# ON THE REGULARITY OF THE SOLUTION AND THE FREE BOUNDARY IN A WEIGHTED $p-$LAPLACIAN PROBLEM

SAMER DWEIK

Abstract. In this paper, we study the regularity of the minimizer in the Alt-Caffarelli type minimum problem for the "weighted" $p-$Laplace operator $(1 < p < \infty)$ with free boundary:

$$\min\left\{ \int_\Omega (w|\nabla u|^p + \psi\,\chi_{\{u>0\}}) \,:\, u \in W^{1,p}(\Omega),\ u \geq 0,\ u = g \ \text{ on } \ \partial\Omega \right\},$$

where $w$ and $\psi$ are two given nonnegative functions on $\Omega$ and $g$ is a nonnegative boundary datum. More precisely, under the assumptions that $w$ is a $C^2$ function with $w \geq w_{\min} > 0$ and $\psi$ belongs to $L^q_{loc}(\Omega)$ for some $q > \frac{N}{p}$, we will show that a minimizer $u$ is locally $\alpha-$Hölderian with $\alpha = 1 - \frac{N}{pq}$. If $\psi$ belongs to $L^\infty_{loc}(\Omega)$ and is bounded away from zero then, thanks to the Lipschitz regularity of $u$, we will be able also to prove that the free boundary $\partial\{u > 0\}$ is locally of finite perimeter.

## 1. Introduction

Given a bounded Lipschitz domain $\Omega \subset \mathbb{R}^N$ and two functions $w,\, \psi : \Omega \mapsto \mathbb{R}^+$, we study the problem of minimizing the energy functional

$$u \mapsto \int_\Omega w|\nabla u|^p + \psi\,\chi_{\{u>0\}}$$

among all functions $u \in W^{1,p}(\Omega)$ with boundary condition $u = g$ on $\partial\Omega$, where $\chi_A$ denotes the characteristic function of set $A$. It is not difficult to see that any minimizer $u$ solves the following $p-$Laplace equation:

$$\Delta_p u = 0 \qquad \text{in} \qquad \{u > 0\}$$

where $\Delta_p u := \nabla \cdot [|\nabla u|^{p-2}\nabla u]$. This problem is referred to as Bernoulli-type free boundary problem and is well studied in the literature. The case $p = 2$ has been studied first by H. W. Alt and L. A. Caffarelli in [1] and later, for any $1 < p < \infty$ in [3].

In this paper, we are interested in studying the Hölder regularity of a minimizer $u$ as well as the regularity of the free boundary $\Gamma = \partial\{u > 0\} \cap \Omega$. The main difference in [3] from what was done in [1, 2] (where the authors consider the Laplacian case $p = 2$) is that the $p-$Laplacian (for $p \neq 2$) is not uniformly elliptic (degenerate for $p > 2$ and singular for $1 < p < 2$). On the other side, the authors of [3] consider Problem (2.1) but in the case when $w = 1$ and $\psi$ is also constant. The presence of non-uniform functions $w$ and $\psi$ in (2.1) makes the problem somehow more complicated. In [3, 1], the Lipschitz regularity of a minimizer $u$ has been proven. Now, suppose that $\psi$ is not constant or even unbounded, what kind of regularity can we prove on the minimizer $u$? Can we still prove Lipschitz (or perhaps Hölder) regularity? Do we also need some regularity assumption on the weight $w$? The answers to these questions do not seem trivial and this is what motivated us to study the problem (2.1) in a much more general setting and to write this article. To be more precise, we will show that the regularity of a minimizer $u$ is related to the $L^q-$summability of $\psi$ as well as the regularity of the weight $w$. More precisely, we will show as soon as $w$ is of class $C^2$ and bounded away from zero that the following statements hold:

$$\psi \in L^q_{loc}(\Omega) \Rightarrow u \in C^{0,\alpha}_{loc}(\Omega), \qquad \alpha = 1 - \frac{N}{pq} > 0,$$

$$\psi \in L^\infty_{loc}(\Omega) \Rightarrow u \in \mathrm{Lip}_{loc}(\Omega).$$

In addition, for any compact set $K \subset \Omega' \subset\subset \Omega$, there is a constant $C$ depending on $p$, $N$, $\min_{\Omega'} w$, $||\psi||_{L^q(\Omega')}$, $||w||_{L^\infty(\Omega')}$, $||\nabla w||_{L^\infty(\Omega')}$, $||D^2 w||_{L^\infty(\Omega')}$, $||u||_{L^\infty(\Omega')}$ and $\mathrm{dist}(K, \partial\Omega')$ such that

$$||u||_{C^{0,1-\frac{N}{pq}}(K)} \leq C.$$

We note that in [3, Section 3] the proof of Lipschitz regularity of $u$ (in the case where $\psi$ is constant) is much complicated. First, the authors show $\alpha-$Hölder regularity on $u$ for some $0 < \alpha < 1$ sufficiently small (depensing on $p$ and $N$). Then, they prove in [3, Lemma 3.2] a uniform bound on the minimizers near to their free boundaries. In this way, they obtain uniform (in $u$) Lipschitz estimates on the minimizers $u$. However, it seems that there is a gap in their proof since all what they can show is that if $x_0$ is a "common" point on the free boundaries of a family of minimizers $u$, then in a neighborhood of $x_0$ these minimizers $u$ are uniformly bounded. Anyway, we are not interested here in proving uniform (in $u$) Lipschitz estimates on the minimizers.

In the case when $\psi \in L^\infty_{loc}(\Omega)$ and $\psi \geq \psi_{\min} > 0$, we will show that the free boundary has zero Lebesgue measure. Moreover, the characteristic function $\chi_{\{u>0\}}$ belongs to $BV_{loc}(\Omega)$. This means that the free boundary $\partial\{u > 0\}$ has locally a finite perimeter. The proofs of these results are based on a nondegeneracy property (see [2, Section 2]). Roughly speaking, there is a uniform constant $c > 0$ such that the following statement holds:

$$\text{If} \quad ||u||_{L^\infty(B_r)} \leq cr \quad \text{then} \quad u = 0 \;\text{ on }\; B_{\frac{r}{2}}.$$

In [3, Theorem 4.4], the authors extend the proof of [1, Lemma 3.7] to the case $p \neq 2$. However, there also the proof seems to be not complete. In fact, the authors in [3] show that there exists a constant $c > 0$ such that for any point $x$ on the free boundary we have

$$c \leq \frac{\mathcal{L}^N(B(x,r) \cap \{u > 0\})}{\mathcal{L}^N(B(x,r))} \leq 1 - c.$$

Yet, according to their proof, it is not clear why this constant $c$ can be taken uniform in $x$. We note that the approach used in [1] is different and based on some estimates on $u - v$ where $v$ is the harmonic replacement of $u$. In Section 2 below, we will also use this argument (the comparison with the weighted $p-$Laplacian replacement of $u$) to prove our Hölder regularity result on the minimizers.

To our knowledge, the dependence of the Hölder regularity of the minimizer $u$ on the $L^q$ summability of the density $\psi$ as well as the regularity of the free boundary in our "weighted" $p-$Laplace version of the Alt-Caffarelli minimum problem seems to be new in the literature and it has not been written anywhere.

