STOCHASTIC HOMOGENIZATION OF INTEGRAL FUNCTIONALS DEFINED ON MANIFOLD-VALUED SOBOLEV MAPS

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Abstract. We prove stochastic homogenization for integral functionals with integrands having p-growth, defined on Sobolev-functions taking values in a given closed C^1-submanifold of $\mathbb{R}^m$ without boundary. We thus extend previous results on relaxation and periodic homogenization. Our approach is flexible enough to also include the analysis of Dirichlet boundary conditions, the latter being non-trivial due to the lack of a fundamental estimate in the manifold-valued setting.

Keywords: stochastic homogenization, manifold-valued maps, integral functionals, p-growth
MSC 2020: 49J45, 49J55, 78A48

1. Introduction

Stochastic homogenization of integral functionals is by now a classical subject in the calculus of variations. First results in the nonlinear setting were obtained by Dal Maso and Modica in [10], where the authors derive an effective, averaged model for integral functionals of the form

$$\int_U f(x/\varepsilon,\omega,\nabla u)\,\mathrm{d}x \tag{1.1}$$

defined on Sobolev functions $u\in W^{1,p}(U)$ and where the stationary integrand f is convex in the last variable and satisfies the two-sided p-growth condition with $p\in(1,+\infty)$ of the form

$$c_1|\xi|^p\le f(\omega,x,\xi)\le c_2(1+|\xi|^p) \tag{1.2}$$

uniformly with respect to x and the realization ω. More precisely, the random functionals in (1.1) Γ-converge almost surely when $\varepsilon\to 0$ towards an autonomous integral functional

$$\int_U f_{\mathrm{hom}}(\omega,\nabla u(x))\,\mathrm{d}x, \tag{1.3}$$

and the function f_{hom} is deterministic provided one assumes also ergodicity of the integrand f (the probabilistic notions are recalled in Definition 2.6). The result was extended to the vectorial (quasiconvex) case in [19]. By now there are many contributions on homogenization where the growth condition (1.2) is weakened in various ways: for instance nonstandard (e.g. $p(x)$, (p,q) or unbounded) growth conditions [7, 14, 18, 22, 24] or degenerate p-growth (that is c_1,c_2 depend on x and $\inf c_1=0$ and $\sup c_2=\infty$) [13, 20, 21] (see also [23] for the case $p=1$).

In this contribution we follow another direction, keeping the p-growth condition (1.2), but restricting the domain of the heterogeneous functional to the space of manifold-valued Sobolev functions $W^{1,p}(U;\mathcal{M})$. Manifold-valued functions play an important role in physics and materials science, e.g., for models of micromagnetism, where typically the magnetization is assumed to take values in the unit sphere (cf. [11] and references therein) or models for liquid crystals (cf. [6]), where either also sphere-valued functions (Oseen-Frank theory) or symmetric, traceless matrices (Landau-de Gennes theory) are used as order parameters.[1] Let us briefly discuss earlier works on related problems involving manifold-valued functions. In [8] the authors study the relaxation problem for integral functionals defined on manifold-valued Sobolev maps with spatially homogeneous integrands satisfying the analogue of (1.2). The analysis was then extended to the case of periodic homogenization in [5] for $p>1$, while the case

[1] These models however are based on integral functionals with integrands also depending on $u(x)$. The generalization of our results to such a setting will be a task for the future.

$p = 1$ is treated in [4]. One of the major issues for manifold-valued Sobolev spaces is the lack (or rather the missing proof) of the so-called fundamental estimate, which allows interpolating between two functions, say u_1 and u_2, with an increase of energy that becomes small when the two functions are close in L^p. In many cases such a property yields that up to subsequences the Γ-limit exists and can be seen as a measure with respect to the set U and this in turn opens the door to a fine analysis via blow-up methods based on differentiation of measures. In the manifold-valued case, interpolating between two maps is more difficult since the standard ansatz of the form $\varphi u_1 + (1-\varphi)u_2$ for some smooth cut-off function φ in general does not preserve the constraint to lie on the manifold. In [8] the authors bypass this problem by constructing a suitable replacement of the convex combination that allows for an interpolation of the form $\Phi(u_1, u_2, \varphi)$, where again φ is a smooth cut-off function and Φ is a map depending on the manifold. However, the composition is globally well-defined only if u_1 and u_2 are close in L^∞, which is incompatible with the topology used for relaxation. Hence the authors consider a modified relaxation with respect to a notion of convergence implying the uniform convergence. This modified functional then is shown to satisfy a fundamental estimate and finally the authors prove that the modification is actually redundant. This approach was then extended to Γ-convergence in the setting of periodic homogenization in [5]. While most of the previously explained techniques could be used in a random setting as well, the subsequent local blow-up analysis uses explicitly the periodic structure and we are not able to find a corresponding argument in the stationary, ergodic setting. Therefore another approach is necessary for proving stochastic homogenization and this is the main contribution of this work.
At this point let us mention that there are already stochastic homogenization results for variational models where manifold-valued Sobolev spaces are the correct setting, the most general (to the best of our knowledge) contained in [11]. However, besides the assumptions on the manifold being more restrictive (bounded, orientable C^2-manifold with tubular neighborhood of uniform thickness),the crucial point in this work is that the energies are quadratic in the gradient variable, so that first of all correctors exist and the problem is accessible with the concept of two-scale convergence ([2] contains a similar approach for the Landau-Lifschitz equation with stationary, ergodic coefficients). In our non-quadratic and even non-convex setting this strategy is not feasible. Instead, we have to rely on the so-called tangentially homogenized multi-cell formula given for $s \in \mathcal{M}$ and $\xi \in [T_s\mathcal{M}]^d$ (the d-fold product of the tangent space at s) by

$$Tf_\infty(s,\xi) = \lim_{r\to+\infty} \frac{1}{r^d} \inf\left\{ \int_{Q_r} f(\omega, y, \xi + \nabla\phi)\, \mathrm{d}y \colon \phi \in W_0^{1,\infty}(Q_r; T_s\mathcal{M}) \right\} \tag{1.4}$$

(here $Q_r = (-r/2, r/2)^d$ and the limit is independent of ω due to ergodicity). The analysis of this formula actually takes a large portion of this work. In contrast to models without manifold constraints the formula depends on s, which then transfers to Γ-limits of the form

$$\int_U Tf_\infty(u(x), \nabla u(x))\, \mathrm{d}x.$$

While for fixed s and ξ the almost sure existence of the limit in (1.4) is a consequence of the subadditive ergodic theorem, a substantial point is to show that the limit exists except for ω belonging to a null set that is independent of s and ξ. Note that the constraint $\varphi(x) \in T_s\mathcal{M}$ is global and therefore comparing the formulas for (s,ξ) and (s',ξ') and arguing by approximation with countably many parameters requires global modifications of (almost) minimizers and a careful estimate of the change in energy.
After establishing the existence of the formula (1.4) it turns out that it is of little use for constructing recovery sequences as there is no control of the $W^{1,\infty}$-norm of small energy configurations when $r \to +\infty$. A crucial observation in this paper is the following: restricting the minimization problem in (1.4) to functions $\varphi \in W_0^{1,\infty}(Q_r; T_s\mathcal{M})$ satisfying $\|\varphi\|_{W^{1,\infty}} \le k$ for some fixed $k \in \mathbb{N}$, gives an auxiliary stochastic process which is shown to convergence almost surely for all $s \in \mathcal{M}$ and $\xi \in [T_s\mathcal{M}]^d$ to a deterministic limit, say $Tf_k(s,\xi)$. When $k \to +\infty$ we can show that this quantity converges to $Tf_\infty(s,\xi)$. Note that this asymptotic behavior involves a change of order of limits which can be

justified due to a double monotonicity once we take expectations in the formulas. As a consequence, for constructing a recovery sequence we can work with Lipschitz functions with a uniform Lipschitz constant. In this way we bypass the explicit use of periodicity exploited in [5]. We also avoid the modified Γ-limit with respect to a stronger topology. Instead, we take another approach, providing first a local construction of recovery sequences in small cubes (Lemma 4.11) and then using a covering argument to obtain the recovery sequence on general open sets (Proposition 4.12). The advantage of this ansatz is that we can (and have to) fix the boundary conditions in the local construction, which is transferred to the global construction and hence allows us to treat the Γ-convergence also under Dirichlet boundary conditions. Note that this is unclear using the other approach since it would require the boundary data to be uniformly close to a recovery sequence maybe not respecting the boundary condition and such a closeness does not follow from energy bounds. The proof of the lower bound for Γ-convergence is achieved via a blow-up argument.

Our findings also generalize the periodic homogenization result of [5] in the sense that we do not assume the manifold $\mathcal{M}$ to be connected and we only require C^1-regularity instead of smoothness.

The paper is structured as follows: in Section 2 we fix some notation, recall the probabilistic setting for stochastic homogenization and introduce the class of integrands we consider along with some preliminary results on such integrands and their modulus of continuity that will be fundamental for our analysis. We state our main result on Γ-convergence with and without Dirichlet boundary conditions in Section 3, while Section 4 contains the proof of the main results.

2. Setting of the problem and preliminary results

In this section we fix some notation employed throughout this work and introduce the relevant function spaces together with the energy functionals and their randomness under consideration.

2.1. Basic Notation.

Throughout this paper $d, m \in \mathbb{N}$ are fixed positive integers and $\mathcal{M} \subset \mathbb{R}^m$ is a closed $m_{\mathcal{M}}$-dimensional C^1-submanifold of $\mathbb{R}^m$ without boundary. For any Lebesgue measurable set $A \subset \mathbb{R}^d$ we write $|A|$ for its d-dimensional Lebesgue measure. We denote by $\mathcal{U}_0$ the class of bounded, open subsets of $\mathbb{R}^d$. The scalar product between two points $x, y \in \mathbb{R}^d$ is denoted by $x \cdot y$, while $|x| = \sqrt{x \cdot x}$ is the Euclidean norm of x. We use the notation $Y = [-1/2, 1/2)^d$ for the half-open unit cube centered at the origin. $T_s\mathcal{M}$ stands for the tangent space to $\mathcal{M}$ at a point $s \in \mathcal{M}$, $\pi_s : \mathbb{R}^m \to T_s\mathcal{M}$ is the orthogonal projection onto $T_s\mathcal{M}$ and for every $\xi = (\xi_1, \dots, \xi_d) \in \mathbb{R}^{m\times d}$ we define $\Pi_s(\xi) \in [T_s\mathcal{M}]^d$ columnwise by setting $\Pi_s(\xi) := (\pi_s(\xi_1), \dots, \pi_s(\xi_d))$.

We recall the following theorem on 'almost' nearest point projections for C^1-manifolds proven on [25, p. 121].

Theorem 2.1 (Almost nearest point projection)**.** *Let $\mathcal{M} \subset \mathbb{R}^m$ be a N-dimensional C^1-submanifold, let $\pi_s : \mathbb{R}^m \to T_s\mathcal{M}$ be the orthogonal projections as above, and $\lambda > 0$. Then there exist a family $(P_s)_{s\in\mathcal{M}}$ of $(m - N)$-dimensional subspaces and a continuous function $r : \mathcal{M} \to (0, +\infty)$ with the following properties:*

(i) $|\pi_s(v)| \leq \lambda |v|$ *for all* $v \in P_s$

(ii) Setting $Q_s = (s + P_s) \cap B_{r(s)}(s)$, these sets fill out a neighborhood U of $\mathcal{M}$ in a one-to-one way. Moreover, defining $\pi^* : U \to \mathcal{M}$ as $\pi^*(u) = s$ if $u \in Q_s$, then $\pi^* \in C^1(U; \mathcal{M})$ and

$$|\pi^*(u) - u| \leq 2 \operatorname{dist}(u, \mathcal{M}) \quad \textit{for all } u \in U.$$

It is also convenient to observe that the mapping $s \mapsto \Pi_s$ with Π_s as above is continuous.

Remark 2.2 (Orthogonal projection onto the tangent space). Let $s \in \mathcal{M}$ and $\pi_s : \mathbb{R}^m \to T_s\mathcal{M}$, $\Pi_s : \mathbb{R}^{m\times d} \to [T_s\mathcal{M}]^d$ be as above. Since $\mathcal{M}$ is a C^1-manifold, the mapping $s \mapsto \Pi_s$ is continuous. This can be seen by writing the orthogonal projection onto $T_s\mathcal{M}$ in local coordinates and using that the gradient of the local chart is continuous. Specifically, for every $s_0 \in \mathcal{M}$ we have that

$$\lim_{s\to s_0} \|\Pi_s - \Pi_{s_0}\|_{\mathrm{op}} = 0\,.$$

Moreover, for any $R > 0$ the modulus of continuity

$$\gamma_R(t) := \sup\left\{ \|\Pi_s - \Pi_{s'}\|_{\mathrm{op}} : s, s' \in \overline{B_R}\,, |s - s'| \leq t \right\} \tag{2.1}$$

converges to zero as $t \to 0$.

Given an open set $U \subset \mathbb{R}^d$ we use the standard notation $W^{1,p}(U;\mathbb{R}^m)$ for the Sobolev space with integrability exponent $p > 1$. Moreover, we set

$$W^{1,p}(U;\mathcal{M}) := \{u \in W^{1,p}(U;\mathbb{R}^m) \colon u(x) \in \mathcal{M} \ \text{ for a.e. } x \in U\}.$$

We recall that any $u \in W^{1,p}(U;\mathcal{M})$ has the property that

$$\frac{\partial u}{\partial x_i}(x) \in T_{u(x)}\mathcal{M} \ \text{ for a.e. } x \in U \text{ and every } i \in \{1, \dots, d\}, \tag{2.2}$$

where $\frac{\partial u}{\partial x_i}$ is the weak partial derivative of u with respect to x_i.
Finally, by C we denote a generic constant whose value might change every time it appears.

2.2. **Ergodic theory.** In this section, we recall some basic notions from probabilistic ergodic theory. Throughout this paper $(\Omega, \mathcal{F}, \mathbb{P})$ denotes a complete probability space. We start by defining measure-preserving group actions.

Definition 2.3 (Measure-preserving group action). *A discrete, measure-preserving, additive group action on $(\Omega, \mathcal{F}, \mathbb{P})$ is a family $(\tau_z)_{z\in\mathbb{Z}^d}$ of mappings $\tau_z : \Omega \to \Omega$ satisfying the following properties:*

1. *(measurability) τ_z is $\mathcal{F}$-measurable for every $z \in \mathbb{Z}^d$;*
2. *(invariance) $\mathbb{P}(\tau_z A) = \mathbb{P}(A)$, for every $A \in \mathcal{F}$ and every $z \in \mathbb{Z}^d$;*
3. *(group property) $\tau_0 = \mathrm{id}_\Omega$ and $\tau_{z_1+z_2} = \tau_{z_2} \circ \tau_{z_1}$ for every $z_1, z_2 \in \mathbb{Z}^d$.*

If, in addition, $(\tau_z)_{z\in\mathbb{Z}^d}$ satisfies the implication

$$\mathbb{P}((\tau_z A)\Delta A) = 0 \quad \forall\, z \in \mathbb{Z}^d \implies \mathbb{P}(A) \in \{0, 1\},$$

then it is called ergodic.

We further recall the basics of subadditive stochastic processes indexed by bounded, open sets:

Definition 2.4 (Subadditive process). *A subadditive process with respect to a discrete, measure-preserving, additive group action $(\tau_z)_{z\in\mathbb{Z}^d}$ is a function $\mu\colon \mathcal{U}_0 \times \Omega \to \mathbb{R}$ satisfying the following properties:*

1. *(measurability) for every $B \in \mathcal{U}_0$ the function $\omega \mapsto \mu(B, \omega)$ is $\mathcal{F}$-measurable;*
2. *(stationarity) for every $B \in \mathcal{U}_0$ and $z \in \mathbb{Z}^d$ we have $\mu(B + z, \omega) = \mu(B, \tau_z(\omega))$ for a.e. $\omega \in \Omega$;*
3. *(subadditivity) for a.e. $\omega \in \Omega$, every $B \in \mathcal{U}_0$ and every finite family $(B_i)_{i\in I} \subset \mathcal{U}_0$ of pairwise disjoint sets with $B_i \subset B$ and $|B \setminus \cup_{i\in I} B_i| = 0$ we have*
$$\mu(B, \omega) \leq \sum_{i\in I} \mu(B_i, \omega);$$

Our analysis heavily relies on the following version of the subadditive ergodic theorem by Dal Maso and Modica [10, Proposition 1], building on the subadditive ergodic theorem by Akcoglu and Krengel [1].

Theorem 2.5 (The subadditive ergodic theorem). *Let $\mu : \mathcal{U}_0 \times \Omega \to \mathbb{R}$ be a subadditive process with respect to a discrete, measure-preserving group action $(\tau_z)_{z\in\mathbb{Z}^d}$ and assume that there exists $C > 0$ such that for all $B \in \mathcal{U}_0$ and a.e. $\omega \in \Omega$*

$$0 \leq \mu(B, \omega) \leq C|B|. \tag{2.3}$$

Then there exists an $\mathcal{F}$-measurable function $\phi : \Omega \to \mathbb{R}$ such that $\mathbb{P}$-almost surely, for any cube $Q \subset \mathbb{R}^d$, it holds that

$$\lim_{r\to+\infty} \frac{1}{|rQ|}\mu(rQ, \omega) = \phi(\omega).$$

If the group action $(\tau_z)_{z\in\mathbb{Z}^d}$ is ergodic, then ϕ is constant.

2.3. **Analytic framework.** We now define the random functionals considered in this paper.

Definition 2.6 (Stationary random integrands)**.** *We say that* $f\colon \Omega\times\mathbb{R}^d\times\mathbb{R}^{m\times d}\to[0,+\infty)$ *is an admissible random integrand, if* f *is* $\mathcal{F}\otimes\mathcal{L}(\mathbb{R}^d)\otimes\mathcal{B}(\mathbb{R}^{m\times d})$ *measurable and there exist constants* $c_1,c_2>0$ *and* $p>1$ *such that for* $\mathbb{P}$*-a.e.* $\omega\in\Omega$ $f(\omega,\cdot\,,\cdot\,)$ *is a Carathéodory-function satisfying*

$$c_1|\xi|^p\leq f(\omega,x,\xi)\leq c_2(1+|\xi|^p) \tag{2.4}$$

for a.e. $x\in\mathbb{R}^d$ *and all* $\xi\in\mathbb{R}^{m\times d}$*. Moreover, we say that* f *is stationary, if there exists a measure preserving group action* $(\tau_z)_{z\in\mathbb{Z}^d}$ *such that for almost all* $\omega\in\Omega$ *we have*

$$f(\omega,x+z,\xi)=f(\tau_z\omega,x,\xi) \tag{2.5}$$

for almost all $x\in\mathbb{R}^d$*, all* $z\in\mathbb{Z}^d$ *and all* $\xi\in\mathbb{R}^{m\times d}$*. If in addition* $(\tau_z)_{z\in\mathbb{Z}^d}$ *is ergodic, then* f *is called ergodic.*
We write $\Omega_f\subset\Omega$ *for the set of full probability where* $f(\omega,\cdot\,,\cdot\,)$ *satisfies all the above properties.*

A crucial quantity in our analysis will be the modulus of continuity associated to a stationary random integrand:

Definition 2.7 (Integrated modulus of continuity)**.** *Let* $f:\Omega\times\mathbb{R}^d\times\mathbb{R}^{m\times d}\to[0,+\infty)$ *be as in Definition 2.6. For any* $R>0$ *and* $\omega\in\Omega_f$ *we define an integrated modulus of continuity* $\alpha_R(\omega,\cdot):[0,+\infty)\to[0,+\infty)$*,* $t\mapsto\alpha_R(\omega,t)$ *by setting*

$$\alpha_R(\omega,t):=\int_Y\sup\left\{\left|f(\omega,y,\xi)-f(\omega,y,\xi')\right|:\xi,\xi'\in\overline{B_R},\ |\xi-\xi'|\leq t\right\}\mathrm{d}y\,.$$

Moreover, we extend $\alpha_R(\cdot\,,t)$ *to* Ω *by setting* $\alpha_R(\omega,t):=0$ *for all* $\omega\in\Omega\setminus\Omega_f$ *and* $R,t>0$*.*

Remark 2.8. We observe that thanks to (2.4) the modulus of continuity $\alpha_R(\omega,t)$ satisfies

$$\alpha_R(\omega,t)\leq 2c_2(1+R^p) \tag{2.6}$$

for all $\omega\in\Omega_f$ and $R,t>0$, while for $\omega\in\Omega\setminus\Omega_f$ the above estimate trivially holds.

