HESSIAN STABILITY AND CONVERGENCE RATES FOR ENTROPIC AND SINKHORN POTENTIALS VIA SEMICONCAVITY

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Abstract. In this paper we determine quantitative stability bounds for the Hessian of entropic potentials, *i.e.*, the dual solution to the entropic optimal transport problem. Up to authors' knowledge this is the first work addressing this second-order quantitative stability estimate in general unbounded settings. Our proof strategy relies on semiconcavity properties of entropic potentials and on the representation of entropic transport plans as laws of forward and backward diffusion processes, known as Schrödinger bridges. Moreover, our approach allows to deduce a stochastic proof of quantitative stability entropic estimates and integrated gradient estimates as well. Finally, as a direct consequence of these stability bounds, we deduce exponential convergence rates for gradient and Hessian of Sinkhorn iterates along Sinkhorn's algorithm, a problem that was still open in unbounded settings. Our rates have a polynomial dependence on the regularization parameter.

1. Introduction

Given two probability measure ρ, $\mu \in \mathcal{P}(\mathbb{R}^d)$ and a regularization parameter $T > 0$, the Entropic Optimal Transport problem (EOT henceforth) reads as

$$\text{minimize } \int_{\mathbb{R}^d\times\mathbb{R}^d} \frac{|x-y|^2}{2}\,\mathrm{d}\pi + T\,\mathscr{H}(\pi|\rho\otimes\mu) \text{ under the constraint } \pi\in\Pi(\rho,\mu)\,,$$

where $\mathscr{H}$ denotes the relative entropy functional (aka Kullback–Leibler divergence) and $\Pi(\rho,\mu)$ is the set of couplings of ρ and μ. This problem can be seen as an entropic regularization of the Optimal Transport (OT) problem, which indeed is recovered in the limit case $T = 0$. For this reason, EOT has been widely studied in the last years and the solutions to its primal and dual formulation are respectively used as proxies for optimal transport plans and Brenier's optimal transport map [Mik04, BGN22, NW22a, CCGT23]. Lastly, EOT is equivalent to a statistical mechanics problem, known as the Schrödinger problem, introduced in [Sch31, Sch32] where E. Schrödinger was interested in the most likely evolution of a cloud of Brownian particles, conditionally to its initial and final distribution at time $s = 0$ and $s = T$ respectively. Therefore EOT has a cutting-edge nature that lies at the interface between analysis and stochastics. Moreover, this problem has recently gained more popularity due to its use in machine learning and generative modeling applications [BTHD21, WJX$^+$21, SDBDD22], mainly due to the possibility of solving EOT via an iterative algorithm, known as *Sinkhorn's algorithm* [Sin64, SK67] or *Iterative Proportional Fitting Procedure*, which can be used to quickly obtain approximate solutions for EOT [Cut13] in a much easier and faster way, compared to standard OT solvers.

In this article, we are interested in analyzing how changes in the marginals ρ,μ affect solutions to EOT. By relying on semiconcavity bounds and stochastic calculus, we are going to show below quantitative stability estimates for EOT potentials up to the second order, namely for

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their gradient and Hessian. To the best of our knowledge, this is the first work where second-order quantitative stability estimates are obtained. This is even more remarkable when compared with unregularized optimal transport, where higher-order quantitative stability estimates are more difficult to obtain, as in general potentials may lack regularity and the Ma–Trudinger–Wang condition [MTW05] is imposed in order to ensure it; without this demanding assumption, only first-order quantitative stability bounds are available in general (see for instance the very recent [LM24, KLM25] and references therein). On the contrary, our main stability theorem is valid under fairly general assumptions, significantly weaker than the Ma–Trudinger–Wang condition, and since EOT is used as a proxy for OT, this highlights the importance of our result.

In order to continue the exposition and state clearly our main contributions, let us collect a few basic facts about EOT and its solutions. First, let us recall that under mild assumptions on the marginals ρ, μ (see for instance [CCGT23, Proposition 2.2]), EOT admits a unique minimizer $\pi^\mu \in \Pi(\rho,\mu)$, referred to as the *entropic plan* (or Schrödinger plan), and there exist two functions $\varphi^\mu \in \mathrm{L}^1(\rho)$ and $\psi^\mu \in \mathrm{L}^1(\mu)$, called *entropic potentials*, such that

$$\pi^\mu(\mathrm{d}x\mathrm{d}y) = (2\pi T)^{-d/2}\,\exp\biggl(-\frac{|x-y|^2}{2T} - \varphi^\mu(x) - \psi^\mu(y)\biggr)\,\mathrm{d}x\,\mathrm{d}y\,.$$

Both the optimal plan π^μ and the entropic potentials φ^μ, ψ^μ depend on T and on ρ, but for ease of notation we omit this dependence, as T and ρ will be kept fixed throughout the whole manuscript, whereas we are interested in stability bounds for changes in the second marginal in EOT. The couple (φ^μ, ψ^μ) is unique up to constant translations $a \mapsto (\varphi^\mu + a, \psi^\mu - a)$ and it is characterized as solution to a system of equations. Indeed, if we suppose that the marginals admit densities of the form

$$\rho(\mathrm{d}x) = \exp(-U_\rho(x))\mathrm{d}x\,, \qquad \mu(\mathrm{d}y) = \exp(-U_\mu(y))\mathrm{d}y\,,$$

then, imposing that $\pi^\mu \in \Pi(\rho,\mu)$ one finds that φ^μ, ψ^μ solve the following system of implicit functional equations, known as *Schrödinger system*

$$\begin{cases} \varphi^\mu = U_\rho + \log P_T \exp(-\psi^\mu) \\ \psi^\mu = U_\mu + \log P_T \exp(-\varphi^\mu)\,, \end{cases} \tag{1.1}$$

where $(P_s)_{s\geq 0}$ is the Markov semigroup generated by the standard d-dimensional Brownian motion $(B_s)_{s\geq 0}$, defined as $P_s f(x) = \mathbb{E}[f(x+B_s)]$ for any non-negative measurable function $f:\mathbb{R}^d \to \mathbb{R}$.

The structure of the Schrödinger system motivates the introduction of the 'interpolated potentials'

$$\varphi^\mu_s = -\log P_{T-s}\exp(-\varphi^\mu)\,, \qquad \psi^\mu_s = -\log P_{T-s}\exp(-\psi^\mu)\,.$$

It is easily seen that they are solutions to the backward Hamilton-Jacobi-Bellman equation

$$\partial_s u_s + \frac{1}{2}\Delta u_s - \frac{1}{2}|\nabla u_s|^2 = 0 \tag{HJB}$$

with final conditions $u_T = \varphi^\mu$ and $u_T = \psi^\mu$ respectively. Such a PDE enjoys a fundamental property of back-propagation of convexity (see Lemma A.1) and this has recently been employed in a stochastic analysis framework in order to prove convexity/concavity estimates for entropic potentials [Con24], providing an entropic version of the celebrated Caffarelli Theorem for Lipschitzianity of transport maps (see also [CP23, FGP20] for a non-stochastic proof). As shown in [CDG23, CCGT24], semiconcavity estimates play a pivotal role in establishing entropic quantitative stability results. In this work, we continue the research line started there, where semiconcavity was used for entropic stability of entropic plans and exponential convergence of Sinkhorn's algorithm; here, we focus on quantitative stability bounds for gradient and Hessian of entropic potentials. For these reasons, let us introduce the notion of semiconcavity that we employ in our paper. We say that a function $f:\mathbb{R}^d \to \mathbb{R}$ is Λ-semiconcave if for all $z, y \in \mathbb{R}^d$ we have

$$f(z) - f(y) \leq \langle \nabla f(y),\, z-y\rangle + \frac{\Lambda}{2}\,|z-y|^2\,.$$

As already observed in [CDG23, CCGT24], a crucial role is played by the semiconcavity of the function

$$g_h^y(z) \coloneqq \frac{|z-y|^2}{2} - T\,h(z) \tag{1.2}$$

where $h \in \{\varphi_0^\mu,\, \psi_0^\mu\}$ is a backpropagated entropic potential along HJB. We will denote with $\Lambda(h)$ a semiconcavity parameter[1] of g_h^y (uniformly in y).

We are now ready to state our main assumptions and results:

***H*1.** Let us assume that $\rho,\, \mu \in \mathcal{P}_2(\mathbb{R}^d)$ have finite relative entropy, namely $\mathscr{H}(\rho|\mathrm{Leb}) < +\infty$ and $\mathscr{H}(\mu|\mathrm{Leb}) < +\infty$.

This first assumption is standard in EOT when considering its Schrödinger problem formulation and it guarantees the existence and uniqueness of optimal plan, entropic potentials as well as the validity of the stochastic representation via forward backward Schrödinger bridge processes, as described in Section 1.2 below. The second assumption is needed when introducing a different target marginal $\nu \in \mathcal{P}(\mathbb{R}^d)$.

***H*2.** Assume that $\nu \in \mathcal{P}_2(\mathbb{R}^d)$ has finite relative entropy, namely $\mathscr{H}(\nu|\mathrm{Leb}) < +\infty$. Moreover, let us assume that: (a) either $\mathscr{H}(\mu|\nu) < +\infty$; (b) or $\mu \ll \nu$ and $\Lambda(\varphi_0^\mu)$ is finite.

Remark 1. Let us stress that, despite the finiteness of $\Lambda(\varphi_0^\mu)$ in **H**2 may seem as a condition on μ, there exist sufficient conditions on ρ that ensure its validity without any extra assumption on μ. For instance, the compactness of the support of ρ or the log-concavity of its Radon-Nikodým derivative, as discussed in Appendix A.

Under these assumptions, we will prove a general Hessian (and gradients) stability result which builds upon semiconcavity estimates for $\Lambda(\varphi_0^\nu)$. In order to show its wide validity, we will further specialize these general estimates in two landmark examples: compactly supported and log-concave marginals. By building upon estimates obtained in [CCGT24] our quantitative stability estimates could be applied to weakly log-concave marginals or could be further specialized to the more regular Caffarelli's setting (namely when the Hessian of marginals' log-densities are both upper and lower bounded). For sake of exposition, we have omitted these two applications where the constants are less readable.

Our main stability result reads as follows.

Theorem 1.1 (Informal main result)**.** *Assume **H**1 and **H**2. We have*

$$\|\nabla\varphi^\nu - \nabla\varphi^\mu\|^2_{\mathrm{L}^2(\rho)} \lesssim \mathbf{W}_2^2(\mu,\nu) \qquad \text{and} \qquad \|\nabla^2\varphi^\mu - \nabla^2\varphi^\nu\|_{\mathrm{L}^1(\rho)} \lesssim \mathbf{W}_2(\mu,\nu) + \mathbf{W}_2^2(\mu,\nu)\,,$$

up to multiplicative constants that depend polynomially only on ρ, ν, T (and not on μ). These constants are explicit and in particular, up to numerical universal constants, we have

- *if $\mathrm{supp}(\rho)$, $\mathrm{supp}(\nu) \subseteq B_R(0)$ (for some radius big enough, i.e., $R^2 \geq T$) then*
$$\|\nabla\varphi^\nu - \nabla\varphi^\mu\|^2_{\mathrm{L}^2(\rho)} \lesssim {}^{R^4}\!/\!{}_{T^4}\,\mathbf{W}_2^2(\mu,\nu)\,,$$
$$\|\nabla^2\varphi^\mu - \nabla^2\varphi^\nu\|_{\mathrm{L}^1(\rho)} \lesssim ({}^{R^4}\!/\!{}_{T^{7/2}} + {}^{d}\!/\!{}_{T})\,\mathbf{W}_2(\mu,\nu) + {}^{R^6}\!/\!{}_{T^5}\,\mathbf{W}_2^2(\mu,\nu)\,,$$
- *if both ρ and ν are log-concave, i.e., their (negative) log-densities satisfy $\nabla^2 U_\rho \geq \alpha_\rho$ and $\nabla^2 U_\nu \geq \alpha_\nu$ for some $\alpha_\rho,\, \alpha_\nu > 0$ (wlog such that $\alpha_\rho \vee \alpha_\nu < T^{-1}$), then*

$$\|\nabla\varphi^\nu - \nabla\varphi^\mu\|^2_{\mathrm{L}^2(\rho)} \lesssim \frac{1}{\alpha_\rho\,\alpha_\nu\,T^4}\,\mathbf{W}_2^2(\mu,\nu)\,,$$

$$\|\nabla^2\varphi^\mu - \nabla^2\varphi^\nu\|_{\mathrm{L}^1(\rho)} \lesssim \left(\frac{1}{\alpha_\nu\,\sqrt{\alpha_\rho}\,T^3} + \frac{d}{\sqrt{\alpha_\rho\,\alpha_\nu}\,T^2}\right)\mathbf{W}_2(\mu,\nu) + \frac{1}{\alpha_\rho\,\alpha_\nu\,T^4}\,\mathbf{W}_2^2(\mu,\nu)\,.$$

[1] To be more precise, in our examples and in the explicit computations we will fix a parameter $\Lambda \in \mathbb{R}$ such that (1) holds. We do not assume it to be the optimal parameter choice.

In this paper whenever we write the L^1-norm between matrices we are considering the L^1-norm induced by the Hilbert-Schmidt norm of their difference, which is defined as

$$\|A\|_{\mathrm{HS}}^2 = \sum_{i,j} A_{i,j}^2 \,.$$

Let us comment here on the dimensionality dependence in the previous bounds. More precisely, all our estimates are dimension-free up to being able to control the Hilbert-Schmidt norm of Hessian of backpropagated potentials $(\nabla^2\psi_s^\nu)_{s\in[0,T)}$. In order to bound these last norms, we then rely on the known identity $\|\nabla^2\psi_s^\nu\|_{\mathrm{HS}} \le \sqrt{d}\left\|\nabla^2\psi_s^\nu\right\|_2$ in (B.4) which allow us to efficiently bound this Hilbert-Schmidt in terms of the semiconcavity parameter $\Lambda(\psi_0^\nu)$.

Let us also comment on a hidden technical point and a first reason why the previous statement is "informal". Rather than the differences $\nabla\varphi^\nu - \nabla\varphi^\mu$ and $\nabla^2\varphi^\nu - \nabla^2\varphi^\mu$, in Theorem 1.1 we control $\nabla(\varphi^\nu - \varphi^\mu)$ and $\nabla^2(\varphi^\nu - \varphi^\mu)$. Note indeed that no regularity assumptions are formulated on ρ, so that φ^ν and φ^μ may lack the required regularity, since they are not evolved along HJB. However, their difference $\varphi^\nu - \varphi^\mu$ is equal to $\psi_0^\mu - \psi_0^\nu$, which is instead the difference of two solutions to HJB, hence of two regular functions. Moreover, under some regularity assumption on ρ (e.g. $\rho \in C^2(\mathbb{R}^d)$), gradient and Hessian of φ^μ, φ^ν are in fact well defined. For this reason, in what follows we will always write $\nabla\varphi^\nu - \nabla\varphi^\mu$ and $\nabla^2\varphi^\nu - \nabla^2\varphi^\mu$, meaning $\nabla(\varphi^\nu - \varphi^\mu)$ and $\nabla^2(\varphi^\nu - \varphi^\mu)$ respectively whenever gradient/Hessian of the single potentials are not defined.

For readers' sake, we collect here the references within this article where our informal main result is stated and proven. The quantitative stability bound for gradients is proven in Theorem 3.1 whereas the Hessian stability bound is proven in Theorem 3.8, where the explicit constants are expressed in terms of T and of the semiconcavity and geometric parameters of ρ, ν. The above statement is informal also for a second reason: solely under **H**1 and **H**2, it is not clear whether these constants are finite, although we are able to show it and compute their asymptotics in our specialized setting. In particular, the compact setting bounds are given in Corollary 3.2 and Corollary 3.9, while for the log-concave setting in Corollary 3.3 and Corollary 3.10. These explicit bounds rely on the computations performed in Appendix A and Appendix B. Lastly, let us remark here that the above general result specifies also to different settings and our computations allow to derive explicit bounds also in the case when solely one marginal is compactly supported and the other one is log-concave. For sake of exposition we prefer not to insist on this, though these explicit bounds are a straightforward computation based on Appendix A and Appendix B and our general theorems.

Finally, it is clear that given the above the interested reader can deduce quantitative stability bounds when both marginals vary.

1.1. **Exponential convergence of Hessian of Sinkhorn's iterates.** As already mentioned in the introduction, most of the popularity EOT has recently gotten is due to the possibility of rapidly computing its solutions via an iterative algorithm, known as *Sinkhorn's algorithm* [Sin64, SK67] or *Iterative Proportional Fitting Procedure*. Given any initialization $\varphi^0\colon \mathbb{R}^d \to \mathbb{R}$, this algorithm solves (1.1) as a fixed point problem by generating two sequences $\{\varphi^n, \psi^n\}_{n\in\mathbb{N}}$, called Sinkhorn potentials, defined recursively as:

$$\begin{cases} \varphi^{n+1} = U_\rho + \log P_T \exp(-\psi^n) \\ \psi^{n+1} = U_\mu + \log P_T \exp(-\varphi^{n+1}) \,. \end{cases}$$

As pointed out in [BCC+15], this is also equivalent to the Bregman's iterated projection algorithm for relative entropy. Indeed, in the current setup Bregman's iterated projection algorithm produces two sequences of plans $(\pi^{n,n}, \pi^{n+1,n})_{n\in\mathbb{N}}$ starting from a positive measure $\pi^{0,0}$ according to the following recursion:

$$\pi^{n+1,n} := \arg\min_{\Pi(\mu,\star)} \mathscr{H}(\cdot|\pi^{n,n})\,, \qquad \pi^{n+1,n+1} := \arg\min_{\Pi(\star,\nu)} \mathscr{H}(\cdot|\pi^{n+1,n})\,,$$

where $\Pi(\rho,\star)$ (resp. $\Pi(\star,\mu)$) is the set of probability measures π on $\mathbb{R}^{2d}$ such that the first marginal is ρ, *i.e.*, $(\mathrm{proj}_x)_\sharp\pi = \rho$ (resp. the second marginal is μ, *i.e.*, $(\mathrm{proj}_y)_\sharp\pi = \mu$). It is relatively easy (cf.