The paper is organized as follows. In Section 2, we show existence of a minimizer $u$ to Problem (2.1). Moreover, we study the $\alpha-$Hölder regularity of $u$. Section 3 is devoted to prove that the free boundary has zero Lebesgue measure and that it has locally a finite perimeter.

## 2. Existence and Hölder regularity of minimizers

Let $w$ and $\psi$ be two nonnegative functions over an open bounded Lipschitz domain $\Omega \subset \mathbb{R}^N$ and $g$ be a nonnegative function on $\partial\Omega$. Then, we consider the following minimization problem:

$$\min\left\{ \int_\Omega (w|\nabla u|^p + \psi\, \chi_{\{u>0\}}) \,:\, u \in W^{1,p}(\Omega),\; u = g \;\text{ on } \partial\Omega \right\}. \tag{2.1}$$

It is clear that we can restrict (2.1) to the set of nonnegative functions $u$ since if $u^+ := \max\{u, 0\}$ then we have $u^+ = g$ on $\partial\Omega$ and

$$\int_\Omega w|\nabla u^+|^p + \int_\Omega \psi\,\chi_{\{u^+>0\}} \le \int_\Omega w|\nabla u|^p + \int_\Omega \psi\,\chi_{\{u>0\}}.$$

Throughout this paper, we assume that $w \in L^\infty(\Omega)$, $\psi \in L^1(\Omega)$ and there is a function $\tilde{g} \in W^{1,p}(\Omega)$ such that $\tilde{g} = g$ on $\partial\Omega$ (so, we have inf (2.1) $< \infty$). First of all, we start by the following result that guarantees the existence of a minimizer for Problem (2.1).

**Proposition 2.1.** *Assume $w \ge w_{\min} > 0$. Then, there exists a minimizer $u$ for Problem (2.1). In addition, every minimizer $u$ is weighted $p-$subharmonic in the sense that*

$$\int_\Omega w\,|\nabla u|^{p-2}\nabla u\cdot\nabla\varphi \le 0, \qquad \text{for all } \ \varphi\in C_0^\infty(\Omega) \ \text{ with } \ \varphi\ge 0.$$

*If $u$ is continuous, then $u$ is also weighted $p-$harmonic inside the positivety set $\{u > 0\}$, i.e. for any function $\varphi \in C_0^\infty(\Omega)$ such that $\mathrm{spt}(\varphi) \subset \{u > 0\}$, we have*

$$\int_\Omega w\,|\nabla u|^{p-2}\nabla u\cdot\nabla\varphi = 0.$$

*Proof.* Let $\{u_n\}_n$ be a minimizing sequence in Problem (2.1). Then, there is a uniform constant $C < \infty$ such that for all $n \in \mathbb{N}$, we have

$$\int_\Omega (w|\nabla u_n|^p + \psi\,\chi_{\{u_n>0\}}) \le C.$$

Since $w \ge w_{\min}$ and $\psi \ge 0$, then one has

$$\int_\Omega |\nabla u_n|^p \le \frac{C}{w_{\min}}.$$

But, $u_n = g$ on $\partial\Omega$. So, this implies that $u_n - \tilde{g} \in W_0^{1,p}(\Omega)$; we recall that $\tilde{g}$ is a $W^{1,p}-$extension of $g$ to $\Omega$. By the Poincaré inequality, this yields that

$$||u_n||_{L^p(\Omega)} \le ||u_n - \tilde{g}||_{L^p(\Omega)} + ||\tilde{g}||_{L^p(\Omega)} \le C||\nabla u_n - \nabla\tilde{g}||_{L^p(\Omega)} + ||\tilde{g}||_{L^p(\Omega)} \le C.$$

Hence, the sequence $\{u_n\}_n$ is bounded in $W^{1,p}(\Omega)$ and so, up to a subsequence, $u_n \rightharpoonup u$ weakly in $W^{1,p}(\Omega)$, for some function $u \in W^{1,p}(\Omega)$ with $u = g$ on $\partial\Omega$ and $u \ge 0$. Thanks to the lower semicontinuity of the $L^p-$norm, one has

$$\int_\Omega w|\nabla u|^p \le \liminf_n \left[\int_\Omega w|\nabla u_n|^p\right]. \tag{2.2}$$

On the other hand, $u_n \to u$ in $L^p(\Omega)$ and so, $u_n(x) \to u(x)$ at almost everywhere point $x \in \Omega$. Hence, if $x \in \{u > 0\}$ then for $n$ large enough, $u_n(x) > 0$ and so,

$$\chi_{\{u>0\}}(x) = 1 = \liminf_n \chi_{\{u_n>0\}}(x).$$

If $x \in \{u = 0\}$, then we always have

$$\chi_{\{u>0\}}(x) = 0 \le \liminf_n \chi_{\{u_n>0\}}(x).$$

By Fatou's Lemma, we get

$$\int_\Omega \psi\,\chi_{\{u>0\}} \le \int_\Omega \liminf_n [\psi\,\chi_{\{u_n>0\}}] \le \liminf_n \left[\int_\Omega \psi\,\chi_{\{u_n>0\}}\right]. \tag{2.3}$$

Combining (2.2) & (2.3), we get

$$\int_\Omega w|\nabla u|^p + \int_\Omega \psi\,\chi_{\{u>0\}} \le \liminf_n \left[\int_\Omega w|\nabla u_n|^p + \int_\Omega \psi\,\chi_{\{u_n>0\}}\right].$$

This implies that $u$ minimizes Problem (2.1).

Fix $\varphi \in C_0^\infty(\Omega)$ with $\varphi \geq 0$. Hence, $u - \varepsilon\varphi \in W^{1,p}(\Omega)$ with $u - \varepsilon\varphi = g$ on $\partial\Omega$ and $u - \varepsilon\varphi \leq u$, for all $\varepsilon > 0$. In particular, $u - \varepsilon\varphi$ is admissible in Problem (2.1). From the minimality of $u$, we have

$$\int_\Omega w|\nabla u|^p + \int_\Omega \psi\, \chi_{\{u>0\}} \leq \int_\Omega w|\nabla u - \varepsilon\nabla\varphi|^p + \int_\Omega \psi\, \chi_{\{u-\varepsilon\varphi>0\}} \leq \int_\Omega w|\nabla u - \varepsilon\nabla\varphi|^p + \int_\Omega \psi\, \chi_{\{u>0\}}.$$

Therefore, we get

$$\int_\Omega w|\nabla u|^p \leq \int_\Omega w|\nabla u - \varepsilon\nabla\varphi|^p, \qquad \text{for all } \varepsilon > 0.$$

Hence,

$$\int_\Omega w|\nabla u|^{p-2}\nabla u \cdot \nabla\varphi \leq 0.$$

Finally, assume that $u$ is continuous (we will show later that this assumption is always satisfied; see Proposition 2.3). Let $\varphi \in C_0^\infty(\Omega)$ be such that $\mathrm{spt}(\varphi) \subset \{u > 0\}$. Since $u$ is continuous, $\mathrm{spt}(\varphi)$ is compact and $u > 0$ on $\mathrm{spt}(\varphi)$, then we have $\{u + \varepsilon\varphi > 0\} = \{u > 0\}$, for any $\varepsilon > 0$ small enough. Yet, $u + \varepsilon\varphi = g$ on $\partial\Omega$. So again, by the minimality of $u$ in Problem (2.1), we get that

$$\int_\Omega w|\nabla u|^p + \int_\Omega \psi\, \chi_{\{u>0\}} \leq \int_\Omega w|\nabla u + \varepsilon\nabla\varphi|^p + \int_\Omega \psi\, \chi_{\{u+\varepsilon\varphi>0\}}.$$