The following lemma justifies the notion of modulus of continuity.

Lemma 2.9. *Let* $f:\Omega\times\mathbb{R}^d\times\mathbb{R}^{m\times d}\to[0,+\infty)$ *be as in Definition 2.6 and for every* $R>0$ *and* $\omega\in\Omega$ *let* $\alpha_R(\omega,\cdot\,)$ *be as in Definition 2.7. Then the mapping* $\omega\mapsto\alpha_R(\omega,t)$ *is* $\mathcal{F}$*-measurable for every* $R>0$ *and every* $t>0$ *and*

$$\lim_{t\to 0}\alpha_R(\cdot\,,t)=0 \tag{2.7}$$

pointwise and in $L^1(\Omega)$*.*

Proof. To establish the measurability, we start observing that it suffices to show that $\omega\mapsto\alpha_R(\omega,t)$ is $\mathcal{F}$-measurable on Ω_f. For every $\xi,\xi'\in\overline{B_R}$ the mapping

$$(\omega,y)\mapsto\left|f(\omega,y,\xi)-f(\omega,y,\xi')\right|$$

is $\mathcal{F}\otimes\mathcal{L}(\mathbb{R}^d)$-measurable. Moreover, for $\omega\in\Omega_f$ it suffices to consider the supremum over countably many ξ,ξ', so that the expression inside the integral defining $\alpha_R(\omega,t)$ is $\mathcal{F}\otimes\mathcal{L}(\mathbb{R}^d)$-measurable. Thanks to Fubini's theorem this implies the $\mathcal{F}$-measurability of $\omega\mapsto\alpha_R(\omega,t)$ on Ω_f and we conclude.

Next we show the pointwise convergence in (2.7). For $\omega\in\Omega\setminus\Omega_f$ this follows directly from the definition. Let now $\omega\in\Omega_f$; then for almost every $y\in Y$ the function $\xi\mapsto f(\omega,y,\xi)$ is continuous and hence uniformly continuous on the compact set $\overline{B_R}$. In particular, the integrand defining $\alpha_R(\omega,t)$ converges to zero as $t\to 0$ for almost every $y\in Y$. Finally, (2.6) allows us to apply the dominated convergence theorem to obtain the pointwise convergence in (2.7). The L^1-convergence is again a consequence of (2.6) and the dominated convergence theorem for the expectation. □

In the proof of the main result we will encounter weighted sums of the modulus of continuity that we can control with the help of the next result.

Lemma 2.10. *Let $f:\Omega\times\mathbb{R}^d\times\mathbb{R}^{m\times d}\to[0,+\infty)$ be as in Definition 2.6. Then $\mathbb{P}$-almost surely for every cube $Q\subset\mathbb{R}^d$, every $t>0$ and every $R>0$,*

$$0\leq\limsup_{r\to+\infty}\frac{1}{|rQ|}\sum_{\substack{z\in\mathbb{Z}^d\\(Y+z)\cap rQ\neq\emptyset}}\alpha_R(\tau_z\omega,t)\leq\inf_{R'>R,t'>t}\mathbb{E}[\alpha_{R'}(\cdot,t')].\tag{2.8}$$

In particular,

$$\lim_{t\to0}\limsup_{r\to+\infty}\frac{1}{|rQ|}\sum_{\substack{z\in\mathbb{Z}^d\\(Y+z)\cap rQ\neq\emptyset}}\alpha_R(\tau_z\omega,t)=0.\tag{2.9}$$

Proof. For the moment we consider arbitrary $R,t\in\mathbb{Q}_+$. Let us define a stochastic process indexed by bounded, open sets $B\in\mathcal{U}_0$ via

$$\mu_{t,R}(B,\omega)=\sum_{z\in\mathbb{Z}^d}|B\cap(Y+z)|\alpha_R(\tau_z\omega,t).$$

From Lemma 2.9 we infer that $\mu_{t,R}(B,\cdot)$ is $\mathcal{F}$-measurable. Moreover, via a change of indices one verifies that for any $z\in\mathbb{Z}^d$ we have

$$\begin{aligned}\mu_{t,R}(B+z,\omega)&=\sum_{z'\in\mathbb{Z}^d}|(B+z)\cap(Y+z')|\alpha(\tau_{z'}\omega,t)\\&=\sum_{z'-z\in\mathbb{Z}^d}|B\cap(Y+z'-z)|\alpha(\tau_{z'-z}\tau_z\omega,t)=\mu_{t,R}(B,\tau_z\omega),\end{aligned}$$

so that $\mu_{t,R}$ is stationary. Next, if $B\in\mathcal{U}_0$ and $(B_i)_{i\in I}\subset\mathcal{U}_0$ is a finite family of pairwise disjoint sets that cover B up to a null set, then by the additivity of the Lebesgue-measure

$$\mu_{t,R}(B,\omega)=\sum_{z\in\mathbb{Z}^d}\sum_{i\in I}|B_i\cap(Y+z)|\alpha_R(\tau_z\omega,t)=\sum_{i\in I}\mu_{t,R}(B_i,\omega),$$

which yields (sub)additivity. Finally, using the bound (2.6) and the non-negativity of $\alpha_R(\cdot,t)$, we obtain the bound

$$0\leq\mu_{t,R}(B,\omega)\leq 2c_2(1+R^p)|B|.$$

Hence we are in a position to use Theorem 2.5 and, since in the additive case the limit is known to agree with the expectation, conclude that there exists a set $\Omega_+\subset\Omega$ of full probability such that for all $\omega\in\Omega_+$ and every cube $Q\subset\mathbb{R}^d$ it holds that

$$\lim_{r\to+\infty}\frac{1}{|rQ|}\sum_{z\in\mathbb{Z}^d}|rQ\cap(Y+z)|\alpha_R(\tau_z\omega,t)=\mathbb{E}[\alpha_R(\cdot,t)]\tag{2.10}$$

for all $t,R\in\mathbb{Q}_+$.

As a next step, we extend the convergence to the sum in (2.8), which one the hand does not take into account the measure of $|rQ\cap(Y+z)|$ and on the other hand might count terms such that $|rQ\cap(Y+z)|=0$. Consider first $z\in\mathbb{Z}^d$ such that $(Y+z)\cap rQ\neq\emptyset$, but $|(Y+z)\cap rQ|\neq1$. Since $|Y+z|=1$, this implies that $(Y+z)\cap\partial rQ\neq\emptyset$ and therefore $Y+z\in\partial rQ+[-1,1]^d$. Second, if $0<|rQ\cap(Y+z)|<1$, the same conclusion holds. Recalling again the bound (2.6), it follows that

$$\begin{aligned}\Bigg|\frac{1}{|rQ|}\sum_{\substack{z\in\mathbb{Z}^d\\(Y+z)\cap rQ(x)\neq\emptyset}}\alpha_R(\tau_z\omega,t)-\mathbb{E}[\alpha_R(\cdot,t)]\Bigg|\leq&\left|\frac{1}{|rQ|}\mu_{t,R}(rQ,\omega)-\mathbb{E}[\alpha_R(\cdot,t)]\right|\\&+C(1+R^p)|rQ|^{-1}|(\partial rQ(x)+[-1,1]^d)|\end{aligned}$$

The term in the last line vanishes as $r\uparrow+\infty$. Hence, taking into account (2.10), we infer that

$$\lim_{r\to+\infty}\frac{1}{|rQ|}\sum_{\substack{z\in\mathbb{Z}^d\\(Y+z)\cap rQ(x)\neq\emptyset}}\alpha_R(\tau_z\omega,t)=\mathbb{E}[\alpha_R(\cdot,t)]$$

for all $\omega \in \Omega_+$ and $t, R \in \mathbb{Q}_+$. Since $\alpha_R(\cdot, t) \geq 0$, the estimate (2.8) is a consequence of the monotonicity of $\alpha_R(\omega, t)$ with respect to R and t and the density of rational numbers. Formula (2.9) further follows from the L^1-convergence $\alpha_R(\cdot, t) \to 0$ as $t \to 0$ (cf. Lemma 2.9). $\square$

3. Statement of the main results

We are now in a position to state the main results of this paper Given a random integrand f as in Definition 2.6, an open and bounded set $U \subset \mathbb{R}^d$ and a parameter $\varepsilon > 0$ we consider the random integral functionals $F_\varepsilon(\omega)(\cdot\,, U)\colon L^p(U;\mathbb{R}^m) \to [0, +\infty]$ given by

$$F_\varepsilon(\omega)(u, U) := \begin{cases} \displaystyle\int_U f(\omega, \tfrac{x}{\varepsilon}, \nabla u)\,\mathrm{d}x & \text{if } u \in W^{1,p}(U;\mathcal{M}), \\ +\infty & \text{otherwise in } L^p(U;\mathbb{R}^m). \end{cases} \tag{3.1}$$

In all that follows $\varepsilon > 0$ varies in a strictly decreasing family of positive parameters converging to 0.

Theorem 3.1. *Let f be an admissible ergodic random integrand in the sense of Definition 2.6 and let F_ε be as in (3.1). Assume that f is stationary with respect to a discrete, measure-preserving, ergodic group action $(\tau_z)_{z\in\mathbb{Z}^d}$. There exists a set $\Omega' \subset \Omega$ with $\mathbb{P}(\Omega') = 1$ such that for all $\omega \in \Omega'$ and for every $U \subset \mathbb{R}^d$ open and bounded the functionals $F_\varepsilon(\omega)(\cdot\,, U)$ Γ-converge with respect to the strong convergence in $L^p(U;\mathbb{R}^m)$ to the functional $F_{\hom}(\cdot\,, U)\colon L^p(U;\mathbb{R}^m) \to [0, +\infty]$ given by*

$$F_{\hom}(u, U) := \begin{cases} \displaystyle\int_U Tf_\infty(u, \nabla u)\,\mathrm{d}x & \text{if } u \in W^{1,p}(U;\mathcal{M}), \\ +\infty & \text{otherwise in } L^p(U;\mathbb{R}^m), \end{cases} \tag{3.2}$$

where for every $s \in \mathcal{M}$ and every $\xi \in [T_s\mathcal{M}]^d$ the integrand $Tf_\infty(s, \xi)$ is given by

$$Tf_\infty(s, \xi) := \lim_{r\to+\infty} \frac{1}{r^d} \inf\left\{ \int_{Q_r} f(\omega, y, \xi + \nabla\phi)\,\mathrm{d}y \colon \phi \in W_0^{1,\infty}(Q_r; T_s\mathcal{M}) \right\}. \tag{3.3}$$

In particular, the limit in (3.3) exists and is independent of $\omega \in \Omega'$.

We can also consider the functionals $F_\varepsilon(\omega)(\cdot, D)$ restricted to functions $u \in W^{1,p}(D;\mathcal{M})$ such that $u = g$ on ∂D in the sense of traces whenever D is sufficiently regular to admit a trace operator on ∂D (say, e.g., D having Lipschitz boundary). We do not bother with the precise structure of the trace space but just consider boundary data $g \in W^{1,p}(D;\mathcal{M})$, i.e., boundary data that are attained by some Sobolev-map. Let us set $F_{\varepsilon,g}(\omega)(\cdot\,, D)\colon L^p(U;\mathbb{R}^m) \to [0, +\infty]$ as

$$F_{\varepsilon,g}(\omega)(u, D) := \begin{cases} \displaystyle\int_D f(\omega, \tfrac{x}{\varepsilon}, \nabla u)\,\mathrm{d}x & \text{if } u \in W^{1,p}(U;\mathcal{M}) \text{ and } u = g \text{ on } \partial D, \\ +\infty & \text{otherwise in } L^p(U;\mathbb{R}^m). \end{cases} \tag{3.4}$$

Theorem 3.2. *Let f, F_ε and $\omega \in \Omega'$ be as in Theorem 3.1. Assume that $g \in W^{1,p}(D;\mathcal{M})$ with $D \subset \mathbb{R}^d$ a bounded, open set with Lipschitz boundary. Then the functionals $F_{\varepsilon,g}(\omega)(\cdot, D)$ Γ-converge with respect to the strong convergence in $L^p(D;\mathbb{R}^m)$ to the functional $F_{\hom,g}(\cdot, D) : L^p(D;\mathbb{R}^m) \to [0, +\infty]$ given by*

$$F_{\hom,g}(u, U) := \begin{cases} \displaystyle\int_U Tf_\infty(u, \nabla u)\,\mathrm{d}x & \text{if } u \in W^{1,p}(U;\mathcal{M}) \text{ and } u = g \text{ on } \partial D, \\ +\infty & \text{otherwise in } L^p(U;\mathbb{R}^m), \end{cases} \tag{3.5}$$

where Tf_∞ is as in Theorem 3.1. Moreover, if $(u_\varepsilon)_{\varepsilon>0}$ is such that $\sup_{\varepsilon\in(0,1)} F_{\varepsilon,g}(\omega)(u_\varepsilon, D) < +\infty$, then up to a subsequence $u_\varepsilon \rightharpoonup u$ in $W^{1,p}(D;\mathbb{R}^m)$ for some $u \in W^{1,p}(D;\mathcal{M})$ with $u = g$ on ∂D.

4. Proof of the main results

4.1. **Existence of the homogenized integrand.** As a preliminary step towards the proof of Theorem 3.1 we prove that the limit defining Tf_∞ in (3.3) exists almost surely. Moreover, we provide an alternative characterization of Tf_∞ in terms of a more restricted class of minimization problems. In doing so we will make use of the following version of the Decomposition Lemma [17, Lemma 1.2].

Lemma 4.1 (Decomposition Lemma preserving boundary conditions). *Let $D \subset \mathbb{R}^d$ be open, bounded, and with Lipschitz boundary. Let $V \subset \mathbb{R}^m$ be a linear subspace and $u_0 \in W^{1,p}(D,V)$. Let finally $(u_n) \subset u_0 + W_0^{1,p}(D,V)$ with $\sup_{n\in\mathbb{N}} \|\nabla u_n\|_{L^p} < +\infty$. Then there exist a subsequence (n_k) and a sequence $(\tilde{u}_k) \subset u_0 + W_0^{1,\infty}(D,V)$ such that $|\nabla \tilde{u}_k|^p$ is equi-integrable and*

$$\left|\left\{x \in D \colon u_{n_k}(x) \neq \tilde{u}_k(x) \text{ or } \nabla u_{n_k}(x) \neq \nabla \tilde{u}_k(x)\right\}\right| \to 0$$

as $k \to \infty$.

Proof. Up to a rotation we can assume that $V = \mathbb{R}^\ell$ with $\ell = \dim V$. Thus, without preserving the boundary condition the statement follows from [17, Lemma 1.2], noting that due to Poincaré's inequality the sequence u_n is bounded in $W^{1,p}(D;V)$. The proof of this lemma relies on a Lipschitz truncation argument (see [17, Lemma 4.1]). It is in general not clear whether this Lipschitz truncation preserves boundary conditions. However, if instead one uses the Lipschitz truncation provided in [12, Theorem 13], then Step 1 in the proof of [17, Lemma 1.2] yields the claim for $u_0 = 0$. Finally, the case of a general boundary datum $u_0 \in W^{1,p}(D;V)$ follows by considering the sequence $(u_n - u_0) \subset W_0^{1,p}(D;V)$. □

In order to analyze the limit in (3.3) we use the subadditive ergodic Theorem 2.5. To this end, we need to define suitable stochastic processes indexed by bounded, open sets. For every $B \in \mathcal{U}_0$, every $k \in \mathbb{N}$, and every $s \in \mathcal{M}$ we set

$$\operatorname{Adm}_k^s(B) := \left\{\phi \in \operatorname{Lip}(\mathbb{R}^d; T_s\mathcal{M}) \colon \|\phi\|_{W^{1,\infty}(\mathbb{R}^d)} \leq k,\ \phi \equiv 0 \text{ on } \mathbb{R}^d \setminus B\right\}. \tag{4.1}$$

We also set $\operatorname{Adm}_\infty^s(U) := \{\phi \in \operatorname{Lip}(\mathbb{R}^d; T_s\mathcal{M}) : \phi \equiv 0 \text{ on } \mathbb{R}^d \setminus B\}$. For $k \in \mathbb{N} \cup \{+\infty\}$ and $(s,\xi) \in \mathcal{M} \times [T_s\mathcal{M}]^d$ we then define $\mu_k^{s,\xi} : \mathcal{U}_0 \times \Omega \to \mathbb{R}$ by

$$\mu_k^{s,\xi}(B,\omega) := \inf\left\{\int_B f(\omega, x, \xi + \nabla\phi)\,\mathrm{d}x \colon \phi \in \operatorname{Adm}_k^s(B)\right\}. \tag{4.2}$$

Remark 4.2 (Alternative Minimization Problems). Consider the auxiliary function $\hat{f} : \Omega \times \mathbb{R}^d \times \mathcal{M} \times \mathbb{R}^{m\times d} \to [0,+\infty)$ given by

$$\hat{f}(\omega, x, s, \xi) := f\big(\omega, x, \Pi_s\xi\big) + |\xi - \Pi_s\xi|^p. \tag{4.3}$$

Observe that $\hat{f}(\omega, x, s, \xi) = f(\omega, x, \xi)$ for every $\xi \in [T_s\mathcal{M}]^d$, every $\omega \in \Omega$, and all $x \in \mathbb{R}^d$. Moreover, for every $(s,\xi) \in \mathcal{M} \times [T_s\mathcal{M}]^d$, every $B \in \mathcal{U}_0$, and every $\omega \in \Omega$ it holds that

$$\mu_\infty^{s,\xi}(B,\omega) = \inf\left\{\int_B \hat{f}(\omega, x, s, \xi + \nabla\phi) \colon \phi \in \operatorname{Lip}(\mathbb{R}^d;\mathbb{R}^m),\ \phi \equiv 0 \text{ on } \mathbb{R}^d \setminus B\right\} \tag{4.4}$$

and

$$\mu_k^{s,\xi}(B,\omega) = \inf\left\{\int_B \hat{f}(\omega, x, s, \xi + \nabla\phi) \colon \phi \in \operatorname{Lip}(\mathbb{R}^d;\mathbb{R}^m),\ \|\phi\|_{W^{1,\infty}(\mathbb{R}^d)} \leq k,\ \phi \equiv 0 \text{ on } \mathbb{R}^d \setminus B\right\} \tag{4.5}$$

for every $k \in \mathbb{N}$. To show this, we first observe that since $\operatorname{Adm}_k^s(B) \subset \operatorname{Lip}(\mathbb{R}^d;\mathbb{R}^m)$ for all $k \in \mathbb{N} \cup \{\infty\}$ and $\hat{f}(\omega, x, s, \xi) = f(\omega, x, s, \xi)$ for all $\xi \in [T_s\mathcal{M}]^d$, all $\omega \in \Omega$ and all $x \in \mathbb{R}^d$, we have that $\mu_\infty^{s,\xi}(B,\omega)$ and $\mu_k^{s,\xi}(B,\omega)$ are greater or equal than the right-hand sides in (4.4) and (4.5), respectively. To show that also the opposite inequality holds true, for an arbitrary competitor $\phi \in \operatorname{Lip}(\mathbb{R}^d;\mathbb{R}^m)$ with $\phi \equiv 0$ on $\mathbb{R}^d \setminus B$ we define $\psi := \pi_s \circ \phi$ as the composition of ϕ with the orthogonal projection π_s onto $T_s\mathcal{M}$. Then $\psi \in \operatorname{Lip}(\mathbb{R}^d; T_s\mathcal{M})$ and $\psi \equiv 0$ on $\mathbb{R}^d \setminus B$. Moreover, since π_s is an orthogonal projection, we have that $\|\pi_s\|_{\mathrm{op}} = 1$. Together with the chain rule this implies that $\|\psi\|_{W^{1,\infty}(\mathbb{R}^d)} \leq \|\phi\|_{W^{1,\infty}(\mathbb{R}^d)}$. Hence, $\psi \in \operatorname{Adm}_k^s(B)$ for $k = \infty$ or $k \in \mathbb{N}$ if ϕ is a competitor for the right-hand side of (4.4) or (4.5), respectively. Finally, we have that

$$f(\omega, x, \nabla\psi) = f(\omega, x, \Pi_s\nabla\phi) \leq \hat{f}(\omega, x, s, \nabla\phi)\,.$$

This yields the opposite inequalities by integration and passing to the infimum over all admissible ϕ. Note that the term $|\xi - \Pi_s\xi|^p$ is only added to ensure that the integrands $\hat{f}$ satisfy the two-sided p-growth condition

$$2^{1-p}\min\{c_1, 1\}|\xi|^p \leq \hat{f}(\omega, x, s, \xi) \leq (c_2 + 1)|\xi|^p\,. \tag{4.6}$$

This in turn is crucial to obtain a fundamental estimate for the corresponding functionals that we will use in the proof of the liminf-inequality.

