QUANTITATIVE AND EXACT CONCAVITY PRINCIPLES FOR PARABOLIC AND ELLIPTIC EQUATIONS

MARCO GALLO, RICCARDO MORASCHI, AND MARCO SQUASSINA

Abstract. Goal of this paper is to study classes of Cauchy-Dirichlet problems which include parabolic equations of the type

$$u_t - \Delta u = a(x,t)f(u) \quad \text{in } \Omega \times (0,T)$$

with $\Omega \subset \mathbb{R}^N$ bounded, convex domain and $T \in (0, +\infty]$. Under suitable assumptions on a and f, we show logarithmic or power concavity (in space, or in space-time) of the solution u; under some relaxed assumptions on a, we show moreover that u enjoys concavity properties up to a controlled error. The results include relevant examples like the torsion $f(u) = 1$, the Lane-Emden equation $f(u) = u^q$, $q \in (0,1)$, the eigenfunction $f(u) = u$, the logarithmic equation $f(u) = u\log(u^2)$, and the saturable nonlinearity $f(u) = \frac{u^2}{1+u}$. The logistic equation $f(x,u) = a(x)u - u^2$ can be treated as well.

Some exact results give a different approach, as well as generalizations, to [38, 39]. Moreover, some quantitative results are valid also in the elliptic framework $-\Delta u = a(x)f(u)$ and refine [13, 29].

Contents

1. Introduction

In the 1970s and 1980s, significant progress was made in understanding the convexity and concavity properties of solutions to certain elliptic differential equations on convex domains and Dirichlet boundary conditions. In the pivotal papers by Brascamp and Lieb [12] and Makar-Limanov [54], the authors showed that solutions of the eigenfunction problem and of the torsion problem are concave up to a composition, respectively, with the logarithm function or the square root function.

2020 *Mathematics Subject Classification.* 26B25, 35B09, 26B25, 35B99, 35E10, 35J60, 35K20, 35K91.

Key words and phrases. Evolutionary semilinear equations, Convexity of solutions, Perturbation results, Nonautonomous equations, Uniqueness of critical point, Nondegenerate critical point.

Fundings. All the authors are member of INdAM-GNAMPA. M. Gallo is supported by INdAM-GNAMPA Project "Metodi variazionali per problemi dipendenti da operatori frazionari isotropi e anisotropi", codice CUP #E5324001950001#. R. Moraschi was partially funded by PRIN 2022 project 2022R537CS NO^3, CUP F53D23002810006, granted by the MUR and funded by the European Union - Next Generation EU.

The seminal papers by Korevaar [47] and Kennington [45] gave a new boost to this research, by introducing general *concavity maximum principles* to deal with general class of equations. We refer to [3, 29] and references therein for a general overview on the topic. Let us mention, among various mathematical applications, that in [60] the authors exploit the log-concavity of the eigenfunction to show notable results about the eigenvalue gap, i.e. the difference between the first and the second Dirichlet eigenvalue.

In the framework of parabolic equations and Cauchy-Dirichlet problems

$$\begin{cases} u_t - \Delta u = b(x,u,t) & \text{in } \Omega \times (0,T), \\ u = 0 & \text{on } \partial\Omega \times [0,T), \\ u = u_0 & \text{on } \overline{\Omega} \times \{0\}, \end{cases}$$

$T \in (0, +\infty]$, several questions arise. A first problem is to study very particular examples of b and discuss if, under suitable assumptions on u_0, the solution $u(x,t)$ becomes logarithmic or power concave (i.e. $\log(u)$ or u^α for some $\alpha > 0$ is concave) after some amount of time ($t > 0$ or possibly $t \gg 0$): these kinds of results are strongly related to the particular shape of b which allows, for instance, explicit representation of the solution, and have been investigated for example in [35, 50, 51]. We highlight that these results allow, under suitable assumptions, to get interesting properties also in the elliptic case, by considering the limit of $u(x,t)$ as $t \to +\infty$ which eventually coincide with the solution of the corresponding stationary problem.

A second question is the following: what is the optimal (in some sense) concavity on u_0 which is preserved by the heat flow? In this problem, the logarithmic concavity plays a very special role. This and similar questions have been addressed in a series of papers [37, 40, 41, 42]. See also [17, 24] for similar questions regarding power concavity and porous medium flow.

In this paper we are interested in a different question: under which assumptions on b and u_0 can we obtain that $u(x,t)$ enjoys some concavity properties for each $t > 0$? To the authors knowledge, only few papers deal with this problem. A first result can be found in Korevaar [47], where the authors adapts the elliptic concavity maximum principles to study classes of parabolic equations and show that, for example, the parabolic eigenfunctions $u(\cdot,t)$ are spatially logarithmic concave for each $t \in [0,T]$ if u_0 is so. Let us mention that in [6] a stronger notion of log-concavity is proved and it is exploited to prove a sharp result on the eigenvalue gap; it is notable to observe that the technique adopted in [6] relies on the introduction and analysis of some suitable multivariable function in a spirit relatable to the one introduced in [45, 47] and used in the present paper.

Some partial generalizations of the results in [47] can be found in [58] (see also [32]), by allowing the inclusion of other interesting models. Other results are contained in [40] (see e.g. Proposition 4.2 in there) by means of *spatial log-concave envelopes* as viscosity subsolutions of the problem and corresponding comparison principles; similar techniques go back to [4] (see also [22]).

In this paper we recover these results, obtaining among others the following statement. To fix notations, we say that u is *α-concave* if $\alpha > 0$ and u^α is concave, or $\alpha = 0$ and $\log(u)$ is concave (see (2.4)).

Theorem 1.1 (Weighted eigenfunctions and logarithmic equations)**.** *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded, smooth, strongly convex domain. Let $T > 0$, $a \colon \Omega \times (0,T] \to \mathbb{R}$, $u_0 \in C^1(\overline{\Omega})$ with $u_0 = 0$ on $\partial\Omega$ and $u \in C^2(\overline{\Omega}) \cap C^1(\overline{\Omega} \times [0,T])$ be a positive solution of*

$$\begin{cases} u_t - \Delta u = a(x,t) f(u) & \text{in } \Omega \times (0,T], \\ u = 0 & \text{on } \partial\Omega \times [0,T], \\ D_x u \cdot \nu > 0 & \text{on } \partial\Omega \times [0,T], \\ u = u_0 & \text{on } \overline{\Omega} \times \{0\}, \end{cases}$$

where ν is the interior normal to $\partial\Omega$. Assume moreover that u_0 is log*-concave and one of the following:*

i) $f(s) = s$, *and* a *is concave;*

ii) $f(s) = s\log^q(1+s)$ *or* $f(s) = \frac{s^{q+1}}{1+s^q}$, $q \in (0,1)$, *and* a *is* $\frac{1}{1-q}$*-concave;*

iii) $f(s) = s\log(1+s)$, $f(s) = s\log(s)$ *or* $f(s) = \frac{s^2}{1+s}$, *and* a *is a positive constant.*

Then $\log(u(\cdot,t))$ *is strictly concave for each* $t \in (0,T)$. *As a consequence, for each* $t \in (0,T)$ *the function* $u(\cdot,t)$ *has a unique, nondegenerate, critical point.*

Remark 1.2. We observe the following facts.

- Theorem 1.1 is a particular case of the more general Corollary 4.3; we highlight the dependence of a in x and t. We also mention that in this paper we actually deduce generalizations of Theorem 1.1 also in the sense of perturbed concavity, see Theorem 1.7 below and Corollary 4.5.
- We highlight that we do not require a to be nonnegative in point (i); this allows us to cover a broader range of problems, including applicative ones (see also Theorem 1.3 below).
- Once concavity is obtained, the strict concavity of solutions is usually gained, in elliptic PDEs, through the use of a constant rank theorem. Parabolic versions of the constant rank theorem have been obtained in a space version in [8, 14], and in a spacetime version in [18, 19] (see Proposition 2.10). We mention that, in our case, the conclusion for the weighted eigenfunction (case (i)) can be obtained in a more straightforward way by assuming a strictly concave in x, see Remark 4.4; similar properties appear in [10] in the setting of periodic solutions.
- The condition $D_x u \cdot \nu > 0$ already appears in the papers [32, 47]; we point out the requirement also in $t = 0$ (that is, $u_0 \in C^1(\overline{\Omega})$ and $D_x u_0 \cdot \nu > 0$) which, as highlighted in [32, Section 6], is essential but erroneously neglected in previous papers; in the same section, the authors provide an approximation process in order to avoid this restriction in zero (in particular, allowing $u_0 \in C(\overline{\Omega})$). Moreover, in [32, Remark 2, page 7] the authors comment the validity of such condition for general class of nonlinearities. For example, assume $a \geq 0$ and $D_x u_0 \cdot \nu > 0$. Then $f(s) \geq 0$ is sufficient to ensure this condition thanks to the parabolic Hopf Lemma A.4; this covers almost every case of Theorem 1.1. To treat $f(s) = s\log(s)$, assuming moreover $a \geq m > 0$, we see that it sufficient to choose u_0 such that $u_0 \geq w$ where w is a positive solution to the stationary problem $-\Delta w = f(w)$ which, by [61, Theorem 2] verifies $D_x w \cdot \nu > 0$; then the comparison principle (Proposition A.1) and $u = w = 0$ on $\partial\Omega$ imply $D_x u \cdot \nu \geq D_x w \cdot \nu > 0$.
- The requirement on the smoothness of Ω (not required in Theorem 1.4 below) is here related to the control of the concavity function of $\log(u)$ near the boundary, by means of the positivity of the curvatures. Anyway this condition can be relaxed through some standard approximation on the domain Ω, see e.g. [11, 29].
- We highlight that the logarithmic equation $f(s) = s\log(s)$, being superlinear and sign-changing, is a problem not easy to tackle in the elliptic framework, see [28]; in the parabolic framework, on the other hand, it can be addressed through a suitable trick in time. Notice moreover that the result is obtained on $[0,T]$ and, generally, the solutions of this problem do not converge to the one of the stationary problem (see Remark 4.6), thus it is not straightforward to obtain the elliptic result from the parabolic one.

As an additional example, we may consider logistic models of population dynamics in heterogeneous environments with lethal boundary. That is, the equation is governed by a logistic law $b(x,s) = a(x)s - s^2$ where $\{a > 0\}$ is the favorable habitat while $\{a < 0\}$ is the hostile one, see [7, 15, 56] (see also Remark 4.9 for bang-bang weights).

Theorem 1.3. *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded, smooth, strongly convex domain. Let $T > 0$, $a\colon \Omega \times (0,T] \to \mathbb{R}$, $u_0 \in C^1(\overline{\Omega})$ with $u_0 = 0$ on $\partial\Omega$ and $u \in C^2(\overline{\Omega}) \cap C^1(\overline{\Omega} \times [0,T])$ be a positive solution of*

$$\begin{cases} u_t - \Delta u = a(x)u - u^2 & \text{in } \Omega \times (0,T], \\ u = 0 & \text{on } \partial\Omega \times [0,T], \\ D_x u \cdot \nu > 0 & \text{on } \partial\Omega \times [0,T], \\ u = u_0 & \text{on } \overline{\Omega} \times \{0\}, \end{cases}$$

where ν is the interior normal to $\partial\Omega$. Assume moreover that u_0 is log*-concave and a is concave. Then $\log(u(\cdot,t))$ is strictly concave for each $t \in (0,T)$.*

Parabolic generalizations for equations in the spirit of the torsion and Lane-Emden problems, which allow to obtain power concavity (stronger than logarithmic one), appear first in [44]: here the author studies the case $a = 1$ and $f(u) = (1-u)^q$, $q \in (0,1)$, obtaining that a suitable transformation of u is concave and thus u is quasi-concave (i.e., the superlevel sets of u are convex). To gain the claim the author exploits again some concavity maximum principle and he strongly uses that $f(0) \neq 0$ coupled with some comparison principle; moreover, he relies on the *information at infinity* given by the stationary problem.

Other results have been more recently obtained in [38, 39] (see also [36]): here the authors rely on the use of *parabolic concave envelopes* as viscosity subsolutions. In such a way the authors are able the study the concavity of nondecreasing solutions with controlled behaviour on the boundary, in the case of nonnegative nonlinearities satisfying a priori some comparison principle, set in smooth domains: for example, they cover the cases $f = 1$, or $a = 1$ and $f(u) = u^q$, $q \in (0,1)$. Moreover, the authors study a *rescaled* concavity in space and time, namely they show that $(x,t) \mapsto u(x,t^\beta)$ is power concave for suitable values of β.

In this paper, we get inspiration from the concavity principle approach in [44] to show some general results, which include also some examples presented in [38, 39]. As a particular case of our results, we obtain the following.

Theorem 1.4 (Weighted Lane-Emden)**.** *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded, convex domain which satisfies the interior sphere property. Let $q \in [0,1)$ and $\gamma \in [0,1]$. Let $u \in C^2(\Omega) \cap C^1(\overline{\Omega}) \cap C^1((0,+\infty)) \cap C(\overline{\Omega} \times [0,+\infty))$ be a positive solution of*

$$\begin{cases} u_t - \Delta u = a(x)t^\gamma u^q & \text{in } \Omega \times (0,+\infty), \\ u = 0 & \text{on } \partial\Omega \times [0,+\infty), \\ u = 0 & \text{on } \overline{\Omega} \times \{0\}. \end{cases}$$

Suppose $a\colon \Omega \to \mathbb{R}$ to be measurable and that there exists $m > 0$ such that $a(x) \geq m$ for all $x \in \Omega$. If $\beta \in [1, \frac{1}{\gamma}) \cap [1,2]$ and a is θ-concave with $\theta \geq \frac{1}{1-\beta\gamma}$, then $u(\cdot,\star^\beta)$ is $\frac{(1-q)\theta}{2\theta+\beta\gamma\theta+1}$-concave. If $\beta \in [1, \min\{\frac{1}{\gamma}, 2\}]$ and a is constant, then $u(\cdot,\star^\beta)$ is $\frac{1-q}{2+\beta\gamma}$-concave.

Remark 1.5. We observe the following facts.

- The theorem is a particular case of the more general Theorem 4.17, where we focus on giving explicit sufficient assumptions on f (rather than on u) to ensure the result; moreover, our method allows to keep track of the error in concavity and deduce some quantitative results, see Theorem 1.8 below and Corollary 4.16.
- We see that the exponent $\alpha = \frac{(1-q)\theta}{2\theta+\beta\gamma\theta+1}$ is coherent with the results in [29, 45], where $\alpha = \frac{(1-q)\theta}{2\theta+1}$ if the problem is elliptic (and thus $\gamma = 0$), and in [38, 39], where $\alpha = \frac{\theta}{2\theta+\beta\gamma\theta+1}$ if the nonlinearity does not depend on u (i.e. $q = 0$).
- If $q = 0$, the condition $a(x) \geq m > 0$ can be relaxed into $a(x) \geq 0$, see Proposition 5.5.

- The initial condition $u(\cdot,0)=0$ appears also in [38, 39, 44], and we mention that, for such a sublinear problem, nontrivial solutions generally may exist (see Remark 2.1). See also Remark 4.25 for some comments on the case $u(\cdot,0)\neq 0$.
- If u is nondecreasing in time (and this is the case of our setting, see Theorem 4.15), then if $u(\cdot,\star^{\beta_1})$ is α-concave then $u(\cdot,\star^{\beta_2})$ is α-concave for all $\beta_2\leq\beta_1$, see [39, Section 2, Property (e)] (see also (2.7)).

Our approach allows to recover also the quasiconcavity obtained by [44], see Proposition 4.18 and the subsequent remark. To mention another consequence of our general setting, we propose here the concavity of solutions for nonhomogeneous nonlinearities.

Theorem 1.6 (Sum of powers). *Let $\Omega\subset\mathbb{R}^n$, $n\geq 2$, be a bounded, convex domain which satisfies the interior sphere property. Let $q\in(\frac{1}{3},1)$ and $p\in(\frac{3q-1}{2},q)$. Let $u\in C^2(\Omega)\cap C^1(\overline{\Omega})\cap C^1((0,+\infty))\cap C(\overline{\Omega}\times[0,+\infty))$ be a positive solution of*

$$\begin{cases} u_t-\Delta u=a(x)u^p+u^q & \text{in } \Omega\times(0,+\infty),\\ u=0 & \text{on } \partial\Omega\times[0,+\infty),\\ u=0 & \text{on } \overline{\Omega}\times\{0\}. \end{cases}$$

Suppose $a\colon\Omega\to\mathbb{R}$ to be measurable and that there exists $m>0$ such that $a(x)\geq m$ for all $x\in\Omega$; assume moreover a to be θ-concave with $\theta\in[\frac{1-q}{2(q-p)},+\infty]$. Then we have that $u(\cdot,\star^2)$ is $\frac{1-q}{2}$-concave.

A second goal of this paper is to deal with quantitative versions of the concavity maximum principles. Recently, quantitative versions of symmetry results have been considered for example in [5, 21, 26, 30]. In the realm of concavity properties, instead, similar results have been recently investigated in the elliptic case in [13] (see also [2, 29]); such estimations are made in terms of some parameter of the weight $a(x)$: more precisely, in these papers the authors show that, if a is close to a constant function, then some power of u is close to a concave function, up to some explicit error. In this paper, we deal with estimations of the *(space) concavity function* of a solution v (see (2.5))

$$C_v^*(x_1,x_3,\lambda,t):=v(x_2,t)-\lambda v(x_3,t)-(1-\lambda)v(x_1,t),$$

or the *(spacetime) concavity function*

$$\mathcal{C}_v(x_1,x_3,t_1,t_3,\lambda):=v(x_2,t_2)-\lambda v(x_3,t_3)-(1-\lambda)v(x_1,t_1),$$

where $x_1,x_3\in\Omega$, $t_1,t_3\in[0,+\infty)$, $\lambda\in[0,1]$ and $x_2=\lambda x_3+(1-\lambda)x_1$ and $t_2=\lambda t_3+(1-\lambda)t_1$; notice that $\mathcal{C}_v\geq 0$ if and only if v is concave. Our goal is to generalize the abovementioned results also to the parabolic setting, which seems to be new in the literature. As a consequence of our general results, we get in particular the following statement. Here $f^-:=\min\{0,f\}$ stands for the negative part of a function f, while $\operatorname{osc}(a):=\sup(a)-\inf(a)$ stands for the oscillation of a.

Theorem 1.7 (Quantitative concavity, I). *Let $\Omega\subset\mathbb{R}^n$, $n\geq 2$, be a bounded, smooth, strongly convex domain. Let $T>0$, $a\colon\Omega\times(0,T]\to\mathbb{R}$, $u_0\in C^1(\overline{\Omega})$ with $u_0=0$ on $\partial\Omega$ and $u\in C^2(\Omega)\cap C^1(\overline{\Omega}\times[0,T])$ be a positive solution of*

$$\begin{cases} u_t-\Delta u=a(x,t)u & \text{in } \Omega\times(0,T],\\ u=0 & \text{on } \partial\Omega\times[0,T],\\ D_xu\cdot\nu>0 & \text{on } \partial\Omega\times[0,T],\\ u=u_0 & \text{on } \overline{\Omega}\times\{0\}, \end{cases}$$

where ν is the interior normal to $\partial\Omega$. Assume moreover that u_0 is log*-concave. Then*

$$\inf_{\Omega\times\Omega\times[0,1]\times[0,T]}\mathcal{C}^*_{\log(u)}\geq -eT\sup_{\Omega\times\Omega\times[0,1]\times(0,T]}\left(\mathcal{C}^*_{a(\cdot,\star)}\right)^-.$$

Theorem 1.8 (Quantitative concavity, II)**.** *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded, convex domain which satisfies the interior sphere property. Let $q \in [0,1)$. Let $u \in C^2(\Omega) \cap C^1(\overline{\Omega}) \cap C^1((0,+\infty)) \cap C(\overline{\Omega} \times [0,+\infty))$ be a positive solution of*

$$\begin{cases} u_t - \Delta u = a(x)u^q & \text{in } \Omega \times (0,+\infty), \\ u = 0 & \text{on } \partial\Omega \times [0,+\infty), \\ u = 0 & \text{on } \overline{\Omega} \times \{0\}. \end{cases}$$

Suppose $a\colon \Omega \to \mathbb{R}$ to be measurable and $0 < m \leq a \leq M$. Then u is increasing in time and

$$\min_{\overline{\Omega}\times\overline{\Omega}\times[0,+\infty]\times[0,+\infty]\times[0,1]} \mathcal{C}_{u^{\frac{1-q}{2}}} \geq -\|u(\cdot,\infty)\|_{L^\infty(\Omega)}^{\frac{1-q}{2}} \frac{\operatorname{osc}(a^2)}{m^2}, \tag{1.1}$$

where $u(\cdot,\infty)$ is the solution of the corresponding stationary problem

If moreover $\theta \geq 1$ and $m^\theta \geq M^\theta/2$, then

$$\min_{\overline{\Omega}\times\overline{\Omega}\times[0,+\infty]\times[0,+\infty]\times[0,1]} \mathcal{C}_{u^{\frac{\theta(1-q)}{2\theta+1}}} \geq -\frac{2\theta}{2\theta+1}\frac{1}{m}\|u(\cdot,\infty)\|_{L^\infty(\Omega)}^{\frac{(\theta-1)(1-q)}{2\theta+1}} \sup_{\Omega\times\Omega\times[0,1]} (\mathcal{C}^-_{a^\theta})^{1/\theta}. \tag{1.2}$$

Remark 1.9. We observe the following facts.

- In Theorem 1.7 one may consider also other nonlinearities, like the ones in Theorem 1.1 (see Corollary 4.5). Moreover, if u_0 is not assumed to be log-concave, than an additional error related to $\mathcal{C}_{\log(u_0)}$ appears (see Remark 4.2).
- We observe that $\|u(\cdot,\infty)\|_{L^\infty(\Omega)}$ could be bounded in terms of known quantities of the problems, i.e. n, $|\Omega|$, q, m and M, see e.g. [21, proof of Corollary 1.4(b)]. Notice that, choosing $\theta = 1$ in (1.2), we can drop the dependence on the L^∞-norm and the statement become
$$\min_{\overline{\Omega}\times\overline{\Omega}\times[0,+\infty]\times[0,+\infty]\times[0,1]} \mathcal{C}_{u^{\frac{1-q}{3}}} \geq -\frac{2}{3}\frac{1}{m} \sup_{\Omega\times\Omega\times[0,1]} \mathcal{C}^-_a.$$
- The condition $m^\theta \geq M^\theta/2$ comes into play in showing quantitative concavity properties of products (see Lemma B.1), but we believe it is technical.
- We highlight that estimate (1.1) is slightly finer if compared with the ones obtained in the elliptic setting in [13, 29]. Moreover, the estimate (1.2) does not involve the oscillation of a and is totally new, even for the elliptic setting.