[Nut21, Section 6]) to show that, starting from $\pi^{0,0}(\mathrm{d}x\mathrm{d}y) \propto \exp(-{|x-y|^2}/{2T} - \psi^0(y) - \varphi^0(x))\mathrm{d}x\mathrm{d}y$, the iterates in (1.1) are related to Sinkhorn potentials through

$$\begin{aligned}\pi^{n+1,n}(\mathrm{d}x\mathrm{d}y) &\propto \exp(-{|x-y|^2}/{2T} - \varphi^{n+1}(x) - \psi^n(y))\mathrm{d}x\mathrm{d}y\,,\\ \pi^{n+1,n+1}(\mathrm{d}x\mathrm{d}y) &\propto \exp(-{|x-y|^2}/{2T} - \varphi^{n+1}(x) - \psi^{n+1}(y))\mathrm{d}x\mathrm{d}y\,.\end{aligned}$$

In the sequel, we will refer to the couplings $(\pi^{n,n}, \pi^{n+1,n})_{n\in\mathbb{N}}$ as Sinkhorn plans. By definition $\pi^{n+1,n}$ has the correct first marginal, but wrong second marginal, which we denote with $\mu^{n+1,n}$. Similarly, the second marginal of $\pi^{n,n}$ is fitted, however the first one might not be correct and hereafter we will denote it as $\rho^{n,n}$. Moreover, $\pi^{n+1,n}$ is the optimal entropic plan associated to the EOT problem with marginals ρ, $\mu^{n+1,n}$ whereas $\pi^{n,n}$ is the optimal EOT associated to the problem with marginals $\rho^{n,n}$, μ. Due to this partial marginal fitting nature of the algorithm, since we can see Sinkhorn plans $\{\pi^{n+1,n}\}_{n\in\mathbb{N}}$ as a sequence of entropic plans where the first marginal is always fixed and the second one changes according to $\{\mu^{n+1,n}\}$, we see that proving the exponential convergence of the algorithm boils down to apply quantitative stability estimates and to control the sequence of wrong marginals. For these reasons, Sinkhorn's algorithm and quantitative convergence bounds for EOT are two problems tightly related and both have been addressed from a vast literature (see literature review below). Despite this, in the unbounded settings, much less has been known until the recent contributions of [CDG23, Eck25, CCGT24], where this problem has been addressed in full generality and where exponential convergence rates were shown to hold in relative entropy for Sinkhorn plans and in L^p-norm (with $p \in \{1, 2\}$) for gradients of Sinkhorn potentials. Here our Hessian stability estimates allow us to deduce also a second order convergence result, *i.e.*, that the Hessian of Sinkhorn potentials converges exponentially fast with the same rate obtained in [CCGT24] for Sinkhorn plans. To state it, let us recall that a probability measure $\nu \in \mathcal{P}(\mathbb{R}^d)$ is said to satisfy a Talagrand's inequality with constant τ, TI(τ) for short, if

$$\mathbf{W}_2^2(\mu,\nu) \le 2\tau\,\mathscr{H}(\mu|\nu)\,, \quad \forall \mu \in \mathcal{P}(\mathbb{R}^d)\,. \tag{TI(τ)}$$

Theorem 1.2. *Assume **H1** and that there exist $\Lambda \in (0, +\infty)$ and $N \ge 2$ such that*

$$y \mapsto g^x_{\psi_0^n}(y) = \frac{|x-y|^2}{2} - T\,\varphi_0^n(y)$$

is Λ-semiconcave[2] uniformly in $x \in \mathrm{supp}(\rho)$ and $n \ge N$. If $\mu^{n,n-1}$ satisfies TI(τ) for some $\tau \in (0, +\infty)$ and for all $n \ge N$, then

$$\begin{aligned}\|\nabla\varphi^{n+1} - \nabla\varphi^\mu\|^2_{\mathrm{L}^2(\rho)} &\lesssim \tau\left(1 - \frac{T}{T+\tau\Lambda}\right)^{(n-N+1)}\mathscr{H}(\pi^\mu|\pi^{0,0})\,,\\ \|\nabla^2\varphi^{n+1} - \nabla^2\varphi^\mu\|_{\mathrm{L}^1(\rho)} &\lesssim \sqrt{\tau}\left(1 - \frac{T}{T+\tau\Lambda}\right)^{\frac{n-N+1}{2}}\sqrt{\mathscr{H}(\pi^\mu|\pi^{0,0})}\\ &\qquad + \tau\left(1 - \frac{T}{T+\tau\Lambda}\right)^{(n-N+1)}\mathscr{H}(\pi^\mu|\pi^{0,0})\,.\end{aligned}$$

hold for all $n \ge N$ up to multiplicative constants that depend polynomially only on ρ, μ, T (and not on the iterates). These constants are explicit and in particular, up to numerical universal constants, we have

[2]This is the same assumption considered in [CCGT24] when proving the exponential convergence of Sinkhorn's plans. Here we have a $-T$ in front of ψ_0 due to notational difference.

- *if* $\mathrm{supp}(\rho)$, $\mathrm{supp}(\mu) \subseteq B_R(0)$ *(for some radius big enough, i.e.,* $R^2 \geq T$*) then*

$$\|\nabla\varphi^{n+1} - \nabla\varphi^\mu\|^2_{\mathrm{L}^2(\rho)} \lesssim \tau\, {R^4}/{T^4} \left(1 - \frac{T}{T+\tau\Lambda}\right)^{(n-N+1)} \mathscr{H}(\pi^\mu|\pi^{0,0})\,,$$

$$\|\nabla^2\varphi^{n+1} - \nabla^2\varphi^\mu\|_{\mathrm{L}^1(\rho)} \lesssim \sqrt{\tau}({R^4}/{T^{7/2}} + {d}/{T}) \left(1 - \frac{T}{T+\tau\Lambda}\right)^{\frac{n-N+1}{2}} \sqrt{\mathscr{H}(\pi^\mu|\pi^{0,0})}$$
$$+ \tau\, {R^6}/{T^5} \left(1 - \frac{T}{T+\tau\Lambda}\right)^{(n-N+1)} \mathscr{H}(\pi^\mu|\pi^{0,0})\,,$$

- *if both* ρ *and* μ *are log-concave, i.e., their (negative) log-densities satisfy* $\nabla^2 U_\rho \geq \alpha_\rho$ *and* $\nabla^2 U_\mu \geq \alpha_\mu$ *for some* α_ρ, $\alpha_\mu > 0$ *(wlog such that* $\alpha_\rho \vee \alpha_\mu < T^{-1}$*), then*

$$\|\nabla\varphi^{n+1} - \nabla\varphi^\mu\|^2_{\mathrm{L}^2(\rho)} \lesssim \frac{\tau}{\alpha_\rho\,\alpha_\mu\,T^4} \left(1 - \frac{T}{T+\tau\Lambda}\right)^{(n-N+1)} \mathscr{H}(\pi^\mu|\pi^{0,0})\,,$$

$$\|\nabla^2\varphi^{n+1} - \nabla^2\varphi^\mu\|_{\mathrm{L}^1(\rho)} \lesssim \sqrt{\tau}\left(\frac{1}{\alpha_\mu\sqrt{\alpha_\rho}\,T^3} + \frac{d}{\sqrt{\alpha_\rho\,\alpha_\mu}\,T^2}\right) \left(1 - \frac{T}{T+\tau\Lambda}\right)^{\frac{n-N+1}{2}} \sqrt{\mathscr{H}(\pi^\mu|\pi^{0,0})}$$
$$+ \frac{\tau}{\alpha_\rho\,\alpha_\mu\,T^4} \left(1 - \frac{T}{T+\tau\Lambda}\right)^{(n-N+1)} \mathscr{H}(\pi^\mu|\pi^{0,0})\,.$$

Lastly, let us remark that the multiplicative constants appearing in this theorem are the same ones obtained in Theorem 1.1 (exchanging the role between ν and μ).

1.2. **The Schrödinger bridges point of view.** Our proof strategy relies on the stochastic control representation of entropic plans as laws of solutions to time-inhomogeneous SDEs. More precisely, we are going to consider the forward Schrödinger bridge process (from ρ to μ) defined as the SDE driven by $-\nabla\psi^\mu_s$, that is the stochastic process $(X^{\psi^\mu,\rho})_{s\in[0,T]}$ solution to (cf. [Con24])

$$\mathrm{d}X^{\psi^\mu,\rho}_s = -\nabla\psi^\mu_s(X^{\psi^\mu,\rho}_s)\mathrm{d}s + \mathrm{d}B_s\,, \quad X^{\psi^\mu,\rho}_0 \sim \rho\,. \tag{1.3}$$

Then the joint law $\mathcal{L}(X^{\psi^\mu,\rho}_0, X^{\psi^\mu,\rho}_T)$ coincides with the optimal entropic coupling π^μ, *i.e.*, the solution to EOT with marginals ρ,μ.

Similarly, we will consider its time reversal corresponding process, *i.e.*, the (backward) Schödinger bridge (from μ to ρ) which solves

$$\mathrm{d}X^{\varphi^\mu,\mu}_s = -\nabla\varphi^\mu_s(X^{\varphi^\mu,\mu}_s)\mathrm{d}s + \mathrm{d}B_s\,, \quad X^{\varphi^\mu,\mu}_0 \sim \mu\,. \tag{1.4}$$

Let us recall here that the bridge $X^{\varphi^\mu,\mu}_\cdot$ is the time-reversal process of the forward bridge $X^{\psi^\mu,\rho}_\cdot$, *i.e.*, for any $s\in[0,T]$ the following identities in law hold

$$X^{\varphi^\mu,\mu}_s \overset{\text{law}}{=} X^{\psi^\mu,\rho}_{T-s} \text{ and } X^{\varphi^\mu,\mu}_s \overset{\text{law}}{=} X^{\psi^\mu,\rho}_{T-s}\,, \tag{1.5}$$

and clearly that $\mathcal{L}(X^{\varphi^\mu,\mu}_T, X^{\varphi^\mu,\mu}_0) = \mathcal{L}(X^{\psi^\mu,\rho}_0, X^{\psi^\mu,\rho}_T) = \pi^\mu$.

In light of these representation, it is clear that semiconcavity and functional properties of the EOT plan π^μ are affected by convexity properties of the drifts appearing in (1.3) and (1.4), as already noticed in [Con24]. For this reason, alongside the semiconcavity parameter $\Lambda(\varphi^\mu_0)$ our constants appearing below will depend on lower bounds on the Hessians of propagated potentials along HJB, *i.e.*, for any $h\in\{\varphi^\mu,\psi^\mu\}$ we will consider the lower bounds $\nabla^2 h_s \geq \lambda(h_s)$ with $\lambda(h_s)\in\mathbb{R}$. Our general results are stated for any given sequence $(\lambda(h_s))_{s\in[0,T)}$ satisfying this lower bound (and we do not assume it to be the optimal one, as we did for $\Lambda(\varphi^\mu_0)$). In Appendix A and Appendix B we will provide explicit lower bounds for the examples considered here.

1.3. **Literature review.**

Quantitative stability. In recent years a rich literature has flourished around quantitative stability questions for primal and dual solutions of the EOT problem.

At the level of the primal solutions, namely entropic plans, let us mention [CCGT23] and [EN22]. In the former, the difference in (symmetric) entropy between the solutions to two different EOT problems is controlled in terms of a negative Sobolev norm, for a wide class of problems with costs induced by diffusions on Riemannian manifolds with Ricci curvature bounded from below (which includes the quadratic cost on $\mathbb{R}^d$). The latter obtains instead a quantitative Hölder estimate between the Wasserstein distance of optimal plans and that of their marginals. This result applies not only to the quadratic cost, but also to more general costs. We refer the reader to [CCGT23, Section 1.2.1] for a comparison between the two references. Let us further cite [GNB22], where a more qualitative stability result is proven under mild hypotheses.

Finally, [CCGT24] provides a control on the entropy between two entropic plans in terms of the (squared) Wasserstein distance between the marginals. The peculiarity of this last work is the approach, since it exploits for the first time the propagation of semiconcavity along HJB to obtain a quantitative stability result for primal solutions. The second-order quantitative stability bounds on entropic and Sinkhorn potentials that we will show in this manuscript build upon this previous contribution. For this reason and for sake of completeness, we prove the entropic stability estimate via semiconcavity also in the present manuscript, but we provide a different proof, based on the stochastic representation of Schrödinger bridges (see Theorem 2.1 below).

As concerns the dual solutions, *i.e.*, entropic potentials, in [CL20] an L^∞-Lipschitz bound is obtained; it applies to multimarginal problems, but it requires either the space or the cost to be bounded. In [DdBD24] the L^∞-norm of the difference between entropic potentials associated to two EOT problems is controlled by the Wasserstein distance of order one between the corresponding marginals, using an approach based on Hilbert's metric; but again, among their hypotheses, both the cost function and the marginals' supports are assumed to be bounded. On the other hand, [CCL24] succeeds in controlling the same difference with the Wasserstein distance of order two of the respective marginals, provided the cost is bounded with two bounded derivatives, *i.e.*, $C^{2,\infty}$; if the regularity of the cost is higher, say $C^{k+2,\infty}$, then the L^∞-norm of the difference between entropic potentials can be replaced by the $C^{k,\infty}$-norm. Let us comment that the interest in higher-order stability results for entropic potentials is motivated by the fact that the gradient of entropic potentials provide good proxies for OT maps ([Gre24, MS23, CCGT23] in unbounded settings and [PNW21] in semidiscrete ones) and entropic estimates can be leveraged to obtain in the T vanishing limit estimates for Kantorovich potentials and OT maps (e.g. [FGP20, CP23, KLM25]). In particular, in the very recent work [KLM25] the authors rely on estimates for regularized potentials combined with gluing arguments in the vanishing T limit, in order to get quantitative stability estimates for OT maps.

Finally, in [DNWP24] the L^2-norm of the difference of the gradients of entropic potentials is controlled in a Lipschitz way by the Wasserstein distance between the corresponding marginals by leveraging a functional inequality for tilt-stable probability measures, see [CE22] and [BBD24, Lemma 3.21], and under the assumption that both entropic potentials have a bounded Hessian. The dependence of the Lipschitz constant on the regularization parameter is polynomial, thus improving on earlier results, and marginals may have unbounded support. Among those just mentioned, this is the closest contribution to ours, since the authors of [DNWP24] use Lipschitzianity of the Schrödinger maps (and hence concavity/convexity bounds for entropic potentials) in order to prove stability bounds for the gradient $\nabla\varphi^\mu$. Therefore, our work can be seen as an extension to second-order quantitative bounds. Moreover, the stability bounds for the gradients that we state here behave as theirs (our potentials and theirs differ from a multiplicative prefactor $-T$). In the compact setting we get the same asymptotic behavior in R and T, whereas if we put ourselves in the Caffarelli's setting (*i.e.*, Hessian of marginals upper and lower bounded), then our general estimate would not depend on T and would behave as their stability result when assuming bi-Lipschitzianity. We have not stated this corollary since already covered by their results and because assuming this extra regularity would not improve the dependence in T in the Hessian stability estimate. Lastly, let us mention here that our approach can be extended also to weakly log-concave marginals, by building upon semiconcavity estimates obtained in [CCGT24].

Sinkhorn's algorithm. Contributions to Sinkhorn's algorithm in the literature date back to the works of Sinkhorn [Sin64] and Sinkhorn and Knopp [SK67], from which the algorithm takes its name. It was originally considered in a discrete setting framework for doubly stochastic matrices and the first exponential convergence result was given in [FL89, BLN94]. There, the authors studied the contraction properties of the algorithm in the Hilbert's projective metric. After the seminal work of [Cut13], which opened up to possible application of EOT to machine learning, multiple papers dealt with the convergence of the algorithm. Particularly in bounded settings (*i.e.*, compact spaces or bounded costs) this has already been well established in [CGP16, DMG20, Car22]. In particular [CGP16] obtained the first exponential convergence results in the continuous setting using the Hilbert's metric approach. However, this approach provides rates that depend exponentially from the regularizing parameter T and cannot be extended to unbounded settings.

On the other hand, much less was known for unbounded settings (including the most iconic and simple quadratic cost setting with log-concave marginals). In fact, the only widely general known qualitative convergence result was due to Rüschendorf [Rus95], who established qualitative convergence for Sinkhorn plans in relative entropy. Recently, this result has been improved in [NW22b] where the authors have managed to show qualitative convergence on the primal and dual sides, under mild assumptions on cost and marginals. The first quantitative convergence result in unbounded settings we are aware of is [EN22], subsequently improved in [GN25], where the authors prove a polynomial convergence rate. These works are based on quantitative stability estimates for EOT and the insightful interpretation of Sinkhorn's algorithm as a block-coordinate descent algorithm on the dual problem [Lég21, AFKL22, LAF23].

Only very recently, it has been established the exponential convergence for unbounded costs and marginals. Up to the authors' knowledge, the first contribution in this setting is given in [CDG23], which studies the quadratic cost. There, the main result is that if the marginals are weakly log-concave and the regularization parameter T is large enough, exponential convergence of the gradients of the iterates holds (and their results work for any $T > 0$ for Gaussian marginals). This kind of convergence is particularly useful as $\nabla\varphi^\mu$ approximates the Brenier map in the $T \to 0$ limit, see [CCGT23, PNW21]. Moreover, this is the first contribution that has highlighted how geometric assumptions on the marginals (such as log-concavity) can be leveraged to improve the dependence in the convergence rates, from exponential to polynomial in T. Later, following similar considerations, [CDV24] has improved the exponential convergence results in the bounded setting, showing that the exponential rate of convergence deteriorates polynomially in T. With regard to the unbounded setting, the article [Eck25] has subsequently managed to construct a suitable version of Hilbert's metric for general unbounded costs. In contrast with [CDG23], exponential convergence is shown for all values of T, under a growth condition assumption. Roughly speaking, therein the author assumes that the tails of the marginals decay (strictly) faster than the cost function considered. When applied to the quadratic cost, this assumption does not completely cover log-concave distributions and their perturbations, leaving out Gaussian marginals for example.

The first paper that finally has managed to provide exponential convergence rates in general (possibly unbounded) settings, working for any regularization parameter $T > 0$ and with polynomial dependence in T, is [CCGT24]. There, together with our coauthors, we show how semiconvexity and semiconcavity bounds on Sinkhorn potentials can be leveraged to obtain exponential convergence. Our geometric approach is broadly general and covers as particular cases the bounded settings as well as the (anisotropic) quadratic costs, which includes, for instance, also the case when considering as cost function the transition kernel induced by an Ornstein–Uhlenbeck process (*i.e.*, the framework of the Schrödinger bridge problem with a non-Gaussian reference process). The key observation employed there is that the semiconcavity of the function defined in (1.2) is enough to deduce quantitative stability estimates and exponential convergence rates depending on the semiconcavity parameter $\Lambda(\varphi_0^\mu)$.

Lastly, it is worth mentioning different contributions that over the past few years have focused on different asymptotic properties of Sinkhorn's algorithm. Let us just mention [Ber20] for the relation with Monge-Ampère equation, [DKPS23] for the construction of Wasserstein mirror gradient flows, [SABP22] for construction of a Transformer variant inspired by Sinkhorn's algorithm, and the very recent series of contributions [ADMM24, ADMM25, DM25] that focus on the relation of Schrödinger bridges and Sinkhorn's algorithm with the Riccati matrix difference equations, and the impact of these results in the context of multivariate linear Gaussian models and statistical finite mixture models (including Gaussian-kernel density estimation). We conclude by mentioning the recent work [EL25], where the authors investigate the convergence of iterative proportional fitting procedures for a more general class of problems (which includes EOT), whose proof is based on strong convexity arguments for the dual problem, which particularly highlights the role of the geometric interplay between the subspaces defining the constraints.

We would like to conclude this review by mentioning results that have inspired us or are related to ours. As we have already stated, our strategy is based on stochastic analysis and second-order estimates along Hamilton–Jacobi–Bellman equations. This approach has been initially introduced in [Con24] where Conforti has proved weak semiconcavity estimates for entropic potentials by studying how this property propagates along HJB equations. In [CDG23] and [CCGT24], this has already been employed for proving the exponential convergence of Sinkhorn's algorithm and for showing stability estimates of entropic plans. Here we further extend its use to show second-order stability estimates. In order to prove the convergence of Sinkhorn's algorithm, a similar approach has been employed also in [GNCD23, Gre24] where we have studied how Lipschitzianity propagates along HJB equations, leading to a more perturbative convergence result (instead of a geometric one). Lastly, we would like to mention [CC24], though not directly applied to EOT, where the authors provide third-order estimates propagated along HJB in order to prove stability estimates for stochastic optimal control problems. These new ideas open up to further investigation of third-order estimates for the entropic potentials.