Remark 4.3. Let $(s,\xi) \in \mathcal{M} \times [T_s\mathcal{M}]^d$ be arbitrary. By definition, we have that

$$\mu_\infty^{s,\xi}(B,\omega) \leq \mu_{k+1}^{s,\xi}(B,\omega) \leq \mu_k^{s,\xi}(B,\omega)$$

for all $\omega \in \Omega$, $k \in \mathbb{N}$, $B \in \mathcal{U}_0$. This in turn implies that

$$\mu_\infty^{s,\xi}(B,\omega) \leq \inf_{k\in\mathbb{N}} \mu_k^{s,\xi}(B,\omega) = \lim_{k\to\infty} \mu_k^{s,\xi}(B,\omega)\,. \tag{4.7}$$

On the contrary, for any $\eta > 0$ we can find $\phi_\eta \in \mathrm{Adm}_\infty^s(B)$ with $\int_B f(\omega, x, \xi + \nabla\phi_\eta)\,\mathrm{d}x < \mu_\infty^{s,\xi}(B,\omega) + \eta$. Since $\phi_\eta \in \mathrm{Adm}_k^s(B)$ for any $k \in \mathbb{N}$ with $k \geq \|\phi_\eta\|_{W^{1,\infty}(\mathbb{R}^d)}$ (the latter being finite since B is bounded), we deduce that

$$\inf_{k\in\mathbb{N}} \mu_k^{s,\xi}(B,\omega) \leq \int_B f(\omega, x, \xi + \nabla\phi_\eta)\,\mathrm{d}x < \mu_\infty^{s,\xi}(B,\omega) + \eta\,.$$

Together with (4.7) and the arbitrariness of $\eta > 0$ this implies that

$$\mu_\infty^{s,\xi}(B,\omega) = \inf_{k\in\mathbb{N}} \mu_k^{s,\xi}(B,\omega) = \lim_{k\to\infty} \mu_k^{s,\xi}(B,\omega)\,. \tag{4.8}$$

As a next step we verify that the quantities $\mu_k^{s,\xi}$ with $k \in \mathbb{N} \cup \{\infty\}$ satisfy the assumptions of Theorem 2.5.

Lemma 4.4. *Let $f : \Omega \times \mathbb{R}^d \times \mathbb{R}^{m\times d}$ be an admissible, ergodic random integrand in the sense of Definition 2.6. For any $k \in \mathbb{N} \cup \{\infty\}$ and $(s,\xi) \in \mathcal{M} \times [T_s\mathcal{M}]^d$ the function $\mu_k^{s,\xi} : \mathcal{U}_0 \times \Omega \to \mathbb{R}$ defined as above is a subadditive process that is stationary with respect to the group action $(\tau_z)_{z\in\mathbb{Z}^d}$. Moreover, for $\omega \in \Omega_f$ it holds that*

$$|\mu_k^{s,\xi}(B,\omega)| \leq c_2(1 + |\xi|^p)|B| \tag{4.9}$$

for any $k \in \mathbb{N} \cup \{\infty\}$, $B \in \mathcal{U}_0$, and $(s,\xi) \in \mathcal{M} \times [T_s\mathcal{M}]^d$.

Proof. Let $k \in \mathbb{N} \cup \{\infty\}$ and $(s,\xi) \in \mathcal{M} \times [T_s\mathcal{M}]^d$ be fixed. We verify the defining properties of a subadditive process in several steps.

Step 1. Measurability. Due to (4.8) it suffices to prove the measurability for all $k \in \mathbb{N}$ as the pointwise limit of a sequence of measurable functions remains measurable. Therefore, let us consider $k \in \mathbb{N}$. We equip $\mathrm{Adm}_k^s(B)$ with the $W^{1,p}(\mathbb{R}^d)$-norm, which makes sense since B is a bounded set. In this way $\mathrm{Adm}_k^s(B)$ becomes a separable, complete metric space. Indeed, separability can be seen from identifying $\mathrm{Adm}_k^s(B)$ as a subset of the separable space $W^{1,p}(\mathbb{R}^d;\mathbb{R}^m)$, while completeness comes from the fact that the constraint $\phi \equiv 0$ on $\mathbb{R}^d \setminus B$ is closed under strong convergence in $W^{1,p}(\mathbb{R}^d)$ and from the weak*-lower semicontinuity of the L^∞-norm, which together with the pointwise convergence (via the Arzela-Ascoli theorem) ensures that any limit is also Lipschitz. Due to the continuity of f in the last variable and the growth condition (2.4), the functional

$$\Omega \times \mathrm{Adm}_k^s(B) \ni (\omega,\phi) \mapsto \int_B f(\omega, x, \xi + \nabla\phi)\,\mathrm{d}x$$

is continuous with respect to strong $W^{1,p}$-convergence of ϕ, while the joint measurability of f and Fubini's theorem yield the $\mathcal{F}$-measurability with respect to ω. It is well-known that these properties imply the joint measurability when we equip $\mathrm{Adm}_k^s(B)$ with its Borel σ-algebra. In order to show the measurability of the infimum, we rely on the measurable projection theorem: for every $t\in\mathbb{R}$ we know that

$$\left\{(\omega,\phi)\in\Omega\times\mathrm{Adm}_k^s(B):\int_B f(\omega,x,\xi+\nabla\phi)\,\mathrm{d}x<t\right\}\in\mathcal{F}\otimes\mathcal{B}(\mathrm{Adm}_k^s(B)). \tag{4.10}$$

By assumption $(\Omega,\mathcal{F},\mathbb{P})$ is a complete probability space. Since $\mathrm{Adm}_k^s(B)$ is a complete, separable, metric space, the projection theorem [16, Theorem 1.136] yields the $\mathcal{F}$-measurability of the projection of the set in (4.10) onto Ω. Therefore

$$\left\{\omega\in\Omega:\inf_{\phi\in\mathrm{Adm}_k^s(B)}\int_B f(\omega,x,\xi+\nabla\phi)\,\mathrm{d}x=\mu_k^{s,\xi}(B,\omega)<t\right\}\in\mathcal{F},$$

which proves the $\mathcal{F}$-measurability of $\mu_k^{s,\xi}(B,\cdot)$.

Step 2. Stationarity. Let $B\in\mathcal{U}_0$ and $z\in\mathbb{Z}^d$ be arbitrary and let $\omega\in\Omega_f$. For any $\phi\in\mathrm{Adm}_k^s(B)$ define $\phi_z\in\mathrm{Adm}_k^s(B+z)$ via $\phi_z(x):=\phi(x-z)$. The stationarity of f ensures that

$$\int_B f(\tau_z\omega,x,\xi+\nabla\phi)\,\mathrm{d}x=\int_{B+z}f(\omega,x,\xi+\nabla\phi_z)\,\mathrm{d}x\,.$$

Passing to the infimum over all such ϕ and using a symmetric argument we obtain that $\mu_k^{s,\xi}(B+z,\omega)=\mu_k^{s,\xi}(B,\tau_z\omega)$, which is the stationarity of $\mu_k^{s,\xi}$.

Step 3. Subadditivity. Let $B\in\mathcal{U}_0$ and $(B_i)_{i\in I}\subset\mathcal{U}_0$ be a finite family of pairwise disjoint subsets of B that cover the latter up to a null set. Fix further $\omega\in\Omega$ and let $\eta>0$ be arbitrary. Consider $\phi_i\in\mathrm{Adm}_k^s(B_i)$ such that

$$\int_{B_i}f(\omega,x,\xi+\nabla\phi_i)\,\mathrm{d}x<\mu_k^{s,\xi}(B_i,\omega)+\frac{\eta}{\#I}\,. \tag{4.11}$$

Let us define $\phi\in\mathrm{Lip}(\mathbb{R}^d;T_s\mathcal{M})$ as $\phi:=\sum_{i\in I}\phi_i$. Since $\phi_i\equiv 0$ on $\mathbb{R}^d\setminus B_i$, we have that $\phi\equiv 0$ on $\mathbb{R}^d\setminus(\cup_{i\in I}B_i)$ and by continuity also on $\mathbb{R}^d\setminus B$. Since the $B_i's$ are pairwise disjoint, it further follows that $\|\phi\|_{W^{1,\infty}(\mathbb{R}^d)}=\max_{i\in I}\{\|\phi_i\|_{W^{1,\infty}(\mathbb{R}^d)}\}$, from which we deduce that $\phi\in\mathrm{Adm}_k^s(B)$. Hence

$$\mu_k^{s,\xi}(B,\omega)\leq\int_B f(\omega,x,\xi+\nabla\phi)\,\mathrm{d}x=\sum_{i\in I}\int_{B_i}f(\omega,x,\xi+\nabla\phi_i)\overset{(4.11)}{\leq}\sum_{i\in I}\mu_k^{s,\xi}(B_i,\omega)+\eta.$$

The arbitrariness of η then implies the subadditivity of the processes.

Step 4. Boundedness. For $U\in\mathcal{U}_0$ and $\omega\in\Omega_f$ the claimed bound (4.9) is a consequence of the non-negativity of f and testing the zero function in the minimization problem defining $\mu_k^{s,\xi}$ and then inserting the upper bound (2.4). □

The previous lemma allows us to apply Theorem 2.5 to the processes $\mu_k^{s,\xi}$, but since we need a common set of full probability such that the convergence holds, we first prove a stability result with respect to the parameters s and ξ which we can use to extend to convergence from countably many subsets to the whole range of parameters. Note that this auxiliary result is only used for the case of finite $k\in\mathbb{N}$. For $k=\infty$ we will argue in a different way.

Lemma 4.5. *Let $f:\Omega\times\mathbb{R}^d\times\mathbb{R}^{m\times d}$ be an admissible, ergodic random integrand in the sense of Definition 2.6. For any $k\in\mathbb{N}$ and $(s,\xi)\in\mathcal{M}\times[T_s\mathcal{M}]^d$ let $\mu_k^{s,\xi}$ be as in (4.2). Let $\omega\in\Omega_f$; then we have that*

$$\left|\mu_k^{s,\xi}(B,\omega)-\mu_k^{s',\xi'}(B,\omega)\right|\leq\sum_{\substack{z\in\mathbb{Z}^d\\(Y+z)\cap B\neq\emptyset}}\alpha_{k+|\xi|+|\xi'|}\big(\tau_z\omega,k\,\|\Pi_s-\Pi_{s'}\|_{\mathrm{op}}+|\xi-\xi'|\big) \tag{4.12}$$

for all $(s,\xi)\in\mathcal{M}\times[T_s\mathcal{M}]^d$ and $(s',\xi')\in\mathcal{M}\times[T_{s'}\mathcal{M}]^d$, and all $B\in\mathcal{U}_0$.

Proof. Fix $\omega \in \Omega_f$ and $k \in \mathbb{N}$. Let $\eta > 0$ be arbitrary and for $(s,\xi) \in \mathcal{M} \times [T_s\mathcal{M}]^d$, $B \in \mathcal{U}_0$ fixed, let $\phi \in \mathrm{Adm}_k^s(B)$ be such that

$$\int_B f(\omega, x, \xi + \nabla\phi)\,\mathrm{d}x \le \mu_k^{s,\xi}(B,\omega) + \eta\,. \tag{4.13}$$

Let now $(s',\xi') \in \mathcal{M} \times [T_{s'}\mathcal{M}]^d$ and define $\psi \in \mathrm{Adm}_k^{s'}(B)$ as $\psi := \pi_{s'} \circ \phi$ (cf. Remark 4.2). Using that $\nabla\psi = \Pi_{s'}(\nabla\phi)$ and (4.13) we deduce that

$$\begin{aligned} \mu_k^{s',\xi'}(B,\omega) &\le \int_B f\big(\omega, x, \xi' + \Pi_{s'}(\nabla\phi)\big)\,\mathrm{d}x \\ &\le \mu_k^{s,\xi}(B,\omega) + \eta + \left| \int_B f\big(\omega, x, \xi' + \Pi_{s'}(\nabla\phi)\big) - f\big(\omega, x, \xi + \Pi_s(\nabla\phi)\big)\,\mathrm{d}x \right|. \end{aligned} \tag{4.14}$$

Observe that $|\xi' + \Pi_{s'}\nabla\phi| \le |\xi'| + k$ and $|\xi + \Pi_s\nabla\phi| \le |\xi| + k$. Moreover,

$$\big|\xi' + \Pi_{s'}(\nabla\phi) - \xi - \Pi_s(\nabla\phi)\big| \le |\xi - \xi'| + k\,\|\Pi_{s'} - \Pi_s\|_{\mathrm{op}}\,.$$

Together with the stationarity of f this implies that

$$\begin{aligned} &\left| \int_B f\big(\omega, x, \xi' + \Pi_{s'}(\nabla\phi)\big) - f\big(\omega, x, \xi + \Pi_s(\nabla\phi)\big)\,\mathrm{d}x \right| \\ &\le \sum_{z\in\mathbb{Z}^d} \int_{Y\cap(B-z)} \big| f\big(\tau_z\omega, y, \xi' + \Pi_{s'}(\nabla\phi(y+z))\big) - f\big(\tau_z\omega, y, \xi + \Pi_s(\nabla\phi(y+z))\big)\big|\,\mathrm{d}y \\ &\le \sum_{\substack{z\in\mathbb{Z}^d \\ (Y+z)\cap B\neq\emptyset}} \alpha_{k+|\xi|+|\xi'|}\big(\tau_z\omega, |\xi - \xi'| + k\,\|\Pi_{s'} - \Pi_s\|_{\mathrm{op}}\big)\,. \end{aligned}$$

Thus, (4.12) follows from (4.14) by the arbitrariness of $\eta > 0$ and exchanging the roles of (s,ξ) and (s',ξ'). □

Now we are in a position to prove the existence of the homogenized integrand as well as its approximation using only k-Lipschitz maps in the minimization problem. We start with the case $k = \infty$. For later reference we first make the following observation.

Remark 4.6. The manifold $\mathcal{M}$ is separable as a subset of a separable metric space. Let us fix a countable dense subset $\mathcal{M}' \subset \mathcal{M}$. Moreover, for any $s \in \mathcal{M}'$ we let D_s be a countable dense subset of $[T_s\mathcal{M}]^d \cap \mathbb{R}^{m\times d}$. Several times we will use the countable set

$$\mathcal{N} = \{(s,\xi) :\, s \in \mathcal{M}',\, \xi \in \mathcal{D}_s\}\,. \tag{4.15}$$

In this way, for every $s \in \mathcal{M}$ and $\xi \in [T_s\mathcal{M}]^d$ there exists a sequence $(s_n,\xi_n)_{n\in\mathbb{N}} \subset \mathcal{N}$ such that $(s_n,\xi_n) \to (s,\xi)$. This can be seen as follows. By the density of $\mathcal{M}'$ in $\mathcal{M}$, we find a sequence $(s_n)_{n\in\mathbb{N}}$ such that $s_n \to s$ when $n \to +\infty$. Defining $\widetilde{\xi}_n = \Pi_{s_n}(\xi) \in [T_{s_n}\mathcal{M}]^d$, we have that $\widetilde{\xi}_n \to \xi$ as $n \to +\infty$. Due to the density of $\mathcal{D}_{s_n}$ in $[T_{s_n}\mathcal{M}]^d$, via a diagonal argument one finds $\xi_n \in \mathcal{D}_{s_n}$ such that $\xi_n \to \xi$.

Proposition 4.7. *Let f be an admissible, ergodic random integrand in the sense of Definition 2.6. Then $\mathbb{P}$-a.s. the following holds: for every cube $Q \subset \mathbb{R}^d$ and every $(s,\xi) \in \mathcal{M} \times [T_s\mathcal{M}]^d$ there exists the limit*

$$Tf_\infty(s,\xi) := \lim_{r\to+\infty} \frac{\mu_\infty^{s,\xi}(rQ,\omega)}{|rQ|} \tag{4.16}$$

and is independent of Q and ω. Moreover, the mapping $(s,\xi) \mapsto Tf_\infty(s,\xi)$ is continuous.

