 4.7 yields that

$$\limsup_{T\to\infty} \widetilde{g}_T(z, \nu) \le \limsup_{T\to\infty} g_T(z, \nu) = g_{\sup}(z, \nu)\,,$$

and the claim in Step 1 follows by taking also into account (4.51).

Step 2. $g_{\inf}(z, \nu) = g_{\sup}(z, \nu)$. By the trivial inequality in (4.9) and Step 1, it is sufficient to show that $g_{\inf}(z, \nu) \ge g(z, \nu)$. To this aim, by taking into account Proposition 4.5, we may consider any sequence $(u_j, v_j) \to (z\chi_{\{x\cdot\nu>0\}}, 1)$ in $L^q(Q^\nu;\mathbb{R}^{m+1})$ in the definition of $g_{\inf}$ to be satisfying additionally (4.19) (cf. (4.30)). Hence, we may argue as above to deduce that

$$\begin{aligned}\mathcal{F}^{M_j}_{\varepsilon_j}(u_j, v_j; Q^\nu) &\ge \int_{\tilde{Q}^\nu} \widetilde{\mathcal{F}}^{M_j}_{\varepsilon_j}((u_j)^\nu_y, (v_j)^\nu_y; (-1/2, 1/2))\mathrm{d}\mathcal{H}^{n-1}(y) \\ &\ge \inf_{\mathcal{V}^1_z} \widetilde{\mathcal{F}}^{M_j}_{\varepsilon_j}(\alpha, \beta; (-1/2, 1/2))\,.\end{aligned}$$

Therefore,

$$g_{\inf}(z, \nu) \ge \liminf_{j\to\infty} \inf_{\mathcal{V}^1_z} \widetilde{\mathcal{F}}^{M_j}_{\varepsilon_j}(\alpha, \beta; (-1/2, 1/2))\,. \tag{4.53}$$

Up to extracting a subsequence not relabeled, we may assume the latter inferior limit to be a limit, and consider $(\alpha_j, \beta_j) \in \mathcal{V}^1_z$ to be such that

$$\widetilde{\mathcal{F}}^{M_j}_{\varepsilon_j}(\alpha_j, \beta_j; (-1/2, 1/2)) \le \inf_{\mathcal{V}^1_z} \widetilde{\mathcal{F}}^{M_j}_{\varepsilon_j}(\alpha, \beta; (-1/2, 1/2)) + \frac{1}{j}\,.$$

Denote by $\lambda_j \in (0, 1)$ the unique root in $(0, 1)$ of the equation $\frac{\ell t}{(1-t)^p} = (M_j/\varepsilon_j)^{1/q'}$. Clearly, $\lambda_j \to 1$ as $j \to \infty$, and more precisely $(1-\lambda_j)^p(M_j/\varepsilon_j)^{1/q'} \to \ell$ as $j \to \infty$. Then, as

$$\sup_j \widetilde{\mathcal{F}}^{M_j}_{\varepsilon_j}(\alpha_j, \beta_j; (-1/2, 1/2)) \le C\,,$$

using the q-homogeneity of Ψ_∞ we infer

$$M_j^{q-1}\int_{\{\beta_j>\lambda_j\}}|\alpha_j'|^q\mathrm{d}t\le C \tag{4.54}$$

so that, by Jensen inequality we deduce that

$$\int_{\{\beta_j>\lambda_j\}}|\alpha_j'|\mathrm{d}t\le C\Big(\frac{1}{M_j}\mathcal{L}^1(\{\beta_j>\lambda_j\})\Big)^{1-\frac{1}{q}}\le CM_j^{\frac{1}{q}-1}\,. \tag{4.55}$$

Define

$$\eta_j(t):=\int_{-1/2}^t\alpha_j'(x)\chi_{\{\beta_j\le\lambda_j\}}(x)\mathrm{d}x\,,$$

then $\eta_j\in W^{1,q}((-1/2,1/2);\mathbb{R}^m)$ with $\eta_j(-1/2)=0$. Clearly, $\eta_j'=\alpha_j'$ $\mathcal{L}^1$-a.e. on $\{\beta_j\le\lambda_j\}$ and $\eta_j'=0$ $\mathcal{L}^1$-a.e. on $\{\beta_j>\lambda_j\}$. Therefore, the q-homogeneity of Ψ_∞ implies

$$\begin{aligned}\widetilde{\mathcal{F}}^\infty_{\varepsilon_j}(\eta_j,\beta_j;(-1/2,1/2))&=\widetilde{\mathcal{F}}^\infty_{\varepsilon_j}(\alpha_j,\beta_j;\{\beta_j\le\lambda_j\})+\widetilde{\mathcal{F}}^\infty_{\varepsilon_j}(0,\beta_j;\{\beta_j>\lambda_j\})\\&=\widetilde{\mathcal{F}}^{M_j}_{\varepsilon_j}(\alpha_j,\beta_j;\{\beta_j\le\lambda_j\})+\widetilde{\mathcal{F}}^{M_j}_{\varepsilon_j}(0,\beta_j;\{\beta_j>\lambda_j\})\\&\le\widetilde{\mathcal{F}}^{M_j}_{\varepsilon_j}(\alpha_j,\beta_j;(-1/2,1/2))\le\inf_{\mathcal{V}^1_z}\widetilde{\mathcal{F}}^{M_j}_{\varepsilon_j}(\alpha,\beta;(-1/2,1/2))+\frac{1}{j}\,.\end{aligned} \tag{4.56}$$

Finally, let $e_j:=z-\eta_j(1/2)$. Being $\eta_j(1/2)=z-\int_{\{\beta_j>\lambda_j\}}\alpha_j'\mathrm{d}t$, by (4.55) we infer that $e_j\to0$ as $j\to\infty$. Since $|z|\neq0$, for j large enough we have $|e_j|\le|z|/2$, and we can choose a matrix $A_j\in\mathbb{R}^{m\times m}$ such that $A_j(z-e_j)=e_j$ and $|A_j|\le2|e_j|/|z|$. We set

$$\zeta_j:=\eta_j+A_j\eta_j,$$

check that $\zeta_j\in\mathcal{V}^1_z$, and estimate with (4.49), for any $\delta>0$,

$$\Psi_\infty(\zeta_j'\otimes\nu)\le(1+\delta+C(\delta)|A_j|^q)\Psi_\infty(\eta_j'\otimes\nu)$$

and therefore, since $A_j\to0$,

$$\limsup_{j\to\infty}\widetilde{\mathcal{F}}^\infty_{\varepsilon_j}(\zeta_j,\beta_j;(-1/2,1/2))\le(1+\delta)\limsup_{j\to\infty}\widetilde{\mathcal{F}}^\infty_{\varepsilon_j}(\eta_j,\beta_j;(-1/2,1/2)) \tag{4.57}$$

for any $\delta>0$ and therefore for $\delta=0$. Hence, the existence of the limit defining $g(z,\nu)$ established in Step 1 yields

$$\begin{aligned}g(z,\nu)&\le\liminf_{j\to\infty}\widetilde{\mathcal{F}}^\infty_{\varepsilon_j}(\zeta_j,\beta_j;(-1/2,1/2))\overset{(4.57)}{\le}\liminf_{j\to\infty}\widetilde{\mathcal{F}}^\infty_{\varepsilon_j}(\eta_j,\beta_j;(-1/2,1/2))\\&\overset{(4.56)}{\le}\liminf_{j\to\infty}\inf_{\mathcal{V}^1_z}\widetilde{\mathcal{F}}^{M_j}_{\varepsilon_j}(\alpha,\beta;(-1/2,1/2))\overset{(4.53)}{\le}g_{\inf}(z,\nu)\,.\end{aligned}$$

Step 3. If $\Psi_\infty = |\cdot|^q$, then $g(z,\nu) = g_{\mathrm{scal}}(|z|)$**.** We first claim that

$$g(z,\nu) = \widetilde{g}(|z|)\,,$$

for every $(z,\nu) \in \mathbb{R}^m \times S^{n-1}$, where for every $s \geq 0$

$$\widetilde{g}(s) := \lim_{T\uparrow\infty} \inf_{(\gamma,\beta)\in\mathcal{U}_s^T} \mathcal{F}_1^\infty(\gamma,\beta;(-{}^T\!/2, {}^T\!/2))\,,$$

and the space of test functions rewrites as

$$\mathcal{U}_s^T := \{(\gamma,\beta) \in W^{1,q}((-{}^T\!/2,{}^T\!/2);\mathbb{R}^2)\colon 0 \leq \beta \leq 1,\ \beta(\pm{}^T\!/2) = 1,\\ \gamma(-{}^T\!/2) = 0\,,\ \gamma({}^T\!/2) = s\}\,.$$

Note that the existence of the limit defining $\widetilde{g}$ is guaranteed by Proposition 4.7 in case $n = 1$, and that the claim is trivial if $z = 0$.