Thus,

$$\int_\Omega w|\nabla u|^p \leq \int_\Omega w|\nabla u + \varepsilon\nabla\varphi|^p, \qquad \text{for all } \varepsilon > 0.$$

This yields that

$$\int_\Omega w|\nabla u|^{p-2}\nabla u \cdot \nabla\varphi \geq 0. \quad \square$$

For any ball $B \subset \Omega$, we will denote by $v = v_B$ the unique solution of the following weighted $p-$Laplacian problem:

$$\begin{cases} \nabla \cdot [w|\nabla v|^{p-2}\nabla v] = 0 & \text{in } B \\ v = u & \text{on } \partial B. \end{cases}$$

Notice that, by the comparison principle (see [4, Lemma 3.18]), $u \leq v$ on $B$. In the sequel, we will call this function $v$ the weighted $p-$harmonic replacement of $u$ in $B$. We recall that for any function $\phi \in W^{1,p}(B)$ such that $v = u$ on $\partial B$, we have

$$\int_B w|\nabla v|^p \leq \int_B w|\nabla\phi|^p. \tag{2.4}$$

Our aim is to show that a minimizer $u$ in Problem (2.1) is locally $\alpha-$Hölderian, for some $\alpha > 0$ that depends on $N$, $p$ and $q$. In order to prove this local Hölder regularity on $u$, we need the following crucial lemma that we use often in the rest of the paper.

**Lemma 2.2.** *Assume $\psi \in L^q_{loc}(\Omega)$, for some $q > \frac{N}{p}$. Let $u$ be a minimizer of Problem (2.1) and $v$ be the weighted $p-$harmonic replacement of $u$ in $B \subset \Omega$. Then, there is a universal constant $C = C(p)$ such that*

$$\begin{cases} \int_B w|\nabla u - \nabla v|^p \leq C||\psi||_{L^q(B)}|B \cap \{u=0\}|^{1-\frac{1}{q}} & \text{if } p \geq 2, \\ \int_B w|\nabla u - \nabla v|^p \leq C\Big(\int_B w + \int_B |\nabla u|^p\Big)^{1-\frac{p}{2}} [||\psi||_{L^q(B)}|B \cap \{u=0\}|^{1-\frac{1}{q}}]^{\frac{p}{2}} & \text{if } 1 < p < 2. \end{cases}$$

*Proof.* First of all, let us extend $v$ to a function $\tilde{v}$ on $\Omega$ by setting

$$\tilde{v} = \begin{cases} v & \text{in } B, \\ u & \text{on } \Omega \backslash B. \end{cases}$$

Clearly, $\tilde{v} \in W^{1,p}(\Omega)$ with $\tilde{v} = u$ on $\partial\Omega$ (so, $\tilde{v}$ is admissible in (2.1)). From the minimality of $u$ in Problem (2.1), we have

$$\int_\Omega w|\nabla u|^p + \int_\Omega \psi\, \chi_{\{u>0\}} \leq \int_\Omega w|\nabla \tilde{v}|^p + \int_\Omega \psi\, \chi_{\{\tilde{v}>0\}}.$$

Thus,

$$\int_B w|\nabla u|^p + \int_{\Omega\backslash B} w|\nabla u|^p + \int_B \psi\, \chi_{\{u>0\}} + \int_{\Omega\backslash B} \psi\, \chi_{\{u>0\}}$$
$$\leq \int_B w|\nabla v|^p + \int_{\Omega\backslash B} w|\nabla u|^p + \int_B \psi\, \chi_{\{v>0\}} + \int_{\Omega\backslash B} \psi\, \chi_{\{u>0\}}.$$

We infer that

$$\int_B w|\nabla u|^p + \int_B \psi\, \chi_{\{u>0\}} \leq \int_B w|\nabla v|^p + \int_B \psi\, \chi_{\{v>0\}}.$$

Hence,

$$\int_B w[|\nabla u|^p - |\nabla v|^p] \leq \int_B \psi\, \chi_{\{v>0\}} - \int_B \psi\, \chi_{\{u>0\}} \leq \int_B \psi\, \chi_{\{u=0\}} \leq ||\psi||_{L^q(B)} |B \cap \{u=0\}|^{1-\frac{1}{q}}. \tag{2.5}$$

Now, set

$$u_t = (1-t)v + tu, \qquad \text{for every } \ 0 \leq t \leq 1.$$

We have

$$\int_B w[|\nabla u|^p - |\nabla v|^p] = \int_B w[|\nabla u_1|^p - |\nabla u_0|^p] = \int_B w\left[\int_0^1 \frac{d}{dt}|\nabla u_t|^p\right] = p\int_0^1 \int_B w|\nabla u_t|^{p-2}\nabla u_t \cdot [\nabla u - \nabla v].$$

Yet,

$$\int_B w|\nabla v|^{p-2}\nabla v \cdot [\nabla u - \nabla v] = 0 \qquad \text{and} \qquad \nabla u_t - \nabla v = t[\nabla u - \nabla v].$$

Then, we get that

$$\int_B w[|\nabla u|^p - |\nabla v|^p] = p\int_0^1 t^{-1} \int_B w[|\nabla u_t|^{p-2}\nabla u_t - |\nabla v|^{p-2}\nabla v] \cdot [\nabla u_t - \nabla v]. \tag{2.6}$$

From [6], we have the following inequalities:

$$\begin{cases} |a-b|^2(1+|a|^2+|b|^2)^{\frac{p-2}{2}} \leq \frac{1}{p-1}[|a|^{p-2}a - |b|^{p-2}b]\cdot[a-b] & \text{if } \ 1<p<2, \\ |a-b|^p \leq 2^{p-2}[|a|^{p-2}a - |b|^{p-2}b]\cdot[a-b] & \text{if } \ p \geq 2. \end{cases}$$

Recalling (2.6), we get that

$$\begin{cases} \int_B w[|\nabla u|^p - |\nabla v|^p] \geq p(p-1)\int_0^1 t^{-1}\int_B w|\nabla u_t - \nabla v|^2(1+|\nabla u_t|^2+|\nabla v|^2)^{\frac{p-2}{2}}, & \text{if } \ 1<p<2, \\ \int_B w[|\nabla u|^p - |\nabla v|^p] \geq p2^{2-p}\int_0^1 t^{-1}\int_B w|\nabla u_t - \nabla v|^p, & \text{if } \ p \geq 2. \end{cases}$$

Assume $p \geq 2$. Then,

$$\int_B w[|\nabla u|^p - |\nabla v|^p] \geq p2^{2-p}\int_0^1 t^{p-1}\int_B w|\nabla u - \nabla v|^p = 2^{2-p}\int_B w|\nabla u - \nabla v|^p.$$

By (2.5), we infer that

$$\int_B w|\nabla u - \nabla v|^p \leq 2^{p-2}||\psi||_{L^q(B)}|B \cap \{u=0\}|^{1-\frac{1}{q}}.$$