Proof. Let $\mathcal{N}$ be as in 4.15. Lemma 4.4 allows us to apply Theorem 2.5 to the subadditive processes $(\mu_\infty^{s,\xi})_{(s,\xi)\in\mathcal{N}}$ to find deterministic constants $Tf_\infty(s,\xi)$ such that for every cube $Q \subset \mathbb{R}^d$, for every $(s,\xi) \in \mathcal{N}$ and $\mathbb{P}$-a.e. $\omega \in \Omega$ it holds that

$$\lim_{r\to+\infty} \frac{1}{|rQ|}\mu_\infty^{s,\xi}(rQ,\omega) = Tf_\infty(s,\xi)\,, \tag{4.17}$$

and the limit is independent of Q and ω. It remains to extend the existence of the limit to the remaining couples $(s,\xi)\in\mathcal{M}\times[T_s\mathcal{M}]^d$ for the same set of ω's. This will be done by approximation. Hence fix $\omega\in\Omega_f$ such that (4.17) and the statement of Lemma 2.10 hold and let $s\in\mathcal{M}$ and $\xi\in[T_s\mathcal{M}]^d$. Due to Remark 4.6 we find $(s_n,\xi_n)_{n\in\mathbb{N}}\subset\mathcal{N}$ converging to (s,ξ). In what follows we will always assume that $|(s_n,\xi_n)-(s,\xi)|\leq 1$. We use this sequence to compare the limsup and liminf of $\frac{1}{|rQ|}\mu_\infty^{s,\xi}(rQ,\omega)$ as $r\to+\infty$. In what follows, Q,s,ξ and ω will be fixed, so we do not indicate when quantities depend on those parameters. Let us choose a sequence $r_j\to+\infty$ such that

$$\liminf_{r\to+\infty}\frac{1}{|rQ|}\mu_\infty^{s,\xi}(rQ,\omega)=\lim_{j\to+\infty}\frac{1}{|r_jQ|}\mu_\infty^{s,\xi}(r_jQ,\omega). \tag{4.18}$$

Given $r_j\gg 1$, we let $\phi_j\in\mathrm{Adm}_\infty^s(r_jQ)$ be such that

$$\fint_{r_jQ}f(\omega,x,\xi+\nabla\phi_j)\,\mathrm{d}x\leq\frac{1}{|r_jQ|}\mu_\infty^{s,\xi}(r_jQ,\omega)+\frac{1}{j}.$$

We further define $\psi_j\in\mathrm{Adm}_\infty^s(Q)$ via $\psi_j=\frac{1}{r_j}\phi_j(r_j\cdot)$. Then due to the lower bound in (2.4)

$$\fint_Q|\nabla\psi_j|^p\,\mathrm{d}x=\fint_{r_jQ}|\nabla\phi_j|^p\,\mathrm{d}x\leq\frac{C}{|r_jQ|}\mu_\infty^{s,\xi}(r_jQ,\omega)+1\leq C(|\xi|^p+1).$$

Since $\psi_j\equiv 0$ on $\mathbb{R}^d\setminus Q$, the above estimate allows us to apply Lemma 4.1 to find a subsequence (not relabeled) and another sequence $\widetilde{\psi}_j\in\mathrm{Adm}_\infty^s(Q)$[2] such that $|\nabla\widetilde{\psi}_j|^p$ is equi-integrable and the set $A_j:=\{\psi_j\neq\widetilde{\psi}_j\text{ or }\nabla\psi_j\neq\nabla\widetilde{\psi}_j\}\to 0$ satisfies $|A_j|\to 0$ as $j\to+\infty$. Then by the choice of ϕ_j, a change of variables and (2.4) we have that

$$\begin{aligned}\frac{1}{|r_jQ|}\mu_\infty^{s,\xi}(r_jQ,\omega)+\frac{1}{j}&\geq\fint_{r_jQ}f(\omega,x,\xi+\nabla\phi_j)\,\mathrm{d}x=\fint_Q f(\omega,r_jx,\nabla\psi_j)\,\mathrm{d}x\\&\geq\frac{1}{|Q|}\int_{Q\setminus A_j}f(\omega,r_jx,\xi+\nabla\widetilde{\psi}_j)\,\mathrm{d}x\\&\geq\fint_Q f(\omega,r_jx,\xi+\nabla\widetilde{\psi}_j)\,\mathrm{d}x-\frac{C}{|Q|}\int_{A_j}(1+|\nabla\widetilde{\psi}_j|^p)\,\mathrm{d}x.\end{aligned}\tag{4.19}$$

Note that the very last integral is negligible when $j\to+\infty$ due to the convergence $|A_j|\to 0$ and the equi-integrability of $|\nabla\widetilde{\psi}_j|^p$. Thus let us continue to bound the remaining integral on the right-hand side from below. We have that

$$\begin{aligned}\fint_Q f(\omega,r_jx,\xi+\nabla\widetilde{\psi}_j)\,\mathrm{d}x&\geq\fint_Q f(\omega,r_jx,\xi_n+\Pi_{s_n}\nabla\widetilde{\psi}_j)\,\mathrm{d}x\\&\quad-\left|\fint_Q f(\omega,r_jx,\xi+\nabla\widetilde{\psi}_j)-f(\omega,r_jx,\xi_n+\Pi_{s_n}\nabla\widetilde{\psi}_j)\,\mathrm{d}x\right|.\end{aligned}$$

In the first right-hand side integral, we can perform a change of variables and noting that $x\mapsto\pi_{s_n}\circ r_j\widetilde{\psi}_j(x/r_j)\in\mathrm{Adm}_\infty^{s_n}(r_jQ)$, we can bound it from below by $\frac{1}{|r_jQ|}\mu_\infty^{s_n,\xi_n}(r_jQ,\omega)$, so that

$$\begin{aligned}\fint_{Q+x}f(\omega,r_jx,\xi+\nabla\widetilde{\psi}_j)\,\mathrm{d}x&\geq\frac{1}{|r_jQ|}\mu_\infty^{s_n,\xi_n}(r_jQ,\omega)\\&\quad-\underbrace{\left|\fint_Q f(\omega,r_jx,\xi+\nabla\widetilde{\psi}_j)-f(\omega,r_jx,\xi_n+\Pi_{s_n}\nabla\widetilde{\psi}_j)\,\mathrm{d}x\right|}_{=:e_{j,n}}.\end{aligned}\tag{4.20}$$

We argue that the error term $e_{j,n}$ is negligible when we let first $j\to+\infty$ and then $n\to+\infty$. To this end, let $\eta>0$ and define the set

$$L_{j,\eta}:=\{|\xi+\nabla\widetilde{\psi}_j|+|\nabla\widetilde{\psi}_j|\geq N_\eta\},$$

[2] Recall that on convex sets $W^{1,\infty}$ can be identified with the space of Lipschitz functions.

where we chose $N_\eta \in \mathbb{N}$ such that

$$\sup_{j\in\mathbb{N}} \int_{Q\cap L_{j,\eta}} (1+|\xi+\nabla\widetilde{\psi}_j|^p)\,\mathrm{d}y \leq \eta. \tag{4.21}$$

The existence of such an N_η is ensured by the equi-integrability of $|\nabla\widetilde{\psi}_j|^p$ since $|L_{j,\eta}|\to 0$ uniformly in j when $N_\eta\to+\infty$. Next, on the complement of $L_{j,\eta}$ we have that

$$|\xi+\nabla\widetilde{\psi}_j-\xi_n-\Pi_{s_n}\nabla\widetilde{\psi_j}| \leq |\xi-\xi_n|+|\Pi_s\nabla\widetilde{\psi}_j-\Pi_{s_n}\nabla\widetilde{\psi}_j| \leq |\xi-\xi_n|+\|\Pi_s-\Pi_{s_n}\|_{\mathrm{op}}\,N_\eta.$$

Recalling the definition of the integrated modulus of continuity $\alpha_R(\omega,t)$ (cf. Definition 2.7), the stationarity of f and the upper bound in (2.4), we can thus estimate

$$\begin{aligned} e_{j,n} &\leq \frac{C}{|Q|}\int_{Q\cap L_{j,\eta}} (1+|\xi+\nabla\widetilde{\psi}_j|^p)\,\mathrm{d}x \\ &\quad + \frac{1}{|r_jQ|}\int_{r_j(Q\setminus L_{j,\eta})} |f(\omega,x,\xi+\nabla\widetilde{\psi}_j(x/r_j))-f(\omega,x,\xi_n+\Pi_{s_n}\nabla\widetilde{\psi}_j(x/r_j))|\,\mathrm{d}x \\ &\leq \frac{C}{|Q|}\eta+\frac{1}{|r_jQ|}\sum_{\substack{z\in\mathbb{Z}^d\\ (Y+z)\cap r_jQ\neq\emptyset}} \alpha_{N_\eta+1}(\tau_z\omega,|\xi-\xi_n|+\|\Pi_s-\Pi_{s_n}\|_{\mathrm{op}}\,N_\eta). \end{aligned}$$

Due to Lemma 2.10 the last sum vanishes when we let first $j\to+\infty$ and then $n\to+\infty$. The arbitrariness of $\eta>0$ then implies that $\limsup_{n\to+\infty}\limsup_{j\to+\infty} e_{j,n}=0$. In combination with (4.18), (4.19) and (4.20) we thus showed that

$$\liminf_{r\to+\infty}\frac{1}{|rQ|}\mu_\infty^{s,\xi}(rQ,\omega) \geq \limsup_{n\to+\infty}\lim_{r\to+\infty}\mu_\infty^{s_n,\xi_n}(rQ,\omega) \overset{(4.17)}{=} \limsup_{n\to+\infty} Tf_\infty(s_n,\xi_n). \tag{4.22}$$

To conclude, we need to derive a suitable upper bound for the lim sup. First, we choose a sequence $r_j\to+\infty$ such that

$$\limsup_{r\to+\infty}\frac{1}{|rQ|}\mu_\infty^{s,\xi}(rQ,\omega) = \lim_{j\to+\infty}\frac{1}{|r_jQ|}\mu_\infty^{s,\xi}(r_jQ,\omega). \tag{4.23}$$

By a diagonal argument, we find a sequence $(n_j)_{j\in\mathbb{N}}$ that diverges to $+\infty$ and such that

$$\liminf_{n\to+\infty}\lim_{j\to+\infty}\frac{1}{|r_jQ|}\mu_\infty^{s_n,\xi_n}(r_jQ,\omega) = \lim_{j\to+\infty}\frac{1}{|r_jQ|}\mu_\infty^{s_{n_j},\xi_{n_j}}(r_jQ,\omega). \tag{4.24}$$

For every $j\in\mathbb{N}$ we let $\phi_j\in\mathrm{Adm}_\infty^{s_{n_j}}(r_jQ)$ be such that

$$⨍_{r_jQ} f(\omega,x,\xi_{n_j}+\nabla\phi_j)\,\mathrm{d}x \leq \frac{1}{|r_jQ|}\mu_\infty^{s_{n_j},\xi_{n_j}}(r_jQ,\omega)+1/j. \tag{4.25}$$

As above we define $\psi_j\in\mathrm{Adm}_\infty^{s_{n_j}}(Q)$ via $\psi_j=\frac{1}{r_j}\phi_j(r_j\cdot)$, which by the same argument gives rise to a subsequence (not relabeled) and a sequence $\widetilde{\psi}_j\in\mathrm{Adm}_\infty^{s_{n_j}}(Q)$ such that $|\nabla\widetilde{\psi}_j|^p$ is equi-integrable and $A_j:=\{\psi_j\neq\widetilde{\psi}_j \text{ or } \nabla\psi_j\neq\nabla\widetilde{\psi}_j\}\to 0$ in measure. By a change of variables we then have

$$\begin{aligned} ⨍_{r_jQ} f(\omega,x,\xi_{n_j}+\nabla\phi_j)\,\mathrm{d}x &= ⨍_Q f(\omega,r_jx,\xi_{n_j}+\nabla\psi_j)\,\mathrm{d}x \geq \frac{1}{|Q|}\int_{Q\setminus A_j} f(\omega,r_jx,\xi_{n_j}+\nabla\widetilde{\psi}_j)\,\mathrm{d}x \\ &\geq ⨍_Q f(\omega,r_jx,\xi_{n_j}+\nabla\widetilde{\psi}_j)\,\mathrm{d}y-\frac{C}{|Q|}\int_{A_j}(1+|\nabla\widetilde{\psi}_j|^p)\,\mathrm{d}x \end{aligned}$$

and the last integral vanishes when $j \to +\infty$ due to the equi-integrability of $|\nabla \widetilde{\psi}_j|^p$. Hence we continue to estimate the other right-hand side term: with the same argument as for (4.20) one shows that

$$\begin{aligned}\fint_Q f(\omega, r_j x, \xi_{n_j} + \nabla\widetilde{\psi}_j)\,\mathrm{d}x &\geq \frac{1}{|r_jQ|}\mu_\infty^{s,\xi}(r_jQ,\omega)\\ &\quad - \underbrace{\left|\fint_Q f(\omega, r_j x, \xi_{n_j} + \nabla\widetilde{\psi}_j) - f(\omega, r_j x, \xi + \Pi_s\nabla\widetilde{\psi}_j)\,\mathrm{d}x\right|}_{=:e_j}\end{aligned} \tag{4.26}$$

and we argue that the error term e_j is negligible when $j \to +\infty$. Given $\eta > 0$, we choose $N_\eta \in \mathbb{N}$ such that the sets

$$G_{j,\eta} := \{|\xi_{n_j} + \nabla\widetilde{\psi}_j| + |\nabla\widetilde{\psi}_j| \geq N_\eta\}$$

satisfy

$$\int_{Q\cap G_{j,\eta}} (1 + |\xi_{n_j} + \nabla\widetilde{\psi}_j|^p)\,\mathrm{d}x \leq \eta.$$

Again this is possible since the sequence $|\nabla\widetilde{\psi}_j|^p$ is equi-integrable, the sequence ξ_{n_j} is converging and the sets $G_{j,\eta}$ converge to zero in measure when $N_\eta \to +\infty$ uniformly with respect to j. Similarly to the previous step we then have

$$\begin{aligned}e_j &\leq \frac{C}{|Q|}\eta + \frac{1}{|r_jQ|}\int_{r_j(Q\setminus G_{j,\eta})} |f(\omega, x, \xi_{n_j} + \nabla\widetilde{\psi}_j(x/r_j)) - f(\omega, x, \xi + \Pi_s\nabla\widetilde{\psi}_j(x/r_j))|\,\mathrm{d}x\\ &\leq \frac{C}{|Q|}\eta + \frac{1}{|r_jQ|}\sum_{\substack{z\in\mathbb{Z}^d\\ (Y+z)\cap r_jQ\neq\emptyset}} \alpha_{N_\eta+1}(\tau_z\omega, |\xi_{n_j} - \xi| + \|\Pi_{s_{n_j}} - \Pi_s\|_{\mathrm{op}}\,N_\eta)\end{aligned} \tag{4.27}$$

Using the monotonicity of $t \mapsto \alpha_R(\omega, t)$, it follows from Lemma 2.10 that the second right-hand side term vanishes when $j \to +\infty$. The arbitrariness of $\eta > 0$ then yields that $\lim_{j\to+\infty} e_j = 0$. Taking into account (4.23), (4.24), (4.25) and (4.26) we thus showed that

$$\liminf_{n\to+\infty} Tf_\infty(s_n, \xi_n) = \liminf_{n\to+\infty}\lim_{r\to+\infty}\frac{1}{|rQ|}\mu_\infty^{s_n,\xi_n}(rQ,\omega) \geq \limsup_{r\to+\infty}\frac{1}{|rQ|}\mu_\infty^{s,\xi}(rQ,\omega). \tag{4.28}$$

Combined with (4.22) and the obvious inequality $\liminf \leq \limsup$, this estimate shows that

$$\liminf_{n\to+\infty} Tf_\infty(s_n,\xi_n) = \limsup_{n\to+\infty} Tf_\infty(s_n,\xi_n) =: Tf_\infty(s,\xi),$$

$$\liminf_{r\to+\infty}\frac{1}{|rQ|}\mu_\infty^{s,\xi}(rQ,\omega) = \limsup_{r\to+\infty}\frac{1}{|rQ|}\mu_\infty^{s,\xi}(rQ,\omega) = Tf_\infty(s,\xi).$$

In particular, the limit of $\frac{1}{|rQ|}\mu_\infty^{s,\xi}(rQ,\omega)$ exists, is deterministic and independent of Q as claimed. Finally, the continuity of the map Tf_∞ can be proven with the same argument used to show the last two estimates. □

In the next proposition we prove the same result for the processes $\mu_k^{s,\xi}$ with finite $k \in \mathbb{N}$.

Proposition 4.8. *Let f be an admissible, ergodic random integrand in the sense of Definition 2.6. Then for $\mathbb{P}$-a.s. the following holds: for every cube $Q \subset \mathbb{R}^d$, every $(s,\xi) \in \mathcal{M} \times [T_s\mathcal{M}]^d$, and every $k \in \mathbb{N}$ there exists the limit*

$$Tf_k(s,\xi) := \lim_{r\to+\infty}\frac{\mu_k^{s,\xi}(rQ,\omega)}{|rQ|} \tag{4.29}$$

and is independent of Q and ω.
Moreover, the mapping $(s,\xi) \mapsto Tf_k(s,\xi)$ is continuous for all $k \in \mathbb{N}$.

Proof. For fixed $k \in \mathbb{N}$ and $(s,\xi) \in \mathcal{M} \times [T_s\mathcal{M}]^d$ we have shown in Lemma 4.4 that $\mu_k^{s,\xi}$ is a bounded subadditive process. Thus, applying Theorem 2.5 to the countable family of subadditive processes $\mu_k^{s,\xi}$ with $k \in \mathbb{N}$ and $(s,\xi) \in \mathcal{N}$ (the set defined in (4.15)) ensures the existence of deterministic constants $Tf_k(s,\xi)$ such for $\mathbb{P}$-a.e. $\omega \in \Omega$ the convergence (4.29) holds for every cube $Q \subset \mathbb{R}^d$, all $k \in \mathbb{N}$ and $(s,\xi) \in \mathcal{N}$. Moreover, upon intersecting two sets of full probability it is not restrictive to assume that Lemma 2.10 holds for the same set of ω's.

Let now $\omega \in \Omega$ be such a realization and fix $k \in \mathbb{N}$; we claim that then (4.29) holds for any $(s,\xi) \in \mathcal{M} \times [T_s\mathcal{M}]^d$. The proof of this claim will be established in several steps.

Step 1. *Locally uniform continuity.* Let $R > 0$ and let us show that the mapping $(s,\xi) \mapsto Tf_k(s,\xi)$ is uniformly continuous on the set

$$\mathcal{N}_R := \big\{(s,\xi)\colon s \in \mathcal{N}\,,\ |s| \leq R\,,\ |\xi| \leq R\big\}\,.$$

This will allow us to extend Tf_k in a uniformly continuous way to the closure $\overline{\mathcal{N}_R}$.
Let $\eta > 0$ be arbitrary; thanks to Lemma 2.10 there exists $t_0 > 0$ such that for all $t \in (0,t_0]$ we have that

$$\limsup_{r\to+\infty} \frac{1}{|rQ|} \sum_{\substack{z\in\mathbb{Z}^d \\ (Y+z)\cap rQ}} \alpha_{k+2R+1}(\tau_z\omega, t) < \frac{\eta}{2}\,. \tag{4.30}$$

Moreover, since $\mathcal{M}$ is a C^1-manifold, the mapping $s \mapsto \Pi_s$ is uniformly continuous on the compact set $\mathcal{M} \cap \overline{B_R(0)}$. In particular, there exists $\delta \in (0,1)$ such that for all (s,ξ), $(s',\xi') \in \overline{\mathcal{N}_R}$ with $|(s,\xi) - (s',\xi')| < \delta$ we have $k\,\|\Pi_s - \Pi_{s'}\|_{\mathrm{op}} + |\xi - \xi'| < t_0$. Thanks to Lemma 4.5 this in turn implies that for all such $(s,\xi),(s',\xi')$ the estimate

$$\frac{1}{|rQ|}\big|\mu_k^{s,\xi}(rQ,\omega) - \mu_k^{s',\xi'}(rQ,\omega)\big| \leq \frac{1}{|rQ|} \sum_{\substack{z\in\mathbb{Z}^d \\ (Y+z)\cap rQ\neq\emptyset}} \alpha_{k+2R+1}(\tau_z\omega, t_0)$$

holds for all all $r > 0$ and all cubes $Q \subset \mathbb{R}^d$. Letting $r \to +\infty$ we thus deduce from (4.30) that

$$\limsup_{r\to+\infty} \frac{1}{|rQ|}\big|\mu_k^{s,\xi}(rQ,\omega) - \mu_k^{s',\xi'}(rQ,\omega)\big| < \frac{\eta}{2} \tag{4.31}$$

for all $(s,\xi),(s',\xi') \in \overline{\mathcal{N}_R}$ with $|(s,\xi) - (s',\xi')| < \delta$.
Let now $(s,\xi),(s',\xi') \in \mathcal{N}_R$ with $|(s,\xi) - (s',\xi')| < \delta$. Applying (4.31) together with (4.29) (which holds on $\mathcal{N}_R$) finally gives

$$|Tf_k(s,\xi) - Tf_k(s',\xi')| \leq \frac{\eta}{2}\,.$$

This yields the uniform continuity of the mapping $(s,\xi) \mapsto Tf_k(s,\xi)$ on the set $\mathcal{N}_R$ which in turn implies that Tf_k can be extended to a uniformly continuous mapping on the closure $\overline{\mathcal{N}_R}$.