As just mentioned in Remark 1.9, some results of this paper can be easily adapted to the elliptic setting, allowing to improve and generalize some previous results [13, 29]. By way of example, with the same techniques, we obtain the following statement. We leave the details to the reader.

Theorem 1.10 (θ-concavity estimate)**.** *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded, convex domain which satisfies the interior sphere property, and let $q \in [0,1)$. Let $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ be a positive solution of*

$$\begin{cases} -\Delta u = a(x)u^q & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Suppose $a\colon \Omega \to \mathbb{R}$ to be measurable and $0 < m \leq a \leq M$ and $1 \leq \theta \leq \log(2)/\log(\frac{M}{m})$. Then

$$\min_{\overline{\Omega}\times\overline{\Omega}\times[0,1]} \mathcal{C}_{u^{\frac{\theta(1-q)}{1+2\theta}}} \geq -\frac{2\theta}{1+2\theta}\frac{1}{m}||u||_\infty^{\frac{(\theta-1)(1-q)}{1+2\theta}} \sup_{\Omega\times\Omega\times[0,1]} (\mathcal{C}^-_{a^\theta})^{1/\theta}.$$

As already mentioned, the main tool of this paper is composed by *concavity* and *harmonic concavity maximum principles*, which allow to give an estimate on the concavity function $\mathcal{C}_v$ of a solution of a general PDE in terms of the concavity/harmonic concavity function of the nonlinear term (see Definition 2.8). The equation treated do not rely on the boundary conditions and can

have very general form (see Theorems 3.1, 3.3), and in particular include the equations satisfied by $v = \log(u)$ or $v = u^\alpha$, $\alpha \in (0,1)$, where u is the solution of the equation of interest. To further treat equations like the logarithmic one, we need to implement a correction in time of the type $e^{\mu t}$, μ large, which allows to decrease the monotonicity of the nonlinearity involved (see Theorem 3.1). We highlight that the concavity principle approach is flexible since it can be used, for example, to deal with *weak* solutions of quasilinear equations, see [11, 29, 53] (see also Remark 4.9 for another nonsmooth setting).

Such maximum principles require that the minimum of the concavity function is in the *interior* of the domain: to ensure such a property we will exploit both the information at the boundary of Ω and the information at $t = 0$; when treating equations in $(0, +\infty)$, we will exploit also the information on the stationary problem (i.e. $t = +\infty$). When $a(x,t)$ depends on t in a coercive way (for instance, $a(x,t) = a(x)t^\gamma$) we also implement a truncation argument in order to be able to compare the problem with a stationary one at infinity. In such a way, we control all the information on $\partial(\Omega \times (0,+\infty))$ and we can apply the concavity maximum principles. We notice that, when dealing with $T < +\infty$, the nonlinearity is concave and concavity is discussed only in the spatial sense, no a priori information on $t = T$ is needed.

To gain the above results, we make some technical assumptions: on the nonlinearity $b(x,s,t)$ we require some conditions to ensure that $u(x,\cdot)$ is monotone nondecreasing, and it converges to the stationary problems (see conditions $(H2)$, $(H3)$ and (i)–(iv) in Remark 2.3). Moreover, we need to ensure the validity of some comparison principles (notice that $b(x,\cdot,t)$, in this setting, cannot be generally assumed locally Lipschitz) and that u can be compared through such a tool with some suitable known subsolution, whose behaviour near $t = 0$ and near $\partial\Omega$ is known (see conditions $(H1)$-$(H2)$); in particular we highlight that, compared with [44], the nonlinearities of interest in this paper vanish at zero, thus a finer analysis is needed. We notice again that, when logarithmic transformations are considered and the concavity is discussed only in the spatial sense, then the analysis on the boundary essentially relies on the strong convexity of the domain.

Finally, in order to adapt the above proofs to some quantitative version and keep track of all the interested quantities in play, a finer and technical analysis of the relations among concavity and harmonic concavity functions is needed.

The paper is organized as follows. In Section 2 we recall some basic facts on parabolic equations, and we introduce the main assumptions that we will require on the nonlinearity. Moreover, we recall notations and properties about concavity functions; more technical results are instead collected in Appendix B. Section 3 is devoted to the proof of the main concavity principles, which take into account only the PDE and not the boundary/initial conditions; these theorems will lead both to exact and quantitative results. In particular, we show both concavity principles in space (in presence of nonlinearities which are concave but not necessarily decreasing, and set in $(0,T]$) and in space-time (in the case of harmonic concave and decreasing nonlinearities set in $(0,+\infty)$); the latter will require the use of the harmonic concavity function. In Section 4 we take into account the boundary conditions and the initial condition, employing a suitable analysis of the solution and of the concavity function near the boundary of the parabolic cylinder; this analysis exploits also comparison principles and Hopf lemmas which are collected in Appendix A. Some applications and examples follow, including the proofs of the main theorems presented in the introduction. When the nonlinearity does not depend on u, finally, we present some further results (both simplifications and improvements of the previous ones) in Section 5.

2. Preliminaries

In what follows, we will denote by $\Omega \subset \mathbb{R}^n$ an open, bounded and convex domain. By smooth we will mean $\partial\Omega \in C^{2,\alpha}$, while by strongly convex we will mean that the second fundamental form with respect to its interior normal is positive definite everywhere. We also set $d_\Omega(x) := \operatorname{dist}(x, \partial\Omega)$

and

$$\Omega_\rho := \{x \in \Omega : d_\Omega(x) > \rho\}$$

for $\rho > 0$. We will also denote $f^+ := \max\{0, f\}$ and $f^- := \max\{0, -f\}$ the positive and negative parts of a function $f\colon \Omega \to \mathbb{R}$, so that $f = f^+ - f^-$, and by

$$\operatorname{osc}(f) := \sup_\Omega f - \inf_\Omega f$$

the oscillation of f, whenever well defined.

In the whole paper, $\cdot$ represents spatial dependence and $\star$ time dependence, e.g. we write $u(\cdot, \star)$ for $u\colon \overline{\Omega} \times [0,T] \to \mathbb{R}$. We will also use the following notation.

Notation. To say that $u\colon \overline{\Omega} \times [0,T] \to \mathbb{R}$ verifies $u(\cdot, t) \in C^k(\overline{\Omega})$ for each $t \in (0,T]$ (or $t > 0$ if $T = +\infty$) and $u(x, \star) \in C^h([0,T])$ for each $x \in \Omega$, we will simply write $u \in C^k(\overline{\Omega}) \cap C^h([0,T])$.

We finally denote

$$Du := (D_x u, \partial_t u) = (D_x u, u_t)$$

where D_x is the spatial gradient and ∂_t is the time derivative of u; in this regard, we will often decompose vectors as $p = (\tilde{p}, p^{n+1}) \in \mathbb{R}^n \times \mathbb{R} = \mathbb{R}^{n+1}$.

2.1. On parabolic equations

In this subsection we recall some facts on the parabolic problem

$$\begin{cases} u_t - \Delta u = b(x, u, Du, t) & \text{in } \Omega \times (0, +\infty), \\ u > 0 & \text{in } \Omega \times (0, +\infty), \\ u = 0 & \text{on } \partial\Omega \times (0, +\infty), \\ u = u_0 & \text{on } \Omega \times \{0\}, \end{cases} \tag{2.1}$$

where $b\colon \Omega \times \mathbb{R} \times \mathbb{R}^{n+1} \times \mathbb{R} \to \mathbb{R}$.

Remark 2.1. We briefly comment existence and regularity results for problem (2.1) when $b(x,s,p,t) = a(x)f(s)$ and $u_0 = 0$. Assumed $a \in C(\overline{\Omega}) \cap C^\sigma(\Omega)$, $\sigma \in (0,1]$, $a \not\equiv 0$, $a \geq 0$, $f \in C([0,+\infty))$, non decreasing, $f(0) = 0$ and $f > 0$ on $(0,+\infty)$ we have that (2.1) admits a non trivial solution if (and only if) [27, 49]

$$\int_0^\varepsilon \frac{1}{f(s)} ds < +\infty \quad \text{for some } \varepsilon > 0.$$

Observe that $f(s) = s^q$ with $q \in [0,1)$ satisfies such condition. For regularity results we refer again to [27, 49]. For local and global existence in the case $u_0 \neq 0$ see e.g. [43, Lemma 3.5 and Theorem 2.1]. We further mention the existence of weighted eigenfunctions with periodic conditions, see [23, 31] and references therein; see also Remark 4.6 for some comments on the logarithmic equation.

To give concavity information on the boundary of $(0,+\infty)$, and in particular for $t = +\infty$, we need to be able to compare the evolutive problem with the stationary one. We give thus the following definition.

Definition 2.2. Let $\Omega \subset \mathbb{R}^n$ be open and consider $u\colon \Omega \times (0,+\infty) \to \mathbb{R}$ such that (2.1) holds. We will say that b satisfies the *stability parabolic condition* if

$$\lim_{t \to +\infty} ||u(\cdot, t) - v||_{L^\infty(\Omega)} = 0 \tag{2.2}$$

for some $v\colon \Omega \to \mathbb{R}$ solution of

$$\begin{cases} -\Delta v = b(x, v, Dv, \infty) & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega; \end{cases} \tag{2.3}$$

here $b(x,s,p,\infty) := \lim_{t \to +\infty} b(x,s,p,t)$ is assumed to exist for all $(x,s,p) \in \Omega \times (0,+\infty) \times \mathbb{R}^{n+1}$.

Remark 2.3. We briefly comment the validity of the stability parabolic condition: in [1, Theorem 1.1] it is proved that such property is satisfied by the power $b(x,s,p,t) = |s|^{q-1}s$ where $0 < q < 1$. Other examples can be deduced by [43]: suppose indeed that $b(x,s,p,t) = b(x,s)$ satisfies the following facts

i) $b \in C(\overline{\Omega} \times \mathbb{R}) \cap C^{\sigma}_{loc}(\mathbb{R})$ for some $\sigma \in (0,1)$ and $b(x,\cdot)$ is odd for each $x \in \Omega$;

ii) there exist $C > 0$ such that

$$|b(x,s)| \leq C(|s|^q + 1) \quad \text{for } s \in \mathbb{R},\ x \in \overline{\Omega},$$

where $q > 1$ if $n = 1, 2$ and $q \in (1, \frac{n}{n-2})$ if $n \geq 3$;

iii) it holds

$$\limsup_{|s| \to +\infty} \left(\max_{x \in \overline{\Omega}} \frac{b(x,s)}{s} \right) < \lambda_1,$$

where λ_1 is the first eigenvalue of the Laplacian.

Then, for any $u_0 \in H^1_0(\Omega)$, [43, Theorem 2.1] states that the orbit of $u(\cdot,t)$ for all $t \in (0,+\infty)$ is relatively compact in $C^{1,\theta}(\overline{\Omega})$ for some $\theta \in (0,1)$, and the limit points are stationary solutions.[1] Suppose moreover that

iv) $s \in (0,+\infty) \mapsto \frac{b(x,s)}{s}$ is strictly decreasing for each $x \in \Omega$

so that the limit problem (2.3) admits a unique solution v by a Brezis-Oswald result (see e.g. [29, Lemma A.1]), then $u(\cdot,t) \to v$ in $C^{1,\theta}(\overline{\Omega})$, which in particular implies (2.2). Notice that, if *(i)* and *(iv)* hold, then *(iii)* means $\inf_{s>0} \left(\max_{x \in \overline{\Omega}} \frac{b(x,s)}{s} \right) < \lambda_1$.

Some functions that satisfy *(i)*–*(iv)* are given by $b(x,s) = a(x)f(s)$ with $f(s) = |s|^{q-1}s$, $f(s) = -s \log |s|$ or $f(s) = (1-s)^q \chi_{(0,1)}(s)$, $q \in (0,1)$ (observed that the solutions verify $u(\Omega) \subset [0,1]$), and $a \in C(\overline{\Omega})$ is positive. Moreover, also the sum of these functions satisfies *(i)*–*(iv)*.

Our problems will often involve a nonlinear term $b = b(x,s,t)$; we present here some main assumptions. We will require $b \colon \Omega \times (0,+\infty) \times (0,+\infty) \to \mathbb{R}$ to satisfy:

$(H1)$ There exists $T > 0$, $M > 0$, $k > 0$, $q \in [0,1)$, and $\gamma \in [0,1]$, such that

$$b(x,s,t) \geq k t^{\gamma} s^q \quad \text{for all } (x,s,t) \in \Omega \times (0,M] \times (0,T].$$

$(H2)$ There exists $T > 0$ such that $b(\cdot,s,t)$ is measurable and nonnegative for $(s,t) \in (0,+\infty) \times (0,T)$, and for all $M > 0$ there exists $L = L(M,T) > 0$ verifying

$$b(x,s,t) - b(x,r,t) \leq \frac{L}{r}(s-r) \quad \text{for all } 0 < r \leq s \leq M \text{ and } (x,t) \in \Omega \times (0,T].$$

$(H3)$ The function $b(x,s,\cdot)$ is nondecreasing for each $(x,s) \in \Omega \times (0,+\infty)$.

With no loss of generality, whenever assumed together, we can choose the constant T in $(H1)$-$(H2)$ to be the same. In practical situations, the constant M in $(H2)$ will actually be given by the L^∞-norm of the solution involved. Moreover, since we will apply $(H1)$ only for small values of t, if $u_0 = 0$ then for t small $\|u(\cdot,t)\|_{L^\infty(\Omega)}$ can be chosen to be smaller than the fixed M in $(H1)$ (see also condition $(H1^*)$).

Remark 2.4. Observe that the function $b(x,s,t) = a(x,t)s^q$, with $q \in (0,1)$ and $a \in L^\infty(\Omega \times \mathbb{R})$ nonnegative, satisfies $(H2)$. Indeed, for each $0 < r \leq s \leq M$ by Lagrange theorem there exists

[1] We are confident that this relative compactness to stationary solutions – which is weaker than the stability parabolic condition, and does not require *(iv)* below – is sufficient to prove Theorem 4.15. On the other hand, in order to make use of the notation $v(\cdot,\infty)$ and keep the presentation simple, we prefer not to improve the generality of the result.

$\xi \in (r,s)$ such that

$$b(x,s,t) - b(x,r,t) = a(x,t)\frac{q}{\xi^{1-q}}(s-r) \le \|a\|_\infty \frac{qs^q}{r}(s-r) \le \|a\|_\infty \frac{qM^q}{r}(s-r).$$

In a similar way, we see that also $b(x,s,t) = a(x,t)s^q \log(s)$ satisfies such condition for $q \in [0,1]$ (notice that for $q = 0$ b is not continuous in $s = 0$). Notice finally that (H2) is closed under summation.

Remark 2.5. In our arguments, it is possible to substitute (H2) with the following:

($H2^*$) There exists $T > 0$ and $\omega \in [\frac{1}{2}, 1]$ such that $b(\cdot, s, t)$ is measurable and nonnegative for $(s,t) \in (0,+\infty) \times (0,T)$, and for all $M > 0$ there exists $L = L(M,T) > 0$ verifying

$$b(x,s,t) - b(x,r,t) \le L(s-r)^\omega \quad \text{for all } 0 < r \le s \le M,\ x \in \Omega \text{ and } t \in (0,T).$$

Notice that any ω-Hölder function with $\omega \in [\frac{1}{2}, 1]$ satisfies ($H2^*$); moreover $b(s) = \sqrt{s|1-s|}$ satisfies ($H2^*$) but not (H2). On the other hand, from a practical point of view related to the applications in the present paper, we see that ($H2^*$) is somehow stronger than (H2), since $b(s) = s^q$ with $q \in (0, \frac{1}{2})$ satisfies (H2) but not ($H2^*$). See also Remark 4.25.

Remark 2.6. As a consequence of the uniqueness result in Corollary A.1, we see that fixed $T > 0$ and set

$$b_T(x,s,t) := \begin{cases} b(x,s,t) & \text{for } t \in [0,T], \\ b(x,s,T) & \text{for } t \in [T,+\infty), \end{cases}$$

then, if b satisfies (H2), then b_T satisfies (H2) as well. Since they coincide for $t \in [0,T]$, we see that positive solutions of

$$\begin{cases} u_t - \Delta u = b(x,u,t) & \text{in } \Omega \times (0,+\infty), \\ u = 0 & \text{on } \partial\Omega \times (0,\infty), \\ u = u_0 & \text{on } x \in \Omega \times \{0\}, \end{cases} \quad \begin{cases} v_t - \Delta v = b_T(x,v,t) & \text{in } \Omega \times (0,+\infty), \\ v = 0 & \text{on } \partial\Omega \times (0,\infty), \\ v = u_0 & \text{on } x \in \Omega \times \{0\}, \end{cases}$$

coincide on $[0,T]$. As a consequence, to study (exact) concavity of the first equation, we can study concavity of the second one and obtain the result thanks to the arbitrariness of T. Notice that the second problem has the advantage of possibly satisfying the parabolic stability condition even if $b(x,s,t) \to +\infty$ as $t \to +\infty$ (for instance, $b(x,s,t) = t^\gamma a(x) u^q$). Indeed, consider v solution of the second system and define $\tilde{v}(x,t) := v(x,t+T)$. Then $\tilde{v}$ satisfies

$$\begin{cases} \tilde{v}_t - \Delta\tilde{v} = b_T(x,\tilde{v},t+T) = b(x,\tilde{v},T) & \text{in } \Omega \times (0,+\infty), \\ \tilde{v} = 0 & \text{on } \partial\Omega \times (0,\infty), \\ \tilde{v} = v(T) \in H^1_0(\Omega) & \text{on } x \in \Omega \times \{0\}. \end{cases}$$

Thus, if $b(x,s,T)$ satisfies the parabolic stability condition and $\tilde{v}(\cdot,t) \to \bar{v}$ uniformly, $\bar{v}$ solution of a stationary problem $-\Delta\bar{v} = b(x,\bar{v},T)$, then $v(\cdot,t) \to \bar{v}$ uniformly as well, and thus $b_T(x,s,t)$ satisfies the parabolic stability condition as well.

2.2. **Power concavity and concavity functions**

Let us recall some notions about α-concavity, see [45] for details.

Notation. Throughout the paper, whenever z_1, z_3 are picked in a convex domain together with a $\lambda \in [0,1]$, we will denote by z_2 the convex combination

$$z_2 \equiv \lambda z_3 + (1-\lambda) z_1.$$

Moreover, whenever a function g is in play, we will denote

$$g_i \equiv g(z_i) \quad \text{for } i = 1,2,3,$$

if it creates no ambiguity. Finally if $x \in \mathbb{R}^n$, then x^i denotes the i-th components of x.

Let $\alpha \in [-\infty, +\infty]$. We recall that a nonnegative function $u\colon \Omega \to \mathbb{R}$ is *α-concave* if

$$\begin{cases} u \text{ is constant} & \text{if } \alpha = +\infty, \\ u^\alpha \text{ is concave} & \text{if } 0 < \alpha < +\infty, \\ \log(u) \text{ is concave} & \text{if } \alpha = 0, \\ u^\alpha \text{ is convex} & \text{if } -\infty < \alpha < 0, \\ \{x \in \Omega : u(x) > t\} \text{ are convex } \forall t \in \mathbb{R} & \text{if } \alpha = -\infty; \end{cases} \tag{2.4}$$

if $-\infty < \alpha \leq 0$ we additionally require $u > 0$. Moreover, we say that u is *log-concave* if it is 0-concave, *harmonic concave* if it is (-1)-concave, and *quasi-concave* if it is $(-\infty)$-concave. In what follows, for the sake of brevity, we adopt the conventions $0^{-1} = +\infty$, $(+\infty)^{-1} = 0$ and $0^0 = 1$.

Proposition 2.7 ([45]). *Let $u\colon \Omega \to \mathbb{R}$ be a nonnegative function.*

i) If u is α-concave and $\beta \leq \alpha$, then u is β-concave.

ii) Let u be quasi-concave. Set $\alpha(u) := \sup\{\beta \in \mathbb{R} : u \text{ is } \beta\text{-concave}\}$, we have that u is $\alpha(u)$-concave.

iii) For $\alpha \geq 1$, the α-concave functions are closed under positive addition and positive scalar multiplication.

iv) If $\alpha, \beta \in [0, +\infty]$, u is α-concave and v is β-concave, then the product $u\,v$ is γ-concave for $\gamma^{-1} = \alpha^{-1} + \beta^{-1}$.

v) If u is harmonic concave, then $u - k$ is harmonic concave for each nonnegative $k \in \mathbb{R}$.

vi) If u is concave and $u > 0$, then $x \mapsto \frac{(x^1)^2}{u(x)}$ is convex.

With the aim of treating perturbations of concavity, we introduce two operators which allow to quantify the concavity of a function.

Definition 2.8. Let $u\colon \Omega \times (0, +\infty) \to \mathbb{R}$, then for each $x_1, x_3 \in \Omega$, t_1, $t_3 \in (0, +\infty)$ and $\lambda \in [0, 1]$, then the *concavity function* $\mathcal{C}_u$ is defined as

$$\mathcal{C}_u(x_1, x_3, t_1, t_3, \lambda) := u(x_2, t_2) - \lambda u(x_3, t_3) - (1 - \lambda)u(x_1, t_1) \tag{2.5}$$

where we recall $x_2 = \lambda x_3 + (1 - \lambda)x_1$ and $t_2 = \lambda t_3 + (1 - \lambda)t_1$.

Let $g\colon \Omega \times (0, +\infty) \to \mathbb{R}$, then for each $x_1, x_3 \in \Omega$, $s_1, s_3 \in (0, +\infty)$, $\lambda \in [0, 1]$ such that

$$\lambda g(x_1, s_1) + (1 - \lambda)g(x_3, s_3) > 0 \quad \text{or} \quad g(x_1, s_1) = g(x_3, s_3) = 0$$

the *harmonic concavity function* $\mathcal{HC}_g$ is defined as

$$\mathcal{HC}_g((x_1, s_1), (x_3, s_3), \lambda) := \begin{cases} g(x_2, s_2) - \dfrac{g(x_1, s_1)g(x_3, s_3)}{\lambda g(x_1, s_1) + (1 - \lambda)g(x_3, s_3)} \\ \qquad\qquad \text{if } \lambda g(x_1, s_1) + (1 - \lambda)g(x_3, s_3) > 0, \\ g(x_2, s_2) \qquad \text{if } g(x_1, s_1) = g(x_3, s_3) = 0. \end{cases}$$

We say that g is *harmonic concave* if $\mathcal{HC}_g \geq 0$. Notice that, if $g\colon \Omega \times (0, +\infty) \to [0, +\infty)$, then $\mathcal{HC}_g$ is defined everywhere.