To prove the above claim for $|z| > 0$, first notice that if $(\alpha,\beta) \in \mathcal{V}_z^T$, then $(|\alpha|,\beta) \in \mathcal{U}_{|z|}^T$, with $\widetilde{\mathcal{F}}_1^\infty(\alpha,\beta;(-{}^T\!/2,{}^T\!/2)) = \mathcal{F}_1^\infty(|\alpha|,\beta;(-{}^T\!/2,{}^T\!/2))$, so that $g(z,\nu) \geq \widetilde{g}(|z|)$.

To establish the opposite inequality, given $T > 0$ and $(\gamma,\beta) \in \mathcal{U}_{|z|}^T$, we obtain a competitor (α,β) for the problem defining g by setting $\alpha(t) := \gamma(t)\frac{z}{|z|}$. Moreover, $\widetilde{\mathcal{F}}_1^\infty(\alpha,\beta;(-{}^T\!/2,{}^T\!/2)) = \mathcal{F}_1^\infty(\gamma,\beta;(-{}^T\!/2,{}^T\!/2))$ so that the inequality follows.

Finally, the equality $\widetilde{g}(s) = g_{\mathrm{scal}}(s)$ is established in [CFI16, Proposition 7.3]. □

4.3 Structural properties of the surface energy densities

In this section, we establish some structural properties of $g_{\inf}$ and $g_{\sup}$.

Lemma 4.10. *Let $g_{\inf}, g_{\sup} : \mathbb{R}^m \times S^{n-1} \to [0,\infty)$ be defined by (4.7), (4.8), respectively. Then, the following properties hold.*

(i) There is $C > 0$ such that, for all $z,\nu \in \mathbb{R}^m \times S^{n-1}$,

$$\frac{1}{C}(|z|^{\frac{2}{p+1}} \wedge 1) \leq g_{\inf}(z,\nu) \leq g_{\sup}(z,\nu) \leq C(|z|^{\frac{2}{p+1}} \wedge 1)\,.$$

(ii) For any $\nu \in S^{n-1}$ and z, $z' \in \mathbb{R}^m$ one has

$$g_{\sup}(z+z',\nu) \leq g_{\sup}(z,\nu) + g_{\sup}(z',\nu).$$

(iii) $g_{\inf}, g_{\sup} \in C^0(\mathbb{R}^m \times S^{n-1})$.

Note that, by (i) above there is $C > 0$ such that, for all $\nu \in S^{n-1}$,

$$\frac{1}{C} \leq \liminf_{z\to 0} \frac{g_{\inf}(z,\nu)}{|z|^{\frac{2}{p+1}}} \leq \limsup_{z\to 0} \frac{g_{\sup}(z,\nu)}{|z|^{\frac{2}{p+1}}} \leq C\,. \tag{4.58}$$

Proof. The bounds in (i) may be derived estimating $\mathcal{F}_\varepsilon$ by its 1D counterpart (see [CFI16, Prop. 7.3]), the properties in (ii) and (iii) for $g_{\sup}$ arguing analogously to [CFI24, Lemma 3.8] that deals with the case $p = 1$).

To conclude we establish property (iii) for $g_{\inf}$, the same argument working for $g_{\sup}$, as well. Clearly, $g_{\inf}$ is continuous in $(0, \nu)$, for every $\nu \in S^{n-1}$, because of (i). Therefore, let (z, ν), $(z', \nu') \in \mathbb{R}^m \times S^{n-1}$, with $|z| \cdot |z'| \neq 0$, and consider functions $(u_j, v_j) \to z'\chi_{\{x\cdot\nu'>0\}}$ in $L^q(Q^{\nu'}; \mathbb{R}^{m+1})$ and sequences $\varepsilon_j \to 0$, $M_j \to \infty$. Let $(R, T) \in SO(m) \times SO(n)$ be such that $R\frac{z'}{|z'|} = \frac{z}{|z|}$, $T\nu' = \nu$, and $|R - \mathrm{Id}_m| \le C|\frac{z'}{|z'|} - \frac{z}{|z|}|$, $|T - \mathrm{Id}_n| \le C|\nu - \nu'|$. By Lemma 4.3 we can assume that $TQ^{\nu'} = Q^\nu$. We define on Q^ν the maps $(\widetilde{u}_j, \widetilde{v}_j)$ by

$$\widetilde{u}_j(x) := \frac{|z|}{|z'|} R u_j(T^{-1}x), \quad \widetilde{v}_j(x) := v_j(T^{-1}x)\,.$$

Then, a straightforward change of variable leads to

$$\int_{Q^\nu} |\widetilde{u}_j - z\chi_{\{x\cdot\nu>0\}}|^q \mathrm{d}x = \left(\frac{|z|}{|z'|}\right)^q \int_{Q^{\nu'}} |u_j - z'\chi_{\{x\cdot\nu'>0\}}|^q \mathrm{d}x\,,$$

to $\|\widetilde{v}_j - 1\|_{L^q(Q^\nu)} = \|v_j - 1\|_{L^q(Q^{\nu'})}$, and moreover to

$$\mathcal{F}^{M_j}_{\varepsilon_j}(\widetilde{u}_j, \widetilde{v}_j; Q^\nu) = \left(\frac{|z|}{|z'|}\right)^q \int_{Q^{\nu'}} (M_j^{q-1} \wedge f_p^q(v_j))\Psi_\infty(R\nabla u_j T^{-1})\mathrm{d}x + \mathcal{F}^{M_j}_{\varepsilon_j}(0, v_j; Q_{\nu'})\,. \tag{4.59}$$

Therefore, for any $\delta > 0$ using (4.49) we may estimate $\mathcal{L}^n$-a.e. on $Q^{\nu'}$

$$\Psi_\infty(R\nabla u_j T^{-1}) \le (1 + \delta + C_\delta|T^{-1} - \mathrm{Id}_n|^q + C_\delta|R - \mathrm{Id}_m|^q)\Psi_\infty(\nabla u_j). \tag{4.60}$$

In particular, the latter estimate together with (4.59) imply

$$\begin{aligned}&\mathcal{F}^{M_j}_{\varepsilon_j}(\widetilde{u}_j, \widetilde{v}_j; Q^\nu)\\ &\le \left\{1 \vee \left(\frac{|z|}{|z'|}\right)^q (1 + \delta + C_\delta(|T^{-1} - \mathrm{Id}_n|^q + |R - \mathrm{Id}_m|^q))\right\} \mathcal{F}^{M_j}_{\varepsilon_j}(u_j, v_j; Q_{\nu'})\end{aligned}$$

in turn implying

$$g_{\inf}(z, \nu) \le \left\{1 \vee \left(\frac{|z|}{|z'|}\right)^q (1 + \delta + C_\delta(|T^{-1} - \mathrm{Id}_n|^q + |R - \mathrm{Id}_m|^q))\right\} g_{\inf}(z', \nu')\,. \tag{4.61}$$

