

ON THE REGULARITY OF THE SOLUTION AND THE FREE BOUNDARY IN A WEIGHTED p -LAPLACIAN PROBLEM

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ABSTRACT. In this paper, we study the regularity of the minimizer in the Alt-Caffarelli type minimum problem for the “weighted” p -Laplace operator ($1 < p < \infty$) with free boundary:

$$\min \left\{ \int_{\Omega} (w|\nabla u|^p + \psi \chi_{\{u>0\}}) : u \in W^{1,p}(\Omega), u \geq 0, u = g \text{ on } \partial\Omega \right\},$$

where w and ψ are two given nonnegative functions on Ω and g is a nonnegative boundary datum. More precisely, under the assumptions that w is a C^2 function with $w \geq w_{\min} > 0$ and ψ belongs to $L^q_{loc}(\Omega)$ for some $q > \frac{N}{p}$, we will show that a minimizer u is locally α -Hölderian with $\alpha = 1 - \frac{N}{pq}$. If ψ belongs to $L^\infty_{loc}(\Omega)$ and is bounded away from zero then, thanks to the Lipschitz regularity of u , we will be able also to prove that the free boundary $\partial\{u > 0\}$ is locally of finite perimeter.

1. INTRODUCTION

Given a bounded Lipschitz domain $\Omega \subset \mathbb{R}^N$ and two functions $w, \psi : \Omega \mapsto \mathbb{R}^+$, we study the problem of minimizing the energy functional

$$u \mapsto \int_{\Omega} w|\nabla u|^p + \psi \chi_{\{u>0\}}$$

among all functions $u \in W^{1,p}(\Omega)$ with boundary condition $u = g$ on $\partial\Omega$, where χ_A denotes the characteristic function of set A . It is not difficult to see that any minimizer u solves the following p -Laplace equation:

$$\Delta_p u = 0 \quad \text{in } \{u > 0\}$$

where $\Delta_p u := \nabla \cdot [|\nabla u|^{p-2} \nabla u]$. This problem is referred to as Bernoulli-type free boundary problem and is well studied in the literature. The case $p = 2$ has been studied first by H. W. Alt and L. A. Caffarelli in [1] and later, for any $1 < p < \infty$ in [3].

In this paper, we are interested in studying the Hölder regularity of a minimizer u as well as the regularity of the free boundary $\Gamma = \partial\{u > 0\} \cap \Omega$. The main difference in [3] from what was done in [1, 2] (where the authors consider the Laplacian case $p = 2$) is that the p -Laplacian (for $p \neq 2$) is not uniformly elliptic (degenerate for $p > 2$ and singular for $1 < p < 2$). On the other side, the authors of [3] consider Problem (2.1) but in the case when $w = 1$ and ψ is also constant. The presence of non-uniform functions w and ψ in (2.1) makes the problem somehow more complicated. In [3, 1], the Lipschitz regularity of a minimizer u has been proven. Now, suppose that ψ is not constant or even unbounded, what kind of regularity can we prove on the minimizer u ? Can we still prove Lipschitz (or perhaps Hölder) regularity? Do we also need some regularity assumption on the weight w ? The answers to these questions do not seem trivial and this is what motivated us to study the problem (2.1) in a much more general setting and to write this article. To be more precise, we will show that the regularity of a minimizer u is related to the L^q -summability of ψ as well as the regularity of the weight w . More precisely, we will show as soon as w is of class C^2 and bounded away from zero that the following statements hold:

$$\psi \in L^q_{loc}(\Omega) \Rightarrow u \in C^{0,\alpha}_{loc}(\Omega), \quad \alpha = 1 - \frac{N}{pq} > 0,$$

$$\psi \in L_{loc}^\infty(\Omega) \Rightarrow u \in \text{Lip}_{loc}(\Omega).$$

In addition, for any compact set $K \subset \Omega' \subset\subset \Omega$, there is a constant C depending on $p, N, \min_{\Omega'} w, \|\psi\|_{L^q(\Omega')}, \|w\|_{L^\infty(\Omega')}, \|\nabla w\|_{L^\infty(\Omega')}, \|D^2 w\|_{L^\infty(\Omega')}, \|u\|_{L^\infty(\Omega')}$ and $\text{dist}(K, \partial\Omega')$ such that

$$\|u\|_{C^{0,1-\frac{N}{pq}}(K)} \leq C.$$

We note that in [3, Section 3] the proof of Lipschitz regularity of u (in the case where ψ is constant) is more complicated. First, the authors show α -Hölder regularity on u for some $0 < \alpha < 1$ sufficiently small (depending on p and N). Then, they prove in [3, Lemma 3.2] a uniform bound on the minimizers near to their free boundaries. In this way, they obtain uniform (in u) Lipschitz estimates on the minimizers u . However, it seems that in their proof all what they can show is that if x_0 is a “common” point on the free boundaries of a family of minimizers u , then in a neighborhood of x_0 these minimizers u are uniformly bounded. Anyway, we are not interested here in proving uniform (in u) Lipschitz estimates on the minimizers.

In the case when $\psi \in L_{loc}^\infty(\Omega)$ and $\psi \geq \psi_{\min} > 0$, we will show that the free boundary has zero Lebesgue measure. Moreover, the characteristic function $\chi_{\{u>0\}}$ belongs to $BV_{loc}(\Omega)$. This means that the free boundary $\partial\{u > 0\}$ has locally a finite perimeter. The proofs of these results are based on a nondegeneracy property (see [2, Section 2]). Roughly speaking, there is a uniform constant $c > 0$ such that the following statement holds:

$$\text{If } \|u\|_{L^\infty(B_r)} \leq cr \text{ then } u = 0 \text{ on } B_{\frac{r}{2}}.$$

In [3, Theorem 4.4], the authors extend the proof of [1, Lemma 3.7] to the case $p \neq 2$, so they show that there exists a constant $c > 0$ such that for any point x on the free boundary we have

$$c \leq \frac{\mathcal{L}^N(B(x, r) \cap \{u > 0\})}{\mathcal{L}^N(B(x, r))} \leq 1 - c.$$

Yet, according to their proof, it is not clear why this constant c can be taken uniform in x . We note that the approach used in [1] is different and based on some estimates on $u - v$, where v is the harmonic replacement of u . In Section 2 below, we will use the same argument as in [1] (the comparison with the “weighted” p -Laplacian replacement of u) to prove our Hölder regularity result on the minimizers.

To our knowledge, the Alt-Caffarelli minimization problem for the “weighted” p -Laplacian with free boundary has not been previously studied. Moreover, the dependence of the Hölder regularity of the minimizer u on the L^q summability of the density ψ as well as the regularity of the free boundary in the “weighted” p -Laplace version of the Alt-Caffarelli minimum problem seems to be new in the literature and it has not been written anywhere.

The paper is organized as follows. In Section 2, we show existence of a minimizer u to Problem (2.1). Moreover, we study the α -Hölder regularity of u . Section 3 is devoted to prove that the free boundary has zero Lebesgue measure and that it has locally a finite perimeter.