It can be easily seen that, whenever $g > 0$, the above definition of harmonic concavity coincide with the (-1)-concavity. Moreover it holds [13, page 4]

$$\mathcal{HC}_g \geq \mathcal{C}_g, \tag{2.6}$$

and, for any $\beta \in (0, 1]$, (see [39, equation (2.2)])

$$\mathcal{HC}_{u(\cdot, \star^\beta)}(x_1, x_3, t_1, t_3, \lambda) \geq \mathcal{HC}_{u(\cdot, \star)}(x_1, x_3, t_1^\beta, t_3^\beta, \lambda). \tag{2.7}$$

In Appendix B we show some generalizations of Proposition 2.7 (*iv*), (*v*) and (*vi*) in terms of the concavity and harmonic concavity functions.

We highlight that $\mathcal{C}$ and $\mathcal{HC}$ measure the concavity also with respect to t. When the concavity in time is not taken into account, we consider

$$C^*_u(x_1, x_3, \lambda, t) := u_2(t) - \lambda u_3(t) - (1-\lambda)u_1(t),$$

where

$$u_i(t) := u(x_i, t).$$

Notice again that $\mathcal{C}^*$ differs from $\mathcal{C}$ since there is no variation in time. Additionally, we will use the following notation.

Notation. We will write

$$\mathcal{HC}_{g(\cdot,u(\cdot,\star))}(x_1, x_3, t_1, t_3, \lambda) := g(x_2, \lambda u_3 + (1-\lambda)u_1) - \frac{g(x_1, u_1)g(x_3, u_3)}{\lambda g(x_1, u_1) + (1-\lambda)g(x_3, u_3)},$$

$$\mathcal{C}^*_{g(\cdot,u(\cdot,\star),\star)}(x_1, x_3, \lambda, t) := g(x_2, \lambda u_3(t) + (1-\lambda)u_1(t), t) - \lambda g(x_3, u_3(t), t) - (1-\lambda)g(x_1, u_1(t), t),$$

for $g\colon \Omega \times (0,+\infty) \to \mathbb{R}$ and $u\colon \Omega \times (0,+\infty) \to \mathbb{R}$.

Furthermore, we recall the following classical result [34, Theorem 2].

Proposition 2.9 (Hyers-Ulam, [34])**.** *Let X be a space of finite dimension and $K \subset X$ convex. Assume that $f\colon K \to \mathbb{R}$ is δ-concave, i.e.*

$$\mathcal{C}_f \geq -\delta \quad \text{in } K \times K \times [0,1].$$

Then there exists a concave function $g : K \to \mathbb{R}$ such that

$$\|f - g\|_{L^\infty(D)} \leq k_n \delta,$$

where $k_n > 0$ depends only on $n = dim(X)$.

We highlight that the optimal constant k_n has been explicitly estimated, for instance by [34] and [48] we know that $\frac{1}{4}\log_2(\frac{n}{2}) \leq k_n \leq \frac{n(n+3)}{4(n+1)}$. Notice that X can be chosen as the minimal hyperplane containing K.

We finally state the following constant rank theorem, consequence of [8, 14] and [18, 19].

Proposition 2.10 (Constant rank theorem)**.** *Let $w \in C^2(\Omega) \times C^1((0,+\infty))$ be a spatially convex solution of*

$$w_t - \Delta w = \widetilde{\mathscr{B}}(x, w, D_x w, t) \quad \text{in } \Omega \times (0,T] \tag{2.8}$$

and assume $\widetilde{\mathscr{B}}(\cdot,\cdot,\tilde{p},t)$ locally convex for each $(\tilde{p},t) \in \mathbb{R}^n \times (0,T)$. Then, for each $t > 0$, the rank of $D^2_x w(\cdot,t)$ is constant. Moreover, called $\ell(t)$ such value, we have that $t \in (0,+\infty) \mapsto \ell(t) \in \mathbb{N}$ is nondecreasing.

If moreover w is regular and spacetime-convex and $\widetilde{\mathscr{B}}(\cdot,\cdot,\tilde{p},\cdot)$ locally convex for each $\tilde{p} \in \mathbb{R}^n$, then the same conclusion as above applies to $D^2 w(\cdot,t)$.

Remark 2.11. We observe the following facts.

- The above result will be applied in the following way: let u be a solution of our Dirichlet problem, and $v = \varphi(u)$ its concave transformation. Then $w := -v$ is convex and satisfies an equation as (2.8). By [28, Proposition 2.6] and the Dirichlet boundary condition, for each t the matrix $D^2_x w(x,t)$ has full rank in a point (equal to n). Thus, by the above constant rank theorem, it is maximum in the whole Ω, and hence here w is strictly convex for each t. That is, $v(\cdot,t)$ is strictly concave for each t.

  If w is also spacetime-convex, then $D^2 v(x,t)$ has constant rank for each t, and it can be equal to n or $n+1$; more precisely, by the monotonicity, it is equal to n in $(0,t_0)$ and equal to $n+1$ in $(t_0,+\infty)$ for some $t_0 \in [0,+\infty]$.

- In the elliptic case [46], the condition $\widetilde{\mathscr{B}}(\cdot,\cdot,\tilde{p},\cdot)$ convex can be substituted by $\widetilde{\mathscr{B}}(\cdot,\cdot,\tilde{p},\cdot)$ positive and harmonic convex: as highlighted in [9], this is essentially related to the fact that the equation $0=\Delta w+\widetilde{\mathscr{B}}(x,w,Dw,t)=:F(D_x^2w,x,w,Dw,t)$ can be rewritten as $0=\frac{1}{\Delta w}+\frac{1}{\widetilde{\mathscr{B}}(x,w,Dw,t)}=:\bar{F}(D_x^2w,x,w,Dw,t)$. This is clearly not the case for parabolic equations $w_t=F(D_x^2w,x,w,Dw,t)$; this is the reason why we obtain the strict concavity in Theorem 1.1 but not in Theorem 1.4. A weaker parabolic counterpart of this generalization can anyway be found in [9, Theorem 2.3].

3. Concavity and harmonic concavity maximum principles

We start by showing a maximum principle which allows to relate the interior minimum of the concavity function of a solution v to the concavity function of the source; in applications, v will be the transformation of a solution of a PDE. In this section we consider a generalization of the result by Korevaar [47], by dealing with equations which may include weighted eigenfunctions $u_t-\Delta u=a(x,t)u$. Differently from Theorem 3.3, the corresponding $\mathscr{B}$ is not strictly decreasing; actually, we will allow $\partial_s\mathscr{B}$ to be also positive, but bounded, in the spirit of [58] (see also [32]): this permits to study also equations of the type $u_t-\Delta u=a(x,t)u\log(u)$ (see [28]). In these results, we look at some concavity of $\mathscr{B}$, and we study concavity of u in x for each t.

We recall the notation $p=(\tilde{p},p^{n+1})\in\mathbb{R}^{n+1}$.

Theorem 3.1 (Concavity maximum principle)**.** *Let $T>0$ and $v\in C^2(\Omega)\cap C^1((0,T])$ be such that*

$$-\sum_{i,j=1}^n a_{ij}(D_xv,t)D^2_{ij}v=\mathscr{B}(x,v,Dv,t)\quad \text{in } \Omega\times(0,T]$$

where $\mathscr{B}\colon\Omega\times\mathbb{R}\times\mathbb{R}^{n+1}\times(0,T]\to\mathbb{R}$ and $(a_{ij}(\tilde{p},t))_{ij}$ is symmetric and positive definite for all $(\tilde{p},t)\in\mathbb{R}^n\times(0,T]$. Assume that $(s,p^{n+1})\in v(\Omega)\times\mathbb{R}\mapsto\mathscr{B}(x,s,\tilde{p},p^{n+1},t)$ is differentiable for all $(x,\tilde{p},t)\in\Omega\times\mathbb{R}^n\times(0,T]$. Let $\mu\in[0,+\infty)$ and suppose $e^{-\mu(\star)}\mathcal{C}^*_v$ admits a negative interior minimum at $(x_1,x_3,\lambda,t)\in\Omega\times\Omega\times[0,1]\times(0,T]$. Then

$$\tilde{\xi}:=D_xv(x_1,t)=D_xv(x_2,t)=D_xv(x_3,t).\tag{3.1}$$

Assume moreover

$$\partial_{p^{n+1}}\mathscr{B}\le 0$$

and that there exists $\sigma>0$ such that[2]

$$\partial_s\mathscr{B}+\mu\partial_{p^{n+1}}\mathscr{B}\le-\sigma<0.$$

Then

$$\inf_{\Omega\times\Omega\times[0,1]\times(0,T]}\left(e^{-\mu(\star)}\mathcal{C}^*_v\right)\ge\frac{e^{-\mu t}}{\sigma}\mathcal{C}^*_{\mathscr{B}(\cdot,v(\cdot,\star),\tilde{\xi},v_t(\cdot,\star),\star)}(x_1,x_3,\lambda,t).$$

As a consequence, set $\rho:=\min\{d(x_1,\partial\Omega),d(x_3,\partial\Omega)\}>0$, we have

$$\inf_{\Omega\times\Omega\times[0,1]\times(0,T]}\mathcal{C}^*_v\ge-\frac{e^{\mu T}}{\sigma}\sup_{\Omega_\rho\times\Omega_\rho\times[0,1]\times(0,T]}\left(\mathcal{C}^*_{\mathscr{B}(\cdot,v(\cdot,\star),\tilde{\xi},v_t(\cdot,\star),\star)}\right)^-.$$

Proof. Let us define

$$w(x,t):=e^{-\mu t}v(x,t)$$

[2] It is indeed sufficient to assume $\left(\partial_s\mathscr{B}+\mu\partial_{p^{n+1}}\mathscr{B}\right)\left(x_2,e^{\mu t}z,e^{\mu t}\tilde{\xi},e^{\mu t}(\mu z+\zeta),t\right)\le\Lambda$ for any $z\in[w_2(t),\lambda w_3(t)+(1-\lambda)w_1(t)]$ and $\zeta\in[w_t(x_2,t),\lambda w_t(x_3,t)+(1-\lambda)w_t(x_1,t)]$.

which satisfies

$$-\sum_{i,j=1}^{n} a_{ij}(e^{\mu t}D_x w,t)D^2_{ij}w = e^{-\mu t}\mathscr{B}(x,e^{\mu t}w,e^{\mu t}D_x w,e^{\mu t}(\mu w+w_t),t)$$
$$=:\widetilde{\mathscr{B}}(x,w,Dw,t) \quad \text{in } \Omega\times(0,T].$$

We notice that (x_1,x_3,λ,t) is an interior minimum for $\mathcal{C}^*_w$; since it is an interior critical point (with respect to the spatial variable), we obtain

$$\bar\xi := D_x w(x_1,t) = D_x w(x_2,t) = D_x w(x_3,t),$$

and thus (3.1) holds, with $\tilde\xi = e^{\mu t}\bar\xi$. Exploiting that it is an interior minimum (thus the Hessian matrix with respect to the spatial variable is positive definite) and the definite positiviness of $a_{ij}(e^{\mu t}\bar\xi,t)$, by [45, Lemma A.1] we obtain

$$\sum_{i,j=1}^{n} a_{ij}(e^{\mu t}\bar\xi,t)D^2_{ij}w(x_2,t) - \lambda\sum_{i,j=1}^{n} a_{ij}(e^{\mu t}\bar\xi,t)D^2_{ij}w(x_3,t) - (1-\lambda)\sum_{i,j=1}^{n} a_{ij}(e^{\mu t}\bar\xi,t)D^2_{ij}w(x_1,t)\geq 0$$

and thus, by the equation,

$$\widetilde{\mathscr{B}}(x_2,w_2(t),\bar\xi,w_t(x_2,t),t)\leq \lambda\widetilde{\mathscr{B}}(x_3,w_3(t),\bar\xi,w_t(x_3,t),t)+(1-\lambda)\widetilde{\mathscr{B}}(x_1,w_1(t),\bar\xi,w_t(x_1,t),t).$$

Subtracting $\widetilde{\mathscr{B}}(x_2,\lambda w_3(t)+(1-\lambda)w_1(t),\bar\xi,\lambda w_t(x_3,t)+(1-\lambda)w_t(x_1,t),t)$ on both sides and applying Lagrange theorem, we know that there exists $z\in[w_2(t),\lambda w_3(t)+(1-\lambda)w_1(t)]$ and $\zeta\in[w_t(x_2,t),\lambda w_t(x_3,t)+(1-\lambda)w_t(x_1,t)]$ such that

$$\nabla_{s,p^{n+1}}\widetilde{\mathscr{B}}(x_2,z,\bar\xi,\zeta,t)\cdot(\mathcal{C}^*_w(x_1,x_3,\lambda,t),\partial_t\mathcal{C}^*_w(x_1,x_3,\lambda,t))\leq -\mathcal{C}^*_{\widetilde{\mathscr{B}}(\cdot,w(\cdot,\star),\bar\xi,w_t(\cdot,\star),\star)}(x_1,x_3,\lambda,t).$$

Using that (x_1,x_3,λ,t) is a minimum (with respect to the time variable), we obtain

$$\partial_t\mathcal{C}^*_w(x_1,x_3,\lambda,t)\leq 0,$$

which plugged in the previous inequality together with $\partial_{p^{n+1}}\mathscr{B}\leq 0$, implies

$$\partial_s\widetilde{\mathscr{B}}(x_2,z,\bar\xi,\zeta,t)\mathcal{C}^*_w(x_1,x_3,\lambda,t)\leq -\mathcal{C}^*_{\widetilde{\mathscr{B}}(\cdot,w(\cdot,\star),\bar\xi,w_t(\cdot,\star),\star)}(x_1,x_3,\lambda,t).$$

By the definition of $\widetilde{\mathscr{B}}$, the assumptions and $\mathcal{C}^*_w(x_1,x_3,\lambda,t)<0$, we achieve

$$-\sigma\mathcal{C}^*_w(x_1,x_3,\lambda,t)\leq -\mathcal{C}^*_{\widetilde{\mathscr{B}}(\cdot,w(\cdot,\star),\tilde\xi,w_t(\cdot,\star),\star)}(x_1,x_3,\lambda,t) \tag{3.2}$$

and thus

$$e^{-\mu t}\mathcal{C}^*_v(x_1,x_3,\lambda,t)\geq\frac{e^{-\mu t}}{\sigma}\mathcal{C}^*_{\mathscr{B}(\cdot,v(\cdot,\star),\tilde\xi,v_t(\cdot,\star),\star)}(x_1,x_3,\lambda,t)$$

which is the first claim. As a consequence

$$\inf_{\Omega\times\Omega\times[0,1]\times(0,T]}\left(e^{-\mu(\star)}\mathcal{C}^*_v\right)\geq -\frac{e^{-\mu t}}{\sigma}\sup_{\Omega_\rho\times\Omega_\rho\times[0,1]\times(0,T]}\left(\mathcal{C}^*_{\mathscr{B}(\cdot,v(\cdot,\star),\tilde\xi,v_t(\cdot,\star),\star)}\right)^-$$
$$\geq -\frac{1}{\sigma}\sup_{\Omega_\rho\times\Omega_\rho\times[0,1]\times(0,T]}\left(\mathcal{C}^*_{\mathscr{B}(\cdot,v(\cdot,\star),\tilde\xi,v_t(\cdot,\star),\star)}\right)^-$$

and hence the second claim, observed that (recall that $\inf_{\Omega\times\Omega\times[0,1]\times(0,T]}\left(e^{-\mu(\star)}\mathcal{C}^*_v\right)<0$)

$$\inf_{\Omega\times\Omega\times[0,1]\times(0,T]}\mathcal{C}^*_v=\inf_{\Omega\times\Omega\times[0,1]\times(0,T]}\left(e^{\mu(\star)}e^{-\mu(\star)}\mathcal{C}^*_v\right)\geq e^{\mu T}\inf_{\Omega\times\Omega\times[0,1]\times(0,T]}\left(e^{-\mu(\star)}\mathcal{C}^*_v\right).$$ □

Remark 3.2. If $\mathscr{B}(x,v,Dv,t)\equiv\mathscr{B}(x,v,D_x v,t)-v_t$, as in the heat equation, we see that $\partial_{p^{n+1}}\mathscr{B}=-1\leq 0$, while

$$\partial_s\mathscr{B}+\mu\partial_{p^{n+1}}\mathscr{B}\equiv\partial_s\mathscr{B}-\mu\leq\Lambda-\mu\leq-\sigma<0$$

is satisfied if $\partial_s\mathscr{B}$ is bounded from above by some $\Lambda\in\mathbb{R}$ and μ is sufficiently large ($\mu>\Lambda$).

We show now a maximum principle which allows to relate the interior minimum of the concavity function of solution v to the *harmonic* concavity function of the source. Notice that here we need the source to enjoy some strict monotonicity (differently from Theorem 3.1). In these results, we look at some harmonic concavity (rather than concavity) of $\mathscr{B}$, and we study concavity of u in (x,t) (rather than concavity in x for each t).

Theorem 3.3 (Harmonic concavity maximum principle)**.** *Let $v \in C^2(\Omega)\cap C^1((0,+\infty))$ be such that*

$$-\sum_{i,j=1}^{n} a_{ij}(Dv)D^2_{ij}v = \mathscr{B}(x,v,Dv,t) \quad \text{in } \Omega\times(0,+\infty),$$

where $\mathscr{B}\colon \Omega\times\mathbb{R}\times\mathbb{R}^{n+1}\times(0,+\infty)\to\mathbb{R}$ and $(a_{ij}(p))_{ij}$ is symmetric and positive definite for all $p\in\mathbb{R}^{n+1}$. Assume that $s\in v(\Omega)\mapsto \mathscr{B}(x,s,p,t)$ is differentiable for all $(x,p,t)\in\Omega\times\mathbb{R}^{n+1}\times(0,+\infty)$ Suppose $\mathcal{C}_v$ admits a negative interior minimum at $(x_1,x_3,t_1,t_3,\lambda)\in\Omega\times\Omega\times(0,+\infty)\times(0,+\infty)\times[0,1]$. Then

$$\xi := Dv(x_1,t_1) = Dv(x_2,t_2) = Dv(x_3,t_3). \tag{3.3}$$

Assume moreover that there exists $\sigma>0$ such that

$$\sup_{z\in[v(x_2,t_2),\lambda v(x_3,t_3)+(1-\lambda)v(x_1,t_1)]} \partial_s\mathscr{B}(x_2,z,\xi,t_2) \le -\sigma. \tag{3.4}$$

Then

$$\inf_{\Omega\times\Omega\times(0,+\infty)\times(0,+\infty)\times[0,1]} \mathcal{C}_v \ge \frac{1}{\sigma}\mathcal{HC}_{\mathscr{B}(\cdot,v(\cdot,\star),\xi,\star)}(x_1,x_3,t_1,t_3,\lambda). \tag{3.5}$$

Proof. Notice first that $(x_1,t_1)\neq(x_3,t_3)$ and $\lambda\in(0,1)$, otherwise $\mathcal{C}_v(x_1,x_3,t_1,t_3,\lambda)=0$ which is a contradiction. Being $(x_1,x_3,t_1,t_3,\lambda)$ in the interior of the domain we obtain

$$D\mathcal{C}_v(x_1,x_3,t_1,t_3,\lambda)=0$$

and in particular $D_{y_1}\mathcal{C}_v = D_{y_3}\mathcal{C}_v = D_{\tau_1}\mathcal{C}_v = D_{\tau_3}\mathcal{C}_v = 0$ imply (3.3). Set now $A := (a_{ij}(\xi))_{ij}$ and define the $2n\times 2n$ matrices

$$C := \begin{bmatrix} D^2_{y_1,y_1}\mathcal{C}_v(x_1,x_3,t_1,t_3,\lambda) & D^2_{y_1,y_3}\mathcal{C}_v(x_1,x_3,t_1,t_3,\lambda) \\ D^2_{y_1,y_3}\mathcal{C}_v(x_1,x_3,t_1,t_3,\lambda) & D^2_{y_3,y_3}\mathcal{C}_v(x_1,x_3,t_1,t_3,\lambda)\end{bmatrix},$$

which is positive semidefinite since $(x_1,x_3,t_1,t_3,\lambda)$ is a minimum for $\mathcal{C}_v$ and

$$B := \begin{bmatrix} s^2a_{ij}(\xi) & sra_{ij}(\xi) \\ sra_{ij}(\xi) & r^2a_{ij}(\xi)\end{bmatrix},$$

which is positive semidefinite by hypothesis. So by [45, Lemma A.1] we have $\mathrm{Tr}(BC)\ge 0$, that is

$$\alpha s^2 + 2\beta sr + \gamma r^2 \ge 0 \quad \text{for all } s,r\in\mathbb{R} \tag{3.6}$$

where

$$\alpha := \mathrm{Tr}(AD^2_{y_1,y_1}\mathcal{C}_v), \quad \beta := \mathrm{Tr}(AD^2_{y_1,y_3}\mathcal{C}_v), \quad \gamma := \mathrm{Tr}(AD^2_{y_3,y_3}\mathcal{C}_v),$$

which rewrite as

$$\alpha = (1-\lambda)^2Q_2 - (1-\lambda)Q_1, \qquad \beta = \lambda(1-\lambda)Q_2, \qquad \gamma = \lambda^2Q_2 - \lambda Q_3,$$

where we have set for $k\in\{1,2,3\}$

$$Q_k := \sum_{i,j=1}^{n} a_{ij}D^2_{ij}v(x_k,t_k).$$

By (3.6) we obtain $\alpha \geq 0, \gamma \geq 0$ and $\beta^2 - \alpha\gamma \leq 0$, i.e.

$$Q_2 \geq \frac{Q_1}{1-\lambda}, \tag{3.7}$$

$$Q_2 \geq \frac{Q_3}{\lambda}, \tag{3.8}$$

$$Q_1 Q_3 \geq Q_2((1-\lambda)Q_3 + \lambda Q_1), \tag{3.9}$$

where we justify the last by

$$\begin{aligned} 0 &\geq \lambda^2(1-\lambda)^2 Q_2^2 - ((1-\lambda)^2 Q_2 - (1-\lambda)Q_1)(\lambda^2 Q_2 - \lambda Q_3) \\ &= \lambda(1-\lambda)((1-\lambda)Q_2 Q_3 + \lambda Q_2 Q_1 - Q_1 Q_3). \end{aligned}$$

Claim: $(1-\lambda)Q_3 + \lambda Q_1 < 0$ or $Q_1 = Q_3 = 0$.