2. Preliminaries

In this paper we are interested in the behavior of the forward process $(Y^\theta_s)_{s\in[0,T]}$ and backward process $(Y^\eta_s)_{s\in[0,T]}$ defined as $Y^\theta_s \coloneqq \nabla\theta_s(X^{\psi^\mu,\rho}_s)$ where $\theta_s \coloneqq \psi^\nu_s - \psi^\mu_s$ and similarly as $Y^\eta \coloneqq \nabla\eta_s(X^{\varphi^\mu,\mu}_s)$ with $\eta_s \coloneqq \varphi^\nu_s - \varphi^\mu_s$. Since both $\varphi^\nu_\cdot$ and $\varphi^\mu_\cdot$ solve (HJB) it is immediate to see that $\eta_\cdot$ solves

$$\partial_s\eta_s + \frac{1}{2}\Delta\eta_s - \nabla\varphi^\mu_s\cdot\nabla\eta_s - \frac{1}{2}|\nabla\eta_s|^2 = 0\,,$$

and hence from Itô's formula we may further deduce that

$$\mathrm{d}Y^\eta_s = \nabla^2\varphi^\nu_s(X^{\varphi^\mu,\mu}_s)Y^\eta_s\,\mathrm{d}s + \nabla^2\eta_s(X^{\varphi^\mu,\mu}_s)\,\mathrm{d}B_s\,. \tag{2.1}$$

Similarly, when considering the backward processes we have

$$\partial_s\theta_s + \frac{1}{2}\Delta\theta_s - \nabla\psi^\mu_s\cdot\nabla\theta_s - \frac{1}{2}|\nabla\theta_s|^2 = 0\,,$$

and hence Itô's formula implies

$$\mathrm{d}Y^\theta_s = \nabla^2\psi^\nu_s(X^{\psi^\mu,\rho}_s)Y^\theta_s\,\mathrm{d}s + \nabla^2\theta_s(X^{\psi^\mu,\rho}_s)\,\mathrm{d}B_s\,. \tag{2.2}$$

Finally, notice that

$$\mathbb{E}[|Y^\eta_T|^2] = \|\nabla\varphi^\nu - \nabla\varphi^\mu\|^2_{\mathrm{L}^2(\rho)} = \|\nabla\psi^\nu_0 - \nabla\psi^\mu_0\|^2_{\mathrm{L}^2(\rho)} = \mathbb{E}[|Y^\theta_0|^2]\,.$$

Besides the relation at initial and terminal times with the integrated difference between the gradients of the potentials, the processes $(Y^\eta_s)_{s\in[0,T]}$ and $(Y^\theta_s)_{s\in[0,T]}$ play a crucial role since their integrated in time mean squares measure the entropic distance between π^ν and π^μ. Namely, from Girsanov's theory we know that

$$\frac{1}{2}\int_0^{\delta T}\mathbb{E}[|Y^\eta_s|^2]\mathrm{d}s = \mathscr{H}(\mathcal{L}(X^{\varphi^\mu,\mu}_{[0,\delta T]})|\mathcal{L}(X^{\varphi^\nu,\mu}_{[0,\delta T]}))$$

and in particular whenever $\mu \ll \nu$ and for $\delta = 1$ we then have

$$\frac{1}{2}\int_0^{T}\mathbb{E}[|Y^\eta_s|^2]\mathrm{d}s = \mathbb{E}_\mu[\mathscr{H}(\pi^\mu(\cdot|X)|\pi^\nu(\cdot|X))]\,, \tag{2.3}$$

which gives rise to

$$\mathscr{H}(\mu|\nu) + \frac{1}{2}\int_0^{T}\mathbb{E}[|Y^\eta_s|^2]\mathrm{d}s = \mathscr{H}(\mathcal{L}(X^{\varphi^\mu,\mu}_{[0,T]})|\mathcal{L}(X^{\varphi^\nu,\nu}_{[0,T]})) = \mathscr{H}(\pi^\mu|\pi^\nu) \tag{2.4}$$

whenever $\mathscr{H}(\mu|\nu)$ is finite. Similarly we have

$$\frac{1}{2}\int_0^{\delta T}\mathbb{E}[|Y_s^\theta|^2]\mathrm{d}s = \mathscr{H}(\mathcal{L}(X_{[0,\delta T]}^{\psi^\mu,\rho})|\mathcal{L}(X_{[0,\delta T]}^{\psi^\nu,\rho}))\,, \tag{2.5}$$

which equals $\mathscr{H}(\pi^\mu|\pi^\nu)$ for $\delta=1$.

Let us start by showing how the above stochastic control point of view can be employed in studying entropic stability for Schrödinger plans.

2.1. **Entropic and gradients' stability.** Our Hessian stability result will rely on the following result regarding the stability of plans and gradients of (backward evolved) potentials.

Theorem 2.1. *Assume **H**1 and let π^ν, π^μ denote the optimal plans associated to EOT with marginals (ρ,ν) and (ρ,μ) respectively. Then it holds*

$$\mathscr{H}(\pi^\mu|\pi^\nu)\le\mathscr{H}(\mu|\nu)+\frac{\Lambda(\varphi_0^\nu)}{2T}\,\mathbf{W}_2^2(\mu,\nu)\,.$$

*Moreover, if **H**2 holds then we have*

$$\mathbb{E}_\mu[\mathscr{H}(\pi^\mu(\cdot|X)|\pi^\nu(\cdot|X))]\le\frac{\Lambda(\varphi_0^\nu)}{2T}\,\mathbf{W}_2^2(\mu,\nu) \tag{2.6}$$

and

$$\mathbb{E}[|Y_0^\eta|^2]=\|\nabla\varphi_0^\mu-\nabla\varphi_0^\nu\|^2_{\mathrm{L}^2(\mu)}\le\frac{\Lambda(\varphi_0^\nu)\,C^{\varphi^\nu}}{T^2}\,\mathbf{W}_2^2(\mu,\nu)\,, \tag{2.7}$$

where the positive constant C^{φ^ν} is defined as

$$C^{\varphi^\nu}\coloneqq T\biggl(\int_0^T e^{\int_0^s 2\lambda(\varphi_t^\nu)\mathrm{d}t}\mathrm{d}s\biggr)^{-1}\,. \tag{2.8}$$

The entropic stability estimate stated above has already been proven by the authors and collaborators in [CCGT24, Theorem 1.1]. We report it here since the proof of this entropic stability estimate can be employed in order to get (2.6) and (2.7), which will play a crucial role in the rest of the paper. Finally, let us also remark that in here we provide a stochastic analysis proof of the entropic stability estimate by building a suitable competitor using a modified Schrödinger bridge process (see (2.10) below).

The above result will follow from combining the following technical bounds. In the first one we bound the relative entropy between $\pi_s^\mu(\cdot|y)=\mathcal{L}(X_s^{\varphi^\mu,y})$ and $\pi_s^\nu(\cdot|z)=\mathcal{L}(X_s^{\varphi^\nu,z})$.

Lemma 2.2. *Assume **H**1. For any $s\in(0,T]$ and any $y\in\mathrm{supp}(\mu)$ and $z\in\mathrm{supp}(\nu)$ it holds*

$$\mathscr{H}(\pi_s^\mu(\cdot|y)|\pi_s^\nu(\cdot|z))\le\frac{\Lambda(\varphi_0^\nu)}{2T}|z-y|^2+(s^{-1}-T^{-1})\frac{|z-y|^2}{2}+\mathbb{E}[\eta_s(X_s^{\varphi_\mu,y})-\eta_0(y)]-\langle\nabla\eta_0(y),z-y\rangle\,.$$

Proof. Firstly, observe that the conditional probability measure $\pi_s^\mu(\cdot|y)$ admits a density of the form

$$\pi_s^\mu(\mathrm{d}x|y)=(2\pi s)^{-d/2}\,\exp\biggl(-\varphi_s^\mu(x)+\varphi_0^\mu(y)-\frac{|x-y|^2}{2s}\biggr)\mathrm{d}x\,,$$

and a similar expression holds for $\pi_s^\nu(\cdot|z)$. Therefore we may rewrite the relative entropy as

$$\begin{aligned}\mathscr{H}(\pi_s^\mu(\cdot|y)|\pi_s^\nu(\cdot|z))&=\varphi_0^\mu(y)-\varphi_0^\nu(z)+\int(\varphi_s^\nu-\varphi_s^\mu)(x)+\frac{|x-z|^2-|x-y|^2}{2s}\pi_s^\mu(\mathrm{d}x|y)\\&=\varphi_0^\mu(y)-\varphi_0^\nu(z)+\frac{|z|^2-|y|^2}{2s}+\int\eta_s(x)+s^{-1}\langle x,y-z\rangle\,\pi_s^\mu(\mathrm{d}x|y)\\&=\varphi_0^\mu(y)-\varphi_0^\nu(z)+\frac{|z|^2-|y|^2}{2s}+\mathbb{E}[\eta_s(X_s^{\varphi_\mu,y})+s^{-1}\langle X_s^{\varphi^\mu,y},y-z\rangle]\,.\end{aligned}$$

Next, since $(\nabla\varphi_s^\nu(X_s^{\varphi^\nu,y}))_{s\in[0,T]}$ is a martingale (cf. [Con24, Proof of Theorem 2.1], namely it follows from Itô's formula combined with (HJB) and (1.4)), we have

$$\langle\mathbb{E}[\nabla\varphi_s^\nu(X_s^{\varphi^\nu,y})],y-z\rangle=\langle\nabla\varphi_0^\nu(y),y-z\rangle\,,$$

so that if we integrate from 0 to s the dynamics of $X_s^{\varphi^\nu,y}$ and take expectations, we get

$$\begin{aligned}\mathbb{E}[\langle X_s^{\varphi^\nu,y}, y-z\rangle] =&\mathbb{E}[\langle X_0^{\varphi^\nu,y}, y-z\rangle] - \int_0^s \langle \mathbb{E}[\nabla\varphi_t^\nu(X_t^{\varphi^\nu,y})], y-z\rangle\,\mathrm{d}t\\ =&\langle y, y-z\rangle + s\,\langle \nabla\varphi_0^\nu(y), z-y\rangle\,.\end{aligned}$$

Hence we conclude that

$$\begin{aligned}\mathscr{H}(\pi_s^\mu(\cdot|y)|\pi_s^\nu(\cdot|z)) &= \varphi_0^\mu(y) - \varphi_0^\nu(z) + \frac{|z-y|^2}{2s} + \mathbb{E}[\eta_s(X_s^{\varphi_\mu,y})] + \langle\nabla\varphi_0^\mu(y), z-y\rangle\\ &= \varphi_0^\nu(y) - \varphi_0^\nu(z) + \langle\nabla\varphi_0^\nu(y), z-y\rangle + \frac{|z-y|^2}{2s} + \mathbb{E}[\eta_s(X_s^{\varphi_\mu,y}) - \eta_0(y)]\\ &\qquad -\langle\nabla\eta_0(y), z-y\rangle\\ \leq \frac{\Lambda(\varphi_0^\nu)}{2T}|z-y|^2 &+ (s^{-1}-T^{-1})\frac{|z-y|^2}{2} + \mathbb{E}[\eta_s(X_s^{\varphi_\mu,y}) - \eta_0(y)] - \langle\nabla\eta_0(y), z-y\rangle\,,\end{aligned}$$

where in the last step we have noticed that

$$T\left(\varphi_0^\nu(y) - \varphi_0^\nu(z) + \langle\nabla\varphi_0^\nu(y), z-y\rangle + \frac{|z-y|^2}{2T}\right) = g^y_{\varphi_0^\nu}(z) - g^y_{\varphi_0^\nu}(y) - \langle\nabla g^y_{\varphi_0^\nu}(y), z-y\rangle\,,$$

with $g^y_{\varphi_0^\nu}$ defined as in (1.2), and we have used its $\Lambda(\varphi_0^\nu)$-semiconcavity. □

In the particular case in which $s=T$ and we take $\mu=\nu$ (henceforth $\eta_\cdot=0$), the above result simply reads as

Corollary 2.3. *Assume **H**1. For any $y,z\in\operatorname{supp}(\nu)$ we have*

$$\mathscr{H}(\pi^\nu(\cdot|y)|\pi^\nu(\cdot|z)) \leq \frac{\Lambda(\varphi_0^\nu)}{2T}\,|z-y|^2\,.$$

The above corollary has already been proven by the authors and their collaborators in [CCGT24, Lemma 2.1], which can thus be seen as a particular case of the more general Lemma 2.2.

Proof of Theorem 2.1. Let us focus on the entropic stability bound first. Without loss of generality we may assume $\mathscr{H}(\mu|\nu)$, $\Lambda(\varphi_0^\nu)$, $\mathbf{W}_2^2(\mu,\nu)$ to be all finite, otherwise there is nothing to prove. Next, observe that π^μ can be seen as the entropic optimal plan w.r.t. the reference measure π^ν for the EOT problem

$$\mathscr{H}(\pi^\mu|\pi^\nu) = \min_{\pi\in\Pi(\rho,\mu)} \mathscr{H}(\pi|\pi^\nu)\,. \tag{2.9}$$

This directly follows from [Nut21, Theorem 2.1.b] after noticing that

$$\frac{\mathrm{d}\pi^\mu}{\mathrm{d}\pi^\nu} = \exp((\varphi^\nu-\varphi^\mu)\oplus(\psi^\nu-\psi^\mu))\quad \mathrm{R}_{0T}-a.s.$$

and hence also π^ν-a.s. (since $\mathscr{H}(\pi^\nu|\mathrm{R}_{0T})<\infty$). Notice that [Nut21, Theorem 2.1.b] further implies $\mathscr{H}(\pi^\mu|\pi^\nu)<\infty$.

We now proceed to bound $\mathscr{H}(\pi^\mu|\pi^\nu)$ exhibiting a suitable admissible plan in (2.9). In view of that, let us consider a transport map $\mathcal{T}$ from μ to ν, that is such that $\mathcal{T}_\#\mu=\nu$ and take $X_0\sim\mu$ and define $X_0^\nu := \mathcal{T}(X_0)\sim\nu$. The existence of such map follows from [San15, Corollary 1.29] since $\mu\ll$ Leb. For any x we also define the process $(X_s^{\varphi^\nu,x})_{s\in[0,T]}$ by

$$\mathrm{d}X_s^{\varphi^\nu,x} = -\nabla\varphi_s^\nu(X_s^{\varphi^\nu,x})\mathrm{d}s + \mathrm{d}B_s\,,\quad X_0^{\varphi^\nu,x}=x, \tag{2.10}$$

where $B_\cdot$ is a Brownian motion independent of (X_0, X_0^ν). Finally, let $\gamma_s=(1-s/T)X_0+(s/T)X_0^\nu$ and consider now the stochastic process $X_\cdot$ given by

$$X_s = X_s^{\varphi^\nu,\gamma_s},\quad s\in[0,T]$$

and note that if we call π_{comp} the law of (X_T, X_0), then $\pi_{\mathrm{comp}}\in\Pi(\rho,\,\mu)$.

Then, by optimality of π^μ in (2.9) and by considering π_{comp} as a competitor we may deduce that

$$\mathscr{H}(\pi^\mu|\pi^\nu) \leq \mathscr{H}(\pi_{\mathrm{comp}}|\pi^\nu) = \mathscr{H}(\mu|\nu) + \int \mathscr{H}\Big(\pi_{\mathrm{comp}}(\cdot|z)|\pi^\nu(\cdot|z)\Big)\,\mathrm{d}\mu(z). \tag{2.11}$$

Next, notice that $X_T = X_T^{\varphi^\nu, X_0^\nu} = X_T^{\varphi^\nu, \mathcal{T}(X_0)}$ while $X_0 = X_0^{\varphi^\nu, X_0} = X_0$, and hence the conditional probabilities appearing in the last display are one the translation of the other, that is

$$\pi_{\text{comp}}(\cdot|z) = \mathcal{L}(X_T|X_0 = z) = \mathcal{L}(X_T^{\varphi^\nu, \mathcal{T}(X_0)}|X_0 = z) = \mathcal{L}(X_T^{\varphi^\nu, \mathcal{T}(z)}) = \pi^\nu(\cdot|\mathcal{T}(z))\,.$$

In particular, this combined with Corollary 2.3 implies for μ-a.e. $z \in \mathbb{R}^d$ that

$$\mathscr{H}(\pi_{\text{comp}}(\cdot|z)|\pi^\nu(\cdot|z)) = \mathscr{H}(\pi^\nu(\cdot|\mathcal{T}(z))|\pi^\nu(\cdot|z)) \leq \frac{\Lambda(\varphi_0^\nu)}{2T}\,|\mathcal{T}(z) - z|^2\,,$$

which integrated yields to

$$\mathscr{H}(\pi^\mu|\pi^\nu) \leq \mathscr{H}(\mu|\nu) + \frac{\Lambda(\varphi_0^\nu)}{2T}\int |\mathcal{T}(z) - z|^2\,\mathrm{d}\mu(z).$$

By minimizing the last display over all the measurable transport maps from μ to ν we conclude the proof of the entropic stability bound (see [San15, Theorem 1.22] for the existence of the optimal transport map for $\mathbf{W}_2(\mu,\nu)$).

Let us now focus on the proof of (2.6). If in **H2** we assume $\mathscr{H}(\mu|\nu) < +\infty$, then the conclusion follows from the disintegration property of the relative entropy (see for instance [Nut21, Lemma 1.6] and [Léo14, Appendix A]) since

$$\mathbb{E}_\mu[\mathscr{H}(\pi^\mu(\cdot|X)|\pi^\nu(\cdot|X))] = \mathscr{H}(\pi^\mu|\pi^\nu) - \mathscr{H}(\mu|\nu)\,,$$

which combined with the above entropic stability bound concludes the proof of (2.6) under a finite entropy assumption.