Clearly, $T^{-1} \to \mathrm{Id}_n$ as $\nu' \to \nu$, and $R \to \mathrm{Id}_m$ as $z' \to z$, so that

$$g_{\inf}(z, \nu) \le (1 + \delta) \liminf_{(z',\nu')\to(z,\nu)} g_{\inf}(z', \nu')$$

for any $\delta > 0$, and hence for $\delta = 0$. Exchanging the roles of (z, ν) and (z', ν') in the construction performed above yields the continuity of $g_{\inf}$ in (z, ν). □

4.4 Γ-liminf

We start off determining the domain of the eventual Γ-limit and proving a lower bound inequality which will turn out to be optimal in case the projection property holds for Ψ_∞. To this aim consider the functional $\mathcal{F}_{\rm inf}(\cdot,\cdot;\cdot) : L^1(\Omega;\mathbb{R}^{m+1})\times\mathcal{A}(\Omega)\to[0,\infty]$ defined by

$$\mathcal{F}_{\rm inf}(u,v;A) := \int_\Omega \Psi^{\rm qc}(\nabla u)\mathrm{d}x + \int_{J_u\cap A} g_{\rm inf}([u],\nu_u)\mathrm{d}\mathcal{H}^{n-1}, \tag{4.62}$$

if $A\in\mathcal{A}(\Omega)$, $u\in(GSBV\cap L^1(\Omega))^m$ with $\nabla u\in L^q(\Omega;\mathbb{R}^{m\times n})$, $v=1$ $\mathcal{L}^n$-a.e. on Ω, and $\mathcal{F}_{\rm inf}(u,v;A):=\infty$ otherwise.

We follow in part the approach developed in [CFI24, Section 4].

One important ingredient in the lower bound is the fact that the energy density $\Psi^{\rm qc}$ is recovered in the limit thanks to Proposition 2.3. We stress that no extra hypothesis on Ψ is assumed besides (1.15) and (4.2). We are now ready to identify the domain of the eventual Γ-limit and to prove the lower bound.

Proposition 4.11. *Let $(u_\varepsilon,v_\varepsilon)\to(u,v)$ in $L^1(\Omega;\mathbb{R}^{m+1})$ with*

$$\liminf_{\varepsilon\to0}\mathcal{F}_\varepsilon(u_\varepsilon,v_\varepsilon)<\infty \tag{4.63}$$

then $v=1$ $\mathcal{L}^n$-a.e. in Ω and $u\in(GSBV\cap L^1(\Omega))^m$ with $\nabla u\in L^q(\Omega;\mathbb{R}^{m\times n})$. Moreover, for all $A\in\mathcal{A}(\Omega)$

$$\mathcal{F}_{\rm inf}(u,v;A)\le\Gamma(L^1)\text{-}\liminf_{\varepsilon\to0}\mathcal{F}_\varepsilon(u,1;A)\,. \tag{4.64}$$

Proof. We divide the proof into several steps: in the first we argue as in [CFI24, Sections 4.3 and 4.4] to show that $u\in(GBV(\Omega))^m$, in the second we establish $u\in(GSBV(\Omega))^m$ and a lower bound inequality for the diffuse part, in the third we discuss the lower bound for the surface energy, and in the last step we infer (4.64).

Step 1. $u\in(GBV(\Omega))^m$, $v=1$ $\mathcal{L}^n$-a.e. on Ω.

We fix $A\in\mathcal{A}(\Omega)$, $\delta\in(0,1)$ and $\varepsilon>0$. We compute, for any pair $(u,v)\in$

$W^{1,q}(\Omega;\mathbb{R}^m \times [0,1])$,

$$\begin{aligned}
\mathcal{F}_\varepsilon(u,v;A) &\geq \int_{A\cap\{\varepsilon^{q-1}f_p^q(v)>1\}} \Psi(\nabla u)\mathrm{d}x \\
&+ \int_{A\cap\{\varepsilon^{q-1}f_p^q(v)\leq 1\}} \Big(\varepsilon^{q-1}f_p^q(v)\Psi(\nabla u) + \delta^{q'}\frac{(1-v)^{q'}}{q'q^{q'/q}\varepsilon}\Big)\mathrm{d}x \\
&+ \int_A \Big((1-\delta^{q'})\frac{(1-v)^{q'}}{q'q^{q'/q}\varepsilon} + \varepsilon^{q-1}|\nabla v|^q\Big)\mathrm{d}x \\
&\geq \int_{A\cap\{\varepsilon^{q-1}f_p^q(v)>1\}} \Psi(\nabla u)\mathrm{d}x + \delta\int_{A\cap\{\varepsilon^{q-1}f_p^q(v)\leq 1\}} \frac{\ell v}{(1-v)^{p-1}}\Psi^{1/q}(\nabla u)\mathrm{d}x \\
&+ (1-\delta^{q'})^{1/q'}\int_A |\nabla(\Phi(v))|\mathrm{d}x \\
&\geq \delta\int_A \Big(\Psi(\nabla u)\wedge\frac{\ell v}{(1-v)^{p-1}}\Psi^{1/q}(\nabla u)\Big)\mathrm{d}x + (1-\delta^{q'})^{1/q'}\int_A|\nabla(\Phi(v))|\mathrm{d}x\,,
\end{aligned} \tag{4.65}$$

where $\Phi : [0,1] \to [0,\frac{1}{2}]$ is defined by

$$\Phi(t) := \int_0^t (1-s)\mathrm{d}s = t - \frac{1}{2}t^2\,. \tag{4.66}$$

We observe that Φ is strictly increasing, $\Phi(1) = \frac{1}{2}$ and in particular Φ is bijective. By the coarea formula there is $\bar{t} \in (\Phi(\delta^{q'}), \Phi(\delta))$ such that

$$(\Phi(\delta) - \Phi(\delta^{q'}))\mathcal{H}^{n-1}(A\cap\partial^*\{\Phi(v) > \bar{t}\}) \leq \int_A |\nabla(\Phi(v))|\mathrm{d}x\,,$$

thus if we define $\tilde{u} := u\chi_{\{\Phi(v)>\bar{t}\}} \in (GSBV(A))^m$ (not highlighting the dependence on A, ε and δ) we obtain from (4.65) and the monotonicity of $[0,1) \ni t \mapsto \frac{t}{(1-t)^{p-1}}$,

$$\mathcal{F}_\varepsilon(u,v;A) \geq \delta^{q'+1}\int_A h_\delta(\nabla\tilde{u})\mathrm{d}x + \beta_\delta\mathcal{H}^{n-1}(A\cap J_{\tilde{u}}) - h_\delta(0)\mathcal{L}^n(\{v\leq\delta\})\,, \tag{4.67}$$

where we have set $\beta_\delta := (1-\delta^{q'})^{1/q'}(\Phi(\delta) - \Phi(\delta^{q'}))$, and $h_\delta : \mathbb{R}^m \to [0,\infty)$ is defined in (2.52). We may thus argue as in [CFI24, Remark 4.7] to infer that if $(u_\varepsilon, v_\varepsilon) \to (u,v)$ in $L^1(\Omega;\mathbb{R}^{m+1})$ and $\sup_\varepsilon \mathcal{F}_\varepsilon(u_\varepsilon, v_\varepsilon) < \infty$, then necessarily $v = 1$ $\mathcal{L}^n$-a.e. on Ω and $u \in (GBV(\Omega))^m$.