If $(1-\lambda)Q_3 + \lambda Q_1 \geq 0$, then by (3.7) and (3.9)

$$\begin{aligned} Q_2((1-\lambda)Q_3 + \lambda Q_1) &\geq \frac{Q_1}{1-\lambda}((1-\lambda)Q_3 + \lambda Q_1), \\ Q_2((1-\lambda)Q_3 + \lambda Q_1) &\geq Q_1 Q_3 + \frac{\lambda}{1-\lambda} Q_1^2, \\ Q_2((1-\lambda)Q_3 + \lambda Q_1) &\geq Q_2((1-\lambda)Q_3 + \lambda Q_1) + \frac{\lambda}{1-\lambda} Q_1^2, \end{aligned}$$

which imply $Q_1 = 0$. Similarly (exploiting (3.8) and (3.9)) we have $Q_3 = 0$. Then if $(1-\lambda)Q_3 + \lambda Q_1 < 0$ and from (3.9), it holds

$$Q_2 \geq \frac{Q_1 Q_3}{(1-\lambda)Q_3 + \lambda Q_1}.$$

Being $Q_k = \mathscr{B}(x_k, v_k, \xi, t_k)$, we get

$$-\mathscr{B}(x_2, v_2, \xi, t_2) \geq -\frac{\mathscr{B}(x_1, v_1, \xi, t_1)\mathscr{B}(x_3, v_3, \xi, t_3)}{(1-\lambda)\mathscr{B}(x_3, v_3, \xi, t_3) + \lambda\mathscr{B}(x_1, v_1, \xi, t_1)}.$$

Adding on both sides $\mathscr{B}(x_2, \lambda v_3 + (1-\lambda)v_1, \xi, t_2)$ and applying Lagrange theorem, we obtain

$$-\partial_s \mathscr{B}(x_2, z, \xi, t_2)\mathcal{C}_v(x_1, x_3, t_1, t_3, \lambda) \geq \mathcal{HC}_{\mathscr{B}(\cdot, v(\cdot,\star), \xi, \star)}(x_1, x_3, t_1, t_3, \lambda)$$

for some $z \in [v_2, \lambda v_3 + (1-\lambda)v_1]$. Otherwise if $Q_1 = Q_3 = 0$, then $Q_2 \geq 0$ and by the same arguments as before it holds

$$-\partial_s \mathscr{B}(x_2, z, \xi, t_2)\mathcal{C}_v(x_1, x_3, t_1, t_3, \lambda) \geq \mathcal{HC}_{\mathscr{B}(\cdot, v(\cdot,\star), \xi, \star)}(x_1, x_3, t_1, t_3, \lambda).$$

Since $\mathcal{C}_v(x_1, x_3, t_1, t_3, \lambda) < 0$ and $\partial_s \mathscr{B}(x_2, z, \xi, t_2) \leq -\sigma$, in any case it follows that

$$\sigma\mathcal{C}_v(x_1, x_3, t_1, t_3, \lambda) \geq -\partial_s \mathscr{B}(x_2, z, \xi)\mathcal{C}_v(x_1, x_3, t_1, t_3, \lambda) \geq \mathcal{HC}_{\mathscr{B}(\cdot, v(\cdot,\star), \xi, \star)}(x_1, x_3, t_1, t_3, \lambda),$$

and thus the claim. □

We see that, in Theorem 3.3, the nonlinear term $\mathscr{B}$ actually includes v_t in any form. In the special case of heat type equations, we can remove the dependence on the right side of (3.5) from v_t by assuming v monotone.

Corollary 3.4. *In the setting of Theorem 3.3, assume that $\mathscr{B}$ has the form*

$$\mathscr{B}(x, v, Dv, t) \equiv \widetilde{\mathscr{B}}(x, v, D_x v, t) - k(t)v_t$$

with $\mathcal{HC}_k \leq 0$ (for instance, k is constant). Assume moreover that $\xi^{n+1} \geq 0$ (recall $\xi^{n+1} = v_t(x_1, t_1) = v_t(x_2, t_2) = v_t(x_3, t_3)$). Then

$$\inf_{\Omega\times\Omega\times(0,+\infty)\times(0,+\infty)\times[0,1]} \mathcal{C}_v \geq \frac{1}{\sigma}\mathcal{HC}_{\widetilde{\mathscr{B}}(\cdot, v(\cdot,\star), \tilde{\xi}, \star)}(x_1, x_3, t_1, t_3, \lambda).$$

Proof. It is sufficient to apply Theorem 3.3 and Lemma B.4. □

Remark 3.5. We notice in Corollary 3.4 the essential role of v_t being constant on (x_1,t_1), (x_2,t_2), (x_3,t_3). This is a consequence of the fact that we study the space-time concavity in $(0,+\infty)$ (and not only space concavity in $(0,T]$).

4. Maximum principles with boundary conditions

In Section 3 we dealt with the value of the concavity function in the interior of the domain. In this section we focus on the behaviour on the boundary: to this aim, we consider a Dirichlet boundary condition and a Cauchy initial datum. For the sake of simplicity, we focus now on equations of heat type, that is with main operator $\partial_t - \Delta$.

4.1. Concavity

In this subsection we apply Theorem 3.1 to v, transformation of a solution u of a Cauchy-Dirichlet problem.

Theorem 4.1. *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded, smooth, strongly convex domain. Let $T > 0$, $b\colon \Omega \times (0,+\infty) \times (0,T]$, $b(x,\cdot,t)$ differentiable for all $(x,t) \in \Omega \times (0,T]$, $u_0 \in C^1(\overline{\Omega})$ with $u_0 = 0$ on $\partial\Omega$ and $u \in C^2(\Omega) \cap C^1(\overline{\Omega} \times [0,T])$, be such that*

$$u_t - \Delta u = b(x,u,t) \quad \text{in } \Omega \times (0,T]$$

with

$$\begin{cases} u > 0 & \text{in } \Omega \times (0,T], \\ u = 0 & \text{on } \partial\Omega \times [0,T], \\ D_x u \cdot \nu > 0 & \text{on } \partial\Omega \times [0,T], \\ u = u_0 & \text{on } \overline{\Omega} \times \{0\}. \end{cases} \tag{4.1}$$

Assume there exists $\Lambda \in \mathbb{R}$ such that

$$\partial_s(e^{-s} b(x,e^s,t)) \leq \Lambda \quad \text{for each } (x,t) \in \Omega \times (0,T].$$

Assume moreover that u_0 is log*-concave. Then there exists $\rho > 0$ such that*

$$\inf_{\Omega\times\Omega\times[0,1]\times[0,T]} \mathcal{C}^*_{\log(u)} \geq -Te^{1+\Lambda T} \sup_{\Omega_\rho\times\Omega_\rho\times[0,1]\times(0,T]} \left(\mathcal{C}^*_{\frac{b(\cdot,u(\cdot,\star),\star)}{u(\cdot,\star)}}\right)^-.$$

Proof. We see that $v := \log(u)$ solves

$$v_t - \Delta v = e^{-v} b(x,e^v,t) + |D_x v|^2 =: \mathscr{B}(x,v,D_x v,t) \quad \text{in } \Omega \times (0,T]$$

with $\partial_s \mathscr{B} \leq \Lambda$. If $\mathcal{C}^*_v \geq 0$ the claim holds. Otherwise, assume that $\mathcal{C}^*_v$ is somewhere negative. Let $\mu > \Lambda$. By the assumption on the initial condition, and by the assumption on the boundary coupled with [47, Lemma 2.4], the function $e^{-\mu t}\mathcal{C}^*_v(x_1,x_3,\lambda,t)$ cannot get negative for $t \to 0$ nor $(x_1,x_3) \to \partial(\Omega\times\Omega)$. Then it must admit an interior negative minimum. Thus, by Theorem 3.1, we obtain

$$\inf_{\Omega\times\Omega\times[0,1]\times(0,T]} \mathcal{C}^*_v \geq -\frac{e^{\mu T}}{\Lambda - \mu} \sup_{\Omega_\rho\times\Omega_\rho\times[0,1]\times(0,T]} \left(\mathcal{C}^*_{e^{-v(\cdot,\star)}b(\cdot,e^{v(\cdot,\star)},\star)}\right)^-.$$

Since the choice of $\mu > \Lambda$ is arbitrary, we maximize the right hand side (which is the case for $\mu = \frac{1}{T} + \Lambda$) and conclude. □

Remark 4.2. If in Theorem 4.1 we do not assume u_0 to be log-concave, then we take into account also $\mathcal{C}_{\log(u_0)}$, obtaining

$$\inf_{\Omega\times\Omega\times[0,1]\times[0,T]} \mathcal{C}^*_{\log(u)} \geq -\max\left\{ Te^{1+\Lambda T} \sup_{\Omega_\rho\times\Omega_\rho\times[0,1]\times(0,T]} \left(\mathcal{C}^*_{\frac{b(\cdot,u(\cdot,\star),\star)}{u(\cdot,\star)}}\right)^-, \sup_{\Omega\times\Omega\times[0,1]} \mathcal{C}^-_{\log(u_0)} \right\}.$$

Corollary 4.3. *In the assumptions of Theorem 4.1 assume moreover that*

- $(s,x) \mapsto s^{-1}b(x,s,t)$ *is concave for each* $t>0$.

Then $u(\cdot,t)$ *is strictly* log*-concave for each* $t>0$.

Proof. By Theorem 4.1 we obtain that $v=\log(u)$ is concave. Arguing as in Remark 2.11 and applying Proposition 2.10 to $w=-v$, we obtain the strict concavity. □

4.1.1. Applications

We start observing the following.

Proof of Theorem 1.1 and Theorem 1.3. They are a consequence of Corollary 4.3 and Proposition 2.7 (*iv*). □

Remark 4.4. We deduced strict concavity in Theorem 1.1 thanks to the constant rank theorem. In the eigenfunction case $b(x,s,t)=a(x,t)s$, by assuming $a(\cdot,t)$ strictly concave for each $t>0$, we can reach the same claim in a more straightforward way.

Indeed, assume by contradiction that there exists $t>0$ and $x_1\neq x_3$, $\lambda\in(0,1)$ such that $\mathcal{C}^*_{\log(u)}(x_1,x_3,\lambda,t)=0$; recalled that we already proved $\mathcal{C}^*_{\log(u)}\geq 0$, in the notations of the proof of Theorem 3.1 – fixed a whatever $\mu>0$ – we have $w=e^{-\mu t}v=e^{-\mu t}\log(u)$, thus $\mathcal{C}^*_w\geq 0=\mathcal{C}^*_w(x_1,x_3,\lambda,t)$, and hence (x_1,x_3,λ,t) is a point of minimum for $\mathcal{C}^*_w$. Thus by (3.2) we obtain

$$0\leq -\mathcal{C}^*_{\widetilde{\mathscr{B}}(\cdot,w(\cdot,\star),\xi,\star)}(x_1,x_3,\lambda,t)=-e^{-\mu t}\mathcal{C}^*_{a(x,t)}(x_1,x_3,\lambda,t)<0,$$

thanks to the strict concavity of a and the assumption $x_1\neq x_3$, $\lambda\in(0,1)$. This contradiction concludes the proof.

We focus now on perturbed concavity as consequence of Theorem 4.1. Exploiting it together with Propositions B.1 and B.2 we obtain the following statement. Notice that $\{\partial_r(e^{-r}f(e^r))\mid r\in\mathbb{R}\}\equiv\{s\partial_s(s^{-1}f(s))\mid s>0\}$.

Corollary 4.5. *Let* $\Omega\subset\mathbb{R}^n$, $n\geq 2$, *be a bounded, smooth, strongly convex domain. Let* $T>0$, $a\colon\Omega\times(0,T]\to\mathbb{R}$ *locally bounded,* $f\colon(0,+\infty)\to\mathbb{R}$ *differentiable,* $u_0\in C^1(\overline{\Omega})$ *with* $u_0=0$ *on* $\partial\Omega$ *and* $u\in C^2(\overline{\Omega})\cap C^1(\overline{\Omega}\times[0,T])$ *be such that*

$$u_t-\Delta u=a(x,t)f(u)\quad \text{in } \Omega\times(0,T]$$

with (4.1). *Set* $\bar f(s):=\frac{f(s)}{s}$ *for* $s>0$ *and assume*

$$\Lambda:=\sup_{s>0}\left(s\bar f'(s)\right)\in\mathbb{R}.$$

Assume moreover that u_0 *is* log*-concave. Then there exists* $\rho>0$ *such that the following facts hold.*

i) If $\bar f$ *is constant, then*

$$\inf_{\Omega\times\Omega\times[0,1]\times[0,T]}\mathcal{C}^*_{\log(u)}\geq -Te^{1+\Lambda T}\sup_{\Omega_\rho\times\Omega_\rho\times[0,1]\times(0,T]}\left(\mathcal{C}^*_{a(\cdot,\star)}\right)^-.$$

ii) If $\bar f$ *is* θ'*-concave, with* $\theta'>1$, *and* $m\leq a\leq M$ *with* $m^\theta\geq\frac{1}{2}M^\theta$, $\frac{1}{\theta}+\frac{1}{\theta'}=1$, *then*

$$\inf_{\Omega\times\Omega\times[0,1]\times[0,T]}\mathcal{C}^*_{\log(u)}\geq -Te^{1+\Lambda T}\|\bar f(u)\|_{L^\infty(\Omega_\rho)}\sup_{\Omega_\rho\times\Omega_\rho\times[0,1]\times(0,T]}\left(\left(\mathcal{C}^*_{a^\theta(\cdot,\star)}\right)^-\right)^{\frac{1}{\theta}}.$$

iii) If $\bar f$ *is concave and* a *is bounded, then*

$$\inf_{\Omega\times\Omega\times[0,1]\times[0,T]}\mathcal{C}^*_{\log(u)}\geq -Te^{1+\Lambda T}\|\bar f(u)\|_{L^\infty(\Omega_\rho)}\mathrm{osc}(a).$$

We note that Corollary 4.5 applies to the following sources $f(s) = s$ (case *i*), $f(s) = s\log^q(1+s)$, $f(s) = \frac{s^{q+1}}{1+s^q}$ with $q \in (0,1)$ (case *ii*), $f(s) = \frac{s^2}{1+s}$, $f(s) = s\log(1+s)$, $f(s) = s\log(s)$ (case *iii*).

Proof of Theorem 1.7. It is a particular case of Corollary 4.5. □

Remark 4.6. Consider $b(x,s,t) = s\log(s)$. In [28] the authors show the existence of a log-concave solution for the stationary problem; here, in Theorem 1.1, we show the log-concavity of the evolutionary solution in each $[0,T]$, whenever such solution exists. We discuss the possible (pointwise) convergence of $u(x,t)$ to a stationary solution as $t \to +\infty$ and, thus, the possibility of achieving the result in [28] through a parabolic approach. We start noticing that the assumptions in Remark 2.3 are not fulfilled. Let u_0 be a log-concave function. If $I(u_0) := |\nabla u_0|_2^2 - \int_\Omega u_0^2 \log(u_0) < 0$, then by [33] we obtain that $u(x,t) \to +\infty$ as $t \to +\infty$. If $I(u_0) = 0$, then exploiting the logarithmic Sobolev inequality and arguing as in [62, Lemma 2.3], we obtain $|u_0|_2^2 \geq (2\pi)^{n/2}e^n$; if equality holds, then by [20] we have $u(x,t) \to 0$ as $t \to +\infty$. The cases $I(u_0) = 0$ with $|u_0|_2^2 > (2\pi)^{n/2}e^n$, and $I(u_0) > 0$ are – up to the authors' knowledge – still open problems. We further observe that condition (*H*2) is verified, thus if u_0 is also a subsolution of the stationary problem (in particular, $I(u_0) \leq 0$), then by Corollary A.3 we have that $u(x,\cdot)$ is nondecreasing, thus a limit must exists.

Example 4.7. As particular cases of concave a we may consider $a(x,t) = t^\gamma d_\Omega(x)^\omega$, $\gamma, \omega \in [0,1]$, with $\gamma + \omega = 1$ (see Proposition 2.7). In such a case the solutions of

$$u_t - \Delta u = t^\gamma d(x,\partial\Omega)^\omega u$$

are log-concave.

As an additional consequence of Theorem 4.1, we may consider also population models [15].

Corollary 4.8. *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded, smooth, strongly convex domain. Let $T > 0$, $a\colon \Omega \times (0,T] \to \mathbb{R}$ bounded, $u_0 \in C(\overline{\Omega})$ with $u_0 = 0$ on $\partial\Omega$ and $u \in C^2(\overline{\Omega}) \cap C^1((0,T]) \cap C(\overline{\Omega} \times [0,T])$ be such that*

$$u_t - \Delta u = a(x,t)u - u^2 \quad \text{in } \Omega \times (0,T]$$

with (4.1). Assume moreover that u_0 is log*-concave. Then there exists $\rho > 0$ such that*

$$\inf_{\Omega\times\Omega\times[0,1]\times[0,T]} \mathcal{C}^*_{\log(u)} \geq -eT \sup_{\Omega_\rho\times\Omega_\rho\times[0,1]\times(0,T]} \left(\mathcal{C}^*_{a(\cdot,\star)}\right)^- .$$

Remark 4.9. We see that some results in Corollary 4.8 can be extended to piecewise constant weights $a = a(x)$ rising in bang-bang optimization theory (see [15, Theorem 3.9]), i.e. equal to $a_1 > 0$ on some region of Ω and to $-a_2 < 0$ on some other region. We give just a sketch of the proof. Let indeed a be as above, and let $a_\varepsilon \in L^\infty(\mathbb{R}^N)$ be a suitable equibounded regular approximating sequence for a. Let u_ε be positive solutions of of $\partial_t u_\varepsilon - \Delta u_\varepsilon = a_\varepsilon(x)u_\varepsilon - u_\varepsilon^2$, $u_\varepsilon = 0$ on $\partial\Omega$, $u_\varepsilon(0) = u_0$. Exploiting that $a_\varepsilon(x)t - t^2 \leq 0$ for $t \geq \sup_\varepsilon \|a_\varepsilon\|_\infty$ and the maximum principle we obtain that $\|u_\varepsilon\|_\infty$ is equibounded; then standard regularity theory implies that u_ε is also equibounded in some $C^{0,\sigma}$, and hence it converges uniformly up to a subsequence by Ascoli-Arzelà theorem. We can thus pass to the limit the weak formulation (see e.g. [16]) of the equations and obtain that $u_\varepsilon \to u$ solution of the equation with weight $a(x)$. Since u_ε are regular, then they satisfy the assumptions of Corollary 4.8 and thus, in particular, $\inf \mathcal{C}^*_{\log(u_\varepsilon)} \geq -eT \sup \left(\mathcal{C}^*_{a_\varepsilon(\cdot,\star)}\right)^- \geq -eT\mathrm{osc}(a_\varepsilon)$. Passing to the limit, we obtain

$$\inf \mathcal{C}^*_{\log(u)} \geq -eT(a_1 + a_2).$$

We leave the details to the interested reader.

4.2. **Harmonic concavity**

In this subsection we consider applications of Cauchy-Dirichlet type to Theorem 3.3. The main assumption we assume are the following ones.

- Domain:

$$\Omega \subset \mathbb{R}^n,\ n \geq 2, \text{ bounded convex domain which satisfies the interior sphere property.} \tag{4.2}$$

- Regularity:

$$u \in C^2(\Omega) \cap C^1(\overline{\Omega}) \cap C^1((0,+\infty)) \cap C(\overline{\Omega} \times [0,+\infty)). \tag{4.3}$$

- Equation:

$$u_t - \Delta u = b(x,u,t) \quad \text{in } \Omega \times (0,+\infty). \tag{4.4}$$

- Boundary conditions:

$$\begin{cases} u > 0 & \text{in } \Omega \times (0,+\infty), \\ u = 0 & \text{on } \partial\Omega \times [0,+\infty), \\ u = 0 & \text{on } \overline{\Omega} \times \{0\}. \end{cases} \tag{4.5}$$

Our aim is to ensure that a negative minimum of the concavity function is attained in the interior of $\Omega \times (0,+\infty)$ or at infinity. In the following propositions, thus, we run out the possibility of having a minimum on $\partial\Omega$ or $t = 0$.

We start with some boundary estimates. We refer also to Lemma 5.3 for a version including $f(x_0,t) = 0$ for $x_0 \in \partial\Omega$. We mention also [55, Theorem 5].

Lemma 4.10 (Boundary estimates)**.** *Let Ω satisfy (4.2), and u be a function such that (4.3), (4.4) and (4.5) hold. Suppose $b\colon \Omega \times (0,+\infty) \times (0,+\infty) \to \mathbb{R}$ satisfies (H1), and consider M, T, q, γ therein; we can assume T small enough in such a way $\|u\|_{L^\infty(\Omega\times[0,T])} < M$. The following facts hold.*

(i) [*Dealing with $\Omega \times \{0\}$*]*: let φ_1 be the first Dirichlet eigenfunction on Ω with eigenvalue λ_1, normalized by $\|\varphi_1\|_\infty = 1$. Then,*

$$u(x,t) \geq Ce^{-\lambda_1 t} t^{\frac{1+\gamma}{1-q}} \varphi_1(x) \quad \text{in } \Omega \times (0,T)$$

for some explicit $C = C(\|u\|_{L^\infty(\Omega\times[0,T])}, q, \gamma, T) > 0$.

(ii) [*Dealing with $\partial\Omega \times \{0\}$*]*: for any $x_0 \in \partial\Omega$ and $\beta \in (0,2]$, there exists $\delta = \delta(x_0, T, \beta) > 0$ such that*

$$u(x_0 + t\nu, t^\beta) \geq \delta t^{\frac{2\beta(1+\gamma)+(2-\beta)(1-q)}{2(1-q)}} \quad \text{for } t \in (0,T) \text{ small,}$$

where ν is the interior normal to $\partial\Omega$ in x_0.

Proof. From (H1) with $M = \|u\|_{L^\infty(\Omega\times[0,T])}$, we know that there exists $k = k(M,T) > 0$ such that u satisfies (4.5) and

$$u_t - \Delta u = b(x,u,t) \geq kt^\gamma u^q \quad \text{in } \Omega \times (0,T).$$

Set

$$w(x,t) := Ce^{-\lambda_1 t} t^{\frac{1+\gamma}{1-q}} \varphi_1(x)$$

and $C := (\frac{1-q}{1+\gamma}k)^{\frac{1}{1-q}}$. A straightforward computation shows that w satisfies (4.5) together with

$$w_t - \Delta w = C\frac{1+\gamma}{1-q} e^{-\lambda_1 t} t^{\frac{\gamma+q}{1-q}} \varphi_1(x) \quad \text{in } \Omega \times (0,+\infty)$$

and hence, since $q < 1$ and by definition of C,

$$w_t - \Delta w = \frac{1}{k} C^{1-q} \frac{1+\gamma}{1-q} e^{-(1-q)\lambda_1 t} \varphi_1^{1-q}(x)(kt^\gamma w^q) \leq kt^\gamma w^q \quad \text{in } \Omega \times (0,+\infty).$$

Then by Proposition A.1, $u \geq w$ in $\Omega \times (0,T)$, which is the first claim.