On the other hand, if we consider in **H2** the case $\mu \ll \nu$ with $\Lambda(\varphi_0^\mu)$ finite (e.g. ρ with compact support or log-concave density), then we can use an approximation argument and consider the sequence of probability measures $\mu^n \in \mathcal{P}(\mathbb{R}^d)$ whose densities are defined as

$$\frac{\mathrm{d}\mu^n}{\mathrm{d}\nu} = C_n^{-1}\left(\frac{\mathrm{d}\mu}{\mathrm{d}\nu} \wedge n\right), \qquad \text{with } C_n = \int\left(\frac{\mathrm{d}\mu}{\mathrm{d}\nu} \wedge n\right)\mathrm{d}\nu\,.$$

Clearly, μ^n converges in $\mathbf{W}_2$-distance towards μ and $C_n \uparrow 1$. Moreover, notice that

$$\mathscr{H}(\mu^n|\nu) \leq \log(n) - \log(C_n) < +\infty\,,$$

and that

$$\begin{aligned}\mathscr{H}(\mu^n|\mu) &= -\log(C_n) + \int \log\left(\mathbf{1}_{\{\mathrm{d}\mu/\mathrm{d}\nu \leq n\}} + n\,\mathbf{1}_{\{\mathrm{d}\mu/\mathrm{d}\nu > n\}}\frac{\mathrm{d}\nu}{\mathrm{d}\mu}\right)\mathrm{d}\mu^n \\ &\leq -\log(C_n) + \int n\,\mathbf{1}_{\{\mathrm{d}\mu/\mathrm{d}\nu > n\}}\frac{\mathrm{d}\nu}{\mathrm{d}\mu}\,\mathrm{d}\mu^n \leq 1 - \log(C_n) < +\infty\,.\end{aligned}$$

As a first consequence of this, we may deduce from the finite entropy case that

$$\mathbb{E}_\nu\left[\frac{\mathrm{d}\mu^n}{\mathrm{d}\nu}(X)\,\mathscr{H}(\pi^{\mu^n}(\cdot|X)|\pi^\mu(\cdot|X))\right] = \mathbb{E}_{\mu^n}[\mathscr{H}(\pi^{\mu^n}(\cdot|X)|\pi^\mu(\cdot|X))] \leq \frac{\Lambda(\varphi_0^\mu)}{2T}\,\mathbf{W}_2^2(\mu,\mu^n),$$

which vanishes as n diverges. Therefore

$$\lim_{n\to\infty}\left(\frac{\mathrm{d}\mu}{\mathrm{d}\nu}(X) \wedge n\right)\mathscr{H}(\pi^{\mu^n}(\cdot|X)|\pi^\mu(\cdot|X)) = 0 \quad \nu - a.s.$$

and a fortiori also

$$\lim_{n\to\infty}\mathscr{H}(\pi^{\mu^n}(\cdot|X)|\pi^\mu(\cdot|X)) = 0 \quad \mu - a.s.\,.$$

This implies that μ-a.s. $\pi^{\mu^n}(\cdot|X)$ converges to $\pi^\mu(\cdot|X)$ in total variation (via Pinsker's inequality), henceforth also weakly. From the lower semicontinuity of relative entropy we then deduce that

$$\mathscr{H}(\pi^\mu(\cdot|X)|\pi^\nu(\cdot|X)) \leq \liminf_{n\to\infty}\left(\frac{\mathrm{d}\mu^n}{\mathrm{d}\mu}\,\mathscr{H}(\pi^{\mu^n}(\cdot|X)|\pi^\nu(\cdot|X))\right) \quad \mu - a.s.\,.$$

By combining this last bound with Fatou's lemma and with the entropic stability estimate already proven above we get

$$\begin{aligned}
\mathbb{E}_\mu[\mathscr{H}(\pi^\mu(\cdot|X)|\pi^\nu(\cdot|X))] &\leq \liminf_{n\to\infty} \mathbb{E}_\mu\left[\frac{\mathrm{d}\mu^n}{\mathrm{d}\mu}\,\mathscr{H}(\pi^{\mu^n}(\cdot|X)|\pi^\nu(\cdot|X))\right] \\
&= \liminf_{n\to\infty} \mathbb{E}_{\mu^n}[\mathscr{H}(\pi^{\mu^n}(\cdot|X)|\pi^\nu(\cdot|X))] \overset{(2.4)}{=} \liminf_{n\to\infty} \mathscr{H}(\pi^{\mu^n}|\pi^\nu) - \mathscr{H}(\mu^n|\nu) \\
&\leq \liminf_{n\to\infty} \frac{\Lambda(\varphi_0^\nu)}{2T}\,\mathbf{W}_2^2(\mu^n,\nu) = \frac{\Lambda(\varphi_0^\nu)}{2T}\,\mathbf{W}_2^2(\mu,\nu)\,.
\end{aligned}$$

Finally, the proof of (2.7) follows from (2.6) since from Itô's formula and (2.1) we immediately see that

$$\mathrm{d}\mathbb{E}[|Y_s^\eta|^2] \geq 2\,\mathbb{E}[Y_s^\eta \cdot \nabla^2\varphi_s^\nu(X_s^{\varphi^\mu,\mu})Y_s^\eta]\mathrm{d}s \geq 2\lambda(\varphi_s^\nu)\,\mathbb{E}[|Y_s^\eta|^2]\mathrm{d}s\,,$$

which combined with Grönwall's lemma gives for any $s \leq \delta T$

$$\mathbb{E}[|Y_0^\eta|^2]\,e^{\int_0^s 2\lambda(\varphi_t^\nu)\mathrm{d}t} \leq \mathbb{E}[|Y_s^\eta|^2]\,.$$

When integrated over $s \in [0,T]$, this inequality reads as

$$\mathbb{E}[|Y_0^\eta|^2] \leq \left(\int_0^T e^{\int_0^s 2\lambda(\varphi_t^\nu)\mathrm{d}t}\mathrm{d}s\right)^{-1}\int_0^T \mathbb{E}[|Y_s^\eta|^2]\mathrm{d}s = \frac{C^{\varphi^\nu}}{T}\int_0^T \mathbb{E}[|Y_s^\eta|^2]\mathrm{d}s\,,$$

and combining it with the energy-entropic identity (2.3) gives

$$\mathbb{E}[|Y_0^\eta|^2] \leq \frac{2C^{\varphi^\nu}}{T}\,\mathbb{E}_\mu[\mathscr{H}(\pi^\mu(\cdot|X)|\pi^\nu(\cdot|X))]\,.$$

□

From the bound for conditional relative entropies proven in Corollary 2.3 and the gradients' stability bound in Theorem 2.1, we may deduce an entropic stability bound between π_s^ν and π_s^μ.

Corollary 2.4. *Assume **H**1 and **H**2. Let $C^{\varphi^\nu} > 0$ as defined in (2.8), then we have*

$$\mathscr{H}(\pi_s^\mu|\pi_s^\nu) \leq \left(\frac{\Lambda(\varphi_0^\nu)}{T} + \frac{s^{-1}-T^{-1}}{2} + \frac{\sqrt{\Lambda(\varphi_0^\nu)\,C^{\varphi^\nu}}}{T}\right)\mathbf{W}_2^2(\mu,\nu)\,.$$

Proof. Firstly, fix a coupling $\tau \in \Pi(\mu,\nu)$ between our two target marginals and consider the probability measures on $(\mathbb{R}^d)^3$ defined by the densities $\pi_s^\mu(\mathrm{d}x|y)\tau(\mathrm{d}y,\mathrm{d}z)$ and $\pi_s^\nu(\mathrm{d}x|z)\tau(\mathrm{d}y,\mathrm{d}z)$ (for notations' sake we indicate these two probabilities respectively with $\pi_s^\mu(\cdot|y)\otimes\tau$ and $\pi_s^\nu(\cdot|z)\otimes\tau$). Clearly we have

$$\pi_s^\mu(\mathrm{d}x) = \int\int \pi_s^\mu(\mathrm{d}x|y)\tau(\mathrm{d}y,\mathrm{d}z) \quad \text{and} \quad \pi_s^\nu(\mathrm{d}x) = \int\int \pi_s^\nu(\mathrm{d}x|z)\tau(\mathrm{d}y,\mathrm{d}z)\,,$$

therefore, from the data processing inequality and from the disintegration property of relative entropy (cf. [Nut21, Lemma 1.6] and [Léo14, Appendix A]) we deduce that

$$\mathscr{H}(\pi_s^\mu|\pi_s^\nu) \leq \mathscr{H}(\pi_s^\mu(\cdot|y)\otimes\tau|\pi_s^\nu(\cdot|z)\otimes\tau) = \int \mathscr{H}(\pi_s^\mu(\cdot|y)|\pi_s^\nu(\cdot|z))\,\tau(\mathrm{d}y,\mathrm{d}z)\,.$$

Recalling the upper bound given in Lemma 2.2 we get

$$\begin{aligned}
\mathscr{H}(\pi_s^\mu|\pi_s^\nu) \leq &\left(\frac{\Lambda(\varphi_0^\nu)}{2T} + \frac{s^{-1}-T^{-1}}{2}\right)\int |z-y|^2\mathrm{d}\tau + \int \mathbb{E}[\eta_s(X_s^{\varphi_\mu,y}) - \eta_0(y)]\mathrm{d}\mu(y) \\
&- \int \langle\nabla\eta_0(y), z-y\rangle\mathrm{d}\tau \\
\leq &\left(\frac{\Lambda(\varphi_0^\nu)}{2T} + \frac{s^{-1}-T^{-1}}{2}\right)\int |z-y|^2\mathrm{d}\tau + \mathbb{E}[\eta_s(X_s^{\varphi_\mu,\mu}) - \eta_0(X_0^{\varphi^\mu,\mu})] \\
&+ \|\nabla\eta_0\|_{\mathrm{L}^2(\mu)}\left(\int |z-y|^2\mathrm{d}\tau\right)^{1/2}.
\end{aligned}$$

Now, since $\|\nabla\eta_0\|^2_{\mathrm{L}^2(\mu)} = \mathbb{E}[|Y_0^\eta|^2]$, from (2.7) we deduce that

$$\mathscr{H}(\pi_s^\mu|\pi_s^\nu) \le \left(\frac{\Lambda(\varphi_0^\nu)}{2T} + \frac{s^{-1}-T^{-1}}{2}\right)\int |z-y|^2 \mathrm{d}\tau + \mathbb{E}[\eta_s(X_s^{\varphi_\mu,\mu}) - \eta_0(X_0^{\varphi^\mu,\mu})] \\ + \frac{\sqrt{\Lambda(\varphi_0^\nu)\, C^{\varphi^\nu}}}{T}\,\mathbf{W}_2(\mu,\nu)\left(\int |z-y|^2\mathrm{d}\tau\right)^{1/2}.$$

and by minimizing over $\tau\in\Pi(\mu,\nu)$, we obtain

$$\mathscr{H}(\pi_s^\mu|\pi_s^\nu) \le \left(\frac{\Lambda(\varphi_0^\nu)}{2T} + \frac{s^{-1}-T^{-1}}{2} + \frac{\sqrt{\Lambda(\varphi_0^\nu)\, C^{\varphi^\nu}}}{T}\right)\mathbf{W}_2^2(\mu,\nu) + \mathbb{E}[\eta_s(X_s^{\varphi_\mu,\mu}) - \eta_0(X_0^{\varphi^\mu,\mu})]\,.$$

In order to conclude, it is enough noticing that from Itô's formula it follows

$$\mathrm{d}\eta_s(X_s^{\varphi^\mu,\mu}) = \frac{1}{2}|\nabla\eta_s(X_s^{\varphi^\mu,\mu})|^2\mathrm{d}s + \nabla\eta_s(X_s^{\varphi^\mu,\mu})\mathrm{d}B_s\,,$$

and hence that

$$\mathbb{E}[\eta_s(X_s^{\varphi_\mu,\mu}) - \eta_0(X_0^{\varphi^\mu,\mu})] = \frac{1}{2}\int_0^s \mathbb{E}[|Y_t^\eta|^2]\mathrm{d}t \overset{(2.3)}{\le} \mathbb{E}_\mu[\mathscr{H}(\pi^\mu(\cdot|X)|\pi^\nu(\cdot|X))]\,,$$

which combined with Theorem 2.1 leads to our thesis. □

Notice that the above bound diverges as $s\downarrow 0$. This should not be surprising since for $s=0$ we are trying to bound $\mathscr{H}(\mu|\nu)$ solely with the Wasserstein distance (and the latter might still be finite while the former diverges). Therefore, the previous lemma shows that the contribution of $\mathscr{H}(\mu|\nu)$ is involved only for $s=0$, while as soon as $s>0$ the Wasserstein distance is enough.

We conclude this section with a useful integral bound which will be employed in the proof of our main result in the next section.

Proposition 2.5. *Assume **H1** and **H2**. For any fixed $\delta\in[0,1)$ we have*

$$\int_0^{\delta T}\mathbb{E}[|Y_s^\theta|^2]\mathrm{d}s \le \left(\frac{3\,\Lambda(\varphi_0^\nu)}{T} + \frac{\delta}{1-\delta}\,\frac{1}{T} + \frac{2\sqrt{\Lambda(\varphi_0^\nu)\, C^{\varphi^\nu}}}{T}\right)\mathbf{W}_2^2(\mu,\nu)\,.$$

Proof. In view of Girsanov's Theorem identity (2.5), we clearly have

$$\begin{aligned}\int_0^{\delta T}\mathbb{E}[|Y_s^\theta|^2]\mathrm{d}s = 2\mathscr{H}(\mathcal{L}(X_{[0,\delta T]}^{\psi^\mu,\rho})|\mathcal{L}(X_{[0,\delta T]}^{\psi^\nu,\rho})) &= \int_{(1-\delta)T}^T \mathbb{E}[|Y_s^\eta|^2]\mathrm{d}s + 2\mathscr{H}(\mathcal{L}(X_{(1-\delta)T}^{\varphi^\mu,\mu})|\mathcal{L}(X_{(1-\delta)T}^{\varphi^\nu,\nu}))\\ &\le \int_0^T \mathbb{E}[|Y_s^\eta|^2]\mathrm{d}s + 2\mathscr{H}(\mathcal{L}(X_{(1-\delta)T}^{\varphi^\mu,\mu})|\mathcal{L}(X_{(1-\delta)T}^{\varphi^\nu,\nu}))\\ &\overset{(2.3)}{=} 2\mathbb{E}_\mu[\mathscr{H}(\pi^\mu(\cdot|X)|\pi^\nu(\cdot|X))] + 2\mathscr{H}(\mathcal{L}(X_{(1-\delta)T}^{\varphi^\mu,\mu})|\mathcal{L}(X_{(1-\delta)T}^{\varphi^\nu,\nu}))\end{aligned}$$

where we have relied on a second application of Girsanov's Theorem (as we did for (2.4)), combined with the time-reversal identities (1.5). By applying Theorem 2.1 and Corollary 2.4 we finally conclude that

$$\int_0^{\delta T}\mathbb{E}[|Y_s^\theta|^2]\mathrm{d}s \le \left(\frac{3\,\Lambda(\varphi_0^\nu)}{T} + \frac{\delta}{1-\delta}\,\frac{1}{T} + \frac{2\sqrt{\Lambda(\varphi_0^\nu)\, C^{\varphi^\nu}}}{T}\right)\mathbf{W}_2^2(\mu,\nu)\,.$$

□

3. Proof of main results

Given the preliminary results of the previous section we are now ready to prove our quantitative stability estimates for gradient and Hessian of the entropic potentials.

3.1. **Quantitative stability estimates of gradients.**

Theorem 3.1. *Assume* $\boldsymbol{H}1$ *and* $\boldsymbol{H}2$*, fix* $\delta \in (0,1)$*, and let*

$$C_\delta^{\psi^\nu} := T\bigg(\int_0^{\delta T} e^{\int_0^s 2\lambda(\psi_t^\nu)\mathrm{d}t}\mathrm{d}s\bigg)^{-1}.$$

Then we have

$$\|\nabla\varphi^\nu - \nabla\varphi^\mu\|^2_{\mathrm{L}^2(\rho)} \le \frac{C_\delta^{\psi^\nu}}{T}\bigg(2\,\mathscr{H}(\pi^\mu_{(1-\delta)T}|\pi^\nu_{(1-\delta)T}) + \int_{(1-\delta)T}^T \mathbb{E}[|Y_s^\eta|^2]\mathrm{d}s\bigg).$$

As a corollary, if we define

$$C_{\rho\nu}^\delta := C_\delta^{\psi^\nu}\left(\frac{\delta}{1-\delta} + 3\,\Lambda(\varphi_0^\nu) + 2\sqrt{\Lambda(\varphi_0^\nu)\,C^{\varphi^\nu}}\right),$$

then we have

$$\|\nabla\varphi^\nu - \nabla\varphi^\mu\|^2_{\mathrm{L}^2(\rho)} \le \frac{C_{\rho\nu}^\delta}{T^2}\,\mathbf{W}_2^2(\mu,\nu)\,.$$

Proof. From Itô's formula and (2.2), for all $s \le \delta T$ we have

$$\mathrm{d}\mathbb{E}[|Y_s^\theta|^2] \ge 2\,\mathbb{E}[Y_s^\theta\cdot\nabla^2\psi_s^\nu(X_s^{\psi^\mu,\rho})Y_s^\theta]\mathrm{d}s \ge 2\lambda(\psi_s^\nu)\,\mathbb{E}[|Y_s^\theta|^2]\mathrm{d}s\,,$$

which combined with Grönwall's lemma gives for any $s \le \delta T$

$$\mathbb{E}[|Y_0^\theta|^2]\,e^{\int_0^s 2\lambda(\psi_t^\nu)\mathrm{d}t} \le \mathbb{E}[|Y_s^\theta|^2]\,,$$

that integrated over $s \in [0,\delta T]$ reads as

$$\mathbb{E}[|Y_0^\theta|^2] \le \bigg(\int_0^{\delta T} e^{\int_0^s 2\lambda(\psi_t^\nu)\mathrm{d}t}\mathrm{d}s\bigg)^{-1}\int_0^{\delta T}\mathbb{E}[|Y_s^\theta|^2]\mathrm{d}s = \frac{C_\delta^{\psi^\nu}}{T}\int_0^{\delta T}\mathbb{E}[|Y_s^\theta|^2]\mathrm{d}s\,.$$

Next, notice that from Girsanov's theory (namely, the energy entropy identity (2.5)) we may recognize in the above right-hand side the relative entropy on the path space between the Schrödinger bridge from ρ to μ and the Schrödinger bridge from ρ to ν, restricted on the time interval $[0,\delta T]$, that is

$$\mathbb{E}[|Y_0^\theta|^2] \le \frac{2\,C_\delta^{\psi^\nu}}{T}\,\mathscr{H}(\mathcal{L}(X_{[0,\delta T]}^{\psi^\mu,\rho})|\mathcal{L}(X_{[0,\delta T]}^{\psi^\nu,\rho}))\,.$$

By recalling the time-reversal identities (1.5) and by applying the disintegration property of relative entropies (cf. [Nut21, Lemma 1.6] and [Léo14, Appendix A]) and Girsanov's Theorem (w.r.t. the backward corrector process $Y^{\cdot\eta}$) we deduce that

$$\begin{aligned}
\mathbb{E}[|Y_0^\theta|^2] &\le \frac{2\,C_\delta^{\psi^\nu}}{T}\,\mathscr{H}(\mathcal{L}(X_{[(1-\delta)T,T]}^{\varphi^\mu,\mu})|\mathcal{L}(X_{[(1-\delta)T,T]}^{\varphi^\nu,\nu}))\\
&= \frac{C_\delta^{\psi^\nu}}{T}\bigg(2\,\mathscr{H}(\mathcal{L}(X_{(1-\delta)T}^{\varphi^\mu,\mu})|\mathcal{L}(X_{(1-\delta)T}^{\varphi^\nu,\nu})) + \int_{(1-\delta)T}^T\mathbb{E}[|Y_s^\eta|^2]\mathrm{d}s\bigg)\\
&= \frac{C_\delta^{\psi^\nu}}{T}\bigg(2\,\mathscr{H}(\pi^\mu_{(1-\delta)T}|\pi^\nu_{(1-\delta)T}) + \int_{(1-\delta)T}^T\mathbb{E}[|Y_s^\eta|^2]\mathrm{d}s\bigg)\,.
\end{aligned}$$

This proves our first claim.