Step 2. $u \in (GSBV(\Omega))^m$, and for every $A \in \mathcal{A}(\Omega)$

$$\int_A \Psi^{\mathrm{qc}}(\nabla u)\mathrm{d}x \leq \liminf_{\varepsilon\to 0} \mathcal{F}_\varepsilon(u_\varepsilon, v_\varepsilon; A)\,. \tag{4.68}$$

Let $A \in \mathcal{A}(\Omega)$ be given. By taking into account (4.67) and the usual averaging and truncation argument (cf. (4.35)), we infer that for every $N \in \mathbb{N}$ there is $k_N \in \{N+1, \dots, 2N\}$ such that

$$\mathcal{F}_\varepsilon(u_\varepsilon, v_\varepsilon; A) \geq \delta^{q'+1} \int_A h_\delta(\nabla(\mathcal{T}_{k_N}(\widetilde{u}_\varepsilon)))\mathrm{d}x \\ + \beta_\delta \mathcal{H}^{n-1}(A \cap J_{\mathcal{T}_{k_N}(\widetilde{u}_\varepsilon)}) - h_\delta(0)\mathcal{L}^n(\{v_\varepsilon \leq \delta\}) - \frac{c}{k_N}\,.$$

In addition, it is not restrictive to assume k_N indipendent from ε, up to passing to a subsequence realizing the inferior limit for $\mathcal{F}_\varepsilon(u_\varepsilon, v_\varepsilon; A)$. Therefore, for every $\delta \in (0,1)$, using formulas (4.23) and (4.24) in [CFI24, Proposition 4.2] yields that

$$\liminf_{\varepsilon \to 0} \mathcal{F}_\varepsilon(u_\varepsilon, v_\varepsilon; A) \geq \delta^{q'+1} \int_A h_\delta^{\mathrm{qc}}(\nabla(\mathcal{T}_{k_N}(u)))\mathrm{d}x \\ + \delta^{q'+1} \int_A h_\delta^{\mathrm{qc},\infty}(\mathrm{d}D^c \mathcal{T}_{k_N}(u)) - \frac{c}{k_N}\,. \tag{4.69}$$

Here $h_\delta^{\mathrm{qc},\infty}$ denotes the (linear) recession function of h_δ^{qc} defined by

$$h_\delta^{\mathrm{qc},\infty}(\xi) := \limsup_{t \to \infty} \frac{h_\delta^{\mathrm{qc}}(t\xi)}{t}. \tag{4.70}$$

Indeed, note that the growth conditions imposed on Ψ in (1.15) and the very definition of h_δ in (2.52) yield that h_δ itself is linear for large $|\xi|$ and, more precisely that $h_\delta(\xi) = \frac{\ell}{(1-\delta^{q'})^{p-1}} \Psi^{1/q}(\xi)$ for $|\xi|$ large enough. Hence, by (4.70) we conclude that

$$h_\delta^{\mathrm{qc},\infty}(\xi) \geq \frac{\ell}{c^{1/q}(1-\delta^{q'})^{p-1}}|\xi|.$$

In particular, we conclude that $|D^c \mathcal{T}_{k_N}(u)|(A) = 0$ by letting $\delta \uparrow 1$ in (4.69). Thus, $\mathcal{T}_{k_N}(u) \in (SBV(A))^m$ for every $N \in \mathbb{N}$, in turn implying $u \in (GSBV(A))^m$. By taking this into account, (4.69) rewrites as

$$\liminf_{\varepsilon \to 0} \mathcal{F}_\varepsilon(u_\varepsilon, v_\varepsilon; A) \geq \delta^{q'+1} \int_A h_\delta^{\mathrm{qc}}(\nabla(\mathcal{T}_{k_N}(u)))\mathrm{d}x - \frac{c}{k_N}\,.$$

Recalling that $\delta \mapsto h_\delta(\xi)$, and hence $\delta \mapsto h_\delta^{\mathrm{qc}}(\xi)$, are nondecreasing, with Beppo Levi's theorem and Proposition 2.3 we obtain

$$\int_A \Psi^{\mathrm{qc}}(\nabla(\mathcal{T}_{k_N}(u)))\mathrm{d}x = \lim_{\delta \uparrow 1} \int_A h_\delta^{\mathrm{qc}}(\nabla(\mathcal{T}_{k_N}(u)))\mathrm{d}x \\ = \lim_{\delta \uparrow 1} \delta^{q'+1} \int_A h_\delta^{\mathrm{qc}}(\nabla(\mathcal{T}_{k_N}(u)))\mathrm{d}x \leq \liminf_{\varepsilon \to 0} \mathcal{F}_\varepsilon(u_\varepsilon, v_\varepsilon; A) + \frac{c}{k_N}\,,$$

and finally (4.68) follows at once by letting $N \to \infty$ in the latter inequality.

Clearly, as $u \in (GSBV(A))^m$ for every $A \in \mathcal{A}(\Omega)$, $u \in (GSBV(\Omega))^m$. Finally, as the growth condition (1.15) holds also for Ψ^{qc}, we conclude in addition that $\nabla u \in L^q(A; \mathbb{R}^{m \times n})$ with a uniform bound with respect to A, so that $\nabla u \in L^q(\Omega; \mathbb{R}^{m \times n})$.

Step 3. If $(u_\varepsilon, v_\varepsilon) \to (u, v)$ in $L^1(\Omega;\mathbb{R}^{m+1})$, then for every $A \in \mathcal{A}(\Omega)$

$$\int_{J_u\cap A} g_{\inf}([u],\nu_u)\mathrm{d}\mathcal{H}^{n-1} \le \liminf_{\varepsilon\to 0} \mathcal{F}_\varepsilon(u_\varepsilon, v_\varepsilon; A)\,. \tag{4.71}$$

Up to subsequences and with a small abuse of notation, we can assume the previous inferior limit to be in fact a limit. Let us define the measures $\mu_\varepsilon \in \mathcal{M}_b^+(A)$

$$\mu_\varepsilon := \left(f_\varepsilon^q(v_\varepsilon)\Psi(\nabla u_\varepsilon) + \frac{(1-v_\varepsilon)^{q'}}{q' q^{q'/q}\varepsilon} + \varepsilon^{q-1}|\nabla v_\varepsilon|^q \right) \mathcal{L}^n \llcorner A\,.$$

Extracting a further subsequence, we can assume that

$$\mu_\varepsilon \rightharpoonup \mu \quad \textit{weakly}^* \text{ in } \mathcal{M}(A) = (C_c^0(A))' \tag{4.72}$$

as $\varepsilon \to 0$, for some $\mu \in \mathcal{M}_b^+(A)$, so that

$$\liminf_{\varepsilon\to 0} \mathcal{F}_\varepsilon(u_\varepsilon, v_\varepsilon; A) \ge \mu(A)\,.$$

Equation (4.71) will follow once we have proved that

$$\frac{\mathrm{d}\mu}{\mathrm{d}\mathcal{H}^{n-1}\llcorner J_u}(x_0) \ge g_{\inf}([u](x_0), \nu_u(x_0)), \quad \mathcal{H}^{n-1}\text{-a.e. } x_0 \in J_u \cap A\,. \tag{4.73}$$

We will prove the last inequality for points $x_0 \in J_u \cap A$ such that

$$\frac{\mathrm{d}\mu}{\mathrm{d}\mathcal{H}^{n-1}\llcorner J_u}(x_0) = \lim_{\rho\to 0} \frac{\mu(Q_\rho^\nu(x_0))}{\mathcal{H}^{n-1}(J_u \cap Q_\rho^\nu(x_0))} \quad \text{exists finite}\,,$$

and

$$\lim_{\rho\to 0} \frac{\mathcal{H}^{n-1}(J_u \cap Q_\rho^\nu(x_0))}{\rho^{n-1}} = 1,$$

where $\nu := \nu_u(x_0)$ and $Q_\rho^\nu(x_0) := x_0 + \rho Q^\nu$ is the cube centred in x_0, with side length ρ, and one face orthogonal to ν. We remark that such conditions define a set of full measure in $J_u \cap A$. First note that

$$\frac{\mathrm{d}\mu}{\mathrm{d}\mathcal{H}^{n-1}\llcorner J_u}(x_0) = \lim_{\rho\to 0} \frac{\mu(Q_\rho^\nu(x_0))}{\rho^{n-1}} = \lim_{\substack{\rho\in I\\ \rho\to 0}} \lim_{\varepsilon\to 0} \frac{\mu_\varepsilon(Q_\rho^\nu(x_0))}{\rho^{n-1}}\,, \tag{4.74}$$

where $I := \{\rho \in (0, \frac{2}{\sqrt{n}}\operatorname{dist}(x_0, \partial A)) : \mu(\partial Q_\rho^\nu(x_0)) = 0\}$, and we have used (4.72) for the second equality.