Finally, assume $1 < p < 2$. Then, we have

$$||\psi||_{L^q(B)}|B\cap\{u=0\}|^{1-\frac{1}{q}} \geq \int_B w[|\nabla u|^p - |\nabla v|^p] \geq C(p)\int_B w|\nabla u-\nabla v|^2(1+|\nabla u|^2+|\nabla v|^2)^{\frac{p-2}{2}}.$$

By Hölder inequality, one has

$$\int_B w|\nabla u-\nabla v|^p \leq \left(\int_B w|\nabla u-\nabla v|^2(1+|\nabla u|^2+|\nabla v|^2)^{\frac{p-2}{2}}\right)^{\frac{p}{2}}\left(\int_B w(1+|\nabla u|^2+|\nabla v|^2)^{\frac{p}{2}}\right)^{1-\frac{p}{2}}.$$

Then, we get

$$\begin{aligned}\int_B w|\nabla u-\nabla v|^p &\leq \left[C(p)\,||\psi||_{L^q(B)}|B\cap\{u=0\}|^{1-\frac{1}{q}}\right]^{\frac{p}{2}}\left(\int_B w(1+|\nabla u|^2+|\nabla v|^2)^{\frac{p}{2}}\right)^{1-\frac{p}{2}}\\ &\leq C(p)\,[||\psi||_{L^q(B)}|B\cap\{u=0\}|^{1-\frac{1}{q}}]^{\frac{p}{2}}\left(\int_B w(1+|\nabla u|^p+|\nabla v|^p)\right)^{1-\frac{p}{2}}\\ &\leq C(p)\left(\int_B w+\int_B w|\nabla u|^p\right)^{1-\frac{p}{2}}[||\psi||_{L^q(B)}|B\cap\{u=0\}|^{1-\frac{1}{q}}]^{\frac{p}{2}},\end{aligned}$$

where the last inequality follows from (2.4). □

Since $u \geq 0$ is weighted $p-$subharmonic, then thanks to [4, Theorem 3.41], $u$ is locally bounded. In the next proposition, we show that $u$ is locally $\alpha-$Hölderian.

**Proposition 2.3.** *Assume $w \in C^2(\Omega)$ and $\psi \in L^q_{loc}(\Omega)$ with $q > \frac{N}{p}$. Let $u$ be a minimizer in Problem (2.1). For any open subset $\Omega' \subset\subset \Omega$ and a compact set $K \subset \Omega'$, there exists a constant $C$ depending on $p$, $N$, $w_{\min}$, $||\psi||_{L^q(\Omega')}$, $||w||_{L^\infty(\Omega')}$, $||\nabla w||_{L^\infty(\Omega')}$, $||D^2 w||_{L^\infty(\Omega')}$, $||u||_{L^\infty(\Omega')}$ and, dist$(K,\partial\Omega')$ such that*

$$||u||_{C^{0,1-\frac{N}{pq}}(K)} \leq C.$$

*Proof.* Fix $x \in K$ and $0 < 2r < R < \text{dist}(K,\partial\Omega')$ (so, we have $B(x,r) \subset B(x,\frac{R}{2}) \subset B(x,R) \subset \Omega'$). Let $v$ be the weighted $p-$harmonic replacement of $u$ on $B = B(x,r)$. Then, we clearly have

$$\left(\int_{B(x,r)} w|\nabla u|^p\right)^{\frac{1}{p}} \leq \left(\int_{B(x,r)} w|\nabla u-\nabla v|^p\right)^{\frac{1}{p}} + \left(\int_{B(x,r)} w|\nabla v|^p\right)^{\frac{1}{p}}. \tag{2.7}$$

But, thanks to Lemma 2.2, one has

$$\begin{cases}\left(\int_{B(x,r)} w|\nabla u-\nabla v|^p\right)^{\frac{1}{p}} \leq C(p,N)\,||\psi||^{\frac{1}{p}}_{L^q(B)}\, r^{\frac{N}{p}(1-\frac{1}{q})}, & \text{if } \ p\geq 2,\\ \left(\int_{B(x,r)} w|\nabla u-\nabla v|^p\right)^{\frac{1}{p}} \leq C(p,N)\left(\int_B w+\int_B|\nabla u|^p\right)^{\frac{1}{p}-\frac{1}{2}}||\psi||^{\frac{1}{2}}_{L^q(B)}\, r^{\frac{N}{2}(1-\frac{1}{q})}, & \text{if } \ 1<p<2.\end{cases} \tag{2.8}$$

On the other hand, we have

$$\int_{B(x,r)} w|\nabla v|^p \leq ||w||_\infty\,|B(x,r)|\,||\nabla v||^p_{L^\infty(B(x,r))}. \tag{2.9}$$

By [9, Theorem 1.1],

$$||\nabla v||_{L^\infty(B(x,r))} \leq C(p,N)\left(\frac{1+\sqrt{\kappa}R}{R}\right)||v||_{L^\infty(B(x,\frac{R}{2}))},$$

where $0 \leq \kappa < \infty$ (depends on $N$, $w_{\min}$, $||\nabla w||_{L^\infty(B(x,\frac{R}{2}))}$, $||D^2 w||_{L^\infty(B(x,\frac{R}{2}))}$) is such that the Ricci tensor $\mathrm{Ric}(w)$ of the Riemannian conformal metric $(\mathbb{R}^N, w)$ satisfies $\mathrm{Ric}(w) \geq -(N-1)\kappa$; it is well known that the Ricci tensor in this case is given by (see, for instance, [8])

$$\mathrm{Ric}(w) = -\frac{N-2}{2}\left[\frac{D^2 w}{w} - \frac{3}{2}\frac{\nabla w \otimes \nabla w}{w^2}\right] - \frac{1}{2}\left[\frac{\Delta w}{w} - \frac{(N-4)}{2}\frac{|\nabla w|^2}{w^2}\right].$$

Recalling (2.9), we get that

$$\int_{B(x,r)} w|\nabla v|^p \leq C(p,N)\,||w||_\infty\, r^N \left(\frac{1+\sqrt{\kappa}R}{R}\right)^p ||v||^p_{L^\infty(B(x,\frac{R}{2}))}. \tag{2.10}$$

Since $u = v$ on $\partial B$, then thanks to the comparison principle [4, Lemma 3.18], we have the following estimate:

$$||v||_{L^\infty(B(x,\frac{R}{2}))} \leq ||u||_{L^\infty(B(x,R))}.$$

For $0 < \delta < 1$, set $R_\delta = r^{1-\delta}$. Now, choose $r > 0$ small enough so that $2r < R_\delta < \mathrm{dist}(K, \partial\Omega')$. This yields that

$$\int_{B(x,r)} w|\nabla v|^p \leq C\, r^N \left(\frac{1+\sqrt{\kappa}\, r^{1-\delta}}{r^{1-\delta}}\right)^p ||u||^p_{L^\infty(B(x,\frac{R_\delta}{2}))} \leq C r^{N-p+\delta p}.$$