To show the second claim, assume first that $x_0 = 0 \in \partial\Omega$. Let $\eta \in (0, \min\{1, T^\beta\})$ and define w_η and Ω_η by

$$w_\eta(x,t) := \eta^{\frac{1+\gamma}{1-q}} w(\eta^{-1/2}x, \eta^{-1}t), \quad \Omega_\eta := \sqrt{\eta}\Omega$$

which satisfy

$$\begin{cases} \partial_t w_\eta - \Delta w_\eta \le k t^\gamma (w_\eta)^q & \text{in } \Omega_\eta \times (0,+\infty), \\ w_\eta > 0 & \text{in } \Omega_\eta \times (0,+\infty), \\ w_\eta = 0 & \text{on } \partial\Omega_\eta \times (0,+\infty), \\ w_\eta = 0 & \text{on } \Omega_\eta \times \{0\}. \end{cases}$$

Noticed that $\Omega_\eta \subset \Omega$, by Proposition A.1 we obtain $u \ge w_\eta$ in $\overline{\Omega}_\eta \times (0,T)$. Now let $t = \eta^{\frac{1}{\beta}} \in (0,T)$ and, observed $0 \in \partial\Omega_\eta$, we choose $x = t\nu \in \Omega_\eta$, so that

$$u(t\nu, t^\beta) \ge t^{\frac{\beta(1+\gamma)}{1-q}} w(t^{\frac{2-\beta}{2}}\nu, 1). \tag{4.6}$$

If $\beta = 2$ we obtain $u(t\nu, t^2) \ge t^{\frac{2(1+\gamma)}{1-q}} w(\nu, 1)$, and the claim follows choosing $\delta := w(\nu, 1)$. If $\beta \in (0,2)$, by Lagrange theorem there exists $\xi_t \in (0, t^{\frac{2-\beta}{2}}\nu)$ such that (recall $w(0,1) = 0$)

$$w(t^{\frac{2-\beta}{2}}\nu, 1) = t^{\frac{2-\beta}{2}} \nu \cdot Dw(\xi_t, 1).$$

Being $\delta := \frac{1}{2}\nu \cdot Dw(0,1) > 0$ thanks to Lemma A.4, for t small enough we have $\nu \cdot Dw(\xi_t, 1) > \delta$ and hence we obtain

$$w(t^{\frac{2-\beta}{2}}\nu, 1) \ge \delta t^{\frac{2-\beta}{2}} \quad \text{for } t \text{ small.} \tag{4.7}$$

Thus, by (4.6) and (4.7) we obtain $u(t\nu, t^\beta) \ge \delta t^{\frac{2\beta(1+\gamma)+(2-\beta)(1-q)}{2(1-q)}}$.

The case of general x_0 can be obtained straightforwardly, noticed the equation related to $u \mapsto kt^\gamma u^q$ is x-translation invariant. □

Proposition 4.11. *Let Ω satisfy (4.2), and u be such that (4.3), (4.4) and (4.5) hold. Suppose $b\colon \Omega \times (0,+\infty) \times (0,+\infty) \to \mathbb{R}$ is such that $b(x,\cdot,t)$ is differentiable for all $(x,t) \in \Omega \times (0,+\infty)$ and satisfies (H1), and let $\alpha \in \big(0, \frac{2(1-q)}{2\beta(1+\gamma)+(2-\beta)(1-q)}\big)$ and $\beta \in (0,2]$. Then $\mathcal{C}_{u^\alpha(\cdot,\star^\beta)}$ cannot achieve any negative minimum at $(x_1,x_3,t_1,t_3,\lambda) \in \overline{\Omega} \times \overline{\Omega} \times [0,+\infty) \times [0,+\infty) \times (0,1)$ such that one $t_1, t_3 \in \{0\}$, or one of $x_1, x_2, x_3 \in \partial\Omega$.*

Proof. Assume now by contradiction that $\mathcal{C}_{u^\alpha}$ admits a negative minimum at $(x_1,x_3,t_1,t_3,\lambda)$ such that one $t_1, t_3 \in \{0\}$ or one $x_1, x_2, x_3 \in \partial\Omega$. Set $v(x,t) := u^\alpha(x,t^\beta)$. If $t_1, t_3 \in [0,+\infty)$ and $x_1, x_3 \in \partial\Omega$, then $\mathcal{C}_v(x_1,x_3,t_1,t_3,\lambda) = v(x_2,t_2) \ge 0$, which is not possible. If $t_1, t_3 = 0$ and $x_1, x_3 \in \overline{\Omega}$, $C_v(x_1,x_3,t_1,t_3,\lambda) = 0$ which again is not possible.

If $t_1, t_3 \in (0,+\infty)$, $x_1 \in \partial\Omega$ and $x_3 \in \Omega$, the parabolic Hopf Lemma A.4 implies $D_{\bar\nu_1} u(x_1, t_1^\beta) > 0$, where ν_1 is the interior normal at (x_1,t_1) and $\bar\nu_1 := (\tilde\nu_1, \beta t_1^{\beta-1}\nu_1^{n+1})$ thus

$$\lim_{(y_1,y_3,\tau_1,\tau_3,\eta)\to(x_1,x_3,t_1,t_3,\lambda)} D_{\nu_1}\mathcal{C}_v(y_1,y_3,\tau_1,\tau_3,\eta)$$

$$= \lim_{(y_1,y_3,\tau_1,\tau_3,\eta)\to(x_1,x_3,t_1,t_3,\lambda)} \Big((1-\eta) D_{\nu_1} v(y_2,\tau_2) - (1-\eta)\alpha u^{\alpha-1}(y_1,\tau_1^\beta) D_{\bar\nu_1} u(y_1,\tau_1^\beta) \Big) = -\infty,$$

and this is a contradiction since it is a minimum point. The same argument holds if $t_1 \in (0,+\infty)$, $t_3 = 0$, $x_1 \in \partial\Omega$, $x_3 \in \Omega$.

If $t_1 = 0$, $t_3 \in (0,+\infty)$, $x_1 \in \Omega$ and $x_3 \in \Omega$, by Lemma 4.10 (i) we have

$$\lim_{\tau\to 0} v_t(x_1,\tau) \ge \lim_{\tau\to 0} \frac{v(x_1,\tau)}{\tau} \ge C^\alpha \lim_{\tau\to 0} \frac{e^{-\alpha\lambda_1\tau^\beta}}{\tau^{1-\frac{1+\gamma}{1-q}\alpha\beta}} (\varphi(x_1))^\alpha = +\infty$$

by the assumption $\alpha < \frac{2(1-q)}{2\beta(1+\gamma)+(2-\beta)(1-q)} \leq \frac{1-q}{\beta(1+\gamma)}$, which implies

$$\lim_{(y_1,y_3,\tau_1,\tau_3,\eta)\to(x_1,x_3,t_1,t_3,\lambda)} D_{t_1}\mathcal{C}_v(y_1,y_3,\tau_1,\tau_3,\eta) = -\infty$$

again a contradiction.

We are left with the case $t_1 = 0$, $t_3 \in (0,+\infty)$, $x_1 \in \partial\Omega$, $x_3 \in \Omega$. By Lemma 4.10 (ii) there exists $\delta > 0$ such that

$$\frac{u^\alpha(x_1 + t\nu, t^\beta)}{t} \geq \delta^\alpha t^{\alpha\frac{2\beta(1+\gamma)+(2-\beta)(1-q)}{2(1-q)}-1} \quad \text{for } t > 0 \text{ small},$$

which positively diverges as $t \to 0^+$ since $\alpha < \frac{2(1-q)}{2\beta(1+\gamma)+(2-\beta)(1-q)}$ by assumption. Thus $(\nu,1)\cdot D_{(x_1,t_1)}\mathcal{C}_v$ negatively diverges while approaching (x_1,t_1) and this implies $(x_1,x_3,t_1,t_3,\lambda)$ can not be a minimum. □

To deal with the case $t = \infty$ we need to be able to compare the evolutive problem with the stationary one, and to this aim we will exploit the stability condition in Definition 2.2.

Let us consider a function $u\colon \overline{\Omega} \to \mathbb{R}$ which satisfies (4.5) and the equation (4.4). For $\alpha \in (0,1)$ and $\beta \in (0,2]$, set

$$v(x,t) := u^\alpha(x,t^\beta).$$

By some straightforward computations, and recalled that $v > 0$, we have that the equation solved by v is

$$-\Delta v = \mathscr{B}(x,v,Dv,t) \quad \text{in } \Omega\times(0,+\infty), \tag{4.8}$$

where

$$\begin{aligned}\mathscr{B}(x,s,p,t) &:= \widetilde{\mathscr{B}}(x,s,\tilde{p},t) - \frac{1}{\beta}t^{1-\beta}p^{n+1}\\ &:= \frac{1-\alpha}{\alpha}\frac{|\tilde{p}|^2}{s} + \alpha s^{\frac{\alpha-1}{\alpha}} b\left(x, s^{1/\alpha}, t^\beta\right) - \frac{1}{\beta}t^{1-\beta}p^{n+1}.\end{aligned} \tag{4.9}$$

From now on with $\widetilde{\mathscr{B}}(x,s,\tilde{p},\infty)$ and $v(x,\infty)$ we mean their pointwise limit in time whenever the stability condition is assumed.

Remark 4.12. Assuming $v_t \geq 0$ in $\Omega\times(0,+\infty)$ and $v(x,t) \to v(x,\infty) < +\infty$ as $t \to +\infty$, if $(x_1,x_3,t_1,t_3,\lambda) \in \overline{\Omega}\times\overline{\Omega}\times[0,+\infty)\times\{+\infty\}\times(0,1)$ is a global minimum for $\mathcal{C}_v$, then also $(x_1,x_3,+\infty,+\infty,\lambda)$ is a global minimum for $\mathcal{C}_v$. Indeed

$$\begin{aligned}\mathcal{C}_v(x_1,x_3,t_1,t_3,\lambda) &= v(x_2,\infty) - \lambda v(x_3,t_3) - (1-\lambda)v(x_1,\infty)\\ &\geq v(x_2,\infty) - \lambda v(x_3,\infty) - (1-\lambda)v(x_1,\infty).\end{aligned}$$

Remark 4.13. Let us discuss the case when $\lambda \in \{0,1\}$. Assume $v_t \geq 0$ in $\Omega\times(0,+\infty)$ and $v(x,t) \to v(x,\infty) < +\infty$ as $t \to +\infty$. If $t_1,t_3 \in [0,+\infty)$, then $\mathcal{C}_{u^\alpha(\cdot,\star^\beta)} = 0$. If $t_3 = +\infty$ and $\lambda = 1$, then $\mathcal{C}_{u^\alpha(\cdot,\star^\beta)} = 0$. If $\lambda = 0, t_3 = +\infty$ let us consider a minimizing sequence $(x_1^n,x_3^n,t_1^n,t_3^n,\lambda^n)$ converging to $(x_1,x_3,t_1,+\infty,0)$ where $(x_1,x_3,t_1) \in \overline{\Omega}\times\overline{\Omega}\times[0,+\infty]$. We can assume, up to subsequence, that t_2^n converges to $t_2 \in [t_1,+\infty]$. Since x_2^n tends to x_1, then

$$\lim_{n\to+\infty} \mathcal{C}_v(x_1^n,x_3^n,t_1^n,t_3^n,\lambda^n) = v(x_1,t_2) - v(x_1,t_1) \geq 0.$$

In light of Remarks 4.12 and 4.13, we will use the following convention.

Convention. If $v_t \geq 0$, then by a global point of minimum for C_v on $\overline{\Omega}\times\overline{\Omega}\times[0,+\infty]\times[0,+\infty]\times[0,1]$ we mean a point picked in $(\overline{\Omega}\times\overline{\Omega}\times[0,+\infty)\times[0,+\infty)\times[0,1])\cup(\overline{\Omega}\times\overline{\Omega}\times\{+\infty\}\times\{+\infty\}\times(0,1))$.

Proposition 4.14. *Under the same assumption of Proposition 4.11, assume moreover $v_t \geq 0$ and b satisfying the stability parabolic condition (e.g. i)–iv) in Remark 2.3 hold). Then $\mathcal{C}_{u^\alpha(\cdot,\star^\beta)}$ cannot achieve any negative minimum at $(x_1,x_3,t_1,t_3,\lambda) \in \overline{\Omega}\times\overline{\Omega}\times[0,+\infty]\times[0,+\infty]\times(0,1)$ such that one $t_1,t_3 \in \{0\}$, or one of $x_1,x_2,x_3 \in \partial\Omega$.*

Proof. In light of Proposition 4.11 and Remark 4.12, we need to exclude only the stationary case $t_1 = t_3 = +\infty$ and $x_1 \in \partial\Omega$, $x_3 \in \overline{\Omega}$; this can be done as in the proof of Proposition 4.11. See also [13, 45]. □

Theorem 4.15. *Let Ω satisfy (4.2), and u be a function such that (4.3), (4.4) and (4.5) hold. Let $\beta \in [1,2]$ and suppose $b\colon \Omega \times (0,+\infty) \times (0,+\infty) \to \mathbb{R}$ is such that $b(x,\cdot,t)$ is differentiable in $\mathbb{R}\setminus\{0\}$ for all $(x,t) \in \Omega \times (0,+\infty)$, (H1), (H2) hold and*

i) $s \mapsto s^{\alpha-1}b(x,s,t)$ is strictly decreasing for all $(x,t) \in \Omega \times (0,+\infty)$ for some $\alpha \in (0,1)$ as in Proposition 4.11.

ii) b satisfies the stability parabolic condition (e.g. i)–iv) in Remark 2.3 hold).

iii) $b(x,s,\cdot)$ is nondecreasing for each $(x,s) \in \Omega \times (0,+\infty)$.

Then $u_t \geq 0$. Let $(x_1,x_3,t_1,t_3,\lambda) \in \overline{\Omega}\times\overline{\Omega}\times[0,+\infty]\times[0,+\infty]\times[0,1]$ be a global minimum for $\mathcal{C}_{u^\alpha}$ (with the above convention). Then (3.4) holds true for some $\sigma > 0$ and

$$\min_{\overline{\Omega}\times\overline{\Omega}\times[0,+\infty]\times[0,+\infty]\times[0,1]} \mathcal{C}_{u^\alpha(\cdot,\star^\beta)} \geq -\frac{1}{\sigma}\mathcal{HC}^-_{\widetilde{\mathscr{B}}(\cdot,u^\alpha(\cdot,\star^\beta),D_xu^\alpha(x_1,t_1^\beta),\star)}(x_1,x_3,t_1,t_3,\lambda).$$

Proof. By (H2) we can apply Corollary A.3 and obtain $u_t \geq 0$ in $\Omega\times(0,+\infty)$, which in turn implies $v_t \geq 0$ (so we are in the setting of the above convention). If the minimum is nonnegative, the statement holds. Assume $\mathcal{C}_{u^\alpha(\cdot,\star^\beta)}$ admits a negative minimum. By Remark 4.13 and Proposition 4.14, we know that $(x_1,x_3,t_1,t_3,\lambda) \in \Omega\times\Omega\times(0,+\infty]\times(0,+\infty]\times(0,1)$.

Let us verify the hypothesis of Corollary 3.4 on the equation (4.8) satisfied by v. As first $\alpha^{-1}(1-\alpha)s^{-1}|p|^2$ is decreasing with respect to s. Then denoting r for $s^{\frac{1}{\alpha}}$, the second term of $\mathscr{B}$ becomes $\alpha r^{\alpha-1}b(x,r,t^\beta)$ which is strictly decreasing for hypothesis with respect to r. Hence $\mathscr{B}(x,\cdot,p,t)$ is strictly decreasing for all $p \in \mathbb{R}^n$, $x \in \Omega$ and $t \in (0,+\infty)$, which implies the existence of $\sigma > 0$ such that (3.4) holds.

If $(x_1,x_3,t_1,t_3,\lambda) \in \Omega\times\Omega\times(0,+\infty)\times(0,+\infty)\times(0,1)$, then the claim follows by Theorem 3.3. If $t_1 = t_3 = +\infty$ the claim follows by [13, Lemma 2.9]. We conclude by observing that, being $\beta \in [1,2]$, we have $\mathcal{HC}_{\star^{1-\beta}} \leq 0$. □

Corollary 4.16. *In the assumptions of Theorem 4.15, we also have*

$$\min_{\overline{\Omega}\times\overline{\Omega}\times[0,+\infty]\times[0,+\infty]\times[0,1]} \mathcal{C}_{u^\alpha(\cdot,\star^\beta)} \geq -\frac{\alpha}{\sigma}\frac{\mathcal{C}^-_{\tilde{g}(\cdot,u^\alpha(\cdot,\star^\beta),\star)}(x_1,x_3,t_1,t_3,\lambda)}{(\lambda u^\alpha(x_3,t_3^\beta)+(1-\lambda)u^\alpha(x_1,t_1^\beta))^2}$$

where $\tilde{g}(x,s,t) := s^{\frac{3\alpha-1}{\alpha}}b(x,s^{1/\alpha},t^\beta)$.

Proof. Since Theorem 4.15 holds, let us give lower bound for the $\mathcal{HC}^-_{\widetilde{\mathscr{B}}}$. Set

$$g(x,s,t) := s^2\widetilde{\mathscr{B}}(x,s,\tilde{\xi},t) = \frac{1-\alpha}{\alpha}|\tilde{\xi}|^2 s + \alpha s^{\frac{3\alpha-1}{\alpha}}b(x,s^{1/\alpha},t^\beta)$$

where $\tilde{\xi} = D_xu^\alpha(x_1,t_1^\beta)$, and moreover $v(x,t) = u^\alpha(x,t^\beta)$. By Proposition B.3 we have

$$\mathcal{HC}_{\widetilde{\mathscr{B}}(\cdot,v(\cdot,\star),\tilde{\xi},\star)}(x_1,x_3,t_1,t_3,\lambda) \geq \frac{\mathcal{C}_{g(\cdot,v(\cdot,\star),\star)}(x_1,x_3,t_1,t_3,\lambda)}{(\lambda v_3+(1-\lambda)v_1)^2}.$$

Since $\mathcal{C}_g = \alpha\mathcal{C}_{\tilde{g}(\cdot,v(\cdot,\star),\star)}$, we obtain

$$\mathcal{HC}_{\widetilde{\mathscr{B}}(\cdot,v(\cdot,\star),\tilde{\xi},\star)}(x_1,x_3,t_1,t_3,\lambda) \geq \alpha\frac{\mathcal{C}_{\tilde{g}(\cdot,v(\cdot,\star),\star)}(x_1,x_3,t_1,t_3,\lambda)}{(\lambda v_3+(1-\lambda)v_1)^2}$$

and thus the claim. □

As a consequence of Corollary 4.16 we obtain the following result.

Theorem 4.17. *In the assumptions of Theorem 4.15, assume moreover*

iv) $(x,s,t)\mapsto s^{\frac{3\alpha-1}{\alpha}}b(x,s^{\frac{1}{\alpha}},t^{\beta})$ *is a concave function.*

Then $u^{\alpha}(\cdot,\star^{\beta})$ *is a concave function in* $\overline{\Omega}\times[0,+\infty]$.

4.2.1. **Applications**

Let Ω be a convex bounded domain of $\mathbb{R}^n$ and suppose u is a function such that (4.3) and (4.5) hold together with (4.4) in the case $b(x,u,t)=a(x,t)u^q$, namely

$$u_t-\Delta u=a(x,t)u^q \quad \text{in } \Omega\times(0,+\infty), \tag{4.10}$$

where $q\in[0,1)$, $a\colon \Omega\times(0,+\infty)\to\mathbb{R}$ such that $a(\cdot,t)$ measurable for all $t\in(0,+\infty)$.

Let us start from some exact results. For the sake of simplicity, we focus on $a(x,t)=a(x)t^{\gamma}$, but more general behaviours in time may be considered.

Proof of Theorem 1.4. Let us verify hypothesis of Theorem 4.17 with

$$\alpha=\frac{\theta(1-q)}{1+2\theta+\beta\gamma\theta}<\frac{2(1-q)}{2\beta(1+\gamma)+(2-\beta)(1-q)}<1-q$$

and $b(x,s,t)=a(x)t^{\gamma}s^q$. Notice $s\mapsto s^{\alpha-1}b(x,s,t)=a(x)t^{\gamma}s^{\alpha-(1-q)}$ is strictly decreasing for all $x\in\Omega$ and $t>0$. Moreover $s\mapsto s^{\frac{3\alpha-1}{\alpha}}b(x,s^{1/\alpha},t^{\beta})=a(x)t^{\gamma\beta}s^{3-\frac{1-q}{\alpha}}$ is concave as a product since $\frac{1}{\theta}+\beta\gamma+3-\frac{1-q}{\alpha}=1$, by the assumptions we have $3-\frac{1-q}{\alpha}\geq 0$ and thus the claim in light of Proposition 2.7 (*iv*). To get the parabolic condition[3], we argue as in Remark 2.6 thanks to Remark 2.3, while we see that (*H*1) is satisfied as shown in Remark 2.4. Then we can apply Theorem 4.17 and get the thesis. Moreover if a is a constant, then a is θ-concave for all $\theta\geq 1$, thus the claim by Proposition 2.7 (*ii*). Finally, we see that, if $\beta=\frac{1}{\gamma}\leq 2$ and a is constant, we can repeat the previous arguments with $\alpha=\frac{1-q}{2+\beta\gamma}$ and obtain the claim. □

Proof of Theorem 1.6. Let us verify that $b(x,s)=a(x)s^p+s^q$ satisfies the hypothesis of Theorem 4.17 with $\alpha=\frac{1-q}{2}$. Since $s^{\frac{1-q}{2}-1}(a(x)s^p+s^q)=a(x)s^{\frac{-1-q+2p}{2}}+s^{\frac{q-1}{2}}$, it is strictly decreasing by the hypothesis on p. Moreover the stability parabolic condition is guaranteed by Remark 2.3 and $b(x,s)=a(x)s^p+s^q$ satisfies (*H*1) as shown in Remark 2.4. Regarding the concavity we observe that $s\mapsto a(x)s^{\frac{1-3q+2p}{1-q}}+s$ is concave by the assumptions. Thus we have the claim. □

Proposition 4.18. *Let* Ω *satisfy* (4.2)*, and* u *be a function such that* (4.3) *and* (4.5) *hold. Suppose*

$$u_t-\Delta u=(1-u)^p \quad \text{in } \Omega\times(0,+\infty)$$

for some $p\in(0,1)$*. Then* $u(\cdot,\star^2)$ *is* $\frac{1}{2}$*-concave in* $\overline{\Omega}\times[0,+\infty)$.