By recalling the identity (2.3) we then have

$$\mathbb{E}[|Y_0^\theta|^2] \le \frac{C_\delta^{\psi^\nu}}{T}\Big(2\,\mathscr{H}(\pi^\mu_{(1-\delta)T}|\pi^\nu_{(1-\delta)T}) + 2\,\mathbb{E}_\mu[\mathscr{H}(\pi^\mu(\cdot|X)|\pi^\nu(\cdot|X))]\Big)\,,$$

which can be bounded with Corollary 2.4 and with Theorem 2.1, yielding to

$$\mathbb{E}[|Y_0^\theta|^2] \le \frac{C_\delta^{\psi^\nu}}{T}\bigg(\frac{2\,\Lambda(\varphi_0^\nu)}{T} + T^{-1}\Big(\frac{1}{1-\delta} - 1\Big) + 2\frac{\sqrt{\Lambda(\varphi_0^\nu)\,C^{\varphi^\nu}}}{T}\bigg)\mathbf{W}_2^2(\mu,\nu)\,.$$

$\square$

Hereafter we specify Theorem 3.1 to two different settings. What follows is based on the explicit computations performed in Appendix A and Appendix B. From a notational viewpoint, we will write $a \lesssim b$ whenever there exists a numerical constant $C > 0$ (independent of T, ν, ρ, μ) such that $a \le C\, b$.

Corollary 3.2 (Gradients stability for compactly supported marginals)**.** *Assume* ***H****1, that $\mu \ll \nu$, and that both ρ and ν are compactly supported in a ball of radius R (big enough so that $R^2 \ge T$). Then we have*

$$\|\nabla\varphi^\nu - \nabla\varphi^\mu\|^2_{\mathrm{L}^2(\rho)} \lesssim \frac{R^4}{T^4}\, \mathbf{W}_2^2(\mu,\nu)\,.$$

Proof. Since ρ has compact support, the same computations performed in Appendix A.1 guarantee $\Lambda(\varphi_0^\mu) < \infty$ and hence the validity of **H**2. Our choice of δ in (B.2) and the following computations yield to $C^\delta_{\rho\nu} \lesssim R^4\, T^{-2}$. □

Corollary 3.3 (Gradients stability for log-concave marginals)**.** *Assume* ***H****1, that $\mu \ll \nu$, and that both ρ and ν are log-concave, i.e., that their (negative) log-densities satisfy $\nabla^2 U_\rho \ge \alpha_\rho$ and $\nabla^2 U_\nu \ge \alpha_\nu$ for some α_ρ, $\alpha_\nu > 0$ (w.l.o.g. such that $\alpha_\rho \vee \alpha_\nu < T^{-1}$). Then we have*

$$\|\nabla\varphi^\nu - \nabla\varphi^\mu\|^2_{\mathrm{L}^2(\rho)} \lesssim \frac{1}{\alpha_\rho\, \alpha_\nu\, T^4}\, \mathbf{W}_2^2(\mu,\nu)\,.$$

Proof. Since ρ has log-concave density, the same computations performed in Appendix A.2 guarantee $\Lambda(\varphi_0^\mu) < \infty$ and hence the validity of **H** 2. Our choice of δ in (B.2) and the following computations yield to $C^\delta_{\rho\nu} \lesssim \alpha_\rho^{-1}\, \alpha_\nu^{-1}\, T^{-2}$. □

As already mentioned in the introduction, our explicit computations allow to straightforwardly deduce bounds akin Corollary 3.2 and Corollary 3.3 when considering one marginal log-concave and the other one with compact support.

3.2. **Quantitative stability estimates of Hessian.** Let us consider once again the function $\theta_s \coloneqq \psi_s^\nu - \psi_s^\mu$ introduced in Section 2 and the forward process $(Y_s^\theta)_{s\in[0,T]}$ defined as $Y_s^\theta \coloneqq \nabla\theta_s(X_s^{\psi^\mu,\rho})$ where $(X^{\psi^\mu,\rho})_{s\in[0,T]}$ is the Schrödinger bridge (1.3) (from ρ to μ) and recall that

$$\begin{cases}\partial_s\theta_s + \frac12\Delta\theta_s - \nabla\psi_s^\mu\cdot\nabla\theta_s - \frac12|\nabla\theta_s|^2 = 0\,,\\ \mathrm{d}X_s^{\psi^\mu,\rho} = -\nabla\psi_s^\mu(X_s^{\psi^\mu,\rho})\mathrm{d}s + \mathrm{d}B_s\,, \quad X_0^{\psi^\mu,\rho}\sim\rho\,,\\ \mathrm{d}Y_s^\theta = \nabla^2\psi^\nu(X_s^{\psi^\mu,\rho})Y_s^\theta\,\mathrm{d}s + \nabla^2\theta_s(X_s^{\psi^\mu,\rho})\,\mathrm{d}B_s\,.\end{cases}$$

Next, let $Z_s^\theta \coloneqq \nabla^2\theta_s(X_s^{\psi^\mu,\rho})$ and notice that

$$\begin{aligned}\mathrm{d}Z_s^\theta &= \Big[2\,\mathrm{sym}(Z_s^\theta\nabla^2\psi_s^\nu(X_s^{\psi^\mu,\rho})) - (Z_s^\theta)^2 + \nabla^3\psi_s^\nu(X_s^{\psi^\mu,\rho})Y_s^\theta\Big]\,\mathrm{d}s + \nabla^3\theta_s(X_s^{\psi^\mu,\rho})\mathrm{d}B_s\\ &= \Big[2\,\mathrm{sym}(Z_s^\theta\nabla^2\psi_s^\mu(X_s^{\psi^\mu,\rho})) + \nabla^3\psi_s^\nu(X_s^{\psi^\mu,\rho})Y_s^\theta\Big]\,\mathrm{d}s + \nabla^3\theta_s(X_s^{\psi^\mu,\rho})\mathrm{d}B_s\,,\end{aligned}$$

where for any matrix M the symbol $\mathrm{sym}(M) \coloneqq (M + M^\intercal)/2$ denotes its symmetrized version and where for any $h \in \{\psi_s^\mu,\, \theta_s\}$ and for any vector $v\in\mathbb{R}^d$ we have defined the product $\nabla^3 h\, v$ as the matrix

$$(\nabla^3 h\, v)_{ij} \coloneqq \langle\nabla(\partial_i\partial_j h),\, v\rangle\,.$$

Clearly, our goal when proving the Hessian stability result is getting a bound on $\mathbb{E}\|Z_0^\theta\|_{\mathrm{HS}}$ since

$$\|\nabla^2\varphi^\nu - \nabla^2\varphi^\mu\|_{\mathrm{L}^1(\rho)} = \|\nabla^2\theta_0\|_{\mathrm{L}^1(\rho)} = \mathbb{E}\|\nabla^2\theta_0(X_0^{\psi^\mu,\rho})\|_{\mathrm{HS}} = \mathbb{E}\|Z_0^\theta\|_{\mathrm{HS}}\,.$$

In view of that, let us firstly prove some lemmata where we are able to bound $\mathbb{E}\|Z_0^\theta\|_{\mathrm{HS}}$ by means of the process $Y_\cdot$ and its norm.

Lemma 3.4. *Assume* ***H****1 and fix $\tau_\ell\in(0,T)$. Then we have*

$$\begin{aligned}\mathbb{E}\|Z_0^\theta\|_{\mathrm{HS}} \le \Big[{\tau_\ell}^{-1/2} + 2\,{\tau_\ell}^{1/2}\,(\inf_{s\in[0,\tau_\ell]}\lambda(\psi_s^\nu))^-\Big]\left(\int_0^{\tau_\ell}\mathbb{E}\|Z_s^\theta\|^2_{\mathrm{HS}}\mathrm{d}s\right)^{1/2}&\\ + \int_0^{\tau_\ell}\mathbb{E}\|Z_s^\theta\|^2_{\mathrm{HS}}\mathrm{d}s + \int_0^{\tau_\ell}\mathbb{E}\|\nabla^3\psi_s^\nu(X_s^{\psi^\mu,\rho})\,Y_s^\theta\|_{\mathrm{HS}}\mathrm{d}s\,,&\end{aligned}$$

where the negative part of $a \in \mathbb{R}$ is defined as $a^- := \max\{-a, 0\}$.

Proof. For notation's sake let $\Gamma_s^\theta = \nabla^3\theta_s(X_s^{\psi^\mu,\rho})$ and note that by Itô's formula we have

$$\mathrm{d}\|Z_s^\theta\|_{\mathrm{HS}}^2 = 2Z_s^\theta\,\mathrm{d}Z_s^\theta + \sum_{ijk}|\Gamma_s^{\theta,ijk}|^2\mathrm{d}s\,.$$

Hence, for any $\varepsilon \in (0,1)$ Itô's formula for the function $r_\varepsilon(a) = \sqrt{a+\varepsilon}$ yields

$$\begin{aligned}
\mathrm{d}r_\varepsilon(\|Z_s^\theta\|_{\mathrm{HS}}^2) &= \frac{Z_s^\theta \mathrm{d}Z_s^\theta}{r_\varepsilon(\|Z_s^\theta\|_{\mathrm{HS}}^2)} + \frac{\sum_{ijk}|\Gamma_s^{\theta,ijk}|^2}{2r_\varepsilon(\|Z_s^\theta\|_{\mathrm{HS}}^2)}\mathrm{d}s - \frac{\|Z_s^\theta\cdot\Gamma_s^\theta\|_{\mathrm{HS}}^2}{2\,r_\varepsilon^3(\|Z_s^\theta\|_{\mathrm{HS}}^2)}\mathrm{d}s \\
&= \left[-\frac{Z_s^\theta\cdot(Z_s^\theta)^2}{r_\varepsilon(\|Z_s^\theta\|_{\mathrm{HS}}^2)} + 2\frac{Z_s^\theta\cdot\mathrm{sym}(Z_s^\theta\nabla^2\psi_s^\nu(X_s^{\psi^\mu,\rho}))}{r_\varepsilon(\|Z_s^\theta\|_{\mathrm{HS}}^2)} + \frac{Z_s^\theta\cdot\nabla^3\psi_s^\nu(X_s^{\psi^\mu,\rho})Y_s^\theta}{r_\varepsilon(\|Z_s^\theta\|_{\mathrm{HS}}^2)}\right]\mathrm{d}s \\
&\quad + \frac{Z_s^\theta\cdot\nabla^3\theta_s(X_s^{\psi^\mu,\rho})}{r_\varepsilon(\|Z_s^\theta\|_{\mathrm{HS}}^2)}\mathrm{d}B_s + \left[\frac{\sum_{ijk}|\Gamma_s^{\theta,ijk}|^2}{2r_\varepsilon(\|Z_s^\theta\|_{\mathrm{HS}}^2)} - \frac{\|Z_s^\theta\cdot\Gamma_s^\theta\|_{\mathrm{HS}}}{2\,r_\varepsilon^3(\|Z_s^\theta\|_{\mathrm{HS}}^2)}\right]\mathrm{d}s\,.
\end{aligned}$$

Next observe that from Cauchy–Schwarz inequality the last term above is almost surely non-negative since

$$\frac{\|Z_s^\theta\cdot\Gamma_s^\theta\|_{\mathrm{HS}}^2}{2\,r_\varepsilon^3(\|Z_s^\theta\|_{\mathrm{HS}}^2)} = \frac{\sum_k|\sum_{ij}Z_s^{\theta,ij}\Gamma_s^{\theta,ijk}|^2}{2\,r_\varepsilon^3(\|Z_s^\theta\|_{\mathrm{HS}}^2)} \le \frac{\|Z_s^\theta\|_{\mathrm{HS}}^2\sum_{ijk}|\Gamma_s^{\theta,ijk}|^2}{2\,r_\varepsilon^3(\|Z_s^\theta\|_{\mathrm{HS}}^2)} \le \frac{\sum_{ijk}|\Gamma_s^{\theta,ijk}|^2}{2\,r_\varepsilon(\|Z_s^\theta\|_{\mathrm{HS}}^2)}\,.$$

Let us now provide a lower bound for each of the terms. For the first one, we use first Cauchy–Schwarz inequality and the sub-multiplicative property of the HS norm to obtain

$$\frac{Z_s^\theta\cdot(Z_s^\theta)^2}{r_\varepsilon(\|Z_s^\theta\|_{\mathrm{HS}}^2)} = \frac{\sum_{ij}Z_s^{\theta,ij}(Z_s^\theta\cdot Z_s^\theta)^{ij}}{r_\varepsilon(\|Z_s^\theta\|_{\mathrm{HS}}^2)} \le \frac{\|Z_s^\theta\|_{\mathrm{HS}}\,\|(Z_s^\theta)^2\|_{\mathrm{HS}}}{r_\varepsilon(\|Z_s^\theta\|_{\mathrm{HS}}^2)} \le \|Z_s^\theta\|_{\mathrm{HS}}^2\,.$$

For the second one, we first use the fact that Z^θ and $\nabla^2\psi$ are symmetric, the permutation identities and the monotonicity of the trace in order to rewrite it as

$$\begin{aligned}
&2\frac{Z_s^\theta\cdot\mathrm{sym}(Z_s^\theta\nabla^2\psi_s^\nu(X_s^{\psi^\mu,\rho}))}{r_\varepsilon(\|Z_s^\theta\|_{\mathrm{HS}}^2)} \\
&= \frac{1}{r_\varepsilon(\|Z_s^\theta\|_{\mathrm{HS}}^2)}\Big(\mathrm{Tr}(Z_s^\theta\cdot Z_s^\theta\nabla^2\psi_s^\nu(X_s^{\psi^\mu,\rho})) + \mathrm{Tr}(Z_s^\theta\cdot\nabla^2\psi_s^\nu(X_s^{\psi^\mu,\rho})Z_s^\theta)\Big) \\
&= \frac{2}{r_\varepsilon(\|Z_s^\theta\|_{\mathrm{HS}}^2)}\ \mathrm{Tr}(Z_s^\theta\cdot\nabla^2\psi_s^\nu(X_s^{\psi^\mu,\rho})Z_s^\theta) \ge \frac{2\,\lambda(\psi_s^\nu)\,\|Z_s^\theta\|_{\mathrm{HS}}^2}{r_\varepsilon(\|Z_s^\theta\|_{\mathrm{HS}}^2)} \\
&= 2\,\lambda(\psi_s^\nu)\,r_\varepsilon(\|Z_s^\theta\|_{\mathrm{HS}}^2) - 2\,\lambda(\psi_s^\nu)\,\varepsilon \ge 2\,\lambda(\psi_s^\nu)\,\|Z_s^\theta\|_{\mathrm{HS}} - 2\,\lambda(\psi_s^\nu)\,\varepsilon\,.
\end{aligned}$$

For the third term we use again Cauchy–Schwarz inequality to obtain

$$\begin{aligned}
\frac{Z_s^\theta\cdot\nabla^3\psi_s^\nu(X_s^{\psi^\mu,\rho})Y_s^\theta}{r_\varepsilon(\|Z_s^\theta\|_{\mathrm{HS}}^2)} &= \frac{\sum_{ijk}Z_s^{\theta,ij}\partial_{ijk}\psi_s^\nu(X_s^{\psi^\mu,\rho})Y_s^{\theta,k}}{r_\varepsilon(\|Z_s^\theta\|_{\mathrm{HS}}^2)} \\
&\ge -\|\nabla^3\psi_s^\nu(X_s^{\psi^\mu})\,Y_s^\theta\|_{\mathrm{HS}}\,\frac{\|Z_s^\theta\|_{\mathrm{HS}}}{r_\varepsilon(\|Z_s^\theta\|_{\mathrm{HS}}^2)} \ge -\|\nabla^3\psi_s^\nu(X_s^{\psi^\mu})\,Y_s^\theta\|_{\mathrm{HS}}\,.
\end{aligned}$$

We have thus shown that for any $\varepsilon \in (0,1)$ almost surely it holds

$$\begin{aligned}
\mathrm{d}r_\varepsilon(\|Z_s^\theta\|_{\mathrm{HS}}^2) \ge \Big(-\|Z_s^\theta\|_{\mathrm{HS}}^2 + 2\,\lambda(\psi_s^\nu)\,\|Z_s^\theta\|_{\mathrm{HS}} - \|\nabla^3\psi_s^\nu(X_s^{\psi^\mu})\,Y_s^\theta\|_{\mathrm{HS}}\Big)\mathrm{d}s \\
-2\,\lambda(\psi_s^\nu)\,\varepsilon\mathrm{d}s + \frac{Z_s^\theta\cdot\nabla^3\theta_s(X_s^{\psi^\mu,\rho})}{r_\varepsilon(\|Z_s^\theta\|_{\mathrm{HS}}^2)}\mathrm{d}B_s\,.
\end{aligned}$$

Taking expectation and integrating for $s \in [0,t]$ we get

$$\begin{aligned}
\mathbb{E}\|Z_0^\theta\|_{\mathrm{HS}} \le \mathbb{E}[r_\varepsilon(\|Z_0^\theta\|_{\mathrm{HS}}^2)] \le \mathbb{E}[r_\varepsilon(\|Z_t^\theta\|_{\mathrm{HS}}^2)] + 2\varepsilon\int_0^t\lambda(\psi_s^\nu)\mathrm{d}s + \int_0^t\mathbb{E}\|Z_s^\theta\|_{\mathrm{HS}}^2\mathrm{d}s \\
-2\int_0^t\lambda(\psi_s^\nu)\mathbb{E}\|Z_s^\theta\|_{\mathrm{HS}}\mathrm{d}s + \int_0^t\mathbb{E}\|\nabla^3\psi_t^\nu(X_s^{\psi^\mu,\rho})\,Y_s^\theta\|_{\mathrm{HS}}\mathrm{d}s\,,
\end{aligned}$$

which combined with the Dominated Convergence Theorem, for $\varepsilon \downarrow 0$, implies

$$\mathbb{E}\|Z_0^\theta\|_{\mathrm{HS}} \leq \mathbb{E}\|Z_t^\theta\|_{\mathrm{HS}} + \int_0^t \mathbb{E}\|Z_s^\theta\|_{\mathrm{HS}}^2 \mathrm{d}s - 2\int_0^t \lambda(\psi_s^\nu)\mathbb{E}\|Z_s^\theta\|_{\mathrm{HS}}\mathrm{d}s + \int_0^t \mathbb{E}\|\nabla^3\psi_t^\nu(X_s^{\psi^\mu,\rho})\, Y_s^\theta\|_{\mathrm{HS}}\mathrm{d}s\,.$$

Finally, by integrating over $t \in [0, \tau_\ell]$ we conclude that

$$\begin{aligned}\tau_\ell\, \mathbb{E}\|Z_0^\theta\|_{\mathrm{HS}} \leq{}& \int_0^{\tau_\ell} \mathbb{E}\|Z_t^\theta\|_{\mathrm{HS}}\mathrm{d}t + \int_0^{\tau_\ell}\int_0^t \mathbb{E}\|Z_s^\theta\|_{\mathrm{HS}}^2\mathrm{d}s\mathrm{d}t - 2\int_0^{\tau_\ell}\int_0^t \lambda(\psi_s^\nu)\mathbb{E}\|Z_s^\theta\|_{\mathrm{HS}}\mathrm{d}s\mathrm{d}t \\ &+ \int_0^{\tau_\ell}\int_0^t \mathbb{E}\|\nabla^3\psi_t^\nu(X_s^{\psi^\mu,\rho})\, Y_s^\theta\|_{\mathrm{HS}}\mathrm{d}s\mathrm{d}t \\ \leq{}& \big(1 + 2\,\tau_\ell\,(\inf_{s\in[0,\tau_\ell]} \lambda(\psi_s^\nu))^-\,\big)\int_0^{\tau_\ell} \mathbb{E}\|Z_s^\theta\|_{\mathrm{HS}}\mathrm{d}s + \tau_\ell \int_0^{\tau_\ell} \mathbb{E}\|Z_s^\theta\|_{\mathrm{HS}}^2\mathrm{d}s \\ &+\tau_\ell \int_0^{\tau_\ell} \mathbb{E}\|\nabla^3\psi_t^\nu(X_s^{\psi^\mu,\rho})\, Y_s^\theta\|_{\mathrm{HS}}\mathrm{d}s\,,\end{aligned}$$

where the negative part of a real number a is defined as $a^- \coloneqq \max\{-a, 0\}$. In conclusion, by applying Jensen's inequality we conclude that

$$\mathbb{E}\|Z_0^\theta\|_{\mathrm{HS}} \leq \left[{\tau_\ell}^{-1/2} + 2\,{\tau_\ell}^{1/2}\,(\inf_{s\in[0,\tau_\ell]} \lambda(\psi_s^\nu))^-\right]\left(\int_0^{\tau_\ell} \mathbb{E}\|Z_s^\theta\|_{\mathrm{HS}}^2\mathrm{d}s\right)^{1/2} + \int_0^{\tau_\ell} \mathbb{E}\|Z_s^\theta\|_{\mathrm{HS}}^2\mathrm{d}s + \int_0^{\tau_\ell} \mathbb{E}\|\nabla^3\psi_t^\nu(X_s^{\psi^\mu,\rho})\, Y_s^\theta\|_{\mathrm{HS}}\mathrm{d}s\,,$$

which ends our proof. □

Let us now fix $\delta' < \delta \in [0, T]$ arbitrary and consider the constant

$$C_{\delta',\delta}^{\psi^\nu} \coloneqq T\left(\int_{\delta' T}^{\delta T} e^{\int_{\delta' T}^s 2\lambda(\psi_t^\nu)\mathrm{d}t}\mathrm{d}s\right)^{-1}, \tag{3.1}$$

which generalizes the constant $C_\delta^{\psi^\nu}$ considered in Theorem 3.1. By repeating the same argument employed in Theorem 2.1 when proving the upper bound (2.7) for $\mathbb{E}[|Y_T^\theta|^2] = \mathbb{E}[|Y_0^\eta|^2]$, we can prove the following generalization.