2. EXISTENCE AND HÖLDER REGULARITY OF MINIMIZERS

Let w and ψ be two nonnegative functions over an open bounded Lipschitz domain $\Omega \subset \mathbb{R}^N$ and g be a nonnegative function on $\partial\Omega$. Then, we consider the following minimization problem:

$$(2.1) \quad \min \left\{ \int_{\Omega} (w|\nabla u|^p + \psi \chi_{\{u>0\}}) : u \in W^{1,p}(\Omega), u = g \text{ on } \partial\Omega \right\}.$$

It is clear that we can restrict (2.1) to the set of nonnegative functions u since if $u^+ := \max\{u, 0\}$ then we have $u^+ = g$ on $\partial\Omega$ and

$$\int_{\Omega} w |\nabla u^+|^p + \int_{\Omega} \psi \chi_{\{u^+ > 0\}} \leq \int_{\Omega} w |\nabla u|^p + \int_{\Omega} \psi \chi_{\{u > 0\}}.$$

Throughout this paper, we assume that $w \in L^\infty(\Omega)$, $\psi \in L^1(\Omega)$ and there is a function $\tilde{g} \in W^{1,p}(\Omega)$ such that $\tilde{g} = g$ on $\partial\Omega$ (so, we have $\inf(2.1) < \infty$). First of all, we start by the following result that guarantees the existence of a minimizer for Problem (2.1).

Proposition 2.1. *Assume $w \geq w_{\min} > 0$. Then, there exists a minimizer u for Problem (2.1). In addition, every minimizer u is weighted p -subharmonic in the sense that*

$$\int_{\Omega} w |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi \leq 0, \quad \text{for all } \varphi \in C_0^\infty(\Omega) \text{ with } \varphi \geq 0.$$

If u is continuous, then u is also weighted p -harmonic inside the positivity set $\{u > 0\}$, i.e. for any function $\varphi \in C_0^\infty(\Omega)$ such that $\text{spt}(\varphi) \subset \{u > 0\}$, we have

$$\int_{\Omega} w |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi = 0.$$

Proof. Let $\{u_n\}_n$ be a minimizing sequence in Problem (2.1). Then, there is a uniform constant $C < \infty$ such that for all $n \in \mathbb{N}$, we have

$$\int_{\Omega} (w |\nabla u_n|^p + \psi \chi_{\{u_n > 0\}}) \leq C.$$

Since $w \geq w_{\min}$ and $\psi \geq 0$, then one has

$$\int_{\Omega} |\nabla u_n|^p \leq \frac{C}{w_{\min}}.$$

But, $u_n = g$ on $\partial\Omega$. So, this implies that $u_n - \tilde{g} \in W_0^{1,p}(\Omega)$; we recall that \tilde{g} is a $W^{1,p}$ -extension of g to Ω . By the Poincaré inequality, this yields that

$$\|u_n\|_{L^p(\Omega)} \leq \|u_n - \tilde{g}\|_{L^p(\Omega)} + \|\tilde{g}\|_{L^p(\Omega)} \leq C \|\nabla u_n - \nabla \tilde{g}\|_{L^p(\Omega)} + \|\tilde{g}\|_{L^p(\Omega)} \leq C.$$

Hence, the sequence $\{u_n\}_n$ is bounded in $W^{1,p}(\Omega)$ and so, up to a subsequence, $u_n \rightharpoonup u$ weakly in $W^{1,p}(\Omega)$, for some function $u \in W^{1,p}(\Omega)$ with $u = g$ on $\partial\Omega$ and $u \geq 0$. Thanks to the lower semicontinuity of the L^p -norm, one has

$$(2.2) \quad \int_{\Omega} w |\nabla u|^p \leq \liminf_n \left[\int_{\Omega} w |\nabla u_n|^p \right].$$

On the other hand, $u_n \rightarrow u$ in $L^p(\Omega)$ and so, $u_n(x) \rightarrow u(x)$ at almost everywhere point $x \in \Omega$. Hence, if $x \in \{u > 0\}$ then for n large enough, $u_n(x) > 0$ and so,

$$\chi_{\{u > 0\}}(x) = 1 = \liminf_n \chi_{\{u_n > 0\}}(x).$$

If $x \in \{u = 0\}$, then we always have

$$\chi_{\{u > 0\}}(x) = 0 \leq \liminf_n \chi_{\{u_n > 0\}}(x).$$

By Fatou's Lemma, we get

$$(2.3) \quad \int_{\Omega} \psi \chi_{\{u > 0\}} \leq \int_{\Omega} \liminf_n [\psi \chi_{\{u_n > 0\}}] \leq \liminf_n \left[\int_{\Omega} \psi \chi_{\{u_n > 0\}} \right].$$

Combining (2.2) & (2.3), we infer that

$$\int_{\Omega} w |\nabla u|^p + \int_{\Omega} \psi \chi_{\{u > 0\}} \leq \liminf_n \left[\int_{\Omega} w |\nabla u_n|^p + \int_{\Omega} \psi \chi_{\{u_n > 0\}} \right].$$

This implies that u minimizes Problem (2.1).

Fix $\varphi \in C_0^\infty(\Omega)$ with $\varphi \geq 0$. Hence, $u - \varepsilon\varphi \in W^{1,p}(\Omega)$ with $u - \varepsilon\varphi = g$ on $\partial\Omega$ and $u - \varepsilon\varphi \leq u$, for all $\varepsilon > 0$. In particular, $u - \varepsilon\varphi$ is admissible in Problem (2.1). From the minimality of u , we have

$$\int_{\Omega} w|\nabla u|^p + \int_{\Omega} \psi \chi_{\{u>0\}} \leq \int_{\Omega} w|\nabla u - \varepsilon\nabla\varphi|^p + \int_{\Omega} \psi \chi_{\{u-\varepsilon\varphi>0\}} \leq \int_{\Omega} w|\nabla u - \varepsilon\nabla\varphi|^p + \int_{\Omega} \psi \chi_{\{u>0\}}.$$

Therefore, we get

$$\int_{\Omega} w|\nabla u|^p \leq \int_{\Omega} w|\nabla u - \varepsilon\nabla\varphi|^p, \quad \text{for all } \varepsilon > 0.$$

Hence,

$$\int_{\Omega} w|\nabla u|^{p-2}\nabla u \cdot \nabla\varphi \leq 0.$$

Finally, assume that u is continuous (we will show later that this assumption is always satisfied; see Proposition 2.3). Let $\varphi \in C_0^\infty(\Omega)$ be such that $\text{spt}(\varphi) \subset \{u > 0\}$. Since u is continuous, $\text{spt}(\varphi)$ is compact and $u > 0$ on $\text{spt}(\varphi)$, then we have $\{u + \varepsilon\varphi > 0\} = \{u > 0\}$, for any $\varepsilon > 0$ small enough. Yet, $u + \varepsilon\varphi = g$ on $\partial\Omega$. So again, by the minimality of u in Problem (2.1), we get that

$$\int_{\Omega} w|\nabla u|^p + \int_{\Omega} \psi \chi_{\{u>0\}} \leq \int_{\Omega} w|\nabla u + \varepsilon\nabla\varphi|^p + \int_{\Omega} \psi \chi_{\{u+\varepsilon\varphi>0\}}.$$