Proof. Observe that $b(s)=(1-s)^p$ satisfies (*H*2) (being nonincreasing), and it satisfies (*H*1) with $q=0$ for $M<1$. To conclude it is sufficient to observe that $s^{\alpha-1}b(s)$ is strictly decreasing for each $\alpha\in(0,1)$, while $s^{\frac{3\alpha-1}{\alpha}}b(s^{\frac{1}{\alpha}})$ is concave for each $\alpha\in[\frac{1}{3},\frac{1}{2}]$. Noticed that $\alpha=\frac{1}{2}$, satisfies the restriction of Proposition 4.11, then the claim holds by Theorem 4.17. □

Remark 4.19. In [44] the author shows that, for $\beta=1$, the solution u is such that $\varphi_p(u)$ is concave, where $\varphi_p(s):=\int_0^s\frac{1}{\sqrt{1-(1-r)^{p+1}}}dr$. Set $\psi(s):=\sqrt{s}$ the transformation found in Proposition 4.18, by the fact that $\psi\circ\varphi_p^{-1}$ is (strictly) concave, we see that, generally, the information given by φ_p is better. This fact is due to the choice of a transformation not necessarily of power type, but tailed on the nonlinearity, that is, up to constants, $\varphi_p(s)=\int_0^t\frac{1}{\sqrt{F(r)}}dr$, where $F(s)=\int_0^t f(r)dr$, $f(s)=(1-s)^p$. This choice is coherent with the approach in [3, 11, 29]. In the present paper, we decided not to pursue this approach in order to focus on the difficulties given by the parabolic

[3] Notice that the solution of the stationary solution of $-\Delta u=a(x)u^q$ is $\frac{\theta(1-q)}{1+2\theta}$-concave, where $\frac{\theta(1-q)}{1+2\theta}\geq\frac{\theta(1-q)}{1+2\theta+\beta\gamma\theta}$.

framework and avoid technicalities. On the other hand, similar arguments could be developed, and we leave the details to the interested reader.

Let us now move to some perturbative result. For the sake of simplicity, we consider $a = a(x)$ locally bounded (see also Remark 4.24), and $\beta = 1$.

Let $(x_1, x_3, t_1, t_3, \lambda) \in \Omega \times \Omega \times (0, +\infty] \times (0, +\infty] \times (0, 1)$ be a global interior minimum of $\mathcal{C}_{u^{\frac{1-q}{2}}}$, then by (3.3) in Theorem 3.3 we have $\xi := D_x u(x_1, t_1) = D_x u(x_2, t_2) = D_x u(x_3, t_3)$. We define, for any $\rho > 0$ small enough,

$$\mathfrak{m}_\rho := \frac{2}{1-q} \inf_{x \in \overline{\Omega_\rho}} f^\xi(x), \quad \mathfrak{M}_\rho := \frac{2}{1-q} \sup_{x \in \overline{\Omega_\rho}} f^\xi(x), \tag{4.11}$$

where

$$f^p(x) := \frac{1+q}{1-q}|p|^2 + \frac{1-q}{2}a(x), \quad \text{for } p \in \mathbb{R}^n \text{ and } x \in \Omega. \tag{4.12}$$

Notice that

$$\mathfrak{M}_\rho - \mathfrak{m}_\rho = \sup_{\overline{\Omega_\rho}} a - \inf_{\overline{\Omega_\rho}} a, \quad \mathfrak{m}_\rho \geq \inf_{\overline{\Omega_\rho}} a. \tag{4.13}$$

Proposition 4.20 (Oscillation estimate)**.** *Let Ω satisfy (4.2), and u be a function such that (4.3), (4.5) and (4.10) hold. Suppose moreover $a \colon \Omega \to \mathbb{R}$ measurable, locally bounded and $a \geq m > 0$. Then u is increasing in time and there exists $\rho > 0$ such that*

$$\min_{\overline{\Omega} \times \overline{\Omega} \times [0,+\infty] \times [0,+\infty] \times [0,1]} \mathcal{C}_{u^{\frac{1-q}{2}}} \geq -\frac{\|u(\cdot,\infty)\|_{L^\infty(\overline{\Omega})}^{\frac{1-q}{2}}}{\mathfrak{m}_\rho} \left(\inf_{\overline{\Omega_\rho} \times \overline{\Omega_\rho} \times [0,1]} \mathcal{C}_a - \frac{\mathfrak{M}_\rho}{\mathfrak{m}_\rho} \varepsilon \right)^-.$$

where $\varepsilon := \mathfrak{M}_\rho - \mathfrak{m}_\rho \geq 0$ and $\mathfrak{M}_\rho, \mathfrak{m}_\rho$ are defined in (4.11). In particular

$$\min_{\overline{\Omega} \times \overline{\Omega} \times [0,+\infty] \times [0,+\infty] \times [0,1]} \mathcal{C}_{u^{\frac{1-q}{2}}} \geq -\|u(\cdot,\infty)\|_{L^\infty(\overline{\Omega_\rho})}^{\frac{1-q}{2}} \left(2 + \frac{\varepsilon}{\inf_{\overline{\Omega_\rho}} a} \right) \frac{\varepsilon}{\inf_{\overline{\Omega_\rho}} a}.$$

Proof. We want to apply Theorem 4.15 with $\alpha = \frac{1-q}{2}$ and $b(x,s) = a(x)s^q$. Since a is bounded from below, $b(x,s) \geq ms^q$. Moreover $s^{\alpha-1}b(x,s) = s^{\frac{1-q}{2}-1}a(x)s^q = a(x)s^{\frac{q-1}{2}}$ is strictly decreasing function of s, the parabolic condition on b is satisfied by Remark 2.3 and $(H1)$ is satisfied as shown in Remark 2.4. Thus we can apply Theorem 4.15. Notice that $\widetilde{\mathscr{B}}$ defined in (4.9) takes the form

$$\widetilde{\mathscr{B}}(x,s,p) = \frac{f^p(x)}{s}$$

where f^p is given in (4.12). Let $(x_1, x_3, t_1, t_3, \lambda) \in \overline{\Omega} \times \overline{\Omega} \times [0, +\infty] \times [0, +\infty] \times [0, 1]$ be a global minimum for $\mathcal{C}_v$. By Proposition 4.11, we know that $(x_1, x_3, t_1, t_3, \lambda) \in \Omega \times \Omega \times (0, +\infty] \times (0, +\infty] \times [0, 1]$. If the minimum is positive, the statement holds. Suppose $\mathcal{C}_v$ admits a negative minimum at $(x_1, x_3, t_1, t_3, \lambda) \in \Omega \times \Omega \times (0, +\infty) \times (0, +\infty) \times (0, 1)$, then we need to estimate of the harmonic concavity function. Define

$$\rho := \min \{ d(x_1, \partial\Omega), d(x_3, \partial\Omega) \} > 0.$$

Then by (4.11), it holds (recall (2.6))

$$
\begin{aligned}
&\mathcal{HC}_{\mathscr{B}(\cdot,v(\cdot,\star),\xi)}(x_1,x_3,t_1,t_3,\lambda)\\
&=\frac{f_2^\xi}{\lambda v_3+(1-\lambda)v_1}-\frac{f_1^\xi}{v_1}\frac{f_3^\xi}{v_3}\frac{v_1v_3}{\lambda f_1^\xi v_3+(1-\lambda)f_3^\xi v_1}\\
&=\frac{1}{\lambda v_3+(1-\lambda)v_1}\left(f_2^\xi-f_1^\xi f_3^\xi\frac{\lambda v_3+(1-\lambda)v_1}{\lambda f_1^\xi v_3+(1-\lambda)f_3^\xi v_1}\right)\\
&\geq\frac{1}{\lambda v_3+(1-\lambda)v_1}\left(\mathcal{HC}_{f^\xi}(x_1,x_3,t_1,t_3,\lambda)+f_1^\xi f_3^\xi\left(\frac{1}{\lambda f_1^\xi+(1-\lambda)f_3^\xi}-\frac{2}{(1-q)\mathfrak{m}_\rho}\right)\right)\\
&\geq\frac{1}{\lambda v_3+(1-\lambda)v_1}\left(\mathcal{HC}_{f^\xi}(x_1,x_3,t_1,t_3,\lambda)+\frac{1-q}{2}\mathfrak{M}_\rho^2\left(\frac{1}{\mathfrak{M}_\rho}-\frac{1}{\mathfrak{m}_\rho}\right)\right)\\
&\geq\frac{1}{\lambda v_3+(1-\lambda)v_1}\left(\mathcal{C}_{f^\xi}(x_1,x_3,t_1,t_3,\lambda)-\frac{1-q}{2}\frac{\mathfrak{M}_\rho}{\mathfrak{m}_\rho}(\mathfrak{M}_\rho-\mathfrak{m}_\rho)\right)\\
&=\frac{1-q}{2}\frac{1}{\lambda v_3+(1-\lambda)v_1}\left(\mathcal{C}_a(x_1,x_3,t_1,t_3,\lambda)-\frac{\mathfrak{M}_\rho}{\mathfrak{m}_\rho}\varepsilon\right).
\end{aligned}
$$

In the case as $t_1=+\infty$ or $t_3=+\infty$, the previous inequality holds by the same arguments for $\mathcal{HC}_{\mathscr{B}(\cdot,v(\cdot,\star),\xi)}(x_1,x_3,+\infty,+\infty,\lambda)$. Thus it holds

$$
\mathcal{HC}_{\mathscr{B}(\cdot,v(\cdot,\star),\xi)}(x_1,x_3,+\infty,+\infty,\lambda)\geq\frac{1-q}{2}\frac{1}{\lambda v_3+(1-\lambda)v_1}\left(\mathcal{C}_a(x_1,x_3,\lambda)-\frac{\mathfrak{M}_\rho}{\mathfrak{m}_\rho}\varepsilon\right).
$$

Let us write explicitly the constant σ of (3.4) recalling that $v_2<\lambda v_3+(1-\lambda)v_1$:

$$
\begin{aligned}
\sup_{z\in[v_2,\lambda v_3+(1-\lambda)v_1]}\partial_s\mathscr{B}(x_2,z,D_xv_2)&\leq f^{D_xv_2}(x_2)\sup_{z\in[v_2,\lambda v_3+(1-\lambda)v_1]}\left(-\frac{1}{z^2}\right)\\
&\leq-\frac{1-q}{2}\frac{\mathfrak{m}_\rho}{(\lambda v_3+(1-\lambda)v_1)^2}
\end{aligned}
$$

which leads to $\sigma:=\frac{1-q}{2}\frac{\mathfrak{m}_\rho}{(\lambda v_3+(1-\lambda)v_1)^2}$. Finally by Theorem 4.15, it holds

$$
\min_{\overline{\Omega}\times\overline{\Omega}\times[0,+\infty]\times[0,+\infty]\times[0,1]}\mathcal{C}_{u^{\frac{1-q}{2}}}\geq\min\left\{0,\frac{\lambda v_3+(1-\lambda)v_1}{\mathfrak{m}_\rho}\left(\mathcal{C}_a(x_1,x_3,\lambda)-\frac{\mathfrak{M}_\rho}{\mathfrak{m}_\rho}\varepsilon\right)\right\}
$$

which implies the claim. We conclude by observing that $\mathcal{C}_a\geq\inf_{\overline{\Omega}_\rho}a-\sup_{\overline{\Omega}_\rho}a$ on $\overline{\Omega}_\rho$, (4.13) and the trivial identity $\frac{\mathfrak{M}_\rho}{\mathfrak{m}_\rho}=\frac{\mathfrak{M}_\rho-\mathfrak{m}_\rho}{\mathfrak{m}_\rho}+1$. □

Remark 4.21. We highlight that, in the previous proof, we exploited $\inf_\Omega a>0$. On the other hand, in regard of the conclusion, we observe that if $a\to0$ on $\partial\Omega$ the final expression is still well defined; we refer to Section 5 for some generalization for the torsion problem in this direction.

Assumed a bounded and set $M:=\sup_\Omega a$, $m:=\inf_\Omega a$, the final statement in Proposition 4.20 implies

$$
\min_{\overline{\Omega}\times\overline{\Omega}\times[0,+\infty]\times[0,+\infty]\times[0,1]}\mathcal{C}_{u^{\frac{1-q}{2}}}\geq-\|u(\cdot,\infty)\|_\infty^{\frac{1-q}{2}}\left(2+\frac{\mathrm{osc}(a)}{m}\right)\frac{\mathrm{osc}(a)}{m}
$$

where we recall $\mathrm{osc}(a)=M-m$. Finally, if $a\in C^1(\Omega)$, it is possible to express ε in terms of the gradient of a. Indeed for some $\bar{x},\tilde{x}\in\overline{\Omega_\rho}$ and $\bar{z}$ on the segment $[\bar{x},\tilde{x}]$ lying in $\overline{\Omega_\rho}$, it holds

$$
\varepsilon=\mathfrak{M}_\rho-\mathfrak{m}_\rho=\nabla a(\bar{z})\cdot(\bar{x}-\tilde{x})\leq\|\nabla a\|_\infty\mathrm{diam}(\Omega).
$$

As a consequence of Proposition 2.9, Proposition 4.20 and Remark 4.21 we obtain the following result.

Corollary 4.22 (Hyers-Ulam approximation)**.** *Under the assumptions of Proposition 4.20, there exist a* $C = C(\|u(\cdot,\infty)\|_\infty, \inf_\Omega a, \mathrm{osc}(a), \gamma, n) > 0$ *(with a linear dependence with respect to* $\mathrm{osc}(a)$*) and a concave function* $w : \Omega \times (0,+\infty) \to \mathbb{R}$ *such that*

$$\|u^{\frac{1-q}{2}} - w\|_{L^\infty(\Omega\times(0,+\infty))} \le C\mathrm{osc}(a).$$

Proposition 4.23 (θ-concavity estimate)**.** *Let* Ω *satisfy* (4.2)*,* $\theta \ge 1$*, and* u *be a function such that* (4.3)*,* (4.5) *and* (4.10) *hold. Suppose moreover* $a\colon \Omega \to \mathbb{R}$ *measurable,* $0 < m \le a \le M$ *and* $m^\theta \ge M^\theta/2$*. Let* $(x_1, x_3, t_1, t_3, \lambda) \in \overline{\Omega} \times \overline{\Omega} \times [0,+\infty] \times [0,+\infty] \times [0,1]$ *be a global minimum for* $\mathcal{C}_{u^{\frac{\theta(1-q)}{1+2\theta}}}$*. Then there exists* $\rho > 0$ *such that*

$$\min_{\overline{\Omega}\times\overline{\Omega}\times[0,+\infty]\times[0,+\infty]\times[0,1]} \mathcal{C}_{u^{\frac{\theta(1-q)}{1+2\theta}}} \ge -\frac{2\theta}{1+2\theta}\frac{||u(\cdot,\infty)||_{L^\infty(\Omega)}^{\frac{(\theta-1)(1-q)}{1+2\theta}}}{\mathfrak{m}_\rho}(\mathcal{C}^-_{a^\theta})^{1/\theta}(x_1,x_3,\lambda).$$

Proof. As in Proposition 4.20, the hypothesis imply Corollary 4.16. Being $b(x,s) = a(x)s^q$, we need to we evaluate $\mathcal{C}_{\tilde g(\cdot,v(\cdot,\star))}$ with $\tilde g(x,s) := a(x)s^{\frac{3\alpha-1+q}{\alpha}}$ and $\alpha = \frac{\theta(1-q)}{1+2\theta}$. We see that $s \mapsto s^{\frac{3\alpha-1+q}{\alpha}}$ is $\left(1-\frac{1}{\theta}\right)^{-1}$-concave. If $\theta > 1$, applying Proposition B.1 it holds

$$\mathcal{C}_{\tilde g(\cdot,v(\cdot,\star))}(x_1,x_3,\lambda) \ge -(\mathcal{C}^-_{a^\theta})^{1/\theta}(x_1,x_3,\lambda)(\lambda v_3^{1-1/\theta} + (1-\lambda)v_1^{1-1/\theta}).$$

Instead if $\theta = 1$, the previous inequality is obvious. So by Corollary 4.16

$$\begin{aligned}\mathcal{C}_{u^\alpha}(x_1,x_3,t_1,t_3,\lambda) &\ge -\frac{\alpha}{\sigma}\frac{\mathcal{C}^-_{\tilde g(\cdot,v(\cdot,\star))}(x_1,x_3,t_1,t_3,\lambda)}{(\lambda v_2 + (1-\lambda)v_1)^2}\\ &\ge -\frac{\theta(1-q)}{1+2\theta}\frac{1}{\sigma}\frac{(\lambda v_3^{1-1/\theta} + (1-\lambda)v_1^{1-1/\theta})}{(\lambda v_3 + (1-\lambda)v_1)^2}(\mathcal{C}^-_{a^\theta})^{1/\theta}(x_1,x_3,\lambda).\end{aligned}$$

Define $\rho := \min\{d(x_1,\partial\Omega), d(x_3,\partial\Omega)\} > 0$. Then as in Proposition 4.20, we can take $\sigma := \frac{1-q}{2}\frac{\mathfrak{m}_\rho}{(\lambda v_3+(1-\lambda)v_1)^2}$ and this leads to

$$\begin{aligned}\mathcal{C}_{u^\alpha}(x_1,x_3,t_1,t_3,\lambda) &\ge -\frac{2\theta}{1+2\theta}\frac{(\lambda v_3^{1-1/\theta} + (1-\lambda)v_1^{1-1/\theta})}{\mathfrak{m}_\rho}(\mathcal{C}^-_{a^\theta})^{1/\theta}(x_1,x_3,\lambda)\\ &\ge -\frac{2\theta}{1+2\theta}\frac{||u||_\infty^{\frac{(\theta-1)(1-q)}{1+2\theta}}}{\mathfrak{m}_\rho}(\mathcal{C}^-_{a^\theta})^{1/\theta}(x_1,x_3,\lambda).\end{aligned}$$

□

Proof of Theorem 1.8. It is a consequence of Proposition 4.23 and Remark 4.21. □

Remark 4.24. The above perturbative results can be extended also to time-dependent nonlinearities $a(x,t)$. If, for instance, $a(x,t)$ is bounded and $a(x,t) \to \bar a(x)$ as $t \to +\infty$, it makes sense to investigate the parabolic stability condition for b. Moreover, if $a > 0$ in $\Omega \times (0,+\infty)$, one can consider the infimum over $\Omega_\rho \times [\rho,+\infty)$, where $\rho := \min\{d(x_1,\partial\Omega), d(x_3,\partial\Omega), t_1, t_3\} > 0$.

If instead $a(x,t) \sim a(x)t^\gamma$, then one may consider a truncation argument (see Remark 2.6) so that a becomes bounded. In such a case, anyway, the perturbative estimate will depend on T. We omit the the details, and leave them to the interested reader.

Remark 4.25 (Comments on $u(\cdot,0) \neq 0$)**.** In the whole Subsection 4.2 we considered only zero initial conditions $u(\cdot,0) = u_0 = 0$, in accordance to [36, 38, 39, 44]. Many of our arguments, anyway, could be adapted also to nonzero initial conditions. First, assume, in place of (*H*1), the following slightly stronger assumption:

($H1^*$) There exist $T > 0$, $q \in [0,1)$ and $\gamma \in [0,1]$ such that for all $M > 0$ there exists $k = k(M,T) > 0$ verifying

$$b(x,s,t) \geq kt^\gamma s^q \quad \text{for all } (x,s,t) \in \Omega \times (0,M] \times (0,T].$$

As in ($H2$), $\|u\|_\infty$ takes the place of M. Notice that that $f(s) = (1-s)^p$ (studied in Proposition 4.18) satisfies ($H1$) but not ($H1^*$). Assuming

- $u_0 \in C(\overline{\Omega})$, $u_0 = 0$ on $\partial\Omega$,[4]

$$u_0 \geq \delta d_\Omega \quad \text{for some } \delta > 0, \tag{4.14}$$

u_0 is α-concave for some (suitable) α, and

$$u(\cdot,t) \geq u_0 \quad \text{for each } t > 0$$

(for instance, when ($H3$) holds, we can require u_0 to be a subsolution of the stationary problem, see Corollary A.3),[5]

all the arguments still hold, except for the ones in Lemma 4.11, case $t_1 = 0$, $t_3 \in (0,+\infty)$ and $x_1, x_3 \in \Omega$. As a matter of facts, in this case we have

$$\limsup_{\tau\to 0} v_t(\tau) \geq \limsup_{\tau\to 0} \frac{v(x,\tau) - v(x,0)}{\tau},$$

with $v(x,0) = u_0^\alpha(x) > 0$, and we are not able to show that this limit explodes to $+\infty$, no matter who α is. More in details, by assuming for simplicity $b(x,s,t) = a(x)s^q$, $q \in [0,1)$, $a(x) \geq m > 0$, then an improvement of Proposition A.1 leads to

$$u(x,t) \geq v(x,t) + e^{Lt}S(t)(u(x,0) - v(x,0)),$$

for any supersolution u and any subsolution v; here $L = \frac{qm}{\|u\|_\infty^{1-q}} \geq 0$ and $S(t)$ is the heat semigroup. Exploiting this improvement in the proof of Lemma 4.10 (i), we obtain

$$u(x,t) \geq Ce^{-\lambda_1 t}t^{\frac{1}{1-q}}\varphi_1(x) + e^{Lt}S(t)u_0$$

with $C = ((1-q)m)^{\frac{1}{1-q}}$. As a consequence, straightforward computations imply, if $\beta = 1$ and $u_0 \in C^2(\overline{\Omega})$,

$$\limsup_{\tau\to 0} v_t(x,\tau) \geq \alpha u_0^{\alpha-1}(x)\Delta u_0(x) + \alpha L u_0^\alpha(x) + Cu_0^{\alpha-1}(x)\varphi_1(x) \limsup_{\tau\to 0} \tau^{\frac{1}{1-q}-1},$$

where the right hand side corresponds to $\alpha u_0^{\alpha-1}(x)h(x)$ with

$$h := \Delta u_0 + \frac{m}{\alpha}\varphi_1 \ \text{ if } q = 0, \quad h := \Delta u_0 + Lu_0 \ \text{ if } q \in (0,1).$$

As abovementioned, these are finite quantities (positive under suitable assumptions on u_0), and thus do not allow to conclude in Lemma 4.11.

With similar computations we see that, if $\beta < 1$, then we actually obtain $\limsup_{\tau\to 0} v_t(\tau) = +\infty$: with such an information at disposal, we can conclude Lemma 4.11 and apply it to obtain (arguing as in Theorem 4.15 and Corollary 4.16)

$$\min_{\overline{\Omega}\times\overline{\Omega}\times[0,+\infty]\times[0,+\infty]\times[0,1]} \mathcal{C}_{u^\alpha(\cdot,\star^\beta)} \geq -\frac{\alpha}{\sigma}\frac{\mathcal{C}^-_{\tilde{g}(\cdot,u^\alpha(\cdot,\star^\beta))}(x_1,x_3,t_1,t_3,\lambda)}{(\lambda u^\alpha(x_3,t_3^\beta) + (1-\lambda)u^\alpha(x_1,t_1^\beta))^2} - \xi^{n+1}\mathcal{HC}_{\star^{1-\beta}}(t_1,t_3,\lambda),$$

[4] We highlight that the requirement (4.14) is related to the application of the comparison principles, see Proposition A.1. By requiring more regularity on b (see ($H2^*$)), this condition can be dropped (see Proposition A.2).