Step 2. *Proof of* (4.29) *for* $(s_0,\xi_0) \notin \mathcal{N}$. Let $(s_0,\xi_0) \in (\mathcal{M} \times [T_s\mathcal{M}]^d) \setminus \mathcal{N}$ be fixed. As shown at the beginning of the proof of Proposition 4.7 we can approximate (s_0,ξ_0) with elements in $\mathcal{N}$, so that $(s_0,\xi_0) \in \overline{\mathcal{N}_R}$ with $R \geq \max\{|s_0|,|\xi_0|\} + 1$. Let now $\eta > 0$ be arbitrary and $\delta \in (0,1)$ such that (4.31) is satisfied and such that

$$|Tf_k(s,\xi) - Tf_k(s',\xi')| < \frac{\eta}{2} \ \text{ for all } (s,\xi),(s'\xi') \in \overline{\mathcal{N}_R} \text{ with } |(s,\xi) - (s',\xi')| < \delta. \tag{4.32}$$

Let now $(s,\xi) \in \mathcal{N}_R$ with $|(s_0,\xi_0) - (s,\xi)| < \delta$. For any $r > 0$ and any cube $Q \subset \mathbb{R}^d$ we have that

$$\Big|\frac{\mu_k^{s_0,\xi_0}(rQ,\omega)}{|rQ|} - Tf_k(s_0,\xi_0)\Big| \leq \Big|\frac{\mu_k^{s,\xi}(rQ,\omega)}{|rQ|} - Tf_k(s,\xi)\Big| + \frac{1}{|rQ|}\big|\mu_k^{s_0,\xi_0}(rQ,\omega) - \mu_k^{s,\xi}(rQ,\omega)\big| + \frac{\eta}{2}\,,$$

where we have used (4.32). Since (4.29) holds for $(s,\xi)\in\mathcal{N}_R$, letting $r\to+\infty$ and using (4.31) yields

$$\limsup_{r\to+\infty}\Big|\frac{\mu_k^{s_0,\xi_0}(rQ,\omega)}{|rQ|}-Tf_k(s_0,\xi_0)\Big|\leq\eta\,.$$

We conclude by the arbitrariness of $\eta>0$. The continuity of Tf_k is a consequence of the first step and the arbitrariness of $R>0$. □

Finally, we prove the crucial fact that Tf_k provides an approximation for Tf_∞ for large k.

Proposition 4.9. *Let f be an admissible, ergodic random integrand in the sense of Definition 2.6. Let moreover Tf_k and Tf_∞ be as in Propositions 4.7 and 4.8, respectively. Then*

$$Tf_\infty(s,\xi)=\lim_{k\to\infty}Tf_k(s,\xi)=\inf_{k\in\mathbb{N}}Tf_k(s,\xi)\tag{4.33}$$

for every $(s,\xi)\in\mathcal{M}\times[T_s\mathcal{M}]^d$.

Proof. Let $(s,\xi)\in\mathcal{M}\times[T_s\mathcal{M}]^d$ be fixed. Since $\mu_k^{s,\xi}(\omega,B)\geq\mu_{k+1}^{s,\xi}(\omega,B)$ for every $B\in\mathcal{U}_0$ and every $k\in\mathbb{N}$, we deduce that

$$Tf_\infty(s,\xi)\leq\lim_{k\to\infty}Tf_k(s,\xi)=\inf_{k\in\mathbb{N}}Tf_k(s,\xi)\,.\tag{4.34}$$

Moreover, applying Propositions 4.7 and 4.8 with $Q=(0,1)^d$ and $r_\ell=2^\ell$ with $\ell\in\mathbb{N}$ and using the dominated convergence theorem yield

$$Tf_k(s,\xi)=\lim_{\ell\to+\infty}\frac{1}{2^{\ell d}}\mathbb{E}[\mu_k^{s,\xi}((0,2^\ell)^d,\cdot)]\tag{4.35}$$

for any $k\in\mathbb{N}\cup\{+\infty\}$. We now show that for any $k\in\mathbb{N}\cup\{+\infty\}$ the terms $\frac{1}{2^{\ell d}}\mathbb{E}[\mu_k^{s,\xi}((0,2^\ell)^d,\cdot)]$ are monotone decreasing in ℓ. To this end, let $k\in\mathbb{N}$ and $\ell\in\mathbb{N}$ be fixed. Then the cube $(0,2^{\ell+1})^d$ can be partitioned into $n_d=2^d$ sub-cubes of the form $(0,2^\ell)^d+z_n$ with $z_1,\dots,z_{n_d}\in\mathbb{Z}^d$. Thus, using the subadditivity and stationarity of $\mu_k^{s,\xi}$ we deduce that

$$\mu_k^{s,\xi}((0,2^{(\ell+1)})^d,\omega)\leq\sum_{n=1}^{n_d}\mu_k^{s,\xi}((0,2^\ell)^d+z_n,\omega)=\sum_{n=1}^{n_d}\mu_k^{s,\xi}((0,2^\ell)^d,\tau_{z_n}\omega)\,.$$

By taking the expectation this simplifies to the claimed monotonicity

$$\frac{1}{2^{(\ell+1)d}}\mathbb{E}[\mu_k^{s,\xi}((0,2^{(\ell+1)})^d,\cdot)]\leq\frac{n_d}{2^{(\ell+1)d}}\mathbb{E}[\mu_k^{s,\xi}((0,2^\ell)^d,\cdot)]=\frac{1}{2^{\ell d}}\mathbb{E}[\mu_k^{s,\xi}((0,2^\ell)^d,\cdot)]\,.\tag{4.36}$$

Using (4.36) for $k\in\mathbb{N}$ and gathering (4.34)-(4.35) leads to

$$Tf_\infty(s,\xi)\leq\inf_{k\in\mathbb{N}}Tf_k(s,\xi)=\inf_{k\in\mathbb{N}}\inf_{\ell\in\mathbb{N}}\frac{1}{2^{\ell d}}\mathbb{E}[\mu_k^{s,\xi}((0,2^\ell)^d,\cdot)]=\inf_{\ell\in\mathbb{N}}\inf_{k\in\mathbb{N}}\frac{1}{2^{\ell d}}\mathbb{E}[\mu_k^{s,\xi}((0,2^\ell)^d,\cdot)]\,.\tag{4.37}$$

Moreover, we have $\inf_{k\in\mathbb{N}}\mathbb{E}[\mu_k^{s,\xi}((0,2^\ell)^d,\cdot)]=\mathbb{E}[\mu_\infty^{s,\xi}((0,2^\ell)^d,\cdot)]$ thanks to (4.8) and the monotone convergence theorem. Thus, using finally (4.36) for $k=\infty$ and again (4.35) we can continue the estimate in (4.37) to

$$Tf_\infty(s,\xi)\leq\inf_{k\in\mathbb{N}}Tf_k(s,\xi)=\inf_{\ell\in\mathbb{N}}\frac{1}{2^{\ell d}}\mathbb{E}[\mu_\infty^{s,\xi}((0,2^\ell)^d,\cdot)]=Tf_\infty(s,\xi)\,,$$

which yields (4.33). □

4.2. **Proof of the liminf-inequality.** In this section we prove the liminf-inequality for the Gamma-convergence statement in Theorem 3.1. This will be done by applying a blow-up procedure, using the auxiliary integrands $\hat{f}$ defined in (4.3) and carrying out a careful local analysis based on the fundamental estimate [9, Theorem 19.1], Lemma 4.1 and Lemma 2.10. Finally, we will use Proposition 4.7. To be able to do so, throughout this section we fix an element $\omega\in\Omega_f$ such that both Lemma 2.10 and Proposition 4.8 hold. In particular, the following statement holds $\mathbb{P}$-almost surely.

Proposition 4.10. *Let f be an admissible, ergodic random integrand in the sense of Definition 2.6 and for every $\omega \in \Omega$ let $F_\varepsilon(\omega)(\cdot, U) : L^p(U;\mathbb{R}^m) \to [0,+\infty]$ be as in (3.1). Then the following holds for almost every $\omega \in \Omega$ and every bounded, open set $U \subset \mathbb{R}^d$: if $(u_\varepsilon) \subset L^p(U;\mathbb{R}^m)$ and $u \in L^p(U;\mathbb{R}^m)$ are such that $u_\varepsilon \to u$ strongly in $L^p(U;\mathbb{R}^m)$, then*

$$\liminf_{\varepsilon\to 0} F_\varepsilon(\omega)(u_\varepsilon, U) \geq F_{\rm hom}(\omega)(u, U)\,. \tag{4.38}$$

Proof. Let $(u_\varepsilon) \subset L^p(U;\mathbb{R}^m)$ and $u \in L^p(U;\mathbb{R}^m)$ be such that $u_\varepsilon \to u$ strongly in $L^p(U;\mathbb{R}^m)$ as $\varepsilon \to 0$. To prove (4.38) it suffices to consider the case $\liminf_{\varepsilon\to 0} F_\varepsilon(\omega)(u_\varepsilon, U) < +\infty$. Let then (ε_n) be a sequence satisfying

$$\liminf_{\varepsilon\to 0} F_\varepsilon(\omega)(u_\varepsilon, U) = \lim_{n\to\infty} F_{\varepsilon_n}(\omega)(u_{\varepsilon_n}, U) < +\infty \tag{4.39}$$

and such that $u_n := u_{\varepsilon_n}$ converges to u pointwise. Since $\mathcal{M}$ is closed, we deduce that $u(x) \in \mathcal{M}$ for a.e. $x \in U$. Moreover, (4.39) together with (2.4) ensures that (u_n) is uniformly bounded in $W^{1,p}(U;\mathcal{M})$. We thus deduce that $u \in W^{1,p}(U;\mathcal{M})$ and it remains to show that

$$\lim_{n\to\infty} \int_U f\big(\omega, \tfrac{x}{\varepsilon_n}, \nabla u_n\big)\,{\rm d}x \geq \int_U Tf_\infty(u, \nabla u)\,{\rm d}x\,. \tag{4.40}$$

We establish (4.40) by means of the blow-up method. To this end, we consider the sequence of non-negative finite Radon measures $\mu_n := f\big(\omega, \tfrac{\cdot}{\varepsilon_n}, \nabla u_n(\,\cdot\,)\big)\mathcal{L}^d \llcorner U$. Then (4.39) implies that

$$\sup_{n\in\mathbb{N}} |\mu_n|(U) < +\infty\,.$$

From the compactness result [3, Teorem 1.59] we deduce that there exist a further subsequence (not relabeled) and a non-negative finite Radon measure μ such that $\mu_n \overset{*}{\rightharpoonup} \mu$ as $n \to +\infty$. Since $U \subset \mathbb{R}^d$ is open, the weak$*$-convergence together with [3, Proposition 1.62] implies that

$$\mu(U) \leq \liminf_{n\to\infty} \mu_n(U) = \lim_{n\to 0} \int_U f\big(\omega, \tfrac{x}{\varepsilon_n}, \nabla u_n\big)\,{\rm d}x\,. \tag{4.41}$$

Thanks to the Besicovitch Derivation Theorem [16, Theorem 1.153] the measure μ can be decomposed as $\mu = \mu_a + \mu_s$ into a pair of non-negative finite Radon measures μ_a, μ_s with $\mu_a \ll \mathcal{L}^d$ and $\mu_s \perp \mathcal{L}^d$. Denoting by $\frac{d\mu_a}{d\mathcal{L}^d}$ the Radon-Nikodym derivative of μ_a with respect to $\mathcal{L}^d$ it thus suffices to show that

$$\frac{d\mu_a}{d\mathcal{L}^d}(x) \geq Tf_\infty\big(u(x), \nabla u(x)\big) \ \text{ for a.e. } x \in U\,. \tag{4.42}$$

In fact, if (4.42) holds true, then (4.41) together with the fact that $\mu_a(U) \leq \mu(U)$ yields (4.40). We now establish (4.42) in several steps.

Step 1. Choice of x_0
To obtain (4.42) we fix a point $x_0 \in U$ with $s_0 := u(x_0) \in \mathcal{M}$, $\xi_0 := \nabla u(x_0) \in [T_{s_0}\mathcal{M}]^d$, and such that

$$\lim_{\rho\to 0+} \fint_{Q_\rho(x_0)} |u(x) - s_0|^p\,{\rm d}x = 0 = \lim_{\rho\to 0+} \fint_{Q_\rho(x_0)} |\nabla u(x) - \xi_0|^p\,{\rm d}x\,, \tag{4.43}$$

$$\lim_{\rho\to 0+} \frac{1}{\rho^p} \fint_{Q_\rho(x_0)} |u(x) - s_0 - \xi_0\cdot(x - x_0)|^p\,{\rm d}x = 0\,, \tag{4.44}$$

$$\frac{d\mu_a}{d\mathcal{L}^d}(x_0) = \lim_{\rho\to 0^+} \frac{\mu(Q_\rho(x_0))}{\rho^d}\,. \tag{4.45}$$

The fact that $u \in W^{1,p}(D;\mathcal{M})$ and the Besicovitch derivation theorem ensure that these properties hold for $\mathcal{L}^d$-a.e. $x_0 \in U$. Thus, to obtain (4.42) it suffices to show that

$$\frac{d\mu_a}{d\mathcal{L}^d}(x_0) \geq Tf_\infty\big(u(x_0), \nabla u(x_0)\big) \tag{4.46}$$

with x_0 satisfying (4.43)–(4.45). Since μ is a finite Radon measure, we have that $\mu(\partial Q_\rho(x_0)) = 0$ except for countably many $\rho > 0$. In particular, we can find a sequence $\rho_k \to 0+$ such that $\mu(\partial Q_{\rho_k}(x_0)) = 0$ for all $k \in \mathbb{N}$. Thus, combining (4.45) with [3, Proposition 1.62] and the fact that $\mu_n \overset{*}{\rightharpoonup} \mu$ we find that

$$\frac{d\mu_a}{d\mathcal{L}^d}(x_0) = \lim_{k\to+\infty} \frac{\mu(Q_{\rho_k}(x_0))}{\rho_k^d} = \lim_{k\to+\infty}\lim_{n\to+\infty} \fint_{Q_{\rho_k}(x_0)} f\big(\omega, \tfrac{x}{\varepsilon_n}, \nabla u_n\big)\,\mathrm{d}x =: \Lambda(x_0) \tag{4.47}$$

and it remains to estimate $\Lambda(x_0)$. This will be done via a diagonal argument. For later use it is convenient to define the constant $C_{\xi_0} := \big(\frac{|\xi_0|}{2} d^{\frac12} + 1\big)$.

<u>*Step 2.*</u> Choice of a diagonal sequence
For fixed $k, n \in \mathbb{N}$ the change of variables $y = \frac{x - x_0}{\rho_k}$ leads to

$$\fint_{Q_{\rho_k}(x_0)} f\big(\omega, \tfrac{x}{\varepsilon_n}, \nabla u_n\big)\,\mathrm{d}x = \fint_{Q_1} f\big(\omega, \tfrac{x_0+\rho_k y}{\varepsilon_n}, \nabla v_{n,k}(y)\big)\,\mathrm{d}y\,, \tag{4.48}$$

where $v_{n,k}(y) := \frac{1}{\rho_k}\big(u(x_0 + \rho_k y) - s_0\big)$ for every $y \in Q_1$. Moreover, denoting by u_0 the affine function $u_0(y) := \xi_0 \cdot y$, we have that

$$\begin{aligned}\fint_{Q_1} |v_{n,k} - u_0|^p\,\mathrm{d}y &= \rho_k^{-p} \fint_{Q_{\rho_k}(x_0)} |u_n(x) - s_0 - \xi_0\cdot(x - x_0)|^p\,\mathrm{d}x \\ &\le 2^{p-1}\rho_k^{-p}\left(\fint_{Q_{\rho_k}(x_0)} |u_n(x) - u(x)|^p\,\mathrm{d}x + \fint_{Q_{\rho_k}(x_0} |u(x) - s_0 - \xi_0\cdot(x - x_0)|^p\,\mathrm{d}x\right),\end{aligned}$$

so that (4.44) ensures that

$$\lim_{k\to\infty}\limsup_{n\to\infty} \int_{Q_1} |v_{n,k} - u_0|^p\,\mathrm{d}y = 0\,. \tag{4.49}$$

In addition, we deduce from Lemma 2.9 and Remark 2.2 that

$$\lim_{k\to\infty} \mathbb{E}\big[\alpha_N(\,\cdot\,, M\gamma_R(C_{\xi_0}\rho_k))\big] = 0 \quad\text{for all } M, N, R > 0\,, \tag{4.50}$$

where γ_R is the modulus of continuity defined in (2.1). Moreover, Lemma 2.10 together with the monotonicity of $R \mapsto \gamma_R$ implies that

$$\limsup_{n\to\infty}\left(\frac{\varepsilon_n}{\rho_k}\right)^d \sum_{\substack{z\in\mathbb{Z}^d\\ (Y+z)\cap\frac{\rho_k}{\varepsilon_n}Q_1\left(\frac{x_0}{\rho_k}\right)\neq\emptyset}} \alpha_N(\tau_z\omega, \gamma_R(C_{\xi_0}\rho_k)) \le \inf_{\substack{R'>R\\ M'>M\\ N'>N}} \mathbb{E}\big[\alpha_{N'}(\,\cdot\,, M'\gamma_{R'}(C_{\xi_0}\rho_k))\big] \tag{4.51}$$

for every $k \in \mathbb{N}$ and $R, N, M > 0$. We finally observe that for any fixed $k \in \mathbb{N}$ Proposition 4.7 together with the choice of ω ensures that

$$\lim_{n\to\infty}\left(\frac{\varepsilon_n}{\rho_k}\right)^d \mu_\infty^{s_0,\xi_0}\left(\frac{\rho_k}{\varepsilon_n}Q_1(x_0/\rho_k), \omega\right) = Tf_\infty(s_0, \xi_0)\,. \tag{4.52}$$