Thus,

$$\int_{\Omega} w|\nabla u|^p \leq \int_{\Omega} w|\nabla u + \varepsilon\nabla\varphi|^p, \quad \text{for all } \varepsilon > 0.$$

This yields that

$$\int_{\Omega} w|\nabla u|^{p-2}\nabla u \cdot \nabla\varphi \geq 0. \quad \square$$

For any ball $B \subset \Omega$, we will denote by $v = v_B$ the unique solution of the following weighted p -Laplacian problem:

$$\begin{cases} \nabla \cdot [w|\nabla v|^{p-2}\nabla v] = 0 & \text{in } B \\ v = u & \text{on } \partial B. \end{cases}$$

Notice that, by the comparison principle (see [4, Lemma 3.18]), $u \leq v$ on B . In the sequel, we will call this function v the weighted p -harmonic replacement of u in B . We recall that for any function $\phi \in W^{1,p}(B)$ such that $v = u$ on ∂B , we have

$$(2.4) \quad \int_B w|\nabla v|^p \leq \int_B w|\nabla\phi|^p.$$

Our aim is to show that a minimizer u in Problem (2.1) is locally α -Hölderian, for some $\alpha > 0$ that depends on N , p and q . In order to prove this local Hölder regularity on u , we need the following crucial lemma that we use often in the rest of the paper.

Lemma 2.2. *Assume $\psi \in L_{loc}^q(\Omega)$, for some $q > \frac{N}{p}$. Let u be a minimizer of Problem (2.1) and v be the weighted p -harmonic replacement of u in $B \subset \Omega$. Then, there is a universal constant $C = C(p)$ such that*

$$\begin{cases} \int_B w|\nabla u - \nabla v|^p \leq C \|\psi\|_{L^q(B)} |B \cap \{u = 0\}|^{1-\frac{1}{q}} & \text{if } p \geq 2, \\ \int_B w|\nabla u - \nabla v|^p \leq C \left(\int_B w + \int_B |\nabla u|^p \right)^{1-\frac{p}{2}} \|\psi\|_{L^q(B)} |B \cap \{u = 0\}|^{1-\frac{1}{q}} \frac{p}{2} & \text{if } 1 < p < 2. \end{cases}$$

Proof. First of all, let us extend the weighted p -harmonic replacement v to a function \tilde{v} on Ω by setting

$$\tilde{v} = \begin{cases} v & \text{in } B, \\ u & \text{on } \Omega \setminus B. \end{cases}$$

Clearly, $\tilde{v} \in W^{1,p}(\Omega)$ with $\tilde{v} = u$ on $\partial\Omega$ (so, \tilde{v} is admissible in (2.1)). From the minimality of u in Problem (2.1), we have

$$\int_{\Omega} w|\nabla u|^p + \int_{\Omega} \psi \chi_{\{u>0\}} \leq \int_{\Omega} w|\nabla \tilde{v}|^p + \int_{\Omega} \psi \chi_{\{\tilde{v}>0\}}.$$

Thus,

$$\begin{aligned} & \int_B w|\nabla u|^p + \int_{\Omega \setminus B} w|\nabla u|^p + \int_B \psi \chi_{\{u>0\}} + \int_{\Omega \setminus B} \psi \chi_{\{u>0\}} \\ & \leq \int_B w|\nabla v|^p + \int_{\Omega \setminus B} w|\nabla u|^p + \int_B \psi \chi_{\{v>0\}} + \int_{\Omega \setminus B} \psi \chi_{\{u>0\}}. \end{aligned}$$

We infer that

$$\int_B w|\nabla u|^p + \int_B \psi \chi_{\{u>0\}} \leq \int_B w|\nabla v|^p + \int_B \psi \chi_{\{v>0\}}.$$

Hence,

$$(2.5) \quad \int_B w[|\nabla u|^p - |\nabla v|^p] \leq \int_B \psi \chi_{\{v>0\}} - \int_B \psi \chi_{\{u>0\}} \leq \int_B \psi \chi_{\{u=0\}} \leq \|\psi\|_{L^q(B)} |B \cap \{u=0\}|^{1-\frac{1}{q}}.$$

Now, set

$$u_t = (1-t)v + tu, \quad \text{for every } 0 \leq t \leq 1.$$

We have

$$\int_B w[|\nabla u|^p - |\nabla v|^p] = \int_B w[|\nabla u_1|^p - |\nabla u_0|^p] = \int_B w \left[\int_0^1 \frac{d}{dt} |\nabla u_t|^p \right] = p \int_0^1 \int_B w|\nabla u_t|^{p-2} \nabla u_t \cdot [\nabla u - \nabla v].$$

Yet,

$$\int_B w|\nabla v|^{p-2} \nabla v \cdot [\nabla u - \nabla v] = 0 \quad \text{and} \quad \nabla u_t - \nabla v = t[\nabla u - \nabla v].$$

Then, we get that

$$(2.6) \quad \int_B w[|\nabla u|^p - |\nabla v|^p] = p \int_0^1 t^{-1} \int_B w[|\nabla u_t|^{p-2} \nabla u_t - |\nabla v|^{p-2} \nabla v] \cdot [\nabla u_t - \nabla v].$$

From [6], we have the following inequalities:

$$\begin{cases} |a-b|^2(1+|a|^2+|b|^2)^{\frac{p-2}{2}} \leq \frac{1}{p-1}[|a|^{p-2}a - |b|^{p-2}b] \cdot [a-b] & \text{if } 1 < p < 2, \\ |a-b|^p \leq 2^{p-2}[|a|^{p-2}a - |b|^{p-2}b] \cdot [a-b] & \text{if } p \geq 2. \end{cases}$$

Recalling (2.6), we get that

$$\begin{cases} \int_B w[|\nabla u|^p - |\nabla v|^p] \geq p(p-1) \int_0^1 t^{-1} \int_B w|\nabla u_t - \nabla v|^2(1+|\nabla u_t|^2+|\nabla v|^2)^{\frac{p-2}{2}}, & \text{if } 1 < p < 2, \\ \int_B w[|\nabla u|^p - |\nabla v|^p] \geq p2^{2-p} \int_0^1 t^{-1} \int_B w|\nabla u_t - \nabla v|^p, & \text{if } p \geq 2. \end{cases}$$

Assume $p \geq 2$. Then,

$$\int_B w[|\nabla u|^p - |\nabla v|^p] \geq p2^{2-p} \int_0^1 t^{p-1} \int_B w|\nabla u - \nabla v|^p = 2^{2-p} \int_B w|\nabla u - \nabla v|^p.$$