By (4.2), for every $\delta \in (0,1)$ one has $\Psi(\xi) \ge (1-\delta)\Psi_\infty(\xi)$ for ξ sufficiently large. As Ψ_∞ is continuous, there is $C(\delta) > 0$ such that

$$\Psi(\xi) + C(\delta) \ge (1-\delta)\Psi_\infty(\xi) \qquad \text{for all } \xi \in \mathbb{R}^{m\times n}\,.$$

We choose $\delta_\rho \to 0$ such that $\rho C(\delta_\rho) \to 0$. As $f_\varepsilon^q(v_\varepsilon) \le 1$ we have

$$f^q_{\varepsilon,p,q}(v_\varepsilon)\Psi(\nabla u_\varepsilon) \ge (1-\delta_\rho) f_\varepsilon^q(v_\varepsilon)\Psi_\infty(\nabla u_\varepsilon) - C(\delta_\rho)\,.$$

As $\rho^{1-n}\mathcal{L}^n(Q^\nu_\rho)C(\delta_\rho) = \rho C(\delta_\rho) \to 0$ as $\rho \to 0$, we conclude by (4.74) that

$$\frac{\mathrm{d}\mu}{\mathrm{d}\mathcal{H}^{n-1}\llcorner J_u}(x_0) \ge \limsup_{\substack{\rho\in I\\ \rho\to 0}} \limsup_{\varepsilon\to 0} \frac{\mathcal{F}^1_\varepsilon(u_\varepsilon, v_\varepsilon; Q^\nu_\rho(x_0))}{\rho^{n-1}}\,, \tag{4.75}$$

where we recall that $\mathcal{F}^1_\varepsilon$ is the functional defined in (4.5) for $M=1$. Setting $y := (x-x_0)/\rho \in Q^\nu$, changing variable to $w^\rho(y) := w(\rho y + x_0)$ for $y \in Q^\nu$, and setting $M_\rho := {}^1/_\rho$, we obtain

$$\mathcal{F}^1_\varepsilon(u_\varepsilon, v_\varepsilon; Q^\nu_\rho(x_0)) = \rho^{n-1}\mathcal{F}^{M_\rho}_{\varepsilon/\rho}(u^\rho_\varepsilon, v^\rho_\varepsilon; Q^\nu).$$

The previous expression becomes

$$\frac{\mathrm{d}\mu}{\mathrm{d}\mathcal{H}^{n-1}\llcorner J_u}(x_0) \ge \limsup_{\substack{\rho\in I\\ \rho\to 0}} \limsup_{\varepsilon\to 0} \mathcal{F}^{M_\rho}_{\varepsilon/\rho}(u^\rho_\varepsilon, v^\rho_\varepsilon; Q^\nu).$$

Recalling that $u_\varepsilon \to u$ in $L^1(\Omega;\mathbb{R}^m)$, by diagonalization we can find subsequences $\{\rho_k\}_k$ and $\{\varepsilon_k\}_k$ such that $u^{\rho_k}_{\varepsilon_k} \to [u](x_0)\chi_{\{y\cdot\nu>0\}} + u^-(x_0)$ in $L^1(Q^\nu;\mathbb{R}^m)$, ${}^{\varepsilon_k}/_{\rho_k} \to 0$, and

$$\frac{\mathrm{d}\mu}{\mathrm{d}\mathcal{H}^{n-1}\llcorner J_u}(x_0) \ge \lim_{k\to\infty} \mathcal{F}^{M_{\rho_k}}_{\varepsilon_k/\rho_k}(u^{\rho_k}_{\varepsilon_k}, v^{\rho_k}_{\varepsilon_k}; Q^\nu).$$

Being $\mathcal{F}^M_\varepsilon$ invariant for translations of the first argument, we find

$$\frac{\mathrm{d}\mu}{\mathrm{d}\mathcal{H}^{n-1}\llcorner J_u}(x_0) \ge \liminf_{k\to\infty} \mathcal{F}^{M_{\rho_k}}_{\varepsilon_k/\rho_k}(u^{\rho_k}_{\varepsilon_k}, v^{\rho_k}_{\varepsilon_k}; Q^\nu) \ge g_{\inf}([u](x_0), \nu_u(x_0)),$$

that is (4.73), and this concludes the proof.

Conclusion. The lower bound inequality in (4.64) is a consequence of (4.68), of (4.71), and of [Bra98, Proposition 1.16] by taking into account that the terms on the left-hand side in (4.68) and in (4.71) define mutually singular measures. □

4.5 Γ-limsup

The next proposition establishes an upper bound for the Γ-limits. To this aim consider the functional $\mathcal{F}_{\sup}(\cdot,\cdot;\cdot) : L^1(\Omega;\mathbb{R}^{m+1}) \times \mathcal{A}(\Omega) \to [0,\infty]$ defined by

$$\mathcal{F}_{\sup}(u,v;A) := \int_A \Psi^{\mathrm{qc}}(\nabla u)\mathrm{d}x + \int_{J_u\cap A} (g_{\sup})_{BV}([u],\nu_u)\mathrm{d}\mathcal{H}^{n-1}, \tag{4.76}$$

if $A \in \mathcal{A}(\Omega)$, $u \in (GSBV \cap L^1(\Omega))^m$ with $\nabla u \in L^q(\Omega;\mathbb{R}^{m\times n})$, $v = 1$ $\mathcal{L}^n$-a.e. on Ω, and $\mathcal{F}_{\sup}(u,v;A) := \infty$ otherwise.

We recall that by (4.2), we have that for $\delta > 0$ there exists $C(\delta) > 0$ such that

$$\Psi(\xi) \le (1+\delta)\Psi_\infty(\xi) + C(\delta), \text{ for all } \xi \in \mathbb{R}^{m\times n}. \tag{4.77}$$

Proposition 4.12. *Let $\mathcal{F}_\varepsilon$ be the functional defined in* (4.4). *Then for all* $(u,v) \in L^1(\Omega;\mathbb{R}^{m+1})$ *it holds*

$$\Gamma(L^1)\text{-}\limsup_{\varepsilon\to 0} \mathcal{F}_\varepsilon(u,v) \le \mathcal{F}_{\sup}(u,v). \tag{4.78}$$

Proof. It is enough to prove (4.78) when $v = 1$ $\mathcal{L}^n$-a.e. and u is such that $\mathcal{F}_{\sup}(u,1) < \infty$.