Using (4.49)–(4.52) we now construct a diagonal sequence as follows. For $j \in \mathbb{N}$ we first choose $k_j \in \mathbb{N}$ sufficiently large such that

$$\left|\Lambda(x_0) - \lim_{n\to\infty}\int_{Q_1} f\big(\omega, \tfrac{x_0+\rho_{k_j}y}{\varepsilon_n}, \nabla v_{n,k_j}(y)\big)\,\mathrm{d}y\right| < \frac{1}{2j} \tag{4.53}$$

and

$$\limsup_{n\to\infty}\int_{Q_1} |v_{n,k_j} - u_0|^p\,\mathrm{d}y < \frac{1}{2j} \quad\text{and}\quad \mathbb{E}\big[\alpha_{2j}\big(\,\cdot\,, 2j\gamma_{2j}(C_{\xi_0}\rho_{k_j})\big)\big] < \frac{1}{2j}\,, \tag{4.54}$$

which is possible thanks to (4.49) and (4.50). Subsequently, we choose $n_j \in \mathbb{N}$ such that $\varepsilon_{n_j} \le \rho_{k_j}^2$ and such that for all $n \ge n_j$ we have that

$$\int_{Q_1} f\big(\omega, \tfrac{x_0+\rho_{k,j}y}{\varepsilon_n}, \nabla v_{n,k_j}(y)\big)\,\mathrm{d}y \le \lim_{n\to\infty}\int_{Q_1} f\big(\omega, \tfrac{x_0+\rho_{k_j}y}{\varepsilon_n}, \nabla v_{n,k_j}(y)\big)\,\mathrm{d}y + \frac{1}{2j} \le \Lambda(x_0) + \frac{1}{j} \tag{4.55}$$

$$\int_{Q_1} |v_{n,k_j} - u_0|^p\,\mathrm{d}y \le \limsup_{n\to\infty}\int_{Q_1} |v_{n,k_j} - u_0|^p\,\mathrm{d}y + \frac{1}{2j} < \frac{1}{j}\,, \tag{4.56}$$

$$\left|\left(\frac{\varepsilon_n}{\rho_{k_j}}\right)^d \mu_\infty^{s_0,\xi_0}\left(\frac{\rho_{k_j}}{\varepsilon_n}Q_1\big(x_0/\rho_{k_j}\big),\omega\right) - Tf_\infty(s_0,\xi_0)\right| < \frac{1}{j}\,, \tag{4.57}$$

$$\left(\frac{\varepsilon_n}{\rho_{k_j}}\right)^d \sum_{\substack{x\in\mathbb{Z}^d \\ (Y+z)\cap\frac{\rho_{k_j}}{\varepsilon_n}Q_1\left(\frac{x_0}{\rho_{k_j}}\right)\neq\emptyset}} \alpha_j(\tau_z\omega, j\gamma_j(C_{\xi_0}\rho_{k_j})) \le \mathbb{E}\big[\alpha_{2j}(\,\cdot\,,2j\gamma_{2j}(C_{\xi_0}\rho_{k_j}))\big] + \frac{1}{2j} < \frac{1}{j}\,, \tag{4.58}$$

which is possible thanks to (4.49) and (4.51)–(4.54). Note that to obtain (4.58) we have applied (4.51) with $R = N = j$ and $R' = N' = 2j$. Let us now define the diagonal sequence $(\bar\varepsilon_j, \bar\rho_j) := (\varepsilon_{n_j}, \rho_{n_j})$; then $\frac{\bar\varepsilon_j}{\bar\rho_j} \to 0$ as $j\to\infty$ and thanks to (4.55) and (4.56) the functions $v_j := v_{n_j,k_j}$ satisfy

$$\frac{d\mu_a}{d\mathcal{L}^d}(x_0) = \Lambda(x_0) \ge \limsup_{j\to\infty}\int_{Q_1} f\big(\omega, \tfrac{x_0+\bar\rho_j y}{\bar\varepsilon_j}, \nabla v_j\big)\,\mathrm{d}y \tag{4.59}$$

and

$$\lim_{j\to\infty}\|v_j - u_0\|_{L^p(Q_1)} = 0\,. \tag{4.60}$$

Moreover, (4.57) ensures that

$$\lim_{j\to\infty}\left(\frac{\bar\varepsilon_j}{\bar\rho_j}\right)^d \mu_\infty^{s_0,\xi_0}\left(\frac{\bar\rho_j}{\bar\varepsilon_j}Q_1\big(x_0/\bar\rho_j\big),\omega\right) = Tf_\infty(s_0,\xi_0)\,. \tag{4.61}$$

Finally, for all $M,N,R>0$ the monotonicity of α_R and γ_R with respect to R and t implies that

$$\left(\frac{\bar\varepsilon_j}{\bar\rho_j}\right)^d \sum_{\substack{x\in\mathbb{Z}^d \\ (Y+z)\cap\frac{\bar\rho_j}{\bar\varepsilon_j}Q_1\left(\frac{x_0}{\bar\rho_j}\right)\neq\emptyset}} \alpha_N(\tau_z\omega, M\gamma_R(C_{\xi_0}\rho_{k_j})) \le \left(\frac{\bar\varepsilon_j}{\bar\rho_j}\right)^d \sum_{\substack{x\in\mathbb{Z}^d \\ (Y+z)\cap\frac{\bar\rho_j}{\bar\varepsilon_j}Q_1\left(\frac{x_0}{\bar\rho_j}\right)\neq\emptyset}} \alpha_j(\tau_z\omega, j\gamma_j(C_{\xi_0}\bar\rho_j)) \ \text{ for all } j\ge M,N,R.$$

Thus, from (4.58) we finally deduce that

$$\lim_{j\to\infty}\left(\frac{\bar\varepsilon_j}{\bar\rho_j}\right)^d \sum_{\substack{x\in\mathbb{Z}^d \\ (Y+z)\cap\frac{\bar\rho_j}{\bar\varepsilon_j}Q_1\left(\frac{x_0}{\bar\rho_j}\right)\neq\emptyset}} \alpha_N(\tau_z\omega, M\gamma_R(C_{\xi_0}\rho_{k_j})) = 0 \ \text{ for all } M,N,R>0. \tag{4.62}$$

Step 3. Modification of boundary values and equi-integrable gradients.
In this step we will modify the sequence (v_j) obtained in Step 2 in a suitable way to obtain a competitor for the minimization problem defining $\mu_\infty^{s_0,\xi_0}$. This is done by rewriting the right-hand side of (4.48) using the auxiliary function $\hat f$ defined in (4.3). This in turn will allow us to first apply the fundamental estimate [9, Theorem 19.1] to change the value of v_j to u_0 on ∂Q_1 and subsequently apply the Decomposition Lemma 4.1 to pass to a sequence with p-equi-integrable gradients.

By the definition of $\hat f$ and the fact that $\nabla v_j(y) = \nabla u_{n_j}(x_0+\bar\rho_j y) \in [T_{u_{n_j}(x_0+\bar\rho_j y)}\mathcal{M}]^d$ for a.e. $y\in Q_1$ we have that

$$\int_{Q_1} f\big(\omega, \tfrac{x_0+\bar\rho_j y}{\bar\varepsilon_j}, \nabla v_j(y)\big)\,\mathrm{d}y = \int_{Q_1} \hat f\big(\omega, \tfrac{x_0+\bar\rho_j y}{\bar\varepsilon_j}, \underbrace{s_0 + \bar\rho_j v_j(y)}_{=u_{n_j}(x_0+\bar\rho_j y)}, \nabla v_j(y)\big)\,\mathrm{d}y\,. \tag{4.63}$$

Thanks to (4.6) the functions $\hat{f}(\omega,\cdot,s,\cdot)$ satisfy for all $s\in\mathcal{M}$ the hypotheses of [9, Theorem 19.1] and thus the fundamental estimate [9, Definition 18.2] holds. This implies in particular that for arbitrary $\eta>0$ and $\delta\in(0,1/2)$ there exists a cut-off function $\varphi=\varphi_{\eta,\delta}$ between $Q_{1-2\delta}$ and $Q_{1-\delta}$ such that for all $j\in\mathbb{N}$ the functions $\hat{v}_j:=\varphi v_j+(1-\varphi)u_0\in W^{1,p}(Q_1,\mathbb{R}^m)$ satisfy

$$\begin{aligned}\int_{Q_1}\hat{f}\big(\omega,\tfrac{x_0+\bar{\rho}_j y}{\bar{\varepsilon}_j},s_0+\bar{\rho}_j v_j(y),\nabla\hat{v}_j(y)\big)\,\mathrm{d}y\leq{}&(1+\eta)\int_{Q_1}\hat{f}\big(\omega,\tfrac{x_0+\bar{\rho}_j y}{\bar{\varepsilon}_j},s_0+\bar{\rho}_j v_j(y),\nabla v_j(y)\big)\,\mathrm{d}y\\&+(1+\eta)\hat{c}_2|Q_1\setminus Q_{1-2\delta}|(1+|\xi_0|^p)+\frac{C}{(\delta\eta)^p}\|v_j-u_0\|^p_{L^p(Q_1)}\end{aligned}\tag{4.64}$$

for some constant $\hat{c}_2$ depending only p and the constants c_1 and c_2 in (2.4). We observe that the functions $\hat{v}_j$ do not take values in $\mathcal{M}$ anymore, but by construction they coincide with u_0 on $Q_1\setminus\overline{Q}_{1-\delta}$, so that $(\hat{v}_j)\subset u_0+W^{1,p}_0(Q_1;\mathbb{R}^m)$. Moreover, the boundedness of (v_j) in $W^{1,p}(Q_1;\mathbb{R}^m)$ ensures that $\sup_{j\in\mathbb{N}}\|\nabla\hat{v}_j\|_{L^p}<+\infty$. Thus Lemma 4.1 yields the existence of a subsequence (j_ℓ) and a sequence $(\tilde{v}_\ell)\subset u_0+W^{1,\infty}_0(Q_1;\mathbb{R}^m)$ such that $|\nabla\tilde{v}_\ell|^p$ is equi-integrable and the measure of the sets

$$A_\ell:=\big\{y\in Q_1\colon\hat{v}_{j_\ell}(y)\neq\tilde{v}_\ell(y)\text{ or }\nabla\hat{v}_{j_\ell}(y)\neq\nabla\tilde{v}_\ell(y)\big\}$$

converges to zero as $\ell\to\infty$. Moreover, by Chebyshev's inequality also the measure of the set

$$B_\ell:=\big\{y\in Q_1\colon|v_{j_\ell}(y)-u_0(y)|>1\big\}$$

converges to zero as $\ell\to\infty$. Thanks to the non-negativity of the integrand f the fundamental estimate (4.64) now yields

$$\begin{aligned}\int_{Q_1}\hat{f}\big(\omega,\tfrac{x_0+\bar{\rho}_{j_\ell}y}{\bar{\varepsilon}_{j_\ell}},s_0+\bar{\rho}_{j_\ell}v_{j_\ell}(y),\nabla v_{j_\ell}(y)\big)\,\mathrm{d}y\geq{}&\frac{1}{1+\eta}\int_{Q_1\setminus A_\ell\setminus B_\ell}\hat{f}\big(\omega,\tfrac{x_0+\bar{\rho}_{j_\ell}y}{\bar{\varepsilon}_{j_\ell}},s_0+\bar{\rho}_{j_\ell}v_{j_\ell}(y),\nabla\tilde{v}_{j_\ell}(y)\big)\,\mathrm{d}y\\&-\hat{c}_2|Q_1\setminus Q_{1-2\delta}|(1+|\xi_0|^p)-\frac{C}{(\delta\eta)^p}\|v_{j_\ell}-u_0\|^p_{L^p(Q_1)}\,.\end{aligned}\tag{4.65}$$

It remains to estimate the first term on the right-hand side of (4.65). This is done in the last step.

Step 4. Conclusion.
Since $|\nabla\tilde{v}_\ell|^p$ is equi-integrable, there exists $N_\eta\in\mathbb{N}$ sufficiently large such that for

$$G_{\ell,\eta}:=\big\{x\in Q_1\colon|\nabla\tilde{v}_\ell(x)|^p\geq N_\eta\big\}$$

we have that

$$\sup_{\ell\in\mathbb{N}}\hat{c}_2\int_{G_{\ell,\eta}}(1+|\nabla\tilde{v}_\ell|^p)\,\mathrm{d}y\leq\eta\,.$$

Moreover, since $|A_\ell\cup B_\ell|\to0$ as $\ell\to\infty$, there exists $\ell_0\in\mathbb{N}$ such that

$$\hat{c}_2\int_{A_\ell\cup B_\ell}(1+|\nabla\tilde{v}_\ell|^p)\,\mathrm{d}y<\eta\ \text{ for all }\ell\geq\ell_0\,.\tag{4.66}$$

Recalling that $\hat{f}(\omega,x,s,\xi)=f(\omega,x,\Pi_s(\xi))+|\xi-\Pi_s(\xi)|^p$ for all $s\in\mathcal{M}$ and all $(x,\xi)\in\mathbb{R}^d\times\mathbb{R}^{m\times d}$ we thus obtain

$$\begin{aligned}\int_{Q_1\setminus A_\ell\setminus B_\ell\setminus G_{\ell,\eta}}\hat{f}\big(\omega,\tfrac{x_0+\bar{\rho}_{j_\ell}y}{\bar{\varepsilon}_{j_\ell}},s_0+\bar{\rho}_{j_\ell}v_{j_\ell}(y),\nabla\tilde{v}_{j_\ell}(y)\big)\,\mathrm{d}y\geq{}&\int_{Q_1}\hat{f}\big(\omega,\tfrac{x_0+\bar{\rho}_{j_\ell}y}{\bar{\varepsilon}_{j_\ell}},s_0,\nabla\tilde{v}_{j_\ell}(y)\big)\,\mathrm{d}y-2\eta\\&-\underbrace{\left|\int_{Q_1\setminus A_\ell\setminus B_\ell\setminus G_{\ell,\eta}}\Big(f\Big(\omega,\tfrac{x_0+\bar{\rho}_{j_\ell}y}{\bar{\varepsilon}_{j_\ell}},\Pi_{s_0+\bar{\rho}_{j_\ell}v_{j_\ell}(y)}\big(\nabla\tilde{v}_{j_\ell}(y)\big)\Big)-f\Big(\omega,\tfrac{x_0+\bar{\rho}_{j_\ell}y}{\bar{\varepsilon}_{j_\ell}},\Pi_{s_0}\big(\nabla\tilde{v}_{j_\ell}(y)\big)\Big)\Big)\,\mathrm{d}y\right|}_{=:e^1_{\ell,\eta}}\\&-\underbrace{\int_{Q_1\setminus A_\ell\setminus B_\ell\setminus G_{\ell,\eta}}\left|\Big|\Pi_{s_0+\bar{\rho}_{j_\ell}v_{j_\ell}(y)}\big(\nabla\tilde{v}_\ell(y)\big)-\nabla\tilde{v}_\ell(y)\Big|^p-\Big|\Pi_{s_0}\big(\nabla\tilde{v}_\ell(y)\big)-\nabla\tilde{v}_\ell(y)\Big|^p\right|\mathrm{d}y}_{=:e^2_{\ell,\eta}}\end{aligned}\tag{4.67}$$

for all $\ell \geq \ell_0$. We finally set $\varphi_\ell(x) := \frac{\bar\rho_{j_\ell}}{\bar\varepsilon_{j_\ell}}\tilde v_\ell\big(\frac{\bar\varepsilon_{j_\ell}x-x_0}{\bar\rho_{j_\ell}}\big)+u_0\big(\frac{x_0}{\bar\varepsilon_{j_\ell}}\big)$ for every $x\in\frac{\bar\rho_{j_\ell}}{\bar\varepsilon_{j_\ell}}Q_1(x_0/\bar\rho_{j_\ell})$, so that

$$\nabla\varphi_\ell(x)=\nabla\tilde v_\ell\Big(\frac{\bar\varepsilon_{j_\ell}x-x_0}{\bar\rho_{j_\ell}}\Big)\quad\text{and}\quad(\varphi_\ell-u_0)\in W^{1,\infty}_0\big(\tfrac{\bar\rho_{j_\ell}}{\bar\varepsilon_{j_\ell}}Q_1(x_0/\bar\rho_{j_\ell});\mathbb{R}^m\big)$$

for every $\ell\in\mathbb{N}$. Recalling that $\nabla u_0\equiv\xi_0\in[T_{s_0}\mathcal{M}]^d$, the alternative characterization of $\mu_\infty^{s_0,\xi_0}$ in (4.4) together with a change of variables yields

$$\begin{aligned}\int_{Q_1}\hat f\big(\omega,\tfrac{x_0+\bar\rho_{j_\ell}y}{\bar\varepsilon_{j_\ell}},s_0,\nabla\tilde v_{j_\ell}(y)\big)\,\mathrm{d}y&=\fint_{\frac{\bar\rho_{j_\ell}}{\bar\varepsilon_{j_\ell}}Q_1\big(\frac{x_0}{\bar\rho_{j_\ell}}\big)}\hat f\big(\omega,x,s_0,\xi_0+\nabla(\varphi_\ell-u_0)(x)\big)\,\mathrm{d}x\\&\geq\left(\frac{\bar\rho_{j_\ell}}{\bar\varepsilon_{j_\ell}}\right)^d\mu_\infty^{s_0,\xi_0}\left(\frac{\bar\rho_{j_\ell}}{\bar\varepsilon_{j_\ell}}Q_1\big(x_0/\bar\rho_{j_\ell}\big)\right).\end{aligned}$$

Thus, (4.61) implies that

$$\liminf_{\ell\to\infty}\int_{Q_1}\hat f\big(\omega,\tfrac{x_0+\bar\rho_{j_\ell}y}{\bar\varepsilon_{j_\ell}},s_0+\bar\rho_{j_\ell}v_{j_\ell}(y),\nabla\tilde v_{j_\ell}(y)\big)\,\mathrm{d}y\geq Tf_{\mathrm{hom}}(s_0,\xi_0)\tag{4.68}$$

and it remains to estimate the two error terms $e^1_{\ell,\eta},e^2_{\ell,\eta}$ in (4.67). To estimate $e^2_{\ell,\eta}$ it suffices to observe that the local lipschitzianity of the mapping $\xi\mapsto|\xi|^p$ and the projection estimate $|\xi-\Pi_s\xi|\leq|\xi|$ for all $s\in\mathcal{M}$ and $\xi\in\mathbb{R}^{m\times d}$ yield

$$\Big||\xi_1-\Pi_{s_1}(\xi_1)|^p-|\xi_2-\Pi_{s_2}(\xi_2)|^p\Big|\leq p\Big(|\xi_1|^{p-1}+|\xi_2|^{p-1}\Big)\Big(\big|\Pi_{s_1}(x_1)-\Pi_{s_2}(\xi_2)\big|+\big|\xi_1-\xi_2\big|\Big)$$

for all $s_1,s_2\in\mathcal{M}$ and $\xi_1,\xi_2\in\mathbb{R}^{m\times d}$. In particular, for all $\ell\in\mathbb{N}$ and $y\in Q_1\setminus G_{\ell,\eta}$ we have that

$$\begin{aligned}&\left|\Big|\Pi_{s_0+\bar\rho_{j_\ell}v_{j_\ell}(y)}\big(\nabla\tilde v_\ell(y)\big)-\nabla\tilde v_\ell(y)\Big|^p-\Big|\Pi_{s_0}\big(\nabla\tilde v_\ell(y)\big)-\nabla\tilde v_\ell(y)\Big|^p\right|\\&\quad\leq2p|\nabla\tilde v_\ell(y)|^{p-1}\Big|\Pi_{s_0+\bar\rho_{j_\ell}v_{j_\ell}(y)}\big(\nabla\tilde v_\ell(y)\big)-\Pi_{s_0}\big(\nabla\tilde v_\ell(y)\big)\Big|\leq2pN_\eta^p\Big\|\Pi_{s_0+\bar\rho_{j_\ell}v_{j_\ell}(y)}-\Pi_{s_0}\Big\|_{\mathrm{op}}.\end{aligned}\tag{4.69}$$