Assume $p \geq 2$. Then, by (2.7) & (2.8), we get

$$\left(\int_{B(x,r)} |\nabla u|^p\right)^{\frac{1}{p}} \leq \frac{1}{w_{\min}^{\frac{1}{p}}}\left(\int_{B(x,r)} w|\nabla u|^p\right)^{\frac{1}{p}} \leq C r^{\frac{N}{p}(1-\frac{1}{q})} + C r^{\frac{N}{p}-1+\delta} \leq C r^{\frac{N}{p}(1-\frac{1}{q})}$$

as soon as $\delta \geq 1 - \frac{N}{pq} > 0$. Thanks to Morrey's Lemma, we conclude that $u \in C^{0,1-\frac{N}{pq}}(K)$. Moreover, we have

$$||u||_{C^{0,1-\frac{N}{pq}}(K)} \leq C(p, N, w_{\min}, ||\psi||_{L^q(\Omega')}, ||w||_{C^2(\Omega')}, ||u||_{L^\infty(\Omega')}, \mathrm{dist}(K, \partial\Omega')).$$

Finally, assume that $1 < p < 2$. So, by (2.8), we recall that

$$\begin{aligned}\int_\Omega w|\nabla u - \nabla v|^p &\leq C\left(|B| + \int_B |\nabla u|^p\right)^{1-\frac{p}{2}}\left(||\psi||_{L^q(B)} r^{N(1-\frac{1}{q})}\right)^{\frac{p}{2}} \\ &\leq C r^{N(1-\frac{p}{2q})} + C r^{\frac{Np}{2}(1-\frac{1}{q})}\left(\int_B |\nabla u|^p\right)^{1-\frac{p}{2}}.\end{aligned}$$

Hence,

$$\begin{aligned}\left(\int_{B(x,r)} w|\nabla u|^p\right)^{\frac{1}{p}} &\leq C r^{\frac{N}{p}(1-\frac{p}{2q})} + C r^{\frac{N}{2}(1-\frac{1}{q})}\left(\int_B |\nabla u|^p\right)^{\frac{1}{p}-\frac{1}{2}} + C r^{\frac{N}{p}-1+\delta} \\ &\leq C r^{\frac{N}{p}(1-\frac{p}{2q})} + \frac{C}{w_{\min}^{\frac{1}{p}-\frac{1}{2}}} r^{\frac{N}{2}(1-\frac{1}{q})}\left(\int_B w|\nabla u|^p\right)^{\frac{1}{p}-\frac{1}{2}}.\end{aligned}$$

Using Young's inequality, we get

$$\left(\int_{B(x,r)} w|\nabla u|^p\right)^{\frac{1}{p}} \leq C r^{\frac{N}{p}(1-\frac{1}{q})}.$$

Consequently, $u$ is locally $\alpha-$Hölder in $\Omega$ with $\alpha = 1 - \frac{N}{pq}$. In addition, we also have the following estimate:

$$||u||_{C^{0,\alpha}(K)} \leq C(p, N, w_{\min}, ||\psi||_{L^q(\Omega')}, ||w||_{C^2(\Omega')}, ||u||_{L^\infty(\Omega')}, \mathrm{dist}(K, \partial\Omega')).$$

□

## 3. Regularity of the free boundary

In this section, we prove under the assumptions $q = \infty$ and $\psi \geq \psi_{\min} > 0$ that the free boundary $\partial\{u > 0\}$ of any minimizer $u$ in Problem (2.1) has locally a finite perimeter. First, we start by the following nondegeneracy result:

**Lemma 3.1.** *Assume $\psi \geq \psi_{\min} > 0$. For any $\kappa \in (0, \frac{1}{2})$, there exists a constant $c > 0$ depending only on $p$, $N$, $w_{\min}$, $\psi_{\min}$, $||\psi||_{L^\infty(\Omega')}$ and $||w||_{C^2(\Omega')}$ such that the following statement holds*

$$||u||_{L^\infty(B_r)} < cr \qquad \textit{implies that} \qquad u = 0 \ \textit{ in } \ B_{\kappa r}$$

*for any ball $B_r \subset \Omega' \subset\subset \Omega$.*

*Proof.* Fix $\kappa' \in (k, 1)$. Let $\phi$ be the weighted $p-$harmonic function in $B_{\kappa' r \setminus \kappa r}$ with $\phi = 1$ on $\partial B_{\kappa' r}$ and $\phi = 0$ on $\partial B_{\kappa r}$. From [5, 7], this function $\phi$ is of class $C^{1,\beta}$ on $\overline{B_{\kappa' r \setminus \kappa r}}$. Let us extend $\phi$ by zero in $B_{\kappa r}$. Moreover, one has $0 \leq \phi \leq 1$.

Set $m = ||u||_{L^\infty(B_{k' r})}$. Then, we define $\Phi = \min\{u, m\phi\}$. Since $m\,\phi = m \geq u$ on $\partial B_{k' r}$, then $\Phi = u$ on $\partial B_{k' r}$. Moreover, $\Phi = 0$ in $B_{\kappa r}$. Thanks to the minimality of $u$ in Problem (2.1), we have

$$\int_{B_{k' r}} (w|\nabla u|^p + \psi\chi_{\{u>0\}}) \leq \int_{B_{k' r}} (w|\nabla \Phi|^p + \psi\chi_{\{\Phi>0\}}) = \int_{B_{k' r} \setminus B_{\kappa r}} (w|\nabla \Phi|^p + \psi\chi_{\{\Phi>0\}}).$$

Hence,

$$\int_{B_{kr}} (w|\nabla u|^p + \psi\chi_{\{u>0\}}) + \int_{B_{k' r} \setminus B_{\kappa r}} (w|\nabla u|^p + \psi\chi_{\{u>0\}}) \leq \int_{B_{k' r} \setminus B_{\kappa r}} (w|\nabla \Phi|^p + \psi\chi_{\{\Phi>0\}}).$$

But, $\{\Phi > 0\} \subset \{u > 0\}$. Then, we get that

$$\int_{B_{kr}} (w|\nabla u|^p + \psi\chi_{\{u>0\}}) \leq \int_{B_{k' r} \setminus B_{\kappa r}} (w[|\nabla \Phi|^p - |\nabla u|^p] + \psi[\chi_{\{\Phi>0\}} - \chi_{\{u>0\}}]) \leq \int_{B_{k' r} \setminus B_{\kappa r}} w[|\nabla \Phi|^p - |\nabla u|^p].$$

Since the map $\xi \mapsto |\xi|^p$ is convex, one has

$$|\nabla u|^p - |\nabla \Phi|^p \geq p\,|\nabla \Phi|^{p-2}\nabla \Phi \cdot [\nabla u - \nabla \Phi].$$

Hence,

$$\begin{aligned}
\int_{B_{kr}} (w|\nabla u|^p + \psi\chi_{\{u>0\}}) &\leq p \int_{B_{k' r} \setminus B_{\kappa r}} w|\nabla \Phi|^{p-2}\nabla \Phi \cdot [\nabla \Phi - \nabla u] \\
&= pm^{p-1} \int_{B_{k' r} \setminus B_{\kappa r}} w|\nabla \phi|^{p-2}\nabla \phi \cdot [\nabla \Phi - \nabla u] \\
&= -pm^{p-1} \int_{B_{k' r} \setminus B_{\kappa r}} \nabla \cdot [w|\nabla \phi|^{p-2}\nabla \phi][\Phi - u] + pm^{p-1} \int_{\partial B_{\kappa r}} w|\nabla \phi|^{p-2}\nabla \phi \cdot \mathbf{n}[\Phi - u] \\
&= pm^{p-1} \int_{\partial B_{\kappa r}} [\Phi - u]w|\nabla \phi|^{p-2}\nabla \phi \cdot \mathbf{n} \\
&\leq pm^{p-1}||\nabla \phi||_\infty^{p-1}\,||w||_\infty \int_{\partial B_{\kappa r}} u.
\end{aligned}$$