Lemma 3.5. *Assume* ***H1*** *and* ***H2****. For any fixed $\delta' < \delta \in [0,1]$ we have*

$$\mathbb{E}[|Y_{\delta' T}^\theta|^2] \leq \frac{C_{\delta',\delta}^{\psi^\nu}}{T}\left(\frac{3\,\Lambda(\varphi_0^\nu)}{T} + \frac{\delta}{1-\delta}\frac{1}{T} + \frac{2\sqrt{\Lambda(\varphi_0^\nu)\,C^{\varphi^\nu}}}{T}\right)\mathbf{W}_2^2(\mu,\nu)\,.$$

Proof. By reasoning as in the proof of Theorem 2.1, from Itô's formula and (2.2), for all $s \leq \delta T$ we have

$$\mathrm{d}\mathbb{E}[|Y_s^\theta|^2] \geq 2\,\mathbb{E}[Y_s^\theta \cdot \nabla^2\psi_s^\nu(X_s^{\psi^\mu,\rho})Y_s^\theta]\mathrm{d}s \geq 2\lambda(\psi_s^\nu)\,\mathbb{E}[|Y_s^\theta|^2]\mathrm{d}s\,,$$

which combined with Grönwall's lemma gives for any $s \geq \delta' T$

$$\mathbb{E}[|Y_{\delta' T}^\theta|^2]\, e^{\int_{\delta' T}^s 2\lambda(\psi_t^\nu)\mathrm{d}t} \leq \mathbb{E}[|Y_s^\theta|^2]\,,$$

and that integrated over $s \in [\delta' T, \delta T]$ reads as

$$\mathbb{E}[|Y_{\delta' T}^\theta|^2] \leq \left(\int_{\delta' T}^{\delta T} e^{\int_{\delta' T}^s 2\lambda(\psi_t^\nu)\mathrm{d}t}\mathrm{d}s\right)^{-1}\int_{\delta' T}^{\delta T}\mathbb{E}[|Y_s^\theta|^2]\mathrm{d}s \leq \frac{C_{\delta',\delta}^{\psi^\nu}}{T}\int_0^{\delta T}\mathbb{E}[|Y_s^\theta|^2]\mathrm{d}s\,.$$

Given the above, the thesis follows from Proposition 2.5. □

Next, we give a bound for the time integral of $\mathbb{E}\|Z_s^\theta\|_{\mathrm{HS}}$ appearing in Lemma 3.4.

Lemma 3.6. *Assume **H**1 and **H**2. For any fixed $\delta' \leq \delta \in [0,T]$ we have*

$$\int_0^{\delta' T} \mathbb{E}\|Z_s^\theta\|_{\mathrm{HS}}^2 \mathrm{d}s \leq \frac{K_{\delta'\delta}^{\rho\nu}}{T^2}\,\mathbf{W}_2^2(\mu,\nu)\,,$$

where the constant is defined as

$$\begin{aligned} K_{\delta'\delta}^{\rho\nu} \coloneqq\, & 2C_{\delta',\delta}^{\psi^\nu}\left(3\,\Lambda(\varphi_0^\nu) + \frac{\delta}{1-\delta} + 2\sqrt{\Lambda(\varphi_0^\nu)\,C^{\varphi^\nu}}\right) \\ & + 4\,T\,(\inf_{s\in[0,\delta' T]} \lambda(\psi_s^\nu))^-\left(3\,\Lambda(\varphi_0^\nu) + \frac{\delta'}{1-\delta'} + 2\sqrt{\Lambda(\varphi_0^\nu)\,C^{\varphi^\nu}}\right). \end{aligned} \tag{3.2}$$

Proof. From Itô's formula and (2.2), by taking expectation we have

$$\frac{\mathrm{d}}{\mathrm{d}s}\mathbb{E}|Y_s^\theta|^2 \geq 2\lambda(\psi_s^\nu)\,\mathbb{E}|Y_s^\theta|^2 + \frac{1}{2}\mathbb{E}\|Z_s^\theta\|_{\mathrm{HS}}^2\,,$$

which integrated over $s\in[0,\delta' T]$ yields to

$$\int_0^{\delta' T} \mathbb{E}\|Z_s^\theta\|_{\mathrm{HS}}^2 \mathrm{d}s \leq 2\,\mathbb{E}|Y_{\delta' T}^\theta|^2 + 4\,(\inf_{s\in[0,\delta' T]}\lambda(\psi_s^\nu))^- \int_0^{\delta' T}\mathbb{E}|Y_s^\theta|^2\mathrm{d}s\,.$$

Then our thesis can be obtained by bounding the first term with Lemma 3.5 and the second term as already done in Proposition 2.5. ☐

Our last ingredient is an upper bound for the time integral of the third derivative term $\mathbb{E}\|\nabla^3\psi_t^\nu(X_s^{\psi^\mu,\rho})\,Y_s^\theta\|_{\mathrm{HS}}$.

Proposition 3.7. *Assume **H**1. Fix $\tau_u \in (0,T]$. Then for all $t\in(0,\tau_u]$,*

$$\|\nabla^3\psi_t^\nu(x)[v]\|_{\mathrm{HS}} \leq |v|\left(\frac{1}{\tau_u - t} + 2\gamma_{\tau_u}\right)\frac{2\gamma_{\tau_u}}{\sqrt{2\pi}}\int_t^{\tau_u}\mathcal{I}^{\psi^\nu}(t,s)^{-1/2}\mathrm{d}s\,,$$

where $\gamma_{\tau_u} \coloneqq \sup_{s\in[0,\tau_u]}\sup_{x\in\mathbb{R}^d}\|\nabla^2\psi_s^\nu\|_{\mathrm{HS}}$ and

$$\mathcal{I}^{\psi^\nu}(t,s) \coloneqq \int_t^s \exp\Big(\int_t^u 2\lambda(\psi_l^\nu)\mathrm{d}l\Big)\mathrm{d}u\,.$$

Proof. Fix $x,\widehat{x}\in\mathbb{R}^d$. Our aim is controlling with $|x-\widehat{x}|$ the following quantity

$$\|\nabla^2\psi_t^\nu(x) - \nabla^2\psi_t^\nu(\widehat{x})\|_{\mathrm{HS}}\,.$$

In view of this, let us consider the processes $X_\cdot^{t,x}$ and $X_\cdot^{t,\widehat{x}}$ satisfying for $s\in[t,\tau_u]$

$$\begin{cases} \mathrm{d}X_s^{t,x} = -\nabla\psi_s^\nu(X_s^{t,x})\,\mathrm{d}s + \mathrm{d}B_s\,,, \\ \mathrm{d}X_s^{t,\widehat{x}} = -\nabla\psi_s^\nu(X_s^{t,\widehat{x}})\,\mathrm{d}s + \mathrm{d}\hat{B}_s \quad \forall\, t\in[0,\tau_{\mathrm{st}}) \ \text{ and } X_s^{t,\widehat{x}} = X_s^{t,x} \quad \forall\, s\geq\tau_{\mathrm{st}} \\ X_t^{t,x} = x \text{ and } X_t^{t,\widehat{x}} = \widehat{x}\,, \end{cases}$$

where $\tau_{\mathrm{st}} \coloneqq \inf\{s\geq t : X_s^{t,x} = X_s^{t,\widehat{x}}\}\wedge\tau_u$, and $(\hat{B}_s)_{s\geq t}$ is defined as

$$\mathrm{d}\hat{B}_s \coloneqq (\mathrm{Id} - 2\,e_s\,e_s^\mathsf{T}\,\mathbf{1}_{\{s<\tau_{\mathrm{st}}\}})\,\mathrm{d}B_s \qquad \text{where} \quad e_s \coloneqq \begin{cases} \frac{X_s^{t,x}-X_s^{t,\widehat{x}}}{|X_s^{t,x}-X_s^{t,\widehat{x}}|} & \text{when } r_t>0\,, \\ u & \text{when } r_t = 0\,, \end{cases}$$

where $r_t \coloneqq |X_s^{t,x} - X_s^{t,\widehat{x}}|$ and $u\in\mathbb{R}^d$ is a fixed (arbitrary) unit-vector. By Lévy's characterization, $(\hat{B}_t)_{t\geq 0}$ is a d-dimensional Brownian motion, therefore $X_\cdot^{t,x}$ and $X_\cdot^{t,\widehat{x}}$ are two Schrödinger bridge processes (from ρ to ν) started respectively in x and $\widehat{x}$, coupled via the coupling by reflection.

Let us also consider the processes $Z_s = \nabla^2\psi_s^\nu(X_s^{t,x})$ and $\widehat{Z}_s = \nabla^2\psi_s^\nu(X_s^{t,\widehat{x}})$. Since

$$\partial_s\nabla^2\psi_s^\nu + \frac{1}{2}\Delta\nabla^2\psi_s^\nu - \nabla^3\psi_s^\nu\nabla\psi_s^\nu - (\nabla^2\psi_s^\nu)^2 = 0\,,$$

by means of Itô's formula we have

$$\mathrm{d}Z_s = Z_s^2\mathrm{d}s + \nabla^3\psi^\nu(X_s^{t,x})\mathrm{d}B_s\,, \qquad \mathrm{d}\widehat{Z}_s = \widehat{Z}_s^2\mathrm{d}t + \nabla^3\psi^\nu(X_s^{t,\widehat{x}})\mathrm{d}\hat{B}_s\,.$$

Therefore, if we set $\mathrm{d}M_s \coloneqq \nabla^3\psi^\nu(X_s^{t,x})\mathrm{d}B_s - \nabla^3\psi^\nu(X_s^{t,\widehat{x}})\mathrm{d}\hat{B}_s$, from Itô's formula we may firstly deduce that

$$\mathrm{d}\|Z_s - \widehat{Z}_s\|_{\mathrm{HS}}^2 = 2(Z_s - \widehat{Z}_s)\cdot(Z_s^2 - \widehat{Z}_s^2)\mathrm{d}s + \sum_{i,j}\mathrm{d}[M_\cdot^{ij}]_s + 2(Z_s - \widehat{Z}_s)\cdot \mathrm{d}M_s$$

where the $A\cdot B$ corresponds to the Hilbert–Schmidt scalar product between the two matrices A, B that is the scalar $\sum_{i,j} A^{ij}B^{ij}$.

From another application of Itô's formula[3] we then have

$$\begin{aligned}\|Z_s - \widehat{Z}_s\|_{\mathrm{HS}} = {} & \frac{(Z_s - \widehat{Z}_s)\cdot(Z_s^2 - \widehat{Z}_s^2)}{\|Z_s - \widehat{Z}_s\|_{\mathrm{HS}}}\mathrm{d}s + \frac{Z_s - \widehat{Z}_s}{\|Z_s - \widehat{Z}_s\|_{\mathrm{HS}}}\cdot \mathrm{d}M_s \\ & + \frac{\sum_{i,j}\mathrm{d}[M_\cdot^{ij}]_s}{2\,\|Z_s - \widehat{Z}_s\|_{\mathrm{HS}}} - \frac{(Z_s - \widehat{Z}_s)^2\cdot \mathrm{d}[M_\cdot]_s}{2\|Z_s - \widehat{Z}_s\|_{\mathrm{HS}}^3}\,.\end{aligned}$$

Since Z_s and $\widehat{Z}_s$ are symmetric matrices we have $Z_s\cdot(\widehat{Z}_sZ_s) = Z_s\cdot(Z_s\widehat{Z}_s)$ and, by recalling $\|\nabla^2\psi_s^\nu\|_{\mathrm{HS}} \le \gamma_{\tau_u}$ for any $s\in(0,\tau_u]$, we then have from Cauchy–Schwarz inequality that

$$\begin{aligned}(Z_s - \widehat{Z}_s)\cdot(Z_s^2 - \widehat{Z}_s^2) &= (Z_s - \widehat{Z}_s)\cdot(Z_s - \widehat{Z}_s)(Z_s + \widehat{Z}_s) + (Z_s - \widehat{Z}_s)\cdot(\widehat{Z}_sZ_s - Z_s\widehat{Z}_s) \\ &= (Z_s - \widehat{Z}_s)^2\cdot(Z_s + \widehat{Z}_s) \ge -\|Z_s - \widehat{Z}_s\|_{\mathrm{HS}}\,\|Z_s + \widehat{Z}_s\|_{\mathrm{HS}} \ge -2\gamma_{\tau_u}\|Z_s - \widehat{Z}_s\|_{\mathrm{HS}}\,.\end{aligned}$$

Moreover the two quadratic covariation terms cancel out since

$$(Z_s - \widehat{Z}_s)^2\cdot \mathrm{d}[M_\cdot]_s = \sum_{i,j}(Z_s^{ij} - \widehat{Z}_s^{ij})^2\mathrm{d}[M_\cdot^{ij}]_s \le \|Z_s - \widehat{Z}_s\|_{\mathrm{HS}}^2\sum_{i,j}\mathrm{d}[M_\cdot^{ij}]_s\,.$$

Putting these two remarks together yields to

$$\|Z_s - \widehat{Z}_s\|_{\mathrm{HS}} \ge -2\gamma_{\tau_u}\|Z_s - \widehat{Z}_s\|_{\mathrm{HS}}\mathrm{d}s + \frac{Z_s - \widehat{Z}_s}{\|Z_s - \widehat{Z}_s\|_{\mathrm{HS}}}\mathrm{d}M_s\,,$$

which implies

$$\frac{\mathrm{d}}{\mathrm{d}s}\mathbb{E}\|Z_s - \widehat{Z}_s\|_{\mathrm{HS}} \ge -2\gamma_{\tau_u}\mathbb{E}\|Z_s - \widehat{Z}_s\|_{\mathrm{HS}}\mathrm{d}s\,,$$

and hence that

$$\|\nabla^2\psi_t^\nu(x) - \nabla^2\psi_t^\nu(\widehat{x})\|_{\mathrm{HS}} = \mathbb{E}\|Z_t - \widehat{Z}_t\|_{\mathrm{HS}} \le \mathbb{E}\|Z_{\tau_u} - \widehat{Z}_{\tau_u}\|_{\mathrm{HS}} + 2\gamma_{\tau_u}\int_t^{\tau_u}\mathbb{E}\|Z_s - \widehat{Z}_s\|_{\mathrm{HS}}\mathrm{d}s\,. \tag{3.3}$$

Next, notice that for any $s\in[t,\tau_u]$ we can write

$$\|Z_s - \widehat{Z}_s\|_{\mathrm{HS}} = \mathbb{E}\Big[\|Z_s - \widehat{Z}_s\|_{\mathrm{HS}}\,\mathbf{1}_{\{X_s^{t,x}\neq X_s^{t,\widehat{x}}\}}\Big] \le 2\gamma_{\tau_u}\mathbb{P}(X_s^{t,x}\neq X_s^{t,\widehat{x}})\,. \tag{3.4}$$

Henceforth, the rest of the proof deals with estimating $\mathbb{P}(X_s^{t,x}\neq X_s^{t,\widehat{x}})$ for any $s\in[t,\tau_u]$. To do so we look at the one-dimensional process $r_s = |X_s^{t,x} - X_s^{t,\widehat{x}}|$, so that $\mathbb{P}(X_s^{t,x}\neq X_s^{t,\widehat{x}}) = \mathbb{P}(r_s>0)$. From Itô's formula we get

$$\mathrm{d}r_s^2 = (-2(X_s^{t,x} - X_s^{t,\widehat{x}})(\nabla\psi_s^\nu(X_s^{t,x}) - \nabla\psi_s^\nu(X_s^{t,\widehat{x}})) + 4)\mathrm{d}s + 4\,r_s\,\mathrm{d}W_s$$

where $\mathrm{d}W_s = e_s^\intercal\mathrm{d}B_s$ is a one-dimensional Brownian motion. Therefore another application of Itô's formula yields to

$$\mathrm{d}r_s = -e_s(\nabla\psi_s^\nu(X_s^{t,x}) - \nabla\psi_s^\nu(X_s^{t,\widehat{x}}))\mathrm{d}s + 2\,\mathrm{d}W_s \le -\lambda(\psi_s^\nu)\,r_s\,\mathrm{d}s + 2\,\mathrm{d}W_s\,.$$

Therefore the process $r_\cdot$ is dominated from above by the process $\widetilde{r}_\cdot$ which solves for $s\in[t,\tau_u]$

$$\mathrm{d}\widetilde{r}_s = -\lambda(\psi_s^\nu)\,\widetilde{r}_s\,\mathrm{d}s + 2\,\mathrm{d}W_s\,,\qquad \widetilde{r}_t = |x-\widehat{x}|\,.$$

[3]Here we would apply Itô's formula to the square root function $r(a) = \sqrt{a}$. Since there is a singularity in the origin, to be more precise we should apply a standard approximation argument by firstly applying Itô's formula to $r_\varepsilon(a) \coloneqq \sqrt{a+\varepsilon}$ and then let $\varepsilon\downarrow 0$ and use the Dominated Convergence Theorem. Since we have already portrayed this approximation in the proof of Lemma 3.4, we omit it here.