[5] The set of functions satisfying these conditions is nonempty. For example, consider $\Omega = B_1$, $b(x,s,t) = a(x) \in L^\infty(\Omega)$ with $a \geq m > 0$. Let $u_0(x) = \delta d_\Omega(x) = \delta(1-|x|)$, δ small enough, on $B_1 \setminus B_{\frac{1}{2}}$ and extended smoothly and in a concave way to $B_{\frac{1}{2}}$. Then $-\Delta u_0 = \delta\frac{N-1}{|x|} \leq a(x)$ in $B_1 \setminus B_{\frac{1}{2}}$ and we can require also $-\Delta u_0 \leq a(x)$ in $B_{\frac{1}{2}}$. Thus u_0 satisfies all the assumptions ($\alpha = 1$). Less trivial examples (such as perturbations of the present examples) hold as well.

where $\tilde{g}(x,s) := a(x)s^{\frac{3\alpha-1+q}{\alpha}}$. While the first term, as in the case $u_0 = 0$ (see the proof of Theorem 1.4), can be positively estimated under suitable assumptions on α, q and $a(x)$, we see that for any $\beta < 1$ the second term remains negative (notice that t_1 or t_3 may go to zero as $\beta \to 1^-$). Thus we are not able to deduce any exact concavity in such a case.

Overall, we observe that conservation of concavity for nontrivial initial data is a delicate issue: in the case of porous medium equations, indeed, it has been proved that $\frac{1}{2}$-concavity is preserved [51], while there are counterexamples in the case of α-concavity, $\alpha < \frac{1}{2}$, [17]; this is in contrast with the general spirit of the present paper, where information on some α implies information also on smaller exponents.

5. Further results on the weighted torsion problem

Let us deal with the case $q = 0$ in (4.10). First, we can find more explicit constant and comparison function in Corollary 4.22; moreover, we have more freedom in the choice of the sign of a.

Proposition 5.1. *Let Ω satisfy (4.2), and u, v be some functions such that (4.3) and (4.5) hold. Suppose*

$$u_t - \Delta u = a(x,t) \quad \text{in } \Omega \times (0,+\infty), \qquad v_t - \Delta v = K \quad \text{in } \Omega \times (0,+\infty)$$

where $K > 0$, $a\colon \Omega \times \mathbb{R} \to \mathbb{R}$ and there exist $m, M \in \mathbb{R}$ such that $m \le a \le M$. Then v is $\frac{1}{2}$-concave and

$$\|u^{1/2} - v^{1/2}\|_\infty \le e^R \sqrt{\max\{|m-K|, |M-K|\}},$$

where R is such that $\Omega \subseteq B(0,R)$.

Proof. Let us subtract the two equations obtaining that $w := u - v$ satisfies the problem

$$\begin{cases} w_t - \Delta w = a(x,t) - K & \text{in } \Omega \times (0,+\infty), \\ w = 0 & \text{on } \partial\Omega \times (0,+\infty), \\ w = 0 & \text{on } \Omega \times \{0\}. \end{cases}$$

Set $\psi := e^{2R} - e^{x_1+R}$. Then

$$\psi_t - \Delta\psi = e^{x_1+R} \ge 1 \quad \text{in } \Omega \times (0,+\infty).$$

Set $f(x,t) := a(x,t) - K$, hence we have $(\partial_t - \Delta)(w - \psi \sup_{\Omega\times(0,+\infty)} f^+) \le 0$ in $\Omega \times (0,+\infty)$ and $w - \psi \sup_{\Omega\times(0,+\infty)} f^+ \le 0$ on the parabolic boundary (see (A.4)). Then by the maximum principle [52, Corollary 2.5], it holds

$$\sup_{\Omega\times(0,+\infty)} (w - \psi \sup_{\Omega} f^+) \le 0,$$

so

$$\sup_{\Omega\times(0,+\infty)} w \le e^{2R} \sup_{\Omega\times(0,+\infty)} |f|.$$

In a similar way we obtain

$$\inf_{\Omega\times(0,+\infty)} w \ge -e^{2R} \sup_{\Omega\times(0,+\infty)} |f|.$$

Then it holds

$$\|u^{1/2} - v^{1/2}\|_\infty \le \|u - v\|_\infty^{1/2} \le e^R \sqrt{\sup_{\Omega\times(0,+\infty)} |f|} = e^R \sqrt{\max\{|m-K|, |M-K|\}}.$$

Finally, v is $\frac{1}{2}$-concave by Proposition 4.20. □

When dealing with the torsion problem ($q = 0$), more general classes of concave weights $a(x)$ can be considered, in particular by allowing $a(x) = 0$ on $\partial\Omega$. In particular, instead of (*H*1) we can consider the following condition

$(H1')$ There exists $T>0$, $k>0$, $\omega\geq 0$ and $\gamma\in[0,1)$, such that,

$$b(x,t)\geq k d_\Omega^\omega(x)t^\gamma \quad \text{for all } (x,t)\in\Omega\times(0,T].$$

Remark 5.2. The presence of d_Ω in assumption $(H1')$ is due to the following fact: if $a(x)$ is a nonnegative concave function, then it is positive in Ω (but not necessarily in $\overline{\Omega}$), and there exists $C>0$ such that

$$a(x)\geq C d_\Omega(x) \quad \text{in } \overline{\Omega}.$$

The fact that, differently from $(H1)$, we focus on $q=0$ in $(H1')$, allows us to compare our solutions, time-scaled by $\beta=2$, with the one of the equation $U_t-\Delta U+U=kd_\Omega^\omega(x)t^\gamma$ for which a representation formula holds. Thus combining [39, Lemma 3] and Proposition A.1, we obtain an alternative version of Lemma 4.10.

Lemma 5.3. *Let $n\geq 2$, Ω be bounded, convex and smooth, and u be a function such that (4.3), (4.4) and (4.5) hold. Suppose $b\colon\Omega\times(0,+\infty)\to(0,+\infty)$ satisfies $(H1')$, and let T,γ,ω given therein. Then*

(i) there exists $C>0$ such that

$$u(x,t)\geq Ct^{\frac{2+2\gamma+\omega}{2}} \quad \text{in } \Omega\times(0,T).$$

(ii) for any $x_0\in\partial\Omega$ there exists $\delta>0$ such that

$$u(x_0+t\nu,t^2)\geq \delta t^{2+2\gamma+\omega} \quad \text{for } t\in(0,T) \text{ small},$$

where ν is the interior normal to $\partial\Omega$ in x_0.

Thanks to this result, we are able to show the counterpart of Proposition 4.11 in the case $\beta=2$, that is

Proposition 5.4. *In the setting of Proposition 4.11, suppose b satisfies $(H1')$, and let $\alpha\in(0,\frac{1}{2+2\gamma+\omega})$. Then $\mathcal{C}_{u^\alpha(\cdot,\star^2)}$ cannot achieve any negative minimum at $(x_1,x_3,t_1,t_3,\lambda)\in\overline{\Omega}\times\overline{\Omega}\times[0,+\infty]\times[0,+\infty]\times(0,1)$ such that one $t_1,t_3\in\{0\}$, or one $x_1,x_2,x_3\in\partial\Omega$.*

Proof. The proof proceed similarly to the one of Proposition 4.11. In particular if $t_1=0$, $t_3\in(0,+\infty)$, $x_1\in\Omega$ and $x_3\in\Omega$, by Lemma 5.3 (i),

$$\lim_{\tau\to 0} v_t(x_1,\tau)\geq C^\alpha\lim_{\tau\to 0}\frac{1}{\tau^{1-(2+2\gamma+\omega)\alpha}}=+\infty$$

by the assumption $\alpha<\frac{1}{2+2\gamma+\omega}$ which implies the contradiction. If $t_1=0$, $t_3\in(0,+\infty)$, $x_1\in\partial\Omega$, $x_3\in\Omega$, by Lemma 5.3 (ii)

$$\frac{u^\alpha(x_1+t\nu,t^2)}{t}\geq\delta^\alpha t^{\alpha(2+2\gamma+\omega)-1} \quad \text{for } t>0 \text{ small},$$

which positively diverges as $t\to 0^+$ again by assumptions and thus again a contradiction. □

With this tool, we can prove a refinement of Theorem 1.4 when $q=0$.

Proposition 5.5 (Weighted torsion)**.** *Let $\gamma\in[0,\frac{1}{2}]$, Ω satisfy (4.2), and u be a function such that (4.3), (4.5) and*

$$u_t-\Delta u=c(t)a(x) \quad \text{in } \Omega\times(0,+\infty).$$

We assume c nondecreasing, $\frac{1}{\gamma}$-concave, and

$$c(t)\geq t^\gamma.$$

If $\gamma<\frac{1}{2}$ and $a\geq 0$ is θ-concave for some $\theta\geq\frac{1}{1-2\gamma}$, then $u(\cdot,\star^2)$ is $\frac{\theta}{2\theta+2\theta\gamma+1}$-concave. If a is constant, then $u(\cdot,\star^2)$ is $\frac{1}{2+2\gamma}$-concave.

Proof. By Remark 5.2 we may assume $a \geq Cd_\Omega^\omega$ for $\omega = \frac{1}{\theta}$. By Remark 2.6 we can restrict to study

$$u_t - \Delta u = c_T(t)a(x) \quad \text{in } \Omega \times (0, +\infty)$$

where T is fixed and $c_T(t) = c(t)$ for $t \in [0, T]$ and $c_T(t) = c(T)$ for $t \leq T$. Clearly $b(x,t) := c_T(t)a(x)$ satisfies $(H1')$ and $(H2)$, as well as the stability parabolic condition thanks again to Remark 2.6. To apply Theorem 4.17 with $\alpha \in (0, \frac{1}{2+2\gamma+\omega}) \subset (0,1)$, we observe that $s \mapsto s^{\alpha-1}b(x,t)$ is strictly decreasing and moreover $(x,s,t) \mapsto s^{\frac{3\alpha-1}{\alpha}}b(x,t^2) = s^{\frac{3\alpha-1}{\alpha}}c(t^2)a(x)$ is concave if $\frac{3\alpha-1}{\alpha} \geq 0$ and $\frac{3\alpha-1}{\alpha} + 2\gamma + \frac{1}{\theta} = 1$ by Proposition 2.7 (iv). Both conditions imply that the optimal α is given by $\alpha = \frac{\theta}{2\theta+2\theta\gamma+1}$. Noticed that the solution of the stationary problem $-\Delta u = a(x)$ is $\frac{\theta}{2\theta+1}$-concave, and $\frac{\theta}{2\theta+2\theta\gamma+1} \leq \frac{\theta}{2\theta+1}$, then we obtain u is $\frac{\theta}{2\theta+2\theta\gamma+1}$-concave. □

Appendix A. Comparison principles

We state now a comparison principle, which is the nonautonomous counterpart of [25, Corollary 2.3] (see also [16]). We recall that $w = w(x,t)$ is a subsolution (resp. supersolution) of

$$\begin{cases} w_t - \Delta w = b(x,w,t) & \text{in } \Omega \times (0,T), \\ w = 0 & \text{on } \partial\Omega \times (0,T), \\ w = u_0 & \text{on } \Omega \times \{0\}, \end{cases} \tag{A.1}$$

if the above three equalities are substituted by $\leq$ (resp. $\geq$).

Proposition A.1 (Comparison and uniqueness, I)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded open domain with the interior sphere property. Let $T > 0$ and $b\colon \Omega \times (0,+\infty) \times (0,T) \to [0,+\infty)$ satisfying $(H2)$ and assume $b(x,\cdot,t)$ continuous for all $(x,t) \in \Omega \times (0,T)$. Let $u_0 \in C(\overline{\Omega})$ with $u_0 \geq \delta d_\Omega$ for some $\delta > 0$ or $u_0 \equiv 0$. Let $u, v \in C^2(\Omega) \cap C^1((0,T]) \cap C(\overline{\Omega} \times [0,T])$ and $(u-v)^+ \in H^1(\Omega)$. Assume u is a positive supersolution and v is a subsolution of (A.1). Then $u(\cdot,t) \geq v(\cdot,t)$ for all $0 \leq t \leq T$. In particular, there exists at most one positive solution of (A.1).*

Proof. Consider first the case $u_0 \geq \delta d_\Omega$. Based on Hardy's inequality, it holds

$$\int_\Omega \frac{\varphi^2}{d_\Omega^2} \leq C \int_\Omega |\nabla \varphi|^2, \tag{A.2}$$

for all $\varphi \in H_0^1(\Omega)$. Being b nonnegative, we have $u_t - \Delta u \geq 0$ and thus by comparison and the semigroup representation of the heat equation we obtain $u(\cdot,t) \geq S(t)u(\cdot,0)$, $S(t) = e^{t\Delta}$. In particular there exists $\gamma > 0$ such that $u(\cdot,t) \geq \gamma d_\Omega$ for all $t \in [0,T]$. Taking $t \in (0,T)$, testing the inequalities satisfied by u and v with $(v(\cdot,t) - u(\cdot,t))^+$ and taking the difference we obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_\Omega |(v(x,t)-u(x,t))^+|^2 dx + \int_\Omega |D_x(v(x,t)-u(x,t))^+|^2 dx \\ \leq \int_{\{v(\cdot,t)>u(\cdot,t)\}} (b(x,v,t) - b(x,u,t))(v(x,t)-u(x,t))^+ dx \\ \leq L_M \int_{\{v(\cdot,t)>u(\cdot,t)\}} \frac{|(v(x,t)-u(x,t))^+|^2}{u(x,t)} dx, \end{aligned}$$

with $M := \max\left\{||u||_{L^\infty((0,T)\times\Omega)}, ||v||_{L^\infty((0,T)\times\Omega)}\right\}$. Since

$$\frac{1}{u(\cdot,t)} \leq \frac{1}{\gamma d_\Omega} \leq \frac{\varepsilon}{d_\Omega^2} + C(\varepsilon) \quad \text{for all } \varepsilon > 0,$$

we deduce that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |(v(x,t)-u(x,t))^+|^2dx + \int_\Omega |D_x(v(x,t)-u(x,t))^+|^2dx \\ \leq \varepsilon \int_\Omega \frac{|(v(x,t)-u(x,t))^+|^2}{d_\Omega^2(x)}dx + C(\varepsilon)\int_\Omega |(v(x,t)-u(x,t))^+|^2dx.$$

Applying (A.2) and choosing $\varepsilon > 0$ sufficiently small, we then obtain

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |(v(x,t)-u(x,t))^+|^2dx \leq C\int_\Omega |(v(x,t)-u(x,t))^+|^2dx,$$

from which the result follows by Gronwall inequality

$$\int_\Omega |(v(x,t)-u(x,t))^+|^2dx \leq e^{2C}\int_\Omega |(v(x,0)-u(x,0))^+|^2dx.$$

Now let us deal with the case $u_0 \equiv 0$. Since $u_t - \Delta u \geq 0$, by [16, Remark 6.1] (see also Lemma A.4) we have that $u(\cdot,t) \geq S(t-r)u(\cdot,r)$ for all $0 < r < t \leq T$ and hence, being $u(\cdot,r) > 0$, we obtain $u(\cdot,t) \geq \delta(t)d_\Omega$ for all $t \in (0,T)$ with $\delta(t) > 0$. Let us fix $t \in (0,T)$. By the previous arguments, applied with initial time t to $u(\cdot, \star + t)$ and v, we conclude that $u(\cdot, s+t) \geq v(\cdot,t)$ for all $0 < s < T - t$. The result follows by letting s tend to 0. □

By requiring a more restrictive regularity of $b(x,\cdot,t)$, we show now that the request $u \geq \delta d_\Omega$ can be dropped in Proposition A.1; notice anyway that such regularity is satisfied by $b(x,s,t) = a(x)s^q$ only for $q \in [\frac{1}{2}, 1]$. We refer also to [36, Section A.1] and references therein for other comparison principles which involve nonincreasing $b(x,\cdot,t)$.

Proposition A.2 (Comparison and uniqueness, II)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded open domain with the interior sphere property. Let $T > 0$ and $b\colon \Omega \times (0,+\infty) \times (0,T) \to [0,+\infty)$ satisfying (H2*) and $b(x,\cdot,t)$ continuous for all $(x,t) \in \Omega \times (0,T)$. Let $u_0 \in C(\overline{\Omega})$. Let $u, v \in C^2(\Omega) \cap C^1((0,T]) \cap C(\overline{\Omega} \times [0,T])$ and $(u-v)^+ \in H^1(\Omega)$. Assume u is a positive supersolution and v is a subsolution of (A.1). Then $u(\cdot,t) \geq v(\cdot,t)$ for all $0 \leq t \leq T$. In particular, there exists at most one positive solution of (A.1).*

Proof. The proof follows the lines of the one in Proposition A.1, with no need of the Hardy inequality, but by exploiting $D^{1,2}(\Omega) \hookrightarrow L^{2^*}(\Omega) \hookrightarrow L^{2\omega}(\Omega)$ and Hölder inequality, where we highlight that $L^{2\omega}(\Omega)$ is well defined as Lebesgue space by the assumption $2\omega \geq 1$. □

We state now a condition for monotonicity of solutions (see also [16, Proposition 6.12] and [57, Lemma 4.1, pag 199]).

Corollary A.3 (Monotonicity)**.** *In the assumptions of Proposition A.1 or Proposition A.2, assume in addition (H3). Let u be a positive solution to*

$$\begin{cases} u_t - \Delta u = b(x,u,t) & \text{in } \Omega \times (0,+\infty), \\ u = 0 & \text{on } \partial\Omega \times (0,+\infty), \\ u = u_0 & \text{on } \Omega \times \{0\}. \end{cases}$$

Assume moreover that

$$u(\cdot,t) \geq u_0 \quad \text{for each } t > 0. \tag{A.3}$$

Then $u_t \geq 0$ in $\Omega \times (0,+\infty)$. A condition to ensure (A.3) is that u_0 is a subsolution of the stationary problem

$$\begin{cases} -\Delta w = b(x,w,0) & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega. \end{cases}$$

Proof. Fix $\tau > 0$ and set $\tilde{u}(x,t) := u(x,t+\tau)$, then $\tilde{u}$ satisfies

$$\begin{cases} \tilde{u}_t - \Delta\tilde{u} = b(x,\tilde{u},t+\tau) \geq b(x,\tilde{u},t) & \text{in } \Omega\times(0,+\infty), \\ \tilde{u} = 0 & \text{on } \partial\Omega\times(0,+\infty), \\ \tilde{u} \geq u_0 & \text{on } \Omega\times\{0\}. \end{cases}$$

Since (*H*2) holds, by Proposition A.1 or A.2 we have $\tilde{u}(x,t) \geq u(x,t)$ for all $(x,t) \in \Omega\times(0,+\infty)$, that is $u(x,t+\tau) \geq u(x,t)$, which is the claim. To see the last statement, it is sufficient to apply again Proposition A.1 or A.2 with $v = u_0$, after having observed that $b(x,s,0) \leq b(x,s,t)$ thanks to (*H*3). □

To state the parabolic version of the Hopf lemma, we recall some usual notation for the domain and the boundary. Let Ω be a bounded open set in $\mathbb{R}^n$, $T > 0$ and define the *parabolic boundary* as

$$\partial_p\Omega_T := (\overline{\Omega}\times[0,T]) \setminus (\Omega\times(0,T]). \tag{A.4}$$

We observe that, if Ω satisfies the interior sphere condition, then every point $(\bar{x},\bar{t}) \in \partial_p\Omega_T$ satisfies the *spherical cap* condition, that is there exists an open ball $B_r(x_0,t_0)$, $x_0 \neq \bar{x}$, such that $(\bar{x},\bar{t}) \in \partial B_r(x_0,t_0)$ and

$$C_r(\bar{x},\bar{t}) := B_r(x_0,t_0) \cap \{t < \bar{t}\} \subset \Omega\times(0,T].$$

We recall thus the Hopf lemma for parabolic equations, see e.g. [59, Theorem 3, page 170].

Lemma A.4 (Parabolic Hopf lemma)**.** *Let Ω be an open bounded set in $\mathbb{R}^n$ satisfying the interior sphere condition, $T > 0$ and $u \in C^2(\Omega) \cap C^1((0,T]) \cap C(\overline{\Omega}\times[0,T])$. Suppose further*

$$u_t + Lu \geq 0 \quad \text{in } \Omega\times(0,T]$$

where

$$Lu := -\sum_{i,j=1}^n a_{ij}D^2_{ij}u + \sum_{i=1}^n b^i D_i u$$

with $a_{ij}, b_i \in L^\infty(\Omega\times(0,T]) \cap C(\Omega\times(0,T])$. Moreover, suppose that there exists a point $(\bar{x},\bar{t}) \in \partial_p\Omega_T$ such that

$$u(\bar{x},\bar{t}) < u(x,t) \quad \forall (x,t) \in C_r(\bar{x},\bar{t}).$$

Let $\mathbf{e} \in \mathbb{R}^{n+1}$ be a direction such that $(\bar{x},\bar{t}) + s\mathbf{e} \in \overline{C_r(\bar{x},\bar{t})}$ for some $s > 0$, and assume that $\partial_{\mathbf{e}}u(\bar{x},\bar{t})$ exists. Then $\partial_{\mathbf{e}}u(\bar{x},\bar{t}) > 0$.

Appendix B. Properties of concavity functions

Proposition B.1 (Concavity of the product)**.** *Consider $\alpha,\beta \in (0,+\infty)$ with $\alpha^{-1}+\beta^{-1} = 1$, K a convex set and $f,g\colon K \to \mathbb{R}$. Suppose on f one of the following:*

i) $m_1 \leq f \leq M_1$ for some $m_1^\alpha \geq \frac{1}{2}M_1^\alpha > 0$,
ii) $\mathcal{C}^-_{f^\alpha} = 0$,

and on g one of the following:

i) $m_2 \leq g \leq M_2$ for some $m_2^\beta \geq \frac{1}{2}M_2^\beta > 0$,
ii) $\mathcal{C}^-_{g^\beta} = 0$.