By [9, Theorem 1.1], we have

$$|\nabla \phi| \leq \frac{C}{r}.$$

Therefore, we get

$$\min\{w_{\min}, \psi_{\min}\}\left(\int_{B_{kr}} |\nabla u|^p + |\{u > 0\}|\right) \leq w_{\min} \int_{B_{kr}} |\nabla u|^p + \psi_{\min}|\{u > 0\}| \leq \int_{B_{kr}} (w|\nabla u|^p + \psi\chi_{\{u>0\}})$$

$$\leq C\,\frac{m^{p-1}}{r^{p-1}}\int_{\partial B_{\kappa r}} u.$$

Thanks to the $W^{1,1}$ trace inequality, we have

$$\int_{\partial B_{\kappa r}} u \leq C\Big(\int_{B_{\kappa r}} u + \int_{B_{\kappa r}} |\nabla u|\Big).$$

Hence,

$$\int_{B_{kr}} |\nabla u|^p \,+\, |\{u>0\}| \leq C\,\frac{m^{p-1}}{r^{p-1}}\Big(\int_{B_{\kappa r}} u + \int_{B_{\kappa r}} |\nabla u|\Big).$$

This implies

$$\int_{B_{kr}} |\nabla u|^p \,+\, |\{u>0\}| \leq C\frac{m^{p-1}}{r^{p-1}}\Big(|\{u>0\}| + |\{u>0\}|^{\frac{p-1}{p}}\Big(\int_{B_{\kappa r}} |\nabla u|^p\Big)^{\frac{1}{p}}\Big)$$

$$\leq C\frac{m^{p-1}}{r^{p-1}}\Big(\int_{B_{\kappa r}} |\nabla u|^p + |\{u>0\}|\Big).$$

Hence,

$$\Big[1 - C\frac{m^{p-1}}{r^{p-1}}\Big]\Big(\int_{B_{kr}} |\nabla u|^p \,+\, |\{u>0\}|\Big) \leq 0$$

which implies that $u=0$ on $B_{\kappa r}$ as soon as

$$m \leq \Big(\frac{1}{C}\Big)^{\frac{1}{p-1}} r.$$

□

**Proposition 3.2.** *For $K \subset \Omega' \subset\subset \Omega$, there is a constant $c \in (0,1)$ (depending on $p$, $N$, $w_{\min}$, $\psi_{\min}$, $||\psi||_{L^\infty(\Omega')}$, $||w||_{C^2(\Omega')}$, $||u||_{L^\infty(\Omega')}$ and, dist$(K, \partial\Omega')$) such that for every $x_0 \in K \cap \partial\{u>0\}$ and $r>0$ with $B(x_0,r) \subset K$, we have*

$$c \leq \frac{\mathcal{L}^N(B(x_0,r)\cap\{u>0\})}{\mathcal{L}^N(B(x_0,r))} \leq 1-c.$$

*Proof.* Again thanks to Lemma 3.1, we see that there exist a constant $c>0$ and a point $x \in B(x_0,\frac{r}{2})$ such that $u(x) \geq c\,\frac{r}{2}$. Yet, we claim that there is a uniform constant $\delta \in (0,1)$ such that $B(x,\delta r) \subset B(x_0,r)\cap\{u>0\}$. Indeed, if $y \in B(x,\delta r)$ then one has

$$|y-x_0| \leq |y-x| + |x-x_0| \leq \Big(\frac{1}{2}+\delta\Big)r < r.$$

Thanks to the Lipschitz regularity of $u$, we also have

$$u(y) \geq u(x) - C|y-x| \geq \Big(\frac{c}{2} - C\delta\Big)r > 0$$

as soon as $\delta < \frac{c}{2C}$. Hence, we get

$$\frac{\mathcal{L}^N(B(x_0,r)\cap\{u>0\})}{\mathcal{L}^N(B(x_0,r))} \geq \frac{\mathcal{L}^N(B(x,\delta r))}{\mathcal{L}^N(B(x_0,r))} \geq c\,\delta^N.$$

For the upper bound: assume that there is a sequence of points $x_{0,n}$ in $K\cap\partial\{u>0\}$ and $r_n>0$ with $B(x_{0,n},r_n)\subset K$ such that

$$\frac{\mathcal{L}^N(B(x_{0,n},r_n)\cap\{u=0\})}{\mathcal{L}^N(B(x_{0,n},r_n))} \leq \frac{1}{n} \to 0.$$

Up to a subsequence, $x_{0,n} \to x_0 \in K\cap\partial\{u>0\}$ and $r_n \to r_0$. Assume $r_0>0$. Let $v_n$ be the weighted $p-$harmonic replacement of $u$ in $B_n := B(x_{0,n}, r_n)$. For all $n$ large enough, it is clear that $B := B(x_0, \frac{r_0}{4}) \subset B(x_{0,n}, \frac{r_n}{2})$. Thanks to Lemma 2.2, we have

$$\begin{cases} \int_B w|\nabla u - \nabla v_n|^p \leq C||\psi||_{L^\infty(\Omega)}\,|B_n \cap \{u=0\}| & \text{if } \ p \geq 2, \\ \int_B w|\nabla u - \nabla v_n|^p \leq C\Big(|B_n| + \int_{B_n} w|\nabla u|^p\Big)^{1-\frac{p}{2}} [||\psi||_{L^\infty(\Omega)}|B_n \cap \{u=0\}|]^{\frac{p}{2}} & \text{if } \ 1<p<2. \end{cases}$$

Hence,

$$\int_B w|\nabla u - \nabla v_n|^p \to 0.$$

Yet, $v_n \to v$ uniformly in $\overline{B}$. In particular, we must have $u = v + C$. But, $v$ is weighted $p-$harmonic in $B$ and so, $u$ is also weighted $p-$harmonic there. However, we have $u(x_0) = 0$. By the maximum principle, we infer that $u = 0$ in $B$. Yet, by Lemma 3.1, there is a constant $c > 0$ such that

$$||u||_{L^\infty(B)} \geq c,$$

which is a contradiction.