By (2.5), we infer that

$$\int_B w|\nabla u - \nabla v|^p \leq 2^{p-2} \|\psi\|_{L^q(B)} |B \cap \{u=0\}|^{1-\frac{1}{q}}.$$

Finally, assume $1 < p < 2$. Then, we have

$$\|\psi\|_{L^q(B)} |B \cap \{u = 0\}|^{1-\frac{1}{q}} \geq \int_B w[|\nabla u|^p - |\nabla v|^p] \geq C(p) \int_B w|\nabla u - \nabla v|^2 (1 + |\nabla u|^2 + |\nabla v|^2)^{\frac{p-2}{2}}.$$

By Hölder inequality, one has

$$\int_B w|\nabla u - \nabla v|^p \leq \left(\int_B w|\nabla u - \nabla v|^2 (1 + |\nabla u|^2 + |\nabla v|^2)^{\frac{p-2}{2}} \right)^{\frac{p}{2}} \left(\int_B w(1 + |\nabla u|^2 + |\nabla v|^2)^{\frac{p}{2}} \right)^{1-\frac{p}{2}}.$$

Then, we get

$$\begin{aligned} \int_B w|\nabla u - \nabla v|^p &\leq \left[C(p) \|\psi\|_{L^q(B)} |B \cap \{u = 0\}|^{1-\frac{1}{q}} \right]^{\frac{p}{2}} \left(\int_B w(1 + |\nabla u|^2 + |\nabla v|^2)^{\frac{p}{2}} \right)^{1-\frac{p}{2}} \\ &\leq C(p) \left[\|\psi\|_{L^q(B)} |B \cap \{u = 0\}|^{1-\frac{1}{q}} \right]^{\frac{p}{2}} \left(\int_B w(1 + |\nabla u|^p + |\nabla v|^p) \right)^{1-\frac{p}{2}} \\ &\leq C(p) \left(\int_B w + \int_B w|\nabla u|^p \right)^{1-\frac{p}{2}} \left[\|\psi\|_{L^q(B)} |B \cap \{u = 0\}|^{1-\frac{1}{q}} \right]^{\frac{p}{2}}, \end{aligned}$$

where the last inequality follows from (2.4). \square

Since $u \geq 0$ is weighted p -subharmonic, then thanks to [4, Theorem 3.41], u is locally bounded. In the next proposition, we show that u is locally α -Hölderian.

Proposition 2.3. *Assume $w \in C^2(\Omega)$ and $\psi \in L^q_{loc}(\Omega)$ with $q > \frac{N}{p}$. Let u be a minimizer in Problem (2.1). For any open subset $\Omega' \subset\subset \Omega$ and a compact set $K \subset \Omega'$, there exists a constant C depending on $p, N, w_{\min}, \|\psi\|_{L^q(\Omega')}, \|w\|_{L^\infty(\Omega')}, \|\nabla w\|_{L^\infty(\Omega')}, \|D^2 w\|_{L^\infty(\Omega')}, \|u\|_{L^\infty(\Omega')}$ and, $\text{dist}(K, \partial\Omega')$ such that*

$$\|u\|_{C^{0,1-\frac{N}{pq}}(K)} \leq C.$$

Proof. Fix $x \in K$ and $0 < 2r < R < \text{dist}(K, \partial\Omega')$ (so, we have $B(x, r) \subset B(x, \frac{R}{2}) \subset B(x, R) \subset \Omega'$). Let v be the weighted p -harmonic replacement of u on $B = B(x, r)$. Then, we clearly have

$$(2.7) \quad \left(\int_{B(x,r)} w|\nabla u|^p \right)^{\frac{1}{p}} \leq \left(\int_{B(x,r)} w|\nabla u - \nabla v|^p \right)^{\frac{1}{p}} + \left(\int_{B(x,r)} w|\nabla v|^p \right)^{\frac{1}{p}}.$$

But, thanks to Lemma 2.2, one has

$$(2.8) \quad \begin{cases} \left(\int_{B(x,r)} w|\nabla u - \nabla v|^p \right)^{\frac{1}{p}} \leq C(p, N) \|\psi\|_{L^q(B)}^{\frac{1}{p}} r^{\frac{N}{p}(1-\frac{1}{q})}, & \text{if } p \geq 2, \\ \left(\int_{B(x,r)} w|\nabla u - \nabla v|^p \right)^{\frac{1}{p}} \leq C(p, N) \left(\int_B w + \int_B |\nabla u|^p \right)^{\frac{1}{p}-\frac{1}{2}} \|\psi\|_{L^q(B)}^{\frac{1}{2}} r^{\frac{N}{2}(1-\frac{1}{q})}, & \text{if } 1 < p < 2. \end{cases}$$

On the other hand, we have

$$(2.9) \quad \int_{B(x,r)} w|\nabla v|^p \leq \|w\|_{\infty} |B(x, r)| \|\nabla v\|_{L^\infty(B(x,r))}^p.$$

By [9, Theorem 1.1],

$$\|\nabla v\|_{L^\infty(B(x,r))} \leq C(p, N) \left(\frac{1 + \sqrt{\kappa}R}{R} \right) \|v\|_{L^\infty(B(x, \frac{R}{2}))},$$

where $0 \leq \kappa < \infty$ (depends on N , w_{\min} , $\|\nabla w\|_{L^\infty(B(x, \frac{R}{2}))}$, $\|D^2 w\|_{L^\infty(B(x, \frac{R}{2}))}$) is such that the Ricci tensor $\text{Ric}(w)$ of the Riemannian conformal metric (\mathbb{R}^N, w) satisfies $\text{Ric}(w) \geq -(N-1)\kappa$; it is well known that the Ricci tensor in this case is given by (see, for instance, [8])

$$\text{Ric}(w) = -\frac{N-2}{2} \left[\frac{D^2 w}{w} - \frac{3}{2} \frac{\nabla w \otimes \nabla w}{w^2} \right] - \frac{1}{2} \left[\frac{\Delta w}{w} - \frac{(N-4)}{2} \frac{|\nabla w|^2}{w^2} \right].$$

Recalling (2.9), we get that

$$(2.10) \quad \int_{B(x,r)} w |\nabla v|^p \leq C(p, N) \|w\|_\infty r^N \left(\frac{1 + \sqrt{\kappa} R}{R} \right)^p \|v\|_{L^\infty(B(x, \frac{R}{2}))}^p.$$

Since $u = v$ on ∂B , then thanks to the comparison principle [4, Lemma 3.18], we have the following estimate:

$$\|v\|_{L^\infty(B(x, \frac{R}{2}))} \leq \|u\|_{L^\infty(B(x, R))}.$$

For $0 < \delta < 1$, set $R_\delta = r^{1-\delta}$. Now, choose $r > 0$ small enough so that $2r < R_\delta < \text{dist}(K, \partial\Omega')$. This yields that

$$\int_{B(x,r)} w |\nabla v|^p \leq C r^N \left(\frac{1 + \sqrt{\kappa} r^{1-\delta}}{r^{1-\delta}} \right)^p \|u\|_{L^\infty(B(x, \frac{R_\delta}{2}))}^p \leq C r^{N-p+\delta p}.$$