Then, for each $x_1, x_3 \in K$ and $\lambda \in [0,1]$, it holds

$$\begin{aligned} \mathcal{C}_{fg}(x_1,x_3,\lambda) \geq &-(\mathcal{C}^-_{f^\alpha})^{1/\alpha}(x_1,x_3,\lambda)(\lambda g_3 + (1-\lambda)g_1) - (\mathcal{C}^-_{g^\beta})^{1/\beta}(x_1,x_3,\lambda)(\lambda f_3 + (1-\lambda)f_1) + \\ &+ (\mathcal{C}^-_{f^\alpha})^{1/\alpha}(x_1,x_3,\lambda)(\mathcal{C}^-_{g^\beta})^{1/\beta}(x_1,x_3,\lambda). \end{aligned}$$

Proof. Suppose $\alpha, \beta \in (1, +\infty)$ and set $\mu := \frac{1}{\alpha}$. Assume for a moment that

$$f_i^{1/\mu} - \mathcal{C}^-_{f^{1/\mu}}(x_1, x_3, \lambda) \geq 0, \quad g_i^{1/(1-\mu)} - \mathcal{C}^-_{g^{1/(1-\mu)}}(x_1, x_3, \lambda) \geq 0 \qquad \text{for } i \in 1, 3. \tag{B.1}$$

Then

$$\begin{aligned}
f_2 g_2 &= \left(f_2^{1/\mu}\right)^{\mu} \left(g_2^{1/(1-\mu)}\right)^{1-\mu} = \lim_{p\to 0} \left(\mu f_2^{p/\mu} + (1-\mu) g_2^{p/(1-\mu)}\right)^{1/p} \\
&\geq \lim_{p\to 0} \Big(\mu \left(\lambda f_3^{1/\mu} + (1-\lambda) f_1^{1/\mu} - \mathcal{C}^-_{f^{1/\mu}}(x_1, x_3, \lambda)\right)^p + \\
&\quad + (1-\mu) \left(\lambda g_3^{1/(1-\mu)} + (1-\lambda) g_1^{1/(1-\mu)} - \mathcal{C}^-_{g^{1/(1-\mu)}}(x_1, x_3, \lambda)\right)^p \Big)^{1/p} \\
&\geq \lim_{p\to 0} \lambda \left(\mu (f_3^{1/\mu} - \mathcal{C}^-_{f^{1/\mu}}(x_1, x_3, \lambda))^p + (1-\mu) \left(g_3^{1/(1-\mu)} - \mathcal{C}^-_{g^{1/(1-\mu)}}(x_1, x_3, \lambda)\right)^p\right)^{1/p} + \\
&\quad + (1-\lambda) \left(\mu (f_1^{1/\mu} - \mathcal{C}^-_{f^{1/\mu}}(x_1, x_3, \lambda))^p + (1-\mu) \left(g_1^{1/(1-\mu)} - \mathcal{C}^-_{g^{1/(1-\mu)}}(x_1, x_3, \lambda)\right)^p\right)^{1/p} \\
&= \lambda \left(f_3^{1/\mu} - \mathcal{C}^-_{f^{1/\mu}}(x_1, x_3, \lambda)\right)^{\mu} \left(g_3^{1/(1-\mu)} - \mathcal{C}^-_{g^{1/(1-\mu)}}(x_1, x_3, \lambda)\right)^{1-\mu} + \\
&\quad + (1-\lambda) \left(f_1^{1/\mu} - \mathcal{C}^-_{f^{1/\mu}}(x_1, x_3, \lambda)\right)^{\mu} \left(g_1^{1/(1-\mu)} - \mathcal{C}^-_{g^{1/(1-\mu)}}(x_1, x_3, \lambda)\right)^{1-\mu},
\end{aligned}$$

where we used the inequality [45, Property 7]

$$(1-\lambda)(f_1^p + g_1^p)^{1/p} + \lambda (f_3^p + g_3^p)^{1/p} \leq \left(((1-\lambda) f_1 + \lambda f_3)^p + ((1-\lambda) g_1 + \lambda g_3)^p\right)^{1/p}.$$

Since $\mu \in (0, 1]$, then for all $A, B \geq 0$ it holds $(A - B)^\mu \geq A^\mu - B^\mu$, and hence

$$\begin{aligned}
f_2 g_2 \geq & \lambda (f_3 - (\mathcal{C}^-_{f^{1/\mu}})^\mu(x_1, x_3, \lambda)) (g_3 - (\mathcal{C}^-_{g^{1/(1-\mu)}})^{1-\mu}(x_1, x_3, \lambda)) + \\
& + (1-\lambda)(f_1 - (\mathcal{C}^-_{f^{1/\mu}})^\mu(x_1, x_3, \lambda)) (g_1 - (\mathcal{C}^-_{g^{1/(1-\mu)}})^{1-\mu})(x_1, x_3, \lambda)
\end{aligned}$$

which implies the claim, observed that $\frac{1}{\mu} = \alpha$ and $\frac{1}{1-\mu} = \beta$. We are left to verify (B.1). If $\mathcal{C}^-_{f^{1/\mu}}(x_1, x_3, \lambda) = 0$, then the inequality trivially holds true. Otherwise consider $(y_1, y_3, \lambda) \in \overline{K} \times \overline{K} \times [0, 1]$ the negative minimum of $\mathcal{C}_{f^{1/\mu}}$. By hypothesis on M, m:

$$f_1^{1/\mu} \geq m_1^{1/\mu} \geq M_1^{1/\mu} - m_1^{1/\mu} \geq -\mathcal{C}_{f^{1/\mu}}(y_1, y_3, \lambda) = \mathcal{C}^-_{f^{1/\mu}}(y_1, y_3, \lambda) \geq \mathcal{C}^-_{f^{1/\mu}}(x_1, x_3, \lambda).$$

Similar computations hold for g. This concludes the proof. □

We deal now with the boundary case $\alpha = \infty$ and $\beta = 1$.

Proposition B.2. *Let K be a convex set and $f, g \colon K \to \mathbb{R}$. Then*

$$\mathcal{C}_{fg}(x_1, x_3, \lambda) \geq \mathcal{C}_g(x_1, x_3, \lambda) f_2 - \operatorname{osc}(f)(\lambda |g_3| + (1-\lambda)|g_1|).$$

In particular, if f is nonnegative and g is concave and bounded, then

$$\mathcal{C}_{fg}(x_1, x_3, \lambda) \geq -\|g\|_\infty \operatorname{osc}(a).$$

Proof. We have

$$\begin{aligned}
\mathcal{C}_{fg}(x_1, x_3, \lambda) &= f_2 g_2 - \lambda f_3 g_3 - (1-\lambda) f_1 g_1 \\
&= \mathcal{C}_g(x_1, x_3, \lambda) f_2 + \lambda g_3 (f_2 - f_3) + (1-\lambda) g_1 (f_2 - f_1)
\end{aligned}$$

and thus the two claims. □

Proposition B.3. *Let $K \subseteq \mathbb{R}^n$, $n \geq 1$, be a convex set. Let $g\colon K \to \mathbb{R}$ and $j \in \{1, \dots, n\}$, and consider $f\colon x \in K \mapsto \frac{g(x)}{(x^j)^2} \in \mathbb{R}$. Then for all $x, y \in K$ and $\lambda \in [0,1]$ it holds*

$$\mathcal{HC}_f(x_1, x_3, \lambda) \geq \frac{\mathcal{C}_g(x_1, x_3, \lambda)}{(x_2^j)^2}.$$

Proof. For clarity, let us set $\zeta := x^j$. If $f_1 = f_3 = 0$, then $g_1 = g_3 = 0$ and thus

$$\mathcal{HC}_f(x_1, x_3, \lambda) = f_2 = \frac{g_2}{\zeta_2^2} = \frac{\mathcal{C}_g(x_1, x_3, \lambda)}{\zeta_2^2}.$$

Assume now $\lambda f_1 + (1-\lambda) f_3 > 0$. Then we have

$$\begin{aligned}\mathcal{HC}_f(x_1, x_3, \lambda) = &= \frac{g_2}{\zeta_2^2} - \frac{g_1 g_3}{\lambda g_1 \zeta_3^2 + (1-\lambda) g_3 \zeta_1^2} \\ &= \frac{\lambda g_3 + (1-\lambda) g_1 + \mathcal{C}_g(x_1, x_3, \lambda)}{\zeta_2^2} - \frac{g_1 g_3}{\lambda g_1 \zeta_3^2 + (1-\lambda) g_3 \zeta_1^2} \\ &= \lambda(1-\lambda) \frac{(g_1 \zeta_3 - g_3 \zeta_1)^2}{\zeta_2^2 (\lambda g_1 \zeta_3^2 + (1-\lambda) g_3 \zeta_1^2)} + \frac{\mathcal{C}_g(x_1, x_3, \lambda)}{\zeta_2^2} \geq \frac{\mathcal{C}_g(x_1, x_3, \lambda)}{\zeta_2^2}.\end{aligned}$$

□

Proposition B.4. *Let K be a convex set and $f, g : K \to \mathbb{R}$ with $g > 0$. Then*

$$\mathcal{HC}_{f-g} \geq \mathcal{HC}_f - \mathcal{HC}_g.$$

In particular, if $\mathcal{HC}_g \leq 0$ (e.g., if g is constant or $g(t) = t^\gamma$ with $\gamma \in [-1, 0]$), then

$$\mathcal{HC}_{f-g} \geq \mathcal{HC}_f.$$

Proof. Set for simplicity $h := f - g$. Let $x_1, x_3 \in \Omega$ and $\lambda \in [0,1]$. Suppose $h_1 = h_3 = 0$, then $f_1 = g_1$ and $f_3 = g_3$, which imply $\lambda f_1 + (1-\lambda) f_3 = \lambda g_1 + (1-\lambda) g_3 > 0$. Moreover

$$\mathcal{HC}_f(x_1, x_3, \lambda) - \mathcal{HC}_g(x_1, x_3, \lambda) = g_2 - f_2 = h_2 = \mathcal{HC}_h(x_1, x_3, \lambda).$$

Otherwise, if $\lambda h_1 + (1-\lambda) h_3 > 0$, then $\lambda f_1 + (1-\lambda) f_3 > \lambda g_1 + (1-\lambda) g_3 > 0$ by the positivity of g. Moreover a straightforward computation implies

$$\begin{aligned}&\mathcal{HC}_h(x_1, x_3, \lambda) - \mathcal{HC}_f(x_1, x_3, \lambda) + \mathcal{HC}_g(x_1, x_3, \lambda) = \\ &= \frac{\lambda(1-\lambda)(f_1 g_3 - f_3 g_1)^2}{(\lambda f_1 + (1-\lambda) f_3)(\lambda g_1 + (1-\lambda) g_3)(\lambda(f_1 - g_1) + (1-\lambda)(f_3 - g_3))} \geq 0.\end{aligned}$$

□

References

[1] G. Akagi and R. Kajikiya, *Stability of stationary solutions for semilinear heat equations with concave nonlinearity*, Commun. Contemp. Math. 17 (2015), article ID 1550001, pp. 29. 9

[2] N. M. Almousa, C. Bucur, R. Cornale and M. Squassina, *Concavity principles for nonautonomous elliptic equations and applications*, Asymptot. Anal. 135 (2023), 509–524. 5

[3] N. M. Almousa, J. Assettini, M. Gallo and M. Squassina, *Concavity properties for quasilinear equations and optimality remarks*, Differential Integral Equations 37 (2023), 1–26. 2, 24

[4] O. Alvarez, J. Lasry and P. Lions, *Convex viscosity solutions and state constraints*, J. Math. Pures Appl. (9) 76 (1997), 265–288. 2

[5] V. Amato, R. Barbato, A. L. Masiello and G. Paoli, *The Talenti comparison result in a quantitative form*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (2024), pp. 31, doi.org/10.2422/2036-2145.202404_010. 5

[6] B. Andrews and J. Clutterbuck, *Proof of the fundamental gap conjecture*, J. Amer. Math. Soc. 24 (2011), 899–916. 2

[7] F. Belgacem and C. Cosner, *The effects of dispersal along environmental gradients on the dynamics of populations in heterogeneous environments*, Canad. Appl. Math. Quart. 3 (1995), 379–397. 3

[8] B. Bian and P. Guan, *A microscopic convexity principle for nonlinear partial differential equations*, Invent. math. 177 (2009), 307–335. 3, 12

[9] B. Bian and P. Guan, *A structural condition for microscopic convexity principle*, Discrete Contin. Dyn. Syst. 28 (2010), 789–807. 13

[10] M. Bitterlin, *Log-concavity of the principal eigenfunction of a linear periodic-parabolic eigenvalue problem*, Differential Integral Equations 5 (1992), 1041–1048. 3

[11] W. Borrelli, S. Mosconi and M. Squassina, *Concavity properties for solutions to p-Laplace equations with concave nonlinearities*, Adv. Calc. Var. 17 (2024), 79–97. 3, 7, 24

[12] H. J. Brascamp and E. H. Lieb, *On extensions of the Brunn-Minkowski and Prékopa-Leindler theorems, including inequalities for log concave functions, and with an application to the diffusion equation*, J. Funct. Anal. 22 (1976), 366–389. 1

[13] C. Bucur and M. Squassina, *Approximate convexity principles and applications to PDEs in convex domains*, Nonlinear Anal. 192 (2019), article ID 111661, pp. 21. 1, 5, 6, 11, 23

[14] L. Caffarelli, P. Guan and X.-N. Ma, *A constant rank theorem for solutions of fully nonlinear elliptic equations*, Comm. Pure Appl. Math. 60 (2007), 1769–1791. 3, 12

[15] R. S. Cantrell and C. Cosner, *Diffusive logistic equations with indefinite weights: population models in disrupted environments*, Proc. Roy. Soc. Edinburgh Sect. A 112 (1989), 293–318. 3, 19

[16] T. Cazenave, F. Dickstein and M. Escobedo, *A semilinear heat equation with concave-convex nonlinearity*, Rend. Mat. Appl. 19 (1999), 211–242. 19, 31, 32

[17] A. Chau and B. Weinkove, *Non-preservation of α-concavity for the porous medium equation*, Adv. Math. 440 (2024), article ID 109520, pp. 18. 2, 29

[18] C. Chen, *On the microscopic spacetime convexity principle for fully nonlinear parabolic equations I: Spacetime convex solutions*, Discrete Contin. Dyn. Syst. 34 (2014), 3383–3402. 3, 12

[19] C. Chen and B. W. Hu, *A microscopic convexity principle for spacetime convex solutions of fully nonlinear parabolic equations*, Acta Math. Sin. (Engl. Ser.) 29 (2012), 651–674. 3, 12

[20] H. Chen, P. Luo and G. Liu, *Global solution and blow-up of a semilinear heat equation with logarithmic nonlinearity*, J. Math. Anal. Appl. 422 (2015), 84–98. 19

[21] G. Ciraolo, M. Cozzi, M. Perugini and L. Pollastro, *A quantitative version of the Gidas-Ni-Nirenberg Theorem*, J. Func. Anal. 284 (2024), article ID 110585, pp. 29. 5, 6

[22] A. Colesanti and P. Salani, *Quasi–concave envelope of a function and convexity of level sets of solutions to elliptic equations*, Math. Nachr. 258 (2003), 3–15. 2

[23] D. Daners, *Existence and perturbation of principal eigenvalues for a periodic-parabolic problem*, Electron. J. Differ. Equ. Conf., Proceedings of Nonlinear Differential Equations (Coral Gables, FL, 1999), Conference 5 (2000), 51–67. 8

[24] P. Daskalopoulos, R. Hamilton and K. Lee, *All time C^∞-regularity of the interface in degenerate diffusion: a geometric approach*, Duke Math. J. 108 (2001), 295–327. 2

[25] F. Dickstein, *On semilinear parabolic problem with non-Lipschitz nonlinearities*, Mat. Contemp. 18 (2000), 111–121. 31

[26] S. Dipierro, J. G. da Silva, G. Poggesi and E. Valdinoci, *A quantitative Gidas-Ni-Nirenberg-type result for the p-Laplacian via integral identities* (2024), pp. 30, arXiv:2408.03522. 5

[27] H. Fujita and S. Watanabe, *On the uniqueness and non-uniqueness of solutions of initial value problems for some quasi-linear parabolic equations*, Comm. Pure Appl. Math. 21 (1968), 631–652. 8

[28] M. Gallo, S. Mosconi and M. Squassina, *Power law convergence and concavity for the Logarithmic Schrödinger equation* (2024), pp. 53, arXiv: 2411.01614. 3, 12, 13, 19

[29] M. Gallo and M. Squassina, *Concavity and perturbed concavity for p-Laplace equations* (2024), pp. 55, arXiv:2405.05404. 1–7, 9, 24
[30] M. Gatti, *Approximate radial symmetry for p-Laplace equations via the moving planes method* (2025), pp. 52, arXiv:2502.11759. 5
[31] T. Godoy and U. Kaufmann, *On principal eigenvalues for periodic parabolic problems with optimal condition on the weight function*, J. Math. Anal. Appl. 262 (2001), 208–220. 8
[32] A. Greco and B. Kawohl, *Log-concavity in some parabolic problems*, Electron. J. Differential Equations 1999 (1999), 1–12. 2, 3, 13
[33] Y. Han, *Blow-up at infinity of solutions to a semilinear heat equation with logarithmic nonlinearity*, J. Math. Anal. Appl. 474 (2019), 513–517. 19
[34] D. H. Hyers and S. M. Ulam, *Approximately convex functions*, Amer. Math. Soc. 3 (1952), 821–828. 12
[35] K. Ishige, *Eventual concavity properties of the heat flow*, Math. Ann. 390 (2008), 5883–5922. 2
[36] K. Ishige, Q. Liu and P. Salani, *Parabolic Minkowski convolutions and concavity properties of viscosity solutions to fully nonlinear equations*, J. Math. Pures Appl. (9) 141 (2020), 342–370. 4, 27, 32
[37] K. Ishige and P. Salani, *Is quasi-concavity preserved by heat flow?*, Arch. Math. 90 (2008), 450–460. 2
[38] K. Ishige and P. Salani, *Parabolic Minkowski convolutions of solutions to parabolic boundary value problems*, Adv. Math. 287 (2016), 640–673. 1, 4, 5, 27
[39] K. Ishige and P. Salani, *Parabolic power concavity and parabolic boundary value problems*, Math. Ann. 358 (2013), 1091–1117. 1, 4, 5, 11, 27, 30
[40] K. Ishige, P. Salani and A. Takatsu, *Characterization of F-concavity preserved by the Dirichlet heat flow*, Trans. Amer. Math. Soc. 377 (2024), 5705–5748. 2
[41] K. Ishige, P. Salani and A. Takatsu, *New characterizations of log-concavity via Dirichlet heat flow*, Ann. Mat. Pura Appl. (4) 201 (2022), 1531–1552. 2
[42] K. Ishige, P. Salani and A. Takatsu, *To logconcavity and beyond*, Commun. Contemp. Math. 22 (2020), article ID 1950009, pp. 17. 2
[43] R. Kajikiya, *Stability and instability of stationary solutions for sublinear parabolic equations*, J. Differential Equations 264 (2018), 786–834. 8, 9
[44] A. U. Kennington, *Convexity of level curves for an initial value problem*, J. Math. Anal. Appl. 133 (1988), 324–330. 4, 5, 7, 24, 27
[45] A. U. Kennington, *Power concavity and boundary value problems*, Indiana Univ. Math. J. 34 (1985), 687–704. 2, 4, 10, 11, 14, 15, 23, 34
[46] N. J. Korevaar and J. L. Lewis, *Convex solutions of certain elliptic equations have constant rank Hessians*, Arch. Ration. Mech. Anal. 97 (1987), 19–32. 13
[47] N. J. Korevaar, *Convex solutions to nonlinear elliptic and parabolic boundary value problems*, Indiana Univ. Math. J. 32 (1981), 603–614. 2, 3, 13, 17
[48] M. Laczkovich, *The local stability of convexity, affinity and of the Jensen equation*, Aequationes Math. 58 (1999), 135–142. 12
[49] R. Laister, J. Robinson and M. Sierżęga, *A necessary and sufficient condition for uniqueness of the trivial solution in semilinear parabolic equations*, J. Differential Equations 262 (2017), 4979–4987. 8
[50] K. Lee and J. Vázquez, *Geometrical properties of solutions of the porous medium equation for large times*, Indiana Univ. Math. J. 52 (2003), 991–1016. 2
[51] K. Lee and J. Vázquez, *Parabolic approach to nonlinear elliptic eigenvalue problems*, Adv. Math. 219 (2008), 2006–2028. 2, 29
[52] G. Lieberman, "Second Order Parabolic Differential Equations", World Scientific Publishing Company, Singapore, 1996, ISBN: 978-981-281-979-6. 29

[53] C.-S. Lin, *Uniqueness of least energy solutions to a semilinear elliptic equation in* $\mathbb{R}^2$, Manuscripta Math. 84 (1994), 13–19. 7

[54] L. G. Makar-Limanov, *Solution of Dirichlet's problem for the equation* $\Delta u = -1$ *in a convex region*, Mat. Zametki 9 (1971), 52–53. 1

[55] Y. Martel and P. Souplet, *Small time boundary behavior of solutions of parabolic equations with noncompatible data*, J. Math. Pures Appl. (9) 79 (2000), 603–632. 20

[56] S. Oruganti, J. Shi and R. Shivaji, *Diffusive logistic equation with constant yield harvesting, I: steady states*, Trans. Amer. Math. Soc. 354 (2002), 3601–3619. 3

[57] C. Pao, "Nonlinear Parabolic and Elliptic Equations", Plenum Press, Springer, New York, 1992, ISBN: 978-1-4615-3034-3. 32

[58] G. Porru and S. Serra, *Maximum principles for parabolic equations*, J. Aust. Math. Soc. 56 (1994), 41–52. 2, 13

[59] M. Protter and H. Weinberger, "Maximum Principles in Differential Equations", Springer-Verlag, New York, 2011, ISBN: 978-1-4612-5282-5. 33

[60] I. Singer, B. Wong, S.-T. Yau and S. S.-T. Yau, *An estimate of the gap of the first two eigenvalues in the Schrödinger operator*, Ann. Sc. Norm. Super. Pisa Cl. Sci. 4 12 (1985), 319–333. 2

[61] J. L. Vázquez, *A strong maximum principle for some quasilinear elliptic equations*, Appl. Math. Optim. 12 (1984), 191–202. 3

[62] X. Wang and Y. Wang, *Blowup for semilinear parabolic equation with logarithmic nonlinearity*, Discrete Contin. Dyn. Syst. Ser. S 17 (2024), 2629–2639. 19

(M. Gallo, M. Squassina)
Dipartimento di Matematica e Fisica
Università Cattolica del Sacro Cuore
Italy, Brescia, BS, Via della Garzetta 48, 25133
Email address: marco.gallo1@unicatt.it
Email address: marco.squassina@unicatt.it

(R. Moraschi)
Dipartimento di Matematica
Università degli Studi di Pavia
Italy, Pavia, PV, Via Ferrata 1, 27100
Email address: r.moraschi1@campus.unimib.it