Finally, assume that $r_0 = 0$. For every $n$, we define $u_n(x) = \frac{u(x_{0,n}+r_n x)}{r_n}$ for all $x \in B_1$. It is not difficult to check that $u_n$ minimizes

$$\min\left\{\int_{B_1}(w_n|\nabla u|^p + \psi_n\,\chi_{\{u>0\}})\,:\,u \in W^{1,p}(B_1),\ u \geq 0,\ u = u_n\ \text{ on }\ \partial B_1\right\},$$

where

$$w_n(x) = w(x_{0,n}+r_n x) \qquad \text{and} \qquad \psi_n(x) = \psi(x_{0,n}+r_n x).$$

Now, let $v_n$ be the weighted $p-$harmonic replacement of $u_n$ in $B_1$. Again, by Lemma 2.2, one has the following:

$$\begin{cases} \int_{B_1} w_n|\nabla u_n - \nabla v_n|^p \leq C||\psi||_{L^\infty(\Omega)}\,|B_1 \cap \{u_n=0\}| & \text{if } \ p \geq 2, \\ \int_{B_1} w_n|\nabla u_n - \nabla v_n|^p \leq C\Big(|B_1| + \int_{B_1} w_n|\nabla u_n|^p\Big)^{1-\frac{p}{2}} [||\psi||_{L^\infty(\Omega)}|B_1 \cap \{u_n=0\}|]^{\frac{p}{2}} & \text{if } \ 1<p<2. \end{cases}$$

But, $w_n \geq \frac{w(x_0)}{2} > 0$ on $B_1$ for $n$ large enough. Moreover, we have

$$\mathcal{L}^N(B_1 \cap \{u_n = 0\}) = \frac{\mathcal{L}^N(B(x_{0,n}, r_n) \cap \{u=0\})}{\mathcal{L}^N(B(x_{0,n}, r_n))}\,\mathcal{L}^N(B_1) \to 0.$$

Thus,

$$\int_{B_1} |\nabla u_n - \nabla v_n|^p \to 0.$$

We note that since $u$ is Lipschitz, then $u_n$ are clearly uniformly Lipschitz. In addition, one has $0 \leq u_n \leq C$ for all $n$ since

$$u(x_{0,n}+r_n x) = u(x_{0,n}+r_n x) - u(x_{0,n}) \leq C r_n.$$

So, up to a subsequence, $u_n \to u^\star$ and $v_n \to v$ uniformly in $\overline{B_1}$. And, we have $u^\star = v + C$. However, $v$ is weighted $p-$harmonic in $B_1$. Hence, $u^\star$ is weighted $p-$harmonic. But, we have $u^\star(0) = 0$ (since $u_n(0) = 0$ for every $n$). Thanks again to the maximum principle, we get that $u^\star = 0$ in $B_1$. From Lemma 3.1, we know that there is a uniform constant $c > 0$ such that

$$||u||_{L^\infty(B(x_{0,n},r_n))} \geq c r_n.$$

Then,

$$||u_n||_{L^\infty(B_1)} \geq c.$$

But, this yields again to a contradiction. □

In particular, we obtain the following:

**Theorem 3.3.** *The free boundary has zero Lebesgue meausre.*

*Proof.* This follows from Proposition 3.2 as well as the Lebesgue density theorem. □

Now, let us define the measure $\Lambda$ as follows:

$$\int_\Omega \varphi \,\mathrm{d}\Lambda = -\int_\Omega w|\nabla u|^{p-2}\nabla u\cdot\nabla\varphi, \qquad \text{for all } \varphi\in C_0^\infty(\Omega).$$

Since $u$ is $p-$subharmonic, then $\Lambda$ is a nonnegative Borel measure. Moreover, $u$ is $p-$harmonic in the positivety set $\{u>0\}$ and so, for every smooth function $\varphi$ such that $\mathrm{spt}(\varphi)\subset\{u>0\}$ we have

$$\int_\Omega \varphi\,\mathrm{d}\Lambda = 0.$$

Therefore, $\mathrm{spt}(\Lambda)\subset\partial\{u>0\}$.

**Proposition 3.4.** *For any compact set $K\subset\Omega'\subset\subset\Omega$, there exist two constants $0<c<C<\infty$ (depending on $p$, $N$, $w_{\min}$, $\psi_{\min}$, $||\psi||_{L^\infty(\Omega')}$, $||w||_{C^2(\Omega')}$, $||u||_{L^\infty(\Omega')}$ and, dist$(K,\partial\Omega')$) such that for every $x_0\in K\cap\partial\{u>0\}$, we have*

$$cr^{N-1}\le\Lambda(B(x_0,r))\le Cr^{N-1}, \qquad \textit{for all } 0<r<d(x_0,\partial K).$$

*Proof.* Fix $\delta>0$. Let $0\le\varphi_\delta\in C_0^\infty(B(x_0,r+\delta))$ be a cutoff function such that $\varphi_\delta=1$ on $B(x_0,r)$ and $|\nabla\varphi|\le\frac{C}{\delta}$. Then, we have

$$\Lambda(B(x_0,r))\le\int_\Omega\varphi_\delta\,\mathrm{d}\Lambda=-\int_\Omega w|\nabla u|^{p-2}\nabla u\cdot\nabla\varphi_\delta\le||w||_\infty||\nabla u||_\infty^{p-1}\frac{C}{\delta}|B(x_0,r+\delta)\backslash B(x_0,r)|,$$

for any $\delta>0$. Letting $\delta\to0^+$, we get

$$\Lambda(B(x_0,r))\le Cr^{N-1}.$$

Let us prove the lower estimate. Assume that there is a sequence of points $x_{0,n}$ in $K\cap\partial\{u>0\}$ and $0<r_n<d(x_{0,n},\partial K)$ such that

$$\Lambda(B(x_{0,n},r_n))\le\frac{r_n^{N-1}}{n}.$$

Up to a subsequence, $x_{0,n}\to x_0$ and $r_n\to r_0$. Assume $r_0>0$. For every $\varphi\in C_0^\infty(B(x_0,\frac{r_0}{2}))$, we have

$$-\int_\Omega w|\nabla u|^{p-2}\nabla u\cdot\nabla\varphi=\int_{B(x_{0,n},r_n)}\varphi\,\mathrm{d}\Lambda\le||\varphi||_\infty\Lambda(B(x_{0,n},r_n))\to0.$$

Substituting $\varphi$ with $-\varphi$, we get

$$-\int_\Omega w|\nabla u|^{p-2}\nabla u\cdot\nabla\varphi=0.$$

Hence, $u$ is weighted $p-$harmonic inside $B(x_0,\frac{r_0}{2})$. But, we have $u(x_0)=\lim_n u(x_{0,n})=0$. By the maximum principle, this implies that $u=0$ on $B(x_0,\frac{r_0}{2})$. However, by Lemma 3.1, there is a constant $c>0$ such that

$$||u||_{L^\infty(B(x_0,\frac{r_0}{2}))}\ge c.$$

Yet, this is a contradiction.

Finally, assume that $r_0=0$. For every $n$, set $u_n(x)=\frac{u(x_{0,n}+r_nx)}{r_n}$ for all $x\in B_1$. Then, we define

$$\int_{B_1}\varphi\,\mathrm{d}\Lambda_n=-\int_{B_1}w_n|\nabla u_n|^{p-2}\nabla u_n\cdot\nabla\varphi, \qquad \text{for all } \varphi\in C_0^\infty(B_1),$$

where

$$w_n(x) = w(x_{0,n} + r_n x).$$

Fix $\varphi \in C_0^\infty(B_1)$. We have

$$\begin{aligned}
\int_{B_1} \varphi \, \mathrm{d}\Lambda_n &= -\int_{B_1} w(x_{0,n} + r_n x)|\nabla u(x_{0,n} + r_n x)|^{p-2}\nabla u(x_{0,n} + r_n x) \cdot \nabla\varphi(x)\, \mathrm{d}x \\
&= -\frac{1}{r_n^N}\int_{B(x_{0,n},r_n)} w(z)|\nabla u(z)|^{p-2}\nabla u(z) \cdot \nabla\varphi\left(\frac{z - x_{0,n}}{r_n}\right) \mathrm{d}z \\
&= -\frac{1}{r_n^{N-1}}\int_{B(x_{0,n},r_n)} w(z)|\nabla u(z)|^{p-2}\nabla u(z) \cdot \nabla\varphi_n(z)\, \mathrm{d}z \\
&= \frac{1}{r_n^{N-1}}\int_{\Omega} \varphi_n \, \mathrm{d}\Lambda
\end{aligned}$$