Let us give for granted that

$$\Gamma(L^1(\Omega))\text{-}\limsup_{\varepsilon\to 0} \mathcal{F}_\varepsilon(u,1) \le H_1(u;\Omega') \tag{4.79}$$

for all open sets Ω' with $\overline{\Omega} \subset \Omega'$ and all $u \in PA(\mathbb{R}^n;\mathbb{R}^m)$, where for $A \in \mathcal{B}(\mathbb{R}^n)$

$$H_1(u;A) := \begin{cases} \displaystyle\int_A \Psi(\nabla u)\mathrm{d}x + \int_{J_u\cap A} g_{\sup}([u],\nu_u)\mathrm{d}\mathcal{H}^{n-1}, & \text{if } u \in PA(\mathbb{R}^n;\mathbb{R}^m),\\ \infty, & \text{otherwise.}\end{cases}$$

Then,

$$\Gamma(L^1(\Omega))\text{-}\limsup_{\varepsilon\to 0} \mathcal{F}_\varepsilon(u,1) \le H_1(u;\overline{\Omega}) = H_0(u), \tag{4.80}$$

for any $u \in PA(\mathbb{R}^n;\mathbb{R}^m)$ with $\mathcal{H}^{n-1}(J_u\cap\partial\Omega) = 0$, where H_0 is the functional introduced in (3.12), with bulk density Ψ and surface density $g_{\sup}$. The Relaxation Theorem 3.2 and the lower semicontinuity of $\mathcal{F}''(\cdot) := \Gamma(L^1)\text{-}\limsup_{k\to\infty} \mathcal{F}_\varepsilon(\cdot,1)$ with respect to the $L^1(\Omega;\mathbb{R}^m)$ convergence imply that (4.78) holds for all $u \in (GSBV(\Omega))^m$.

Hence, we only need to show that (4.79) holds. The construction is based on Proposition 4.6 and on a decomposition of the domain similar to the one used in proving Theorem 3.2, for which one covers a large part of the jump set with cubes of scale ρ. The current construction, however, needs to produce Sobolev functions and not functions with jumps, therefore the interfaces need to be regularized. We do this using a mollification on a scale ε close to the jump set, and an interpolation on an intermediate scale $(1-\lambda)\rho$ around the boundaries of the cubes. We use the same notation as in Theorem 3.2 to emphasize the similarities, but since there are several differences in the details, we repeat the common parts of the argument for greater clarity.

Fix $u \in PA(\mathbb{R}^n;\mathbb{R}^m)$ and an open set Ω' with $\Omega \subset\subset \Omega'$. Let $\{F_j\}_{j\in\mathbb{N}}$ denote the faces of the simplexes in the decomposition of $\mathbb{R}^n$ associated to u. Possibly splitting the simplexes, we can ensure that $\mathrm{diam}(F_j) \le \frac{1}{2}\mathrm{dist}(\Omega,\partial\Omega')$ for all j. We set

$$F_* := \bigcup_j F_j.$$

Let

$$M_u := \|\nabla u\|_{L^\infty(\Omega')} + \mathcal{H}^{n-1}(F_*\cap\Omega') + \mathcal{H}^{n-2}(\bigcup_j \partial F_j \cap \Omega'), \tag{4.81}$$

where the boundary operator is intended in the $(n-1)$-dimensional sense; since u is piecewise affine, we have $M_u < \infty$. We first construct a sequence with bounded energy. We recall that $u \in PA(\mathbb{R}^n; \mathbb{R}^m)$ by assumption, and define for any $\varepsilon > 0$ similarly to (4.20)

$$U_\varepsilon := u * \varphi_\varepsilon \,, \qquad V_\varepsilon := \chi_{\mathbb{R}^n \setminus (F_*)_{2\varepsilon}} * \varphi_\varepsilon \,. \tag{4.82}$$

We observe that, since $\varphi_\varepsilon \in C_c^\infty(B_\varepsilon)$ is an even function and u is affine on each connected component of $\mathbb{R}^n \setminus F_*$, we have $U_\varepsilon = u$ outside $(F_*)_\varepsilon$. At the same time, $V_\varepsilon = 0$ in $(F_*)_\varepsilon$ and $V_\varepsilon = 1$ outside $(F_*)_{3\varepsilon}$, so that with $f_\varepsilon(V_\varepsilon) = 0$ in $(F_*)_\varepsilon$ and $f_\varepsilon(V_\varepsilon) \le 1$ everywhere we obtain, for $\varepsilon < \frac{1}{3}\mathrm{dist}(\Omega, \partial\Omega')$,

$$\mathcal{F}_\varepsilon(U_\varepsilon, V_\varepsilon) \le \int_\Omega \Psi(\nabla u)\mathrm{d}x + \mathcal{F}_\varepsilon^\infty(0, V_\varepsilon; \Omega \cap (F_*)_{3\varepsilon}), \tag{4.83}$$

where we recall that

$$\mathcal{F}_\varepsilon^\infty(0, v; A) = \int_A \Big(\frac{(1-v)^{q'}}{q' q^{q'/q} \varepsilon} + \varepsilon^{q-1} |\nabla v|^q \Big) \mathrm{d}x \tag{4.84}$$

only contains the phase field part of the functional.

Using $V_\varepsilon \in [0,1]$ and $|\nabla V_\varepsilon| \le C/\varepsilon$ leads to

$$\mathcal{F}_\varepsilon^\infty(0, V_\varepsilon; \Omega \cap (F_*)_{3\varepsilon}) \le \frac{C}{\varepsilon} \mathcal{L}^n(\Omega \cap (F_*)_{3\varepsilon}). \tag{4.85}$$

Since F_* is locally a finite union of polygons, we conclude that

$$\limsup_{\varepsilon \to 0} \mathcal{F}_\varepsilon^\infty(0, V_\varepsilon; \Omega \cap (F_*)_{3\varepsilon}) < \infty, \tag{4.86}$$

so that the sequence $(U_\varepsilon, V_\varepsilon)$ is bounded in energy, as claimed.

We next modify the sequence in a set covering a large part of $(F_*)_{3\varepsilon}$ in order to replace this estimate with the one with the optimal constant. To do this, we shall introduce a parameter $\rho > 0$ and cover a large part of each face F_j by cubes on a scale ρ, as in the proof of Theorem 3.2. In order to quantify "a large part" it is useful to first introduce another parameter $\delta > 0$, that will be chosen later. From now on, it is convenient to restrict to the indices in the finite set $J := \{j : F_j \cap \overline{\Omega} \neq \emptyset\}$. We first fix relatively open sets $F_j' \subset F_j$ such that $\mathrm{dist}(F_j', \partial F_j) > 0$ and

$$\sum_{j \in J} \mathcal{H}^{n-1}(F_j \setminus F_j') \le \delta. \tag{4.87}$$

Let $A_j : \mathbb{R}^n \to \mathbb{R}^n$ be an affine isometry that maps $\mathbb{R}^{n-1} \times \{0\}$ to the $n-1$-dimensional affine space containing F_j, and set $\nu_j := (\nabla A_j) e_n$. For $y \in \rho\mathbb{Z}^{n-1}$ we let $Q_{j,y} := A_j(y + Q_\rho)$. Let I_j denote the set of $y \in \rho\mathbb{Z}^{n-1}$ such that $Q_{j,y} \cap F_j' \neq \emptyset$. Since $\mathrm{dist}(F_j', F_l) > 0$ for $j \neq l$, for ρ sufficiently small for all $l \neq j \in J$ and all $y \in I_j$, $y' \in I_l$, one has $Q_{j,y} \cap F_l = \emptyset$ and $Q_{j,y} \cap Q_{l,y'} = \emptyset$.

In each cube $Q_{j,y}$, we intend to apply Proposition 4.6. However, this proposition provides a construction for limit functions that are constant on the two halves of the cube, whereas u is affine on each of them. Therefore, we first introduce a piecewise constant approximation in each cube, followed by an additional interpolation step.