Assume $p \geq 2$. Then, by (2.7) & (2.8), we get

$$\left(\int_{B(x,r)} |\nabla u|^p \right)^{\frac{1}{p}} \leq \frac{1}{w_{\min}^{\frac{1}{p}}} \left(\int_{B(x,r)} w |\nabla u|^p \right)^{\frac{1}{p}} \leq C r^{\frac{N}{p}(1-\frac{1}{q})} + C r^{\frac{N}{p}-1+\delta} \leq C r^{\frac{N}{p}(1-\frac{1}{q})}$$

as soon as $\delta \geq 1 - \frac{N}{pq} > 0$. Thanks to Morrey's Lemma, we conclude that $u \in C^{0,1-\frac{N}{pq}}(K)$. Moreover, we have

$$\|u\|_{C^{0,1-\frac{N}{pq}}(K)} \leq C(p, N, w_{\min}, \|\psi\|_{L^q(\Omega')}, \|w\|_{C^2(\Omega')}, \|u\|_{L^\infty(\Omega')}, \text{dist}(K, \partial\Omega')).$$

Finally, assume that $1 < p < 2$. So, by (2.8), we recall that

$$\begin{aligned} \int_{\Omega} w |\nabla u - \nabla v|^p &\leq C \left(|B| + \int_B |\nabla u|^p \right)^{1-\frac{p}{2}} \left(\|\psi\|_{L^q(B)} r^{N(1-\frac{1}{q})} \right)^{\frac{p}{2}} \\ &\leq C r^{N(1-\frac{p}{2q})} + C r^{\frac{Np}{2}(1-\frac{1}{q})} \left(\int_B |\nabla u|^p \right)^{1-\frac{p}{2}}. \end{aligned}$$

Hence,

$$\begin{aligned} \left(\int_{B(x,r)} w |\nabla u|^p \right)^{\frac{1}{p}} &\leq C r^{\frac{N}{p}(1-\frac{p}{2q})} + C r^{\frac{N}{2}(1-\frac{1}{q})} \left(\int_B |\nabla u|^p \right)^{\frac{1}{p}-\frac{1}{2}} + C r^{\frac{N}{p}-1+\delta} \\ &\leq C r^{\frac{N}{p}(1-\frac{p}{2q})} + \frac{C}{w_{\min}^{\frac{1}{p}-\frac{1}{2}}} r^{\frac{N}{2}(1-\frac{1}{q})} \left(\int_B w |\nabla u|^p \right)^{\frac{1}{p}-\frac{1}{2}}. \end{aligned}$$

Using Young's inequality, we get

$$\left(\int_{B(x,r)} w |\nabla u|^p \right)^{\frac{1}{p}} \leq C r^{\frac{N}{p}(1-\frac{1}{q})}.$$

Consequently, u is locally α -Hölder in Ω with $\alpha = 1 - \frac{N}{pq}$. In addition, we also have the following estimate:

$$\|u\|_{C^{0,\alpha}(K)} \leq C(p, N, w_{\min}, \|\psi\|_{L^q(\Omega')}, \|w\|_{C^2(\Omega')}, \|u\|_{L^\infty(\Omega')}, \text{dist}(K, \partial\Omega')).$$

□

3. REGULARITY OF THE FREE BOUNDARY

In this section, we prove under the assumptions $q = \infty$ and $\psi \geq \psi_{\min} > 0$ that the free boundary $\partial\{u > 0\}$ of any minimizer u in Problem (2.1) has locally a finite perimeter. First, we start by the following nondegeneracy result:

Lemma 3.1. *Assume $\psi \geq \psi_{\min} > 0$. For any $\kappa \in (0, \frac{1}{2})$, there exists a constant $c > 0$ depending only on $p, N, w_{\min}, \psi_{\min}, \|\psi\|_{L^\infty(\Omega')}$ and $\|w\|_{C^2(\Omega')}$ such that the following statement holds*

$$\|u\|_{L^\infty(B_r)} < cr \quad \text{implies that} \quad u = 0 \quad \text{in } B_{\kappa r}$$

for any ball $B_r \subset \Omega' \subset \subset \Omega$.

Proof. Fix $\kappa' \in (k, 1)$. Let ϕ be the weighted p -harmonic function in $B_{\kappa'r} \setminus B_{\kappa r}$ with $\phi = 1$ on $\partial B_{\kappa'r}$ and $\phi = 0$ on $\partial B_{\kappa r}$. From [5, 7], this function ϕ is of class $C^{1,\beta}$ on $\overline{B_{\kappa'r} \setminus B_{\kappa r}}$. Let us extend ϕ by zero in $B_{\kappa r}$. Moreover, one has $0 \leq \phi \leq 1$.

Set $m = \|u\|_{L^\infty(B_{\kappa'r})}$. Then, we define $\Phi = \min\{u, m\phi\}$. Since $m\phi = m \geq u$ on $\partial B_{\kappa'r}$, then $\Phi = u$ on $\partial B_{\kappa'r}$. Moreover, $\Phi = 0$ in $B_{\kappa r}$. Thanks to the minimality of u in Problem (2.1), we have

$$\int_{B_{\kappa'r}} (w|\nabla u|^p + \psi\chi_{\{u>0\}}) \leq \int_{B_{\kappa'r}} (w|\nabla\Phi|^p + \psi\chi_{\{\Phi>0\}}) = \int_{B_{\kappa'r} \setminus B_{\kappa r}} (w|\nabla\Phi|^p + \psi\chi_{\{\Phi>0\}}).$$

Hence,

$$\int_{B_{\kappa r}} (w|\nabla u|^p + \psi\chi_{\{u>0\}}) + \int_{B_{\kappa'r} \setminus B_{\kappa r}} (w|\nabla u|^p + \psi\chi_{\{u>0\}}) \leq \int_{B_{\kappa'r} \setminus B_{\kappa r}} (w|\nabla\Phi|^p + \psi\chi_{\{\Phi>0\}}).$$

But, $\{\Phi > 0\} \subset \{u > 0\}$. Then, we get that

$$\int_{B_{\kappa r}} (w|\nabla u|^p + \psi\chi_{\{u>0\}}) \leq \int_{B_{\kappa'r} \setminus B_{\kappa r}} (w[|\nabla\Phi|^p - |\nabla u|^p] + \psi[\chi_{\{\Phi>0\}} - \chi_{\{u>0\}}]) \leq \int_{B_{\kappa'r} \setminus B_{\kappa r}} w[|\nabla\Phi|^p - |\nabla u|^p].$$