Moreover, notice that the above SDE implies that the process defined for any $s \in [t, \tau_u]$ as $N_s := e^{\int_t^s \lambda(\psi_u^\nu)\mathrm{d}u}\,\widetilde{r}_s$ is a martingale, more precisely

$$\mathrm{d}N_s = 2\exp\left(\int_t^s \lambda(\psi_u^\nu)\mathrm{d}u\right)\mathrm{d}W_s, \qquad \text{with } N_t = |x-\widehat{x}|\,.$$

Therefore from the Martingale Representation Theorem we have $N_s = N_t + B_{[N]_s}$ where $B.$ is a Brownian motion and

$$[N]_s = 4\int_t^s \exp\left(\int_t^u 2\lambda(\psi_l^\nu)\mathrm{d}l\right)\mathrm{d}u\,.$$

This information can then be employed in bounding $\mathbb{P}(X_{\tau_u}^{t,x} \neq X_{\tau_u}^{t,\widehat{x}}) = \mathbb{P}(r_{\tau_u} > 0)$ since from the Reflection Principle we may deduce

$$\begin{aligned}
\mathbb{P}(X_s^{t,x} \neq X_s^{t,\widehat{x}}) &= \mathbb{P}(r_s > 0) = \mathbb{P}\Big(\inf_{u\in[t,s]} r_u > 0\Big) \le \mathbb{P}\Big(\inf_{u\in[t,s]} \widetilde{r}_u > 0\Big) \le \mathbb{P}\Big(\inf_{u\in[t,s]} N_u > 0\Big)\\
&= \mathbb{P}\Big(\inf_{u\in[t,s]} B_{[N]_u} > -|x-\widehat{x}|\Big) = \mathbb{P}\Big(\inf_{u\in[t,[N]_s]} B_u > -|x-\widehat{x}|\Big)\\
&= \mathbb{P}\Big(\sup_{u\in[t,[N]_s]} B_u \le |x-\widehat{x}|\Big) = \mathbb{P}\Big(|B_{[N]_s}| \le |x-\widehat{x}|\Big)\\
&\le \sqrt{\frac{2}{\pi}}\,|x-\widehat{x}|\,[N]_s^{-1/2} = \frac{|x-\widehat{x}|}{\sqrt{2\pi}}\left(\int_t^s \exp\left(\int_t^u 2\lambda(\psi_l^\nu)\mathrm{d}l\right)\mathrm{d}u\right)^{-1/2}.
\end{aligned}$$

By combining this last estimate with (3.4) in (3.3) gives

$$\begin{aligned}
\|\nabla^2\psi_t^\nu(x) - \nabla^2\psi_t^\nu(\widehat{x})\|_{\mathrm{HS}} &\le 2\gamma_{\tau_u}\frac{|x-\widehat{x}|}{\sqrt{2\pi}}\left(\mathcal{I}^{\psi^\nu}(t,\tau_u)^{-1/2} + 2\gamma_{\tau_u}\int_t^{\tau_u}\mathcal{I}^{\psi^\nu}(t,s)^{-1/2}\mathrm{d}s\right)\\
&\le 2\gamma_{\tau_u}\left(\frac{1}{\tau_u - t} + 2\gamma_{\tau_u}\right)\frac{|x-\widehat{x}|}{\sqrt{2\pi}}\int_t^{\tau_u}\mathcal{I}^{\psi^\nu}(t,s)^{-1/2}\mathrm{d}s\,,
\end{aligned}$$

and hence the conclusion. □

We are now ready to prove the general quantitative stability result for the Hessians. Our estimates will depend on two free parameters $\delta' < \delta \in [0,1]$. A priori one could simply optimize over their choice; however, this optimization heavily depends on the semiconcavity parameters. We state the general result keeping these two free parameters and then choose them appropriately in Appendix B, in a setting-wise manner.

Theorem 3.8 (Stability of Hessians (with explicit costants))**.** *Assume* ***H****1 and* ***H****2. For any $\delta' < \delta \in [0,1]$ we have*

$$\|\nabla^2\varphi^\mu - \nabla^2\varphi^\nu\|_{\mathrm{L}^1(\rho)} \le A\,\mathbf{W}_2(\mu,\nu) + \frac{K_{\delta'\delta}^{\rho\nu}}{T^2}\,\mathbf{W}_2^2(\mu,\nu)\,,$$

with A defined at (3.5).

Proof. Fix $\delta' < \delta \in [0,1]$, let $\tau_u = \delta T$, $\tau_\ell = \delta' T$, fix the positive constants

$$\gamma_{\tau_u} := \sup_{s\in[0,\tau_u]}\sup_{x\in\mathbb{R}^d}\|\nabla^2\psi_s^\nu\|_{\mathrm{HS}}\,, \qquad \text{and } \bar{\lambda}_{\tau_\ell} := (\inf_{s\in[0,\tau_\ell]}\lambda(\psi_s^\nu))^-$$

and recall from (3.2) (combined with (3.1)) the constant

$$\begin{aligned}
K_{\delta'\delta}^{\rho\nu} = {}& \frac{2\,T}{\mathcal{I}^{\psi^\nu}(\tau_\ell,\tau_u)}\left(3\,\Lambda(\varphi_0^\nu) + \frac{\delta}{1-\delta} + 2\sqrt{\Lambda(\varphi_0^\nu)\,C^{\varphi^\nu}}\right)\\
&+ 4\,T\,\bar{\lambda}_{\tau_\ell}\left(3\,\Lambda(\varphi_0^\nu) + \frac{\delta'}{1-\delta'} + 2\sqrt{\Lambda(\varphi_0^\nu)\,C^{\varphi^\nu}}\right).
\end{aligned}$$

From Proposition 3.7 and Proposition 2.5 we see that

$$\begin{aligned}
&\int_0^{\tau_\ell} \mathbb{E}\|\nabla^3\psi_s^\nu(X_s^{\psi^\mu,\rho})\, Y_s^\theta\|_{\mathrm{HS}}\mathrm{d}s \\
&\quad\le \left(\frac{1}{\tau_u-\tau_\ell}+2\gamma_{\tau_u}\right)\frac{2\gamma_{\tau_u}}{\sqrt{2\pi}}\int_0^{\tau_\ell}\mathbb{E}[|Y_s^\theta|]\int_s^{\tau_u}\mathcal{I}^{\psi^\nu}(s,u)^{-1/2}\mathrm{d}u\,\mathrm{d}s \\
&\quad\le \left(\frac{1}{\tau_u-\tau_\ell}+2\gamma_{\tau_u}\right)\frac{2\gamma_{\tau_u}}{\sqrt{2\pi}}\left(\int_0^{\tau_\ell}\mathbb{E}[|Y_s^\theta|^2]\mathrm{d}s\right)^{1/2}\tau_\ell\sup_{s\in[0,\tau_\ell]}\int_s^{\tau_u}\mathcal{I}^{\psi^\nu}(s,u)^{-1/2}\mathrm{d}u \\
&\quad\le \frac{\mathbf{W}_2(\mu,\nu)}{\sqrt{T}}\left(3\,\Lambda(\varphi_0^\nu)+\frac{\delta'}{1-\delta'}+2\sqrt{\Lambda(\varphi_0^\nu)\,C^{\varphi^\nu}}\right)^{1/2}\left(\frac{1}{\tau_u-\tau_\ell}+2\gamma_{\tau_u}\right)\cdot \\
&\qquad\qquad\cdot\frac{2\gamma_{\tau_u}\,\tau_\ell}{\sqrt{2\pi}}\sup_{s\in[0,\tau_\ell]}\int_s^{\tau_u}\mathcal{I}^{\psi^\nu}(s,u)^{-1/2}\mathrm{d}u\,.
\end{aligned}$$

By combining Lemma 3.4 with the above estimate and with Lemma 3.6 we finally deduce

$$\begin{aligned}
\mathbb{E}\|Z_0^\theta\|_{\mathrm{HS}} &\le \left[{\tau_\ell}^{-1/2}+2\,{\tau_\ell}^{1/2}\,\bar\lambda_{\tau_\ell}\right]\left(\int_0^{\tau_\ell}\mathbb{E}\|Z_s^\theta\|_{\mathrm{HS}}^2\mathrm{d}s\right)^{1/2}+\int_0^{\tau_\ell}\mathbb{E}\|Z_s^\theta\|_{\mathrm{HS}}^2\mathrm{d}s \\
&\qquad+\int_0^{\tau_\ell}\mathbb{E}\|\nabla^3\psi_s^\nu(X_s^{\psi^\mu,\rho})\,Y_s^\theta\|_{\mathrm{HS}}\mathrm{d}s \\
&\le A\,\mathbf{W}_2(\mu,\nu)+\frac{K_{\delta'\delta}^{\rho\nu}}{T^2}\,\mathbf{W}_2^2(\mu,\nu)\,,
\end{aligned}$$

with

$$\begin{aligned}
A &:= \left[{\tau_\ell}^{-1/2}+2\,{\tau_\ell}^{1/2}\,\bar\lambda_{\tau_\ell}\right]\frac{\sqrt{K_{\delta'\delta}^{\rho\nu}}}{T}+\frac{1}{\sqrt{T}}\left(3\,\Lambda(\varphi_0^\nu)+\frac{\delta'}{1-\delta'}+2\sqrt{\Lambda(\varphi_0^\nu)\,C^{\varphi^\nu}}\right)^{1/2}\cdot \\
&\qquad\cdot\left(\frac{1}{\tau_u-\tau_\ell}+2\gamma_{\tau_u}\right)\frac{2\gamma_{\tau_u}\,\tau_\ell}{\sqrt{2\pi}}\sup_{s\in[0,\tau_\ell]}\int_s^{\tau_u}\mathcal{I}^{\psi^\nu}(s,u)^{-1/2}\mathrm{d}u\,.
\end{aligned}\tag{3.5}$$

□

Let us conclude by specifying Theorem 3.8 to diverse settings, relying on the explicit computations performed in Appendix A and Appendix B. Recall that hereafter we write $a\lesssim b$ whenever there exists a numerical constant $C>0$ (independent of T,ν,ρ,μ) such that $a\le C\,b$.

Corollary 3.9 (Hessian stability for compactly supported marginals)**.** *Assume **H**1, that $\mu\ll\nu$, and that both ρ and ν are compactly supported in a ball of radius R (big enough so that $R^2\ge T$). Then we have*

$$\|\nabla^2\varphi^\mu-\nabla^2\varphi^\nu\|_{\mathrm{L}^1(\rho)}\lesssim ({R^4}/{T^{7/2}}+{d}/{T})\,\mathbf{W}_2(\mu,\nu)+{R^6}/{T^5}\,\mathbf{W}_2^2(\mu,\nu)\,,$$

Proof. Since ρ has compact support, the same computations performed in Appendix A.1 guarantee $\Lambda(\varphi_0^\mu)<\infty$ and hence the validity of **H**2. Our computations yield to

$$K_{\delta'\delta}^{\rho\nu}\lesssim {R^6}/{T^3}\qquad\text{and}\qquad A\lesssim {R^4}/{T^{7/2}}+{d}/{T}\,.$$

□

Corollary 3.10 (Hessian stability for log-concave marginals)**.** *Assume **H**1, that $\mu\ll\nu$, and that both ρ and ν are log-concave, i.e., that their (negative) log-densities satisfy $\nabla^2U_\rho\ge\alpha_\rho$ and $\nabla^2U_\nu\ge\alpha_\nu$ for some $\alpha_\rho,\alpha_\nu>0$ (w.l.o.g. such that $\alpha_\rho\vee\alpha_\nu<T^{-1}$). Then we have*

$$\|\nabla^2\varphi^\mu-\nabla^2\varphi^\nu\|_{\mathrm{L}^1(\rho)}\lesssim\left(\frac{1}{\alpha_\nu\sqrt{\alpha_\rho}\,T^3}+\frac{d}{\sqrt{\alpha_\rho\,\alpha_\nu}\,T^2}\right)\mathbf{W}_2(\mu,\nu)+\frac{1}{\alpha_\rho\,\alpha_\nu\,T^4}\,\mathbf{W}_2^2(\mu,\nu)\,,$$

Proof. Since ρ is log-concave, the same computations performed in Appendix A.2 guarantee $\Lambda(\varphi_0^\mu)<\infty$ and hence the validity of **H**2. Our computations yield to

$$K_{\delta'\delta}^{\rho\nu}\lesssim\frac{1}{\alpha_\rho\,\alpha_\nu\,T^2}\qquad\text{and}\qquad A\lesssim\frac{1}{\alpha_\nu\sqrt{\alpha_\rho}\,T^3}+\frac{d}{\sqrt{\alpha_\rho\,\alpha_\nu}\,T^2}\,.$$

□

Lastly, as we have already mentioned in the introduction, our explicit computations allow to straightforwardly deduce bounds akin Corollary 3.9 and Corollary 3.10 when considering one marginal log-concave and the other one with compact support.

3.3. **Exponential convergence of Hessian of Sinkhorn iterates.** We conclude with the proof of the convergence of gradient and Hessian of Sinkhorn iterates. This will be a straightforward application of our quantitative stability estimates and the explicit computations performed in Appendices A and B (this time considering the couple of marginals (ρ,μ) as fixed and considering μ^n as perturbation of μ).

Proof of Theorem 1.2. Under our assumptions, Talagrand inequality (TI(τ)) and the data processing inequality for relative entropy combined with [CCGT24, Theorem 1.2] and guarantee that (3.6)

$$\mathbf{W}_2^2(\mu,\mu^{n+1,n}) \leq 2\tau\,\mathscr{H}(\mu|\mu^{n+1,n}) \leq 2\tau\,\mathscr{H}(\pi^{\mu}|\pi^{n+1,n}) \leq 2\tau\left(1-\frac{T}{T+\tau\Lambda}\right)^{(n-N+1)}\mathscr{H}(\pi^{\mu}|\pi^{0,0})\,.$$

In particular notice that this implies $\mathscr{H}(\mu|\mu^{n+1,n})<+\infty$, which combined with our assumption further guarantees the validity of **H**2 for the marginals $\mu^{n+1,n}$ generated along Sinkhorn's iterates. This allows us to apply Theorems 3.1 and 3.8 (with the couple ν,μ there, replaced here as $\mu,\mu^{n+1,n}$) and deduce that

$$\|\nabla\varphi^{n+1}-\nabla\varphi^{\mu}\|^2_{\mathrm{L}^2(\rho)} \leq \frac{C^{\delta}_{\rho\mu}}{T^2}\,\mathbf{W}_2^2(\mu^{n+1,n},\mu) \tag{3.7}$$

and

$$\|\nabla^2\varphi^{n+1}-\nabla^2\varphi^{\mu}\|_{\mathrm{L}^1(\rho)} \leq A\,\mathbf{W}_2(\mu^{n+1,n},\mu)+\frac{K^{\rho\mu}_{\delta'\delta}}{T^2}\,\mathbf{W}_2^2(\mu^{n+1,n},\mu)\,, \tag{3.8}$$

with $C_{\rho\mu}$, A and $K^{\rho\mu}_{\delta'\delta}$ defined as in the stability results, this time depending solely on T, and on the marginals ρ and μ. Putting together (3.6), (3.7) and (3.8) yields to

$$\begin{aligned}
\|\nabla\varphi^{n+1}-\nabla\varphi^{\mu}\|^2_{\mathrm{L}^2(\rho)} &\lesssim \tau\left(1-\frac{T}{T+\tau\Lambda}\right)^{(n-N+1)}\mathscr{H}(\pi^{\mu}|\pi^{0,0})\,,\\
\|\nabla^2\varphi^{n+1}-\nabla^2\varphi^{\mu}\|_{\mathrm{L}^1(\rho)} &\lesssim \sqrt{\tau}\left(1-\frac{T}{T+\tau\Lambda}\right)^{\frac{n-N+1}{2}}\sqrt{\mathscr{H}(\pi^{\mu}|\pi^{0,0})}\\
&\qquad+\tau\left(1-\frac{T}{T+\tau\Lambda}\right)^{(n-N+1)}\mathscr{H}(\pi^{\mu}|\pi^{0,0})\,.
\end{aligned}$$

Finally, the specific asymptotics of the constants when ρ and μ are compactly supported or log-concave can be obtained as already done in Corollaries 3.2, 3.3, 3.9 and 3.10 for the gradient and Hessian stability estimates. □

Appendix A. Explicit computations for $C^{\varphi^{\nu}}$ and $\Lambda(\varphi^{\nu}_0)$

In this section we specify the constants appearing in the entropic stability bound of Theorem 2.1 to various settings. Before actually doing it, let us preliminary recall the well-known identities [CP23, FGP20, CDG23, Con24]

$$\nabla^2\psi^{\nu}_s(y) = (T-s)^{-1}-(T-s)^{-2}\mathrm{Cov}(X^{\psi^{\nu},\rho}_T|X^{\psi^{\nu},\rho}_s=y)\quad\forall\, s\in[0,T)\,, \tag{A.1}$$

where $(X^{\psi^{\nu},\rho}_s)_{s\in[0,T]}$ is the forward Schrödinger process (from ρ to ν) defined at (1.3), whereas $\mathrm{Cov}(X^{\psi^{\nu},\rho}_0|X^{\psi^{\nu},\rho}_s=y)$ is the covariance of the law of this process at initial time conditioned on being in y at time s. This can be easily seen by recalling that

$$\psi^{\nu}_s(y) = -\log P_{T-s}e^{-\psi^{\nu}_T}(y) = -\log\int\exp\left(-\psi^{\nu}_T(x)-\frac{|x-y|^2}{2(T-s)}\right)\mathrm{d}x+\frac{d}{2}\log(2\pi(T-s))\,,$$

and computing the Hessian as done in [CDG23, Proposition 17] for the case $s=T$.

Similarly, for φ^ν_s, for any $s\in[0,T)$ we have

$$\nabla^2\varphi^\nu_s(y) = (T-s)^{-1} - (T-s)^{-2}\mathrm{Cov}(X^{\varphi^\nu,\nu}_T | X^{\varphi^\nu,\nu}_s = y)\,, \tag{A.2}$$

where $(X^{\varphi^\nu,\nu}_s)_{s\in[0,T]}$ is the backward Schrödinger process defined at (1.4) and $\mathrm{Cov}(X^{\varphi^\nu,\nu}_0 | X^{\varphi^\nu,\nu}_s = y)$ is the covariance of the law of this process at initial time conditioned on being in y at time s.

Furthermore, let us recall here the following convexity backpropagation result along Hamilton-Jacobi-Bellman equations (see for instance [Con24, Lemma 3.1])

Lemma A.1. *Assume that $\nabla^2 h\geq\alpha$ for some $\alpha>-T^{-1}$ uniformly. Then if $(h_s)_{s\in[0,T]}$ denotes the solution of*

$$\begin{cases}\partial_s u_s + \frac12\Delta u_s - \frac12|\nabla u_s|^2 = 0\\ u_s = h\end{cases}$$

then for any $s\in[0,T]$ we have $\nabla^2 h_s \geq (\alpha^{-1}+(T-s))^{-1}$.