where

$$\varphi_n(z) = \varphi\left(\frac{z - x_{0,n}}{r_n}\right).$$

Yet, we see that $u_n$ is uniformly bounded and Lipschitz continuous on $B_1$. Hence, up to a subsequence, $u_n \to u^\star$ uniformly in $\overline{B_1}$. On the other hand, one has $w_n|\nabla u_n|^{p-2}\nabla u_n \rightharpoonup^\star h$ in $L^\infty(B_1)$. We claim that $h = w(x_0)|\nabla u^\star|^{p-2}\nabla u^\star$. If this is the case, so we get that for every $\varphi \in C_0^\infty(B_1)$,

$$-\int_{B_1} w(x_0)|\nabla u^\star|^{p-2}\nabla u^\star \cdot \nabla\varphi$$

$$= \lim_{n\to\infty} -\int_{\Omega} w_n|\nabla u_n|^{p-2}\nabla u_n \cdot \nabla\varphi = \lim_{n\to\infty}\int_{B_1} \varphi\, \mathrm{d}\Lambda_n \le ||\varphi||_\infty \Lambda_n(B_1) \le C\,\frac{\Lambda(B(x_{0,n}, r_n))}{r_n^{N-1}} \to 0.$$

Hence, $u^\star$ is weighted $p-$harmonic inside $B_1$. But, we have $u^\star(0) = \lim_n u_n(0) = 0$. By the maximum principle, this implies that $u^\star = 0$ in $B_1$. However, by Lemma 3.1, there is a constant $c > 0$ (uniform in $n$) such that

$$||u_n||_{L^\infty(B_1)} \ge c.$$

Passing to the limit when $n \to \infty$, we get

$$||u^\star||_{L^\infty(B_1)} \ge c > 0,$$

which is a contradiction.

It remains to prove the claim. Fix $x \in B_1$. First, assume that $u^\star(x) > 0$. Let $r' > 0$ be sufficiently small so that $\overline{B(x, r')} \subset \{u^\star > 0\}$. Since $u_n \to u^\star$ uniformly in $B_1$, then for $n$ large enough we must have $\overline{B(x, r')} \subset \{u_n > 0\}$ and so, $u_n$ is weighted $p-$harmonic in $B(x, r')$. Hence, thanks to [7, Theorem 13.1], $u_n$ is bounded in $C^{1,\alpha}(\overline{B(x, r')})$. Then, up to a subsequence, $w_n|\nabla u_n|^{p-2}\nabla u_n \to w(x_0)|\nabla u^\star|^{p-2}\nabla u^\star$ uniformly and so, $h = w|\nabla u^\star|^{p-2}\nabla u^\star$ on $B(x, r')$. Now, assume that there is a small $r' > 0$ such that $B(x, r') \subset \{u^\star = 0\}$. Thanks to Lemma 3.1, for $n$ large enough, $u_n = 0$ on $B(x, \frac{r'}{2})$. In particular, we get that $h = w(x_0)|\nabla u^\star|^{p-2}\nabla u^\star = 0$ in $B(x, \frac{r'}{2})$.

To conclude the proof, we just need to check that $\mathcal{L}^N(\partial\{u^\star > 0\}) = 0$. Fix $x \in \partial\{u^\star > 0\}$. Hence, it is not difficult to see that there is a constant $c > 0$ such that for every $r > 0$ small one has for $n$ large enough

$$||u_n||_{L^\infty(B(x,r))} \ge cr.$$

Passing to the limit when $n \to \infty$, we infer that

$$||u^\star||_{L^\infty(B(x,r))} \ge cr.$$

In particular, there exists a point $x^\star \in B(x, r)$ such that $u^\star(x^\star) \ge cr$. Moreover, thanks to the Lipschitz regularity of $u^\star$, it is easy to check that $B(x^\star, \delta r) \subset B(x, 2r) \cap \{u^\star > 0\}$, for $\delta \in (0,1)$ small. Hence, we get

$$\frac{\mathcal{L}^N(B(x,2r) \cap \{u^\star > 0\})}{\mathcal{L}^N(B(x,2r))} \geq \frac{\mathcal{L}^N(B(x^\star, \delta r))}{\mathcal{L}^N(B(x,2r))} \geq c.$$

□

Finally, we arrive to the following regularity result on the free boundary; $\partial\{u > 0\}$ is locally of finite perimeter:

$$\text{“}\chi_{\{u>0\}} \in BV_{loc}(\Omega)\text{”}.$$

**Theorem 3.5.** *Let $u$ be a minimizer for Problem (2.1). For any compact set $K \subset \Omega$, we have $\mathcal{H}^{N-1}(\partial\{u > 0\} \cap K) < \infty$.*

*Proof.* Thanks to Proposition 3.4, we infer that

$$c\,\mathcal{H}^{N-1}(\{\partial\{u > 0\} \cap K\}) \leq \Lambda(K) \leq C\,\mathcal{H}^{N-1}(\{\partial\{u > 0\} \cap K\}).$$

Yet, $\Lambda(K) < \infty$. Hence,

$$\mathcal{H}^{N-1}(\{\partial\{u > 0\} \cap K\}) < \infty. \quad \square$$

**• Open Questions: What kind of regularity can be proved on the free boundary in the case when the density $\psi$ is not bounded? Can we show that the free boundary has a finite “fractional” perimeter? Assume $\psi$ is more regular (say Hölder), can we improve the regularity of the free boundary?**

## References


[1] H. W. Alt and L. A. Caffarelli, Existence and regularity for a minimum problem with free boundary, *J. Reine Angew. Math.* 325, 105-144, 1981.

[2] H. W. Alt, L. A. Caffarelli, A. Friedman, A free boundary problem for quasilinear elliptic equations, *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* 11(4), 1-44, 1984.

[3] D. Danielli and A. Petrosyan, A minimum problem with free boundary for a degenerate quasilinear operator, *Calc. Var.* 23, 97-124, 2005.

[4] J. Heinonen, T. Kilpeläinen, O. Martio, *Nonlinear Potential Theory of Degenerate Elliptic Equations*, Oxford University Press, Oxford, 1993.

[5] G. M. Lieberman, Boundary regularity for solutions of degenerate elliptic equations, *Nonlinear Analysis: Theory, Methods & Applications*, 12, 11, 1988, 1203-1219.

[6] P. Lindqvist, Notes on the $p-$Laplace equation, University of Jyvaskyla, Department of Mathematics and Statistics, 2017.

[7] D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer Berlin Heidelberg.

[8] R. Schoen and S. T. Yau, Riemannian geometry, Chinese Science Press, Beijing, 1988.

[9] X. Wang and L. Zhang, Local gradient estimate for p-harmonic functions on Riemannian manifolds, *communications in analysis and geometry* Volume 19, Number 4, 759-771, 2011.



Department of Mathematics, Statistics and Physics, College of Arts and Sciences, Qatar University, 2713, Doha, Qatar.

*Email address*: sdweik@qu.edu.qa