Since the map $\xi \mapsto |\xi|^p$ is convex, one has

$$|\nabla u|^p - |\nabla\Phi|^p \geq p|\nabla\Phi|^{p-2}\nabla\Phi \cdot [\nabla u - \nabla\Phi].$$

Hence,

$$\begin{aligned} \int_{B_{\kappa r}} (w|\nabla u|^p + \psi\chi_{\{u>0\}}) &\leq p \int_{B_{\kappa'r} \setminus B_{\kappa r}} w|\nabla\Phi|^{p-2}\nabla\Phi \cdot [\nabla\Phi - \nabla u] \\ &= pm^{p-1} \int_{B_{\kappa'r} \setminus B_{\kappa r}} w|\nabla\phi|^{p-2}\nabla\phi \cdot [\nabla\Phi - \nabla u] \\ &= -pm^{p-1} \int_{B_{\kappa'r} \setminus B_{\kappa r}} \nabla \cdot [w|\nabla\phi|^{p-2}\nabla\phi][\Phi - u] + pm^{p-1} \int_{\partial B_{\kappa r}} w|\nabla\phi|^{p-2}\nabla\phi \cdot \mathbf{n}[\Phi - u] \\ &= pm^{p-1} \int_{\partial B_{\kappa r}} [\Phi - u]w|\nabla\phi|^{p-2}\nabla\phi \cdot \mathbf{n} \\ &\leq pm^{p-1} \|\nabla\phi\|_\infty^{p-1} \|w\|_\infty \int_{\partial B_{\kappa r}} u. \end{aligned}$$

By [9, Theorem 1.1], we have

$$|\nabla\phi| \leq \frac{C}{r}.$$

Therefore, we get

$$\min\{w_{\min}, \psi_{\min}\} \left(\int_{B_{\kappa r}} |\nabla u|^p + |\{u > 0\}| \right) \leq w_{\min} \int_{B_{\kappa r}} |\nabla u|^p + \psi_{\min} |\{u > 0\}| \leq \int_{B_{\kappa r}} (w|\nabla u|^p + \psi\chi_{\{u>0\}})$$

$$\leq C \frac{m^{p-1}}{r^{p-1}} \int_{\partial B_{\kappa r}} u.$$

Thanks to the $W^{1,1}$ trace inequality, we have

$$\int_{\partial B_{\kappa r}} u \leq C \left(\int_{B_{\kappa r}} u + \int_{B_{\kappa r}} |\nabla u| \right).$$

Hence,

$$\int_{B_{\kappa r}} |\nabla u|^p + |\{u > 0\}| \leq C \frac{m^{p-1}}{r^{p-1}} \left(\int_{B_{\kappa r}} u + \int_{B_{\kappa r}} |\nabla u| \right).$$

This implies

$$\begin{aligned} \int_{B_{\kappa r}} |\nabla u|^p + |\{u > 0\}| &\leq C \frac{m^{p-1}}{r^{p-1}} \left(|\{u > 0\}| + |\{u > 0\}|^{\frac{p-1}{p}} \left(\int_{B_{\kappa r}} |\nabla u|^p \right)^{\frac{1}{p}} \right) \\ &\leq C \frac{m^{p-1}}{r^{p-1}} \left(\int_{B_{\kappa r}} |\nabla u|^p + |\{u > 0\}| \right). \end{aligned}$$

Hence,

$$\left[1 - C \frac{m^{p-1}}{r^{p-1}} \right] \left(\int_{B_{\kappa r}} |\nabla u|^p + |\{u > 0\}| \right) \leq 0$$

which implies that $u = 0$ on $B_{\kappa r}$ as soon as

$$m \leq \left(\frac{1}{C} \right)^{\frac{1}{p-1}} r.$$

□

Proposition 3.2. *For $K \subset \Omega' \subset \subset \Omega$, there is a constant $c \in (0, 1)$ (depending on $p, N, w_{\min}, \psi_{\min}, \|\psi\|_{L^\infty(\Omega')}, \|w\|_{C^2(\Omega')}, \|u\|_{L^\infty(\Omega')}$ and $\text{dist}(K, \partial\Omega')$) such that for every $x_0 \in K \cap \partial\{u > 0\}$ and $r > 0$ with $B(x_0, r) \subset K$, we have*

$$c \leq \frac{\mathcal{L}^N(B(x_0, r) \cap \{u > 0\})}{\mathcal{L}^N(B(x_0, r))} \leq 1 - c.$$

Proof. Again thanks to Lemma 3.1, we see that there exist a constant $c > 0$ and a point $x \in B(x_0, \frac{r}{2})$ such that $u(x) \geq c \frac{r}{2}$. Yet, we claim that there is a uniform constant $\delta \in (0, 1)$ such that $B(x, \delta r) \subset B(x_0, r) \cap \{u > 0\}$. Indeed, if $y \in B(x, \delta r)$ then one has

$$|y - x_0| \leq |y - x| + |x - x_0| \leq \left(\frac{1}{2} + \delta \right) r < r.$$

Thanks to the Lipschitz regularity of u , we also have

$$u(y) \geq u(x) - C|y - x| \geq \left(\frac{c}{2} - C\delta \right) r > 0$$

as soon as $\delta < \frac{c}{2C}$. Hence, we get

$$\frac{\mathcal{L}^N(B(x_0, r) \cap \{u > 0\})}{\mathcal{L}^N(B(x_0, r))} \geq \frac{\mathcal{L}^N(B(x, \delta r))}{\mathcal{L}^N(B(x_0, r))} \geq c\delta^N.$$

For the upper bound: assume that there is a sequence of points $x_{0,n}$ in $K \cap \partial\{u > 0\}$ and $r_n > 0$ with $B(x_{0,n}, r_n) \subset K$ such that

$$\frac{\mathcal{L}^N(B(x_{0,n}, r_n) \cap \{u = 0\})}{\mathcal{L}^N(B(x_{0,n}, r_n))} \leq \frac{1}{n} \rightarrow 0.$$

Up to a subsequence, $x_{0,n} \rightarrow x_0 \in K \cap \partial\{u > 0\}$ and $r_n \rightarrow r_0$. Assume $r_0 > 0$. Let v_n be the

weighted p -harmonic replacement of u in $B_n := B(x_{0,n}, r_n)$. For all n large enough, it is clear that $B := B(x_0, \frac{r_0}{4}) \subset B(x_{0,n}, \frac{r_n}{2})$. Thanks to Lemma 2.2, we have

$$\begin{cases} \int_B w |\nabla u - \nabla v_n|^p \leq C \|\psi\|_{L^\infty(\Omega)} |B_n \cap \{u = 0\}| & \text{if } p \geq 2, \\ \int_B w |\nabla u - \nabla v_n|^p \leq C \left(|B_n| + \int_{B_n} w |\nabla u|^p \right)^{1-\frac{p}{2}} \left[\|\psi\|_{L^\infty(\Omega)} |B_n \cap \{u = 0\}| \right]^{\frac{p}{2}} & \text{if } 1 < p < 2. \end{cases}$$

Hence,

$$\int_B w |\nabla u - \nabla v_n|^p \rightarrow 0.$$

Yet, $v_n \rightarrow v$ uniformly in \bar{B} . In particular, we must have $u = v + C$. But, v is weighted p -harmonic in B and so, u is also weighted p -harmonic there. However, we have $u(x_0) = 0$. By the maximum principle, we infer that $u = 0$ in B . Yet, by Lemma 3.1, there is a constant $c > 0$ such that

$$\|u\|_{L^\infty(B)} \geq c,$$

which is a contradiction.