We also observe that for all $\ell\in\mathbb{N}$ and $y\in Q_1\setminus B_\ell$ we have that

$$\bar\rho_{j_\ell}|v_{j_\ell}(y)|\leq\bar\rho_{j_\ell}(|u_0(y)|+1)\leq C_{\xi_0}\bar\rho_{j_\ell},\tag{4.70}$$

for which we recall that $C_{\xi_0}=\big(\frac{|\xi_0|}{2}d^{\frac12}+1\big)$. Setting $R:=|s_0|+C_{\xi_0}$ a combination of (4.69) and (4.70) thus yields

$$\left|\Big|\Pi_{s_0+\bar\rho_{j_\ell}v_{j_\ell}(y)}\big(\nabla\tilde v_\ell(y)\big)-\nabla\tilde v_\ell(y)\Big|^p-\Big|\Pi_{s_0}\big(\nabla\tilde v_\ell(y)\big)-\nabla\tilde v_\ell(y)\Big|^p\right|\leq2pN_\eta^p\gamma_R\big(C_{\xi_0}\bar\rho_{j_\ell}\big)$$

for all $y\in Q_1\setminus G_{\ell,\eta}\setminus B_\ell$, where γ_R is the modulus of continuity defined in (2.1). We thus deduce that

$$0\leq e^2_{\ell,\eta}\leq2pN_\eta^p\gamma_R\big(C_{\xi_0}\bar\rho_{j_\ell}\big)\ \to0\ \text{ as }\ \ell\to\infty\,.\tag{4.71}$$

To estimate $e^1_{\ell,\eta}$ we apply a similar argument as in (4.27) using the modulus of continuity α_{N_η}. Setting

$$A_{\ell,\eta}(x_0):=\frac{\bar\rho_{j_\ell}}{\bar\varepsilon_{j_\ell}}\big(Q_1\setminus A_\ell\setminus B_\ell\setminus G_{\ell,\eta}\big)+\frac{x_0}{\bar\varepsilon_{j_\ell}}\subset\frac{\bar\rho_{j_\ell}}{\bar\varepsilon_{j_\ell}}Q_1(x_0/\bar\rho_{j_\ell}),$$

the change of variables $x=\frac{x_0+\bar\rho_{j_\ell}y}{\bar\varepsilon_{j_\ell}}$, (4.69) and (4.70) yields

$$\begin{aligned}e^1_{\ell,\eta}&\leq\left(\frac{\bar\varepsilon_{j_\ell}}{\bar\rho_{j_\ell}}\right)^d\sum_{z\in\mathbb{Z}^d}\left|\int_{A_{\ell,\eta}(x_0)\cap(Y+z)}f\Big(\omega,x,\Pi_{s_0+\bar\rho_{j_\ell}v_{j_\ell}(y(x))}\big(\nabla\varphi_\ell(x)\big)\Big)-f\Big(\omega,x,\Pi_{s_0}\big(\nabla\varphi_\ell(x)\big)\Big)\,\mathrm{d}x\right|\\&\leq\left(\frac{\bar\varepsilon_{j_\ell}}{\bar\rho_{j_\ell}}\right)^d\sum_{\substack{z\in\mathbb{Z}^d\\(Y+z)\cap\frac{\bar\rho_{j_\ell}}{\bar\varepsilon_{j_\ell}}Q_1(x_0/\bar\rho_{j_\ell})\neq\emptyset}}\alpha_{N_\eta}\Big(\tau_z\omega,2pN_\eta^p\gamma_R\big(C_{\xi_0}\bar\rho_{j_\ell}\big)\Big).\end{aligned}$$

Applying (4.62) with $N = N_\eta$ and $M = N_\eta^p$ and R as above thus implies that

$$\lim_{\ell\to\infty} e^1_{\ell,\eta} = 0\,. \tag{4.72}$$

Combining (4.71) and (4.72) with (4.59), (4.63), (4.65), (4.67), and (4.68) we thus infer that

$$\frac{d\mu_a}{d\mathcal{L}^d}(x_0) \geq \limsup_{\ell\to\infty} \int_{Q_1} f\big(\omega, \tfrac{x_0+\bar{\rho}_{j_\ell}y}{\bar{\varepsilon}_{j_\ell}} \nabla v_\ell(y)\big)\,\mathrm{d}y \geq \frac{1}{1+\eta} Tf_\infty(s_0,\xi_0) - 2\eta - \hat{c}_2|Q_1 \setminus Q_{1-2\delta}||\xi_0|^p$$

from which we finally deduce (4.46) by letting first $\eta \to 0$ and then $\delta \to 0$. □

4.3. **Construction of recovery sequences.** In this section we provide recovery sequences for all functions $u \in W^{1,p}(U;\mathcal{M})$ on general bounded, open sets $U \subset \mathbb{R}^d$. We first start with a local construction, giving an almost upper bound with the functional $\int Tf_k(u,\nabla u)\,\mathrm{d}x$ with the integrand Tf_k given by Proposition 4.8. Then we use this one and a covering argument to obtain the global recovery sequence with the auxiliary integrand. Finally, we conclude via approximation letting $k \to +\infty$ and a diagonal argument. In this section we fix an element $\omega \in \Omega$ such that Lemma 2.10 and Propositions 4.8 and 4.9 hold, so that all statements hold $\mathbb{P}$-almost surely.

Lemma 4.11. *For all $s \in \mathcal{M}$, $\xi \in [T_s\mathcal{M}]^d$, $k \in \mathbb{N}$ and $\eta \in (0,1]$ there exists $\delta > 0$ satisfying the following property: for every cube $Q = Q_r(x_0) \subset \mathbb{R}^d$ with $0 < r < \delta$ and for every $u \in W^{1,p}(Q;\mathcal{M})$ satisfying*

$$\fint_Q |u - s|^p + |\nabla u - \xi|^p\,\mathrm{d}x \leq \delta, \tag{4.73}$$

there exists a sequence $(z_\varepsilon)_\varepsilon \subset W^{1,p}(Q;\mathcal{M})$ with $z_\varepsilon = u$ on ∂Q and

$$\limsup_{\varepsilon\to 0} \fint_Q f(\omega, \tfrac{x}{\varepsilon}, \nabla z_\varepsilon)\,\mathrm{d}x \leq \fint_Q Tf_k(u,\nabla u)\,\mathrm{d}x + \eta. \tag{4.74}$$

Additionally, there exists $c < +\infty$ depending on the dimensions d, m, the exponent p and the constants c_1, c_2 in (2.4) such that

$$\limsup_{\varepsilon\to 0} \fint_Q |u - z_\varepsilon|^p\,\mathrm{d}x \leq c\, r^p \fint_Q 1 + |\nabla u|^p\,\mathrm{d}x. \tag{4.75}$$

Proof. We first show (4.75) using (4.74). From Poincaré's inequality with its scaling (recall that $u - z_\varepsilon = 0$ on ∂Q) and the lower bound in (2.4) we infer that

$$\begin{aligned}\fint_Q |u - z_\varepsilon|^p\,\mathrm{d}x \leq & Cr^p \fint_Q |\nabla(u - z_\varepsilon)|^p \leq Cr^p \fint_Q |\nabla u|^p + |\nabla z_\varepsilon|^p\,\mathrm{d}x \\ \leq & Cr^p \fint_Q |\nabla u|^p + f(\omega, \tfrac{x}{\varepsilon}, \nabla z_\varepsilon)\,\mathrm{d}x\end{aligned}$$

and thus (4.75) follows with help of (4.74) and the upper bound $Tf_k(s,\xi) \leq c_2(1 + |\xi|^p)$ that one obtains from Lemma 4.4. Thus we are left to show (4.74). Throughout this proof we assume that $\delta \leq 1$ and refine its smallness depending on s, ξ, k and η. Since the latter parameters are fixed, we do not indicate when quantities depend on them.
Due to Proposition 4.8 with $t = 1/\varepsilon$, we find $\varepsilon_0 > 0$ such that for all $0 < \varepsilon < \varepsilon_0$ there exists $\varphi_\varepsilon \in \mathrm{Lip}(\mathbb{R}^d; T_s\mathcal{M})$ satisfying $\varphi_\varepsilon \equiv 0$ on $\mathbb{R}^d \setminus (\varepsilon^{-1}Q)$ with $\|\varphi_\varepsilon\|_{W^{1,\infty}(\mathbb{R}^d)} \leq k$ and

$$\fint_{\varepsilon^{-1}Q} f(\omega, x, \xi + \nabla\varphi_\varepsilon)\,\mathrm{d}x \leq \frac{\varepsilon^d}{|Q|} \mu_k^{s,\xi}(\varepsilon^{-1}Q,\omega) + \varepsilon \leq Tf_k(s,\xi) + \frac{1}{4}\eta. \tag{4.76}$$

Next, let $\kappa > 0$ be such that $\overline{B_{4\kappa}(s)} \subset U$, where U is the neighborhood of $\mathcal{M}$ given by Theorem 2.1 (with, e.g., $\lambda = 1$). Consider a cut-off function $\theta_\kappa \in C_c^\infty(\mathbb{R}^m;[0,1])$ such that

$$\theta_\kappa \equiv 1 \text{ on } B_\kappa(0), \quad \theta_\kappa \equiv 0 \text{ on } \mathbb{R}^m \setminus B_{3\kappa}(0), \quad \|\nabla\theta_\kappa\|_{L^\infty(\mathbb{R}^m)} \leq \frac{1}{\kappa}. \tag{4.77}$$

We define $u_s \in W^{1,p}(Q;\mathbb{R}^m)$ by setting $u_s(x) := u(x) - s$ for every $x \in Q$ and we then define

$$\phi_{\varepsilon,\kappa} := u + (\theta_\kappa \circ u_s)\varepsilon\varphi_\varepsilon(\cdot/\varepsilon) \in u + W_0^{1,p}(Q;T_s\mathcal{M}). \tag{4.78}$$

Our aim is to map $\phi_{\varepsilon,\kappa}$ onto $\mathcal{M}$, so let us estimate its distance to points in $\mathcal{M}$. In what follows we shall always assume that $\varepsilon < \kappa/k$. Since $\theta_\kappa \circ u_s$ vanishes whenever $|u_s(x)| \geq 3\kappa$, we can distinguish two cases:

$$\phi_{\varepsilon,\kappa}(x) = u(x) \in \mathcal{M} \text{ a.e. if } |u_s(x)| \geq 3\kappa, \tag{4.79}$$

$$|\phi_{\varepsilon,\kappa}(x) - s| \leq |\phi_{\varepsilon,\kappa}(x) - u(x)| + |u_s(x)| < k\varepsilon + 3\kappa \leq 4\kappa \text{ if } |u_s(x)| < 3\kappa. \tag{4.80}$$

Hence, up to considering a representative of u that is $\mathcal{M}$-valued everywhere, we can define the map $z_\varepsilon^\kappa = \pi^* \circ \phi_{\varepsilon,\kappa}$, where π^* is as in Theorem 2.1. Below we show that $z_\varepsilon^\kappa \in W^{1,p}(Q;\mathcal{M})$. We first observe that by construction z_ε^k takes values in $\mathcal{M}$ and is weakly differentiable by the chain rule, since $\pi^* \in C^1(U;\mathcal{M})$ with U being the neighborhood of $\mathcal{M}$ given by Theorem 2.1. Moreover, since $\pi^*_{|\mathcal{M}} = \mathrm{Id}$, we have that $z_\varepsilon^\kappa = u$ on the set $\{|u_s| \geq 3\kappa\}$, while on the set $\{|u_s| < 3\kappa\}$ we deduce from (4.80) that

$$\begin{aligned} |z_\varepsilon^\kappa| &\leq |s| + |\pi^*(\phi_{\varepsilon,\kappa,k}) - \pi^*(s)| \leq |s| + \|\nabla\pi^*\|_{L^\infty(B_{4\kappa}(s))}|\phi_{\varepsilon,\kappa} - s| \\ &\leq |s| + 4\|\nabla\pi^*\|_{L^\infty(B_{4\kappa}(s))}\kappa. \end{aligned} \tag{4.81}$$

In particular, we deduce that $z_\varepsilon^\kappa \in L^p(Q;\mathcal{M})$. For the gradient, the locality of the weak derivative yields $\nabla z_\varepsilon^\kappa = \nabla u$ a.e. on $\{|u_s| \geq 3\kappa\}$, while on $\{|u_s| < 3\kappa\}$ the chain rule and the properties of θ_κ summarized in (4.77) yield

$$\begin{aligned} |\nabla z_\varepsilon^\kappa| &\leq |\nabla\pi^*(\phi_{\varepsilon,\kappa})||\nabla\phi_{\varepsilon,\kappa}| \leq \|\nabla\pi^*\|_{L^\infty(B_{4\kappa}(s))}(|\nabla u| + \|\nabla\theta_\kappa\|_{L^\infty(\mathbb{R}^m)}|\nabla u|\varepsilon k + k) \\ &\leq \|\nabla\pi^*\|_{L^\infty(B_{4\kappa}(s))}(2|\nabla u| + k) \end{aligned} \tag{4.82}$$

Hence also $|\nabla z_\varepsilon^\kappa| \in L^p(Q)$, so that $z_\varepsilon^\kappa \in W^{1,p}(Q;\mathcal{M})$. Moreover, since $\phi_{\varepsilon,\kappa} = u$ on ∂Q, it follows that also $z_\varepsilon^\kappa = u$ on ∂Q. At the end we will choose κ small enough, but fixed. Hence we are left to show the energy estimate (4.74) for z_ε^κ.

Using again the chain rule and splitting the domain of integration, by the upper bound on f in (2.4) we have that

$$\begin{aligned} \fint_Q f(\omega, \tfrac{x}{\varepsilon}, \nabla z_\varepsilon^\kappa)\,\mathrm{d}x &\leq \frac{c_2}{|Q|}\int_{Q\cap\{|u-s|\geq\kappa\}}(1 + |\nabla z_\varepsilon^\kappa|^p)\,\mathrm{d}x + \frac{c_2}{|Q|}\int_{Q\cap\{|\nabla u-\xi|\geq\kappa\}}(1 + |\nabla z_\varepsilon^\kappa|^p)\,\mathrm{d}x \\ &\quad + \frac{1}{|Q|}\int_{Q\cap\{|u-s|<\kappa,|\nabla u-\xi|<\kappa\}} f(\omega, \tfrac{x}{\varepsilon}, \nabla z_\varepsilon^\kappa)\,\mathrm{d}x\,. \end{aligned} \tag{4.83}$$

We separately estimate the three right-hand side terms, arguing that the first two are small. In order to reduce notation, let us set $c_{s,\kappa} = \|\nabla\pi^*\|_{L^\infty(B_{4\kappa}(s))}$. Using the bound (4.82) or its alternative $|\nabla z_\varepsilon^k| = |\nabla u|$, we find that

$$\begin{aligned} \frac{c_2}{|Q|}\int_{Q\cap\{|u-s|\geq\kappa\}}(1 + |\nabla z_\varepsilon^\kappa|^p)\,\mathrm{d}x &\leq \frac{2^p c_2}{|Q|}(c_{s,\kappa}^p + 1)\int_{Q\cap\{|u-s|\geq\kappa\}}|\nabla u|^p + k^p\,\mathrm{d}x \\ &\leq \frac{4^p c_2}{|Q|}(c_{s,\kappa}^p + 1)\int_{Q\cap\{|u-s|\geq\kappa\}}|\nabla u - \xi|^p + |\xi|^p + k^p\,\mathrm{d}x \\ &\leq 4^p c_2(c_{s,\kappa}^p + 1)\left(\fint_Q |\nabla u - \xi|^p\,\mathrm{d}x + \frac{|\xi|^p + k^p}{\kappa^p}\fint_Q |u - s|^p\,\mathrm{d}x\right) \\ &\overset{(4.73)}{\leq} 4^p c_2(c_{s,k}^p + 1)\left(1 + \frac{|\xi|^p + k^p}{\kappa^p}\right)\delta. \end{aligned} \tag{4.84}$$

Similarly, with the same bound for $\nabla z_\varepsilon^\kappa$ we can estimate the second term via

$$\begin{aligned}\frac{c_2}{|Q|}\int_{Q\cap\{|\nabla u-\xi|\geq\kappa\}}(1+|\nabla z_\varepsilon^\kappa|^p)\,\mathrm{d}x &\leq \frac{2^p c_2}{|Q|}(c_{s,\kappa}^p+1)\int_{Q\cap\{|\nabla u-\xi|\geq\kappa\}}|\nabla u|^p+k^p\,\mathrm{d}x\\ &\leq \frac{4^p c_2}{|Q|}(c_{s,\kappa}^p+1)\int_{Q\cap\{|\nabla u-\xi|\geq\kappa\}}|\nabla u-\xi|^p+|\xi|^p+k^p\,\mathrm{d}x\\ &\leq 4^p c_2(c_{s,\kappa}^p+1)\left(1+\frac{|\xi|^p+k^p}{\kappa^p}\right)\fint_Q|\nabla u-\xi|^p\,\mathrm{d}x\\ &\leq 4^p c_2(c_{s,\kappa}^p+1)\left(1+\frac{|\xi|^p+k^p}{\kappa^p}\right)\delta. \qquad (4.85)\end{aligned}$$

Finally, we estimate the third term. Here we have to be more careful. When $|u-s|<\kappa$ and $|\nabla u-\xi|<\kappa$, the definition of $\phi_{\varepsilon,\kappa}$ reduces to $\theta_\kappa\equiv 1$ and therefore, by the chain rule and locality of the weak derivative we have that

$$\nabla z_\varepsilon^\kappa=\nabla\pi^*(\phi_{\varepsilon,\kappa})\left(\nabla u+\nabla\varphi_\varepsilon(\cdot/\varepsilon)\right).$$

Moreover, Theorem 2.1 yields that $\pi^*_{|\mathcal{M}}=\mathrm{Id}$, which implies that $\nabla\pi^*(s')\xi'=\xi'$ for all $s'\in\mathcal{M}$ and $\xi'\in T_{s'}\mathcal{M}$. This in turn can be used to estimate

$$\begin{aligned}|\nabla z_\varepsilon^\kappa-\xi-\nabla\varphi_\varepsilon(\cdot/\varepsilon)| &= |\nabla\pi^*(\phi_{\varepsilon,\kappa})\left(\nabla u+\nabla\varphi_{\varepsilon,\kappa}(\cdot/\varepsilon)\right)-\nabla\pi^*(s)\left(\xi+\nabla\varphi_{\varepsilon,\kappa}(\cdot/\varepsilon)\right)|\\ &\leq |\nabla\pi^*(\phi_{\varepsilon,\kappa,k})-\nabla\pi^*(s)|\,|\nabla u+\nabla\varphi_{\varepsilon,k}(\cdot/\varepsilon)|+|\nabla\pi^*(s)||\nabla u-\xi|\\ &\overset{(4.80)}{\leq}\sup_{s'\in B_{4\kappa}(s)}|\nabla\pi^*(s')-\nabla\pi^*(s)|(|\nabla u-\xi|+|\xi|+k)+|\nabla\pi^*(s)||\nabla u-\xi|\\ &\leq\sup_{s'\in B_{4\kappa}(s)}|\nabla\pi^*(s')-\nabla\pi^*(s)|(\kappa+|\xi|+k)+|\nabla\pi^*(s)|\kappa=:e_\kappa. \qquad (4.86)\end{aligned}$$