For each $j \in J$ and $y \in I_j$, let $u^{\pm}_{j,y} := u^{\pm}(A_j(y))$ denote the two traces of u at the center of $Q_{j,y}$ (the traces have point values, since u is piecewise affine). From these two values we construct a piecewise constant function by

$$w_{j,y}(x) := \begin{cases} u^+_{j,y}, & \text{if } x \in A_j(y + \mathbb{R}^{n-1} \times [0, \infty)), \\ u^-_{j,y}, & \text{if } x \in A_j(y + \mathbb{R}^{n-1} \times (-\infty, 0)). \end{cases} \tag{4.88}$$

We remark that $\|w_{j,y} - u\|_{L^\infty(Q_{j,y})} \le \rho\sqrt{n} M_u$, and in particular that $\|[w_{j,y}] - [u]\|_{L^\infty(J_u \cap Q_{j,y}; \mathcal{H}^{n-1})} \le 2\rho\sqrt{n} M_u$.

We fix a cutoff function $\theta_\delta \in C^\infty_c(Q_1; [0, 1])$ with $\theta_\delta = 1$ in $Q_{1-\frac{\delta}{2}}$, for $\delta \in (0, 1/2)$. For each $j \in J$ and $y \in I_j$, we define $\theta_{j,y} \in C^\infty_c(Q_{j,y})$ by $\theta_{j,y}(A_j(y + \rho x)) := \theta_\delta(x)$ and let $q_{j,y} := A_j(y + Q_{(1-\delta)\rho}) \subset\subset Q_{j,y}$. We set

$$W_{j,y} := (w_{j,y}\theta_{j,y} + u(1 - \theta_{j,y})) * \varphi_\varepsilon. \tag{4.89}$$

Then for ε sufficiently small we have

$$W_{j,y} = U_\varepsilon \text{ on } \partial Q_{j,y}, \qquad \text{and} \qquad W_{j,y} = w_{j,y} * \varphi_\varepsilon \text{ on } \partial q_{j,y}. \tag{4.90}$$

For each j and y we use Proposition 4.6, with $\nu = \nu_j$ and $z = u^+_{j,y} - u^-_{j,y}$. By an elementary translation and scaling argument, we obtain functions $(u_{\varepsilon,j,y}, v_{\varepsilon,j,y}) \in W^{1,q}(q_{j,y}; \mathbb{R}^m \times [0, 1])$ such that $u_{\varepsilon,j,y} \to w_{j,y}$ as $\varepsilon \to 0$ in $L^q(q_{j,y}; \mathbb{R}^{m+1})$,

$$\lim_{\varepsilon\to 0} \mathcal{F}^\infty_\varepsilon(u_{\varepsilon,j,y}, v_{\varepsilon,j,y}; q_{j,y}) = (1 - \delta)^{n-1} \rho^{n-1} g_{\sup}([u](A_j(y)), \nu_j), \tag{4.91}$$

and

$$u_{\varepsilon,j,y} = w_{j,y} * \varphi_\varepsilon = W_{j,y}, \quad v_{\varepsilon,j,y} = V_\varepsilon \quad \text{on } \partial q_{j,y}. \tag{4.92}$$

Using (4.77), we have

$$\mathcal{F}_\varepsilon(u_{\varepsilon,j,y}, v_{\varepsilon,j,y}; Q^\nu_\rho) \le (1 + \delta)\mathcal{F}^\infty_\varepsilon(u_{\varepsilon,j,y}, v_{\varepsilon,j,y}; Q^\nu_\rho) + C(\delta)\rho^n,$$

and then

$$\lim_{\varepsilon\to 0} \mathcal{F}_\varepsilon(u_{\varepsilon,j,y}, v_{\varepsilon,j,y}; q_{j,y}) \le (1 + \delta)\rho^{n-1} g_{\sup}([u](A_j(y)), \nu_j) + C_\delta\rho^n. \tag{4.93}$$

Recalling the bounds after the definition of $w_{j,y}$ setting $g_0(s) := s^{\frac{2}{p+1}} \wedge 1$ for every $s \ge 0$, and using (i) and (ii) in Lemma 4.10, we have

$$\begin{aligned} \rho^{n-1} g_{\sup}([u](A_j(y)), \nu_j) &= \int_{J_u \cap Q_{j,y}} g_{\sup}([w_{j,y}], \nu_j) \mathrm{d}\mathcal{H}^{n-1} \\ &\le \int_{J_u \cap Q_{j,y}} \left(g_{\sup}([u], \nu_j) + C g_0(2\rho\sqrt{n} M_u) \right) \mathrm{d}\mathcal{H}^{n-1}. \end{aligned} \tag{4.94}$$

We finally set

$$\begin{aligned}u_\varepsilon^{\delta,\rho} :=& U_\varepsilon + \sum_{j\in J}\sum_{y\in I_j} \left(\chi_{q_{j,y}}(u_{\varepsilon,j,y} - W_{j,y}) + \chi_{Q_{j,y}}(W_{j,y} - U_\varepsilon)\right),\\ v_\varepsilon^{\delta,\rho} :=& V_\varepsilon + \sum_{j\in J}\sum_{y\in I_j} \chi_{q_{j,y}}(v_{\varepsilon,j,y} - V_\varepsilon).\end{aligned} \tag{4.95}$$

The boundary data in (4.92) show that the characteristic functions do not introduce any jump on each $\partial q_{j,y}$; and with the boundary data in (4.90) the same holds on each $\partial Q_{j,y}$. Therefore $(u_\varepsilon^\rho, v_\varepsilon^\rho) \in W^{1,q}(\Omega; \mathbb{R}^m \times [0,1])$.

In order to estimate the energy we decompose the domain in three parts,

$$\Omega^{\text{in}} := \bigcup_{j\in J}\bigcup_{y\in I_j} q_{j,y}, \quad \Omega^{\text{mid}} := \bigcup_{j\in J}\bigcup_{y\in I_j} Q_{j,y}\setminus q_{j,y}, \quad \Omega^{\text{out}} := \Omega \setminus \Omega^{\text{in}} \setminus \Omega^{\text{mid}}. \tag{4.96}$$

As the cubes $Q_{j,y}$ are disjoint, $\sum_{j\in J}\sum_{y\in I_j}\rho^{n-1} \le \mathcal{H}^{n-1}(\Omega\cap F_*)$ implies that their number is bounded by $C_u\rho^{1-n}$, so that for $\delta\in(0,1/2)$ we have

$$\mathcal{L}^n(\Omega^{\text{in}}) \le C_u\rho, \qquad \mathcal{L}^n(\Omega^{\text{mid}}) \le C_u\rho\delta, \tag{4.97}$$

where C_u denotes a constant that may depend on u (and Ω) but not on ε, ρ, and δ.

We start from Ω^{in}. Since in each $q_{j,y}$ we have $(u_\varepsilon^{\delta,\rho}, v_\varepsilon^{\delta,\rho}) = (u_{\varepsilon,j,y}, v_{\varepsilon,j,y})$, by (4.93), (4.94), and the first in (4.97)

$$\begin{aligned}&\limsup_{\varepsilon\to0}\mathcal{F}_\varepsilon(u_\varepsilon^{\delta,\rho}, v_\varepsilon^{\delta,\rho}; \Omega^{\text{in}}) \le (1+\delta)\sum_{j,y}\Big(\rho^{n-1}g_{\sup}([u](A_j(y)),\nu_j) + C_\delta\rho^n\Big)\\ &\le (1+\delta)H_1(u; J_u\cap\Omega^{\text{in}}) + CM_u g_0(2\rho\sqrt{n}M_u) + CC_\delta\mathcal{L}^n(\Omega^{\text{in}})\\ &\le (1+\delta)H_1(u; J_u\cap\Omega') + CM_u g_0(2\rho\sqrt{n}M_u) + C_uC_\delta\rho,\end{aligned} \tag{4.98}$$

and similarly, using $u_{\varepsilon,j,y}\to w_{j,y}$ and the definition of $w_{j,y}$ in (4.88) and that of $W_{j,y}$ in (4.89) we infer,

$$\limsup_{\varepsilon\to0}\|u_\varepsilon^{\delta,\rho}\|_{L^1(\Omega^{\text{in}})} = \sum_{j,y}\|w_{j,y}\|_{L^1(q_{j,y})} \le \mathcal{L}^n(\Omega^{\text{in}})\|u\|_{L^\infty(\Omega')} \le C_u\rho. \tag{4.99}$$