Finally, assume that $r_0 = 0$. For every n , we define $u_n(x) = \frac{u(x_{0,n} + r_n x)}{r_n}$ for all $x \in B_1$. It is not difficult to check that u_n minimizes

$$\min \left\{ \int_{B_1} (w_n |\nabla u|^p + \psi_n \chi_{\{u > 0\}}) : u \in W^{1,p}(B_1), u \geq 0, u = u_n \text{ on } \partial B_1 \right\},$$

where

$$w_n(x) = w(x_{0,n} + r_n x) \quad \text{and} \quad \psi_n(x) = \psi(x_{0,n} + r_n x).$$

Now, let v_n be the weighted p -harmonic replacement of u_n in B_1 . Again, by Lemma 2.2, one has the following:

$$\begin{cases} \int_{B_1} w_n |\nabla u_n - \nabla v_n|^p \leq C \|\psi\|_{L^\infty(\Omega)} |B_1 \cap \{u_n = 0\}| & \text{if } p \geq 2, \\ \int_{B_1} w_n |\nabla u_n - \nabla v_n|^p \leq C \left(|B_1| + \int_{B_1} w_n |\nabla u_n|^p \right)^{1-\frac{p}{2}} \left[\|\psi\|_{L^\infty(\Omega)} |B_1 \cap \{u_n = 0\}| \right]^{\frac{p}{2}} & \text{if } 1 < p < 2. \end{cases}$$

But, $w_n \geq \frac{w(x_0)}{2} > 0$ on B_1 for n large enough. Moreover, we have

$$\mathcal{L}^N(B_1 \cap \{u_n = 0\}) = \frac{\mathcal{L}^N(B(x_{0,n}, r_n) \cap \{u = 0\})}{\mathcal{L}^N(B(x_{0,n}, r_n))} \mathcal{L}^N(B_1) \rightarrow 0.$$

Thus,

$$\int_{B_1} |\nabla u_n - \nabla v_n|^p \rightarrow 0.$$

We note that since u is Lipschitz, then u_n are clearly uniformly Lipschitz. In addition, one has $0 \leq u_n \leq C$ for all n since

$$u(x_{0,n} + r_n x) = u(x_{0,n} + r_n x) - u(x_{0,n}) \leq Cr_n.$$

So, up to a subsequence, $u_n \rightarrow u^*$ and $v_n \rightarrow v$ uniformly in \bar{B}_1 . And, we have $u^* = v + C$. However, v is weighted p -harmonic in B_1 . Hence, u^* is weighted p -harmonic. But, we have $u^*(0) = 0$ (since $u_n(0) = 0$ for every n). Thanks again to the maximum principle, we get that $u^* = 0$ in B_1 . From Lemma 3.1, we know that there is a uniform constant $c > 0$ such that

$$\|u\|_{L^\infty(B(x_{0,n}, r_n))} \geq cr_n.$$

Then,

$$\|u_n\|_{L^\infty(B_1)} \geq c.$$

But, this yields again to a contradiction. \square

In particular, we obtain the following:

Theorem 3.3. *The free boundary has zero Lebesgue measure.*

Proof. This follows from Proposition 3.2 as well as the Lebesgue density theorem. \square

Now, let us define the measure Λ as follows:

$$\int_{\Omega} \varphi d\Lambda = - \int_{\Omega} w |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi, \quad \text{for all } \varphi \in C_0^\infty(\Omega).$$

Since u is p -subharmonic, then Λ is a nonnegative Borel measure. Moreover, u is p -harmonic in the positivity set $\{u > 0\}$ and so, for every smooth function φ such that $\text{spt}(\varphi) \subset \{u > 0\}$ we have

$$\int_{\Omega} \varphi d\Lambda = 0.$$

Therefore, $\text{spt}(\Lambda) \subset \partial\{u > 0\}$.

Proposition 3.4. *For any compact set $K \subset \Omega' \subset\subset \Omega$, there exist two constants $0 < c < C < \infty$ (depending on $p, N, w_{\min}, \psi_{\min}, \|\psi\|_{L^\infty(\Omega')}, \|w\|_{C^2(\Omega')}, \|u\|_{L^\infty(\Omega')}$ and $\text{dist}(K, \partial\Omega')$) such that for every $x_0 \in K \cap \partial\{u > 0\}$, we have*

$$cr^{N-1} \leq \Lambda(B(x_0, r)) \leq Cr^{N-1}, \quad \text{for all } 0 < r < d(x_0, \partial K).$$

Proof. Fix $\delta > 0$. Let $0 \leq \varphi_\delta \in C_0^\infty(B(x_0, r + \delta))$ be a cutoff function such that $\varphi_\delta = 1$ on $B(x_0, r)$ and $|\nabla \varphi| \leq \frac{C}{\delta}$. Then, we have

$$\Lambda(B(x_0, r)) \leq \int_{\Omega} \varphi_\delta d\Lambda = - \int_{\Omega} w |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi_\delta \leq \|w\|_\infty \|\nabla u\|_\infty^{p-1} \frac{C}{\delta} |B(x_0, r + \delta) \setminus B(x_0, r)|,$$

for any $\delta > 0$. Letting $\delta \rightarrow 0^+$, we get

$$\Lambda(B(x_0, r)) \leq Cr^{N-1}.$$

Let us prove the lower estimate. Assume that there is a sequence of points $x_{0,n}$ in $K \cap \partial\{u > 0\}$ and $0 < r_n < d(x_{0,n}, \partial K)$ such that

$$\Lambda(B(x_{0,n}, r_n)) \leq \frac{r_n^{N-1}}{n}.$$

Up to a subsequence, $x_{0,n} \rightarrow x_0$ and $r_n \rightarrow r_0$. Assume $r_0 > 0$. For every $\varphi \in C_0^\infty(B(x_0, \frac{r_0}{2}))$, we have

$$- \int_{\Omega} w |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi = \int_{B(x_{0,n}, r_n)} \varphi d\Lambda \leq \|\varphi\|_\infty \Lambda(B(x_{0,n}, r_n)) \rightarrow 0.$$

Substituting φ with $-\varphi$, we get

$$- \int_{\Omega} w |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi = 0.$$

Hence, u is weighted p -harmonic inside $B(x_0, \frac{r_0}{2})$. But, we have $u(x_0) = \lim_n u(x_{0,n}) = 0$. By the maximum principle, this implies that $u = 0$ on $B(x_0, \frac{r_0}{2})$. However, by Lemma 3.1, there is a constant $c > 0$ such that

$$\|u\|_{L^\infty(B(x_0, \frac{r_0}{2}))} \geq c.$$

Yet, this is a contradiction.