Then, if we assume that there exists some $\alpha>-T^{-1}$ such that $\nabla^2 h\geq\alpha$, the previous result implies that $\nabla^2 h_0\geq(\alpha^{-1}+T)^{-1}$ and hence that the semiconcavity parameter Λ of the function $g^y_{h_0}(z) \coloneqq \frac{|z-y|^2}{2} - T\,h_0(z)$ can be bounded by

$$\Lambda(h_0)\leq 1 - T\,\lambda(h_0) \leq 1-\frac{1}{(T\alpha)^{-1}+1} = \frac{1}{1+T\alpha}\,. \tag{A.3}$$

A.1. **Marginal ρ with compact support.** Clearly if $\mathrm{supp}(\rho)\subseteq B_R(0)$ for some radius $R>0$, then for any $s\in[0,T)$ we have $\mathrm{Cov}(X^{\varphi^\nu,\nu}_T | X^{\varphi^\nu,\nu}_s = y)\leq R^2$ since $X^{\varphi^\nu,\nu}_T\sim\rho$ and as a consequence of (A.2) we can take

$$\lambda(\varphi^\nu_s) = (T-s)^{-1} - (T-s)^{-2}\,R^2\,,\ \text{and hence } \Lambda(\varphi^\nu_0) = {R^2}/{T} \tag{A.4}$$

Next, let us compute C^{φ^ν} defined at (2.8). This can be easily accomplished since for any $l\leq u<T$ we have

$$\begin{aligned}\mathcal{I}^{\varphi^\nu}(l,u) &= \int_l^u \exp\left(2\int_l^s \lambda(\varphi^\nu_t)\mathrm{d}t\right)\mathrm{d}s = \int_l^u \exp\left(2\int_l^s (T-t)^{-1} - (T-t)^{-2}\,R^2\mathrm{d}t\right)\mathrm{d}s\\ &= \int_l^u \exp\left([-2\log(T-t)]_l^s - \left[\frac{2R^2}{T-t}\right]_l^s\right)\mathrm{d}s\\ &=(T-l)^2\,e^{\frac{2R^2}{T-l}}\int_l^u \frac{e^{-\frac{2R^2}{T-s}}}{(T-s)^2}\mathrm{d}s = \frac{(T-l)^2}{2R^2}\left(1-e^{\frac{2R^2}{T-l}-\frac{2R^2}{T-u}}\right).\end{aligned}$$

Therefore we have

$$C^{\varphi^\nu} \coloneqq T\inf_{\delta\in[0.1)}(\mathcal{I}^{\varphi^\nu}(0,\delta T))^{-1} = \frac{2R^2}{T}\inf_{\delta\in[0,1)}\left(1-\exp\left(-\frac{\delta}{1-\delta}\,\frac{2R^2}{T}\right)\right)^{-1} = \frac{2R^2}{T}\,. \tag{A.5}$$

A.2. **Log-concavity of ρ.** Let U_ρ denotes the (negative) log-density of the marginal ρ and let us assume that there exists $\alpha_\rho>0$ such that $\nabla^2 U_\rho\geq\alpha_\rho$. Without loss of generalities, since we are interested in the asymptotics $T\downarrow 0$, we will further assume that $\alpha_\rho<T^{-1}$.

Then, it is well known (cf. [CDG23]) that $\nabla^2\varphi^\nu\geq\alpha_\rho - T^{-1}$ and hence we can take $\lambda(\varphi^\nu_T) = \alpha_\rho - T^{-1}$. This is enough to deduce from Lemma A.1 that

$$\nabla^2\varphi^\nu_s \geq \frac{1}{(\alpha_\rho - T^{-1})^{-1} + T - s}$$

and hence that we can set

$$\lambda(\varphi^\nu_0) = \frac{1}{(\alpha_\rho - T^{-1})^{-1}+T} = \frac{\alpha_\rho - T^{-1}}{\alpha_\rho\,T} < 0 \qquad\text{and hence } \Lambda(\varphi^\nu_0) = (\alpha_\rho\,T)^{-1}\,,$$

and for any $s\in[0,T]$

$$\lambda(\varphi^\nu_s) = \frac{1}{\lambda(\varphi^\nu_0)^{-1} - s} < 0\,.$$

We are now ready to compute C^{φ^ν} defined at (2.8). This can be easily accomplished since for any $l \leq u < T$ we have

$$\begin{aligned}\mathcal{I}^{\varphi^\nu}(l,u) &= \int_l^u \exp\Big(2\int_l^s \lambda(\varphi^\nu_t)\mathrm{d}t\Big)\mathrm{d}s = \int_l^u \exp\Big(-2\int_l^s \frac{1}{t-\lambda(\varphi^\nu_0)^{-1}}\mathrm{d}t\Big)\\ &= \int_l^u \exp\Big([-2\log(t-\lambda(\varphi^\nu_0)^{-1})]_l^s\Big)\mathrm{d}s = \int_l^u \frac{(l-\lambda(\varphi^\nu_0)^{-1})^2}{(s-\lambda(\varphi^\nu_0)^{-1})^2}\mathrm{d}s\\ &= (l-\lambda(\varphi^\nu_0)^{-1})^2\Big(\frac{1}{l-\lambda(\varphi^\nu_0)^{-1}} - \frac{1}{u-\lambda(\varphi^\nu_0)^{-1}}\Big).\end{aligned}$$

Therefore we have

$$C^{\varphi^\nu} := T\inf_{\delta\in[0.1)}(\mathcal{I}^{\varphi^\nu}(0,\delta T))^{-1} = T\mathcal{I}^{\varphi^\nu}(0,T))^{-1} = (\alpha_\rho T)^{-1}. \tag{A.6}$$

Let us conclude this appendix with a table summarizing the values of the constants so far computed (up to numerical prefactors).

Constant	$\Lambda(\varphi^\nu_0)$	C^{φ^ν}
ρ compact support	$R^2\,T^{-1}$	$R^2\,T^{-1}$
ρ log-concave	$\alpha_\rho^{-1}\,T^{-1}$	$\alpha_\rho^{-1}\,T^{-1}$

Appendix B. Explicit computations for Hessians stability

In this section we will compute the constants appearing in our main general result Theorem 3.8 in two specific settings and analyze their behavior w.r.t. the parameters T,ν,ρ . Hereafter we write $a \lesssim b$ whenever there exists a numerical constant $C>0$ (independent of T,ν,ρ) such that $a \leq C\,b$. In order to compute the constants appearing in the Hessian stability bounds recall that $C^{\psi^\nu}_{\delta',\delta}$ was introduced in (3.1) as

$$C^{\psi^\nu}_{\delta',\delta} = T(\mathcal{I}^{\psi^\nu}(\delta' T,\delta T))^{-1}. \tag{B.1}$$

Through this section we always choose

$$\delta = \frac{1}{1+\Lambda(\psi^\nu_0)}, \tag{B.2}$$

so that

$$\frac{\delta}{1-\delta} = \frac{1}{\Lambda(\psi^\nu_0)} \quad \text{and} \quad \frac{1}{1-\delta} = \frac{1+\Lambda(\psi^\nu_0)}{\Lambda(\psi^\nu_0)}. \tag{B.3}$$

Moreover we will pick $\delta' = \delta/2$ so that

$$\frac{\delta'}{1-\delta'} = ((\delta')^{-1}-1)^{-1} = (2/\delta - 1)^{-1} = (1+2\Lambda(\psi^\nu_0))^{-1}$$

Finally, recall that hereafter we choose $\tau_u = \delta T$ and $\tau_\ell = \delta' T$ and note that in general we always have

$$\gamma_{\tau_u} := \sup_{s\in[0,\tau_u]}\sup_{x\in\mathbb{R}^d}\|\nabla^2\psi^\nu_s\|_{\mathrm{HS}} \overset{(A.1)}{\leq} \sup_{s\in[0,\tau_u]}\sqrt{d}\,(T-s)^{-1} = \frac{\sqrt{d}}{T(1-\delta)} = \frac{\sqrt{d}}{T}\,\frac{1+\Lambda(\psi^\nu_0)}{\Lambda(\psi^\nu_0)}. \tag{B.4}$$

B.1. **Marginal ν with compact support.** By reasoning as in Appendix A.1, if $\mathrm{supp}(\nu) \subseteq B_R(0)$ for some radius $R>0$, which we assume to be big enough, *i.e.*, that $R^2 \geq T$. Then for any $s\in[0,T)$ we have $\mathrm{Cov}(X^{\psi^\nu,\rho}_T | X^{\psi^\nu,\rho}_s = y) \leq R^2$ since $X^{\psi^\nu,\rho}_T \sim \nu$ and as a consequence of (A.1) we can take

$$\lambda(\psi^\nu_s) = (T-s)^{-1} - (T-s)^{-2}\,R^2, \tag{B.5}$$

and hence

$$\mathcal{I}^{\psi^\nu}(l,u) = \frac{(T-l)^2}{2R^2}\Big(1 - e^{\frac{2R^2}{T-l} - \frac{2R^2}{T-u}}\Big), \qquad \text{and we take } \Lambda(\psi^\nu_0) = R^2/T. \tag{B.6}$$

This combined with (B.4) already gives

$$\gamma_{\tau_u} \leq \frac{\sqrt{d}}{T}\,\frac{1+\Lambda(\psi_0^\nu)}{\Lambda(\psi_0^\nu)} = \sqrt{d}(R^{-2}+T^{-1}) \leq \frac{2\sqrt{d}}{T}\,.$$

Next let us compute the integral constant term appearing in Theorem 3.8, that is the value

$$\sup_{s\in[0,\tau_\ell]}\int_s^{\tau_u} \mathcal{I}^{\psi^\nu}(s,u)^{-1/2}\mathrm{d}u\,.$$

In view of that, notice that for any $s \in [0,\tau_\ell]$

$$\begin{aligned}
\int_s^{\tau_u}\mathcal{I}^{\psi^\nu}(s,u)^{-1/2}\mathrm{d}u &= \sqrt{2}\,\frac{R}{T-s}\int_s^{\tau_u}\left(1-e^{\frac{2R^2}{T-s}-\frac{2R^2}{T-u}}\right)^{-1/2}\mathrm{d}u\\
&\leq \sqrt{2}\,\frac{R}{T-s}\int_s^{\tau_u}\left(1-e^{-\frac{2R^2}{(T-s)^2}(u-s)}\right)^{-1/2}\mathrm{d}u = \frac{T-s}{\sqrt{2}\,R}\log\left(\frac{1+\sqrt{1-e^{-\frac{2R^2}{(T-s)^2}(\tau_u-s)}}}{1-\sqrt{1-e^{-\frac{2R^2}{(T-s)^2}(\tau_u-s)}}}\right)\\
&\leq \frac{\log 4}{\sqrt{2}}\,\frac{T-s}{R} + \sqrt{2}\,R\,\frac{\tau_u-s}{T-s} \leq \frac{\log 4}{\sqrt{2}}\,\frac{T}{R} + \sqrt{2}\,R\,\frac{\tau_u}{T-\tau_\ell} = \frac{\log 4}{\sqrt{2}}\,\frac{T}{R} + \sqrt{2}\,R\,\frac{2\delta'}{1-\delta'}\\
&= \frac{\log 4}{\sqrt{2}}\,\frac{T}{R} + \frac{2\sqrt{2}\,R}{1+2\Lambda(\psi_0^\nu)} = \frac{\log 4}{\sqrt{2}}\,\frac{T}{R} + \frac{2\sqrt{2}\,T\,R}{T+2\,R^2}\,.
\end{aligned}$$

Therefore

$$\sup_{s\in[0,\tau_\ell]}\int_s^{\tau_u}\mathcal{I}^{\psi^\nu}(s,u)^{-1/2}\mathrm{d}u \lesssim \frac{T}{R} + \frac{T\,R}{T+R^2} \lesssim {}^{T}\!/_{R}\,. \tag{B.7}$$

Now, let us compute $C^{\psi^\nu}_{\delta',\delta}$ from (B.1) and (B.6). We have

$$\begin{aligned}
C^{\psi^\nu}_{\delta',\delta} &= \frac{2\,R^2}{T(1-\delta')^2}\left(1-e^{\frac{2R^2}{T(1-\delta')}-\frac{2R^2}{T(1-\delta)}}\right)^{-1} = \frac{2\,R^2}{T(1-\delta')^2}\left(1-\exp\left(-\frac{R^2}{T}\,\frac{\delta}{(1-\delta)(1-\delta')}\right)\right)^{-1}\\
&\overset{\text{(B.3)}}{=} \frac{2\,R^2}{T(1-\delta')^2}\left(1-\exp\left(-\frac{R^2}{T}\,\frac{1}{\Lambda(\psi_0^\nu)(1-\delta')}\right)\right)^{-1} = \frac{2\,R^2}{T(1-\delta')^2}\left(1-\exp\left(-\frac{1}{1-\delta'}\right)\right)^{-1}\\
&\leq \frac{2\,R^2}{T(1-\delta)^2}\,\frac{1}{1-e^{-1}} = \frac{(1+\Lambda(\psi_0^\nu))^2}{\Lambda(\psi_0^\nu)^2}\,\frac{R^2}{T}\,\frac{2}{1-e^{-1}} = (1+{}^{R^2}\!/_{T})^2\,\frac{2}{1-e^{-1}} \lesssim 1+{}^{R^4}\!/_{T^2} \leq {}^{R^4}\!/_{T^2}\,.
\end{aligned}$$

Similarly, we can compute

$$C^{\psi^\nu}_{\delta} = C^{\psi^\nu}_{0,\delta} = \frac{2}{1-e^{-1}}\,\frac{R^2}{T}$$

Lastly, notice that from $R^2 \geq T$ we know that $\lambda(\psi_s^\nu) \leq 0$ and it is monotone decreasing, which yields to

$$\begin{aligned}
\bar{\lambda}_{\tau_\ell} := (\inf_{s\in[0,\tau_\ell]}\lambda(\psi_s^\nu))^- = -\lambda(\psi_{\tau_\ell}^\nu) = \frac{R^2}{T^2(1-\delta')^2} - \frac{1}{T(1-\delta')} \leq \frac{R^2}{T^2(1-\delta')^2}\\
= \frac{R^2}{T^2}\left(\frac{1+\Lambda(\psi_0^\nu)}{{}^1\!/_2+\Lambda(\psi_0^\nu)}\right)^2 \leq 4\,\frac{R^2}{T^2}\,.
\end{aligned}$$

B.2. **Log-concavity of ν.** By reasoning as in Appendix A.2, if U_ν denotes the (negative) log-density of ν and we assume that $\nabla^2 U_\nu \geq \alpha_\nu$ for some $\alpha_\nu > 0$ (w.l.o.g. such that $\alpha_\nu < T^{-1}$) then we can consider

$$\lambda(\psi_s^\nu) = \frac{1}{\lambda(\psi_0^\nu)^{-1}-s} \qquad \text{where} \qquad \lambda(\psi_0^\nu) = \frac{1}{(\alpha_\psi - T^{-1})^{-1}+T} = \frac{\alpha_\nu - T^{-1}}{\alpha_\nu\,T}\,,$$

since for any $s \in [0,T]$ it holds

$$\nabla^2\psi_s^\nu \geq \frac{1}{(\alpha_\nu - T^{-1})^{-1}+T-s}\,.$$

Moreover, this further implies $\Lambda(\psi_0^\nu) = (\alpha_\nu T)^{-1}$, and since $\alpha_\nu < T^{-1}$ we are guaranteed that $\lambda(\psi_s^\nu)$ is always negative. This combined with (B.4) already gives

$$\gamma_{\tau_u} \leq \frac{\sqrt{d}}{T} \frac{1+\Lambda(\psi_0^\nu)}{\Lambda(\psi_0^\nu)} = \sqrt{d}(\alpha_\nu + T^{-1}) \,.$$

Next, by reasoning as in Appendix A.2 we have

$$\mathcal{I}^{\psi^\nu}(l,u) = (l-\lambda(\psi_0^\nu)^{-1})^2 \left(\frac{1}{l-\lambda(\psi_0^\nu)^{-1}} - \frac{1}{u-\lambda(\psi_0^\nu)^{-1}} \right) = \frac{l-\lambda(\psi_0^\nu)^{-1}}{u-\lambda(\psi_0^\nu)^{-1}} (u-l)\,, \tag{B.8}$$

and hence that for any $s \in [0,\tau_\ell]$

$$\begin{aligned} \int_s^{\tau_u} \mathcal{I}^{\psi^\nu}(s,u)^{-1/2} \mathrm{d}u &= \int_s^{\tau_u} \sqrt{\frac{u-\lambda(\psi_0^\nu)^{-1}}{s-\lambda(\psi_0^\nu)^{-1}}} \frac{1}{\sqrt{u-s}} \mathrm{d}u \leq \sqrt{\frac{\tau_u-\lambda(\psi_0^\nu)^{-1}}{s-\lambda(\psi_0^\nu)^{-1}}} \int_s^{\tau_u} \frac{1}{\sqrt{u-s}} \mathrm{d}u \\ &= 2\sqrt{\frac{\tau_u-\lambda(\psi_0^\nu)^{-1}}{s-\lambda(\psi_0^\nu)^{-1}}} \sqrt{\tau_u - s} \leq 2\sqrt{\tau_u} \sqrt{1-\tau_u \lambda(\psi_0^\nu)} = 2\sqrt{2} \frac{\sqrt{\alpha_\nu}\, T}{1+\alpha_\nu T}\,, \end{aligned}$$

and hence

$$\sup_{s\in[0,\tau_\ell]} \int_s^{\tau_u} \mathcal{I}^{\psi^\nu}(s,u)^{-1/2} \mathrm{d}u \leq \frac{2\sqrt{2}}{\sqrt{\alpha_\nu}}\,.$$

Next, from (B.1) and (B.8) we may compute $C_{\delta',\delta}^{\psi^\nu}$ and $C_\delta^{\psi^\nu} = C_{0,\delta}^{\psi^\nu}$ as

$$C_{\delta',\delta}^{\psi^\nu} = \frac{8}{\alpha_\nu T} \frac{1+\alpha_\nu T}{3+\alpha_\nu T} \leq 8\,(\alpha_\nu T)^{-1} \quad \text{and} \quad C_\delta^{\psi^\nu} = \frac{2}{\alpha_\nu T}\,.$$

Lastly, notice that $\lambda(\psi_s^\nu)$ is a negative monotone increasing sequence and hence

$$\bar{\lambda}_{\tau_\ell} \coloneqq (\inf_{s\in[0,\tau_\ell]} \lambda(\psi_s^\nu))^- = -\lambda(\psi_0^\nu) = \frac{T^{-1}-\alpha_\nu}{\alpha_\nu T}\,.$$

Let us conclude this appendix with a table summarizing the values of the constants so far computed (up to numerical prefactors).

Constant	$\Lambda(\psi_0^\nu)$	$C_{\delta',\delta}^{\psi^\nu}$	$C_\delta^{\psi^\nu}$	γ_{τ_u}	$\sup_{s\in[0,\tau_\ell]} \int_s^{\tau_u} \mathcal{I}^{\psi^\nu}(s,u)^{-1/2}\mathrm{d}u$	$\bar{\lambda}_{\tau_\ell}$
ν compact support	$R^2 T^{-1}$	$R^4 T^{-2}$	$R^2 T^{-1}$	$\sqrt{d}\, T^{-1}$	$T R^{-1}$	$R^2 T^{-2}$
ν log-concave	$\alpha_\nu^{-1} T^{-1}$	$\alpha_\nu^{-1} T^{-1}$	$\alpha_\nu^{-1} T^{-1}$	$\sqrt{d}(\alpha_\nu + T^{-1})$	$\alpha_\nu^{-1/2}$	$\alpha_\nu^{-1} T^{-2} - T^{-1}$

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