Note that e_κ vanishes when $\kappa\to0$, since $\pi^*\in C^1(U;\mathcal{M})$. We also have the bound $|\xi+\nabla\varphi_\varepsilon(\cdot/\varepsilon)|\leq|\xi|+k$. Upon choosing κ sufficiently small such that $e_\kappa\leq1$, by the definition of the modulus of continuity in Definition 2.7 and (4.86), via a change of variables and stationarity we can estimate

$$\begin{aligned}\frac{1}{|Q|}\int_{B\cap\{|u-s|<\kappa,|\nabla u-\xi|<\kappa\}}f(\omega,\tfrac{x}{\varepsilon},\nabla z_\varepsilon^\kappa)\,\mathrm{d}x &=\frac{1}{|Q|}\sum_{z\in\mathbb{Z}^d}\int_{\varepsilon(Y+z)\cap Q\cap\{|u-s|<\kappa,|\nabla u-\xi|<\kappa\}}f(\omega,\tfrac{x}{\varepsilon},\nabla z_\varepsilon^\kappa)\,\mathrm{d}x\\ &\leq\frac{1}{|\varepsilon^{-1}Q|}\sum_{\substack{z\in\mathbb{Z}^d\\(Y+z)\cap\varepsilon^{-1}Q\neq\emptyset}}\alpha_{|\xi|+k+1}(\tau_z\omega,e_\kappa)\\ &\quad+\fint_Q f\big(\omega,\tfrac{x}{\varepsilon},\xi+\nabla\varphi_\varepsilon(\tfrac{x}{\varepsilon})\big)\,\mathrm{d}x.\end{aligned}$$

The term in the last line can be bounded via (4.76), while for the second one we use (2.8) in Lemma 2.10. Gathering these bounds along with (4.84) and (4.85) and the starting estimate (4.83), we find that

$$\begin{aligned}\limsup_{\varepsilon\to0}\fint_Q f(\omega,\tfrac{x}{\varepsilon},\nabla z_\varepsilon^\kappa)\,\mathrm{d}x &\leq 2\cdot4^p c_2(c_{s,k}^p+1)\left(1+\frac{|\xi|^p+k^p}{\kappa^p}\right)\delta+\mathbb{E}[\alpha_{|\xi|+2k+1}(\cdot,2e_\kappa)]\\ &\quad+\fint_Q Tf_k(s,\xi)+\frac{\eta}{4}\,\mathrm{d}x. \qquad (4.87)\end{aligned}$$

Using Lemma 2.9 and the convergence $e_\kappa\to0$ as $\kappa\to0$, we can select κ small enough such that the term involving the expectation is bounded by $\eta/4$ and having fixed such $\kappa=\kappa(s,\xi,\eta)$ we can then select δ small enough such that the first right-hand side term is also bounded by $\eta/4$ (recall that s,ξ and k are fixed). Finally, we have to estimate the integrated difference of $Tf_k(s,\xi)$ and $Tf_k(u,\nabla u)$. We know from Proposition 4.8 that Tf_k is continuous at (s,ξ), so there exists $\rho_\eta>0$ such that

$$|Tf_k(s,\xi)-Tf_k(s',\xi')|\leq\frac{\eta}{8}\quad\text{for all } s'\in\mathcal{M},\ \xi'\in[T_{s'}\mathcal{M}]^d \text{ with } |s-s'|+|\xi-\xi'|\leq\rho_\eta.$$

By the non-negativity of Tf_k and (4.73) we therefore obtain

$$\begin{aligned}\fint_Q Tf_k(s,\xi)-Tf_k(u,\nabla u)\,\mathrm{d}x &\leq \frac{1}{|Q|}\int_{Q\cap\{|u-s|+|\nabla u-\xi|>\rho_\eta\}} Tf_k(s,\xi)-Tf_k(u,\nabla u)\,\mathrm{d}x+\frac{\eta}{8}\\ &\leq 2^{p-1}\frac{Tf_k(s,\xi)}{\rho_\eta^p}\fint_Q |u-s|^p+|\nabla u-\xi|^p\,\mathrm{d}x+\frac{\eta}{8}\\ &\leq 2^{p-1}\frac{Tf_k(s,\xi)}{\rho_\eta^p}\delta+\frac{\eta}{8}\end{aligned}$$

Hence, further refining δ if necessary, the above term is bounded by $\eta/4$, which together with (4.87) concludes the proof. □

Next we use the above local construction to construct recovery sequences on general bounded, open sets.

Proposition 4.12. *Let $U\subset\mathbb{R}^d$ be open and bounded and let $u\in W^{1,p}(U;\mathcal{M})$. Then there exists a sequence $(u_\varepsilon)_\varepsilon\subset W^{1,p}(U;\mathcal{M})$ such that $u_\varepsilon\to u$ in $L^p(U;\mathbb{R}^m)$ and*

$$\limsup_{\varepsilon\to 0} F_\varepsilon(\omega)(u_\varepsilon,U)\leq\int_U Tf_\infty(u,\nabla u)\,\mathrm{d}x.$$

Moreover, $u_\varepsilon=u$ in a neighborhood of ∂U.

Proof. It suffices to construct for every $k\in\mathbb{N}$ and sequence $(u^k_\varepsilon)_\varepsilon\subset W^{1,p}(U;\mathcal{M})$ such that $u^k_\varepsilon=u$ in a neighborhood of ∂U, $u^k_\varepsilon\to u$ in $L^p(U;\mathbb{R}^m)$ as $\varepsilon\to 0$ and

$$\limsup_{\varepsilon\to 0} F_\varepsilon(\omega)(u^k_\varepsilon,U)\leq\int_U Tf_k(u,\nabla u)\,\mathrm{d}x, \tag{4.88}$$

where Tf_k is the integrand given by Proposition 4.8. Indeed, the claim then follows from Proposition 4.9 and the dominated convergence theorem (recall the bound $0\leq Tf_k(s,\xi)\leq c_2(1+|\xi|^p)$) combined with a diagonal argument. Note that from now on k will be fixed, so we do not indicate when quantities depend on k.

<u>Step 1.</u> Fix $\eta\in(0,1]$. For all $j\in\mathbb{N}$ we will construct a sequence $(u^j_\varepsilon)_\varepsilon\subset W^{1,p}(U;\mathcal{M})$ with $u^j_\varepsilon=u$ in a neighborhood of ∂U and an open set $U_j\subset U$ satisfying

$$|U_j|\leq 2^{-j}|U| \quad\text{and}\quad u^j_\varepsilon\equiv u \quad\text{on } U_j, \tag{4.89}$$

$$\limsup_{\varepsilon\to 0}\int_{U\setminus U_j} f(\omega,\tfrac{x}{\varepsilon},\nabla u^j_\varepsilon)\,\mathrm{d}x\leq\int_{U\setminus U_j} Tf_k(u,\nabla u)+\eta\,\mathrm{d}x, \tag{4.90}$$

$$\limsup_{\varepsilon\to 0}\int_{U\setminus U_j}|u^j_\varepsilon-u|^p\,\mathrm{d}x\leq C\eta^p\int_{U\setminus U_j}1+|\nabla u|^p\,\mathrm{d}x, \tag{4.91}$$

where $C\in[1,+\infty)$ is the constant in (4.75) (in particular it does not depend on η).
We construct u^j_ε recursively. For $j=0$, we set $U_0=U$ and $u^0_\varepsilon=u$ and the verification of (4.89)–(4.91) is straightforward as $U\setminus U_0=\emptyset$.
Fix $j\in\mathbb{N}$ and assume that $(u^j_\varepsilon)_\varepsilon\subset W^{1,p}(U;\mathcal{M})$ with $u^j_\varepsilon=u$ in a neighborhood of ∂U and the open set $U_j\subset U$ satisfy (4.89)–(4.91). Let $L\subset U$ be the set of Lebesgue points of u and ∇u[3] and such that $u(x)\in\mathcal{M}$ and $\nabla u(x)\in[T_{u(x)}\mathcal{M}]^d$. For $x\in L$ let $\delta(x)>0$ be given by Lemma 4.11 for the choice $s=u(x)$, $\xi=\nabla u(x)$ and $\eta>0$. For defining u^{j+1}_ε and the open set $U_{j+1}\subset U$, we fix for any $x\in L\cap U_j$ a length $r_j(x)\in(0,\eta)$ such that $\overline{Q_{r_j(x)}(x)}\subset U_j$ and for all $0<r<r_j(x)$

$$\fint_{Q_r(x)}|u^j_\varepsilon-u(x)|^p+|\nabla u^j_\varepsilon-\nabla u(x)|^p\,\mathrm{d}y=\fint_{Q_r(x)}|u-u(x)|^p+|\nabla u-\nabla u(x)|^p\,\mathrm{d}y\leq\delta(x).$$

[3]Actually we need that $\lim_{r\to 0}\fint_{Q_r(x)}|u-u(x)|^p+|\nabla u-\nabla u(x)|^p\,\mathrm{d}y=0$ for all $x\in L$, which is slightly stronger than just x being a Lebesgue point of u and ∇u. Still this set of points has full measure as a consequence of [15, Corollary 1, Ch. 1.7.1].

Applying [16, Theorem 1.149 and Remark 1.151] to the family of closed cubes $\{\overline{Q_{r_j}(x)}\colon x\in L\cap U_j\}$ and the Lebesgue measure we obtain a countable family of disjoint cubes $\{Q_{r_\ell}(x_\ell)\}_{\ell\in\mathbb{N}}$ that covers U_j up to a null set. Then we can pick finitely many cubes $\{Q_{r_\ell}(x_\ell)\}_{\ell=1}^{N_j}$ such that

$$\Big|\bigcup_{\ell=1}^{N_j}Q_{r_\ell}(x_\ell)\Big|\geq\frac{1}{2}|U_j|\,. \tag{4.92}$$

In each of these cubes we apply Lemma 4.11 to $s=u(x)$, $\xi=\nabla u(x)$, the given $\eta>0$ and the function $u\in W^{1,p}(Q_{r_\ell}(x_\ell);\mathcal{M})$ to obtain the corresponding family $z_\varepsilon^\ell\in W^{1,p}(Q_{r_\ell}(x_\ell);\mathcal{M})$ with $z_\varepsilon^\ell=u$ on $\partial Q_{r_\ell}(x_\ell)$ and such that

$$\limsup_{\varepsilon\to0}\int_{Q_{r_\ell}(x_\ell)}f(\omega,\tfrac{x}{\varepsilon},\nabla z_\varepsilon^\ell)\,\mathrm{d}x\leq\int_{Q_{r_\ell}(x_\ell)}Tf_k(u,\nabla u)+\eta\,\mathrm{d}x, \tag{4.93}$$

$$\limsup_{\varepsilon\to0}\int_{Q_{r_\ell}(x_\ell)}|u-z_\varepsilon^\ell|^p\leq Cr_\ell^p\int_{Q_{r_\ell}(x_\ell)}1+|\nabla u|^p\,\mathrm{d}x. \tag{4.94}$$

We set

$$u_\varepsilon^{j+1}:=\begin{cases}u_\varepsilon^j & \text{on } U\setminus\bigcup_{\ell=1}^{N_j}Q_{r_\ell}(x_\ell),\\ z_\varepsilon^\ell & \text{on } Q_{r_\ell}(x_\ell),\ 1\leq\ell\leq N_j,\end{cases}\qquad\text{and}\qquad U_{j+1}:=U_j\setminus\bigcup_{\ell=1}^{N_j}\overline{Q_{r_\ell}(x_\ell)}.$$

In this way U_{j+1} is open and $u_\varepsilon^{j+1}=u_\varepsilon^j=u$ on U_{j+1}. Moreover, since $\overline{Q_{r_\ell}(x_\ell)}\subset U_j$, the property $u_\varepsilon^j=u$ on U_j and $z_\varepsilon^\ell=u$ on $\partial Q_{r_\ell}(x_\ell)$ imply that $u_\varepsilon^{j+1}\in W^{1,p}(U;\mathcal{M})$. Finally, in order to pass from u_ε^j to u_ε^{j+1} we only modified the map on cubes that are compactly contained in U, so that by induction also u_ε^{j+1} in a neighborhood of ∂U. It thus only remains to show that the estimates in (4.89)–(4.91) are satisfied. Thanks to (4.92) we have that

$$|U_{j+1}|\leq|U_j|-\Big|\bigcup_{k=\ell}^{N_j}Q_{r_\ell}(x_\ell)\Big|\leq\frac{1}{2}|U_j|\leq2^{-(j+1)}|U|,$$

which is (4.89) for $j+1$. Next, we establish the estimates (4.90) and (4.91) for $j+1$. By definition

$$U\setminus U_{j+1}=(U\setminus U_j)\cup\bigcup_{\ell=1}^{N_j}\overline{Q_{r_\ell}(x_\ell)}$$

and thus (4.90) applied for j and (4.93) yield

$$\begin{aligned}\limsup_{\varepsilon\to0}\int_{U\setminus U_{j+1}}f(\omega,\tfrac{x}{\varepsilon},\nabla u_\varepsilon^{j+1})\,\mathrm{d}x&\leq\limsup_{\varepsilon\to0}\int_{U\setminus U_j}f(\omega,\tfrac{x}{\varepsilon},\nabla u_\varepsilon^j)\,\mathrm{d}x+\sum_{\ell=1}^{N_j}\limsup_{\varepsilon\to0}\int_{Q_{r_\ell}(x_\ell)}f(\omega,\tfrac{x}{\varepsilon},\nabla z_\varepsilon^\ell)\,\mathrm{d}x\\&\leq\int_{U\setminus U_j}Tf_k(u,\nabla u)+\eta\,\mathrm{d}x+\sum_{\ell=1}^{N_j}\int_{Q_{r_\ell}(x_\ell)}Tf_k(u,\nabla u)+\eta\,\mathrm{d}x\\&=\int_{U\setminus U_{j+1}}Tf_k(u,\nabla u)+\eta\,\mathrm{d}x.\end{aligned}$$

This yields (4.90) for $j+1$. Similarly, using (4.91) for j and (4.94) we infer that

$$\begin{aligned}\limsup_{\varepsilon\to0}\int_{U\setminus U_{j+1}}|u_\varepsilon^{j+1}-u|^p\,\mathrm{d}x&\leq\limsup_{\varepsilon\to0}\int_{U\setminus U_j}|u_\varepsilon^j-u|^p\,\mathrm{d}x+\sum_{\ell=1}^{N_j}\limsup_{\varepsilon\to0}\int_{Q_{r_\ell}(x_\ell)}|z_\varepsilon^\ell-u|^p\,\mathrm{d}x\\&\leq C\eta^p\int_{U\setminus U_j}1+|\nabla u|^p\,\mathrm{d}x+\sum_{\ell=1}^{N_j}Cr_\ell^p\int_{Q_{r_\ell}(x_\ell)}1+|\nabla u|^p\,\mathrm{d}x\\&\leq C\eta^p\int_{U\setminus U_{j+1}}1+|\nabla u|^p\,\mathrm{d}x,\end{aligned}$$

where we used that $r_\ell(x)\in(0,\eta)$ in the last inequality. This concludes Step 1.

Step 2. Conclusion. Appealing to Step 1, for $\eta \in (0,1]$ and every $j \in \mathbb{N}$ we find a sequence $(u_\varepsilon^j)_\varepsilon \subset W^{1,p}(U;\mathcal{M})$ such that $u_\varepsilon^j = u$ in a neighborhood of ∂U and an open set $U_j \subset U$ satisfying (4.89),

$$\begin{aligned}\limsup_{\varepsilon\to 0}\int_U f(\omega,\tfrac{x}{\varepsilon},\nabla u_\varepsilon^j)\,\mathrm{d}x &\le \limsup_{\varepsilon\to 0}\int_{U\setminus U_j} f(\omega,\tfrac{x}{\varepsilon},\nabla u_\varepsilon^j)\,\mathrm{d}x + \limsup_{\varepsilon\to 0}\int_{U_j} f(\omega,\tfrac{x}{\varepsilon},\nabla u)\,\mathrm{d}x\\ &\le \int_{U\setminus U_j} Tf_k(u,\nabla u)+\eta\,\mathrm{d}x + c_2\int_{U_j} 1+|\nabla u|^p\,\mathrm{d}x\end{aligned}$$

and

$$\limsup_{\varepsilon\to 0}\int_U |u_\varepsilon^j - u|^p\,\mathrm{d}x \le C\eta^p\int_U 1+|\nabla u|^p\,\mathrm{d}x.$$

The above two limits in combination with $|\nabla u|^p \in L^1(U)$ and (4.89) yield

$$\begin{aligned}\limsup_{j\to+\infty}\limsup_{\varepsilon\to 0}\int_U f(\omega,\tfrac{x}{\varepsilon},\nabla u_\varepsilon^j)\,\mathrm{d}x &\le \int_U Tf_k(u,\nabla u)+\eta\,\mathrm{d}x,\\ \limsup_{j\to+\infty}\limsup_{\varepsilon\to 0}\int_U |u_\varepsilon^j-u|^p\,\mathrm{d}x &\le C\eta^p\int_U 1+|\nabla u|^p\,\mathrm{d}x,\end{aligned}$$

and the claim (4.88) follows from the arbitrariness of $\eta \in (0,1]$ and a diagonal sequence argument. Note that the latter still gives a sequence agreeing with u in a neighborhood of ∂U. □

4.4. Convergence of boundary-value problems.

We directly start with the proof of the convergence result including Dirichlet boundary conditions.

Proof of Theorem 3.2. We start with the coercivity property. Whenever a family of maps u_ε satisfies $\sup_{\varepsilon\in(0,1)} F_\varepsilon(\omega)(u_\varepsilon, D) < +\infty$, then the lower bound on f in (2.4) implies that ∇u_ε is bounded in $L^p(D;\mathbb{R}^{m\times d})$. Together with Poincaré's inequality (applied to $u_\varepsilon - g$) this implies that u_ε is bounded in $W^{1,p}(D;\mathbb{R}^m)$, so that up to subsequences we have that $u_\varepsilon \rightharpoonup u$ in $W^{1,p}(D;\mathbb{R}^m)$. We may also assume that $u_\varepsilon \to u$ pointwise a.e. in D. Since M is closed, this yields that $u(x) \in \mathcal{M}$ a.e. in D, so that $u \in W^{1,p}(D;\mathcal{M})$. The condition $u = g$ on ∂D is satisfied by the weak continuity of the trace operator.

The Γ-$\liminf$ inequality follows from the corresponding one without boundary conditions in Proposition 4.10 since by the first part of the proof the boundary conditions are stable for sequences with equibounded energy and these are the only ones to consider for the lower bound.

Finally, the existence of recovery sequences for $u \in W^{1,p}(D;\mathcal{M})$ such that $u = g$ on ∂D is a direct consequence of Proposition 4.12 that provides recovery sequences u_ε such that $u_\varepsilon = u$ in a neighborhood of ∂D, so that in particular $u_\varepsilon = g$ on ∂D. □

Acknowledgments

The research of MR is funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) – Project n° 530813503.

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