Finally, assume that $r_0 = 0$. For every n , set $u_n(x) = \frac{u(x_{0,n} + r_n x)}{r_n}$ for all $x \in B_1$. Then, we define

$$\int_{B_1} \varphi d\Lambda_n = - \int_{B_1} w_n |\nabla u_n|^{p-2} \nabla u_n \cdot \nabla \varphi, \quad \text{for all } \varphi \in C_0^\infty(B_1),$$

where

$$w_n(x) = w(x_{0,n} + r_n x).$$

Fix $\varphi \in C_0^\infty(B_1)$. We have

$$\begin{aligned} \int_{B_1} \varphi \, d\Lambda_n &= - \int_{B_1} w(x_{0,n} + r_n x) |\nabla u(x_{0,n} + r_n x)|^{p-2} \nabla u(x_{0,n} + r_n x) \cdot \nabla \varphi(x) \, dx \\ &= - \frac{1}{r_n^N} \int_{B(x_{0,n}, r_n)} w(z) |\nabla u(z)|^{p-2} \nabla u(z) \cdot \nabla \varphi\left(\frac{z - x_{0,n}}{r_n}\right) \, dz \\ &= - \frac{1}{r_n^{N-1}} \int_{B(x_{0,n}, r_n)} w(z) |\nabla u(z)|^{p-2} \nabla u(z) \cdot \nabla \varphi_n(z) \, dz \\ &= \frac{1}{r_n^{N-1}} \int_{\Omega} \varphi_n \, d\Lambda \end{aligned}$$

where

$$\varphi_n(z) = \varphi\left(\frac{z - x_{0,n}}{r_n}\right).$$

Yet, we see that u_n is uniformly bounded and Lipschitz continuous on B_1 . Hence, up to a subsequence, $u_n \rightarrow u^*$ uniformly in $\overline{B_1}$. On the other hand, one has $w_n |\nabla u_n|^{p-2} \nabla u_n \rightharpoonup^* h$ in $L^\infty(B_1)$. We claim that $h = w(x_0) |\nabla u^*|^{p-2} \nabla u^*$. If this is the case, so we get that for every $\varphi \in C_0^\infty(B_1)$,

$$\begin{aligned} & - \int_{B_1} w(x_0) |\nabla u^*|^{p-2} \nabla u^* \cdot \nabla \varphi \\ &= \lim_{n \rightarrow \infty} - \int_{\Omega} w_n |\nabla u_n|^{p-2} \nabla u_n \cdot \nabla \varphi = \lim_{n \rightarrow \infty} \int_{B_1} \varphi \, d\Lambda_n \leq \|\varphi\|_\infty \Lambda_n(B_1) \leq C \frac{\Lambda(B(x_{0,n}, r_n))}{r_n^{N-1}} \rightarrow 0. \end{aligned}$$

Hence, u^* is weighted p -harmonic inside B_1 . But, we have $u^*(0) = \lim_n u_n(0) = 0$. By the maximum principle, this implies that $u^* = 0$ in B_1 . However, by Lemma 3.1, there is a constant $c > 0$ (uniform in n) such that

$$\|u_n\|_{L^\infty(B_1)} \geq c.$$

Passing to the limit when $n \rightarrow \infty$, we get

$$\|u^*\|_{L^\infty(B_1)} \geq c > 0,$$

which is a contradiction.

It remains to prove the claim. Fix $x \in B_1$. First, assume that $u^*(x) > 0$. Let $r' > 0$ be sufficiently small so that $\overline{B(x, r')} \subset \{u^* > 0\}$. Since $u_n \rightarrow u^*$ uniformly in B_1 , then for n large enough we must have $\overline{B(x, r')} \subset \{u_n > 0\}$ and so, u_n is weighted p -harmonic in $B(x, r')$. Hence, thanks to [7, Theorem 13.1], u_n is bounded in $C^{1,\alpha}(\overline{B(x, r')})$. Then, up to a subsequence, $w_n |\nabla u_n|^{p-2} \nabla u_n \rightarrow w(x_0) |\nabla u^*|^{p-2} \nabla u^*$ uniformly and so, $h = w |\nabla u^*|^{p-2} \nabla u^*$ on $B(x, r')$. Now, assume that there is a small $r' > 0$ such that $B(x, r') \subset \{u^* = 0\}$. Thanks to Lemma 3.1, for n large enough, $u_n = 0$ on $B(x, \frac{r'}{2})$. In particular, we get that $h = w(x_0) |\nabla u^*|^{p-2} \nabla u^* = 0$ in $B(x, \frac{r'}{2})$.

To conclude the proof, we just need to check that $\mathcal{L}^N(\partial\{u^* > 0\}) = 0$. Fix $x \in \partial\{u^* > 0\}$. Hence, it is not difficult to see that there is a constant $c > 0$ such that for every $r > 0$ small one has for n large enough

$$\|u_n\|_{L^\infty(B(x, r))} \geq cr.$$

Passing to the limit when $n \rightarrow \infty$, we infer that

$$\|u^*\|_{L^\infty(B(x, r))} \geq cr.$$

In particular, there exists a point $x^* \in B(x, r)$ such that $u^*(x^*) \geq cr$. Moreover, thanks to the Lipschitz

regularity of u^* , it is easy to check that $B(x^*, \delta r) \subset B(x, 2r) \cap \{u^* > 0\}$, for $\delta \in (0, 1)$ small. Hence, we get

$$\frac{\mathcal{L}^N(B(x, 2r) \cap \{u^* > 0\})}{\mathcal{L}^N(B(x, 2r))} \geq \frac{\mathcal{L}^N(B(x^*, \delta r))}{\mathcal{L}^N(B(x, 2r))} \geq c.$$

□

Finally, we arrive to the following regularity result on the free boundary; $\partial\{u > 0\}$ is locally of finite perimeter:

$$“\chi_{\{u>0\}} \in BV_{loc}(\Omega)”.$$

Theorem 3.5. *Let u be a minimizer for Problem (2.1). For any compact set $K \subset \Omega$, we have $\mathcal{H}^{N-1}(\partial\{u > 0\} \cap K) < \infty$.*

Proof. Thanks to Proposition 3.4, we infer that

$$c\mathcal{H}^{N-1}(\{\partial\{u > 0\} \cap K\}) \leq \Lambda(K) \leq C\mathcal{H}^{N-1}(\{\partial\{u > 0\} \cap K\}).$$

Yet, $\Lambda(K) < \infty$. Hence,

$$\mathcal{H}^{N-1}(\{\partial\{u > 0\} \cap K\}) < \infty. \quad \square$$

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- **Data Availability Statement:** The data supporting the findings of this study are available from the author upon reasonable request.

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