We next address Ω^{mid}. In $Q_{j,y}\setminus(F_j)_\varepsilon$, letting $Q'_{j,y}$ be a cube with the same center and twice the side length,

$$\begin{aligned}\|\nabla W_{j,y} - \nabla u\|_{L^\infty(Q_{j,y}\setminus(F_j)_\varepsilon)} \le& \|\nabla((w_{j,y}-u)\theta_{j,y})\|_{L^\infty(Q'_{j,y})}\\ \le& \|\nabla\theta_{j,y}\|_{L^\infty(Q'_{j,y})}\|w_{j,y}-u\|_{L^\infty(Q'_{j,y})} + \|\nabla u\|_{L^\infty(\Omega')},\end{aligned} \tag{4.100}$$

where we used that $U_\varepsilon = u$ outside $(F_*)_\varepsilon$, $\nabla w_{j,y} = 0$ and $\theta_{j,y}\in[0,1]$ pointwise. Recalling that $|\nabla\theta_{j,y}| \le C_\delta/\rho$ and $|\nabla u| \le M_u$, we conclude

$$\|\nabla W_{j,y} - \nabla u\|_{L^\infty(Q_{j,y}\setminus(F_j)_\varepsilon)} \le C_\delta M_u, \tag{4.101}$$

which implies that $\Psi(\nabla W_{j,y}) \le C_\delta(M_u^q + 1)$ pointwise in $Q_{j,y} \setminus (F_j)_\varepsilon$.

Since $v_\varepsilon^{\delta,\rho} = V_\varepsilon = 0$ in $\Omega \cap (F_*)_\varepsilon \setminus \Omega^{\mathrm{in}}$, we conclude

$$\begin{aligned}\int_{\Omega^{\mathrm{mid}}} f_\varepsilon^q(v_\varepsilon^{\delta,\rho})\Psi(\nabla u_\varepsilon^{\delta,\rho})\mathrm{d}x \le& \int_{\Omega^{\mathrm{mid}}\setminus(F_*)_\varepsilon} \Psi(\nabla u_\varepsilon^{\delta,\rho})\mathrm{d}x \\ &\le C_\delta(M_u^q+1)\mathcal{L}^n(\Omega^{\mathrm{mid}}) \le C_\delta(M_u^q+1)C_u\rho.\end{aligned} \tag{4.102}$$

Similarly,

$$\|W_{j,y} - u * \varphi_\varepsilon\|_{L^1(Q_{j,y})} \le \|(w_{j,y}-u)\theta_{j,y}\|_{L^1(Q'_{j,y})} \le \|w_{j,y}-u\|_{L^1(Q'_{j,y})} \le C_u\rho^{n+1}, \tag{4.103}$$

so that for ε small, using that $u * \varphi_\varepsilon \to u$ in $L^1_{\mathrm{loc}}(\mathbb{R}^n;\mathbb{R}^m)$ as $\varepsilon \to 0$, we get

$$\|W_{j,y} - u\|_{L^1(\Omega^{\mathrm{mid}})} \le C_u\rho^2. \tag{4.104}$$

Finally, in $\Omega^{\mathrm{out}} \setminus (F_*)_\varepsilon$ we have $u_\varepsilon^{\delta,\rho} = U_\varepsilon = u$, so that

$$\int_{\Omega^{\mathrm{out}}} f_\varepsilon^q(v_\varepsilon^{\delta,\rho})\Psi(\nabla u_\varepsilon^{\delta,\rho})\mathrm{d}x \le \int_{\Omega^{\mathrm{out}}\setminus(F_*)_\varepsilon} \Psi(\nabla u)\mathrm{d}x \le \int_\Omega \Psi(\nabla u)\mathrm{d}x. \tag{4.105}$$

The last two terms of $\mathcal{F}_\varepsilon$ are estimated jointly in Ω^{mid} and Ω^{out}. Since $v_\varepsilon^{\delta,\rho} = V_\varepsilon$ in $\Omega \setminus \Omega^{\mathrm{in}}$, $V_\varepsilon = 1$ in $\Omega \setminus (F_*)_{3\varepsilon}$, and $|\nabla V_\varepsilon| \le C/\varepsilon$ everywhere,

$$\mathcal{F}_\varepsilon^\infty(0, v_\varepsilon^{\delta,\rho}; \Omega \setminus \Omega^{\mathrm{in}}) = \mathcal{F}_\varepsilon^\infty(0, V_\varepsilon; \Omega\setminus\Omega^{\mathrm{in}}) \le \frac{C}{\varepsilon}\mathcal{L}^n(\Omega \cap (F_*)_{3\varepsilon} \setminus \Omega^{\mathrm{in}}). \tag{4.106}$$

To compute this limit we observe that, since $\Omega \cap F_* \subseteq \cup_j F_j$, the F_j are contained in planes, and Ω^{in} does not touch the $(n-2)$-dimensional boundary of the F_j,

$$\Omega \cap (F_*)_{3\varepsilon} \setminus \Omega^{\mathrm{in}} = \Omega \cap \bigcup_j (F_j)_{3\varepsilon} \setminus \Omega^{\mathrm{in}} \subset \bigcup_{j\in J}(F_j \setminus \Omega^{\mathrm{in}})_{3\varepsilon}. \tag{4.107}$$

As each $F_j \setminus \Omega^{\mathrm{in}}$ is a compact subset of a plane, the Minkowski content equals their $\mathcal{H}^{n-1}$ dimensional measure, so that

$$\limsup_{\varepsilon\to0} \frac{\mathcal{L}^n(\Omega \cap (F_*)_{3\varepsilon} \setminus \Omega^{\mathrm{in}})}{6\varepsilon} \le \sum_{j\in J}\Big(\mathcal{H}^{n-1}(F_j \setminus F'_j) + C\delta\mathcal{H}^{n-1}(F'_j)\Big). \tag{4.108}$$

Combining this with (4.106) and (4.87) leads to

$$\mathcal{F}_\varepsilon^\infty(0, v_\varepsilon^{\delta,\rho}; \Omega \setminus \Omega^{\mathrm{in}}) \le C\delta + C_u\delta. \tag{4.109}$$

Putting together (4.98), (4.102), (4.105), (4.109) shows that for any $\delta \in (0, 1/2)$ there is a sequence with

$$\limsup_{\rho\to0}\limsup_{\varepsilon\to0} \mathcal{F}_\varepsilon(u_\varepsilon^{\delta,\rho}, v_\varepsilon^{\delta,\rho}; \Omega) \le (1+\delta)H_1(u;\Omega') + C\delta + C_u\delta. \tag{4.110}$$

At the same time, from (4.99), (4.104), and $u_\varepsilon^{\delta,\rho} = U_\varepsilon$ on Ω^{out},

$$\limsup_{\rho\to0}\limsup_{\varepsilon\to0} \|u_\varepsilon^{\delta,\rho} - u\|_{L^1(\Omega)} = 0. \tag{4.111}$$

Taking the limit $\delta \to 0$, and finally a diagonal subsequence, concludes the proof. □

We are now ready to prove Theorem 4.1.

Proof of Theorem 4.1. Propositions 4.11 and 4.12 imply in particular that $g_{\inf} \leq (g_{\sup})_{BV}$, so that by (4.10) we have $g_{\sup} \leq (g_{\sup})_{BV}$, and then $g_{\sup} = (g_{\sup})_{BV}$. Hence, $\mathcal{F}_{\inf} = \mathcal{F}_{\sup}$ and the common value $g = g_{\inf} = g_{\sup}$ is BV-elliptic. □

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